Notes on Rotation and Rigid Bodies in Relativity Theory

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Covariant conditions for rigid-body motion are set up. They are equivalent to those proposed by Born and lead to the linear speed-distance law for the case of rotation about an axis. This result and also a discussion of the transformation equations in going over to a rotating frame of reference are used as arguments for the desirability of retaining the concept of rigid rotation with the linear speed-distance law, contrary to the opinion expressed by Hill. The rotation question is also considered from the standpoint of angular velocity, and one is lead rather naturally to two fluid velocity distributions, one of which was found by Hill. The expression for the spatial distance between two points on a rotating disk obtained by Berenda is derived without the use of the assumption introduced by the latter.

THE question of whether the geometry on a rotating disk is Euclidean or not has interested many writers. Berenda[1] has subjected the problem to a careful analysis. However, there are still a number of questions connected with the problem of rotation in relativity theory that require further investigation, and some of them will be considered in the present work.

1. MOTION OF A RIGID BODY

In discussing the rotating disk, one frequently begins with an inertial frame of reference described by a cylindrical polar coordinate system with coordinates r_0, θ_0, z_0, t_0, and one then goes over to a rotating frame of reference attached to the rotating body, with coordinates r, θ, z, t, given by

$$r_0=r, \quad \theta_0=\theta+\omega t, \quad z_0=z, \quad t_0=t, \tag{1}$$

where ω, the angular velocity about the z axis, is constant. Starting with the usual expression for the line element in the non-rotating coordinate system, one gets in the rotating system:

$$ds^2=-(dr^2+r^2d\theta^2+dz^2+2\omega r^2d\theta dt)+(1-\omega^2r^2)dt^2, \tag{2}$$

with units in which the velocity of light in empty space is unity.

Hill[2] pointed out (as von Laue[3] had done earlier) that a body rotating in this way would have a limit to the radius it could have, for the velocity would vary linearly with the radius and would exceed that of light for $r>1/\omega$. For this reason, Hill was led to conclude that the speed-distance law must be non-linear.

While one can conceive of a fluid rotating with an arbitrary speed-distance law, the case of greatest interest is that of the speed-distance law for a rigid body. To investigate this, one must first have some criterion for rigid-body motion. Such a criterion was first given in relativity theory by Born.[4]

In classical physics one can characterize the motion of a rigid body by the fact that the rate of strain vanishes. In a Cartesian coordinate system, the velocity components satisfy the relation

$$\partial v_i/\partial x_k+\partial v_k/\partial x_i=0 \quad (i, k=1, 2, 3). \tag{3}$$

In a relativistic treatment one looks for a covariant equation which reduces to (3) in a Galilean system if the velocity is small (compared to that of light). The obvious generalization is to introduce, in an arbitrary coordinate system, the symmetrical tensor

$$P_{\mu\nu}=\tfrac{1}{2}(u_{\mu;\nu}+u_{\nu;\mu}), \tag{4}$$

where the velocity 4-vector u^λ is given by

$$u^\lambda=dx^\lambda/ds \tag{5}$$

and a semi-colon denotes covariant differentiation, and to take as the condition for rigid-body

[1] C. W. Berenda, Phys. Rev. **62**, 280 (1942). Other references will be found in this paper.
[2] E. L. Hill, Phys. Rev. **69**, 488 (1946).
[3] M. von Laue, *Relativitätstheorie* (Vieweg, Braunschweig, 1921), Vol. 2, p. 24.
[4] M. Born, Ann. d. Physik **30**, 1 (1909). See also W. Pauli, Jr., *Encyklopädie der math. Wissenschaften*, Vol. 5, Part 2, pp. 689–691.

motion

$$P_{\mu\nu}=0. \tag{6}$$

However, this condition represents too severe a restriction. In view of the fact that the vector u^λ satisfies the identity

$$u^\lambda u_\lambda=1, \tag{7}$$

it follows from (6) on multiplication by u^ν that

$$u_{\mu;\nu}u^\nu=0. \tag{8}$$

This expression is just the covariant form of the acceleration vector

$$\delta u^\lambda/ds=du^\lambda/ds+\{{}_\alpha{}^\lambda{}_\beta\}u^\alpha u^\beta, \tag{9}$$

so that every particle of the body moves along a geodesic, or in gravitation-free space, along a straight line.

It is therefore necessary to weaken the condition imposed on the motion. For this purpose we replace $p_{\mu\nu}$ by

$$p_{\mu\nu}=\tfrac{1}{2}(u_{\mu;\nu}+u_{\nu;\mu}-u_{\mu;\alpha}u^\alpha u_\nu-u_{\nu;\alpha}u^\alpha u_\mu), \tag{10}$$

since we then have the identity

$$p_{\mu\nu}u^\nu\equiv 0. \tag{11}$$

Let us now take as the condition for rigid-body motion

$$p_{\mu\nu}=0. \tag{12}$$

In a Galilean coordinate system, at a point where the velocity 3-vector vanishes, this reduces to the classical condition (3). Therefore it is equivalent to the condition proposed by Born.[4]

Since Eq. (12) is a tensor equation it can be applied in any frame of reference. If we take an inertial system with Cartesian coordinates then we can describe rotation about an axis by setting

$$u_1=-u^1=\sigma y,\quad u_2=-u^2=-\sigma x,$$
$$u_3=-u^3=0,\quad u_4=u^4=(1+\sigma^2r^2)^{\frac{1}{2}}, \tag{13}$$

with

$$\sigma=\sigma(r),\quad r^2=x^2+y^2. \tag{14}$$

From Eq. (12) one gets the single equation

$$d\sigma/dr=\sigma^3 r \tag{15}$$

which has for its solution

$$\sigma=\omega/(1-\omega^2r^2)^{\frac{1}{2}}, \tag{16}$$

where ω is a constant. Going over to the three-dimensional velocity v by the relations

$$u=(u_1^2+u_2^2)^{\frac{1}{2}}=\sigma r=v/(1-v^2)^{\frac{1}{2}}, \tag{17}$$

we get

$$v=\omega r. \tag{18}$$

This result was obtained by Herglotz[5] by means of a different procedure.

That it is possible to have a covariant law of rigid-body motion (even if it has only a small number of solutions) appears to be of importance in connection with the inner consistency of the theory of relativity. Since the fundamental concepts of the theory include those of the rigid frame of reference and the rigid measuring rod, it would be unsatisfactory if no rigid bodies or rigid-body motions existed in the theory.

In 1911, von Laue[6] presented an argument of a general nature against the existence of a rigid body in relativity theory. It is based on the fact that a rigid body is expected to have a finite number of degrees of freedom, while, on the other hand, one can set up N disturbances near N separated points of the body, and they will be non-interacting, i.e., independent, for a sufficiently short interval of time (because of the finite propagation time required by relativity theory) so that there are at least N degrees of freedom, where N can be increased indefinitely for a continuous medium. However, this argument assumes the possibility of setting up arbitrary deformations in the body initially. Since however the initial deformations can only be obtained by displacements with velocity components governed by Eq. (12) it follows that it is not possible to have arbitrary initial disturbances, but only those which can be arrived at by means of motions described by Eq. (12), i.e., displacements which can be obtained by integrating this equation, starting from an undeformed state.

It might be remarked here that the difficulty pointed out by Hill does not appear to provide a sufficiently strong argument for giving up the linear speed-distance law. It is true that no rigid body would have a radius equal to, or exceeding, $1/\omega$, but there seems to be no reason why an

[5] G. Herglotz, Ann. d. Physik **31**, 393 (1910).

[6] M. von Laue, *Relativitätstheorie* (1921), fourth edition, Vol. 1, pp. 203–204.

idealized rigid body with a smaller radius could not rotate according to the linear law. That it is desirable to retain the concept of the rigid rotation can be seen from the following considerations:

Let us go back to Eqs. (1) and (2), and let us look for the most general transformation that can be introduced in place of Eq. (1) which (a) will be linear in t and θ, (b) will have axial and cylindrical symmetry, (c) will be single-valued, and (d) will give a static line element. The conditions (a), (b), and (c) lead to the transformation equations

$$r_0=e(r)+f(r)t,\quad \theta_0=\theta+g(r)t+h(r),\quad z_0=z,\quad t_0=p(r)t+q(r). \tag{19}$$

Applying the condition (d), one finds that

$$f=g'=p'=0, \tag{20}$$

where a prime denotes a derivative with respect to r. What is of interest to us here is that unless $g'=0$ ($\theta-\theta_0$ a linear function of t with a constant coefficient), it will not be possible to have a static metric. This condition is, of course, just the linear relation between speed and distance from the axis.

2. ANGULAR VELOCITY IN RELATIVITY THEORY

Although it appears that the motion of a rigid body is best discussed by means of the criterion of the previous section, it is nevertheless interesting to examine the rotation problem by a consideration of the angular velocity distribution. This is the general method that was used by Hill.[2]

In classical physics the angular velocity is defined as the vector

$$\boldsymbol{\omega}=\tfrac{1}{2}\nabla\times\mathbf{v}. \tag{21}$$

The obvious relativistic generalization consists in introducing an antisymmetric tensor

$$\Omega_{\mu\nu}=\tfrac{1}{2}(\partial u_\mu/\partial x^\nu-\partial u_\nu/\partial x^\mu) \tag{22}$$

in an arbitrary four-dimensional coordinate system.

In an inertial system with Cartesian coordinates, if we have rotation with the components u^λ given by Eqs. (13) and (14), the following components of $\Omega_{\mu\nu}$ are obtained:

$$\Omega_{12}=\sigma+\tfrac{1}{2}\sigma' r, \tag{23}$$

$$\Omega_{14}=-Fx,\quad \Omega_{24}=-Fy, \tag{24}$$

with

$$F=\tfrac{1}{2}(\sigma^2+\sigma\sigma' r)/(1+\sigma^2r^2)^{\frac{1}{2}}. \tag{25}$$

If we now set

$$\Omega_{12}=\omega=\text{constant}, \tag{26}$$

then from (23) we find for the singularity-free solution[7]

$$\sigma=\omega. \tag{27}$$

This gives

$$u=\omega r,\quad v=\omega r/(1+\omega^2r^2)^{\frac{1}{2}}. \tag{28}$$

The components ω_{14} and ω_{24} are given by Eq. (24), with

$$F=\tfrac{1}{2}\omega^2/(1+\omega^2r^2)^{\frac{1}{2}}, \tag{29}$$

so that, for small speeds, they are proportional to the centripetal acceleration components.

Now, it is not satisfactory for the fluid motion to be determined by Eq. (26) since the latter is a condition on only one component and is not covariant. One can improve matters by introducing

$$S^\lambda=\Omega^{\lambda\tau}u_\tau=u^\lambda{}_{;\tau}u^\tau=\delta u^\lambda/ds, \tag{30}$$

the acceleration vector. In the present case one finds

$$S^1=-\tfrac{1}{2}\sigma^2x,\quad S^2=-\tfrac{1}{2}\sigma^2y,\quad S^3=S^4=0, \tag{31}$$

so that the condition

$$S^\alpha{}_{;\alpha}=0 \tag{32}$$

would give the same solution as above. Another possibility, not as simple, is to take as the condition for the motion,

$$\Box S^\lambda=0. \tag{33}$$

If, instead of the above coordinate system, we follow Hill[2] in introducing a Galilean system (coordinates x', y', z', t') with a given point as origin and moving so that the fluid there is momentarily at rest with respect to it ($u_1'=u_2'=u_3'=0$, $u_4'=1$), we get for the angular velocity components

$$\Omega_{12}'=(\sigma+\tfrac{1}{2}\sigma' r+\tfrac{1}{2}\sigma^3r^2)/(1+\sigma^2r^2)^{\frac{1}{2}}, \tag{34}$$

$$\Omega_{14}'=\tfrac{1}{2}\sigma^2 r, \tag{35}$$

[7] Nathan Rosen, Phys. Rev. 70, 93 (1946).

provided we take the X' axis in the radial direction and the Z' axis parallel to the Z axis. If one sets

$$\Omega_{12}'=\omega \tag{36}$$

and introduces a new variable

$$\xi=v/r, \tag{37}$$

so that

$$\sigma=\xi/(1-\xi^2r^2)^{\frac{1}{2}}, \tag{38}$$

one gets

$$\xi' r+2\xi=2\omega(1-\xi^2r^2), \tag{39}$$

which is essentially the equation obtained by Hill.

It is possible to put the condition expressed by Eq. (36) into covariant form. For this purpose let us introduce what might be called the "spatial angular velocity tensor"

$$\omega_{\mu\nu}=\Omega_{\mu\nu}-\Omega_{\mu\alpha}u^\alpha u_\nu+\Omega_{\nu\alpha}u^\alpha u_\mu, \tag{40}$$

which satisfies the identity

$$\omega_{\mu\nu}u^\nu\equiv 0. \tag{41}$$

It might be noted that the vanishing of $\Omega_{\mu\nu}$ leads to the absence of all acceleration; that is not the case with $\omega_{\mu\nu}$. Corresponding to Eqs. (34) and (35) one finds as the only non-vanishing components

$$\omega_{12}'=-\omega_{21}'=\Omega_{12}'. \tag{42}$$

Hence, an invariant condition corresponding to Eq. (36) can be written[8]

$$\omega_{\mu\nu}\omega^{\mu\nu}=2\omega^2. \tag{43}$$

We see that, depending on whether one works in the fixed coordinate system or the system instantaneously moving with the fluid, one arrives naturally at one or the other of two velocity distributions, differing from that for the rigid body as discussed in the previous section. Other distributions are also possible, of course. However, at this time there does not appear to be any strong reason for giving up the linear speed-distance law in favor of any other distribution.

[8] In reference 7 the author questioned Hill's procedure because it made use of a uniformly moving coordinate system in describing the accelerated motion of the fluid, so that the acceleration components of the fluid were not taken into account. However, we see now that this is not an objection if the description is given in terms of $\omega_{\mu\nu}$, since the acceleration components in this case are not involved.

3. SPATIAL DISTANCE IN A ROTATING SYSTEM

Returning now to the rigid-body rotation as discussed in Section 1, let us consider a question treated by Berenda in his investigation of the nature of the geometry on the rotating disk. The question was this: given a frame of reference with the line element of Eq. (2), what is the correct expression for the distance dl between two neighboring points at rest in this reference frame? Berenda[1] obtained this distance as the length of a four-vector which had for its space components the three spatial coordinate differentials, but also had for its fourth component a value chosen to make this vector orthogonal to the world-lines of the two points. The identification of "rigid metric rod readings on the disk with spatial dimensions and geometry derived from such orthogonality" Berenda considered a matter of assumption. However since the expression for dl^2 derived from this assumption determines the spatial geometry of the disk, it seems desirable to derive it by another procedure free from such an assumption.

It is well known that in the general relativity theory the coordinates used in locating events in space-time need have no simple relations to the readings of measuring rods and clocks. What requires stressing here is the fact that even the value of the interval or line element need have no simple relation to the readings of scales and clocks in a given frame of reference. This is associated with the fact that the readings of scales and clocks will be influenced by gravitational and inertial forces. Only in a Galilean frame of reference can intervals be obtained simply from clock and scale readings, according to the ideas of special relativity. Hence, considering the two points of the body in the neighborhood of some particular values of r, θ, z, t, let us introduce a local Galilean coordinate system momentarily at rest relative to the rotating system by the relations

$$dx'=dr,\quad dy'=Ad\theta,\quad dz'=dz,\quad dt'=Bdt+Cd\theta, \tag{44}$$

where A, B, and C are to be determined so as to make

$$ds^2=-dx'^2-dy'^2-dz'^2+dt'^2. \tag{45}$$

One readily finds that

$$A=r/(1-\omega^2r^2)^{\frac{1}{2}},\quad B=(1-\omega^2r^2)^{\frac{1}{2}},\quad C=-\omega r^2/(1-\omega^2r^2)^{\frac{1}{2}}. \tag{46}$$

This is essentially the same system as the one used by Hill.[2] From the form of Eq. (45) it follows that

$$dl^2=dx'^2+dy'^2+dz'^2, \tag{47}$$

or, on the basis of the transformation equations,

$$dl^2=dr^2+r^2(1-\omega^2r^2)^{-1}d\theta^2+dz^2, \tag{48}$$

which is the result obtained by Berenda and shows that the spatial geometry on the surface of a rotating disk (z = constant) is non-Euclidean.

It is also interesting to note that the time interval between two events dt' is given by

$$dt'=(1-\omega^2r^2)^{\frac{1}{2}}dt-\omega r^2(1-\omega^2r^2)^{-\frac{1}{2}}d\theta. \tag{49}$$

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Note on Magnetic Energy

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1.

IN a recent paper[1] under this same title E. A. Guggenheim discusses certain results from a paper[2] published by me recently and claims that they are special cases of more general ones obtained by him on a previous occasion.[3] As the results quoted from my paper are themselves special cases of quite general ones contained therein, and his note generally misrepresents the scope and purpose of my work, perhaps I may be permitted a reiteration of the main outlines of my arguments and the opportunity to discuss the relation they bear to the methods employed by him.

The following symbols are used:

- B magnetic induction,
- H magnetic force intensity,
- I_i induced magnetization intensity,
- I_p permanent magnetization intensity,
- i_s linear electric current,
- N_s magnetic flux through i_s,
- μ permeability,
- $\mathcal{L}$ Lagrangian function,
- $\mathcal{H}$ Hamiltonian function.

Guggenheim bases his discussion not on Maxwell's theory in its original form but on the modification given to it by Cohn.[4] The essence of this form of the theory, like that proposed by Hertz which it follows closely, is that it incorporates the induced polarizations and the aether, whatever this may be, into a single transmitting medium whose elastic quality is summed up in the characteristic constant, the permeability μ. This hypothesis of a single medium, excluding as it does the possibility of a displacement of the polarized medium from one position of the field to another, proves however to be a fatal handicap in a theory which has eventually to be extended to cover electromagnetic phenomena in moving media. And it was precisely for this reason that Larmor and Lorentz were forced back to the views held by Kelvin and Maxwell that the only really satisfactory treatment of these affairs interprets them in terms of a separate universal transmitting medium with its own stress on which is superposed the polarized media with their reacting mechanical forces. This implies that it is absolutely essential to distinguish between the parts of the field vectors and energy which belong to the aether and remain with it and the parts which belong to the matter

[1] E. A. Guggenheim, Phys. Rev. **68**, 273 (1945).
[2] G. H. Livens, Phil. Mag. **36**, 1 (1945). Cf. also Proc. Roy. Soc. **A93**, 200 (1916), and Phil. Trans. Roy. Soc. **A220**, 207 (1919).
[3] E. A. Guggenheim, Proc. Roy. Soc. **A155**, 49 (1936).
[4] E. Cohn, *Das elektromagnetische Feld* (1900). My knowledge of this book is derived from this earlier edition.