Cognitive Systems Research 70 (2021) 93–100

Contents lists available at ScienceDirect

Cognitive Systems Research

journal homepage: www.elsevier.com/locate/cogsys

On the dynamics and adaptivity of mental processes: Relating adaptive dynamical systems and self-modeling network models by mathematical analysis

Jan Treur

Social AI Group, Department of Computer Science, Vrije Universiteit, Amsterdam, the Netherlands

ARTICLE INFO

Keywords:
Dynamical system
Adaptive
Network model
Self-modeling network

ABSTRACT

In this paper, it is addressed by mathematical analysis how network-oriented modeling relates to the dynamical systems perspective on mental processes. It has been mathematically proven that any dynamical system can be modeled as a temporal-causal network model and that any adaptive dynamical system (of any order) can be modeled by a self-modeling network (of the same order).

1. Introduction

The scope of applicability of a network-oriented modelling approach covers mental processes described by networks for the interplay of mental states, social (interaction) processes described by social network models, and more; e.g., (Treur, 2016a; Treur, 2016b; Treur, 2017). In fact, any scientific area in which networks of causal relations and causal pathways within them are used to describe theories, hypotheses and findings falls within the scope of applicability of such a network-oriented modeling approach. This covers practically all scientific domains, as causal explanation is used as a main vehicle almost everywhere in science. Besides, as a modeling paradigm it has a longstanding tradition in AI; e.g., (Kuipers, 1984; Kuipers and Kassirer, 1983; Pearl, 2000). Also, from the area of philosophy of mind the important role of causality in mental processes is emphasized; e.g., (Kim, 1996).

In (Port and Van Gelder, 1995; van Gelder, 1998; van Gelder and Port, 1995) it is argued that modeling realistic mental processes needs a dynamical system perspective where the temporal dimension is covered well; here a dynamical system is defined by the notion of a state-determined system as analysed by (Ashby, 1960). For similar positions emphasizing the role of dynamics to model mental processes, see, for example (Beer, 2000; Kelso, 1995; Scherer, 2009; Thelen and Smith, 1994). Given these positions, an important question becomes in how far network-oriented modeling is able to cover dynamics well. Moreover, also adaptivity of mental processes is an important issue in this, what sometimes is silently assumed to be covered by dynamics as well, but deserves its own treatment.

In the first place, the way in which the temporal dynamics of the impacts of network nodes (also called states) on other nodes are handled in (Treur, 2016a; Treur, 2016b) is important here, as for many applications detailed handling of such dynamics is required, for example, since usually there is no global synchrony of everything that happens in the world or in the mind. Secondly, in addition to such dynamics of impacts, in (Treur, 2020) adaptivity of network models is addressed as well. This provides good application possibilities for the many applications that concern learning or other forms of adaptation. However, these are just indications or expectations and some experiences; in addition to this, the question on how well dynamics is covered by network-oriented modeling can also be analysed mathematically. This is what is done in the current paper. Two theorems are presented that provide a positive answer to this question, one for nonadaptive dynamical systems in relation to network models and one for adaptive dynamical systems in relation to self-modeling network models.

More specifically, in this paper, in Section 5 it is shown mathematically that temporal-causal network models can model any dynamical system (Ashby, 1960), or, equivalently, any set of first-order equations. Next, in Section 6 it is shown how this also holds for any adaptive dynamical system or any adaptive set of first-order differential equations. It is shown in particular how any adaptive dynamical system can be modeled by a self-modeling network model. But first, some preparations are made. In Section 2 the notion of state-determined system from (Ashby, 1960; Port and Van Gelder, 1995) is discussed which is the basis for the notion of dynamical system used. In Section 3, it is shown that this notion is equivalent to a set of first-order differential equations.

E-mail address: j.treur@vu.nl.

https://doi.org/10.1016/j.cogsys.2021.07.003
Received 30 April 2021; Accepted 13 July 2021
Available online 20 July 2021

Section 4 briefly summarises the self-modeling network modeling approach. Finally, Section 7 is a discussion.

2. The State-Determined system assumption

The notion of *state-determined system*, adopted from Ashby (1960) is taken as the basis to describe what a dynamical system is in (van Gelder and Port, 1995, p. 6). That a system is state-determined means that its *current state always determines a unique future behaviour*. This property is reflected in modelling and simulation. Three features in particular are (van Gelder and Port, 1995):

- The future behaviour cannot depend on states the system might have been in *before* the current state: past history only can make a difference insofar as it has left an effect on the current state. This means that if you want to make a prediction on a next state, for example by simulation, only the information from the current state is needed, not from earlier states.
- That the current state determines future behaviour implies the existence of some *rule of evolution* describing the behaviour of the system as a function of its current state. The idea is that this rule can be specified in some reasonable succinct and useful fashion. The formats of differential equations (Ashby, 1960; Port and van Gelder, 1995) and of network models as described in (Treur, 2016a; Treur, 2016b) are examples of formats in which such rules of evolution can be expressed.
- That future behaviours are uniquely determined means that state space sequences can *never fork*. This means that when a next state is determined out of a current state, there is only one outcome.

From a wider philosophical perspective, this notion of state-determined system can be related to the notions temporal factorisation and criterial causation as discussed in (Treur, 2007a; Treur, 2007b; Treur, 2021; Tse, 2013).

The possibility of a choice of a proper set of state properties is the crucial factor to obtain a state-determined system that is practically usable. The validity of the assumptions underlying the Dynamical Systems Theory depends on the existence of such sets. For example, if to obtain a proper state-determined system to study some mental process, all states of the universe (including, for example the positions of all planets and stars and even the mental states of all other humans) are needed, then for practical purposes this perspective is useless. The truth is that, even for those who believe in science, for example, concerning Newton's gravitation laws, even for the application of such solid monumental laws, in principle all mass and positions from the universe have to be taken into account, which obviously is infeasible. But in practice, only mass that is not very far away is incorporated in a model, which then in fact provides an approximation. If such an approximation is accurate enough (objects very far away have some effect, but this is a very small effect), then still a useful outcome can be obtained. So, more in general, usually an additional kind of locality assumption is made that to model a specific process (and not the whole universe), a limited set of state variables can be found to get a state-determined system. In Ashby (1960), such a hypothesis is expressed as follows:

> 'Because of its importance, science searches persistently for the state-determined. As a working guide, the scientist has for some centuries followed the hypothesis that, given a set of variables, he can always find a larger set that (1) includes the given variables, and (2) is state-determined. Much research work consists of trying to identify such a larger set, for when it is too small, important variables will be left out of the account, and the behaviour of the set will be capricious. The assumption that such a larger set exists is implicit in almost all science, but, being fundamental, it is seldom mentioned explicitly.' (Ashby, 1960, p. 28).

In this paper it will be analyzed in some more depth in what formats in general such dynamical systems can be described adequately. It will turn out that one such format is by sets of first-order differential equations, and another adequate format is the temporal-causal network format described in (Treur, 2016a; Treur, 2016a). Similarly, adaptive dynamical systems and self-modeling networks (Treur, 2020a; Treur, 2020a) turn out both to be adequate formats for adaptive processes.

3. Dynamical systems and First-Order differential equations

Dynamical systems can be specified in mathematical formats; see (Ashby, 1960, pp. 241–252) for some details. In the first place a finite set of states (or state variables) $X_1, \ldots, X_n$ is assumed describing how the system changes over time via functions $X_1(t), \ldots, X_n(t)$ of time t. As discussed in Section 2, the criteria for a dynamical system can be formalized in a numerical manner by a relation (rule of evolution) that expresses that for each time point t the future value of each state X_i at time $t + s$ uniquely depends on s and on $X_1(t), \ldots, X_n(t)$ and hence can be described via some function $F_i(X_1, \ldots, X_n, s)$ in the following manner (see also Ashby, 1960, pp. 243–244):

$$X_i(t+s) = F_i(X_1(t), \cdots, X_n(t), s) \tag{1}$$

for $\quad s \geq 0$

Assuming continuous processes and smoothness (being differentiable) of the functions $X_i(t)$ and F_i, these relations can be reformulated (see Box 1) into a set of first-order differential equations of the form

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = f_i(X_1(t), \cdots, X_n(t)) \tag{2}$$

for some functions $f_i(X_1, \ldots, X_n)$; see also (Ashby, 1960), pp. 244–246.

Note that X_i may also occur in $f_i(X_1, \ldots, X_n)$. Conversely, such a set of first-order differential equations always describes a state-determined system; so for the smooth continuous numerical case, state-determined systems are the systems that can be described by sets of first-order differential equations (Ashby, 1960, p. 246).

4. Self-Modeling network modeling

According to the network-oriented modeling approach described in (Treur, 2016a) a network model is characterised by:

- **connectivity characteristics**
  Connections from a node (or state) X to a node Y and their *weights* $\boldsymbol{\omega}_{X,Y}$
- **aggregation characteristics**
  For any node Y, some *combination function* $\mathbf{c}_Y$**(..)** defines aggregation that is applied to the single impacts $\boldsymbol{\omega}_{X,Y}X(t)$ on Y through its incoming connections from states X
- **timing characteristics**
  Each node Y has a *speed factor* $\boldsymbol{\eta}_Y$ defining how fast it changes for given (aggregated) impact

The difference (or differential) equations that are useful for simulation purposes and also for analysis of network dynamics incorporate these network characteristics $\boldsymbol{\omega}_{X,Y}$, $\mathbf{c}_Y$**(..)**, $\boldsymbol{\eta}_Y$: it holds

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \cdots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t$$

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \cdots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)] \tag{3}$$

for any state Y and where $X_1, \cdots, X_k$ are the states from which it gets its incoming connections. The above concepts enable to design network models and their dynamics in a declarative manner, based on mathematically defined functions and relations.

To support the design of network models, for any application from a library predefined basic combination functions $\mathrm{bcf}_i(..)$, $i = 1, .., m$ are

Box 1
Why a smooth continuous state-determined system can be represented by a set of first-order differential equations.

Suppose a smooth continuous state-determined system is given. A sketch of why it can be described by a set of first-order differential equations is as follows. For any given time point t the future states $X_i(t+s)$ at some future time point time $t+s$ purely depend on s and the states $X_i(t)$ at t. This can be described by (smooth) mathematical functions $F_i(\ldots)$:

$$X_i(t+s) = F_i(X_1(t), \cdots, X_n(t), s) \text{ for } s \geq 0.$$

In the particular case of $s = 0$ it holds.

$$X_i(t) = F_i(X_1(t), \cdots, X_n(t), 0)$$

Subtracting these two expressions above and dividing by $s>0$ provides:

$$\frac{X_i(t+s) - X_i(t)}{s} = \frac{F_i(X_1(t), \cdots, X_n(t),\ s) - F_i(X_1(t), \cdots, X_n(t),\ 0)}{s}$$

When the limit for s very small, approaching 0 is taken, it follows that:

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = \left[\frac{\partial F_i(X_1(t), \cdots, X_n(t), s)}{\partial s}\right]_{s=0}$$

Now define the function $f_i(X_1, \ldots, X_n)$ by:

$$f_i(X_1, \cdots, X_n) = \left[\frac{\partial F_i(X_1(t), \cdots, X_n(t), s)}{\partial s}\right]_{s=0}$$

Then it holds.

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = f_i(X_1(t), \cdots, X_n(t))$$

This shows that the given state-determined system can be described by a set of first-order differential equations.

selected by assigning *combination function weights* $\boldsymbol{\gamma}_{i,Y}$, where the combination function then becomes the weighted average

$$\mathbf{c}_Y(.) = \frac{\boldsymbol{\gamma}_{1,Y}\mathrm{bcf}_1(..) + \ldots + \boldsymbol{\gamma}_{m,Y}\mathrm{bcf}_m(..)}{\boldsymbol{\gamma}_{1,Y} + \ldots + \boldsymbol{\gamma}_{m,Y}} \quad (4)$$

Furthermore, *combination function parameters* are specified, so that $\mathrm{bcf}_i(..) = \mathrm{bcf}_i(\mathbf{p},\mathbf{v})$ where $\mathbf{p}$ is a list of parameters (sometimes denoted by $\boldsymbol{\pi}_{i,j,Y}$) and $\mathbf{v}$ is a list of values.

A network model is fully defined by its network characteristics. A standardised (and computer-readable) form of specification in table format is used to specify the network characteristics of a model. This format is called the *role matrices* format; role matrices group the different types of characteristics:

- **connectivity characteristics role matrices**
  - o **mb** for base connectivity
  - o **mcw** for connection weights $\boldsymbol{\omega}_{X,Y}$
- **aggregation characteristics role matrices**
  - o **mcfw** for combination function weights $\boldsymbol{\gamma}_{i,Y}$
  - o **mcfp** for combination function parameters $\boldsymbol{\pi}_{i,j,Y}$
- **timing characteristics role matrix**
  - o **ms** for speed factors $\boldsymbol{\eta}_Y$

Examples of specifications of network models in this standard role matrices format will be shown in Sections 5.2 and 6.2.

Realistic network models are usually adaptive: their network characteristics often are adapted over time. Therefore, their dynamics is usually an interaction (sometimes called co-evolution) of these two sorts of dynamics: dynamics of the nodes (or states) in the network (dynamics *within* the network) versus dynamics of the characteristics of the network (dynamics *of* the network). By using *self-models* within the network, a network-oriented conceptualisation can also be applied to *adaptive* networks to obtain a declarative description using mathematically defined functions and relations; see (Treur, 2020a; Treur, 2020b). This works through the addition of new nodes to the network (called *self-model states* or *reification states*) which represent (adaptive) network characteristics. Such nodes are depicted at a next level (*self-model level*), where the original network is at a *base level*. These types of characteristics with their self-model states and their roles are shown in Table 1.

This provides an extended network, also called *self-modeling network*. Like for all network models, a self-modeling network model is specified in a (network-oriented) declarative mathematical manner based on nodes and connections. These include interlevel connections relating nodes at one level to nodes on the other. The outcome is also a network model; so, this whole construction can be applied iteratively to obtain multiple self-model levels that can provide higher-order adaptive networks, and is quite useful to model, for example, plasticity and metaplasticity in the form of a second-order adaptive network with three levels, one base level and a first- and a second-order self-model level; e.g., (Treur, 2020a) or (Treur, 2020b), Ch 4.

5. Relating dynamical systems to network models

Sets of first-order differential equations form a very general format used in computational modeling in many disciplines. For cognitive and neurological modeling in particular, often causal relationships are used in explaining mental processes. But also in many other domains, in a wide variety of scientific disciplines causal relationships play a crucial

Table 1
Different network characteristics and self-model states for them.

Types of characteristics	Concepts	Notations	Self-model states	Role played by the self-model state
Connectivity characteristics	Connections weights	$\boldsymbol{\omega}_{X,Y}$	$\mathbf{W}_{X,Y}$	Connection weight **W**
Aggregation characteristics	Combination functions weights and parameters	$\boldsymbol{\gamma}_{i,Y}$ $\boldsymbol{\pi}_{i,j,Y}$	$\mathbf{C}_{i,Y}$ $\mathbf{P}_{i,j,Y}$	Combination function weight **C** Combination function parameter **P**
Timing characteristics	Speed factors	$\boldsymbol{\eta}_Y$	$\mathbf{H}_Y$	Speed factor **H**

role. In this context it will be useful if it can be explained more explicitly how any state-determined system can be described or transformed into a format that more directly relates to causal relationships between states. This indeed can always be achieved in the temporal-causal network format (when arbitrary combination functions are allowed), in the manner shown below in general in Section 5.1 and illustrated by an example in Section 5.2.

5.1. Transforming a dynamical system model into a network model

In a dynamical system the changes in each state S depend on the other states. Those states R that actually play a role in this dependence relation form a subset D_S of the set of all states (in some special cases this subset may be the set of all states). The states R not in this subset D_S are those states for which never any state change of R has influence on a change of the state of S. The states in this subset D_S can be considered to cause the changes in the state S. Such causal effects of states on each other by causal relationships can be visualized in a graphical manner as, for example, in the network shown in Fig. 3. Such a network model can be simply defined by the following criterion: for any state R and any state S there is a connection from R to S if and only if $R \in D_S$. This provides a conceptual representation of the dynamical system as a causal graph.

Another transformation can be done for the numerical representation on the basis of a set of differential equations representing the dynamical system. Suppose a differential equation for one of the states X_i is given of the form:

$$\frac{dX_i(t)}{dt} = f_i(X_1(t), \cdots, X_n(t)) \tag{5}$$

Then this function $f_i(X_1(t), \ldots, X_n(t))$ will depend on a subset D_{Xi} of the set of states $\{X_1, \ldots, X_n\}$. Note that X_i itself may occur in D_{Xi}. Usually this function f_i will be given as a formula in $X_1, \ldots, X_n$; then this subset can be taken as the set of all states in $\{X_1, \ldots, X_n\}$ that actually occur in this formula. Again, for any two states X_j and X_i with $j \neq i$ a causal connection from X_j to X_i can be defined by the criterion that $X_j \in D_{Xi}$. Now, by defining the function $h_i(X_1, \ldots, X_n)$ by

$$h_i(X_1, \cdots, X_n) = X_i + f_i(X_1, \cdots, X_n) \tag{6}$$

differential equation (5) for X_i always can be rewritten into a differential equation of the form

$$\frac{dX_i(t)}{dt} = \boldsymbol{\eta}_i[\mathbf{c}_i(\boldsymbol{\omega}_{1,i}X_1(t), \cdots, \boldsymbol{\omega}_{n,i}X_n(t)) - X_i(t)] \tag{7}$$

where $\boldsymbol{\eta}_i = 1$, and also $\boldsymbol{\omega}_{j,i} = 1$, and for combination function $\mathbf{c}_i(\ldots)$ it holds $\mathbf{c}_i(\ldots) = h_i(\ldots)$. Note that X_i itself may occur in $\mathbf{c}_i(\ldots)$. So, having started with any arbitrary continuous, smooth dynamical system and its representation (5) in differential equation format, finally a numerical representation of a temporal-causal network model was obtained. This shows that any continuous smooth dynamical system can be described by a specific temporal-causal network model, as long as any type of combination function is allowed. So, the following theorem has been obtained:

Theorem 1 (from dynamical system to network model)

Any continuous smooth dynamical system model can be transformed into a temporal-causal network model. Conversely, any temporal-causal network model is a dynamical system model.

This Theorem 1 and the underlying transformation will be illustrated by an example in Section 5.2.

5.2. Illustration of the transformation for an example dynamical system

As an illustration, consider an arbitrary example of a dynamical system model described by numerical first-order differential equations representation with parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$:

$$\frac{dX_1(t)}{dt} = X_1(t)(X_5(t) - \boldsymbol{\alpha})$$

$$\frac{dX_2(t)}{dt} = X_1(t) - X_2(t) + X_3(t)$$

$$\frac{dX_3(t)}{dt} = X_2(t)(\boldsymbol{\beta} - X_3(t))$$

$$\frac{dX_4(t)}{dt} = X_3(t) - X_4(t)(\boldsymbol{\gamma} - X_5(t))$$

$$\frac{dX_5(t)}{dt} = X_5(t)(\boldsymbol{\delta} - X_4(t)) \tag{8}$$

To transform this dynamical system model (8) into a temporal-causal network model, the five states X_1, X_2, X_3, X_4, X_5 are considered. From each of the equations by inspecting which states occur in the right hand side it can subsequently be determined that:

X_5	affects	X_1
X_1 and X_3	affect	X_2
X_2	affects	X_3
X_3 and X_5	affect	X_4
X_4	affects	X_5

These connections are represented in a graphical network format as shown in Fig. 1.

Note that, when comparing, for example, the first differential equation in (8) to the standard format of differential equations for temporal-causal networks, it can be written as

$$\frac{dX_1(t)}{dt} = X_1(t)(X_5(t) - \boldsymbol{\alpha})$$

$$= X_1(t)(X_5(t) - \boldsymbol{\alpha}) + X_1(t) - X_1(t)$$

$$= [X_1(t)(X_5(t) - \boldsymbol{\alpha}) + X_1(t)] - X_1(t) \tag{9}$$

Here the part $(X_1(t)\ (X_5(t) - \boldsymbol{\alpha}) + X_1(t))$ can be considered the result of a combination function $\mathbf{c}_{X1}(..)$ defined by

$$\mathbf{c}_{X_1}(V_1, V_5) = V_1 + V_1(V_5 - \boldsymbol{\alpha}) \tag{10}$$

applied to $X_1(t)$ (for V_1) and $X_5(t)$ (for V_5). In a similar manner the following combination functions (built from sum and product functions) can be identified from the five differential equations (8):

$$\mathbf{c}_{X_1}(V_1, V_5) = V_1 + V_1(V_5 - \boldsymbol{\alpha}) = (1 - \boldsymbol{\alpha})V_1 + V_1V_5 \tag{11}$$

$$\mathbf{c}_{X_2}(V_1, V_3) = V_2 + V_1 - V_2 + V_3 = V_1 + V_3$$

$$\mathbf{c}_{X_3}(V_2, V_3) = V_3 + V_2(\boldsymbol{\beta} - V_3) = \boldsymbol{\beta}V_2 + V_3 - V_2V_3$$

$$\mathbf{c}_{X_4}(V_3, V_4, V_5) = V_4 + V_3 - V_4(\boldsymbol{\gamma} - V_5) = V_3 + (1 - \boldsymbol{\gamma})V_4 + V_4V_5$$

$$\mathbf{c}_{X_5}(V_4, V_5) = V_5 + V_5(\boldsymbol{\delta} - V_4) = (1 + \boldsymbol{\delta})V_5 - V_4V_5$$

Using these combination functions, the differential equations (8) can

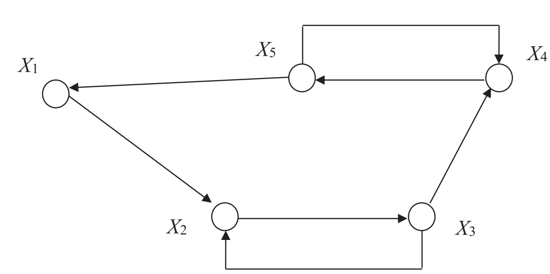

Fig. 1. Graphical network representation for the example model based on the given differential equation representation.

be rewritten into:

$$\frac{dX_1(t)}{dt} = \mathbf{c}_{X_1}(X_1(t), X_5(t)) - X_1(t)$$

$$\frac{dX_2(t)}{dt} = \mathbf{c}_{X_2}(X_1(t), X_3(t)) - X_1(t)$$

$$\frac{dX_3(t)}{dt} = \mathbf{c}_{X_3}(X_2(t), X_3(t)) - X_3(t)$$

$$\frac{dX_4(t)}{dt} = \mathbf{c}_{X_4}(X_3(t), X_4(t), X_5(t)) - X_4(t)$$

$$\frac{dX_5(t)}{dt} = \mathbf{c}_{X_5}(X_4(t), X_5(t)) - X_5(t) \tag{12}$$

This is the numerical representation (3) of a temporal-causal network model with $\boldsymbol{\eta}_{Xi} = 1$ for all i and $\boldsymbol{\omega}_{XiXj} = 1$ for all i and j. It turns out that the dynamical system model described by the differential equations representation (8) can be transformed in an exact manner into an instance of the numerical representation of the more general temporal-causal model described by the following differential equations:

$$\frac{dX_1(t)}{dt} = \boldsymbol{\eta}_{X_1}[\mathbf{c}_{X_1}((\boldsymbol{\omega}_{X_1,X_1}X_1(t), \boldsymbol{\omega}_{X_5,X_1}X_5(t)) - X_1(t)]$$

$$\frac{dX_2(t)}{dt} = \boldsymbol{\eta}_{X_2}[\mathbf{c}_{X_2}(\boldsymbol{\omega}_{X_1,X_2}X_1(t), \boldsymbol{\omega}_{X_3,X_2}X_3(t)) - X_2(t)]$$

$$\frac{dX_3(t)}{dt} = \boldsymbol{\eta}_{X_3}[\mathbf{c}_{X_3}(\boldsymbol{\omega}_{X_2,X_3}X_2(t), \boldsymbol{\omega}_{X_3,X_3}X_3(t)) - X_3(t)]$$

$$\frac{dX_4(t)}{dt} = \boldsymbol{\eta}_{X_4}[\mathbf{c}_{X_4}(\boldsymbol{\omega}_{X_3,X_4}X_3(t), \boldsymbol{\omega}_{X_4,X_4}X_4(t), \boldsymbol{\omega}_{X_5,X_4}X_5(t)) - X_4(t)$$

$$\frac{dX_5(t)}{dt} = \boldsymbol{\eta}_{X_5}[\mathbf{c}_{X_5}(\boldsymbol{\omega}_{X_4,X_5}X_4(t), \boldsymbol{\omega}_{X_5,X_5}X_5(t)) - X_5(t)] \tag{13}$$

Based on these network characteristics $\boldsymbol{\omega}_{X,Y}$, $\mathbf{c}_Y$, $\boldsymbol{\eta}_Y$ found from the given set of differential equations, the role matrix representation of the corresponding network model (with parameter values 1 chosen) can be obtained as shown in Fig. 2. Here the different rows in each matrix address the different states X_j of the network model. Recall that they group the different types of characteristics

Connectivity characteristics

- In role matrix **mb** at each row for the indicated state X_j it is specified from which other states it gets incoming connections. For example, in the second row it is indicated that state X_2 has incoming connections from state X_1 and X_3.
- In role matrix **mcw** at each row for the indicated state X_j the connection weight $\boldsymbol{\omega}_{Xi,Xj}$ is specified for each of the incoming connections for X_j as indicated in **mb**. For example, in the second row it is indicated that state X_2 has connection weights 1 for both incoming connections (as all other states have).

Aggregation characteristics

- In role matrix **mcfw** at each row for the indicated state X_j it is specified which combination function(s) are used and with what weight(s) $\boldsymbol{\gamma}_{i,Y}$. For example, in the fourth row it is indicated that state X_4 uses combination function $\mathbf{c}_{X4}(..)$ with weight 1.
- In role matrix **mcfp** at each row for the indicated state X_j the parameter values $\boldsymbol{\pi}_{i,j,Xi}$ of the combination function(s) are specified for each of the combination functions indicated in **mcfw**. For example, in the fourth row it is indicated that the combination function indicated in **mcfw** for state X_4 uses value 1 for parameter $\boldsymbol{\gamma}$.

Timing characteristics

- In role matrix **ms** at each row for the indicated state X_j it is specified which speed factor is used. It shows that for this network model all speed factors are 1.

6. Relating adaptive dynamical systems to self-modeling network models

In this section, it is shown how the approach described in Section 5 can be extended to obtain a transformation of an adaptive dynamical system into a self-modeling network model. In Section 6.1 the general approach is described and in Section 6.2 it is illustrated for an example adaptive dynamical system.

6.1. *Transforming an adaptive dynamical system model into a self-modeling network model*

Adaptive dynamical systems are usually modeled by two levels of dynamical systems (see Fig. 3) where the higher level dynamical system

mb base connectivity	1	2
X_1	X_5	
X_2	X_1	X_3
X_3	X_2	
X_4	X_3	X_5
X_5	X_4	

mcw connection weights	1	2
X_1	1	
X_2	1	1
X_3	1	
X_4	1	1
X_5	1	

ms speed factors	1
X_1	1
X_2	1
X_3	1
X_4	1
X_5	1

mcfw combination function weights	1	2	3	4	5
	$\mathbf{c}_{X_1}$	$\mathbf{c}_{X_2}$	$\mathbf{c}_{X_3}$	$\mathbf{c}_{X_4}$	$\mathbf{c}_{X_5}$
X_1	1				
X_2		1			
X_3			1		
X_4				1	
X_5					1

mcfp function parameter	$\mathbf{c}_{X_1}$	$\mathbf{c}_{X_2}$	$\mathbf{c}_{X_3}$	$\mathbf{c}_{X_4}$	$\mathbf{c}_{X_5}$
	α		β	γ	δ
X_1	1				
X_2					
X_3			1		
X_4				1	
X_5					1

Fig. 2. Role matrices specification of the example set of differential equations.

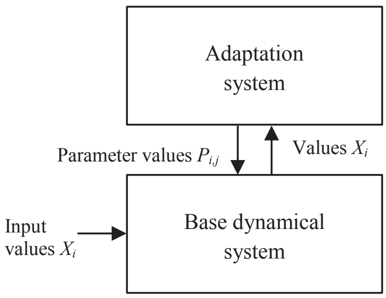

Fig. 3. Architecture of an adaptive dynamical system.

models the dynamics of the parameters $P_{i,j}$ of the lower level dynamical system (the lower level component in Fig. 3) that describes the dynamics of variables X_i, for example by

$$\frac{dX_i(t)}{dt} = f_i(P_{i,1}, \cdots, P_{i,k}, X_1(t), \cdots, X_n(t)) \quad (14)$$

These parameters $P_{i,j}$ then become time-dependent: $P_{i,j}(t)$. In addition, for the dynamics of the $P_{i,j}$ there will also be differential equations (the upper level component in Fig. 3):

$$\frac{dP_{i,j}(t)}{dt} = p_{i,j}(P_{1,1}(t), \cdots, P_{n,k}(t), X_1(t), \cdots, X_n(t)) \quad (15)$$

Applying the transformation described in Section 5 on the base dynamical system (14) for the X_i, these $P_{i,j}$ will also become parameters of the combination functions $h_i(\ldots)$ found for the base level states X_i:

$$h_i(P_{i,1}, \cdots, P_{i,k}, X_1, \cdots, X_n) = X_i + f_i(P_{i,1}, \cdots, P_{i,k}, X_1, \cdots, X_n) \quad (16)$$

Applying a similar transformation to the adaptation level described by (15), in this case define function $q_{i,j}(..)$ by:

$$q_{i,j}(P_{1,1}(t), \cdots, P_{n,k}(t), X_1(t), \cdots, X_n(t)) = P_{i,j}(t) + p_{i,j}(P_{1,1}(t), \cdots, P_{n,k}(t), X_1(t), \cdots, X_n(t)) \quad (17)$$

Then

$$\frac{dP_{i,j}(t)}{dt} = \eta_{i,j}[\mathbf{c}_{i,j}(\omega_{1,1,i,j}P_{1,1}(t), \cdots, \omega_{n,k,i,j}P_{n,k}(t), \omega_{1,i,j}X_1(t), \cdots, \omega_{n,i,j}X_n(t)) - P_{i,j}(t)] \quad (18)$$

where again $\boldsymbol{\eta}_{i,j} = 1$ and all connection weights $\boldsymbol{\omega}$ involved are 1 too and for the combination function $\mathbf{c}_{i,j}(\ldots)$, it simply holds $\mathbf{c}_{i,j}(\ldots) = q_{i,j}(\ldots)$. By this, the states $P_{i,j}$ can be modeled at the first-order self-model level as self-model states for the base level. In a similar manner it can be shown by iteration that any higher-order adaptive dynamical system can be modelled as a higher-order self-modeling network model. Thus the following theorem is obtained:

Theorem 2 (from adaptive dynamical system to self-modeling network model)

Any adaptive continuous smooth dynamical system model can be transformed into a self-modeling temporal-causal network model. Conversely, any self-modeling temporal-causal network model is an adaptive dynamical system model. These also apply to higher-order adaptive dynamical systems in relation to higher-order self-modeling networks.

This Theorem 2 and the underlying transformation will be illustrated by an example in Section 6.2.

6.2. Illustration of the transformation for an example adaptive dynamical system

The transformation described in Section 6.1 can be applied to an adaptive extension of the example described in Section 5.2 by making the four parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ adaptive. To this end, for each of these parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ a differential equation is assumed for the $P_{i,j}$ states used, where:

$$P_{1,1} = \boldsymbol{\alpha} \quad P_{3,1} = \boldsymbol{\beta} \quad P_{4,1} = \boldsymbol{\gamma} \quad P_{5,1} = \boldsymbol{\delta} \quad (19)$$

Extending the example, suppose these are the equations for them:

$$\frac{dP_{1,1}(t)}{dt} = X_1(t) + X_5(t)$$

$$\frac{dP_{3,1}(t)}{dt} = X_2(t) + X_3(t)$$

$$\frac{dP_{4,1}(t)}{dt} = X_3(t) + X_4(t)$$

$$\frac{dP_{5,1}(t)}{dt} = X_4(t) + X_5(t) \quad (20)$$

Now rename them as self-model states X_6, X_7, X_8, X_9, respectively:

$$X_6 = P_{1,1} = \boldsymbol{\alpha} \quad X_7 = P_{3,1} = \boldsymbol{\beta} \quad X_8 = P_{4,1} = \boldsymbol{\gamma} \quad X_9 = P_{5,1} = \boldsymbol{\delta} \quad (21)$$

Then the above equations (20) for the parameters (now denoted by additional network self-model states X_6 to X_9) can be rewritten (as in Section 6.1) in the standard network format using combination functions named $\mathbf{c}_{X6}$, $\mathbf{c}_{X7}$, $\mathbf{c}_{X8}$, $\mathbf{c}_{X9}$, respectively, leading to the equations

$$\frac{dX_6(t)}{dt} = \mathbf{c}_{X_6}(X_1(t), X_5(t), X_6(t)) - X_6(t)$$

$$\frac{dX_7(t)}{dt} = \mathbf{c}_{X_7}(X_2(t), X_3(t), X_7(t)) - X_7(t)$$

$$\frac{dX_8(t)}{dt} = \mathbf{c}_{X_8}(X_3(t), X_4(t), X_8(t)) - X_8(t)$$

$$\frac{dX_9(t)}{dt} = \mathbf{c}_{X_9}(X_4(t), X_5(t), X_9(t)) - X_9(t) \quad (22)$$

where the new combination functions $\mathbf{c}_{X6}(..)$ to $\mathbf{c}_{X9}(..)$ are defined by

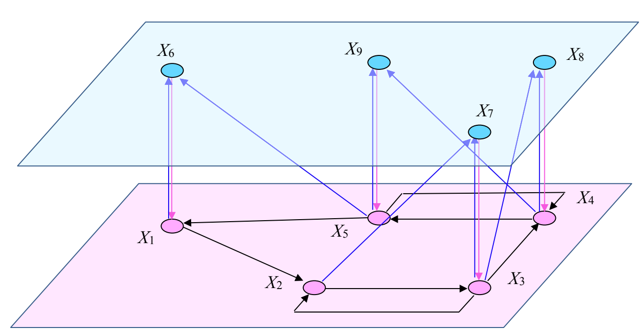

Fig. 4. Graphical representation of the self-modeling network model based on the given differential equation representation of an adaptive dynamical system.

mb base connectivity	1	2
X_1	X_5	
X_2	X_1	X_3
X_3	X_2	
X_4	X_3	X_5
X_5	X_4	
X_6	X_1	X_5
X_7	X_3	X_2
X_8	X_4	X_3
X_9	X_5	X_4

mcw connection weights	1	2
X_1	1	
X_2	1	1
X_3	1	
X_4	1	1
X_5	1	
X_6	1	1
X_7	1	1
X_8	1	1
X_9	1	1

ms speed factors	1
X_1	1
X_2	1
X_3	1
X_4	1
X_5	1
X_6	1
X_7	1
X_8	1
X_9	1

mcfw combination function weights	1	2	3	4	5	6	7	8	9
	$\mathbf{c}_{X_1}$	$\mathbf{c}_{X_2}$	$\mathbf{c}_{X_3}$	$\mathbf{c}_{X_4}$	$\mathbf{c}_{X_5}$	$\mathbf{c}_{X_6}$	$\mathbf{c}_{X_7}$	$\mathbf{c}_{X_8}$	$\mathbf{c}_{X_9}$
X_1	1								
X_2		1							
X_3			1						
X_4				1					
X_5					1				
X_6						1			
X_7							1		
X_8								1	
X_9									1

mcfp function parameter	$\mathbf{c}_{X_1}$	$\mathbf{c}_{X_2}$	$\mathbf{c}_{X_3}$	$\mathbf{c}_{X_4}$	$\mathbf{c}_{X_5}$
	α		β	γ	δ
X_1	X_6				
X_2					
X_3			X_7		
X_4				X_8	
X_5					X_9
X_6					
X_7					
X_8					
X_9					

Fig. 5. Role matrices specification for the self-modeling network model based on the example adaptive dynamical system.

$$\mathbf{c}_{X_6}(V_1, V_5, V_6) = V_1 + V_5 + V_6$$

$$\mathbf{c}_{X_7}(V_2, V_3, V_7) = V_2 + V_3 + V_7$$

$$\mathbf{c}_{X_8}(V_3, V_4, V_8) = V_3 + V_4 + V_8$$

$$\mathbf{c}_{X_9}(V_4, V_5, V_9) = V_4 + V_5 + V_9 \quad (23)$$

Here V_i are variables used for $X_i(t)$. Note that these four functions $\mathbf{c}_{X6}$, $\mathbf{c}_{X7}$, $\mathbf{c}_{X8}$, $\mathbf{c}_{X9}$ are actually the same function, namely the sum function; but for generality of the illustration they will be named by their different names. This then will get the form of the example picture as shown in Fig. 4 and the role matrices in Fig. 5.

7. Discussion

In this paper it was addressed how network-oriented modeling relates to the dynamical systems perspective on mental processes as described by (Ashby, 1960; Beer, 2000; Kelso, 1995; Port and Van Gelder, 1995; Scherer, 2009; Thelen and Smith, 1994; van Gelder, 1998; van Gelder and Port, 1995). It has been mathematically proven that any dynamical system can be modeled as a temporal-causal network model according to the approach described in (Treur, 2016a; Treur, 2016b) and that any adaptive dynamical system (of any order) can be modeled by a self-modeling network (of the same order) as described in (Treur, 2020a; Treur, 2020b). Note that the approach presented here gives exact mathematical transformations, not approximations such as, for example described in (Funahashi and Nakamura, 1993).

In a wider philosophical context, the dynamical systems perspective has its foundation in the notion of state-determined system (Ashby, 1960), and relates to the role the notions of temporal factorisation and criterial causation play in adaptive dynamical systems (Treur, 2007a; Treur, 2007b; Treur, 2021; Tse, 2013) and to the earlier ideas on dynamics introduced by Descartes (1634) and Laplace (1825). What the current paper shows is that from a mathematical analysis perspective, network-oriented modeling provide adequates means to model such dynamics and adaptivity.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Ashby, W.R.: Design for a Brain. Chapman and Hall, London (second extended edition). First edition, 1952 (1960).
Beer, R. D. (2000). Dynamical approaches to cognitive science. *Trends in Cognitive Sciences, 4*(3), 91–99.
Descartes, R. (1634) The World Ch 6: Description of a New World and on the Qualities of the Matter of Which it is Composed.
Funahashi, K.-I., & Nakamura, Y. (1993). Approximation of dynamical systems by continuous time recurrent neural networks. *Neural Networks, 6*(6), 801–806.
Kelso, J. A. S. (1995). *Dynamic patterns: The self-organization of brain and behavior.* Cambridge: MIT Press.
Kim, J. (1996). *Philosophy of Mind.* Westview Press.
Kuipers, B. J. (1984). Commonsense reasoning about causality: Deriving behavior from structure. *Artificial Intelligence, 24*(1984), 169–203.
Kuipers, B.J. and Kassirer, J.P., (1983). How to discover a knowledge representation for causal reasoning by studying an expert physician. In: Proceedings Eighth International Joint Conference on Artificial Intelligence, IJCAI'83, Karlsruhe, F.R.G. (William Kaufman, Los Altos, CA, 1983).
Laplace, P.S. (1825) Philosophical Essays on Probabilities Springer-Verlag New York 1995 Translated by AI Dale from the 5th French edition of 1825.
Pearl, J. (2000). *2000.* Causality: Cambridge University Press.
Port, R. F., & van Gelder, T. (1995). *Mind as motion: Explorations in the dynamics of cognition.* Cambridge, MA: MIT Press.
Scherer, K. R. (2009). Emotions are emergent processes: They require a dynamic computational architecture. *Phil. Trans. R. Soc. B, 364*(1535), 3459–3474.
Thelen, E., & Smith, L. (1994). *A dynamic systems approach to the development of cognition and action.* Cambridge: MIT Press.
Treur, J. (2007a). Temporal Factorisation: A Unifying Principle for Dynamics of the World and of Mental States. *Cognitive Systems Research, 8*(2), 57–74.
Treur J (2007b) Temporal Factorisation: Realisation of mediating state properties for dynamics Cognitive Systems Research 8 (2) (2007): 75-88.
Treur, J. (2016a). Dynamic Modeling Based on a Temporal-Causal Network Modeling Approach. Biologically Inspired Cognitive Architectures, 16, 131–168 (2016).
Treur, J. (2016). *Network-Oriented Modeling: Addressing Complexity of Cognitive, Affective and Social Interactions.* Springer Publishers.
Treur, J. (2017). On the applicability of network-oriented modeling based on temporal-causal networks: Why network models do not just model networks. *J. Inf. Telecommun., 1*, 23–40.

Treur, J. (2020). Modeling Higher-Order Adaptivity of a Network by Multilevel Network Reification. *Network Science, 8*(S1), S110–S144.
Treur, J. (2020). *Network-oriented modeling for adaptive networks: Designing higher-order adaptive biological, mental and social network models*. Springer Nature Publishers.
Treur, J. (2021). Modeling the Emergence of Informational Content by Adaptive Networks for Temporal Factorisation and Criterial Causation. *Cognitive Research Journal, 68*, 34–52.
Tse, P. U. (2013). *The Neural Basis of Free Will: Criterial Causation*. Cambridge: MIT Press.
van Gelder, T. (1998). The dynamical hypothesis in cognitive science. *Behavioral and Brain Sciences, 21*(5), 615–628.
Van Gelder, T. and Port, R.F. (1995). It's about time: An overview of the dynamical approach to cognition. In: Port, R.F., & van Gelder, T. (1995). Mind as motion: Explorations in the dynamics of cognition. Cambridge, MA: MIT Press, pp. 1-43.