NILPOTENT ASPHERICAL SASAKIAN MANIFOLDS
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Abstract. We show that every compact aspherical Sasakian manifold with nilpotent fundamental group is diffeomorphic to a Heisenberg nilmanifold.

1. Introduction

The interaction between topological constraints and geometric structures is a classical topic. If we are interested only in the homotopy type of an underlying manifold then Algebraic Topology techniques are usually sufficient. For example, Benson and Gordon in [5] show that a compact nilpotent aspherical manifold that admits a Kähler structure is homotopy equivalent to a torus. It is known that Sasakian manifolds play the same role in contact geometry as Kähler manifolds in symplectic geometry. Similarly to the result of Benson and Gordon, a compact nilpotent aspherical manifold that admits a Sasakian structure is homotopy equivalent to a Heisenberg nilmanifold [11, 4].

Recall that an aspherical manifold is a manifold whose homotopy groups besides the fundamental group are trivial. A manifold is called nilpotent if its fundamental group is nilpotent. According to the Borel conjecture any two compact aspherical manifolds with the same fundamental group should be homeomorphic. This conjecture is proven for a wide class of groups, including nilpotent groups (cf. [1]). In particular, a compact nilpotent aspherical manifold that admits a Sasakian structure is homeomorphic to a Heisenberg nilmanifold.

One of the most surprising results of the 20th century was Milnor’s discovery in [31] of exotic spheres, i.e. spheres that are homeomorphic but not diffeomorphic to the standard one. Nowadays there are many known examples of topological manifolds that admit non-equivalent smooth structures.

If one fixes a smooth structure on a topological manifold, which is quite natural from a differential geometer’s point of view, then the problems of existence of compatible geometric structures on it become much harder to solve and require ad hoc approaches. For example the

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conjecture of Boyer, Galicki and Kollár [10] that predicts that every parallelizable exotic sphere admits a Sasakian structure is still open.

In this paper we study the existence of Sasakian structures on compact aspherical manifolds with nilpotent fundamental group. Among all aspherical smooth manifolds with nilpotent fundamental group the most studied class is the class of nilmanifolds. A nilmanifold is a compact quotient of a nilpotent Lie group by a discrete subgroup with the smooth structure inherited from the Lie group. One can show that a compact aspherical nilpotent manifold is homotopy equivalent to a nilmanifold. Then, it follows from the truthfulness of the Borel conjecture in the nilpotent case that the manifold is homeomorphic to a nilmanifold. If it is not diffeomorphic to a nilmanifold, it is called an exotic nilmanifold. By [16, Lemma 4] a connected sum of a nilmanifold with an exotic sphere is always an exotic nilmanifold. Thus exotic nilmanifolds do exist. Moreover, it is not difficult to show the existence of contact exotic nilmanifolds. By the main result of Meckert in [30], the connected sum of two contact manifolds carries a contact structure. Moreover, it is shown in [10] that there are infinitely many Sasakian (hence contact) exotic spheres. Thus the connected sum of a contact nilmanifold and a Sasakian exotic sphere provides an example of a contact exotic nilmanifold.

Write $H(1,n)$ for the Heisenberg group of dimension $2n + 1$. The main result of the article is the following theorem.

**Theorem 1.1.** If $M^{2n+1}$ is a compact aspherical Sasakian manifold with nilpotent fundamental group, then $M$ is diffeomorphic to a Heisenberg nilmanifold $\Gamma \backslash H(1,n)$, where $\Gamma$ is a lattice in $H(1,n)$ isomorphic to $\pi_1(M)$. Moreover, there is a second type deformation of a left-invariant normal almost contact structure on $\Gamma \backslash H(1,n)$, such that $M$ and $\Gamma \backslash H(1,n)$ are isomorphic as normal almost contact manifolds.

Equivalently, the above result can be stated as the non-existence of compact Sasakian exotic nilmanifold. Theorem 1.1 can be seen as an odd-dimensional version of [2, 23], where it is shown that every compact Kähler aspherical nilmanifold is biholomorphic to a complex torus. As a corollary, Baues and Kamishima derived in [3] the result of Theorem 1.1 under the stronger assumption that the Sasakian structure is regular. Their approach, based on passing to the quotient of the Reeb vector field action, cannot be extended to the non-regular case.

Baues and Cortés also showed in [2] that if $X$ is a compact aspherical Kähler manifold with virtually solvable fundamental group, then $X$ is biholomorphic to a finite quotient of a complex torus. In the same vain we obtain the following.

**Corollary 1.2.** If $M^{2n+1}$ is a compact aspherical Sasakian manifold with (virtually) solvable fundamental group, then $M$ is diffeomorphic to a finite quotient of a Heisenberg nilmanifold.
Proof. Let $M^{2n+1}$ be a compact aspherical Sasakian manifold with virtually solvable fundamental group. Denote by $\overline{M}$ the finite cover of $M$ with solvable fundamental group. The Sasakian structure on $M$ transfers to a Sasakian structure on $\overline{M}$. By the result of Bieri in [6] (see also [7]), every solvable group, for which Poincaré duality holds, is torsion-free and polycyclic. In [25], Kasuya showed that if the fundamental group of a compact Sasakian manifold is polycyclic, then it is virtually nilpotent. Hence there is a finite (compact) cover $\tilde{M}$ of $\overline{M}$, such that $\pi_1(\tilde{M})$ is nilpotent. The Sasakian structure on $\overline{M}$ transfers to a Sasakian structure on $\tilde{M}$. As $\tilde{M}$ is a finite cover of an aspherical manifold $M$, the manifold $\tilde{M}$ is also aspherical. By Theorem 1.1, the manifold $\tilde{M}$ is diffeomorphic to a Heisenberg nilmanifold of dimension $2n+1$. Hence $M$ is diffeomorphic to a finite quotient of a Heisenberg nilmanifold.

Our main result and the similar one for the Kähler case provide evidence that exotic nilmanifolds do not admit as rigid geometric structures as nilmanifolds. Another result that points in the same direction was proved in a recent article [34], where it is shown that compact exotic nilmanifolds admit no Anosov $\mathbb{Z}^r$-action without rank-one factor.

In the above-mentioned paper [3] on locally homogeneous aspherical Sasakian manifolds, Baues and Kamishima deal with the regular case by showing that any compact regular aspherical Sasakian manifold with solvable fundamental group is finitely covered by a Heisenberg nilmanifold, as its Sasaki structure may be deformed to a locally homogeneous one.

Despite the analogy between Sasakian and Kähler geometry, the proof of Theorem 1.1 is significantly different from the proofs in [2, 23]. The main obstacle to imitate these proofs is that there is no suitable version of the Albanese map for Sasakian manifolds.

The main stages of the proof are the following. In Section 4 we discuss two Tievsky models for compact Sasakian manifolds. One of them is a subcomplex of the de Rham algebra of $M$ and the other is a quotient of the former one. Then we prove that for every compact aspherical Sasakian manifold with nilpotent fundamental group $\Gamma$, the Malcev envelope $G(\Gamma)$ of $\Gamma$ is isomorphic to the Heisenberg group (Proposition 4.2). Further, we show that there exists a quasi-isomorphism $\rho: \Omega^*(G(\Gamma))^G(\Gamma) \to \Omega^*(M)$ such that its image lies in the first Tievsky model of $M$ (Proposition 4.3).

In Section 3, we develop a theory that permits to mitigate the absence of an Albanese map for Sasakian manifolds. The main result of this section implies that by modifying the above $\rho$, we can assume that it is the restriction of $h^*: \Omega^*(G(\Gamma))^G(\Gamma) \to \Omega^*(M)$, where $h: M \to \Gamma \backslash G(\Gamma)$ is a homotopy equivalence.
In Section 5, we prove that $h$ is a diffeomorphism. First, we deduce from Proposition 4.4, that there is a left-invariant 1-form $\eta_h \in \Omega^1(G(\Gamma))$ and a basic function $f$ on $M$ such that $h^*(\eta_h) = \eta + (df) \circ \varphi$.

In Section 2.2, we introduce the notion of a $\beta$-twisted map between Sasakian manifolds, which gives a relative version of second type deformations of Sasakian manifolds. In the same section, we show that a map $\psi: X \to Y$ between two Sasakian manifolds is $(df)$-twisted with $f \in C^\infty(X)$ if and only if the map

$$\psi_f: X \times \mathbb{R} \to Y \times \mathbb{R}$$

$$(x,t) \mapsto (\psi(x), f(x) + t)$$

is holomorphic. We show that $\Gamma \setminus G(\Gamma)$ can be equipped with a Sasakian structure in such way that $h$ becomes $(df)$-twisted (Claims 5.2 and 5.3).

It is straightforward that $h$ is a diffeomorphism if and only if $h_f$ is a diffeomorphism. It is also not very difficult to show that $h_f$ is surjective, universally closed and proper. Then using the embedded Hironaka resolution of singularities we show that $h_f$ is a finite map. This, combined with several deep results from complex analytic geometry, implies that $h_f$ is a biholomorphism, and thus a diffeomorphism.

The paper is organized as follows. Section 2 contains the necessary preliminaries about Sasakian manifolds, Sullivan CDGAs and nilmanifolds. In Section 3 we prove a rather general result on maps from a manifold onto its aspherical nilpotent approximations. In Section 4 we discuss Tievsky models for compact Sasakian manifolds, and establish the existence of the quasi-isomorphism $\rho$ discussed above. In the final section we complete the proof of Theorem 1.1.

2. Preliminaries

2.1. Frölicher-Nijenhuis calculus. For a general treatment of Frölicher-Nijenhuis calculus, we refer to [26]. Given a smooth manifold $M$ and a vector bundle $E$ over $M$, for every $\psi \in \Omega^k(M, TM)$ one defines an operator $i_\psi: \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ of degree $k-1$. If $\xi$ is a vector field and $\phi$ is an endomorphism of $TM$, the general definition specializes to

$$i_\xi \omega(X_1, \ldots, X_{p-1}) = \omega(\xi, X_1, \ldots, X_p)$$

$$i_\phi \omega(X_1, \ldots, X_p) = \sum_{j=1}^p \omega(X_1, \ldots, \phi X_j, \ldots, X_p),$$

where $\omega \in \Omega^p(M, E)$. In the particular case when $E$ is the trivial one-dimensional vector bundle over $M$, we get operators $i_\psi$ on $\Omega^\bullet(M)$. Next, we define the operators $L_\psi$ on $\Omega^\bullet(M)$ by $L_\psi := i_\psi d + (-1)^k di_\psi$. The Frölicher-Nijenhuis bracket on $\Omega^\bullet(M, TM)$ is defined by the characteristic property $[L_{\psi_1}, L_{\psi_2}] = L_{[\psi_1, \psi_2]}$. Here on the left side $[\ , \ ]$ stands for the graded commutator of operators.
2.2. Sasakian manifolds. An almost contact structure on a manifold $M$ is a triple $(\varphi, \xi, \eta)$ where $\varphi \in \Omega^1(M, TM)$, $\xi$ is a vector field and $\eta \in \Omega^1(M)$ such that $\varphi^2 = -\text{Id} + \xi \otimes \eta$ and $\eta(\xi) = 1$. Given an almost contact structure on $M$, one can define an almost complex structure $J$ on $M \times \mathbb{R}$ by

$$(1) \quad J \left( X, a \frac{d}{dt} \right) = \left( \varphi X - a\xi, \eta(X) \frac{d}{dt} \right).$$

For every almost contact structure one has

$$(2) \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0.$$

We say that $(\varphi, \xi, \eta)$ is normal if $J$ is integrable. It can be verified by simple computation (cf. [8, Sec. 6.1]) that the above definition is equivalent to the vanishing of four tensors:

$$(3) \quad \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi \varphi = 0, \quad \mathcal{L}_\varphi \eta = 0, \quad [\varphi, \varphi]_{FN} + 2d\eta \otimes \xi = 0.$$  

Actually, it can be shown that the vanishing of $[\varphi, \varphi]_{FN} + 2d\eta \otimes \xi$ implies that $(\varphi, \xi, \eta)$ is normal.

On every normal almost contact manifold, one also has

$$(4) \quad i_{\varphi}d\eta = 0, \quad i_{\xi}d\eta = 0.$$  

For a normal almost contact manifold $(M, \varphi, \xi, \eta)$ define the symmetric 2-tensor $g$ by

$$g(X, Y) = \frac{1}{2}d\eta(\varphi X, Y) + \eta(X)\eta(Y).$$

A normal almost contact manifold $(M, \varphi, \xi, \eta)$ is called Sasakian if $g$ is a Riemannian metric. It can be checked (cf. [8, Section 6.5]) that $M$ is Sasakian if and only if $\tilde{g} := e^{2t}(g + dt^2)$ and $J$ defined by (1) give a Kähler structure on $M \times \mathbb{R}$.

**Remark 2.1.** The corresponding Kähler form $e^{2t}(\Phi + dt \wedge \eta)$ is exact. Indeed it equals to one-half of $d(e^{2t}\eta)$.

Following the development in [9, Sec. 7.5.1], we say that an almost contact structure $(\varphi', \xi, \eta')$ is a second type deformation of an almost contact structure $(\varphi, \xi, \eta)$ on $M$, if there is a basic 1-form $\beta$ on $M$ such that $\eta' = \eta - i_{\varphi} \beta$ and $\varphi' = \varphi - \xi \otimes \beta$.

Let $M$ and $N$ be an almost contact manifolds and $\beta$ a basic 1-form on $M$. We say that a smooth map $h: M \to N$ is $\beta$-twisted if

$$\eta_M = h^*\eta_N - i_{\varphi_M} \beta, \quad Th \circ \xi_M = \xi_N \circ h,$$

$$Th \circ (\varphi_M + \xi_M \otimes \beta) = \varphi_N \circ Th.$$  

This definition is designed so that if $h$ is a $\beta$-twisted diffeomorphism, then the almost contact structure on $N$, transferred from $M$ via $h$, is a second type deformation of $(\varphi_N, \xi_N, \eta_N)$. Also, if the identity map from $(M, \varphi', \xi, \eta')$ to $(M, \varphi, \xi, \eta)$ is $\beta$-twisted, then $(\varphi', \xi, \eta')$ is a second type deformation of $(\varphi, \xi, \eta)$.  

For \( h: M \to N \) and \( f \in C^\infty(M) \), define
\[
h_f: M \times \mathbb{R} \to N \times \mathbb{R}
\]
\[(x, t) \mapsto (h(x), f(x) + t).\]

**Proposition 2.2.** Let \( M \) and \( N \) be almost contact manifolds, \( h: M \to N \) a smooth map, and \( f \) a smooth function on \( M \). The map \( h_f \) is holomorphic if and only if \( f \), and hence also \( df \), is basic and \( h \) is \( df \)-twisted.

**Proof.** For \( L \in \{M, N\} \) write the elements of \( T(L \times \mathbb{R}) \) in the form \((X, t)\) with \( X \in TL \) and \( t \in \mathbb{R} \). Then for \( x \in M \), \( y \in N \), \( t \in \mathbb{R} \)
\[
J_{M \times \mathbb{R},(x,t)} = \begin{pmatrix}
\frac{\varphi_{M,x}}{dt} & -\xi_{M,x} \otimes dt \\
\frac{\varphi_{N,y}}{dt} & 0
\end{pmatrix},
J_{N \times \mathbb{R},(y,t)} = \begin{pmatrix}
\frac{\varphi_{N,y}}{dt} & 0 \\
\frac{\varphi_{N,y}}{dt} & \eta_{N,y} \otimes dt
\end{pmatrix}.
\]

and
\[
T_{(x,t)}h_f = \begin{pmatrix}
T_xh & 0 \\
T_{f(t)} \tau_f(x) & 1
\end{pmatrix},
\]
where \( \tau_s: \mathbb{R} \to \mathbb{R} \) is the translation by \( s \). Now \( Th_f \circ J_{M \times \mathbb{R}} = J_{N \times \mathbb{R}} \circ Th_f \) at \((x, t)\) if and only if
\[
T_xh \circ \varphi_{M,x} = \varphi_{N,h(x)} \circ T_xh - \xi_{N,h(x)} \otimes (df)_x, \quad T_xh(\xi_{M,x}) = \xi_{N,h(x)}
\]
\[
T_{f(t)} \tau_f(x) \circ T_xh \circ \varphi_M + \frac{d}{dt} \otimes \eta_{M,x} = \frac{d}{dt} \otimes \eta_{N,h(x)} \circ T_xh, \quad T_{f(t)} \tau_f(x) \circ T_xh(\xi_{M,x}) = 0.
\]
The last equation is equivalent to \( f \) being basic. The first three equations are equivalent to the definition \( df \)-twisted map. \( \square \)

Let \((M, \varphi, \xi, \eta)\) be a normal almost contact manifold and \( \mathcal{F} \) be the one-dimensional foliation on \( M \) generated by \( \xi \). The manifold \( M \) is transversely complex with respect to \( \mathcal{F} \). Namely, for each foliated chart \( U \) of \((M, \mathcal{F})\), the endomorphism \( \varphi \) of \( TM \) induces a complex structure on the tangent bundle of the leaf space \( U/\mathcal{F} \). Write \( \pi_U \) for the projection from \( U \) onto \( U/\mathcal{F} \). The induced complex structure \( J \) on \( U/\mathcal{F} \) is uniquely characterized by \( T\pi_U(\varphi X) = J(T\pi_U(X)) \) for all \( x \in U \) and \( X \in T_xU \).

If \( U \) is sufficiently small, the set \( U/\mathcal{F} \) admits holomorphic coordinates \( z_1, \ldots, z_n \). Using these coordinates one can introduce the operators \( \partial \) and \( \bar{\partial} \) on the complexified de Rham complex \( \Omega^*(U/\mathcal{F})_\mathbb{C} \) in the usual way. The operator \( d^c \) on \( \Omega^*(U/\mathcal{F})_\mathbb{C} \) is frequently defined as \( d^c = i(\bar{\partial} - \partial) \). It should be noticed that it can be identified with \((-1)^LJ \). In particular, \( d^c \) preserves \( \Omega^*(U/\mathcal{F}) \).

The pull-back map \( \pi^*_U \) induces an isomorphism between \( \Omega^*(U/\mathcal{F}) \) and the basic de Rham complex \( \Omega^*_B(U, \mathcal{F}) \). These isomorphisms permit to glue the operators \( d^c \) defined on \( \Omega^*(U/\mathcal{F}) \) for different \( U \) to a globally defined operator \( d^c \) on \( \Omega^*_B(M, \mathcal{F}) \). The operator \( d^c \) is uniquely characterized by the property
\[
(d^c_B)(\omega|_U) = \pi^*_U \left( d^c (\pi^*_U)^{-1} \omega|_U \right)
\]
for all $\omega \in \Omega^*_B(M, F)$ and sufficiently small foliated charts $U$. The computation

$$(i_\varphi \pi^*_U \omega)(X) = \omega((T \pi_U)((\varphi(X)) \circ \omega) = \omega((T \pi_U)(\varphi(X))) = \pi^*_U(i_\varphi \omega)(X)$$

implies that $\pi^*_U \circ d^f = -\pi^*_U \circ L_{\varphi} = -\mathcal{L}_{\varphi} \circ \pi^*_U$. We see that $d^f$ coincides with the restriction of $(-1)\mathcal{L}_{\varphi}$ to basic forms.

2.3. **Sullivan models.** A differential graded algebra is a graded algebra $A = \bigoplus_{k \geq 0} A_k$ equipped with a derivation $d$ of degree one such that $d^2 = 0$. It is commutative if $ab = (-1)^{k_l}ba$, for all $a \in A_k$ and $b \in A_l$.

The motivating example of a commutative differential graded algebra (CDGA) is provided by the de Rham algebra $\Omega^*(M)$ of a manifold $M$.

For CDGAs $A$ and $B$, a homomorphism $f : A \to B$ of CDGAs is a degree preserving homomorphism of algebras which commutes with $d$. Each homomorphism of CDGAs $f : A \to B$ induces a homomorphism $H^\bullet(f) : H^\bullet(A) \to H^\bullet(B)$ of graded algebras by $H^\bullet(f)([a]) = [f(a)]$ for $a \in A$ such that $da = 0$. We say that a homomorphism of CDGAs $f : A \to B$ is a quasi-isomorphism if $H^\bullet(f)$ is an isomorphism. Two CDGAs $A$ and $B$ are said to be quasi-isomorphic if there are CDGAs $A_0 = A, A_1, \ldots, A_{2k} = B$, and quasi-isomorphisms $f_j : A_{2j+1} \to A_{2j}$, $h_j : A_{2j+1} \to A_{2j+2}$ for $j$ between $0$ and $k - 1$.

A CDGA $A$ is called a Sullivan algebra, if there is a generating set of homogeneous elements $a_i \in A$ indexed by a well ordered set $I$, such that

(i) $da_k$ lies in the subalgebra generated by the elements $a_j$ with $j < k$;
(ii) $A$ has a basis consisting of the elements

$$a_{j_1}^{r_1} \cdots a_{j_n}^{r_n},$$

with $j_1 < \cdots < j_n$, $r_k = 1$ if degree of $a_k$ is odd, and $r_k \in \mathbb{N}$ if degree of $a_k$ is even.

The motivating example of Sullivan CDGA was the Chevalley-Eilenberg algebra $\bigwedge g^*$ for nilpotent Lie algebra $g$. We will need the following lifting property of Sullivan CDGAs.

**Proposition 2.3** (cf. [17] Proposition 12.9]). Suppose $q : A \to B$ is a quasi-isomorphism of CDGAs and $f : D \to B$ a homomorphism of CDGAs. If $D$ is Sullivan, then there is a homomorphism $h : D \to A$ of CDGAs, such that $H^\bullet(q) \circ H^\bullet(h) = H^\bullet(f)$.

An algebra $A$ quasi-isomorphic to $\Omega^*(M)$ is called a real homotopy model of $M$. Suppose $M$ and $N$ are homotopy equivalent smooth manifolds. Let $F : M \to N$ and $G : N \to M$ be mutually inverse homotopy equivalences. By Whitney Approximation Theorem (cf. [27], Thm. 6.26]), there are smooth maps $\tilde{F} : M \to N$ and $\tilde{G} : N \to M$ homotopy equivalent to $F$ and $G$, respectively. Then $\tilde{F}$ and $\tilde{G}$ are
mutually inverse smooth homotopy equivalences. This implies that 
\( \tilde{F}^* : \Omega^* (N) \to \Omega^* (M) \) and 
\( \tilde{G}^* : \Omega^* (M) \to \Omega^* (N) \) are quasi-isomorphisms. 
Hence the following proposition holds.

**Proposition 2.4.** Suppose \( M \) and \( N \) are homotopy equivalent smooth manifolds. Then there are quasi-isomorphisms 
\( \Omega^* (M) \to \Omega^* (N) \) and 
\( \Omega^* (N) \to \Omega^* (M) \). In particular, \( M \) and \( N \) have the same real homotopy models.

2.4. Nilmanifolds. Let \( \Gamma \) be a torsion-free finitely generated nilpotent group. It was shown in [28, 29], that such \( \Gamma \) can be realized as a lattice in a connected and simply connected nilpotent Lie group \( G(\Gamma) \) which is unique up to isomorphism. We denote the canonical embedding of \( \Gamma \) into \( G(\Gamma) \) by \( \nu_{\Gamma} \).

We will use the following extension principle for lattices in nilpotent Lie groups.

**Theorem 2.5.** Let \( H \) and \( G \) be connected and simply connected nilpotent Lie groups. If \( \Gamma \) is a lattice in \( H \), then every homomorphism of groups \( f : \Gamma \to G \) has a unique extension to a smooth homomorphism \( \tilde{f} : H \to G \).

**Proof.** This is essentially [33, Theorem 2.11], but there the author claims only the existence of continuous homomorphism \( \tilde{f} : H \to G \) that extends \( f \). However, it is known (cf. [28, Theorem 3.39]) that every continuous homomorphism of Lie groups is smooth. \( \square \)

Denote the nilmanifold \( \Gamma \backslash G(\Gamma) \) by \( N_{\Gamma} \). It is a manifold of type \( K(\Gamma,1) \). We will consider \( N_{\Gamma} \) as a pointed manifold with the base point \( \Gamma e \).

3. APPROXIMATION WITH NILMANIFOLDS

The main result of this section is more general than needed for the proof of Theorem [1.1]. For simplicity, the reader may assume that \( M \) is an aspherical nilpotent manifold through the section. We decided to present the results in a more general way in order to be able to compare them to the construction in [13] and with the hope that Theorem 3.4 can be useful in another context.

Let \( (M,x_0) \) be a pointed manifold and \( \Pi \) its fundamental group. For every homomorphism \( q : \Pi \to \Gamma \) there is a unique up to homotopy map \( h_q : M \to N_{\Gamma} \) of pointed manifolds such that \( \pi_1 (h_q) = q \) (cf. [24, Prop. 1B.9]). By Whitney Approximation Theorem (cf. [27, Thm. 6.26]), one can always assume that \( h_q \) is smooth, and any two such maps are connected by a smooth homotopy.

The aim of this section is to give a parametrization of the homotopy class of smooth maps \( h_q \) in the special case when \( M \) is compact and \( q \) is the canonical projection onto \( (\Pi/\Pi_k)_{(c)} \) for a fixed natural number.
Here \( \Pi_k \) denotes as usual the \( k \)th member of lower central series of \( \Pi \) and the subscript \( tf \) indicates the torsion-free part of a nilpotent group. Denote \( (\Pi/\Pi_k)_{tf} \) by \( \Gamma \) and write \( \mathfrak{g}_\Gamma \) for the Lie algebra of \( G(\Gamma) \). We will use \( \mathfrak{g}_\Gamma \)-valued forms on \( M \) for the parametrization.

We start by recalling some facts about Lie algebra-valued 1-forms. Let \( G \) be a Lie group and \( \mathfrak{g} \) its Lie algebra. We identify \( \mathfrak{g}^* \) with the left-invariant 1-forms on \( G \) and \( \bigwedge \mathfrak{g}^* \) with the subcomplex \( \Omega^\bullet(G)^G \) of all left-invariant forms on \( G \).

Let \( \omega \in \Omega^1(M, \mathfrak{g}) \). The form \( \omega \) corresponds to a unique linear map \( \hat{\omega}: \mathfrak{g}^* \to \Omega^1(M) \), which extends to the unique homomorphism \( \hat{\omega}: \bigwedge \mathfrak{g}^* \to \Omega^\bullet(M) \) of graded algebras. If \( h: N \to M \) is a smooth map, then \( h^*\omega \in \Omega^1(N, \mathfrak{g}) \) corresponds to the linear map \( h^* \circ \hat{\omega} \), i.e.

\[
(6) \quad h^*\omega = h^* \circ \hat{\omega}.
\]

The map \( \hat{\omega} \) is a homomorphism of chain complexes if and only if \( \omega \) is flat, i.e. if and only if it satisfies the Maurer-Cartan equation

\[
d\omega + \frac{1}{2}[\omega, \omega] = 0.
\]

Here \([\omega, \omega] \in \Omega^2(M, \mathfrak{g})\) is defined by \([\omega, \omega](X, Y) = 2[\omega(X), \omega(Y)]\).

In particular, the inclusion \( \bigwedge \mathfrak{g}^* = \Omega(G)^G \to \Omega^\bullet(G) \) corresponds to a flat form \( \mu_G \in \Omega^1(G, \mathfrak{g}) \), called the Maurer-Cartan form on \( G \). We will later use that the form \( \mu_G(\Gamma) \) is left invariant.

Given a smooth map \( h: M \to G \), the homomorphism of CDGAs \( h^* \circ \hat{\mu}_G: \bigwedge \mathfrak{g}^* \to \Omega^\bullet(M) \) corresponds to the flat form \( h^*\mu_G \in \Omega^1(M, \mathfrak{g}) \).

The form \( h^*\mu_G \) is called the Darboux derivative of \( h \). The following result shows that in the case \( M \) is simply connected, every flat form is the Darboux derivative of a smooth map from \( M \) to \( G \), unique up to translation.

**Theorem 3.1** ([35] Section 3.7). Let \( (X, x_0) \) be a pointed simply connected manifold and \( \omega \) a flat \( \mathfrak{g} \)-valued 1-form on \( X \). For each \( g \in G \) there is a unique smooth map \( P_{g, \omega}: X \to G \) such that \( P_{g, \omega} \) sends the base point of \( X \) to \( g \) and \( P_{g, \omega}^*\mu_G = \omega \). Moreover, \( L_{g'} \circ P_{g, \omega} = P_{g'g, \omega} \) for any \( g' \in G \).

We write \( \widetilde{M} \) for the universal cover of \( M \) and \( \pi \) for the projection from \( \widetilde{M} \) to \( M \). We fix a point \( \widetilde{x}_0 \in \widetilde{M} \) over \( x_0 \) and consider \( \widetilde{M} \) as a pointed manifold with the base point \( \widetilde{x}_0 \). If \( \omega \) is a flat \( \mathfrak{g} \)-valued 1-form on \( M \), then \( \pi^*\omega \) is a flat 1-from in \( \Omega^1(\widetilde{M}, \mathfrak{g}) \). Then, \( P_{e, \pi^*\omega}^*\mu_G = \pi^*\omega \) and (6) imply that

\[
(7) \quad \pi^* \circ \hat{\omega} = P_{e, \pi^*\omega}^* \bigwedge \mathfrak{g}^*.
\]
Corollary 3.2. Let $\omega$ a flat $\mathfrak{g}$-valued form on a manifold $M$. There is a homomorphism of groups $\sigma_\omega : \pi_1(M) \to G$, such that for all $g \in \pi_1(M)$ and $x \in \tilde{M}$

$$P_{e,\pi^*\omega}(gx) = \sigma_\omega(g)P_{e,\pi^*\omega}(x).$$

Proof. Write $L_g$ for the operator on $\tilde{M}$, that sends $x$ to $gx$. We compute the Darboux derivative of $P_{e,\pi^*\omega} \circ L_g$. We get

$$(P_{e,\pi^*\omega} \circ L_g)^* \mu_G = L^*_g P_{e,\pi^*\omega}^* \mu_G = L^*_g \pi^*\omega = (\pi \circ L_g)^* \omega = \pi^*\omega.$$  

Denote the image of $\tilde{x}_0$ under $P_{e,\pi^*\omega} \circ L_g$ by $\sigma_\omega(g)$. By Theorem 3.1, we get $P_{e,\pi^*\omega} \circ L_g = P_{\sigma_\omega(g),\pi^*\omega} = L_{\omega(g)}P_{e,\pi^*\omega}$. It is a routine to check that $\sigma_\omega$ is a homomorphism of groups. \hfill $\square$

Now we move to the special case when $G$ is the Lie group $G(\Gamma)$ with $\Gamma = (\Pi/\Pi_k)_{tr}$. As the pullback along the projection $G(\Gamma) \to N_\Gamma$ induces an isomorphism between $\Omega^*(N_\Gamma)$ and the set of $\Gamma$-invariant forms on $G(\Gamma)$, we get an embedding of CDGAs $\psi_\Gamma : \bigwedge^\bullet \mathfrak{g}_\Gamma \to \Omega^*(N_\Gamma)$. It was shown in [32] that $\psi_\Gamma$ is a quasi-isomorphism. We denote by $\mu_\Gamma$ the corresponding $\mathfrak{g}_\Gamma$-valued 1-form on $N_\Gamma$.

For every smooth map $h : M \to N_\Gamma$ such that $\pi_1(h)$ coincides with the canonical projection $q : \Pi \to \Gamma$, we get a $\mathfrak{g}_\Gamma$-valued 1-form $h^* \mu_\Gamma$ on $M$. The corresponding homomorphism of CDGAs is $h^* \circ \psi_\Gamma$. The induced map $H^1(h^* \circ \psi_\Gamma) : H^1(\mathfrak{g}_\Gamma) \to H^1(M, \mathbb{R})$ is an isomorphism.

Proposition 3.3. If a flat $\mathfrak{g}_\Gamma$-valued 1-form $\omega$ on $M$ is such that $H^1(\omega) : H^1(\mathfrak{g}_\Gamma) \to H^1(M)$ is an isomorphism, then there is a unique automorphism $A_\omega : G(\Gamma) \to G(\Gamma)$ such that $\sigma_\omega = A_\omega \circ \nu_\Gamma \circ q$.

Proof. The Lie group $G(\Gamma)$ is a nilpotent group with nilpotency class at most $k$. Hence every map from $\Pi$ to $G(\Gamma)$ factors uniquely via the projection from $\Pi$ onto $\Pi/\Pi_k$. Since $G(\Gamma)$ is torsion-free, every map from $\Pi/\Pi_k$ to $G(\Gamma)$ admits a unique factorization via the projection from $\Pi/\Pi_k$ onto $\Gamma = (\Pi/\Pi_k)_{tr}$. Hence there is a unique homomorphism of groups $f : \Gamma \to G(\Gamma)$ such that $\sigma_\omega = f \circ q$.

By Theorem 2.5, there is a unique smooth homomorphism of Lie groups $A_\omega : G(\Gamma) \to G(\Gamma)$ that extends $f$, i.e. $f = A_\omega|_\Gamma = A \circ \nu_\Gamma$. Thus $\sigma_\omega = A_\omega \circ \nu_\Gamma \circ q$. The uniqueness of $A_\omega$ with such property follows from the uniqueness part of Theorem 2.5 and the uniqueness of $f$ such that $\sigma_\omega = f \circ q$.

It is left to show that $A_\omega$ is an automorphism. As $G(\Gamma)$ is a simply connected nilpotent Lie group, it is enough to check that $A_\omega$ is onto.

Write $H$ for the image of $A_\omega$ in $G(\Gamma)$. It is shown in [14, Theorem 7.18], that for a nilpotent group $G$ and a subgroup $N < G$, one has $N[G, G] = G$ if and only if $N = G$. Thus $A_\omega$ is onto if and only if $H[G(\Gamma), G(\Gamma)] = G(\Gamma)$.

The group $H[G(\Gamma), G(\Gamma)]$ is a path-connected subgroup of $G(\Gamma)$. By the result of Yamabe [39] (see also [20]), every path-connected subgroup
of $G(\Gamma)$ is a Lie subgroup. Thus it corresponds to a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}_F$. As $G(\Gamma)$ is a simply connected nilpotent Lie group, the exponential map $\exp: \mathfrak{g}_F \to G(\Gamma)$ is a diffeomorphism. From the Baker-Campbell-Hausdorff formula and nilpotency of $G(\Gamma)$, it follows that $\exp(\mathfrak{k})$ is a Lie subgroup of $G(\Gamma)$. Hence $\exp(\mathfrak{k})$ coincides with $H[G(\Gamma), G(\Gamma)]$. As $\mathfrak{k}$ is a closed subset of $\mathfrak{g}_F$, we conclude that all the homotopy groups of $\mathfrak{g}_F$ are trivial and thus $Q$ is contractible. Hence $Q = (\mathbb{R}^k, +)$ as a Lie group for some $k \geq 0$. Clearly, $k = 0$ if and only if $H[G(\Gamma), G(\Gamma)] = G(\Gamma)$, if and only if $A_\omega$ is onto.

Suppose $k \neq 0$. Then there is a non-trivial homomorphism $\phi: Q \to \mathbb{R}$ of Lie groups. Denote by $\text{pr}_Q$ the canonical projection of $G(\Gamma)$ onto $Q$. Write $\beta$ for $(\phi \circ \text{pr}_Q)^*(dt)$. Since $dt$ is a non-zero left-invariant form on $(\mathbb{R}, +)$ and $\phi \circ \text{pr}_Q$ is a surjective homomorphism of Lie algebras, the form $\beta$ is a non-zero left-invariant form on $G(\Gamma)$. Hence $\beta \in \mathfrak{g}_F^*$. As $\beta$ is closed, we get that $\beta \in \mathfrak{g}_F^* \cap \ker dCE = H^1(\wedge^1 \mathfrak{g}_F^*)$.

As $H^1(\wedge^1 \mathfrak{g}_F^*)$ is injective, the class $[\omega(\beta)]$ is a non-zero element in $H^1(M)$. Hence there is a loop $\gamma$ at $x_0$ such that $\int_\gamma \omega(\beta) \neq 0$. Let $\tilde{\gamma}: \mathbb{R} \to \tilde{M}$ be the unique lifting of $\gamma$ such that $\tilde{\gamma}(0) = \tilde{x}_0$. Using (7), we get

$$0 \neq \int_\gamma \omega(\beta) = \int_0^1 \gamma^*(\omega(\beta)) = \int_0^1 \tilde{\gamma}^* \circ \pi_\ast \circ \omega(\beta)$$

$$= \int_0^1 \tilde{\gamma}^* \circ P_{e,\pi^\ast \omega} \circ \text{pr}_Q^* \circ \phi^*(dt) = \int_c \phi^*(dt),$$

where $c$ is the curve $\text{pr}_Q \circ P_{e,\pi^\ast \omega} \circ \tilde{\gamma}$ in $Q$. Consider the path $P_{e,\pi^\ast \omega} \circ \tilde{\gamma}$ in $G(\Gamma)$. We have $\tilde{\gamma}(1) = [\gamma]x_0$ by definition of the action of $\Pi$ on $\tilde{M}$. By Corollary 3.2, we get

$$P_{e,\pi^\ast \omega}(\tilde{\gamma}(1)) = P_{e,\pi^\ast \omega}([\gamma]x_0) = \sigma_\omega([\gamma]) = A_\omega \circ \nu_\Gamma \circ q([\gamma]).$$

Thus $P_{e,\pi^\ast \omega}(\tilde{\gamma}(1))$ lies in the image of $A_\omega$. Thus $c(1)$ is the neutral element of $Q$. Obviously $c(0)$ is also the neutral element in $Q$. Therefore $c$ is a loop in a contractible space. We get $\int_c \phi^*dt = 0$. This gives a contradiction to the assumption $k \neq 0$. Therefore $k = 0$ and $A_\omega$ is an automorphism.

For a flat 1-form $\omega \in \Omega^1(M, \mathfrak{g}_F)$ such that $H^1(\omega)$ is an isomorphism, denote by $\tilde{h}_\omega$ the smooth map $A_\omega^{-1} \circ P_{e,\pi^\ast \omega}$ from $\tilde{M}$ to $G(\Gamma)$. For every $g \in \Pi$ and $y \in \tilde{M}$, we get $\tilde{h}_\omega(gx) = q(g)\tilde{h}_\omega(x)$. Moreover, $\tilde{h}_\omega(\tilde{x}_0) = e$. Therefore, $\tilde{h}_\omega$ induces a smooth map $h_\omega: M \to N_\Gamma$ of pointed manifolds such that $\pi_\Gamma(h_\omega) = q$. Write $\text{pr}_\Gamma$ for the canonical
projection $G(\Gamma) \to N_\Gamma$. We get

$$\pi^* \circ \hat{\omega} = P^*_{\omega^1 \omega} \mid_{\hat{\gamma}_1} = \tilde{h}_\omega^* \circ A^*_\omega \mid_{\hat{\gamma}_1} = \tilde{h}_\omega^* \mid_{\hat{\gamma}_1} \circ a_\omega,$$

where $a_\omega$ is the restriction of $A^*_{\omega}$: $\Omega(G(\Gamma)) \to \Omega(G(\Gamma))$ to $\hat{\gamma}_1$. The map $pr^*_T \circ \psi_T$ coincides with the canonical inclusion of $\hat{\gamma}_1$ into $\Omega^*(G(\Gamma))$. As $h_\omega \circ \pi = pr^*_T \circ \tilde{h}_\omega$, we get

$$\tilde{h}_\omega^* \mid_{\hat{\gamma}_1} = \hat{h}_\omega^* \circ pr^*_T \circ \psi_T = \pi^* \circ h_\omega^* \circ \psi_T.$$

Combining this equation with (3.3), we obtain

$$\pi^* \circ \hat{\omega} = \pi^* \circ h_\omega^* \circ \psi_T \circ a_\omega.$$

Since $\pi^*: \Omega(M) \to \Omega(\tilde{M})$ is injective, we get $\hat{\omega} = h_\omega^* \circ \psi_T \circ a_\omega$. Hence $\hat{\omega} \circ a_\omega^{-1} = h_\omega^* \circ \psi_T$. Therefore, we proved the existence part of the following theorem.

**Theorem 3.4.** Let $(M, x_0)$ be a pointed compact manifold and $k$ a natural number. Write $\Pi$ for the fundamental group of $M$, $\Gamma$ for $(\Pi/\Pi_t)_t$, and $q$ for the canonical projection from $\Pi$ to $\Gamma$. For every flat $1$-form $\omega \in \Omega^1(M, g_\Gamma)$ such that $H^1(\hat{\omega})$ is an isomorphism, there is a unique smooth map $h_\omega: M \to N_\Gamma$ of pointed manifolds and a unique automorphism $a_\omega$ of $\hat{\gamma}_1$ such that $\pi_1(h_\omega) = q$ and $\hat{\omega} = h_\omega^* \circ \psi_T \circ a_\omega$.

**Proof of the uniqueness.** Suppose $h: M \to N_\Gamma$ and $b \in \text{Aut}(\hat{\gamma}_1)$ are such that $h(x_0) = \Gamma e$, $\hat{\omega} = h^* \circ \psi_T \circ b$, and $\pi_1(h) = q$. There is a unique lifting $\tilde{h}: \tilde{M} \to G(\Gamma)$ of $h$ such that $\tilde{h}(\tilde{x}_0) = e$. Denote by $B$ the automorphism of $G(\Gamma)$ that integrates $b^*|_{\hat{\gamma}_1}: \mathfrak{g}_\Gamma \to \mathfrak{g}_\Gamma$. We claim that $B \circ \tilde{h}$ and $P_{e, \pi^* \omega}$ are equal. As both maps send $\tilde{x}_0$ to $e$, by the uniqueness part of Theorem 3.3 it is enough to show that they have the same Darboux derivative. The $1$-form $(B \circ \tilde{h})^* \mu_{G(\Gamma)}$ corresponds to the homomorphism $\tilde{h}^* \circ B^*|_{\hat{\gamma}_1}: \hat{\gamma}_1 \to \Omega^*(\tilde{M})$ of CDGAs. We have

$$\tilde{h}^* \circ B^*|_{\hat{\gamma}_1} = \tilde{h}^* \circ pr^*_T \circ \psi_T \circ b = \pi^* \circ h^* \circ \psi_T \circ b = \pi^* \circ \hat{\omega} = \pi^* \circ \hat{\sigma}_\omega.$$

Hence $(B \circ \tilde{h})^* \mu_{G(\Gamma)} = \pi^* \omega = P_{e, \pi^* \omega} \mu_{G(\Gamma)}$ and $B \circ \tilde{h} = P_{e, \pi^* \omega}$. In particular, $\sigma_\omega(g) = B(\tilde{h}(g\tilde{x}_0))$. Since $\pi_1(h) = q$, we have $\tilde{h}(gx) = q(g)\tilde{h}(x)$. Thus $\sigma_\omega(g) = B(q(g))$. The uniqueness part of Proposition 3.3 implies that $B = A_\omega$ and, hence, $b = a_\omega$. Now, $B \circ \tilde{h} = P_{e, \pi^* \omega} = A_\omega \circ \tilde{h}_\omega$ implies that $\tilde{h} = \tilde{h}_\omega$. Hence $h = h_\omega$. \qed

**Remark 3.5.** In [13], Chen constructed a smooth homotopy equivalence $h: M \to N_\Gamma$ such that $\pi_1(h) = q$ starting with a splitting of $\Omega^*(M)$. In particular, his construction can be applied to any Riemannian manifold. Our understanding is that the corresponding $\mathfrak{g}_\Gamma$-valued $1$-form can be obtained from the Chen connection in $\Omega^*(M) \otimes \hat{T}(H_*(M))$. We do not know if every smooth homotopy equivalence $h: M \to N_\Gamma$ such that $\pi_1(h) = q$ corresponds to a suitable splitting of $\Omega^*(M)$. 
Remark 3.6. Suppose $M$ aspherical and $\Pi = \Gamma$, i.e. that $\Pi$ is a torsion-free nilpotent group of nilpotency class at most $k$. If a flat $\mathfrak{g}_T$-valued 1-form $\omega$ on $M$ is such that $\hat{\omega}$ is a quasi-isomorphism, then $h_\omega$ is a homotopy equivalence. In particular, the degree of $h_\omega$ is either 1 or $-1$. As a non-zero degree map between compact manifolds, $h_\omega$ is surjective (cf. [22, Prop. I, Sec. 6.1]).

Corollary 3.7. Let $M^n$ be a compact aspherical manifold with nilpotent fundamental group $\Gamma$ and $\mathfrak{g}$ the Lie algebra of $G(\Gamma)$. If $\rho: \wedge \mathfrak{g}^* \to \Omega^*(M)$ is a quasi-isomorphism of CDGAs then $\rho$ is an injective map.

Proof. Let $\omega \in \Omega^1(M, \mathfrak{g})$ be such that $\rho = \hat{\omega}$. By Theorem 3.4, $\rho = h_\omega^* \circ \psi_\Gamma \circ a_\omega$, where $a_\omega$ is an automorphism of $\wedge \mathfrak{g}_T$. It is clear that $\psi_\Gamma: \wedge \mathfrak{g}^* \to \Omega^*(N_\Gamma)$ is injective. The map $h_\omega$ is surjective by the previous remark. Thus $h_\omega^*: \Omega^*(N_\Gamma) \to \Omega^*(M)$ is injective. Hence $\rho$ is a composition of three injective maps.

4. On de Rham algebra of Sasakian manifolds

The main objective of this section is to show that for every compact aspherical Sasakian manifold with nilpotent fundamental group $\Gamma$, there is a quasi-isomorphism $\wedge T^*_e G(\Gamma) \to \Omega^*(M)$ with sufficiently rigid properties that will imply that a corresponding smooth homotopy equivalence $h: M \to N_\Gamma$ is a diffeomorphism.

We will use the following notation. Given linear operators $A_1, \ldots, A_k$ on $\Omega^*(M)$, we will write $\Omega^{A_1,\ldots,A_k}(M)$ for the intersection of the kernels of the operators $A_1, \ldots, A_k$. If each of $A_1, \ldots, A_k$ is a homogeneous graded derivation, then $\Omega^{A_1,\ldots,A_k}(M)$ is a subalgebra of $\Omega^*(M)$. If $R$ is a graded subalgebra of $\Omega^*(M)$ and $\alpha$ is an element of $\Omega^*(M)$, then $R[\alpha]$ denotes the subalgebra of $\Omega^*(M)$ generated by $R$ and $\alpha$. If $\alpha$ has an odd degree, then, of course, $R[\alpha] = R + R \wedge \alpha$.

Now we discuss Tievsky models for the complexified de Rham algebra of a Sasakian manifold constructed in his PhD thesis [37]. Let $(M^{2n+1}, \varphi, \xi, \eta)$ be a Sasakian manifold. Tievsky showed that the embedding

$$\Omega^*_{\xi,e^c,\xi_B}(M)[\eta] \hookrightarrow \Omega^*(M)[\eta]$$

is a quasi-isomorphism. Since the base field change functor is exact, we get that also

$$\Omega^*_{\xi,e^c,\xi_B}(M)[\eta] \hookrightarrow \Omega^*(M)$$

is a quasi-isomorphism. We write $\mathcal{T}^*(M)$ for the CDGA $\Omega^*_{\xi,e^c,\xi_B}(M)[\eta]$ and refer to it as the first Tievsky model.

By [15 Lemma 1], the $d_Bd^c_B$-lemma holds on $\Omega^*_{\xi,e^c,\xi_B}(M)$, i.e.

$$\ker d^c_B \cap \ker d_B \subset \ker d_B = \operatorname{im}(d_Bd^c_B).$$

This implies that

$$\ker d^c_B = \ker d^c_B \cap \ker d_B + \operatorname{im}(d_Bd^c_B).$$
The inclusion “⊂” is obvious. Applying $d_Bd_B^c$-lemma to $d_B\beta$ with $\beta \in \ker d_B^c$, we get that there is $\alpha$ such that $d_B\beta = d_Bd_B^c\alpha$. Thus

\[ \beta = (\beta - d_B^c\alpha) + d_B^c\alpha \in \ker d_B^c \cap d_B + \im d_B^c. \]

This permits to identify the first two components of $T^\bullet(M)$.

**Proposition 4.1.** Let $M^{2n+1}$ be a compact Sasakian manifold. Then

\begin{equation}
T^0(M) = \Omega^0_{i\xi,\ell \xi, d_B^c}(M) = \mathbb{R}
\end{equation}

\begin{equation}
T^1(M) = \Omega^1_{\Delta}(M) \oplus d_B^c\Omega^0_{i\xi, \ell \xi}(M) \oplus \mathbb{R}\eta.
\end{equation}

Moreover, $H^1(T^\bullet(M))$ coincides with $\Omega^1_{\Delta}(M)$.

**Proof.** Applying (10) in degree zero, we get

\[ \mathbb{R} \subset \Omega^0_{i\xi, \ell \xi, d_B^c}(M) = \Omega^0_{i\xi, \ell \xi, d_B^c, d_B}(M) \subset \Omega^0_d(M) = \mathbb{R}. \]

This shows (11). In degree 1, we get

\[ \Omega^1_{i\xi, \ell \xi, d_B^c}(M) = \Omega^1_{i\xi, \ell \xi, d_B^c, d_B}(M) + d_B^c\Omega^0_{i\xi, \ell \xi}(M). \]

The sum on the right side is direct. Indeed, suppose $\beta = d_B^c h \in \Omega^1_{i\xi, \ell \xi, d_B^c, d_B}(M)$ for some basic function $h$. Then by $d_Bd_B^c$-lemma $\beta$ is in the image of $d_Bd_B^c$, and, for the degree reasons, $\beta = 0$. Thus

\begin{equation}
T^1(M) = \Omega^1_{i\xi, \ell \xi, d_B^c}(M) \oplus d_B^c\Omega^0_{i\xi, \ell \xi}(M) \oplus \mathbb{R}\eta.
\end{equation}

As the zeroth component of the model is $\mathbb{R}$, we see that the image of the differential in the first component is zero. Hence the first cohomology group of the Tievsky model coincides with the kernel of $d_B$ restricted to $T^1(M)$. By [36] and [19, Thm. 4.1], it is known that every harmonic 1-form on a Sasakian manifold is basic and that the space $\Omega^1_{\Delta}(M)$ is invariant under the action of $i_{\xi_i}$. The second property implies that for every harmonic 1-form $\alpha$ we have $d_B^c\alpha = -\mathcal{L}_{\xi_i}\alpha = d(i_{\xi_i}\alpha) = 0$. Thus

\[ \Omega^1_{\Delta}(M) \subset \Omega^1_{i\xi, \ell \xi, d_B^c, d_B}(M) \subset H^1(\Omega^\bullet_{i\xi, \ell \xi, d_B^c}(M)[\eta]). \]

For dimension reasons the above inclusions are equalities. This shows the last claim of the proposition. Finally, (12) follows from (13) and $\Omega^1_{\Delta}(M) = \Omega^1_{i\xi, \ell \xi, d_B^c, d_B}(M)$. $\square$

Now we introduce the second Tievsky model. Consider the graded algebra $H_B(M)[t]/t^2$ with $\deg(t) = 1$. It is a CDGA with the differential $d(a + bt) = b[da]_B$. Tievsky showed that the epimorphism

\[ T^\bullet(M)_C = \Omega^\bullet_{i\xi, \ell \xi, d_B^c}(M)_C[\eta] \twoheadrightarrow H_B(M)_C[t]/t^2 \]

\[ \alpha + \beta + \eta \mapsto [\alpha] + [\beta]t \]

is a quasi-isomorphism. This implies that the corresponding map

\[ T^\bullet(M) \rightarrow H_B(M)[t]/t^2 \]

is a quasi-isomorphism. Hence $H_B(M)[t]/t^2$ is a model of $\Omega^\bullet(M)$. We call it the second Tievsky model of $M$. 
It was observed by Bazzoni in [11] that our proof of the main result in [11] can be modified to imply the following proposition. For completeness, we give here a proof based on our results in [12].

**Proposition 4.2.** If $M^{2n+1}$ is a compact aspherical Sasakian manifold with nilpotent fundamental group $\Gamma$, then $G(\Gamma)$ is isomorphic to the Heisenberg group $H(1,n)$.

**Proof.** The nilmanifold $N_\Gamma$ and $M$ are both of type $K(\Gamma,1)$, and, hence, they are homotopy equivalent to each other. By Proposition 2.4 the second Tievsky model of $M$ shown in [12, Thm. 5.3] that if a nilmanifold $N_\Gamma$ has a real homotopy model of the form $A[t]/t^2$ with $\deg(t) = 1$, $dt \in A$, $[dt]^n \neq 0$, and the zero differential on $A$, then $G(\Gamma)$ is isomorphic to $H(1,n)$. As $[dt]^n \neq 0$ by [11] Lemma 3.1, we get the result. □

The following result is a corollary of the theory of Sullivan models.

**Proposition 4.3.** Let $M^{2n+1}$ be a compact aspherical Sasakian manifold with nilpotent fundamental group $\Gamma$. Then there is a quasi-isomorphism of CDGAs $\rho: \bigwedge g^*_\Gamma \to \Omega^*(M)$ such that $\im \rho \subset T^*(M)$.

**Proof.** By Proposition 2.4 there is a quasi-isomorphism $f: \Omega^*(N_\Gamma) \to \Omega^*(M)$. Write $i$ for the embedding of $T^*(M)$ into $\Omega^*(M)$. Since $g^*_\Gamma$ is nilpotent, the CDGA $\bigwedge g^*_\Gamma$ is Sullivan. By the lifting property for Sullivan algebras (see Prop. 2.3) applied to $f \circ \psi_\Gamma: \bigwedge g^*_\Gamma \to \Omega^*(M)$ and $i: T^*(M) \to \Omega^*(M)$, there is a homomorphism of CDGAs

$$j: \bigwedge g^*_\Gamma \to T^*(M)$$

such that $H^*(i) \circ H^*(j) = H^*(f \circ \psi_\Gamma)$. Since $i$ and $f \circ \psi_\Gamma$ are quasi-isomorphisms, we get that also $j$ is a quasi-isomorphism. Thus $\rho := i \circ j: \bigwedge g^*_\Gamma \to \Omega^*(M)$ is a quasi-isomorphism of CDGAs with the claimed property. □

**Proposition 4.4.** Let $M^{2n+1}$ be a compact aspherical Sasakian manifold with nilpotent fundamental group $\Gamma$. If $\rho: \bigwedge g^*_\Gamma \to \Omega^*(M)$ a quasi-isomorphism such that $\im \rho \subset T^*(M)$, then $\rho$ induces an isomorphism between the space of closed elements in $\bigwedge g^*_\Gamma$ and $\Omega^1_\Delta(M)$. Moreover, there is $\eta_\theta \in g^*_\Gamma$ and $f \in \Omega^1_{\xi}(M)$ such that $\rho(\eta_\theta) = \eta + L_\omega f$.

**Proof.** By Proposition 4.1 we know that $\Omega^1_\Delta(M) = H^1(T^*(M))$. Write $Z^1$ for the set of closed elements in $\bigwedge g^*_\Gamma$. As $\bigwedge^0 g^*_\Gamma = \mathbb{R}$, we have $Z^1 = H^1(\bigwedge g^*_\Gamma)$.

Denote by $\dot{\rho}: \bigwedge g^*_\Gamma \to T^*(M)$ the map $\omega \mapsto \rho(\omega)$. As $\rho$ and the inclusion $T^*(M) \hookrightarrow \Omega^*(M)$ are quasi-isomorphisms, also $\dot{\rho}$ is a quasi-isomorphism. Hence the quasi-isomorphism $\dot{\rho}$ induces an isomorphism between $Z^1$ and $\Omega^1_\Delta(M)$.

Now we will show the existence of $\eta_\theta \in g^*_\Gamma$ and $f \in \Omega^1_{\xi}(M)$ with the claimed properties. By Proposition 1.2 the Lie algebra $g^*_\Gamma$ is isomorphic
to the Heisenberg Lie algebra. In particular, the codimension of $Z^1$ in $\mathfrak{g}_r^*$ is one. Choose an arbitrary $\beta \in \mathfrak{g}_r^* \setminus Z^1$. From Proposition 4.1, we get that $\rho(\beta) = \hat{\rho}(\beta) = \omega + L_\beta h + \lambda \eta$ for some $\omega \in \Omega^1_\Delta(M)$, $h \in \Omega^0_{\mathcal{L}_\beta}(M)$ and $\lambda \in \mathbb{R}$. As $\rho|_{Z^1}: Z^1 \to \Omega^1_\Delta(M)$ is an isomorphism, there is $\alpha \in Z^1$ such that $\rho(\alpha) = \omega$. Notice that $\beta - \alpha \notin Z^1$. Hence, replacing $\beta$ with $\beta - \alpha$, we can assume that $\omega = 0$.

Next, we show that $\lambda \neq 0$. The top component of $\hat{\rho}$ can be seen as a map from $\mathbb{R} \beta \wedge (\Lambda^{2n} Z^1)$ to $\mathcal{T}^{2n+1}(M) = \Omega^B_\mathbb{R}(M, \mathcal{F}) \wedge \eta$. If $\lambda = 0$, then $\hat{\rho}(\mathbb{R} \beta) \subset \mathbb{R}(d_B h) \subset \Omega^1_B(M, \mathcal{F})$. As $\hat{\rho}(Z^1)$ is also a subset of $\Omega^B_\mathbb{R}(M, \mathcal{F})$, we get that $\hat{\rho}(\mathbb{R} \beta \wedge (\Lambda^{2n} Z^1))$ should lie in the zero space $\Omega^B_\mathbb{R}(M, \mathcal{F})$. But then $H^{2n+1}(\hat{\rho})$ is a zero map, which contradicts the assumption that $\hat{\rho}$ is a quasi-isomorphism. This shows that $\lambda \neq 0$.

Now define $\eta_\beta = (1/\lambda)\beta$. We get $\rho(\eta_\beta) = L_\beta f + \eta$, where $f = (1/\lambda)h$.

5. Proof of Theorem 1.1

Let $(M^{2n+1}, \varphi, \xi, \eta)$ be a compact nilpotent aspherical Sasakian manifold with nilpotent group $\Gamma$. By Proposition 4.2, the group $G(\Gamma)$ is isomorphic to the Heisenberg group $H(1, n)$, and, hence, $M$ is homotopy equivalent to a Heisenberg nilmanifold $N_\Gamma = \Gamma \backslash G(\Gamma)$.

Proposition 4.3 implies that there is a $\mathfrak{g}_r$-valued 1-form $\omega$ on $M$ such that $\hat{\omega} : \Lambda^* \mathfrak{g}_r^* \to \Omega^*(M)$ is a quasi-isomorphism of CDGAs whose image lies in $\mathcal{T}^*(M)$. By Theorem 3.4, $\hat{\omega} = h^\ast \circ \psi_T \circ \omega$. Write $h$ for $h_\omega$ and $\rho$ for $\hat{\omega} \circ a \omega^{-1} = h^\ast \circ \psi_T$. By Remark 3.6, the smooth map $h : M \to N_\Gamma$ is a homotopy equivalence, and, therefore, it is surjective and has degree $\pm 1$.

The image of $\rho = \hat{\omega} \circ a \omega^{-1}$ lies in $\mathcal{T}^*(M)$. Hence, by Proposition 4.4, there is $\eta_\beta \in \mathfrak{g}_r^*$ and $f \in \Omega^0_{\mathcal{L}_\beta}(M)$, such that $h^\ast \eta_\beta = \rho(\eta_\beta) = \eta + L_\beta f$.

Our next aim is to show that $h$ is $df$-twisted with respect to a suitable left-invariant normal almost contact structure on $N_\Gamma$.

Let $\eta_N = \psi_T(\eta_\beta) \in \Omega^1(N_\Gamma)$. To define $\xi_N$, we have to ensure that $\xi$ is projectable.

For every point $x$ in $N_\Gamma$ the map

$$\psi_{T,x} : \mathfrak{g}_r^* \to T^*_x N_\Gamma$$

$$\alpha \mapsto \psi_T(\alpha)_x$$

is an isomorphism of vector spaces. Write $Z^1$ for the set of closed elements in $\mathfrak{g}_r^*$. As $Z^1 \oplus \langle \eta_\beta \rangle = \mathfrak{g}_r^*$, we get that $T^*_x N_\Gamma = \psi_{T,x}(Z^1) \oplus \langle \eta_{N,x} \rangle$ for all $x \in N_\Gamma$.

Claim 5.1. (i) For all $x \in M$, we have $\eta_N(Th(\xi_x)) = 1$.
(ii) For each $x \in M$, we have $\psi_{T,h(x)}(Z^1) = \text{Ann}(Th(\xi_x))$.
(iii) The vector field $\xi$ is $h$-projectable.

Proof. For every $x \in M$, we have

$$\eta_N(Th(\xi_x)) = h^\ast(\psi_T(\eta_\beta))(\xi_x) = \rho(\eta_\beta)(\xi_x) = (\eta + i_\varphi df)(\xi_x) = 1.$$

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For every $\alpha \in Z^1$, we get
\[
\psi_T(\alpha)(Th(\xi_x)) = (h^* \circ \psi_T)(\alpha)(\xi_x) = \rho(\alpha)(\xi_x) = (i_\xi \rho(\alpha))_x.
\]
By Proposition 4.4, the 1-form $\rho(\alpha)$ is harmonic, and, hence, basic. Thus $\psi_T(\alpha)(Th(\xi_x)) = 0$. This shows that $\psi_{T,x}(Z_1) \subset \text{Ann}(Th(\xi_x))$ for all $x \in M$. It follows from (ii) that $\dim \text{Ann}(Th(\xi_x)) = 2n = \dim Z^1$. Thus $\psi_{T,x}(Z_1) = \text{Ann}(Th(\xi_x))$.

It is left to show that $\xi$ is projectable. We already know that $h$ is surjective. Suppose $x, x' \in M$ are in the preimage of $y \in N_\Gamma$. By (iii), we have
\[
\text{Ann}(Th(\xi_x)) = \psi_{T,y}(Z^1) = \text{Ann}(Th(\xi_{x'})).
\]
Hence $Th(\xi_x)$ and $Th(\xi_{x'})$ are collinear. By (ii), we know that $\eta_N(Th(\xi_x)) = 1 = \eta_N(Th(\xi_{x'}))$. Therefore $Th(\xi_x) = Th(\xi_{x'})$. □

For $y \in N_\Gamma$, define $\xi_{N,y} = Th(\xi_y)$, where $x$ is any point in the preimage of $y$. As $\xi$ is projectable, the vector field $\xi_N$ is well defined.

Instead of defining $\varphi_N$ on $TN_\Gamma$, we define its transpose $\varphi'_N$ on $T^*N_\Gamma$. By Proposition 4.4 the quasi-isomorphism $\rho$ induces an isomorphism $\tau : Z^1 \to \Omega^1_\Delta(M)$. By [19] Thm. 4.1 the operator $i_\varphi$ preserves harmonic forms. For $y \in N_\Gamma$, define $(\varphi'_N)_y : T^*_y N_\Gamma \to T^*_y N_\Gamma$ to be zero on $\langle \eta_{N,y} \rangle$ and to coincide with the composition
\[
\ker \xi_{N,y} \xrightarrow{\psi_{T,y}^{-1}} Z^1 \xrightarrow{\tau} \Omega^1_\Delta(M) \xrightarrow{i_{\varphi'_N}} \Omega^1_\Delta(M) \xrightarrow{\tau^{-1}} Z^1 \xrightarrow{\psi_{T,y}} \ker \xi_{N,y}.
\]
on $\ker(\xi_{N,y})$.

Claim 5.2. The triple $(\varphi_N, \xi_N, \eta_N)$ is a left-invariant normal almost contact structure on $N_\Gamma$.

Proof. We already know that $\eta_N(\xi_N) = 1$ by Claim 5.1[iv]. It is immediate that $(\varphi'_N)^2|_{\ker \xi_N} = -\text{Id}$ for all $x \in N_\Gamma$. Also $(\varphi'_N)^2 \eta_N = 0$. Hence $(\varphi'_N)^2 = -\text{Id} + \eta \otimes \xi$, i.e. $\varphi'_N = -\text{Id} + \xi \otimes \eta$. Hence $(\varphi_N, \xi_N, \eta_N)$ is an almost contact structure.

The form $\eta_N$ is left invariant by its definition. Claim 5.1[iv] and the definition of $\xi_N$ imply that the space of smooth sections of $\text{Ann}(\xi_N) \subset T^*N_\Gamma$ is generated by left-invariant sections in $\psi_T(Z^1)$. Hence $\text{Ann}(\xi_N)$ is a left-invariant subbundle of $T^*N_\Gamma$. Thus $\langle \xi_N \rangle$ is a left-invariant subbundle of $TN_\Gamma$. As $\eta_N$ and $\eta_N(\xi_N) = 1$ are left-invariant, we conclude that $\xi_N$ is a left-invariant vector field on $N_\Gamma$. From the definition of $\varphi'_N$ and the fact that $\text{Ann}(\xi_N)$ and $\langle \eta_N \rangle$ are left-invariant subbundles of $T^*N_\Gamma$, it follows that $\varphi_N$ is left invariant.

To check the normality condition $[\varphi_N, \varphi_N]_{FN} + 2d\eta \otimes \xi = 0$, it is enough to show for every $x \in N_\Gamma$ and $\zeta \in T^*_x N_\Gamma$, we have
\[
\zeta \circ ([\varphi_N, \varphi_N]_{FN})_x + 2\zeta(\xi_N,x)(d\eta_N)_x = 0.
\]
As each vector space $T^*_x N_\Gamma$ is generated by $\psi_{T,x}(Z^1)$ and $\eta_{N,x}$, it is enough to show that $\beta \circ [\varphi_N, \varphi_N]_{FN} + 2\beta(\xi_N)d\eta_N = 0$ just for $\beta \in
ψ_T(Z^1) and for β = η_N. It is easy to check that for all β ∈ Ω^1(N_T), one has
(14) β ∘ [ϕ_N, ϕ_N]_F_N = 2(-d((ϕ^2_N)β) - (ϕ′_N)(dβ) + i_ϕ_N d(ϕ'_N β)).
Suppose β ∈ ψ_T(Z^1). Then by the definition of ϕ'_N also ϕ'_N β and (ϕ'_N)^2 β are elements of ψ_T(Z^1). As all the elements in Z^1 are closed and ψ_T commutes with the differentials, we get that all three forms β, ϕ'_N β, and (ϕ'_N)^2 β are closed. Hence by (14), β ∘ [ϕ_N, ϕ_N]_F_N = 0. From Claim 5.1(ii) and the definition of ξ_N, we get that β(ξ_N) = 0. Hence β ∘ ((ϕ_N, ϕ_N)_F_N + 2dη_N ⊗ ξ_N) = 0 for all β ∈ ψ_T(Z^1).

Now take β = η_N. Using ϕ'Nη_N = 0 and (14), we get
η_N ∘ [ϕ_N, ϕ_N]_F_N = -2(ϕ^2_N) dη_N.

Hence, we have to check that
(15) (ϕ^2_N)dη_N = dη_N.
Write ψ for ψ_T|_{θ^1}. Then ψ_T|_{Λ^2 θ^1} = ϕ^2. We have dη_N = dψ_T(η_θ) = ψ_T(dη_θ) = ϕ^2(dη_θ). As g_θ is isomorphic to the Heisenberg Lie algebra, one gets that dη_θ ∈ Λ^2 Z^1. Hence
(ϕ^2(dψ_N) dη_θ = (ϕ^2(ψ ∘ τ^{-1} ∘ ϕ^t ∘ ϕ^{-1})) dη_θ.
We get that (15) is equivalent to
(16) ψ_T((ϕ^2 τ^{-1} ∘ ϕ^t ∘ ϕ^{-1})(dη_θ)) = ψ_T(dη_θ).
As ψ_T is injective, as well as ρ by Corollary 3.7, (16) is equivalent to
ρ((ϕ^2 τ^{-1} ∘ ϕ^t ∘ ϕ^{-1})(dη_θ)) = ρ(dη_θ).
As ρ|_{Z^1} = τ, we have ρ|_{Λ^2 Z^1} = ϕ^2. Thus the above equation becomes
(17) (ϕ^2 ϕ^t) ρ(dη_θ) = ρ(dη_θ).
From the characteristic property of η_θ, we get
ρ(dη_θ) = dρ(η_θ) = d(η + i_ϕ df) = dη + di_ϕ df.
Notice that i_ϕ df = 0 by (1). As f is basic, we get
i_ϕ d_i_ϕ = L^2_ϕ f + d_i_ϕ df = (d_f + d(df ∘ ϕ^2)) = d(-df + df(ξ_η) = 0.
Thus i_ϕ ρ(dη_θ) = 0. Since dη_θ ∈ Λ^2 Z^1 and ρ(Z_1) is a subset of basic forms on M, we get that ρ(dη_θ) is a basic form.
Now, if ω is a basic 2-form on M such that i_ϕ ω = 0, then (ϕ^2 ϕ^t) ω = ω. Indeed, for any X, Y ∈ TM, we get
0 = (i_ϕ)(ϕ X, Y) = ω(ϕ^2 X, Y) + ϕ(ϕ X, ϕ Y) = -ω(X, Y) + η(X)i_ϕ(ϕ Y) + (ϕ^2 ϕ^t) ω(X, Y) = -ω(X, Y) + (ϕ^2 ϕ^t) ω(X, Y).
This proves (17).

Claim 5.3. The map h is df-twisted.
Claim 5.4. \( (18) \)

The map \( \tilde{h} \circ \tilde{f} \) is a diffeomorphism. The map \( \tilde{h} \) is a diffeomorphism if and only if \( h \) is a diffeomorphism. Hence also \( h_f \) is a diffeomorphism if and only if \( h \) is a diffeomorphism.

Claim 5.4. The map \( h_f \) is surjective, universally closed and proper.

Proof. The map \( h_f \) is surjective being a composition of two surjective maps. Next, the map \( h \) is proper since it is a continuous map between compact topological spaces. Given a Hausdorff topological space \( X \) and a locally compact Hausdorff space \( Y \), a continuous map \( X \to Y \) is proper if and only if it is universally closed. Thus \( h_f \) is universally closed.
closed, and, hence, also $\tilde{h}$ is universally closed and, thus, proper. The same properties hold for $h_f$, as $f$ is a homeomorphism.

By Proposition 2.2 and Claim 5.3, the map $h_f$ is holomorphic.

Recall that a continuous map $\psi: X \to Y$ between two topological spaces is called finite if it is closed and has finite fibers.

Claim 5.5. The map $h_f: M \times \mathbb{R} \to N_\Gamma \times \mathbb{R}$ is finite.

Proof. We already saw in Claim 5.4 that $h_f$ is closed. Thus it is left to show that $h_f$ has finite fibers. Fix $y \in N_\Gamma \times \mathbb{R}$. Since $h_f$ is holomorphic, $h_f^{-1}(y)$ is a complex analytic subvariety of $M \times \mathbb{R}$. By Claim 5.4, $h_f$ is proper. Thus $h_f^{-1}(y)$ is compact. Hence $h_f^{-1}(y)$ is a union of finitely many irreducible complex subvarieties (cf. [21, Sec. 9.2.2]). By Remark 2.1, the Kähler form on $M \times \mathbb{R}$ is exact. We will show in Lemma 5.6, that every compact irreducible subvariety of a Kähler manifold with an exact Kähler form is a point. Hence $h_f^{-1}(y)$ is a union of finitely many points.

Lemma 5.6. Let $X$ be a Kähler manifold and $Z \subset X$ an irreducible compact complex analytic subvariety in $X$. If the Kähler form of $X$ is exact, then $Z$ is a point.

Proof. By embedded Hironaka resolution of singularities for the pair $Z \subset X$, there is a proper birational holomorphic map $\pi: \tilde{X} \to X$ of complex manifolds with exceptional locus $\Sigma$, such that the strict transform

$$\tilde{Z} = \pi^{-1}(Z \setminus \Sigma)$$

of $Z$ is a smooth complex subvariety of $\tilde{X}$ and the restriction of $\pi$ to $\tilde{Z}$ is an immersion. Denote this restriction by $\sigma$. In our case, we have that $\tilde{Z}$ is a compact submanifold of $\tilde{X}$. Indeed, $\tilde{Z}$ is a closed subset of $\pi^{-1}(Z)$, which is compact since $\pi$ is proper and $Z$ is compact.

Write $\omega$ for the Kähler form on $X$. Since $\omega$ is exact, we have $\omega = d\alpha$ for some $\alpha \in \Omega^1(X)$. The form $\sigma^*\omega$ is a Kähler form on $\tilde{Z}$. Indeed, it is obviously closed and it is positive, since for every nonzero $X \in T\tilde{Z}$

$$\sigma^*\omega(JX, X) = \omega(\sigma_*(JX), \sigma_*X) = \omega(J\sigma_*X, \sigma_*X) = g(\sigma_*X, \sigma_*X) > 0.$$ 

We used that $\sigma$ is an immersion in the last step. As $\sigma^*\omega = d(\sigma^*\alpha)$, we get that $\tilde{Z}$ is a compact Kähler manifold with an exact Kähler form, which is possible only if $\tilde{Z}$ is a finite union of points. Hence $Z = \sigma(\tilde{Z})$ is a finite union of points. Since $Z$ is irreducible, we conclude that $Z$ is a point.

Now we are ready to finish the proof of Theorem 1.1

Claim 5.7. The map $h_f$ is a biholomorphism. In particular, $h_f$ and $h$ are diffeomorphisms.
Proof. By [21 Sec. 9.3.3] a finite holomorphic surjection between irreducible complex spaces is an analytic covering. The map $h_f$ is surjective by Claim 5.4, finite by Claim 5.5 and holomorphic by Proposition 2.2 and Claim 5.3. Thus $h_f$ is an analytic covering. Hence there is a nowhere dense closed subset $T$ in $N \Gamma \times \mathbb{R}$ such that the induced map
\[ h_T : (M \times \mathbb{R}) \setminus h_f^{-1}(T) \to (N \Gamma \times \mathbb{R}) \setminus T \]
is locally biholomorphic. To complete the proof it is enough to show that $h_T$ is a biholomorphism. Indeed, this will imply that $h_f$ is a one-sheeted analytic covering and then by a result in [21 Sec. 8.1.2] the map $h_f$ is a bijection. By [18 Corollary 8.6], every holomorphic bijection is a biholomorphism. Hence, we will get that $h_f$ is a biholomorphism.

To show that $h_T$ is a biholomorphism it is enough to show that $h_T$ is bijective. Let $(y, b) \in R := (N \Gamma \times \mathbb{R}) \setminus T$. Then $(y, b)$ is a regular value of $h_T$ and thus a regular value of $h_f$. Let $(x, a)$ be in the preimage of $(y, b)$. As $(x, a)$ is a regular point of the holomorphic map $h_f$, we have $\det T_{(x,a)}h_f > 0$. But $\det T_{(x,a)}h_f = \det T_x h_f$, hence $\det T_x h > 0$ for all $x \in h^{-1}(y)$. In particular, $x$ is a regular point of $h$. Since $(x, a)$ was arbitrary, we conclude that $y$ is a regular value of $h$. By Remark 5.6 the map $h$ has degree $\pm 1$. Hence
\[ \pm 1 = \deg(h) = \sum_{x \in h^{-1}(y)} \text{sign}(\det T_x h) = |h^{-1}(y)|. \]
Therefore the number of points in $h^{-1}(y)$ is one. Let $x$ be the unique point in $h^{-1}(y)$. Then $h_T^{-1}(y, b) = h_f^{-1}(y, b) = \{(x, b - f(x))\}$. Hence $h_T$ is a bijection. \hfill $\square$

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