PROBLEMS ON ELECTRORHEOLOGICAL FLUID FLOWS.

R.H.W. HOPPE, W.G. LITVINOV
LEHRSTUHL FÜR ANGEWANDTE ANALYSIS MIT SCHWERPUNKT NUMERIK
UNIVERSITÄT AUGSBURG, UNIVERSITÄTSSTRASSE, 14
86159 AUGSBURG, GERMANY

Abstract. We develop a model of an electrorheological fluid such that the fluid is considered as an anisotropic one with the viscosity depending on the second invariant of the rate of strain tensor, on the module of the vector of electric field strength, and on the angle between the vectors of velocity and electric field. We study general problems on the flow of such fluids at nonhomogeneous mixed boundary conditions, wherein values of velocities and surface forces are given on different parts of the boundary. We consider the cases where the viscosity function is continuous and singular, equal to infinity, when the second invariant of the rate of strain tensor is equal to zero. In the second case the problem is reduced to a variational inequality. By using the methods of a fixed point, monotonicity, and compactness, we prove existence results for the problems under consideration. Some efficient methods for numerical solution of the problems are examined.

1. Introduction

Electrorheological fluids are smart materials which are concentrated suspensions of polarizable particles in a nonconducting dielectric liquid. In moderately large electric fields, the particles form chains along the field lines, and these chains then aggregate to form columns (see Fig. 1, taken from [18]). These chainlike and columnar structures cause dramatic changes in the rheological properties of the suspensions. The fluids become anisotropic, the apparent viscosity (the resistance to flow) in the direction orthogonal to the direction of electric field abruptly increases, while the apparent viscosity in the direction of the electric field changes not so drastically.

The chainlike and columnar structures are destroyed under the action of large stresses, and then the apparent viscosity of the fluid decreases and the fluid becomes less anisotropic.

Constitutive relations for electrorheological fluids in which the stress tensor \( \sigma \) is an isotropic function of the vector of electric field strength \( E \) and the rate of strain tensor \( \varepsilon \) were derived in [19], and for an incompressible fluid there was obtained the following equation:

\[
\sigma = -p I_1 + \alpha_2 E \otimes E + \alpha_3 \varepsilon + \alpha_4 \varepsilon^2 + \alpha_5 (\varepsilon E \otimes E + E \otimes \varepsilon E) + \alpha_6 (\varepsilon^2 E \otimes E + E \otimes \varepsilon^2 E).
\]

(1.1)

Here \( p \) is the pressure, \( I_1 \) the unit tensor, \( \alpha_i \) are scalar functions of six invariants of the tensors \( \varepsilon, E \otimes E \), and mixed tensors; \( \alpha_i \) are to be determined by experiments.

In the condition of simple shear flow, when the vectors of velocity \( v \) and electric field \( E \) are orthogonal and \( E \) is in the plane of flow, the terms with coefficients \( \alpha_2, \alpha_4, \alpha_5, \alpha_6 \) give rise to two normal stresses differences (see [19]). But these terms lead to incorrectness of the boundary value problems for the constitutive equation (1.1), and very restrictive conditions should be imposed on the coefficients \( \alpha_2, \alpha_4, \alpha_5, \alpha_6 \) in order to get an operator satisfying the

\[1991 \text{ Mathematics Subject Classification. 35Q35.}\]
conditions of coerciveness and monotonicity (the condition of coerciveness is almost similar to the Clausius-Duhem inequality following from the second law of thermodynamics, and the condition of monotonicity denotes that stresses increase as the rate of strains increase).

The constitutive equation (1.1) does not describe anisotropy of the fluid; in the case of simple shear flow (1.1) gives the same values of the shear stresses in the cases, when the vectors of velocity and electric field are orthogonal and parallel ($\sigma$ is an isotropic function of $E$ and $\varepsilon$ in (1.1)).

Stationary and nonstationary mathematical problems for the special case of (1.1) are studied in [20]. It is supposed in [20] that velocities are equal to zero everywhere on the boundary and the stress tensor is given by

$$\sigma = -pI_1 + \gamma_1((1 + |\varepsilon|^2)^{\frac{b+1}{2}} - 1)E \otimes E$$

$$+ (\gamma_2 + \gamma_3|E|^2)(1 + |\varepsilon|^2)^{b+2} \varepsilon + \gamma_4(1 + |\varepsilon|^2)^{b+2} (\varepsilon E \otimes E + E \otimes \varepsilon E),$$

(1.2)

where $|\varepsilon|^2 = \sum_{i,j=1}^n \varepsilon_{ij}^2$, $n$ being the dimension of a domain of flow, $\gamma_1 - \gamma_4$, are constants, and $k$ is a function of $|E|^2$.

The constants $\gamma_1 - \gamma_4$ and the function $k$ are determined by the approximation of flow curves which are obtained experimentally for different values of the vector of electric field $E$ (see Subsection 2.2). But the conditions of coerciveness and monotonicity of the operator $-\text{div} (\sigma + pI_1)$ impose severe constraints on the constants $\gamma_1 - \gamma_4$ and on the function $k$, see [20], such that with these restrictions one cannot obtain a good approximation of a flow curve, to say nothing of approximation of a set of flow curves corresponding to different values of $E$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fibrous_structure.png}
\caption{Fibrous structure formed by the electric field for alumina particles}
\end{figure}

Below in Section 2, we develop a constitutive equation of electrorheological fluids such that a fluid is considered as a viscous one with the viscosity depending on the second invariant of the rate of strain tensor, on the module of the vector of electric field strength, and on the angle between the vectors of velocity and electric field strength. This constitutive equation describes the main peculiarities of electrorheological fluids, and it can be identified so that a set of flow curves corresponding to different values of $E$ is approximated with a high degree of accuracy, and it leads to correct mathematical problems.

In Section 3, we present auxiliary results, and in Sections 4–8 we study problems on stationary flow of such fluids at nonhomogeneous mixed boundary conditions. Here we prescribe values of velocities and surface forces on different parts of the boundary and ignore the inertial forces. The cases where the viscosity function is continuous and singular,
equal to infinity, when the second invariant of the rate of strain tensor is equal to zero, are studied. In the second case the problem is reduced to a variational inequality.

By using the methods of a fixed point, monotonicity, and compactness we prove existence results for the regular and singular viscosity functions. In the second case existence results are obtained at more restrictive assumptions. Here the singular viscosity is approximated by a continuous bounded one with a parameter of regularization, and a solution of the variational inequality is obtained as a limit of the solutions of regularized problems.

Section 9 is concerned with numerical solution of the problems on stationary flows of electrorheological fluids with regular viscosity function. We consider here methods of the augmented Lagrangian, Birger-Kachanov, contraction and gradient.

In Sections 10 and 11 we study problems on flow of electrorheological fluids in which inertial forces are taken into account. Here we consider nonhomogeneous boundary conditions in the case that velocities are given on the whole of the boundary and in the case that velocities and surface forces are prescribed on different parts of the boundary. With some suppositions existence results are proved.

### 2. Constitutive equation.

#### 2.1. The form of the constitutive equation. It has been found experimentally that the shear stress and accordingly the viscosity of electrorheological fluids depend on the shear rate, the module of the vector of electric field strength, and the angle between the vectors of fluid velocity and electric fields strength (see [18, 22]). Thus, on the basis of experimental results we introduce the following constitutive equation

\[
\sigma_{ij}(p, u, E) = -p\delta_{ij} + 2\varphi(I(u), |E|, \mu(u, E))\varepsilon_{ij}(u), \quad i, j = 1, \ldots, n, \quad n = 2 \text{ or } 3.
\]

(2.1)

Here, \(\sigma_{ij}(p, u, E)\) are the components of the stress tensor which depend on the pressure \(p\), the velocity vector \(u = (u_1, \ldots, u_n)\) and the electric field strength \(E = (E_1, \ldots, E_n)\), \(\delta_{ij}\) is the Kronecker delta, and \(\varepsilon_{ij}(u)\) are the components of the rate of strain tensor

\[
\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).
\]

(2.2)

Moreover, \(I(u)\) is the second invariant of the rate of strain tensor

\[
I(u) = \sum_{i,j=1}^{n} (\varepsilon_{ij}(u))^2,
\]

(2.3)

and \(\varphi\) the viscosity function depending on \(I(u), |E|\) and \(\mu(u, E)\), where

\[
(\mu(u, E))(x) = \left(\frac{u(x)}{|u(x)|}, \frac{E(x)}{|E(x)|}\right)^2 = \frac{(\sum_{i=1}^{n} u_i(x)E_i(x))^2}{(\sum_{i=1}^{n} |u_i(x)|^2)(\sum_{i=1}^{n} |E_i(x)|^2)}.
\]

(2.4)

So \(\mu(u, E)\) is the square of the scalar product of the unit vectors \(\frac{u}{|u|}\) and \(\frac{E}{|E|}\). The function \(\mu\) is defined by (2.4) in the case of an immovable frame of reference. If the frame of reference moves uniformly with a constant velocity \(\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)\), then we set:

\[
\mu(u, E)(x) = \left(\frac{u(x) + \tilde{u}}{|u(x) + \tilde{u}|}, \frac{E(x)}{|E(x)|}\right)^2_{\mathbb{R}^n}.
\]

(2.5)

As the scalar product of two vectors is independent of the frame of reference, the constitutive equation (2.1) is invariant with respect to the group of Galilei transformations of the frame
of reference that are represented as a product of time-independent translations, rotations and uniform motions.

It is obvious that $\mu(u,E)(x) \in [0,1]$, and for fixed $y_1, y_2 \in \mathbb{R}_+$, where $\mathbb{R}_+ = \{ z \in \mathbb{R}, \ z \geq 0 \}$, the function $y_3 \to \varphi(y_1, y_2, y_3)$ reaches its maximum at $y_3 = 0$ and its minimum at $y_3 = 1$ when the vectors $u(x) + \tilde{u}$ and $E$ are correspondingly orthogonal and parallel.

The function $\mu$ defined by (2.4), (2.5) is not specified at $E = 0$ and at $u = 0$, and there does not exist an extension by continuity to the values of $u = 0$ and $E = 0$. However, at $E = 0$ there is no influence of the electric field. Therefore,

$$\varphi(y_1, 0, y_3) = \tilde{\varphi}(y_1), \quad y_3 \in [0,1],$$

and the function $\mu(u, E)$ need not be specified at $E = 0$. Likewise, in case that the measure of the set of points $x$ at which $u(x) = 0$ is zero, the function $\mu$ need not also be specified at $u = 0$. But in the general case we should specify $\mu$ for all values of $u$. Because of this we assume that the function $\mu$ is defined as follows:

$$\mu(u,E)(x) = \left( \frac{\alpha \tilde{I} + u(x) + \tilde{u}}{\alpha \sqrt{u} + |u(x) + \tilde{u}|} \frac{E(x)}{|E(x)|} \right)^2 \mathbb{R}^n,$$

where $\tilde{I}$ denotes a vector with components equal to one, and $\alpha$ is a small positive constant. If $u(x) \neq 0$ almost everywhere in $\Omega$, we may choose $\alpha = 0$.

2.2. Assumptions on the viscosity function. Flow curves of electrorheological fluids obtained experimentally for $\mu(u,E) = 0$ have the form as displayed in Fig. 2 (cf.,e.g.,[22]).

These curves define the relationship between the shear stress $\tau = \sigma_{12}$ and the shear rate $\gamma = \varepsilon_{12}(u) = \frac{1}{2} \frac{du_1}{dx_2}$ for a flow that is close to simple shear flow. Line 1 is the flow curve for $|E| = 0$, and lines 2–4 represent the flow curve for increasing $|E|$.

Flow curves are obtained in some region, say $\gamma_0 \leq \gamma \leq \gamma_1$, $\gamma_0 > 0$. Experimental results for small $\gamma$ are not precise, and one has to extend the flow curves to $\mathbb{R}_+$. It is customary to extend flow curves by straight lines over the region $\gamma_1 < \gamma < \infty$. One can prolong flow curves in $(0,\gamma_0)$ such that either $\tau = \tau_0$ for $\gamma = 0$ or $\tau = 0$ for $\gamma = 0$ (see the dash and dot-dash lines in Fig.2).
The viscosity \(\eta(\gamma, E)\) of the fluid is determined as

\[
\eta(\gamma, E) = \frac{1}{2} \frac{\tau}{\gamma}, \tag{2.8}
\]
and it is defined by the approximation of the lines 1–4 extended to \(\mathbb{R}_+\). Generalizing (2.8) to an arbitrary flow, we take

\[
\gamma = \left(\frac{1}{2} I(u)\right)^{\frac{1}{2}}, \quad \varphi(I(u), |E|, \mu(u, E)) = \eta\left(\left(\frac{1}{2} I(u)\right)^{\frac{1}{2}}, E\right). \tag{2.9}
\]

If the flow curve is extended by the straight line \(\tau = c_1 + c_2 \gamma, \gamma \in (\gamma_1, \infty)\), we obtain

\[
\varphi(I(u), |E|, \mu(u, E)) = \begin{cases} 
\varphi_1(I(u), |E|, \mu(u, E)), & I(u) \in [0, 2\gamma_1^2], \\
\frac{1}{2}(c_2 + c_1 \left(\frac{1}{2} I(u)\right)^{-\frac{1}{2}}), & I(u) \in (2\gamma_1^2, \infty).
\end{cases} \tag{2.10}
\]

Here, the coefficients \(c_1, c_2\) depend on \(|E|\) and \(\mu(u, E)\). The viscosity function \(\varphi\) is continuous in \(\mathbb{R}_+^2 \times [0, 1]\), if the flow curve is extended in \([0, \gamma_0]\) by the dot-dash line, and it has the form

\[
\varphi(I(u), |E|, \mu(u, E)) = \frac{b(|E|, \mu(u, E))}{I(u)^{\frac{1}{2}}} + \psi(I(u), |E|, \mu(u, E)) \tag{2.11}
\]

with \(b(|E|, \mu(u, E)) = 2^{-\frac{1}{2}} \tau_0\), if the flow curve is extended by the dash line in \([0, \gamma_0]\), \(\psi\) being a function continuous in \(\mathbb{R}_+^2 \times [0, 1]\).

We note that if \(\psi(I(u), |E|, \mu(u, E)) = b_1(|E|, \mu(u, E))\), then (2.11) is the viscosity function of an extended Bingham electrorheological fluid.

In the case that the flow curve is extended in \([0, \gamma_0]\) by the dot-dash line, the viscosity function can be written as follows:

\[
\varphi(I(u), |E|, \mu(u, E)) = b(|E|, \mu(u, E))(\lambda + I(u))^{-\frac{1}{2}} + \psi(I(u), |E|, \mu(u, E)), \tag{2.12}
\]

where \(\lambda\) is a small positive parameter. Obviously, for \(\lambda = 0\) the function \(\varphi\) defined by (2.12) is the same as the one defined by (2.11). Moreover, for \(I(u) = 0\) we have

\[
\varphi(0, |E|, \mu(u, E)) = \infty \quad \text{for (2.11)},
\]
\[
\varphi(0, |E|, \mu(u, E)) = b(|E|, \mu(u, E))\lambda^{-\frac{1}{2}} + \psi(0, |E|, \mu(u, E)) \quad \text{for (2.12)}.
\]

Flow problems for fluids with a constitutive equation (2.11) reduce to the solution of variational inequalities. Such problems are considerably more complicated than problems for fluids with finite viscosity, in particular, for fluids with a constitutive equation as given by (2.12). From a physical point of view, (2.12) with a finite, but possibly large viscosity for \(I(u) = 0\) seems to be more reasonable than (2.11).

We will study problems in the case that \(\varphi\) is a continuous bounded function of its arguments and in the case that \(\varphi\) is singular and the singular part of the function \(\varphi\) is equal to \(b(|E|, \mu(u, E))I(u)^{-\frac{1}{2}}\). In the first case we assume that \(\varphi\) satisfies one of the following conditions (C1), (C2), (C3):

(C1): \(\varphi : (y_1, y_2, y_3) \rightarrow \varphi(y_1, y_2, y_3)\) is a function continuous in \(\mathbb{R}_+^2 \times [0, 1]\), and for an arbitrarily fixed \((y_2, y_3) \in \mathbb{R}_+ \times [0, 1]\) the function \(\varphi(\cdot, y_2, y_3) : y_1 \rightarrow \varphi(y_1, y_2, y_3)\) is continuously differentiable in \(\mathbb{R}_+\), and the following inequalities hold:
$a_2 \geq \varphi(y_1, y_2, y_3) \geq a_1$ \hspace{1cm} (2.13)

\[ \varphi(y_1, y_2, y_3) + 2 \frac{\partial \varphi}{\partial y_1} (y_1, y_2, y_3)y_1 \geq a_3 \] \hspace{1cm} (2.14)

\[ \left| \frac{\partial \varphi}{\partial y_1} (y_1, y_2, y_3) \right| y_1 \leq a_4, \] \hspace{1cm} (2.15)

where $a_i, 1 \leq i \leq 4$, are positive numbers.

(C2): $\varphi : (y_1, y_2, y_3) \rightarrow \varphi(y_1, y_2, y_3)$ is a function continuous in $\mathbb{R}_+^3 \times [0, 1]$, and for an arbitrarily fixed $(y_2, y_3) \in \mathbb{R}_+ \times [0, 1]$, (2.13) and the following inequality hold:

\[ [\varphi(z_1^2, y_2, y_3)z_1 - \varphi(z_2^2, y_2, y_3)z_2](z_1 - z_2) \geq a_3(z_1 - z_2)^2 \quad \forall (z_1, z_2) \in \mathbb{R}_+^2. \] \hspace{1cm} (2.16)

(C3): $\varphi : (y_1, y_2, y_3) \rightarrow \varphi(y_1, y_2, y_3)$ is a function continuous in $\mathbb{R}_+^3 \times [0, 1]$, and for an arbitrarily fixed $(y_2, y_3) \in \mathbb{R}_+ \times [0, 1]$, (2.13) holds and the function $z \rightarrow \varphi(z^2, y_2, y_3)z$ is strictly increasing in $\mathbb{R}_+$, i.e., the conditions $z_1, z_2 \in \mathbb{R}_+$, $z_1 > z_2$ imply $\varphi(z_1^2, y_2, y_3)z_1 > \varphi(z_2^2, y_2, y_3)z_2$.

Let us dwell on the physical sense of these inequalities. (2.13) indicates that the viscosity is bounded from below and from above by positive constants. The inequality (2.14) implies that for fixed values of $|E|$ and $\mu(u, E)$ the derivative of the function $I(v) \rightarrow G(v)$ is positive, where $G(v)$ is the second invariant of the stress deviator

\[ G(v) = 4[\varphi(I(v), |E|, \mu(u, E))]^2 I(v). \]

This means that in case of simple shear flow the shear stress increases with increasing shear rate. (2.15) is a restriction on $\frac{\partial \varphi}{\partial y_1}$ for large values of $y_1$. These inequalities are natural from a physical point of view.

The assumptions (C2) and (C3) indicate that in case of simple shear flow, the shear stress must increase with increasing shear rate.

Moreover, let $\lambda(z) = \varphi(z^2, y_2, y_3)z$. Assume that the function $\lambda$ is continuously differentiable in $\mathbb{R}_+$. It follows from (C2) that $\frac{d \lambda}{dz} \geq a_3$ for all $z \in \mathbb{R}_+$. Calculating $\frac{d \lambda}{dz}(z)$, we obtain (2.14) from (2.16). On the contrary, (2.16) follows from (2.14). The assumption (C3) indicates that $\frac{d \lambda}{dz}(z) > 0$ for all $z \in \mathbb{R}_+$ or, equivalently

\[ \varphi(y_1, y_2, y_3) + 2 \frac{\partial \varphi}{\partial y_1} (y_1, y_2, y_3)y_1 > 0, \quad y_1 \in \mathbb{R}_+. \]

The following assumption is concerned with the function (coefficient) $b$ in (2.11), (2.12):

(C4): $b : y_1, y_2 \rightarrow b(y_1, y_2)$ is a function continuous in $\mathbb{R}_+ \times [0, 1]$ and in addition

\[ 0 \leq b(y_1, y_2) \leq a_5, \quad (y_1, y_2) \in \mathbb{R}_+ \times [0, 1], \] \hspace{1cm} (2.17)

$a_5$ being a positive number.

Generally, the continuous function $\varphi$ is expressible in the form

\[ \varphi(I(u), |E|, \mu(u, E)) = \sum_{i=1}^{m} e_i(|E|, \mu(u, E))\beta_i(I(u)). \] \hspace{1cm} (2.18)

Polynomials or splines can be used to represent the functions $\beta_i$. The flow curves obtained for various values of $E$ can be approximated with an arbitrary accuracy. We note that (2.18) may also be used for an identification of the function $\psi$. 
In the general case, the coefficients $e_i$ as well as the viscosity function $\varphi$ depend on the temperature and these coefficients can be determined by an approximation of the corresponding flow curves.

2.3. General problems. Fig. 3 below gives an example of an electrorheological fluid flow.

![Figure 3](image)

Here, the domain $\Omega$ of fluid flow consists of three parts $\Omega_1, \Omega_2, \Omega_3$. A fluid flows from the part $\Omega_1$ across $\Omega_2$ in the part $\Omega_3$. Electrodes are placed on parts $\Gamma_0$ and $\Gamma_1$ of the boundary of $\Omega_2$, and an electric field $E$ is generated by applying voltages $\Delta U(t)$ to electrodes at time $t$. Generally it may be a $k$ pairs of electrodes and voltages $\Delta U_i(t)$ are applied to $i$-th pair of electrodes, $i = 1, \ldots, k$. The boundary $S$ of the domain $\Omega$ consists of two parts $S_1$ and $S_2$. Surface forces $F = (F_1, \ldots, F_n)$ act on $S_2$, and the distribution of velocities $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$ is given on $S_1$.

The equations of motion and the incompressibility condition read as follows:

$$
\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial p}{\partial x_i} - 2 \frac{\partial}{\partial x_j} \left[ \varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u) \right] = K_i \quad \text{in} \quad Q = \Omega \times (0, T), \quad i = 1, \ldots, n, \quad (2.19)
$$

$$
\text{div } u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in} \quad Q. \quad (2.20)
$$

Here, $K_i$ are the components of the volume force vector $K$, $\rho$ is the density, $T$ a positive constant. In (2.19) and below Einstein’s convention on summation over repeated index is applied.

We assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3. Suppose that $S_1$ and $S_2$ are open subsets of $S$ such that $S = \overline{S_1} \cup \overline{S_2}$ and $S_1 \cap S_2 = \emptyset$. The boundary and initial conditions are the following:

$$
u_j |_{S_1 \times (0, T)} = \nu_j |_{S_2 \times (0, T)} = F_i. \quad i = 1, \ldots, n, \quad (2.22)
$$

$$
u(\cdot, 0) = \nu^0 \quad \text{in} \quad \Omega. \quad (2.23)
$$

Here, $F_i$ and $\nu_j$ are the components of the vector of surface force $F = (F_1, \ldots, F_n)$ and the unit outward normal $\nu = (\nu_1, \ldots, \nu_n)$ to $S$, respectively.
We consider Maxwell's equations in the following form (see e.g. [8]):

\[
\begin{align*}
\text{curl } E + \frac{1}{c} \frac{\partial B}{\partial t} &= 0, \quad \text{div } B = 0, \\
\text{curl } H - \frac{1}{c} \frac{\partial D}{\partial t} &= 0, \quad \text{div } D = 0.
\end{align*}
\] (2.24)

Here \( E \) is the electric field, \( B \) the magnetic induction, \( D \) the electric displacement, \( H \) the magnetic field, \( c \) the speed of light. One can assume that

\[
D = \varepsilon E, \quad B = \mu H,
\] (2.25)

where \( \varepsilon \) is the dielectric permittivity, \( \mu \) the magnetic permeability.

Since electrorheological fluids are dielectrics the magnetic field \( H \) can be neglected. Then (2.24), (2.25) give the following relations

\[
\begin{align*}
\text{curl } E &= 0, \quad (2.26) \\
\text{div}(\varepsilon E) &= 0. \quad (2.27)
\end{align*}
\]

It follows from (2.26) that there exists a function of potential \( \theta \) such that

\[
E = - \text{grad } \theta,
\] (2.28)

and (2.27) implies

\[
\text{div}(\varepsilon \text{grad } \theta) = 0.
\] (2.29)

The boundary conditions are the following:

\[
\theta = U_i(t) \quad \text{on } \Gamma_i, \quad i = 1, \ldots, k, \quad (2.30)
\]

\[
\theta = 0 \quad \text{on } \Gamma_{i0}, \quad (2.31)
\]

\[
\nu \cdot \varepsilon \text{grad } \theta = 0 \quad \text{on } S \setminus \left( \bigcup_{i=1}^{k} \left( \Gamma_i \cup \Gamma_{i0} \right) \right). \quad (2.32)
\]

Here \( \Gamma_i \) and \( \Gamma_{i0} \) are the surfaces of the \( i \)-th control and null electrodes respectively, and it is supposed that \( \Gamma_i, \Gamma_{i0} \) are open subset of \( S \). We assume

\[
\varepsilon \in L_\infty(\Omega), \quad e_1 \leq \varepsilon \leq e_2 \quad \text{a.e. in } \Omega, \quad (2.33)
\]

\( e_1, e_2 \) being positive constants. Suppose also that

\[
U_i(t) \in H^{\frac{1}{2}}_{00}(\Gamma_i), \quad t \in [0, T], \quad i = 1, \ldots, k, \quad (2.34)
\]

where

\[
H^{\frac{1}{2}}_{00}(\Gamma_i) = \{ \psi | \psi = v|_{\Gamma_i}, \quad v \in H^1(\Omega), \quad v|_{S \setminus \Gamma_i} = 0 \}. \quad (2.35)
\]

The space \( H^{\frac{1}{2}}_{00}(\Gamma_i) \) is normed by

\[
\| \psi \|_{H^{\frac{1}{2}}_{00}(\Gamma_i)} = \inf \{ \| v \|_{H^1(\Omega)}, \quad v \in H^1(\Omega), \quad v|_{\Gamma_i} = \psi, \quad v|_{S \setminus \Gamma_i} = 0 \}. \quad (2.36)
\]

Let \( \bar{\theta} \) be a function such that

\[
\bar{\theta} \in H^1(\Omega), \quad \bar{\theta}|_{\Gamma_i} = U_i(t), \quad \bar{\theta}|_{\Gamma_{i0}} = 0, \quad i = 1, \ldots, k. \quad (2.37)
\]
Define a space $\tilde{V}$ and a bilinear form $a : H^1(\Omega) \times \tilde{V} \to \mathbb{R}$ as follows:

\begin{equation}
\tilde{V} = \{ v \in H^1(\Omega), \quad v \bigg|_{\Gamma_i \cup \Gamma_{10}} = 0 \},
\end{equation}

\begin{equation}
a(v,h) = \int_{\Omega} \epsilon \frac{\partial v_i}{\partial x_i} \frac{\partial h_j}{\partial x_j} \, dx, \quad v \in H^1(\Omega), \quad h \in \tilde{V}.
\end{equation}

Consider the problem: find $u$ satisfying

\begin{equation}
u \in \tilde{V}
\end{equation}

\begin{equation}
a(u,h) = -a(\tilde{\theta},h) \quad h \in \tilde{V}.
\end{equation}

By use of Green’s formula it can be seen that, if $u$ is a solution of problem (2.40), then the function $\theta = u + \tilde{\theta}$ is a solution of (2.29)–(2.32) in the sense of distributions. On the contrary, if $\theta$ is a solution of problem (2.29)–(2.32), then $u = \theta - \tilde{\theta}$ is a solution of the problem (2.40). Therefore, the function $\theta = u + \tilde{\theta}$ is a generalized solution of problem (2.29)–(2.32).

**Theorem 2.1.** Suppose the conditions (2.33), (2.34) are satisfied. Then there exists a unique generalized solution $\theta$ of problem (2.29)–(2.32), and the function $\theta$ is represented in the form $\theta = \tilde{\theta} + u$, where $\tilde{\theta}$ satisfies (2.37) and $u$ is the solution of problem (2.40).

**Proof.** By virtue of (2.33) the bilinear form $a$ is continuous and coercive in $\tilde{V}$. Therefore there exists a unique solution $u$ of problem (2.40), and the function $\theta = u + \tilde{\theta}$ is a generalized solution of problem (2.29)–(2.32).

Let $\theta_1 = \tilde{\theta}_1 + u_1$ and $\theta_2 = \tilde{\theta}_2 + u_2$ be two generalized solutions of problem (2.29)–(2.32), where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ satisfy (2.37), and $u_1$, $u_2 \in \tilde{V}$. Then $\theta_1 - \theta_2 = w \in \tilde{V}$, and (2.40) implies $a(w,w) = 0$. Therefore $w = 0$. □

The functions of volume force $K$ and surface force $F$ in (2.19) and (2.22) are represented in the form

\begin{equation}
K = \tilde{K} + K_e, \quad F = \tilde{F} + F_e,
\end{equation}

where $\tilde{K}$ and $\tilde{F}$ are the main volume and surface forces, $K_e$ and $F_e$ are volume and surface forces generated by the vector of electric field $E$. Considering electrorheological fluid as a liquid dielectric we present the stress tensor $\sigma_e = \{ \sigma_{eik} \}_{i,k=1}^n$ induced by electric field as follows (see [8]):

\begin{equation}
\sigma_{eik} = -\frac{|E|^2}{8\pi} \left( \epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) \delta_{ik} + \frac{\epsilon}{4\pi} E_i E_k.
\end{equation}

Taking into account (2.26), we obtain the following formula for the vector of volume force

\begin{equation}
K_e = (K_{e1}, \ldots, K_{en}),
K_{ei} = \frac{\partial \sigma_{eij}}{\partial x_j} = -\frac{|E|^2}{8\pi} \frac{\partial \epsilon}{\partial x_i} + \frac{1}{8\pi} \frac{\partial}{\partial x_i} \left( \frac{|E|^2}{\rho} \frac{\partial \rho}{\partial x_i} \right), \quad i = 1, \ldots, n.
\end{equation}

The vector of surface forces is given by

\begin{equation}
F_e = (F_{e1}, \ldots, F_{en}), \quad F_{ei} = \sigma_{eik} \nu_k.
\end{equation}

Thus the systems (2.19)–(2.23) and (2.29)–(2.32) are separated, so one can first solve quasi-static system (2.29)–(2.32) and then solve the problem (2.19)–(2.23), (2.41).
3. Auxiliary results.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary $S$, $n = 2$ or 3. Let $S_1$ be an open non-empty subset of $S$. We consider the following spaces:

$$X = \{ u | u \in H^1(\Omega)^n, \ u |_{S_1} = 0 \},$$

$$V = \{ u | u \in X, \ \text{div} \ u = 0 \}.$$  \hfill (3.1)

By means of Korn’s inequality, the expression

$$\|u\|_X = \left( \int_{\Omega} I(u) dx \right)^{\frac{1}{2}}$$

defines a norm on $X$ and $V$ being equivalent to the norm of $H^1(\Omega)^n$.

Everywhere below we use the following notations: If $Y$ is a normed space, we denote by $Y^*$ the dual of $Y$, and by $(f, h)$ the duality between $Y^*$ and $Y$, where $f \in Y^*$, $h \in Y$. In particular, if $f \in L_2(\Omega)$ or $f \in L_2(\Omega)^n$, then $(f, h)$ is the scalar product in $L_2(\Omega)$ or in $L_2(\Omega)^n$, respectively. The sign $\rightharpoonup$ denotes weak convergence in a Banach space.

We further consider three functions $\tilde{v}, v_1, v_2$ such that

$$\tilde{v} \in H^1(\Omega)^n, \ v_1 \in L_2(\Omega), \ v_1(x) \geq 0 \text{ a.e. in } \Omega,$$

$$v_2 \in L_\infty(\Omega), \ v_2(x) \in [0, 1] \text{ a.e. in } \Omega.$$  \hfill (3.4)

We set $v = (\tilde{v}, v_1, v_2)$ and define an operator $L_v : X \to X^*$ as follows:

$$(L_v(u), h) = 2 \int_{\Omega} \varphi(I(\tilde{v} + v_1, v_2)\varepsilon_{ij}(u + \tilde{v})\varepsilon_{ij}(h)) dx \quad u, h \in X.$$  \hfill (3.5)

**Lemma 3.1.** Suppose the conditions (C1) and (3.4) are satisfied. Then the following inequalities hold

$$(L_v(u) - L_v(w), u - w) \geq \mu_1 \|u - w\|_X^2, \quad u, w \in X,$$

$$\|L_v(u) - L_v(w)\|_{X^*} \leq \mu_2 \|u - w\|_X, \quad u, w \in X,$$

where

$$\mu_1 = \min(2a_1, 2a_3), \quad \mu_2 = 2a_2 + 4a_4.$$  \hfill (3.6)

**Proof.** Let $u, w$ be arbitrarily fixed functions in $X$ and 

$$h = u - w.$$  \hfill (3.7)

We introduce the function $\gamma$ as follows:

$$\gamma(t) = \int_{\Omega} \varphi(I(\tilde{v} + w + th), v_1, v_2)\varepsilon_{ij}(\tilde{v} + w + th)\varepsilon_{ij}(e) dx, \quad t \in [0, 1], \quad e \in X.$$  \hfill (3.8)

It is obvious that

$$\gamma(1) - \gamma(0) = \frac{1}{2} (L_v(u) - L_v(w), e).$$  \hfill (3.9)

By classical analysis it follows that $\gamma$ is differentiable at any point $t \in (0, 1)$. Therefore

$$\gamma(1) = \gamma(0) + \frac{d\gamma}{dt}(\xi), \quad \xi \in (0, 1),$$  \hfill (3.10)
where

$$
\frac{d\gamma}{dt}(\xi) = \int_\Omega [\varphi(I(\tilde{v} + w + \xi h), v_1, v_2)\varepsilon_{ij}(h)\varepsilon_{ij}(e) + 2 \frac{\partial \varphi}{\partial y_1}(I(\tilde{v} + w + \xi h), v_1, v_2)\varepsilon_{km}(\tilde{v} + w + \xi h)\varepsilon_{jm}(\tilde{v} + w + \xi h)\varepsilon_{ij}(e)]dx.
$$

(3.13)

Taking note of the inequality

$$
|\varepsilon_{ij}(\tilde{v} + w + \xi h)\varepsilon_{ij}(h)| \leq I(\tilde{v} + w + \xi h)^{\frac{1}{2}} I(h)^{\frac{1}{2}}
$$

and (2.13), (2.15) we get (3.7) as a direct consequence of (3.9)–(3.13).

Define the function $g$ as follows:

$$
g(\alpha, x) = \begin{cases} 
\frac{\partial \varphi}{\partial y_1}(\alpha, v_1(x), v_2(x)), & \text{if } \frac{\partial \varphi}{\partial y_1}(\alpha, v_1(x), v_2(x)) < 0, \\
0, & \text{if } \frac{\partial \varphi}{\partial y_1}(\alpha, v_1(x), v_2(x)) \geq 0,
\end{cases}
$$

where $\alpha \in \mathbb{R}_+, x \in \Omega$.

Then, taking $e = h$ in (3.13) and applying (2.14) we get

$$
\frac{d\gamma}{dt}(\xi) = \int_\Omega [\varphi(I(\tilde{v} + w + \xi h), v_1, v_2)I(h) + 2 \frac{\partial \varphi}{\partial y_1}(I(\tilde{v} + w + \xi h), v_1, v_2)(\varepsilon_{ij}(\tilde{v} + w + \xi h)\varepsilon_{ij}(h))^2]dx \geq \min(a_1, a_3)\|h\|_X^2
$$

(3.14)

and (3.6) follows from (3.14). ■

**Lemma 3.2.** Let the function $\varphi$ satisfy condition (C2) and assume that (3.4) holds true. Then the operator $L_v$ is a continuous mapping from $X$ into $X^*$ and

$$
(L_v(u) - L_v(w), u - w) \geq 2a_3 \int_\Omega [I(u + \tilde{v})^{\frac{1}{2}} - I(w + \tilde{v})^{\frac{1}{2}}]^2dx, \quad u, w \in X.
$$

(3.15)

Moreover,

$$
(L_v(u) - L_v(w), u - w) = 0 \iff u = w.
$$

(3.16)

**Proof.** We set $u^1 = u + \tilde{v}$, $w^1 = w + \tilde{v}$. Taking into account that $[\varepsilon_{ij}(u^1)\varepsilon_{ij}(w^1)] \leq I(u^1)^{\frac{1}{2}} I(w^1)^{\frac{1}{2}}$, we obtain

$$
(L_v(u) - L_v(w), u - w) = (L_v(u) - L_v(w), u^1 - w^1)
$$

$$
= 2 \int_\Omega [\varphi(I(u^1), v_1, v_2)I(u^1) + \varphi(I(w^1), v_1, v_2)I(w^1) - \varphi(I(u^1), v_1, v_2)\varepsilon_{ij}(u^1)\varepsilon_{ij}(w^1) - \varphi(I(w^1), v_1, v_2)\varepsilon_{ij}(u^1)\varepsilon_{ij}(w^1)]dx
$$

$$
\geq 2 \int_\Omega [\varphi(I(u^1), v_1, v_2)I(u^1)^{\frac{1}{2}} - \varphi(I(w^1), v_1, v_2)I(w^1)^{\frac{1}{2}}][I(u^1)^{\frac{1}{2}} - I(w^1)^{\frac{1}{2}}]dx
$$

(3.17)

Observing (2.16), (3.17) yields (3.15). Now, assume

$$
(L_v(u) - L_v(w), u - w) = 0.
$$

(3.18)

Then, by (3.15) we have

$$
I(u + \tilde{v}) = I(w + \tilde{v}) \text{ a.e. in } \Omega, \quad \varphi(I(u^1), v_1, v_2) = \varphi(I(w^1), v_1, v_2) \text{ a.e. in } \Omega.
$$

(3.19)
Taking (2.13), (3.5), (3.18), (3.19) into account we get \( \|u - w\|_X = 0 \).

The continuity of the mapping \( L_v \) follows from the continuity of the Nemytskii operator (see [24], and also [13], Lemma 8.2 Chapter 2). ■

**Lemma 3.3.** Assume that (C3) is satisfied and (3.4) holds true. Then

\[
(L_v(u) - L_v(w), u - w) \geq 0, \quad u, w \in X.
\]  

Moreover, (3.16) is valid, and the operator \( L_v \) is a continuous mapping from \( X \) into \( X^* \).

**Proof.** Indeed, (3.20) follows from (C3) and (3.17). Assume that (3.18) is valid. Then (C3) and (3.17) imply (3.19), and by (2.13) we obtain \( u = w \). ■

Define the set \( U \) as follows

\[
U = \{ h \in L_\infty(\Omega), \quad 0 \leq h(x) \leq a_5 \text{ a.e. in } \Omega \},
\]

and let \( \tilde{v} \in H^1(\Omega)^n \). For a given constant \( \lambda > 0 \) define an operator \( L_\lambda : U \times X \rightarrow X^* \) as follows:

\[
(L_\lambda(h, u), w) = \int_\Omega h(\lambda + I(\tilde{v} + u))^{-\frac{1}{2}} \varepsilon_{ij}(\tilde{v} + u)\varepsilon_{ij}(w)dx.
\]

**Lemma 3.4.** For an arbitrary \( \lambda > 0 \) and an arbitrarily fixed \( h \in U \) the following inequalities hold:

\[
(L_\lambda(h, u_1) - L_\lambda(h, u_2), u_1 - u_2) \geq 0, \quad u_1, u_2 \in X,
\]  

\[
\|L_\lambda(h, u_1) - L_\lambda(h, u_2)\|_{X^*} \leq \alpha\|u_1 - u_2\|_X, \quad \alpha = 2a_5\lambda^{-\frac{1}{2}},
\]  

\[
\|L_\lambda(h, u)\|_{X^*} \leq \alpha_1, \quad \alpha_1 = \left( \frac{\int_{\Omega} h^2 dx}{\lambda} \right)^{\frac{1}{2}}.
\]

Moreover, the conditions

\[
\{ h_m \} \subset U, \quad h_m \rightarrow h \text{ a.e. in } \Omega,
\]  

\[
u_m \rightarrow u \text{ in } X, \quad u_m \rightarrow u \text{ a.e. in } \Omega,
\]  

\[
\frac{\partial u_m}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega, \quad i, j = 1, \ldots, n,
\]

imply

\[
L_\lambda(h_m, u_m) \rightarrow L_\lambda(h, u) \text{ in } X^*.
\]

**Proof.** The viscosity function associated with the operator \( L_\lambda(\cdot, \cdot) : u \rightarrow L_\lambda(h, u) \) has the form

\[
\varphi(y) = \frac{1}{2}h(\lambda + y)^{-\frac{1}{2}}, \quad y \in \mathbb{R}_+,
\]

where \( y \) plays the role of the second invariant of the rate of strain tensor. We have

\[
\varphi(y) + 2\frac{d\varphi}{dy}(y)y = \frac{1}{2}h(\lambda + y)^{-\frac{1}{2}}[1 - (\lambda + y)^{-1}y] > 0, \quad y \in \mathbb{R}_+, \quad \lambda > 0.
\]

The left-hand side of (3.29) represents the derivative of the function \( g : z \rightarrow g(z) = \varphi(z^2)z, \quad z^2 = y \). Therefore, the function \( g \) is increasing, and (3.23) follows from the proof of Lemma 3.3.

By (3.21), (3.22) we obtain

\[
|(L_\lambda(h, u), w)| \leq \int_\Omega h(\lambda + I(\tilde{v} + u))^{-\frac{1}{2}} I(\tilde{v} + u)^{\frac{1}{2}} I(w)^{\frac{1}{2}} dx \leq \left( \int_\Omega h^2 dx \right)^{\frac{1}{2}} \|w\|_X,
\]
Lemma 3.5. Let (3.31) also tends to zero. Thus (3.27) is satisfied, and the lemma is proven.

Moreover,

\[ \int_{\Omega} h_m[(\lambda + I(\tilde{v} + u_m))^{-\frac{1}{2}} - (\lambda + I(\tilde{v} + u))^{-\frac{1}{2}}] |\varepsilon_{ij}(\tilde{v} + u)\varepsilon_{ij}(w)|dx, \]

whence

\[ \|L_\lambda(h_m, u_m) - L_\lambda(h_m, u)\|_{X^*} \leq \left[ \int_{\Omega} h_m^2(\lambda + I(\tilde{v} + u_m))^{-1} I(\tilde{v} + u)dx \right]^\frac{1}{2} \]

\[ + \left\{ \int_{\Omega} h_m^2[(\lambda + I(\tilde{v} + u_m))^{-\frac{1}{2}} - (\lambda + I(\tilde{v} + u))^{-\frac{1}{2}}]^2 I(\tilde{v} + u)dx \right\}^\frac{1}{2}. \]  

Obviously, the first term of the right-hand side in (3.32) tends to zero. By (3.26) we have \( I(\tilde{v} + u_m) \to I(\tilde{v} + u) \) a.e. in \( \Omega \), and by the Lebesgue theorem we obtain that the second term of the right-hand side in (3.32) tends to zero. The second term of the right-hand side in (3.31) also tends to zero. Thus (3.27) is satisfied, and the lemma is proven. \( \blacksquare \)

**Lemma 3.5.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or 3 with a Lipschitz continuous boundary \( S \), and let the operator \( B \in \mathcal{L}(X, L_2(\Omega)) \) be defined as follows:

\[ Bu = \text{div} u. \]  

Then, the inf-sup condition

\[ \inf_{\mu \in L_2(\Omega)} \sup_{v \in X} \frac{(Bv, \mu)}{\|v\|_X \|\mu\|_{L_2(\Omega)}} \geq \beta_1 > 0 \]  

holds true. The operator \( B \) is an isomorphism from \( V^\perp \) onto \( L_2(\Omega) \), where \( V^\perp \) is the orthogonal complement of \( V \) in \( X \), and the operator \( B^* \) that is adjoint to \( B \), is an isomorphism from \( L_2(\Omega) \) onto the polar set

\[ V^0 = \{ f \in X^*, (f, u) = 0, \ u \in V \}. \]  

Moreover,

\[ \|B^{-1}\|_{\mathcal{L}(L_2(\Omega), V^\perp)} \leq \frac{1}{\beta_1}, \]  

\[ \|(B^*)^{-1}\|_{\mathcal{L}(V^0, L_2(\Omega))} \leq \frac{1}{\beta_1}. \]  

For a proof see in [1]. Lemma 3.5 is a generalization of the inf-sup condition in case that the operator \( \text{div} \) acts in the subspace \( H_0^1(\Omega) \) (see [6]). This result was first established in an equivalent form by Ladyzhenskaya and Solonnikov in [7].
Let \( \{X_m\}_{m=1}^{\infty}, \{N_m\}_{m=1}^{\infty} \) be sequences of finite-dimensional subspaces in \( X \) and \( L_2(\Omega) \), respectively, such that
\[
\lim_{m \to \infty} \inf_{z \in X_m} \|u - z\|_X = 0, \quad u \in X, \tag{3.38}
\]
\[
\lim_{m \to \infty} \inf_{y \in N_m} \|w - y\|_{L_2(\Omega)} = 0, \quad w \in L_2(\Omega). \tag{3.39}
\]
Define the operators \( B_m \in \mathcal{L}(X_m, N_m^*) \) as follows:
\[
(B_m u, \mu) = \int_{\Omega} \mu \, \text{div} \, u \, dx, \quad u \in X_m, \quad \mu \in N_m, \tag{3.40}
\]
and let \( B_m^* \in \mathcal{L}(N_m, X_m^*) \) be the adjoint operator of \( B_m \) with \((B_m u, \mu) = (u, B_m^* \mu)\) for all \( u \in X_m \) and all \( \mu \in N_m \).

We introduce the spaces \( V_m \) and \( V_m^0 \) by
\[
V_m = \{u \in X_m, \ (B_m u, \mu) = 0, \ \mu \in N_m\}, \tag{3.41}
\]
\[
V_m^0 = \{q \in X_m^*, \ (q, u) = 0, \ u \in V_m\}. \tag{3.42}
\]

The following Lemma is valid (see [1]).

**Lemma 3.6.** Let \( \{X_m\}_{m=1}^{\infty}, \{N_m\}_{m=1}^{\infty} \) be sequences of finite-dimensional subspaces in \( X \) and \( L_2(\Omega) \) and assume that the discrete inf – sup condition (LBB condition)
\[
\inf_{\mu \in N_m} \sup_{u \in X_m} \frac{(B_m u, \mu)}{\|u\|_X \|\mu\|_{L_2(\Omega)}} \geq \beta > 0, \quad m \in \mathbb{N}, \tag{3.43}
\]
holds true. Then the operator \( B_m^* \) is an isomorphism from \( N_m \) onto \( V_m^0 \), and the operator \( B_m \) is an isomorphism from \( V_m^\perp \) onto \( N_m^* \), where \( V_m^\perp \) is an orthogonal complement of \( V_m \) in \( X_m \). Moreover,
\[
\| (B_m^*)^{-1} \|^\mathcal{L}(V_m^0, N_m) \leq \frac{1}{\beta}, \quad \| B_m^{-1} \|^\mathcal{L}(N_m^*, V_m^\perp) \leq \frac{1}{\beta}, \quad m \in \mathbb{N}. \tag{3.44}
\]

Consider a functional \( \Psi : U \times X \to \mathbb{R}_+ \) of the form
\[
\Psi(h, u) = \int_{\Omega} h I(u)^\frac{1}{2} dx \quad h \in U, \quad u \in X, \tag{3.45}
\]
where \( U \) is as in (3.21).

**Lemma 3.7.** For an arbitrarily fixed \( h \in U \) the functional \( \Psi(h, \cdot) : u \to \Psi(h, u) \) is continuous in \( X \) and the conditions
\[
\{h_m\} \subset U, \quad h_m \to h \ \text{a.e. in} \ \Omega, \quad u_m \rightharpoonup u \ \text{in} \ X \tag{3.46}
\]

imply
\[
\liminf_{m} \Psi(h_m, u_m) \geq \Psi(h, u). \tag{3.47}
\]

Here and below the sign \( \rightharpoonup \) designates the weak convergence.

**Proof of the Lemma 3.7.** Let \( u_m \to u \) in \( X \). We have
\[
\int_{\Omega} h I(u_m - u)^\frac{1}{2} dx \leq \left( \int_{\Omega} h^2 dx \right)^\frac{1}{2} \left( \int_{\Omega} I(u_m - u) dx \right)^\frac{1}{2}.
\]
Therefore,
\[
\lim \int_{\Omega} h I(u_m - u)^\frac{1}{2} dx = 0, \tag{3.48}
\]
and
\[
\int_{\Omega} hI(u_m - u)^{\frac{1}{2}} \, dx \geq \left| \int_{\Omega} hI(u_m)^{\frac{1}{2}} \, dx - \int_{\Omega} hI(u)^{\frac{1}{2}} \, dx \right|.
\]
Consequently, \(\lim \Psi(h, u_m) = \Psi(h, u)\).

Let \(\alpha \in [0, 1]\), \(u, v \in X\). Then
\[
I(\alpha u + (1 - \alpha)v) = I(\alpha u) + 2\alpha(1 - \alpha) \sum_{i,j=1}^{n} \varepsilon_{ij}(u)\varepsilon_{ij}(v) + I((1 - \alpha)v) \leq [\alpha I(u)^{\frac{1}{2}} + (1 - \alpha)I(v)^{\frac{1}{2}}]^2.
\]
(3.49)
Therefore,
\[
\Psi(h, \alpha u + (1 - \alpha)v) = \int_{\Omega} hI(\alpha u + (1 - \alpha)v)^{\frac{1}{2}} \, dx \leq \alpha \Psi(h, u) + (1 - \alpha)\Psi(h, v),
\]
(3.50)
which shows that
\[
\Psi(h, \cdot) : u \to \Psi(h, u) \text{ is a convex functional in } X.
\]
(3.51)
Let now (3.46) be fulfilled. We have
\[
\Psi(h_m, u_m) = \int_{\Omega} \left[hI(u_m)^{\frac{1}{2}} + (h_m - h)I(u_m)^{\frac{1}{2}}\right] \, dx,
\]
(3.52)
\[
\left| \int_{\Omega} (h_m - h)I(u_m)^{\frac{1}{2}} \, dx \right| \leq \|h_m - h\|_{L_2(\Omega)} \|u_m\|_X.
\]
(3.53)
In view of (3.46), the right-hand side of (3.53) tends to zero as \(m \to \infty\). Hence (3.51), (3.52), and the continuity of the functional \(\Psi(h, \cdot)\) imply
\[
\lim \inf \Psi(h_m, u_m) = \lim \inf \Psi(h, u_m) \geq \Psi(h, u),
\]
(3.54)
and the lemma is proved. ■

**Remark 3.1.** Assume that
\[
h_1 \leq h(x) \leq h_2 \quad \text{a.e. in } \Omega,
\]
(3.55)
where \(h_1, h_2\) are positive constants. Then the expression
\[
\int_{\Omega} hI(u)^{\frac{1}{2}} \, dx = \|u\|_h
\]
(3.56)
defines a norm on \(X\). Note that this norm is not equivalent to the norm of the space \(W_1^1(\Omega)^n\). However,
\[
\|u\|_p = \left( \int_{\Omega} hI(u)^{\frac{1}{2}} \, dx \right)^{\frac{1}{2}}
\]
(3.57)
is a norm on \(X\), which is equivalent to the norm of \(W_p^1(\Omega)^n\) for \(p > 1\) (cf., e.g., [15]).
4. The stationary problem.

We consider stationary flow problems of electrorheological fluids under the Stokes approximation, i.e., we ignore inertial forces. Such an approach is reasonable, because the viscosities of electrorheological fluids are large, and the inertial terms have a small impact. We deal with the following problem: find a pair of functions $u, p$ satisfying

\[
\frac{\partial p}{\partial x_i} - 2 \frac{\partial}{\partial x_j} [\varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u)] = K_i \quad \text{in } \Omega, \quad i = 1, \ldots, n, \tag{4.1}
\]

\[
\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{4.2}
\]

\[
u|_{S_1} = \hat{u}, \tag{4.3}
\]

\[-p\delta_{ij} + 2\varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u)]v_j|_{S_2} = F_i, \quad i = 1, \ldots, n. \tag{4.4}
\]

We assume that

\[
\hat{u} \in H^1(S_1)^n. \tag{4.5}
\]

Then there exists a function $\tilde{u}$ such that

\[
\tilde{u} \in H^1(\Omega)^n, \quad \tilde{u}|_{S_1} = \hat{u}, \quad \operatorname{div} \tilde{u} = 0. \tag{4.6}
\]

Suppose also

\[
K = (K_1, \ldots, K_n) \in L^2(\Omega)^n, \quad F = (F_1, \ldots, F_n) \in L^2(\Omega)^n. \tag{4.7}
\]

In line with (2.11), we choose the viscosity function $\varphi$ of the following form:

\[
\varphi(I(u), |E|, \mu(u, E)) = \frac{b(|E|, \mu(u, E))}{I(u)^{\frac{1}{2}}} + \psi(I(u), |E|, \mu(u, E), \mu(u, E)), \tag{4.8}
\]

where $\psi$ is a function satisfying one out of the conditions (C1), (C2), (C3) with $\varphi$ replaced by $\psi$, and $b$ satisfies (C4). We refer to the fluid with the viscosity function $\varphi$ defined by (4.8) as a generalized Bingham electrorheological fluid.

Define a functional $J$ on the set $X \times X$ and an operator $L : X \to X^*$ as follows:

\[
J(v, h) = 2 \int_{\Omega} b(|E|, \mu(\hat{u} + v, E)) I(\hat{u} + h)^{\frac{1}{2}} dx, \quad v, h \in X. \tag{4.9}
\]

\[
(L(v), h) = 2 \int_{\Omega} \psi(I(\hat{u} + v), |E|, \mu(\hat{u} + v, E)) \varepsilon_{ij}(\hat{u} + v) \varepsilon_{ij}(h) dx, \quad v, h \in X. \tag{4.10}
\]

We use the notations

\[
(K, h) = \int_{\Omega} K_i h_i dx, \quad (F, h) = \int_{S_2} F_i h_i ds, \quad h \in X. \tag{4.11}
\]

The following assertion holds.

Theorem 4.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary $S$. Assume (4.6), (4.7) are satisfied and $(u, p)$ with $u = \hat{u} + v$ is a regular solution of (4.1) – (4.4), where the viscosity function $\varphi$ is defined by (4.8) with $\psi$ meeting one out of the conditions (C1), (C2), (C3) ($\varphi$ replaced by $\psi$) and $b$ satisfying (C4). Then

\[
v \in V, \quad J(v, h) - J(v, v) + (L(v), h - v) \geq (K + F, h - v), \quad h \in V. \tag{4.12}
\]
Proof. Let $h = (h_1, \ldots, h_n) \in V$. We multiply (4.1) with $h_i - v_i$, sum over $i$ and integrate over $\Omega$. By Green’s formula and (4.4), (4.8), we obtain

$$2 \int_{\Omega} b(|E|, \mu(\tilde{u} + v, E)) I(\tilde{u} + v)^{-\frac{1}{2}} \varepsilon_{ij}(\tilde{u} + v) \varepsilon_{ij}(h - v)dx$$

$$+ (L(v), h - v) = (K + F, h - v), \quad h \in V. \quad (4.14)$$

We use the relations

$$\varepsilon_{ij}(h - v) = \varepsilon_{ij}(\tilde{u} + h) - \varepsilon_{ij}(\tilde{u} + v), \quad \varepsilon_{ij}(\tilde{u} + v) \varepsilon_{ij}(\tilde{u} + h) \leq I(\tilde{u} + v)^{\frac{1}{2}} I(\tilde{u} + h)^{\frac{1}{2}}, \quad (4.15)$$

so that the first addendum of the left-hand side of (4.14) is majorized by $J(v, h) - J(v, v)$. Then, (4.14) implies (4.13), and the theorem is proved. ■

Let $v$ be a solution of the problem (4.12), (4.13) such that $I(\tilde{u} + v) \neq 0$ a.e. in $\Omega$. (4.16)

We replace $h$ in (4.13) by $v + \lambda h$, $\lambda > 0$, from which

$$\lambda^{-1}[J(v, v + \lambda h) - J(v, v)] + (L(v), h) \geq (K + F, h). \quad (4.17)$$

For $\lambda \to 0$ we get

$$\left( \frac{\partial J}{\partial h}(v, v), h \right) + (L(v), h) \geq (K + F, h), \quad h \in V, \quad (4.18)$$

where $\frac{\partial J}{\partial h}(v, v)$ is the partial Gâteaux derivative of the functional $J$ with respect to the second argument

$$\left( \frac{\partial J}{\partial h}(v, v), h \right) = 2 \int_{\Omega} b(|E|, \mu(\tilde{u} + v, E)) I(\tilde{u} + v)^{-\frac{1}{2}} \varepsilon_{ij}(\tilde{u} + v) \varepsilon_{ij}(h)dx. \quad (4.19)$$

Replacing $h$ by $-h$ in (4.18) we obtain

$$\left( \frac{\partial J}{\partial h}(v, v), h \right) + (L(v), h) = (K + F, h), \quad h \in V, \quad (4.20)$$

and Lemma 3.5 gives

$$\frac{\partial J}{\partial h}(v, v) + L(v) - K - F = B^* p, \quad p \in L_2(\Omega), \quad (4.21)$$

that is

$$\left( \frac{\partial J}{\partial h}(v, v), h \right) + (L(v), h) - (B^* p, h) = (K + F, h) \quad \forall h \in X. \quad (4.22)$$

It follows from (4.22) that the pair $(u, p)$ with $u = \tilde{u} + v$ is a solution of (4.1)–(4.4) in the sense of distributions. Thus, we have proved the following:

**Remark 4.1** If $v$ is a solution of (4.12), (4.13) that satisfies (4.16), then there exists a function $p \in L_2(\Omega)$ such that the pair $(u, p)$ with $u = \tilde{u} + v$ is a solution of (4.1)–(4.4) in the distributional sense. In view of this and Theorem 4.1 it is reasonable to refer to the function $u = \tilde{u} + v$ as a generalized solution of (4.1)–(4.4).
5. Problem for the fluid with constitutive equation (2.12).

5.1. **Existence theorem.** We define the following functional on \( X \times X \):

\[
J_\lambda(v, h) = 2 \int_\Omega b(|E|, \mu(\tilde{u} + v, E))(\lambda + I(\tilde{u} + h))^{\frac{\nu}{2}} dx, \quad \lambda > 0. \tag{5.1}
\]

Obviously, \( J_\lambda(v, h) = J(v, h) \) for \( \lambda = 0 \). Note that the functional \( J_\lambda \) is Gâteaux differentiable in \( X \) with respect to the second argument for \( \lambda > 0 \), but not for \( \lambda = 0 \).

The partial Gâteaux derivative \( \frac{\partial J_\lambda}{\partial h} \) is given by

\[
\left( \frac{\partial J_\lambda}{\partial h}(v, h), w \right) = 2 \int_\Omega b(|E|, \mu(\tilde{u} + v, E))(\lambda + I(\tilde{u} + h))^{-\frac{\nu}{2}} \varepsilon_{ij}(\tilde{u} + h)\varepsilon_{ij}(w) dx, \quad v, h, w \in X, \quad \lambda > 0. \tag{5.2}
\]

Consider the following problem: find \( v_\lambda \) such that

\[
v_\lambda \in V, \tag{5.3}
\]

\[
\left( \frac{\partial J_\lambda}{\partial h}(v_\lambda, \lambda), w \right) + (L(v_\lambda), w) = (K + F, w), \quad w \in V. \tag{5.4}
\]

Lemma 3.5 implies that if \( v_\lambda \) is a solution of (5.3), (5.4), then there exists a function \( p_\lambda \) such that the pair \((v_\lambda, p_\lambda)\) is a solution of the following problem:

\[
(v_\lambda, p_\lambda) \in X \times L_2(\Omega), \tag{5.5}
\]

\[
\left( \frac{\partial J_\lambda}{\partial h}(v_\lambda, \lambda), w \right) + (L(v_\lambda), w) - (B^* p_\lambda, w) = (K + F, w), \quad w \in X, \tag{5.6}
\]

\[
(Bv_\lambda, q) = 0, \quad q \in L_2(\Omega). \tag{5.7}
\]

We remark that (5.5)–(5.7) represent the flow of an electrorheological fluid with the constitutive equation (2.12). We seek an approximate solution of the problem (5.5)–(5.7) of the form

\[
(v_m, p_m) \in X_m \times N_m, \tag{5.8}
\]

\[
\left( \frac{\partial J_\lambda}{\partial h}(v_m, v_m), w \right) + (L(v_m), w) - (B^*_m p_m, w) = (K + F, w), \quad w \in X_m, \tag{5.9}
\]

\[
(B_m v_m, q) = 0, \quad q \in N_m, \tag{5.10}
\]

where \( X_m \) and \( N_m \) are finite dimensional subspaces in \( X \) and \( L_2(\Omega) \), respectively, and \( B_m \) is defined as in (3.40).

**Theorem 5.1.** Suppose that the conditions (C4), (4.6), (4.7) are satisfied and the function \( \psi \) meets one of the conditions (C1), (C2), (C3) (\( \varphi \) replaced by \( \psi \)). Let \( \{X_m\}, \{N_m\} \) be sequences of finite-dimensional subspaces in \( X \) and \( L_2(\Omega) \), respectively, such that (3.38), (3.39), (3.43) hold and

\[
X_m \subset X_{m+1}, \quad N_m \subset N_{m+1}, \quad m \in \mathbb{N}. \tag{5.11}
\]

Then, for an arbitrary \( \lambda > 0 \) there exists a solution \((v_\lambda, p_\lambda)\) of (5.5)–(5.7). Moreover, for \( m \in \mathbb{N} \) and \( \lambda > 0 \) there exists a solution \((v_m, p_m)\) of (5.8)–(5.10), and a subsequence \( \{(v_k, p_k)\} \) can be extracted from the sequence \( \{(v_m, p_m)\} \) such that \( v_k \rightharpoonup v_\lambda \) in \( X \), \( p_k \rightharpoonup p_\lambda \) in \( L_2(\Omega) \).
Thus the pair \((v_m, \partial J/\partial h(v_m))\) is a solution of the problem
\[ v_m \in V_m, \quad \left( \frac{\partial J}{\partial h}(v_m, v_m), w \right) + (L(v_m), w) = (K + F, w), \quad w \in V_m. \tag{5.12} \]

Taking (2.17) into account we obtain
\[
\left| \left( \frac{\partial J}{\partial h}(e, e) \right) \right| = 2 \left| \int_\Omega b(|E|, \mu(\ddot{u} + e, E)) \frac{\varepsilon_{ij}(\ddot{u} + e)\varepsilon_{ij}(e)}{(\lambda + I(\ddot{u} + e))^2} \, dx \right|
\leq 2 \int_\Omega b(|E|, \mu(\ddot{u} + e, E)) I(e)^\frac{2}{7} \, dx \leq c_1\|e\|_X, \quad \lambda > 0, \quad e \in X, \tag{5.13}
\]
where
\[ c_1 = 2a_5 (\text{mes } \Omega)^\frac{2}{7}. \tag{5.14} \]

By (2.13), (4.6), (4.7), (4.10), and (5.13), for an arbitrary \(e \in X\) we get
\[ z(e) = \left( \frac{\partial J}{\partial h}(e, e) \right) + (L(e), e) - (K + F, e) \geq 2a_1\|e\|^2_X - c\|e\|_X, \quad e \in X, \quad \lambda > 0, \tag{5.15} \]
giving \(z(e) \geq 0\) for \(\|e\|_X \geq r = \frac{c}{2a_1}\).

From the corollary of Brouwer’s fixed point theorem (cf.[5]) it follows that there exists a solution of (5.12) with
\[ \|v_m\|_X \leq r, \quad \|L(v_m)\|_{X^*} \leq c_2, \quad m \in \mathbb{N}, \tag{5.16} \]
where the second inequality follows from (2.13) and (4.6). For an arbitrary \(f \in X^*\) we denote by \(Gf\) the restriction of \(f\) to \(X_m\). Then \(Gf \in X_m^*\), and by (3.42), (5.12) we obtain
\[ G\left( \frac{\partial J}{\partial h}(v_m, v_m) + L(v_m) - K - F \right) \in V_m^0. \tag{5.17} \]

Therefore, there exists a unique \(p_m \in N_m\) (see Lemma 3.6) such that
\[ B_m^* p_m = G\left( \frac{\partial J}{\partial h}(v_m, v_m) + L(v_m) - K - F \right). \tag{5.18} \]
Thus the pair \((v_m, p_m)\) is a solution of (5.8)–(5.10). Due to (2.17), (4.7), (5.16) and Lemmas 3.4, 3.6 we get
\[ \|p_m\|_{L^2(\Omega)} \leq c, \quad m \in \mathbb{N}. \tag{5.19} \]

By (5.16), (5.19) we can extract a subsequence \(\{v_{n}, p_{n}\}\) such that
\[ v_n \rightarrow v_0 \quad \text{in } X, \tag{5.20} \]
\[ p_n \rightarrow p_0 \quad \text{in } L^2(\Omega), \tag{5.21} \]
\[ L(v_n) \rightharpoonup \chi \quad \text{in } X^*, \tag{5.22} \]
\[ \frac{\partial J}{\partial h}(v_n, v_n) \rightharpoonup \chi_1 \quad \text{in } X^*. \tag{5.23} \]

Let \(\eta_0\) be a fixed positive number and \(w \in X_{\eta_0}, q \in N_{\eta_0}\). Observing (5.20), (5.22)–(5.24) we pass to the limit in (5.9), (5.10) with \(m\) replaced by \(\eta\), and obtain
\[ (\chi_1 + \chi - B^* p_0, w) = (K + F, w), \quad w \in X_{\eta_0}, \tag{5.25} \]
\[ \int_\Omega q \, \text{div } v_0 \, dx = 0, \quad q \in N_{\eta_0}. \]
Since \( \eta_0 \) is an arbitrary positive integer, by (3.38), (3.39)
\[
\chi_1 + \chi - B^*p_0 = K + F, \quad \text{(5.26)}
\]
\[
div v_0 = 0. \quad \text{(5.27)}
\]
We present the operator \( L(v) \) in the form
\[
L(v) = L(v, v), \quad \text{(5.28)}
\]
where the operator \((v, w) \to L(v, w)\) is considered as a mapping of \( X \times X \) into \( X^* \) according to
\[
(L(v, w), h) = 2 \int_\Omega \psi(I(\tilde{u} + w), |E|, \mu(\tilde{u} + v, E))\varepsilon_{ij}(\tilde{u} + w)\varepsilon_{ij}(h)dx. \quad \text{(5.29)}
\]
We get
\[
X_\eta(w) = \left( \frac{\partial J_\lambda}{\partial h}(v_\eta, v_\eta) + L(v_\eta, v_\eta) - \frac{\partial J_\lambda}{\partial h}(v_\eta, w) - L(v_\eta, w), v_\eta - w \right), \quad w \in X. \quad \text{(5.30)}
\]
Lemmas 3.3, 3.4 imply
\[
X_\eta(w) \geq 0, \quad \eta \in \mathbb{N}, \quad w \in X. \quad \text{(5.31)}
\]
We have
\[
\left\| \frac{\partial J_\lambda}{\partial h}(v_\eta, w) - \frac{\partial J_\lambda}{\partial h}(v_0, w) \right\|_{X^*} \leq 2 \left[ \int_\Omega [b(|E|, \mu(\tilde{u} + v_\eta, E)) - b(|E|, \mu(\tilde{u} + v_0, E))]^2 dx \right]^{\frac{1}{2}}. \quad \text{(5.32)}
\]
(2.17), (5.21), (5.32) and the Lebesgue theorem give
\[
\frac{\partial J_\lambda}{\partial h}(v_\eta, w) \to \frac{\partial J_\lambda}{\partial h}(v_0, w) \quad \text{in } X^*. \quad \text{(5.33)}
\]
Likewise we obtain
\[
L(v_\eta, w) \to L(v_0, w) \quad \text{in } X^*. \quad \text{(5.34)}
\]
Taking into account that \((B_\eta v_\eta, p_\eta) = 0\), by (5.9), (5.20), (5.22) we obtain
\[
\left( \frac{\partial J_\lambda}{\partial h}(v_\eta, v_\eta) + L(v_\eta, v_\eta) \right) = (K + F, v_\eta) \to (K + F, v_0), \quad \text{(5.35)}
\]
and
\[
\lim \left( \frac{\partial J_\lambda}{\partial h}(v_\eta, v_\eta) + L(v_\eta, w) \right) - (B^*p_0, w) = (K + F, w), \quad w \in X. \quad \text{(5.36)}
\]
Observing (5.33)–(5.36) and passing to the limit in (5.30), by (5.27), (5.31) we get
\[
(K + F - \frac{\partial J_\lambda}{\partial h}(v_0, w) - L(v_0, w) + B^*p_0, v_0 - w) \geq 0, \quad w \in X. \quad \text{(5.37)}
\]
We choose \( w = v_0 - \gamma h, \ \gamma > 0, \ h \in X \), and consider \( \gamma \to 0 \). Then, Lemmas 3.3, 3.4 give
\[
(K + F - \frac{\partial J_\lambda}{\partial h}(v_0, v_0) - L(v_0, v_0) + B^*p_0, h) \geq 0. \quad \text{(5.38)}
\]
This inequality holds for any \( h \in X \). Therefore, replacing \( h \) by \(-h\) shows that equality holds true. Consequently, the pair \((v_\lambda, p_\lambda)\) with \( v_\lambda = v_0 \) and \( p_\lambda = p_0 \) solves (5.5)–(5.7). The theorem is proved. ■
5.2. On the uniqueness of the solution. Let \( v_\lambda, w_\lambda \) be two solutions of (5.5)–(5.7) and
\[
e = w_\lambda - v_\lambda.
\](5.39)

Define a function \( \eta \) as follows:
\[
\eta(t) = \left( \frac{\partial J_\lambda}{\partial h}(v_\lambda + t e, v_\lambda + t e), e \right) + (L(v_\lambda + t e), e), \quad t \in [0, 1].
\](5.40)

It follows from (5.6) that
\[
\eta(1) - \eta(0) = 0.
\](5.41)

Assume that the function \( \mu \) is defined by (2.7). Suppose also that the functions \( b \) and \( \psi \) are continuously differentiable and in addition
\[
\left| \frac{\partial \psi}{\partial y_3}(y_1, |E(x)|, y_3) \right| y_1^\frac{1}{2} \leq \tilde{c}, \quad (y_1, y_3) \in \mathbb{R}_+ \times [0, 1], \quad x \in \Omega.
\](5.42)

Note that (5.42) is a restriction on the behaviour of the function \( \frac{\partial \psi}{\partial y_3} \) at large values of \( y_1 \).

Under the above conditions the function \( \eta \) is differentiable, and we have
\[
\eta(1) - \eta(0) = \frac{d\eta}{dt}(\xi), \quad \xi \in (0, 1).
\](5.43)

Here
\[
\frac{d\eta}{dt}(\xi) = \sum_{i=1}^{4} \gamma_i(\xi),
\]
where
\[
\gamma_1(\xi) = 2 \int_\Omega \frac{\partial b}{\partial y_2}(|E|, \mu(\tilde{u} + v_\lambda + \xi e, E)) f_\xi (\lambda + I(\tilde{u} + v_\lambda + \xi e))^{-\frac{1}{2}}
\times \varepsilon_{ij}(\tilde{u} + v_\lambda + \xi e) \varepsilon_{ij}(e) \, dx.
\](5.44)

\[
f_\xi = 2 \left( \frac{\alpha \tilde{I} + \tilde{u} + v_\lambda + \xi e}{\alpha \sqrt{n} + |\tilde{u} + v_\lambda + \xi e| + |E| / \mathbb{R}_n} \left[ \left( \frac{e}{\alpha \sqrt{n} + |\tilde{u} + v_\lambda + \xi e|} \frac{E}{|E|} \right)_n \right]^n \right.
\times \left( (\tilde{u}_i + v_\lambda + \xi e_i) \right) \left( \alpha \tilde{I} + \tilde{u} + v_\lambda + \xi e, E / |E|_n \right)
\times (\tilde{u}_i + v_\lambda + \xi e_i) \left( \alpha \tilde{I} + \tilde{u} + v_\lambda + \xi e, E / |E|_n \right] \right).
\](5.45)

\[
\gamma_2(\xi) = 2 \int_\Omega b(|E|, \mu(\tilde{u} + v_\lambda + \xi e, E))[-(\lambda + I(\tilde{u} + v_\lambda + \xi e))^{-\frac{3}{2}}
\times \varepsilon_{km}(\tilde{u} + v_\lambda + \xi e) \varepsilon_{km}(e) \varepsilon_{ij}(\tilde{u} + v_\lambda + \xi e) \varepsilon_{ij}(e) + (\lambda + I(\tilde{u} + v_\lambda + \xi e))^{-\frac{1}{2}} I(e)] \, dx,
\](5.46)

\[
\gamma_3(\xi) = 2 \int_\Omega \frac{\partial \psi}{\partial y_3}(I(\tilde{u} + v_\lambda + \xi e), |E|, \mu(\tilde{u} + v_\lambda + \xi e, E)) f_\xi
\times \varepsilon_{ij}(\tilde{u} + v_\lambda + \xi e) \varepsilon_{ij}(e) \, dx.
\](5.47)

\[
\gamma_4(\xi) = 2 \int_\Omega \psi(I(\tilde{u} + v_\lambda + \xi e), |E|, \mu(\tilde{u} + v_\lambda + \xi e, E)) I(e)
\times \varepsilon_{km}(\tilde{u} + v_\lambda + \xi e) \varepsilon_{km}(e) \varepsilon_{ij}(\tilde{u} + v_\lambda + \xi e) \varepsilon_{ij}(e) \, dx.
\](5.48)
By using (4.15) it is easy to see that
\[ \gamma_2(\xi) \geq 0, \] \hspace{2cm} (5.49)
and Lemma 3.1 implies
\[ \gamma_4(\xi) \geq \mu_1 \|e\|_X^2, \quad \mu_1 = \min(2a_1, 2a_3). \] \hspace{2cm} (5.50)
(5.45) yields
\[ |f_{\xi}| \leq 4 \frac{|e|}{\alpha \sqrt{n} + |\tilde{u} + v_\lambda + \xi e|}. \] \hspace{2cm} (5.51)
We denote
\[ b_0 = \sup \left| \frac{\partial b}{\partial y_2}(|E(x)|, y_2) \right|, \quad y_2 \in [0, 1], \quad x \in \Omega. \] \hspace{2cm} (5.52)
By (5.44), (5.51), and (5.52) we obtain
\[ |\gamma_1(\xi)| \leq 8b_0 \|e\|_X \left( \int_\Omega |e|^4 \, dx \right)^{\frac{1}{4}} \left( \int_\Omega (\alpha \sqrt{n} + |\tilde{u} + v_\lambda + \xi e|)^{-4} \, dx \right)^{\frac{1}{4}} \leq c_1 b_0 \|e\|^2_X. \] \hspace{2cm} (5.53)
Here
\[ c_1 = 8\tilde{c}\hat{c}, \] \hspace{2cm} (5.54)
where \( \tilde{c} \) is the constant of the inequality
\[ \|e\|_{L_4(\Omega)} \leq \tilde{c}\|e\|_X, \] \hspace{2cm} (5.55)
and
\[ \hat{c} = \sup (\int_\Omega (\alpha \sqrt{n} + |\tilde{u} + v_\lambda + \xi e|)^{-4} \, dx)^{\frac{1}{4}}, \quad \xi \in (0, 1), \quad \|e\|_X \leq 2r, \] \hspace{2cm} (5.56)
r being the constant of (5.16).
(5.42), (5.47), (5.51), and (5.53) yield
\[ |\gamma_3(\xi)| \leq 8\tilde{c}\hat{c}\|e\|_X \left( \int_\Omega |e|^4 \, dx \right)^{\frac{1}{4}} \left( \int_\Omega (\alpha \sqrt{n} + |\tilde{u} + v_\lambda + \xi e|)^{-4} \, dx \right)^{\frac{1}{4}} \leq c_1 \tilde{c}\|e\|^2_X. \] \hspace{2cm} (5.57)
Assume that
\[ \mu_1 - c_1(b_0 + \tilde{c}) = c_0 > 0. \] \hspace{2cm} (5.58)
Then \( \frac{\partial b}{\partial y}(\xi) \geq c_0 \|e\|^2_X \), and by (5.41), (5.43) we obtain that \( e = 0 \). Thus, we proved the following:

**Theorem 5.2.** Suppose that the conditions \((C1) \ (\varphi \ replaced \ by \ \psi), \ (C4), \ (4.6), \ (4.7), \ (5.42), \ (5.58)\) are satisfied. Then there exists a unique solution of \((5.5)-(5.7)\) in the ball
\[ d_r = \{ u \in X, \quad \|u\| \leq r = a_1^{-1}\|K + F\|_{X^*}. \}\]

Note that in the case that the values of the function \(|E|\) are small the constants \(b_0\) and \(\tilde{c}\) are small, (5.58) is satisfied and there exists a unique solution of \((5.5)-(5.7)\).
6. Variational inequality for the extended Bingham electrorheological fluid.

We now consider a problem on stationary flow of the extended Bingham electrorheological fluid. The constitutive equation of this fluid is the following:

\[ \varphi(I(u), |E|, \mu(u, E)) = \frac{b(|E|, \mu(u, E))}{I(u)^{\frac{1}{2}}} + b_1(|E|, \mu(u, E)). \]  \hfill (6.1)

We deal with the problem (4.1)–(4.4) and assume that (4.5) and (4.7) are satisfied. Then, according to Remark 4.1 the generalized solution of our problem is \( u = \tilde{u} + v \), where \( \tilde{u} \) is a function satisfying (4.6) and \( v \) is a solution of the problem

\[ v \in V, \]

\[ J(v, h) - J(v, v) + (L_1(v), h - v) \geq (K + F, h - v), \quad h \in V. \]  \hfill (6.2)

Here, \( J \) is the functional given by (4.9) and the operator \( L_1 : X \to X^* \) is defined as follows

\[ (L_1(v), h) = 2 \int_{\Omega} b_1(|E|, \mu(\tilde{u} + v, E)) \varepsilon_{ij}(\tilde{u} + v) \varepsilon_{ij}(h) dx, \quad v, h \in X. \]  \hfill (6.3)

The function \( b_1 \) is subject to the following condition:

(C5): \( b_1 : (y_1, y_2) \to b_1(y_1, y_2) \) is a continuous function on \( \mathbb{R}_+ \times [0,1] \) and satisfies

\[ a_6 \leq b_1(y_1, y_2) \leq a_7, \quad (y_1, y_2) \in \mathbb{R}_+ \times [0,1], \]  \hfill (6.4)

with positive constants \( a_6 \) and \( a_7 \).

We approximate the functional \( J \) by \( J_\lambda \) as given by (5.1). Replacing \( J \) by \( J_\lambda \), by analogy with the reasoning in the proof of Theorem 4.1 we obtain the following problem: find \( v_\lambda \) such that

\[ v_\lambda \in V, \]  \hfill (6.5)

\[ \left( \frac{\partial J_\lambda}{\partial h}(v_\lambda, v_\lambda), w \right) + (L_1(v_\lambda), w) = (K + F, w), \quad w \in V, \]  \hfill (6.6)

with \( \frac{\partial J_\lambda}{\partial h} \) given by (5.2).

**Theorem 6.1.** Suppose that conditions (4.6), (4.7), (C4), (C5) are satisfied. Then there exists a solution \( v \) of (6.2), (6.3). Moreover, for an arbitrary \( \lambda > 0 \) there exists a solution of (6.6), (6.7), and there exists a function \( p_\lambda \) such that the pair \( (v_\lambda, p_\lambda) \) is a solution of the problem

\[ (v_\lambda, p_\lambda) \in X \times L_2(\Omega), \]  \hfill (6.8)

\[ \left( \frac{\partial J_\lambda}{\partial h}(v_\lambda, v_\lambda), w \right) + (L_1(v_\lambda), w) - (B^* p_\lambda, w) = (K + F, w), \quad w \in X, \]  \hfill (6.9)

\[ (Bv_\lambda, q) = 0, \quad q \in L_2(\Omega). \]  \hfill (6.10)

A subsequence can be extracted from the sequence \( \{v_\lambda\} \), again denoted by \( \{v_\lambda\} \), such that

\[ v_\lambda \to v \text{ in } X \text{ and } v_\lambda \to _{L_2(\Omega)} v \text{ as } \lambda \to 0. \]  \hfill (6.11)

If \( I(\tilde{u} + v) \neq 0 \) almost everywhere in \( \Omega \), then the functional \( h \to J(v, h) \) is Gâteaux differentiable at the point \( v \), and there exists a function \( p \in L_2(\Omega) \) such that the pair \( (v, p) \) is a
solution of the problem

\[ v \in V, \quad p \in L_2(\Omega), \]  
\[ \left( \frac{\partial J}{\partial h}(v, v), h \right) + (L_1(v), h) - (B^*p, h) = (K + F, h), \quad h \in X, \]  
\hspace{1cm} (6.12)

with \( \frac{\partial J}{\partial h} \) given by (4.19).

Proof. The existence of a solution \( v_\lambda \) of (6.6), (6.7) follows from Theorem 5.1 as well as the existence of a function \( p_\lambda \) such that (6.8)–(6.10) hold. It is inferred from the proof of Theorem 5.1 (see (5.15)) that

\[ v_\lambda \text{ remains in a bounded set of } V \text{ independent of } \lambda. \]  
\hspace{1cm} (6.14)

Therefore, from sequence \( \{v_\lambda\} \) we can select a subsequence, again denoted by \( \{v_\lambda\} \), such that

\[ v_\lambda \rightharpoonup v \text{ in } X \text{ as } \lambda \to 0, \]  
\[ v_\lambda \to v \text{ in } L_2(\Omega)^n \text{ and a.e. in } \Omega. \]  
\hspace{1cm} (6.15)

For \( h \in V \) we introduce

\[ Z_\lambda = (L_1(v_\lambda), h - v_\lambda) + J_\lambda(v_\lambda, h) - J_\lambda(v_\lambda, v_\lambda) - (K + F, h - v_\lambda). \]  
\hspace{1cm} (6.17)

Using (6.9), we see that

\[ Z_\lambda = - \left( \frac{\partial J_\lambda}{\partial h}(v_\lambda, v_\lambda), h - v_\lambda \right) + J_\lambda(v_\lambda, h) - J_\lambda(v_\lambda, v_\lambda). \]  
\hspace{1cm} (6.18)

It follows from (5.1), (5.2) and Lemma 3.4 (cf. (3.23)) that for an arbitrarily fixed \( w \in X \) the functional \( u \to J_\lambda(w, u) \) is convex. Therefore

\[ Z_\lambda \geq 0. \]  
\hspace{1cm} (6.19)

(C5), (6.16) and the Lebesgue theorem give

\[ b_1(|E|, \mu(\tilde{u} + v_\lambda, E)) \varepsilon_{ij}(h) \to b_1(|E|, \mu(\tilde{u} + v), E)) \varepsilon_{ij}(h) \text{ in } L_2(\Omega). \]  
\hspace{1cm} (6.20)

(6.15), (6.20) imply

\[ \lim(L_1(v_\lambda), h) = (L_1(v), h). \]  
\hspace{1cm} (6.21)

We have

\[ (L_1(v_\lambda), v_\lambda) = A_{1\lambda} + A_{2\lambda}, \]  
\hspace{1cm} (6.22)

where

\[ A_{1\lambda} = 2 \int_{\Omega} b_1(|E|, \mu(\tilde{u} + v_\lambda, E)) \varepsilon_{ij}(\tilde{u}) \varepsilon_{ij}(v_\lambda) dx, \]  
\hspace{1cm} (6.23)

\[ A_{2\lambda} = 2 \int_{\Omega} b_1(|E|, \mu(\tilde{u} + v_\lambda, E)) I(v_\lambda) dx. \]  
\hspace{1cm} (6.24)

(6.20) still holds true if the function \( h \) is replaced by \( \tilde{u} \). Consequently, (6.15) implies

\[ \lim A_{1\lambda} = 2 \int_{\Omega} b_1(|E|, \mu(\tilde{u} + v, E)) \varepsilon_{ij}(\tilde{u}) \varepsilon_{ij}(v). \]  
\hspace{1cm} (6.25)

It follows from (6.5) and (6.16) that

\[ [b_1(|E|, \mu(\tilde{u} + v_\lambda, E))]^{1/2} w \to [b_1(|E|, \mu(\tilde{u} + v))]^{1/2} w \text{ in } L_2(\Omega), \quad w \in L_2(\Omega), \]
so that (6.15) yields
\[
\lim \int_{\Omega} \left[ b_1(|E|, \mu(\tilde{u} + v, E)) \right] \frac{1}{2} \varepsilon_{ij}(v) w dx = \int_{\Omega} \left[ b_1(|E|, \mu(\tilde{u} + v, E)) \right] \frac{1}{2} \varepsilon_{ij}(v) w dx, \\

w \in L_2(\Omega).
\]
Therefore,
\[
[b_1(|E|, \mu(\tilde{u} + v, E))]^{\frac{1}{2}} \varepsilon_{ij}(v) wdx = \int_{\Omega} [b_1(|E|, \mu(\tilde{u} + v, E))]^{\frac{1}{2}} \varepsilon_{ij}(v) w dx, \\

w \in L_2(\Omega).
\]
(6.26)
(6.24) and (6.26) give
\[
\lim \inf A_{2\lambda} \geq 2 \int_{\Omega} b_1(|E|, \mu(\tilde{u} + v, E)) I(v) dx. \\
\]
By (6.22), (6.25), and (6.27) we obtain
\[
\lim \inf (L_1(v_\lambda), v_\lambda) \geq (L_1(v), v).
\]
(5.1), (6.16) and the Lebesgue theorem give
\[
\lim J_\lambda(v_\lambda, h) = J(v, h).
\]
Setting
\[
b_\lambda = b(|E|, \mu(\tilde{u} + v, E)), \quad b_0 = b(|E|, \mu(\tilde{u} + v, E)), \\
I_\lambda = I(\tilde{u} + v_\lambda), \quad I_0 = I(\tilde{u} + v),
\]
we have
\[
J_\lambda(v_\lambda, v_\lambda) = J_\lambda(v_\lambda, v_\lambda) + B_{1\lambda}, \\
\]
where
\[
B_{1\lambda} = 2 \int_{\Omega} (b_\lambda - b_0)(\lambda + I_\lambda)^{\frac{1}{2}} dx, \\
\]
and
\[
|B_{1\lambda}| \leq 2 \left( \int_{\Omega} (\lambda + I_\lambda) dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |b_\lambda - b_0|^2 dx \right)^{\frac{1}{2}}.
\]
(6.15), (6.16) and (2.17) imply
\[
\lim B_{1\lambda} = 0.
\]
(4.9) and (5.1) yield \(J_\lambda(v, v_\lambda) \geq J(v, v_\lambda)\), whence
\[
\lim \inf J_\lambda(v, v_\lambda) \geq \lim \inf J(v, v_\lambda).
\]
(6.15) and Lemma 3.7 yield
\[
\lim \inf J(v, v_\lambda) \geq J(v, v).
\]
(6.32), (6.35), (6.36), and (6.37) give
\[
\lim \inf J_\lambda(v_\lambda, v_\lambda) \geq J(v, v).
\]
(6.10), (6.15) imply (6.2), and using (6.17), (6.19), (6.21), (6.28), (6.29), (6.38) we obtain
(6.3).

It follows from Remark 4.1 that if \(I(\tilde{u} + v) \neq 0\) almost everywhere in \(\Omega\), then there exists a function \(p\) such that (6.12), (6.13) hold.

**Remark 6.1.** Assume that in (6.3), (6.7), (6.9) the operator \(L_1\) is replaced by the operator \(L\) defined by (4.10), and the function \(\psi\) meets one of the conditions (C1), (C2),
(C3) with \( \varphi \) replaced by \( \psi \). Then, by Theorem 5.1 for an arbitrary \( \lambda > 0 \) there exists a solution \((v_\lambda, p_\lambda)\) of (6.8)-(6.10), and a subsequence \( \{v_\lambda\} \) can be extracted satisfying (6.15), (6.16).

However, (6.15), (6.16) do not imply \( \lim \inf (L(v_\lambda), v_\lambda) \geq (L(v), v) \) (compare with ((6.28))), and we cannot assert that \( v \) is a solution of (4.12), (4.13). In the next section we prove the existence of a solution of (4.12), (4.13) under conditions which are more restrictive than those of Theorem 5.1.

7. General variational inequality.

We assume that the function \( \mu \) in the operator \( L \) defined by (4.10) is replaced by a function \( \mu_1 \) such that

\[
u_m \rightharpoonup u \quad \text{in} \quad X \Rightarrow \mu_1(u_m, E) \rightarrow \mu_1(u, E) \quad \text{in} \quad L_\infty(\Omega).
\]

According to (2.7) we may define \( \mu_1 \) as follows:

\[
\mu_1(u, E)(x) = \left( \frac{\alpha \tilde{I} + Pu(x) + \tilde{u}}{\alpha \sqrt{n} + |Pu(x) + \tilde{u}|}, \frac{\beta \tilde{I} + P E(x)}{\beta \sqrt{n} + |PE(x)|} \right)^2, \quad x \in \overline{\Omega},
\]

where \( \alpha, \beta \) are small positive constants, \( \tilde{I} \) is a vector with components equal to one, and \( P \) an operator of regularization given by

\[
Pu(x) = \int_{\mathbb{R}^n} \omega(|x - x'|)u(x')dx', \quad x \in \overline{\Omega},
\]

where

\[
\omega \in C^\infty(\mathbb{R}_+), \quad \text{supp} \ \omega = [0, a], \quad \omega(z) \geq 0 \quad z \in \mathbb{R}_+,
\]

\[
\int_{\mathbb{R}^n} \omega(|x|)dx = 1, \quad a \quad \text{is a small positive constant}.
\]

In (7.3) we assume that the function \( u \) is extended to \( \mathbb{R}^n \). In case that \( Pu(x) \neq 0 \) a.e. in \( \Omega \) we may choose \( \alpha = 0 \), if \( PE(x) \neq 0 \) a.e. in \( \Omega \) we may choose \( \beta = 0 \).

For the function \( \mu_1 \) condition (7.1) is satisfied. From the physical point of view (7.2) means that the value of the function \( \mu_1 \) and therefore the viscosity of the fluid at a point \( x \) depends on the angle between the vectors of velocity and electric field strength at points belonging to some small vicinity of the point \( x \), implying that the model is not local.

This seems to be natural, since electrorheological properties of the fluid are linked with the presence of small solid particles in the fluid. The mean dimension of these particles can be taken as the regularization parameter \( a \).

In the case under consideration the operator \( L \) is defined as follows:

\[
(L(v), h) = 2 \int_{\Omega} \psi(I(\tilde{u} + v), |E|, \mu_1(\tilde{u} + v, E))\varepsilon_{ij}(\tilde{u} + v)\varepsilon_{ij}(h)dx, \quad v, h \in X.
\]

We assume also that the following condition of uniform continuity of the function \( \psi \) holds:

(C6): for an arbitrary \( \gamma > 0 \) there exists \( \varepsilon > 0 \) such that the conditions

\[
y_3', y_3'' \in [0, 1], \quad |y_3' - y_3''| \leq \varepsilon, \quad y_1, y_2 \in \mathbb{R}_+
\]

imply

\[
|\psi(y_1, y_2, y_3') - \psi(y_1, y_2, y_3'')| \leq \gamma.
\]

The function \( \mu \) can as well be replaced by the function \( \mu_1 \) in the functionals \( J \) and \( J_\lambda \). In the following theorem this is not assumed, although it is also valid in this case.
\textbf{Theorem 7.1.} Suppose that the conditions (C4), (4.6), (4.7) are satisfied and assume that the function \( \psi \) meets one of the conditions (C1), (C2), (C3) (\( \varphi \) replaced by \( \psi \)) and that (C6) holds. Further assume that the function \( \mu_1 \) meets condition (7.1), and the operator \( L \) is given by (7.5). Then, for an arbitrary \( \lambda > 0 \) there exists a solution \( (v_\lambda, p_\lambda) \) of (5.5)–(5.7), and there exists a solution \( v \) of (4.12), (4.13). A subsequence can be selected from the sequence \( \{v_\lambda\} \), again denoted by \( \{v_\lambda\} \), such that
\[ v_\lambda \to v \text{ in } X \text{ and } v_\lambda \to v \text{ in } L_2(\Omega)^n \text{ as } \lambda \to 0. \]

\textit{Proof.} 1) The existence of a solution \( (v_\lambda, p_\lambda) \) of (5.5)–(5.7) follows from Theorem 5.1, and it is inferred from the proof of this theorem (see (5.15)), that \( v_\lambda \) remains in a bounded set of \( V \) independent of \( \lambda \). Therefore, from the sequence \( \{v_\lambda\} \) we can select a subsequence, again denoted by \( \{v_\lambda\} \), such that
\[ v_\lambda \to v \text{ in } V \text{ as } \lambda \to 0, \quad (7.6) \]
\[ v_\lambda \to v \text{ in } L_2(\Omega)^n \text{ and a.e. in } \Omega. \quad (7.7) \]

For every \( v_\lambda \) we define a functional \( \Psi_\lambda \) as follows:
\[ \Psi_\lambda(v) = J_\lambda(v_\lambda, v) + \Phi(v_\lambda, v) - (K + F, v), \quad v \in V, \quad (7.8) \]
where
\[ \Phi(v_\lambda, v) = \int_{\Omega} \left( \int_0^{I(\tilde{u} + v)} \psi(\xi, |E|, \mu_1(\tilde{u} + v_\lambda, E)) d\xi \right) dx. \quad (7.9) \]

Consider the problem: find a function \( \bar{v} \) satisfying
\[ \bar{v} \in V, \quad \Psi_\lambda(\bar{v}) = \min_{v \in V} \Psi_\lambda(v). \quad (7.10) \]
If \( \bar{v} \) is a solution of (7.10), then we have
\[ \bar{v} \in V, \quad \frac{d}{dt} \Psi_\lambda(\bar{v} + th) \bigg|_{t=0} = \left( \frac{\partial J_\lambda}{\partial v}(v_\lambda, \bar{v}), h \right) + (L(v_\lambda, \bar{v}), h) - (K + F, h) = 0, \quad h \in V. \quad (7.11) \]

Here, \( L(v_\lambda, \bar{v}) \) is the Gâteaux derivative of the functional \( \Phi(v_\lambda, \cdot) : u \to \Phi(v_\lambda, u) \) at the point \( \bar{v} \), i.e.
\[ \frac{\partial \Phi}{\partial v}(v_\lambda, \bar{v}) = L(v_\lambda, \bar{v}). \]
The operator \( L(v_\lambda,:) : X \ni u \to L(v_\lambda, u) \in X^* \) has the form
\[ (L(v_\lambda, u), h) = 2 \int_{\Omega} \psi(I(\tilde{u} + u), |E|, \mu_1(\tilde{u} + v_\lambda, E)) \varepsilon_{ij}(\tilde{u} + u) \varepsilon_{ij}(h) dx, \quad u, h \in X, \quad (7.12) \]
and (7.5) yields
\[ L(v, v) = L(v). \quad (7.13) \]
It follows from (5.3), (5.4) that the function \( \bar{v} = v_\lambda \) is a solution of (7.11). By means of Lemmas 3.1–3.4 the functional \( \Psi_\lambda \) is strictly convex. Therefore, there exists a unique solution \( \bar{v} = v_\lambda \) of (7.11), and problems (7.10) and (7.11) are equivalent.

(7.10) implies
\[ J_\lambda(v_\lambda, h) + \Phi(v_\lambda, h) - J_\lambda(v_\lambda, v_\lambda) - \Phi(v_\lambda, v_\lambda) \geq (K + F, h - v_\lambda), \quad h \in V. \quad (7.14) \]
2) It follows from (7.6), (7.7) and the proof of Theorem 6.1 (see (6.29), (6.38)) that
\[
\lim J_\lambda(v_\lambda, h) = J(v, h), \quad \liminf J_\lambda(v_\lambda, v_\lambda) \geq J(v, v).
\] (7.15)

We have
\[
\Phi(v_\lambda, h) = \int_\Omega f_\lambda dx, \quad \Phi(v, h) = \int_\Omega f dx,
\] (7.16)

where
\[
f_\lambda(x) = \int_0^{I(\tilde{u} + h)(x)} \psi(\xi, |E(x)|, \mu_1(\tilde{u} + v_\lambda, E)(x))d\xi,
\]
\[
f(x) = \int_0^{I(\tilde{u} + h)(x)} \psi(\xi, |E(x)|, \mu_1(\tilde{u} + v, E)(x))d\xi.
\]

(7.7) and (C6) imply \(f_\lambda \to f\) almost everywhere in \(\Omega\), and (2.13) yields
\[
|f_\lambda| \leq a_2 I(\tilde{u} + h).
\]

Thus (7.16) and the Lebesgue theorem give
\[
\lim \Phi(v_\lambda, h) = \Phi(v, h).
\] (7.17)

It is obvious that
\[
\Phi(v_\lambda, v_\lambda) = \Phi(v_\lambda) + \alpha_\lambda,
\] (7.18)

\[
\alpha_\lambda = \Phi(v_\lambda, v_\lambda) - \Phi(v_\lambda, v_\lambda) = \int_\Omega \left\{ \int_0^{I(\tilde{u} + v_\lambda)} \psi(\xi, |E|, \mu_1(\tilde{u} + v_\lambda, E))
\]
\[
- \psi(\xi, |E|, \mu_1(\tilde{u} + v, E)) \right\}d\xi dx.
\]

(7.1), (7.6) and (C6) imply
\[
\alpha_\lambda \leq \beta_\lambda \int_\Omega I(\tilde{u} + v_\lambda)dx,
\]
and \(\lim \beta_\lambda = 0\). Therefore \(\lim \alpha_\lambda = 0\).

The functional \(\Phi(v, .) : u \to \Phi(v, u)\) is continuous in \(X\). Indeed, let \(u_m \to u\) in \(X\). We have
\[
|\Phi(v, u_m) - \Phi(v, u)| = \left| \int_\Omega \left( \int_0^{I(\tilde{u} + u_m)} \psi(\xi, |E|, \mu_1(\tilde{u} + v, E))d\xi \right) dx \right|
\]
\[
\leq a_2 \left| \int_\Omega (I(\tilde{u} + u_m) - I(\tilde{u} + u)) dx \right|.
\] (7.19)

and the right hand side of this inequality tends to zero as \(m \to \infty\).

By Lemmas 3.1, 3.2, 3.3 the functional \(\Phi(v, .)\) is convex in \(X\). Therefore, \(\Phi(v, .)\) is lower semi-continuous for the weak topology on \(X\), and (7.6), (7.18) imply
\[
\liminf \Phi(v_\lambda, v_\lambda) \geq \Phi(v, v).
\] (7.20)

By (7.15), (7.17), (7.20) we pass to the limit as \(\lambda \to 0\) in (7.14). This gives
\[
J(v, h) + \Phi(v, h) - J(v, v) - \Phi(v, v) \geq (K + F, h - v), \quad h \in V.
\] (7.21)
Taking into account (7.21) and the convexity of the functional \( J(v,.) : u \to J(v,u) \), we get

\[
J(v,v) + \Phi(v,v) - (K + F,v) \\
\leq J(v,(1-\theta)v + \theta h) + \Phi(v,(1-\theta)v + \theta h) - (K + F,(1-\theta)v + \theta h) \\
\leq (1-\theta)J(v,v) + \theta J(v,h) + \Phi(v,(1-\theta)v + \theta h) - (K + F,(1-\theta)v + \theta h), \theta \in (0,1),
\]

whence

\[
\frac{\Phi(v,(1-\theta)v + \theta h) - \Phi(v,v)}{\theta} + J(v,h) - J(v,v) - (K + F,h - v) \geq 0. \tag{7.22}
\]

For \( \theta \to 0 \) we get (4.13), with the operator \( L \) defined by (7.12), (7.13). \[\blacksquare\]

**Remark 7.1.** We have shown that (7.21) implies (4.13). Let us show that (4.13) yields (7.21), that is problems (4.12), (4.13) and (4.12), (7.21) are equivalent.

Let (4.12), (4.13) be valid. Obviously,

\[
J(v,h) + \Phi(v,h) - J(v,v) - \Phi(v,v) - (K + F,h - v) \\
= J(v,h) - J(v,v) + (L(v,v),h - v) + \Phi(v,h) - \Phi(v,v) \\
- (L(v,v),h - v) - (K + F,h - v) \tag{7.23}
\]

The functional \( u \to \Phi(v,u) \) is convex, whence

\[
\Phi(v,h) - \Phi(v,v) - (L(v,v),h - v) \geq 0. \tag{7.24}
\]

(4.13), (7.13), (7.23), (7.24) give (7.21).

**Remark 7.2.** By comparing Theorems 5.1 and 7.1 we observe that a solution of the operator equations for a fluid with bounded viscosity function (2.12) exists under less restrictive conditions than the conditions for the existence of a solution of the variational inequality (4.13) for a fluid with unbounded viscosity function (2.11). In addition, such an important characteristic of the flow as the function of pressure is defined for (2.11) only in case that (4.16) holds, i.e., when the viscosity function does not take infinite values. But in this case, the variational inequality reduces to operator equations as outlined in Section 4.

Moreover, from a physical point of view a fluid with finite viscosity (2.12) seems to be more reasonable than a fluid with unbounded viscosity (2.11).

### 8. Problems with given function \( \mu \).

In the case that the distance between the electrodes is small compared with the lengths of the electrodes, one can assume that in between the electrodes the velocity vector is orthogonal to the vector of electric field strength, and the electric fields strength is equal to zero in the remaining part of the domain under consideration.

In this case one reckon that \( \mu(u,E) \) is a known function of \( x \), so that the viscosity functions (2.11) and (2.12) take the form

\[
\varphi(I(u),|E|,x) = \frac{e(|E|,x)}{I(u)^{\frac{1}{2}}} + \psi_1(I(u),|E|,x), \tag{8.1}
\]

\[
\varphi(I(u),|E|,x) = e(|E|,x)(\lambda + I(u))^{-\frac{1}{2}} + \psi_1(I(u),|E|,x), \tag{8.2}
\]

and the constitutive equation is defined by (2.1).

Dependence of the viscosity function on \( x \) in (8.1), (8.2) is connected with the anisotropy of the fluid. If the direction of the velocity vector at each point \( x \) at which \( E(x) \neq 0 \) is known, then the viscosity functions (2.11) and (2.12) transform in relations (8.1), (8.2).

We assume the function \( \psi_1 \) to satisfy
(CO): for almost all \( x \in \Omega \) the function \( \psi_1(.,., x) : (y_1, y_2) \to \psi_1(y_1, y_2, x) \) is continuous in \( \mathbb{R}_+^2 \), and for an arbitrarily fixed \( (y_1, y_2) \in \mathbb{R}_+^2 \) the function \( \psi_1(y_1, y_2, .) : x \to \psi_1(y_1, y_2, x) \) is measurable in \( \Omega \).

We also suppose that for almost all \( x \in \Omega \) and all \( y_2 \in \mathbb{R}_+ \), the function \( \psi_1(., y_2, x) : y_1 \to \psi_1(y_1, y_2, x) \) satisfies one of the following conditions (C1a), (C2a), (C3a):

(C1a): \( \psi_1(., y_2, x) \) is continuously differentiable in \( \mathbb{R}_+ \) and the following inequalities hold:

\[
\frac{\partial \psi_1}{\partial y_1}(y_1, y_2, x) |_{y_1} \leq a_4.
\]

(C2a): (8.3) is fulfilled and for an arbitrary \( (z_1, z_2) \in \mathbb{R}_+^2 \) the following inequality is valid:

\[
[\psi_1(z_1^2, y_2, x)z_1 - \psi_1(z_2^2, y_2, x)z_2](z_1 - z_2) \geq a_3(z_1 - z_2)^2.
\]

(C3a): (8.3) is fulfilled and the function \( z \to \psi_1(z^2, y_2, x)z \) is strictly increasing in \( \mathbb{R}_+ \), i.e., the conditions \( z_1, z_2 \in \mathbb{R}_+, z_1 > z_2 \) imply \( \psi_1(z_1^2, y_2, x)z_1 > \psi_1(z_2^2, y_2, x)z_2 \).

(C1a), (C2a), (C3a) are analogies of conditions (C1), (C2), (C3), and an analog of (C4) is the following condition:

(C4a): for almost all \( x \in \Omega \) the function \( e(., x) : y \to e(y, x) \) is continuous in \( \mathbb{R}_+ \) and for an arbitrarily fixed \( y \in \mathbb{R}_+ \), the function \( e(y, .) : x \to e(y, x) \) is measurable in \( \Omega \) and

\[
0 \leq e(y, x) \leq a_5.
\]

Define functionals \( Y \) and \( Y_\lambda, \lambda > 0 \), as follows:

\[
Y(u) = 2 \int_{\Omega} e(|E|, x)I(\bar{u} + u)\frac{1}{2} dx, \quad u \in X,
\]

\[
Y_\lambda(u) = 2 \int_{\Omega} e(|E|, x)(\lambda + I(\bar{u} + u))\frac{1}{2} dx, \quad u \in X.
\]

Define also an operator \( L_2 : X \to X^* \) by means of

\[
(L_2(u), h) = 2 \int_{\Omega} \psi_1(I(\bar{u} + u), |E|, x)\varepsilon_{ij}(\bar{u} + u)\varepsilon_{ij}(h) dx, \quad u, h \in X.
\]

Consider the following two problems:

Problem 1. Find a pair of functions \( (v_\lambda, p_\lambda) \) such that

\[
v_\lambda \in X, \quad p_\lambda \in L_2(\Omega),
\]

\[
\left( \frac{\partial Y_\lambda}{\partial u}(v_\lambda, h) + (L_2(v_\lambda), h) - (B^*p_\lambda, h) = (K + F, h), \quad h \in X,
\]

\[
(Bv_\lambda, q) = 0, \quad q \in L_2(\Omega).
\]
Problem 2. Find a function \( v \) such that

\[
Y(h) - Y(v) + (L_2(v), h - v) \geq (K + F, h - v), \quad h \in V.
\] (8.14)

Here, the operator \( \frac{\partial Y_\lambda}{\partial u} : X \to X^* \) is given by

\[
\left( \frac{\partial Y_\lambda}{\partial u}(u), h \right) = 2 \int_\Omega e(\|E\|, x)(\lambda + I(\tilde{u} + u))^{-\frac{1}{2}} \epsilon_{ij}(\tilde{u} + u)\epsilon_{ij}(h)dx, \quad u, h \in X.
\] (8.16)

If \( (v_\lambda, p_\lambda) \) is a solution of Problem 1, then the pair \( (\tilde{u} + v_\lambda, p_\lambda) \) is a generalized solution of (4.1)–(4.4) with the viscosity function defined by (8.2). If \( v \) is a solution of Problem 2, then \( \tilde{u} + v \) is a generalized solution of (4.1)–(4.4) with the viscosity function defined by (8.1).

**Theorem 8.1.** Suppose that the conditions (4.6), (4.7), (C4a) are satisfied, and let the function \( \psi_1 \) satisfy both (C0) and one of the conditions (C1a), (C2a), (C3a). Then, for an arbitrary \( \lambda > 0 \) there exists a unique solution \( (v_\lambda, p_\lambda) \) of (8.11)–(8.13). Moreover, there exists a unique solution \( v \) of (8.14)–(8.15). In addition \( v_\lambda \to v \) in \( V \) as \( \lambda \to 0 \).

**Proof.** The existence of a solution \( (v_\lambda, p_\lambda) \) of (8.11)–(8.13) for an arbitrary \( \lambda > 0 \) follows from Theorem 5.1. Let \( (v_1^\lambda, p_1^\lambda) \) and \( (v_2^\lambda, p_2^\lambda) \) be two solutions of (8.11)–(8.13). By (8.12) we obtain

\[
\left( \frac{\partial Y_\lambda}{\partial u}(v_1^\lambda) + L_2(v_1^\lambda) - \frac{\partial Y_\lambda}{\partial u}(v_2^\lambda) - L_2(v_2^\lambda), v_1^\lambda - v_2^\lambda \right) = 0.
\] (8.17)

By Lemmas 3.1–3.4, the operator \( \frac{\partial Y_\lambda}{\partial u} + L_2 \) is strictly monotone. Consequently (8.17) implies \( v_1^\lambda = v_2^\lambda \), whence \( p_1^\lambda = p_2^\lambda \).

By Theorem 7.1 from the sequence \( \{v_\lambda\} \) a subsequence, again denoted by \( \{v_\lambda\} \), can be selected such that \( v_\lambda \to v \) in \( X \) where \( v \) is a solution of (8.14), (8.15). Also, Remark 7.1 infers

\[
v \in V, \quad Y(h) + \Phi_1(h) - Y(v) - \Phi_1(v) \geq (K + F, h - v), \quad h \in V,
\] (8.18)

where

\[
\Phi_1(u) = \int_\Omega \left( \int_0^{\|E\|} \psi_1(\xi, |E|, x) d\xi \right) dx, \quad u \in V,
\] (8.19)

and

\[
\left( \frac{\partial \Phi_1}{\partial u}(u), h \right) = (L_2(u), h).
\] (8.20)

In addition, the problems (8.14), (8.15) and (8.18) are equivalent. The functional \( Y \) is convex, and the functional \( \Phi_1 \) is strictly convex. Therefore the functional

\[
\Psi_1(u) = Y(u) + \Phi_1(u) - (K + F, u), \quad u \in V,
\]

is strictly convex, and if \( v_1, v_2 \) are two solutions of the problem (8.18), then we have

\[
\Psi_1\left(\frac{1}{2}(v_1 + v_2)\right) < \frac{1}{2} \Psi_1(v_1) + \frac{1}{2} \Psi_1(v_2) = \inf_{h \in V} \Psi_1(h).
\]

Whence \( v_1 = v_2 \). ■
9. Numerical solution of stationary problems.

9.1. Theorem on convergence. Let \( \{X_m\}, \{N_m\} \) be sequences of finite-dimensional subspaces in \( X \) and \( L_2(\Omega) \), respectively, such that (3.38), (3.39), (3.43) and (5.11) hold. We denote

\[
L_3 = \frac{\partial Y_\lambda}{\partial u} + L_2,
\]

where \( L_2 \) and \( \frac{\partial Y_\lambda}{\partial u} \) are the operators defined by (8.10), (8.16).

We seek an approximate solution \((v_m, p_m)\) of problem (8.11)–(8.13) of the form

\[
(v_m, p_m) \in X_m \times N_m,
\]

\[
(L_3(v_m), h) - (B_m^* p_m, h) = (K + F, h), \quad h \in X_m,
\]

\[
(B_m v_m, q) = 0, \quad q \in N_m.
\]

We remind that the operator \( B_m \) is defined by (3.40) and \( B_m^* \) is the adjoint operator of \( B_m \).

**Theorem 9.1.** Suppose that conditions (4.6), (4.7), (C4a) are satisfied, and let the function \( \psi_1 \) satisfy both (C0) and (C3a). Let also (3.38), (3.39), (3.43) and (5.11) are fulfilled. Then, for an arbitrary \( m \in \mathbb{N} \) there exists a unique solution of (9.2)–(9.4) and

\[
v_m \rightharpoonup v_\lambda \text{ in } X, \quad p_m \rightharpoonup p_\lambda \text{ in } L_2(\Omega),
\]

where \( v_\lambda, p_\lambda \) is the solution of (8.11)–(8.13). If in addition \( \psi_1 \) satisfies (C1a) or (C2a), then

\[
v_m \to v_\lambda \text{ in } X,
\]

\[
p_m \to p_\lambda \text{ in } L_2(\Omega).
\]

**Proof.** The existence of a unique solution \((v_m, p_m)\) of the problem (9.2)–(9.4) and the relations (9.5) follows from Theorems 5.1, 8.1. The following equalities also arise from the proof of Theorem 5.1 (see (5.35), (5.36))

\[
(L_3(v_m), v_m) = (K + F, v_m) \to (K + F, v_\lambda),
\]

\[
\lim (L_3(v_m), h) - (B_m^* p_m, h) = (K + F, h), \quad h \in X.
\]

(8.12), (8.9) yield

\[
\lim (L_3(v_m) - L(v_\lambda), v_m - v_\lambda) = 0.
\]

Assume that (C2a) is satisfied; if \( \psi_1 \) meets (C1a) it meets also (C2a) (see Subsection 2.2). Then observing Lemmas 3.2, 3.4, we obtain

\[
(L_3(v_m) - L_3(v_\lambda), v_m - v_\lambda) \geq 2a_3 \int_\Omega [I(\tilde{u} + v_m)^{1/2} - I(\tilde{u} + v_\lambda)^{1/2}]^2 dx,
\]

and (9.10) implies

\[
\int_\Omega [I(\tilde{u} + v_m)^{1/2} - I(\tilde{u} + v_\lambda)^{1/2}]^2 dx \to 0.
\]

From here, taking into account that the function \( u \to \int_\Omega u^2 dx \) is a continuous mapping from \( L_2(\Omega) \) into \( \mathbb{R} \), we obtain

\[
\int_\Omega I(\tilde{u} + v_m) dx \to \int_\Omega I(\tilde{u} + v_\lambda) dx.
\]
It is obvious that
\[ \|v_m - v_\lambda\|^2_X = \int_\Omega \sum_{i,j=1}^n (\varepsilon_{ij}(v_m - v_\lambda))^2 \, dx = \int_\Omega \sum_{i,j=1}^n [\varepsilon_{ij}(\tilde{u} + v_m) - \varepsilon_{ij}(\tilde{u} + v_\lambda)]^2 \, dx \]
\[ = \int_\Omega I(\tilde{u} + v_m) \, dx + \int_\Omega I(\tilde{u} + v_\lambda) \, dx - 2 \int_\Omega \varepsilon_{ij}(\tilde{u} + v_m) \varepsilon_{ij}(\tilde{u} + v_\lambda) \, dx. \]  
(9.13)

By virtue of (9.5), (9.12) the right-hand side of (9.13) tends to zero. Therefore, (9.6) holds true.

It follows from (8.12) and (9.1) that
\[ (L_3(v_\lambda), h) - (B^* p_\lambda, h) = (K + F, h), \quad h \in X_m, \]  
(9.14)

By (9.3) and (9.14) we get
\[ (B^* (p_m - \mu), h) = (L_3(v_m) - L_3(v_\lambda), h) + (B^*(p_\lambda - \mu), h), \quad h \in X_m, \quad \mu \in N_m. \]

This equality, together with (3.43), yields
\[ \|p_m - \mu\|_{L_2(\Omega)} \leq \sup_{h \in X_m} \frac{(B^*(p_m - \mu), h)}{\beta \|h\|_X} \leq \beta^{-1} \beta^{-1} \|L_3(v_m) - L_3(v_\lambda)\|_{X^*} + c \|p_\lambda - \mu\|_{L_2(\Omega)}, \quad \mu \in N_m, \]  
(9.15)

where
\[ c = \|B^*\|_{L_2(\Omega), X^*} = \|B\|_{L_2(\Omega), X}. \]

Hence
\[ \|p_\lambda - p_m\|_{L_2(\Omega)} \leq \|p_\lambda - \mu\|_{L_2(\Omega)} + \|p_m - \mu\|_{L_2(\Omega)} \leq \beta^{-1} \|L_3(v_m) - L_3(v_\lambda)\|_{X^*} + (c\beta^{-1} + 1) \inf_{\mu \in N_m} \|p_\lambda - \mu\|_{L_2(\Omega)}. \]  
(9.16)

Lemmas 3.2, 3.4 and (9.6) imply \( L_3(v_m) \to L_3(v_\lambda) \) in \( X^* \), and (9.7) follows from (9.16) and (3.39).

9.2. A saddle-point approach. We introduce two functionals
\[ J : X \to \mathbb{R}, \quad \Psi : X \times L_2(\Omega) \to \mathbb{R} \]
defined by
\[ J(u) = \int_\Omega \left( \int_0^{I(\tilde{u} + u)} \left( e(|E|, x)(\lambda + \xi)^{-\frac{1}{2}} + \psi_1(\xi, |E|, x) \right) d\xi \right) \, dx - (K + F, u), \]  
(9.17)
\[ \Psi(u, \nu) = J(u) - (B^* \nu, u). \]  
(9.18)

Problem: find a saddle-point of the Lagrangian \( \Psi \), i.e.
\[ v_\lambda, p_\lambda \in X \times L_2(\Omega), \]
\[ \Psi(v_\lambda, \nu) \leq \Psi(v_\lambda, p_\lambda) \leq \Psi(u, p_\lambda), \quad u \in X, \quad \mu \in L_2(\Omega). \]  
(9.19)

**Theorem 9.2.** Suppose that conditions (4.6), (4.7), (C4a) are satisfied, and let the function \( \psi_1 \) meet (C0) and one of the conditions (C1a), (C2a), (C3a). Then, for an arbitrary \( \lambda > 0 \) there exists a unique solution \( v_\lambda, p_\lambda \) of problem (9.19), (9.20) which is the solution of problem (8.11)–(8.13), and the problems (8.11)–(8.13) and (9.19), (9.20) are equivalent.
Taking $\nu$ By subtracting (8.12) from (9.23) and (8.13) from (9.24), we get (where $X$ is a positive constant).

Here $\rho$ is convex for an arbitrarily fixed $u$. By Lemmas 3.1–3.4 the operator $L_2$ is monotone, and so the functional $u \mapsto \Psi(u, q)$ is convex for an arbitrarily fixed $q \in L_2(\Omega)$ (see e.g. [5]). Therefore, the minimum of the functional $u \mapsto \Psi(u, p)$ is characterized by (8.12).

The first inequality in (9.20) gives $(Bv, p) \leq 0$, for all $\nu \in L_2(\Omega)$, and so we get (8.13). However in this case $\Psi(v, p) \leq 0$ for all $\nu \in L_2(\Omega)$. Thus, the problems (8.11) and (8.13) and (9.19), (9.20) are equivalent. The existence and uniqueness of the solution of (9.19), (9.20) follows from Theorem 8.1. ■

Now we introduce the augmented Lagrangian as follows:

$$
\Psi_1(u, \nu) = \Psi(u, \nu) + \frac{r}{2}(Bu, Bu), \quad (u, \nu) \in X \times L_2(\Omega).
$$

(9.21)

It is obvious that the pair $(v, p)$ is also the saddle point of the functional $\Psi_1$, where $r$ is an arbitrary positive constant.

We study

Algorithm of the augmented Lagrangian: find a sequence $\{v_m, p_m\}$ satisfying

$$(v_{m+1}, p_{m+1}) \in X \times L_2(\Omega),$$

$$L_3(v_{m+1}, h) - (B^* p_m, h) + r(Bv_{m+1}, Bh) = (K + F, h), \quad h \in X,$$

$$p_{m+1} = p_m + \rho(Bu_{m+1} + \nu) = 0, \quad \nu \in L_2(\Omega).$$

(9.22)

(9.23)

(9.24)

Here $\rho_m$ is a positive constant.

**Theorem 9.3.** Suppose that conditions (4.6), (4.7), (C4a) are satisfied, and let the function $\psi_1$ meets (C0) and one of the conditions (C1a), (C2a). Assume that $(v_0, p_0)$ is an arbitrary pair in $X \times L_2(\Omega)$. Then, for any arbitrary $m$ there exists a unique pair $(v_{m+1}, p_{m+1})$ satisfying (9.22)–(9.24). Moreover, if

$$0 < \inf_{m \in \mathbb{N}} \rho_m \leq \sup_{m \in \mathbb{N}} \rho_m < 2r,$$

then

$$v_m \to v, \quad p_m \to p \text{ in } L_2(\Omega),$$

(9.25)

(9.26)

where $(v, p)$ is the solution of (8.11)–(8.13).

**Proof.** We set

$$u_m = v_m - v, \quad q_m = p_m - p.$$

(9.27)

By subtracting (8.12) from (9.23) and (8.13) from (9.24), we get

$$L_3(v_{m+1}) - L_3(v_m) + r(Bu_{m+1} + Bh) = (K + F, h), \quad h \in X,$$

$$q_{m+1} - q_m = -\rho_m(Bu_{m+1} + \nu), \quad \nu \in L_2(\Omega).$$

(9.28)

(9.29)

Taking $\nu = 2q_m$ in (9.29), we obtain

$$\|q_{m+1}\|_{L_2(\Omega)}^2 - \|q_m\|_{L_2(\Omega)}^2 + \|q_{m+1} - q_m\|_{L_2(\Omega)}^2 = -2\rho_m(Bu_{m+1} + q_{m+1}).$$

(9.30)
Take $h = u_{m+1}$ in (9.28). Then (9.28)–(9.30) give
\[
\|q_{m+1}\|^2_{L_2(\Omega)} - \|q_m\|^2_{L_2(\Omega)} + \|q_{m+1} - q_m\|^2_{L_2(\Omega)} + 2\rho_m (L_3(v_{m+1}) - (L_3(v_\lambda), u_{m+1}) + 2\rho_m r\|Bu_{m+1}\|^2_{L_2(\Omega)} = -2\rho_m (B u_{m+1}, q_{m+1} - q_m).
\] (9.31)

(9.31) and Lemma 3.2 (see (3.15)) imply
\[
\|q_{m+1}\|^2_{L_2(\Omega)} - \|q_m\|^2_{L_2(\Omega)} + \|q_{m+1} - q_m\|^2_{L_2(\Omega)} + 4\rho_m a_3 \int_\Omega |I(\tilde{u} + v_{m+1})^{1/2} - I(\tilde{u} + v_\lambda)^{1/2}|^2 \, dx + 2\rho_m r\|Bu_{m+1}\|^2_{L_2(\Omega)} \leq 2\rho_m \|Bu_{m+1}\|_{L_2(\Omega)} \|q_{m+1} - q_m\|_{L_2(\Omega)}. \tag{9.32}
\]
By applying the inequality $2ab \leq a^2 + b^2$ to the right-hand side of (9.32), we obtain
\[
\|q_{m+1}\|^2_{L_2(\Omega)} - \|q_m\|^2_{L_2(\Omega)} + 4\rho_m a_3 \int_\Omega |I(\tilde{u} + v_{m+1})^{1/2} - I(\tilde{u} + v_\lambda)^{1/2}|^2 \, dx + \rho_m (2r - \rho_m) \|Bu_{m+1}\|^2_{L_2(\Omega)} \leq 0. \tag{9.33}
\]
By virtue of (9.25) there exists $\delta > 0$ such that $\rho_m (2r - \rho_m) \geq \delta$ for any $m$, and by (9.33), $\|q_m\|_{L_2(\Omega)} \geq \|q_{m+1}\|_{L_2(\Omega)}$. Thus, the sequence $\{|q_m|^2_{L_2(\Omega)}\}$ converges, i.e. $\lim\|q_m\|_{L_2(\Omega)} = \alpha \geq 0$, and (9.33) yields
\[
\int_\Omega |I(\tilde{u} + v_{m+1})^{1/2} - I(\tilde{u} + v_\lambda)^{1/2}|^2 \, dx \to 0, \tag{9.34}
\]
\[
Bu_m \to 0 \text{ in } L_2(\Omega). \tag{9.35}
\]
Since the function $u \to \int_\Omega u^2 \, dx$ is a continuous mapping from $L_2(\Omega)$ into $\mathbb{R}$ we obtain from (9.34) that
\[
\int_\Omega I(\tilde{u} + v_m) \, dx \to \int_\Omega I(\tilde{u} + v_\lambda) \, dx. \tag{9.36}
\]
Therefore
\[
\|v_m\|_X \leq c, \quad m \in \mathbb{N}, \tag{9.37}
\]
and by (9.23), (3.37) we get $\|p_m\|_{L_2(\Omega)} \leq c$ for all $m$. Therefore, a subsequence $\{v_{\eta}, p_{\eta}\}$ can be extracted such that $v_{\eta} \to v_0$ in $X$, $p_{\eta} \to p_0$ in $L_2(\Omega)$. We pass to the limit by analogy with the above. Then we get $v_0 = v_\lambda$, $p_0 = p_\lambda$. Due to (9.36) and by the uniqueness of the solution of (8.11)–(8.13), we obtain by analogy with the above (see (9.13)) that
\[
v_m \to v_\lambda \text{ in } X. \tag{9.38}
\]
It follows from (9.28) and (3.37) that
\[
\|L_3(v_{m+1}) - L_3(v_\lambda) + r B^* Bu_{m+1}\|_{X^*} = \|B^* q_m\|_{X^*} \geq \beta_1 \|q_m\|_{L_2(\Omega)}.
\]
This inequality, together with (9.35) and (9.38), yields $q_m \to 0$ in $L_2(\Omega)$. Therefore (9.26) holds true. •
9.3. Solving a nonlinear problem. We consider two methods for solving the nonlinear problem (9.23), namely the Birger-Kachanov method and the contraction method. Both methods transform a nonlinear problem into a sequence of linear problems.

We consider the problem: find a function $u$ satisfying

$$ u \in X, \quad (L_3(u), h) + r(Bu, Bh) = (f, h), \quad \forall h \in X, \quad (9.39) $$

where $f \in X^*$. 

For an arbitrary $v \in X$ we define the operator $M(v) \in \mathcal{L}(X, X^*)$ as follows:

$$ (M(v)w, h) = 2\int_{\Omega} [c(|E|, x)(\lambda + I(\tilde{u} + v))^{-\frac{1}{2}} \varepsilon_{ij}(\tilde{u} + w) \varepsilon_{ij}(h) + \psi_1(I(\tilde{u} + v), |E|, x) \varepsilon_{ij}(\tilde{u} + w) \varepsilon_{ij}(h)]dx + r(Bw, Bh). \quad (9.40) $$

The Birger-Kachanov method consists in constructing a sequence $\{u_m\}$ such that

$$ u_{m+1} \in X, \quad (M(u_m)u_{m+1}, h) = (f, h), \quad \forall h \in X. \quad (9.41) $$

The conditions for the convergence of the Birger-Kachanov method in the general situation were established in [4].

From the known results [4], [13], [16], the next theorem follows.

**Theorem 9.4.** Suppose the conditions (4.6), (4.7), (C4a) are satisfied. Assume that $\psi_1$ is a nonincreasing function meeting the conditions (C0) and (C1a). Let also $\lambda > 0$ an $u_0$ be an arbitrary element of $X$. Then for any $m$ there exists a unique solution $v_{m+1}$ of the problem (9.41) and $u_m \rightarrow u$ in $X$, where $u$ is the solution of (9.39).

Consider now the contraction method. Let $A$ be a linear continuous selfadjoint and coercive mapping from $X$ into $X^*$, i.e.

$$ (Au, h) = (u, Ah), \quad |(Au, h)| \leq b_1\|u\|_X \|h\|_X, \quad u, h \in X, $$

$$ (Au, u) \geq b_2\|u\|_X^2, \quad u \in X, \quad (9.42) $$

where $b_1, b_2$ are positive constants. By (9.42) the expression

$$ \|u\|_1 = (Au, u)^{\frac{1}{2}} \quad (9.43) $$

defines that norm in $X$ that is equivalent to the norm $\|\cdot\|_X$ and to the norm of $H^1(\Omega)^n$. By $X_1$ we denote the space $X$ equipped with the scalar product

$$ (u, h)_{X_1} = (Au, h) \quad (9.44) $$

and with the norm (9.43).

We study the following iterative method:

$$ u_{m+1} \in X, \quad (Au_{m+1}, h) = (Au_m, h) - t[L_3(u_m, h) + r(Bu_m, Bh) - (f, h)], \quad \forall h \in X, \quad (9.45) $$

where $t$ is a positive constant.

We may define the operator $A$ by

$$ (Au, h) = \int_{\Omega} g \frac{\partial u_i}{\partial x_j} \frac{\partial h_j}{\partial x_i} dx, \quad u, h \in X, \quad (9.46) $$

where $g \in C(\overline{\Omega})$, $g(x) \geq c_0 > 0$ for all $x \in \overline{\Omega}$. Then, taking $h = (h_1, 0)$ and $h = (0, h_2)$ in the case that $\Omega \subset \mathbb{R}^2$, and $h = (h_1, 0, 0)$, $h = (0, h_2, 0)$, $h = (0, 0, h_3)$ for $\Omega \subset \mathbb{R}^3$, where $h_i$ are arbitrary functions from $U = \{\eta \in H^1(\Omega), \eta|_{\partial \Omega} = 0\}$, we split the problem (9.45) and obtain
independent problems for calculation \( u_{mi} \), \( i = 1, \ldots, n \). Such a split is very convenient for computations.

**Lemma 9.1.** Suppose the conditions (4.6), (C0), (C1a), (C4a), (9.42) are satisfied. Then
\[
(L_3(v) - L_3(h), v - h) \geq q_1\|v - h\|^2_1, \tag{9.47}
\]
\[
\|L_3(v) - L_3(h)\|_{X^*_1} \leq q_2\|v - h\|_1, \tag{9.48}
\]

where
\[
q_1 = \mu_1b_1^{-1},
\]
\[
q_2 = (\mu_2 + 4a_5\lambda^{-\frac{1}{2}})b_2^{-\frac{3}{2}}. \tag{9.49}
\]

\( \mu_1, \mu_2 \) are defined by (3.8).

**Proof.** Taking into account the inequalities of (9.42), and applying Lemmas 3.1 and 3.4, we obtain (9.47), (9.48) with \( q_1, q_2 \) defined by (9.49).

**Theorem 9.5.** Suppose the conditions (4.6), (C0), (C1a), (C4a) are satisfied. Let \( f \in X^* \) and the operator \( A \in \mathcal{L}(X, X^*) \) meets (9.42). Let also \( \lambda > 0 \) and \( u_0 \) be an arbitrary element of \( X \). Then for \( t \in (0, 2q_1q_3^{-2}) \), where
\[
q_3 = q_2 + r\|B^*B\|_{\mathcal{L}(X_1, X^*_1)}, \tag{9.50}
\]

and for any \( m \) there exists a unique solution \( u_{m+1} \) of problem (9.45) and the following estimate holds
\[
\|u_m - u\|_1 \leq \frac{k(t)^m}{1 - k(t)}\|L_3(u_0) + rB^*Bu_0 - f\|_{X_1}, \tag{9.51}
\]

where
\[
k(t) = (1 - 2q_1t + q_3^2t^2)^\frac{1}{2} < 1, \tag{9.52}
\]

and \( u \) is the solution of (9.39).

The function \( k \) takes its minimal value \( k(t_0) = (1 - q_1^2q_3^{-2})^\frac{1}{2} \) at the point \( t_0 = q_1q_3^{-2} \).

**Proof.** Let \( N = L_3 + rB^*B \). By Lemma 9.1 we have
\[
(N(v) - N(h), v - h) \geq q_1\|v - h\|^2_1,
\]
\[
\|N(v) - N(h)\|_{X^*_1} \leq q_3\|v - h\|_1, \quad v, h \in X, \tag{9.53}
\]

where \( q_3 = q_2 + r\|B^*B\|_{\mathcal{L}(X_1, X^*_1)} \).

Denote by \( J \) the Riesz operator \( J \in \mathcal{L}(X^*_1, X_1) \) that is defined as follows
\[
(Jg, h)_{X_1} = (g, h), \quad g \in X^*_1, \quad h \in X_1, \quad \|Jg\|_1 = \|g\|_{X^*_1}. \tag{9.54}
\]

It is obvious that the problem (9.39) is equivalent to finding a fixed point \( u = U_t(u) \), where
\[
U_t : X_1 \to X_1, \quad U_t(h) = h - tJ(N(h) - f). \tag{9.55}
\]

By (9.53)–(9.55) we have
\[
\|U_t(v) - U_t(h)\|_1^2 = \|v - h - tJ(N(v) - N(h))\|_1^2
= \|v - h\|^2_1 - 2t(N(v) - N(h), v - h) + t^2\|N(v) - N(h)\|^2_{X^*_1}
\leq \|v - h\|^2_1 - 2tq_1\|v - h\|_1^2 + t^2q_3^2\|v - h\|^2_1 = (k(t)\|v - h\|_1)^2,
\]
where $k(t)$ is defined by (9.52) and $k(t) < 1$, if $t \in (0, 2q_1q_3^{-2})$. Therefore the mapping $U_t$ is a contraction, and the existence of a unique solution $u$ of problem (9.39) and estimate (9.51) follow from the fixed point theorem (see eq. [21]).

**Remark 9.1.** Equalities (9.49) and (9.50) imply that $q_3 \to \infty$ as $\lambda \to 0$. Therefore $k(t_0) = (1 - q_1^2 q_3^{-2})^{1/2}$ the minimal value of $k(t)$ tends to unit as $\lambda \to 0$, and the iterative method (9.45) provides slow convergence at small value of $\lambda$. The reason of this is that the differentiable functional $Y_\lambda$ tends to nondifferentiable functional $Y$ (see (8.8), (8.9)) as $\lambda \to 0$.

9.4. **Solving the problem** (5.8)–(5.10). For the case that the conditions of Theorem 5.2 are satisfied, the operators $\frac{\partial J}{\partial \lambda}$ and $L$ are strictly monotone and Lipschitz continuous. Therefore the algorithm of the augmented Lagrangian (see (9.22)–(9.24)) can be used for the solution of the problem (5.8)–(5.10), and the corresponding nonlinear systems can be solved by the Birger-Kachanov method and by the contraction method.

Let us consider the general case that the conditions of Theorem 5.2 are not satisfied. Let $\{\chi_i\}_{i=1}^{k_1(m)}$, $\{\eta_i\}_{i=1}^{k_2(m)}$ be bases in the spaces $X_m$ and $N_m$, respectively. Let also $k = k_1(m) + k_2(m)$. Define a mapping $\mathcal{M} : \mathbb{R}^k \to \mathbb{R}^k$ as follows:

$$\mathbb{R}^k \ni c = \{c_i\}_{i=1}^{k} \to \mathcal{M}(c) = \{\mathcal{M}_i(c)\}_{i=1}^{k},$$

where

$$\mathcal{M}_i(c) = \left( \frac{\partial J}{\partial h_i} \left( \sum_{j=1}^{k_1(m)} c_j \chi_j, \sum_{j=1}^{k_2(m)} c_j \chi_j \right), \chi_i \right) + \left( L \left( \sum_{j=1}^{k_1(m)} c_j \chi_j, \chi_i \right), \chi_i \right) - \left( B^*_m \sum_{j=k_1(m)+1}^{k} c_j \eta_j-k_1(m), \chi_i \right) - (K + F, \chi_i), \quad i = 1, \ldots, k_1(m).$$

$$\mathcal{M}_i(c) = \left( B^*_m \sum_{j=1}^{k_2(m)} c_j \chi_j, \eta_i-k_1(m), \chi_i \right), \quad i = k_1(m) + 1, \ldots, k.$$  

(9.57)  

(9.58)

It is obvious that the problem (5.8)–(5.10) is equivalent to the following one: find $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_k)$ such that

$$\tilde{c} \in \mathbb{R}^k, \quad \mathcal{M}(\tilde{c}) = 0.$$  

(9.59)

Define the functional

$$\Phi_2(c) = \sum_{i=1}^{k} (\mathcal{M}_i(c))^2, \quad c \in \mathbb{R}^k.$$  

(9.60)

The problem (9.59) is equivalent to the following one:

$$\tilde{c} \in \mathbb{R}^k, \quad \Phi_2(\tilde{c}) = \min_{c \in \mathbb{R}^k} \Phi_2(c) = 0.$$  

(9.61)

In the case that the functions $\psi$ and $b$ are continuously differentiable the functional $\Phi_2$ is continuously differentiable in $\mathbb{R}^k$, and gradient method can be applied for calculation of a solution of the problem (9.61). Derivative of the mapping $v \to \frac{\partial J}{\partial \lambda}(v, v) + L(v)$ is defined in 5.2 (see (5.43)–(5.48)).
10. Stationary problem with consideration for the inertia forces. Nonhomogeneous problem.

10.1. Basic equations and auxiliary results. The equations of motion with regard for the inertia forces read as follows:

\[\rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} - 2 \frac{\partial}{\partial x_j} \left[ \varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u) \right] = K_i \text{ in } \Omega, \quad i = 1, \ldots, n. \tag{10.1}\]

The condition of incompressibility is

\[\text{div } u = 0. \tag{10.2}\]

We assume that velocities are specified on the boundary \(S\) of \(\Omega\), i.e.

\[u \big|_S = \hat{u}. \tag{10.3}\]

We assume also \((C7)\): \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n = 2\) or \(3\). The boundary \(S\) of \(\Omega\) belongs to the class \(C^2\) and consists of \(l\) connected components \(\Gamma_1, \ldots, \Gamma_l (l \geq 1)\), and suppose that

\[\hat{u} \in H^\frac{1}{2}(S), \int_{\Gamma_i} \hat{u}_i \nu ds = 0, \quad i = 1, \ldots, l, \tag{10.4}\]

and the function \(\varphi\) is defined by (2.12).

The following lemma follows from the known results (see e.g. [11, 23]).

**Lemma 10.1.** Suppose that the conditions \((C7)\) and (10.4) are satisfied. Then there exists a function \(\tilde{u}\) such that

\[\tilde{u} = \text{curl } \eta, \quad \eta = (\eta_1, \ldots, \eta_n) \in H^2(\Omega)^n, \quad \tilde{u} \big|_S = \hat{u}, \tag{10.5}\]

moreover for an arbitrary \(\alpha > 0\) one can choose a vector-valued function \(\eta\) such that

\[\|\tilde{u}_i v_j\|_{L^2(\Omega)} \leq \alpha \|v\|_X, \quad v \in H^1_0(\Omega)^n, \quad i, j = 1, \ldots, n. \tag{10.6}\]

We set

\[q(u, v, w) = \rho \int \sum_{i,k=1}^n u_k \frac{\partial v_i}{\partial x_k} w_i dx, \quad u, v, w \in H^1(\Omega)^n. \tag{10.7}\]

Obviously

\[|q(u, v, w)| \leq \rho \sum_{i,k=1}^n \|u_k\|_{L^4(\Omega)} \left\| \frac{\partial v_i}{\partial x_k} \right\|_{L^2(\Omega)} \|w_i\|_{L^4(\Omega)}, \tag{10.8}\]

Therefore, the trilinear form \(q\) is continuous in \(H^1(\Omega)^n \times H^1(\Omega)^n \times H^1(\Omega)^n\).

We consider the following spaces

\[X = H^1_0(\Omega)^n \quad \text{with the norm } \| \cdot \|_X = \| \cdot \|_X, \tag{10.9}\]

\[\mathcal{V} = \{w \in X, \text{ div } w = 0\} \quad \text{with the norm } \| \cdot \|_{\mathcal{V}} = \| \cdot \|_X, \tag{10.10}\]

\[\mathcal{N} = \{w \in L^2(\Omega), \int_{\Omega} w dx = 0\} \quad \text{with the norm } \| \cdot \|_{\mathcal{N}} = \| \cdot \|_{L^2(\Omega)}. \tag{10.11}\]

It is easy to verify that

\[q(z, w, h) = -q(z, h, w), \quad z \in \mathcal{V}, \quad w, h \in H^1(\Omega)^n, \quad n = 2 \text{ or } 3, \quad q(z, h, h) = 0. \tag{10.12}\]
Define a trilinear form $q_1$ as follows:

$$q_1(v, w, h) = \frac{1}{2} q(v, w, h) - \frac{1}{2} q(v, h, w), \quad v, h, w \in H^1(\Omega)^n. \quad (10.13)$$

It is evident that

$$q_1(v, h, h) = 0, \quad v, h \in H^1(\Omega)^n, \quad (10.14)$$
$$q_1(v, w, h) = -q_1(v, h, w), \quad v, h, w \in H^1(\Omega)^n. \quad (10.15)$$

**Lemma 10.2.** Suppose that

$$v_k \rightharpoonup v \text{ in } X, \quad (10.16)$$
$$v_k \to v \text{ in } L_4(\Omega)^n. \quad (10.17)$$

Then, for an arbitrary fixed $h \in H^1_0(\Omega)^n$ the following relations hold

$$q_1(v_k, v_k, h) \to q(v, v, h), \quad (10.18)$$
$$q_1(\tilde{u}, v_k, h) \to q(\tilde{u}, v, h), \quad (10.19)$$
$$q_1(v_k, \tilde{u}, h) \to q(v, \tilde{u}, h). \quad (10.20)$$

In this case if $v \in V$, then

$$q_1(v, v, h) = q(v, v, h), \quad q_1(\tilde{u}, v, h) = q(\tilde{u}, v, h), \quad q_1(v, \tilde{u}, h) = q(v, \tilde{u}, h). \quad (10.21)$$

**Proof.** We have

$$\int_\Omega (v_{kj}h_i - v_jh_i)^2 \, dx \leq \left( \int_\Omega (v_{kj} - v_j)^4 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega h_i^4 \, dx \right)^{\frac{1}{2}}, \quad (10.22)$$

where $v_{kj}$ are the components of the vector-valued function $v_k$. By (10.17) the left-hand side of (10.22) tends to zero.

Therefore,

$$v_{kj} \to v_jh_i \text{ in } L_2(\Omega) \text{ as } k \to \infty, \quad i, j = 1, \ldots, n. \quad (10.23)$$

(10.16) and (10.23) yield

$$q(v_k, v_k, h) \to q(v, v, h). \quad (10.24)$$

Application Green’s formula gives

$$q(v_k, h, v_k) = \rho \int_\Omega v_{kj} \frac{\partial h_i}{\partial x_j} v_{ki} \, dx = -\rho \int_\Omega h_i \frac{\partial}{\partial x_j} (v_{kj} v_{ki}) \, dx,$$

and by analogy with the stated above we obtain

$$q(v_k, h, v_k) \to q(v, h, v).$$

Therefore (10.18) holds. It is evident that (10.19), (10.20) follows from (10.16) and (10.17).

In the special case that $v \in V$ we have

$$q(v, v, h) = \rho \int_\Omega v_j \frac{\partial v_i}{\partial x_j} h_i \, dx = -\rho \int_\Omega v_j \frac{\partial h_i}{\partial x_j} v_i \, dx = -q(v, h, v). \quad (10.25)$$

(10.13) and (10.25) imply (10.21).

**Lemma 10.3.** Suppose the conditions (C7) and (10.4) are satisfied. Then for an arbitrary $\xi > 0$ one can choose a vector valued function $\eta$ such that (10.5) is satisfied and in addition

$$|q_1(v, \tilde{u}, v)| \leq \xi \|v\|_X^2, \quad v \in X. \quad (10.26)$$
Proof. By Green’s formula we obtain
\[
q_1(v, \tilde{u}, v) = \frac{1}{2} \rho \int_{\Omega} \left[ v_j \frac{\partial \tilde{u}_i}{\partial x_j} v_i - v_j \frac{\partial v_i}{\partial x_j} \tilde{u}_i \right] dx
\]
\[
= \frac{1}{2} \rho \int_{\Omega} \left[ \left( - \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \right) (\tilde{u}_i v_i) - 2 \frac{\partial v_i}{\partial x_j} \tilde{u}_i v_j \right] dx. \quad v \in \mathcal{X}.
\]

It follows from here that
\[
|q_1(v, \tilde{u}, v)| \leq c_1 \|v\|_{\mathcal{X}} \sum_{i,j=1}^{n} \|\tilde{u}_i v_j\|_{L_2(\Omega)},
\]
and Lemma 10.3 follows from this inequality and Lemmas 10.1.

10.2. Boundary value problem. We consider the problem: find \(v\) satisfying
\[
v \in \mathcal{V} \tag{10.27}
\]
\[
\left( \frac{\partial J_\lambda}{\partial h}(v, v), w \right) + (L(v), w) + q(v, v, w) + q(\tilde{u}, v, w) + q(v, \tilde{u}, w) - (y, w) = 0, \quad w \in \mathcal{V}, \tag{10.28}
\]
where
\[
(y, w) = \int_{\Omega} K_i w_i dx - \int_{\Omega} \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} w_i dx. \tag{10.29}
\]
It follows from (4.7) and (10.5) that \(y \in \mathcal{X}^*\). The left-hand side of (10.28) belongs to the polar set
\[
\mathcal{V}^* = \{ f \in \mathcal{X}^*, \quad (f, w) = 0, \quad w \in \mathcal{V} \}.
\]
Therefore, there exists a function \(p \in \mathcal{N}\) such that the pair \((v, p)\) is a solution of the following problem:
\[
(v, p) \in \mathcal{X} \times \mathcal{N}, \quad (10.30)
\]
\[
\left( \frac{\partial J_\lambda}{\partial h}(v, v), w \right) + (L(v), w) + q(v, v, w) + q(\tilde{u}, v, w)
\]
\[
+ q(v, \tilde{u}, w) - (B^* p, w) = (y, w), \quad w \in \mathcal{X}, \tag{10.31}
\]
\[
(Bv, \gamma) = 0, \quad \gamma \in \mathcal{N}. \tag{10.32}
\]

By use of Green’s formula it can be seen that, if \((v, p)\) is a solution of problem (10.30)–(10.32), then \((u, p)\) with \(u = \tilde{u} + v\) is a solution of problem (10.1)–(10.3) in the sense of distributions. On the contrary, if \((u, p)\) is a classical solution of problem (10.1)–(10.3), then the pair \((v, p)\) with \(v = u - \tilde{u}\) is a solution of problem (10.30)–(10.32).

Let \(\{\mathcal{X}_m\}, \{\mathcal{N}_m\}\) be sequences of finite-dimensional subspaces in \(\mathcal{X}\) and \(\mathcal{N}\) which satisfy the following conditions
\[
\lim_{m \to \infty} \inf_{z \in \mathcal{X}_m} \|w - z\|_{\mathcal{X}} = 0, \quad w \in \mathcal{X}, \quad (10.33)
\]
\[
\lim_{m \to \infty} \inf_{y \in \mathcal{N}_m} \|h - y\|_{L_2(\Omega)} = 0, \quad h \in \mathcal{N}, \quad (10.34)
\]
\[
\inf_{\mu \in \mathcal{N}_m} \sup_{w \in \mathcal{X}_m} \frac{(B_m w, \mu)}{\|w\|_{\mathcal{X}} \|\mu\|_{L_2(\Omega)}} \geq \beta > 0, \quad m \in \mathbb{N}, \quad (10.35)
\]
\[
\mathcal{X}_m \subset \mathcal{X}_{m+1}, \quad \mathcal{N}_m \subset \mathcal{N}_{m+1}. \quad (10.36)
\]
We introduce the spaces $\mathcal{V}_m$ and $\mathcal{V}_m^0$ by

$$
\mathcal{V}_m = \{u \in \mathcal{X}_m, \quad (B_m u, \gamma) = 0, \quad \gamma \in \mathcal{N}_m\}, \\
\mathcal{V}_m^0 = \{q \in \mathcal{X}_m^*, \quad (q, u) = 0, \quad u \in \mathcal{V}_m\}.
$$

Define approximate solutions of problem (10.30)–(10.32) as follows:

$$(v_m, p_m) \in \mathcal{X}_m \times \mathcal{N}_m,$$

$$
\left(\frac{\partial J_\lambda}{\partial h}(v_m, v_m), w\right) + (L(v_m), w) + q_1(v_m, v_m, w) + q_1(\tilde{u}, v_m, w)
+ q_1(v_m, \tilde{u}, w) - (B_{m}^* p_m, w) = (y, w), \quad w \in \mathcal{X}_m,
$$

$$(B_m, v_m, \gamma) = 0, \quad \gamma \in \mathcal{N}_m.
$$

**Theorem 10.1.** Suppose that $K \in L^2(\Omega)^n$ and the condition (C4) is satisfied. Let the function $\psi$ meets one of conditions (C1), (C2), (C3) ($\varphi$ replaced by $\psi$). Assume that (C7), (10.4), (10.5) and (10.26) with $\xi \leq a_1$ are fulfilled. Let also $\{\mathcal{X}_m\}$ and $\{\mathcal{N}_m\}$ be sequences of finite-dimensional subspaces in $\mathcal{X}$ and $\mathcal{N}$ respectively, such that (10.33)–(10.36) hold. Then, for an arbitrary fixed $\lambda > 0$ and an arbitrary $m \in \mathbb{N}$ there exists a solution $(v_m, p_m)$ of problem (10.39)–(10.41), and a subsequence $\{v_k, p_k\}$ can be extracted from the sequence $\{v_m, p_m\}$ such that $v_k \rightharpoonup v$ in $\mathcal{X}$, $p_k \rightharpoonup p$ in $\mathcal{N}$, where $v, p$ is a solution of the problem (10.30)–(10.32).

**Proof.** Define an operator $M : \mathcal{X} \rightarrow \mathcal{X}^*$ by

$$
(M(g), w) = \left(\frac{\partial J_\lambda}{\partial h}(g, g), w\right) + (L(g), w) + q_1(g, q, w) + q_1(\tilde{u}, g, w) + q_1(g, \tilde{u}, w), \quad g, w \in \mathcal{X}.
$$

It follows from (10.39)–(10.42) that $v_m$ is a solution of the problem

$$
v_m \in \mathcal{V}_m, \quad (M(v_m), w) = (y, w), \quad w \in \mathcal{V}_m.
$$

Bearing in mind (10.14) and that $\xi \leq a_1$ in (10.26) we obtain by analogy with the proof of Theorem 5.1 (see (5.13), (5.15), that

$$
z(e) = (M(e), e) - (y, e) \geq a_1 \|e\|_{\mathcal{X}}^5 - c_1 \|e\|_{\mathcal{X}}, \quad e \in \mathcal{X}, \quad \lambda > 0.
$$

Therefore, $z(e) \geq 0$ for $\|e\|_{\mathcal{X}} \geq r = \frac{\xi}{a_1}$, and there exists a solution of (10.43) with

$$
\|v_m\|_{\mathcal{X}} \leq r, \quad \|M(v_m)\|_{\mathcal{X}^*} \leq c_1 \quad m \in \mathbb{N}.
$$

For an arbitrary $f \in \mathcal{X}^*$ we denote by $Gf$ the restriction of $f$ to $\mathcal{X}_m$. Then $Gf \in \mathcal{X}_m^*$, and by (10.43) we obtain

$$
G(M(v_m) - y) \in \mathcal{V}_m^0.
$$

Therefore, there exists a unique $p_m \in \mathcal{N}_m$ such that (10.40) is satisfied (see Lemma 3.6), and (10.35) yields

$$
\|p_m\|_{\mathcal{N}} \leq c_2.
$$

By (10.45), (10.47) we can extract a subsequence $\{v_k, p_k\}$ such that

$$
v_k \rightharpoonup v \text{ in } \mathcal{X},
$$

$$
v_k \rightarrow v \text{ in } L_4(\Omega)^n \text{ and a.e. in } \Omega,
$$

$$
M(v_k) \rightharpoonup \eta \text{ in } \mathcal{X}^*,
$$

$$
p_k \rightarrow p \text{ in } \mathcal{N}.
$$
Next we use Lemma 10.2 and by analogy with the proof of Theorem 5.1 we pass to the limit in (10.40), (10.41) with $m$ replaced by $k$, and obtain
\[
(M(v), w) - (B^* p, w) = (y, w), \quad w \in \mathcal{X}.
\]
(10.52)
\[
(Bv, \gamma) = 0, \quad \gamma \in \mathcal{N}.
\]
(10.53)
Taking into consideration that (10.21) is satisfied, for $v \in \mathcal{V}$, we get that pair $(v, p)$ is a solution of the problem (10.30)–(10.32).

11. Stationary problem with consideration for the inertia forces. Mixed problem.

11.1. Formulation of the problem and an existence result. As before we consider that $S_1$ and $S_2$ are open subsets of the boundary $S$ of $\Omega$ such that $S_1$ is non-empty, $S_1 \cap S_2 = \emptyset$ and $\overline{S_1} \cup \overline{S_2} = S$. We study the problem on searching for a pair of functions $(u, p)$ which satisfy the motion equations (10.1), the condition of incompressibility (10.2) and the mixed boundary conditions, wherein velocities are specified on $S_1$ and surface forces are given on $S_2$, i.e.
\[
\left. u \right|_{S_1} = \hat{u}, \quad u \mid_{\partial S_1} = 0, \quad u \mid_{\partial S_2} = F_i, \quad i = 1, \ldots, n.
\]
(11.1)
\[
[-p \delta_{ij} + 2 \varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u)] r_j \mid_{S_2} = F_i, \quad i = 1, \ldots, n.
\]
(11.2)

It is obvious that in the special case that $S_2$ is an empty set this problem transforms into the problem considered in Section 10.

We assume that $\varphi$ is defined by (2.12) and $\hat{u} \in H^\frac{3}{2}(S_1)$. Then there exists a function $\tilde{u} \in \mathcal{X}$ satisfying (4.6).

Let us define operators $M_1 : X \to X^*$, $M_2 : X \to X^*$ and an element $\chi \in X^*$ as follows:
\[
(M_1(w), g) = \left( \frac{\partial J_\lambda}{\partial h}(w, w), g \right) + (L(w), g), \quad w, g \in X,
\]
(11.3)
\[
(M_2(w), g) = q(\tilde{u}, w, g) + q(w, \tilde{u}, g), \quad w, g \in X;
\]
(11.4)
\[
(\chi, g) = \int_{\Omega} K_i g_i \, dx + \int_{S_2} F_i g_i \, ds - q(\tilde{u}, \tilde{u}, g), \quad g \in X.
\]
(11.5)

We consider the problem
\[
(v, p) \in X \times L_2(\Omega),
\]
(11.6)
\[
(M_1(v), w) + (M_2(v), w) + q(v, v, w) - (B^* p, w) = (\chi, w), \quad w \in X,
\]
(11.7)
\[
(Bv, \gamma) = 0, \quad \gamma \in L_2(\Omega).
\]
(11.8)

By using Green’s formula, one may show that, if $(v, p)$ is a solution of the problem (11.6)–(11.8), then $(u, p)$ with $u = \hat{u} + v$ is a solution of the problem (10.1), (10.2), (11.1), (11.2) in the distribution sense. On the contrary, if $(u, p)$ is a solution of (10.1), (10.2), (11.1), (11.2) such that (11.6) holds with $v = u - \hat{u}$, then $(v, p)$ is a solution of the problem (11.6)–(11.8). Define also the following constants
\[
r_1 = \sup_{w \in \mathcal{V}, \|w\|_X \leq 1} q(w, w, w),
\]
\[
r_2 = \inf_{w \in \mathcal{V}, \|w\|_X \leq 1} (M_2(w), w),
\]
\[
r_3 = \sup_{w \in \mathcal{V}, \|w\|_X \leq 1} |(\chi, w)|.
\]
(11.9)
Consider the space
\[ P = \{ w \in H^1(\Omega)^n, \quad \text{div} \, w = 0 \}. \] (11.10)

The space \( P \) is presented in the form \( P = V \oplus V^\perp \), where \( V \) is given by (3.2) and \( V^\perp \) is the orthogonal complement of \( V \) in \( P \). Evidently that the constants \( r_2 \) and \( r_3 \) in (11.9) depend on the function \( \tilde{u} \) satisfying (4.6). Let \( \tilde{u}_1 \) be a function from \( V^\perp \) that satisfies (4.6). Then the function \( \tilde{u} = \tilde{u}_1 + \tilde{u}_2 \), where \( \tilde{u}_2 \) is an arbitrary element of \( V \), meets (4.6). We assume that there exists a function \( \tilde{u}_2 \in V \) such that the following inequalities hold
\[ r_4 = a_1 + \frac{r_2^2}{2} > 0, \quad r_4^2 > r_1 r_3, \] (11.11)
where \( a_1 \) is the positive constants from (2.13). It is evident that (11.11) holds, if the norms of the functions \( K, F, \) and \( \tilde{u} \) are not large.

**Lemma 11.1.** Suppose the condition (C4) is satisfied and the function \( \psi \) meets one of the conditions (C1), (C2), (C3) (\( \varphi \) replaced by \( \psi \)). Let also (11.11) holds.

Then the following inequality is valid:
\[ \beta(w) = (M_1(w), w) + (M_2(w), w) + q(w, w, w) - (\chi, w) \geq 0, \]
if \( w \in V \) and \( \|w\|_X = \frac{r_4 - \sqrt{r_4^2 - r_1 r_3}}{r_1} = \xi. \) (11.12)

**Proof.** It follows from (11.9) and (11.12) that
\[ \beta(w) \geq \beta_1(\|w\|_X) = (2a_1 + r_2)\|w\|_X^2 - r_1 \|w\|_X^3 - r_3 \|w\|_X, \quad w \in V. \] (11.13)

Consider the quadratic equation
\[ 2r_4 y - r_1 y^2 - r_3 = 0, \]
r_4 being defined in (11.11). Its roots are those of the equation \( \beta_1(y) = 0 \) and they are equal to
\[ y_1 = \frac{r_4 - \sqrt{r_4^2 - r_1 r_3}}{r_1}, \quad y_2 = \frac{r_4 + \sqrt{r_4^2 - r_1 r_3}}{r_1}. \]

If (11.11) holds, then \( y_1 \) and \( y_2 \) are real and \( \beta_1(y) \geq 0 \) for \( y \in [y_1, y_2] \). Therefore (11.11) yields (11.12), and the lemma is proved. □

**Theorem 11.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or \( 3 \) with a Lipschitz continuous boundary \( S \). Suppose the condition (C4) is satisfied and the function \( \psi \) meets one of the conditions (C1), (C2), (C3) (\( \varphi \) replaced by \( \psi \)). Let also (11.11) holds. Then, for an arbitrary \( \lambda > 0 \) there exists a solution of the problem (11.6)–(11.8).

**Proof.** It follows from (11.6)–(11.8) that the function \( v \) is a solution of the problem
\[ v \in V, \]
\[ (M_1(v), w) + (M_2(v), w) + q(v, v, w) = (\chi, w), \quad w \in V. \] (11.14)
Let \( \{V_m\} \) be a sequence of finite-dimensional subspaces of \( V \) such that
\[ \lim_{m \to \infty} \inf_{z \in V_m} \|w - z\|_X = 0, \quad w \in V, \] (11.15)
\[ V_m \subset V_{m+1}. \] (11.16)
We search for the Galerkin approximations $v_k$ satisfying

$$v_m \in V_m, \quad (M_1(v_m), w) + (M_2(v_m), w) + q(v_m, v_m, w) = (\chi, w), \quad w \in V_m. \quad (11.17)$$

By virtue of Lemma 11.1 there exists a solution of the problem (11.17) and $\|v_m\|_X \leq \xi$. Thus we can extract a subsequence $\{v_k\}$ such that $v_k \rightharpoonup v_0$ in $V$. We pass to the limit as $k \to \infty$ in (11.17), with $m$ changed by $k$. In this case by analogy with the stated above, (see the proofs of Theorems 5.1, 10.1 and Lemma 10.2), we use the methods of monotonicity and compactness. Thus, we get that the function $v = v_0$ is a solution of the problem (11.14).

11.2. Approximation of the problem (11.6)–(11.8). Let $\{X_m\}, \{N_m\}$ be sequences of finite dimensional subspaces of $X$ and $L_2(\Omega)$ which satisfy the conditions (3.38), (3.39), (3.43) and (5.11). We search for an approximate solutions of the problem (11.6)–(11.8) in the form

$$(v_m, p_m) \in X_m \times N_m, \quad (11.18)$$

$$(M_1(v_m), w) + (M_2(v_m), w) + q(v_m, v_m, w) - (B_m p_m, w) = (\chi, w), \quad w \in X_m, \quad (11.19)$$

$$(B_m v_m, \gamma) = 0, \quad \gamma \in N_m. \quad (11.20)$$

From the point of view of applications, in particular, of computation, the problem (11.18)–(11.20) is considerably more preferable than (11.17). So, we study the question of convergence of the approximations $\{v_m, p_m\}$.

Define constants $\eta_1 - \eta_4$ by

$$\eta_1 = \sup_m \max_{w \in d_m} |q(w, w, w)|,$$

$$\eta_2 = \inf_m \min_{w \in d_m} (M_2(w), w),$$

$$\eta_3 = \sup_m \max_{w \in d_m} |(\chi, w)|,$$

$$\eta_4 = a_1 + \frac{\eta_2}{2}, \quad (11.21)$$

where

$$d_m = \{w \in X_m, \quad (B_m w, \gamma) = 0, \quad \gamma \in N_m, \quad \|w\|_X \leq 1\}. \quad (11.22)$$

Note that $\eta_1 \geq r_1, \eta_2 \leq r_2, \eta_3 \geq r_3$ (see (3.38), (3.39) and (11.9)).

Define a mapping $M_3: X \to X^*$ as follows:

$$(M_3(v), w) = (M_2(v), w) + q(v, v, w) = q(\tilde{u}, v, w) + q(v, \tilde{u}, w) + q(v, v, w), \quad v, w \in X. \quad (11.23)$$

**Lemma 11.2.** Suppose that (4.6) is satisfied and let

$$\{v_k\} \subset X, \quad v_k \rightharpoonup v \text{ in } X. \quad (11.24)$$

Then

$$\lim (M_3(v_k), w) = (M_3(v), w), \quad w \in X, \quad (11.25)$$

$$\lim (M_3(v_k), v_k) = (M_3(v), v). \quad (11.26)$$
Proof. It follows from (11.24) that
\[ v_k \to v \text{ in } L_4(\Omega), \quad (11.27) \]
and (11.25) arises from the proof of Lemma 10.2, see (10.24).
We have
\[
|q(v_k, v_k, v_k) - q(v, v, v)| = \left| \int_{\Omega} (v_{kj} - v_j) \frac{\partial v_{ki}}{\partial x_j} v_{ki} \, dx \right| + \int_{\Omega} v_j \left( \frac{\partial v_{ki}}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \right) v_{ki} \, dx + \int_{\Omega} v_j \frac{\partial v_i}{\partial x_j} (v_{ki} - v_i) \, dx. \quad (11.28)
\]
(11.27) implies (see (10.22)) that
\[ v_j v_{ki} \to v_j v_i \text{ in } L_2(\Omega). \quad (11.29) \]
By (11.24), (11.27), (11.29) each addend in the right-hand side of the equality (11.28) tends
to zero. Therefore,
\[ \lim q(v_k, v_k, v_k) = q(v, v, v) \]
and (11.26) holds true. ■

Theorem 11.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2$ or $3$ with a Lipschitz continuous
boundary $S$. Suppose the condition (C4) is satisfied and the function $\psi$ meets one of the
conditions (C1), (C2), (C3) ($\varphi$ replaced by $\psi$). Let also $\{X_m\}, \{N_m\}$ be sequences of finite
dimensional subspaces of $X$ and $L_2(\Omega)$ which satisfy the conditions (3.38), (3.39), (3.43)
and (5.11). Finally, assume that
\[ \eta_4 > 0, \quad \eta_4^2 > \eta_1 \eta_3. \quad (11.30) \]
Then, for an arbitrary fixed $\lambda > 0$, and for each $m \in \mathbb{N}$, there exists a solution of the
problem (11.18)–(11.20), and a subsequence $\{(v_k, p_k)\}$ can be extracted from the sequence
$\{(v_m, p_m)\}$ such that $v_k \to v$ in $X$, $p_k \to p$ in $L_2(\Omega)$, where $(v, p)$ is a solution of the
problem (11.6)–(11.8).

The proof of this theorem is analogous to that of Theorem 10.1, and it is not given because of
this.

Remark 11.1. In the general case we can consider that the density of an electrorheo-
logical fluid depends on the module of the vector of electric field strength, i.e. $\rho = \rho(|E|)$,
and
\[ \rho_2 \geq \rho(y) \geq \rho_1, \quad y \in \mathbb{R}_+, \quad (11.31) \]
where $\rho_1$, $\rho_2$ are positive constants. It is easy to see that all results of Sections 10 and 11
still stand valid in the case that $\rho$ is a function satisfying the condition (11.31).

References
[1] Belonosov M.S., Litvinov W.G., Finite element methods for nonlinearly viscous fluids, Zangew.
Math.Mech. 76, 307–320, 1996.
[2] Ceccio S., Wineman A., Influence of orientation of electric field on shear flow of electrorheological fluids,
J.Rheol. 38, 453–463, 1994.
[3] Duvaut G., lions J.L., Inequalities in Mechanics and Physics, Springer-Verlag, 1976.
[4] Fučík S., Kratovil A., Nečas J., Kachanov-Galerkin method, Comment. Math. Univ. Carolinae 14,
651–659, 1973.
[5] Gajewski H., Gröger K., Zacharias K., Nichtlineare Operatorgleichungen und Operator
differentialgleichungen, Akademie-Verlag, Berlin, 1974.
[6] Girault V., Raviart P., Finite Element Approximation of the Navier-Stokes Equations, Springer-Verlag, Berlin 1986.
[7] Ladyzhenskaya O., Solonnikov V., Some problems of vector analysis and generalized formulation of boundary value problems for the Navier-Stokes equations, Zap.Nauchn.Sem.Leningrad. Otdel.Mat.Inst. Steklov (LOMI), 59, 81–116, 1976 (in Russian).
[8] Landau L.D., Lifshitz E.M., Electrodynamics of Continuous Media, Pergamon, Oxford, 1984.
[9] Langenbach A., Monotone Potentialoperatoren in Theorie und Anwendung, Berlin, 1975.
[10] Lions J.L., Optimal Control of Systems Governed by Partial Differential Equations, Springer, Berlin, 1971.
[11] Lions J.L., Quelques Méthodes de Résolution des Problèmes aux limites non linéaires, Dunod, Paris, 1969.
[12] Litvinov W.G., Optimization in Elliptic Problems with Applications to Mechanics of Deformable Bodies and Fluid Mechanics, Birkhauser, 2000.
[13] Litvinov W.G., Motion of Nonlinearly Viscous Fluid, Moscow, Nauka, 1982 (in Russian).
[14] Maugin G.A., Continuum Mechanics of Electro-Magnetic Solids, North- Holland, Amsterdam, 1988.
[15] Mosolov P.P., Miasnikov V.P., Proof of the Korn inequality, Dokl.Akad.Nauk SSSR, Matem., 201, 36–39, 1971 (in Russian).
[16] Nečas J., Hlaváček I., Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction, Elsevier Scientific Publishing Company, Amsterdam, 1981.
[17] Panagiotopoulos D., Inequality Problems in Mechanics and Applications, Birkhäuser, 1985.
[18] Parthasarathy M., Klingenberg D.J., Electrorheology: mechanisms and models. Material Science and Engineering, R17, 57–103, 1996.
[19] Rajagopal K., Wineman A., Flow of electrorheological materials, Acta Mechanica 91, 57–75, 1992.
[20] Růžička M., Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, 1748, Springer, 2000.
[21] Schwartz L., Analyse Mathématique 1, Hermann, 1967.
[22] Shulman Z.P., Nosov B.M., Rotation of Nonconducting Bodies in Electrorheological suspensions, Nauka i Technika, Minsk, 1985 (in Russian).
[23] Temam R., Navier-Stokes Equations, North-Holland Publishing Company, Amsterdam, 1979.
[24] Vainberg M.M., Variational Methods for the Study of Nonlinear Operators, Holden Day, San Francisco, 1964.

E-MAILS:
HOPPE.MATH.UNI-AUGSBURG.DE
LITVINOV.MATH.UNI-AUGSBURG.DE