General Relativity and Flat Space. I

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Within the framework of the general theory of relativity, it is proposed to introduce at each point of space-time a Euclidean metric tensor $\gamma_{\mu\nu}$ in addition to the usual Riemannian metric tensor $g_{\mu\nu}$. In this way one imparts tensor character to quantities which in the usual form of the theory do not have it. For example, one can obtain a gravitational energy-momentum density tensor in place of the usual pseudo-tensor. Furthermore one can impose four additional covariant conditions on the gravitational field and thus restrict the form of the solution for the field corresponding to a given physical situation.

§1

IN the general theory of relativity, Einstein[1] makes use of Riemannian geometry, characterized by the existence of an invariant interval or line element ds, given by

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor and (x^1,x^2,x^3,x^4) are the coordinates in an arbitrary coordinate system. The metric tensor is determined by the law of gravitation:

$$G_{\mu\nu}=-8\pi T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum density tensor, and

$$G_{\mu\nu}\equiv R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R, \tag{3}$$

with $R_{\mu\nu}$ the contracted Riemann-Christoffel tensor.

The tensor $G_{\mu\nu}$ satisfies the identity

$$G_{\mu}{}^{\alpha}{}_{;\alpha}\equiv 0, \tag{4}$$

where (;) denotes covariant differentiation based on $g_{\mu\nu}$ (g-differentiation). This permits setting $G_{\mu\nu}$ proportional to $T_{\mu\nu}$ which is taken to satisfy the equation (expressing the laws of motion of matter or energy)

$$T_{\mu}{}^{\alpha}{}_{;\alpha}=0. \tag{5}$$

§2

It is now proposed to introduce at every point of space-time, along with the metric tensor $g_{\mu\nu}$, a second metric tensor $\gamma_{\mu\nu}$ corresponding to flat space, i.e., for which the Riemann-Christoffel tensor vanishes identically everywhere. This may be interpreted in various ways. For instance, we may suppose that we map the Riemannian space with metric $g_{\mu\nu}$ on a flat space with metric $\gamma_{\mu\nu}$ and assign the same values to the coordinates of corresponding points. Or we may say that the two metrics side by side represent a comparison of the given space with the space one would have if the gravitational field were removed. It should be emphasized that in introducing the $\gamma_{\mu\nu}$, we are not postulating here any new properties of the space.

In this way one can define a Euclidean line element analogous to (1)

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu. \tag{6}$$

Moreover one can now define covariant differentiation based on $\gamma_{\mu\nu}$ (γ-differentiation), which will be denoted by (,). Since the Riemann-Christoffel tensor formed from $\gamma_{\mu\nu}$ has been assumed to vanish, it follows that one can interchange the order of γ-differentiation, so that the latter obeys all the rules of ordinary differentiation, except that the γ-derivatives of $\gamma_{\mu\nu}$ vanish. Indeed it is always possible to choose a coordinate system (special relativity) such that the components $\gamma_{\mu\nu}$ are constant and γ-differentiation reduces to ordinary; but it may not always be convenient to choose such a system.

Let us next consider $\Delta^\lambda{}_{\mu\nu}$ defined by

$$\left\{ \begin{matrix} \lambda \\ \mu\ \ \nu \end{matrix} \right\}=\Gamma^\lambda{}_{\mu\nu}+\Delta^\lambda{}_{\mu\nu}, \tag{7}$$

where $\{^{\,\lambda}_{\mu\nu}\}$ is the Christoffel 3-index symbol used in g-differentiation and $\Gamma^\lambda{}_{\mu\nu}$ that occurring in γ-differentiation. This can be shown to be a tensor and is found to be given by

$$\Delta^\lambda{}_{\mu\nu}=\tfrac{1}{2}g^{\lambda\alpha}(g_{\mu\alpha,\nu}+g_{\nu\alpha,\mu}-g_{\mu\nu,\alpha}). \tag{8}$$

[1] A. Einstein, Ann. d. Physik **49**, 769 (1916).

If in the usual expression[2] defining $R_{\mu\nu}$, one substitutes (7), it is found that, as a consequence of the Euclidean character of $\gamma_{\mu\nu}$, one can write

$$R_{\mu\nu}=-\Delta^{\alpha}{}_{\mu\nu,\alpha}+\Delta^{\alpha}{}_{\alpha\mu,\nu}-\Delta^{\alpha}{}_{\alpha\beta}\Delta^{\beta}{}_{\mu\nu}+\Delta^{\alpha}{}_{\beta\mu}\Delta^{\beta}{}_{\alpha\nu}. \quad (9)$$

This relation is interesting in that $R_{\mu\nu}$ is seen to be obtainable from the tensor $\Delta^{\lambda}{}_{\mu\nu}$ by tensor operations, i.e., $R_{\mu\nu}$ is a tensor function of $g_{\mu\nu}$.

§3

Comparing (9) with the usual expression for $R_{\mu\nu}$, one sees that $\{^{\lambda}_{\mu\nu}\}$ has been replaced by $\Delta^{\lambda}{}_{\mu\nu}$ and ordinary differentiation by γ-differentiation. This type of correspondence turns out to be quite general. It appears that one can rewrite all the quantities occurring in relativity theory so that $\{^{\lambda}_{\mu\nu}\}$ is replaced by $\Delta^{\lambda}{}_{\mu\nu}$, an ordinary derivative by a γ-derivative (in particular, $\partial g_{\mu\nu}/\partial x^{\sigma}$ by $g_{\mu\nu,\,\sigma}$) and $(-g)^{\frac{1}{2}}$ by

$$\kappa=(g/\gamma)^{\frac{1}{2}}, \quad (10)$$

where g and γ are the determinants of $g_{\mu\nu}$ and $\gamma_{\mu\nu}$, respectively. Since $g_{\mu\nu,\sigma}$ is a tensor and κ a scalar, it follows that some quantities, on being rewritten in this way, take on a tensor character, although they do not have it in the usual form of the theory.

To illustrate this point, the following example may be useful to compare with the analogous calculation usually made.[3] One can show that the gravitational Eqs. (2) are connected with a variational principle

$$\delta\int L(-g)^{\frac{1}{2}}d\tau\left(\equiv\delta\int L\kappa(-\gamma)^{\frac{1}{2}}d\tau\right)=\int G^{\mu\nu}\delta g_{\mu\nu}(-g)^{\frac{1}{2}}d\tau, \quad (11)$$

where

$$d\tau=dx^1dx^2dx^3dx^4,$$

$$L=g^{\mu\nu}(\Delta^{\alpha}{}_{\beta\mu}\Delta^{\beta}{}_{\alpha\nu}-\Delta^{\alpha}{}_{\alpha\beta}\Delta^{\beta}{}_{\mu\nu}), \quad (12)$$

and only the components of $g_{\mu\nu}$ are varied (with the variations vanishing on the boundaries of the region of integration). This function L is a scalar. From it one can derive a gravitational energy-momentum density tensor (analogous to the pseudo-tensor of Einstein). One writes (Eq. (4))

$$G_{\mu}{}^{\alpha}{}_{,\alpha}+G^{\beta}{}_{\mu}\Delta^{\alpha}{}_{\beta\alpha}-G^{\alpha}{}_{\beta}\Delta^{\beta}{}_{\mu\alpha}=0, \quad (13)$$

or, since

$$\Delta^{\alpha}{}_{\alpha\beta}=\kappa_{,\beta}/\kappa, \quad (14)$$

$$(\kappa G^{\alpha}{}_{\mu})_{,\alpha}-\tfrac{1}{2}\kappa G^{\alpha\beta}g_{\alpha\beta,\mu}=0. \quad (15)$$

Throughout this paper, unless otherwise specified, indices are raised and lowered by means of $g_{\mu\nu}$.

Now (11) can be shown to be equivalent to the relation

$$\kappa G^{\mu\nu}=\partial(L\kappa)/\partial g_{\mu\nu}-[\partial(L\kappa)/\partial g_{\mu\nu,\sigma}]_{,\sigma}, \quad (16)$$

and from this one obtains (using integration by parts)

$$\kappa G^{\alpha\beta}g_{\alpha\beta,\mu}=\{L\kappa\delta^{\lambda}{}_{\mu}-[\partial(L\kappa)/\partial g_{\alpha\beta,\lambda}]g_{\alpha\beta,\mu}\}_{,\lambda}, \quad (17)$$

which is to be substituted in (15). If we now define as the gravitational energy-momentum density tensor,

$$t_{\mu}{}^{\lambda}=\frac{1}{16\pi}\{L\delta_{\mu}{}^{\lambda}-[\partial(L\kappa)/\kappa\partial g_{\alpha\beta,\lambda}]g_{\alpha\beta,\mu}\}, \quad (18)$$

we can write (15) or (4) as

$$[\kappa(T_{\mu}{}^{\lambda}+t_{\mu}{}^{\lambda})]_{,\lambda}=0. \quad (19)$$

Or, taking

$$T'_{\mu}{}^{\lambda}=\kappa T_{\mu}{}^{\lambda}, \quad t'_{\mu}{}^{\lambda}=\kappa t_{\mu}{}^{\lambda}, \quad (20)$$

we can write

$$(T'_{\mu}{}^{\lambda}+t'_{\mu}{}^{\lambda})_{,\lambda}=0. \quad (21)$$

By a suitable coordinate transformation, one can put the left-hand member into the form of an ordinary divergence. The fact that $t'_{\mu}{}^{\lambda}$ is a tensor serves to remove some objections that had been raised in the past.[4]

§4

It is to be seen that, in general, the quantities in the usual form of the relativity theory can be considered as arising from the corresponding ones here, through the special choice of $\gamma_{\mu\nu}$,

$$\gamma_{\mu\nu}=\text{constant}. \quad (22)$$

[2] E.g., L. P. Eisenhart, *Riemannian Geometry* (Princeton, 1926), p. 21.

[3] Reference 1, p. 804, and H. Weyl, *Raum Zeit Materie* (Berlin, 1923), p. 272.

[4] H. Bauer, Physik. Zeits. **19**, 163 (1918).

However, if (22) is to hold for all coordinate systems, $\gamma_{\mu\nu}$ cannot be a tensor. Consequently the quantities which depend on $\gamma_{\mu\nu}$ lose their tensor character. Thus the gravitational tensor $t_\mu{}^\lambda$ of (18) goes over into the pseudo-tensor in the Einstein theory. On the other hand, $R_{\mu\nu}$ is independent of $\gamma_{\mu\nu}$ and hence remains a tensor. Obviously, if one wishes to maintain the tensor character of quantities depending on $\gamma_{\mu\nu}$ one must give up (22) and allow $\gamma_{\mu\nu}$ to transform as a tensor under coordinate transformations.

§5

Although $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ have been considered as existing side by side, so far nothing has been said about any relation between them. Actually one would expect to have some relation between them, for it seems reasonable that, if the gravitational field is made to vanish, $g_{\mu\nu}$ should go over into $\gamma_{\mu\nu}$. Such a relation can be obtained by imposing four additional covariant conditions on the field. This can be done because of the set of identities (4) existing among the field equations. In working with the linear approximation of the gravitational equations, Einstein[5] similarly added four conditions on the field for the purpose of eliminating (or reducing) apparent fields arising from infinitesimal coordinate transformations. It has hitherto not been possible to set up corresponding (covariant) conditions for the exact equations. In the present form of the theory this can be done, and it is these equations that provide the relation between $g_{\mu\nu}$ and $\gamma_{\mu\nu}$. They should serve to remove, or at least restrict, the ambiguity in the form of a solution arising from the possibility of coordinate transformations which go over into the identical transformation in the absence of a field.

For example, the static spherically symmetric solution of the gravitational equations in free space can be written in a number of forms depending on the choice of the radial variable r, it being asserted that one cannot know which choice corresponds to the variable r in flat space. The additional conditions would serve to single out one of the various possibilities.

One reasonable set of conditions can be obtained from the following considerations:

One can rewrite the expression (9) for $R_{\mu\nu}$ in the form

$$R_{\mu\nu}=\tfrac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta}-\tfrac{1}{2}S_{\mu;\nu}-\tfrac{1}{2}S_{\nu;\mu}-\Delta^{\alpha}{}_{\mu\beta}\Delta^{\alpha^*}{}_{\nu\beta^*}-\Delta^{\alpha}{}_{\mu\beta}\Delta^{\nu^*}{}_{\alpha\beta^*}-\Delta^{\alpha}{}_{\nu\beta}\Delta^{\mu^*}{}_{\alpha\beta^*}, \quad (23)$$

where

$$S_\mu\equiv\Delta^{\mu}{}_{\alpha\alpha^*}\equiv g^{\alpha\beta}g_{\alpha\mu,\beta}-\kappa_{,\mu}/\kappa, \quad (24)$$

and an asterisk with an index indicates raising or lowering it with $g_{\mu\nu}$.

This suggests taking as additional conditions

$$S_\mu=0. \quad (25)$$

In consequence, the right-hand member of (23) is simplified, only the first term now containing second derivatives. It can be verified that the conditions (25) in first approximation are the same as those imposed by Einstein in the case of the linear equations.

The static spherically symmetric solution of the field equations in free space

$$R_{\mu\nu}=0 \quad (26)$$

together with (25), is found to be

$$ds^2=-\frac{r+m}{r-m}dr^2-(r+m)^2(d\theta^2+\sin^2\theta d\phi^2)+\frac{r-m}{r+m}dt^2, \quad (27)$$

with the usual notation, if one takes

$$d\sigma^2=-dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+dt^2. \quad (28)$$

Another possible set of conditions can be obtained based upon taking

$$\kappa=1. \quad (29)$$

This serves to simplify many of the formulas. It should be noted that this differs very essentially from the condition $g=-1$, often used, in that it is not affected by coordinate transformations. Three further conditions are required. One might take (as being equivalent to three conditions)

$$S_{\mu,\nu}-S_{\nu,\mu}=0. \quad (30)$$

In this case if one takes for $d\sigma^2$ the expression (28), the static spherically symmetric solution has the Schwarzschild[6] form:

$$ds^2=-(1-2m/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+(1-2m/r)dt^2. \quad (31)$$

However this solution leads to the difficulty pointed out by Schrödinger[7] (a not very serious difficulty, however[8]) that

[5] A. Einstein, Berl. Ber., p. 688 (1916).

[6] K. Schwarzschild, Berl. Ber., p. 189 (1916).

[7] E. Schrödinger, Physik. Zeits. **19**, 4 (1918).

[8] A. Einstein, Physik. Zeits. **19**, 115 (1918); Berl. Ber., p. 448 (1918).

the components of $t_\mu{}^\nu$ all vanish for $r>2m$. If one wishes to avoid this, one must give up (29) and (30).

§6

Enough has been given to show that there are advantages from the formal point of view in introducing the Euclidean $\gamma_{\mu\nu}$ into the general relativity theory. It imparts tensor character to quantities which otherwise do not have it, and allows additional conditions to be imposed on the field so as to restrict the form of the solution for a given physical situation.

In conclusion, it is necessary to point out that, having once introduced $\gamma_{\mu\nu}$ into the theory, one has a great number of new tensors and scalars at one's disposal. One can, therefore, set up other field equations than (2). It is possible that some of these may be more satisfactory for the description of nature. Further investigation is here required.

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General Relativity and Flat Space. II

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The possibility is considered of interpreting the formalism of the general theory of relativity in terms of flat space, the fundamental tensor $g_{\mu\nu}$ being regarded as describing the gravitational field but having no direct connection with geometry. The resulting theory in general leads to the same predictions as the Einstein theory, but there are cases where the predictions differ. The present theory may explain the principal results obtained by D. C. Miller in his "ether-drift" experiments. The implications of the theory for cosmology are briefly touched upon.

§1

In a previous paper[1] (hereafter referred to as I) it was shown that it is useful to introduce into the general theory of relativity the concept of the existence at each point of space-time of a Euclidean metric tensor $\gamma_{\mu\nu}$ in addition to the usual Riemannian metric tensor $g_{\mu\nu}$. From the standpoint of the general theory of relativity, one must look upon $\gamma_{\mu\nu}$ as a fiction introduced for mathematical convenience. However, the question arises whether it may not be possible to adopt a different point of view, one in which the metric tensor $\gamma_{\mu\nu}$ is given a real physical significance as describing the geometrical properties of space, which is therefore taken to be flat, whereas the tensor $g_{\mu\nu}$ is to be regarded as describing the gravitational field.[2]

It has been pointed out in I that the introduction of $\gamma_{\mu\nu}$ leads to the possibility of other laws than those adopted in general relativity. In the present paper, however, no attempt will be made to change the laws of the latter, since they form a self-consistent system and have proved to be quite satisfactory for the description of large scale phenomena, at any rate.

§2

As far as the field equations are concerned, it is immaterial what interpretation one gives to the variables involved. This is not the case with the equations of motion. Let us therefore consider the law of motion for a particle in the field. The latter is given in the general theory of relativity by the equation of the geodesic[3]

$$\frac{d^2x^\mu}{ds^2}+\left\{\begin{matrix}\mu \\ \alpha\ \beta\end{matrix}\right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0, \tag{1}$$

where ds is the line element, defined in terms of the tensor $g_{\mu\nu}$, (I (1)). Let us now introduce as independent variable the Euclidean line element

[1] N. Rosen, Phys. Rev. **57**, 146 (1940).

[2] In some respects this resembles the theory of gravitation proposed by Nordström (cf. report by M. v. Laue, Jahrbuch f. Rad. u. El. **14**, 263 (1917)). It will be seen that there are important differences, however.

[3] A. Einstein, Ann. d. Physik **49**, 769 (1916).