Local invariants for mixed qubit-qutrit states

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Abstract

In the present paper few steps are undertaken towards the description of the “qubit-qutrit” pair – quantum bipartite system composed of two and three level subsystems. The computational difficulties with the construction of the “local unitary polynomial invariants” are discussed. Calculations of the Molien functions and Poincaré series for the qubit-qubit and qubit-qutrit local unitary invariants are outlined and compared with the known results. The requirement of positive semi-definiteness of the density operator is formulated explicitly as a set of inequalities in five Casimir invariants of the algebra $\text{su}(6)$.

Key words: entanglement, polynomial invariants, Molien function, positive definiteness
1 Introduction

The present article discusses several computational aspects of a pure quantum effects in composite systems playing an important role in the modern theory of quantum computing and quantum information [1, 2].

The cornerstone of these latest trends is an extraordinary quantum phenomenon – the “entanglement” of quantum states. Basically, under the entanglement it is assumed an exposition of diverse non-local correlations in a composite multiparticle quantum system, which have no classical analogue. From the mathematical standpoint of view characteristics of entanglement can be understood within the classical theory of invariants (cf. [3, 4]). The central object in these studies is the ring of G-invariant polynomials, called local invariants, in elements of the density matrix with the group G consisting from the so-called local unitary transformations acting separately on every part of the multipartite composite system. The program of description of this ring for multipartite mixed states was outlined in [5] and during the last decade has been intensively developed. Over this time many interesting physical and pure mathematical results have been obtained. Particularly, for the simplest bipartite system of two qubits, the structure of the corresponding ring has been clarified (see e.g. [6, 7, 8]). However, comparative less is known for multipartite states, as well as for bipartite mixed states, composed from arbitrary d-level subsystems, the so-called qudits [9, 10]. The reason is first of all in a big computational difficulties we are faced. Indeed, even dealing with 3-level subsystem, qutrit, the large number of independent elements of the density matrix leads to the wide variety of the local polynomial invariants and makes non-effective the direct usage of the modern computer algebra software.

Below, attempting to construct the polynomial ring of local invariants for qubit-qutrit pair, i.e. invariants against the action of SU(2) ⊗ SU(3) group, we got added evidence of the complexity of the problem. The known results [12] and our calculation of the Molien function and Poincaré series show that the number of local invariants grows up significantly compared with the two qubits case. Nevertheless the derived information is very useful for the analysis of the polynomial ring of local invariants. As a preliminary result we present here a set of linearly independent local invariants up to the fourth order constructed via trace operation from the non-commutative monomials in three elements of a special decomposition of qubit-qutrit density matrix. Using the subset of the local invariants, consisting from the Casimir invariants
of the enveloping algebra $\mathfrak{U}(\mathfrak{su}(6))$, the positive semi-definiteness of density matrix of qubit-qutrit pair is derived in the form of a system of algebraic inequalities.

2 The SU(n) Casimir invariants

Here the basic statements on the unitary symmetry in quantum mechanics and its role in the description of composite multipartite states is given.

- **Density operator and SU(n)-invariants**

According to the conventional quantum theory, a complete information about a generic $n$-dimensional system is accumulated in the self-adjoint positive semi-definite density operator $\varrho$ with the unit trace, $\varrho \in \mathfrak{P}_+$. For a closed quantum system, this description is highly redundant, the equivalence relation between elements of $\mathfrak{P}_+$, due to the invariance of observables under the adjoint action of SU(n) group

$$(\text{Ad } g) \varrho = g \varrho g^{-1}, \quad g \in \text{SU}(n),$$

(2.1)

guarantees that the physically relevant knowledge about quantum states can be extracted from the orbit space $\mathfrak{P}_+ / \text{SU}(n)$. Relaxing for a moment condition of semi-definiteness, the density operator $\varrho$ can be expressed via the Lie algebra $\mathfrak{su}(n)$ of SU(n) group $[11]$:

$$\varrho = \frac{1}{n} \mathbb{I}_n + \tilde{\kappa} i \mathfrak{g}, \quad \mathfrak{g} \in \mathfrak{su}(n), \quad i^2 = -1.$$  (2.2)

with some normalization factor $\tilde{\kappa}$. Therefore the density operator can be decomposed over $n^2 - 1$ basis elements, $e_i$, of the Lie algebra $\mathfrak{su}(n)$

$$\mathfrak{g} = \sum_{i=1}^{n^2-1} \xi_i e_i,$$  (2.3)

and any other operator $\mathcal{A}[\varrho]$, constructed from the density operator $\varrho$, admits a representation in the graded power series:

$$\mathcal{A}(\mathbf{e}) = A^{(0)} \mathbb{I} + A^{(1)}_i e_i + \frac{1}{2!} A^{(2)}_{ij} e_i e_j + \frac{1}{3!} A^{(3)}_{ijk} e_i e_j e_k + \ldots.$$  (2.4)

\footnote{The orbit space $\mathfrak{P}_+ / \text{SU}(n)$ of SU(n) is defined as the set of all SU(n)-orbits, endowed with the quotient topology and differentiable structure, the subset of all the SU(n)-orbits with the same orbit-type forms a stratum of $\mathfrak{P}_+ / \text{SU}(n)$.}
According to the Poincaré-Birkhoff-Witt theorem \[13\] the ordered monomials
\[ e_0 = 1, \quad e_{i_1 i_2 \ldots i_k} = e_{i_1} e_{i_2} \ldots e_{i_k}, \quad e_{i_1} < e_{i_2} < \cdots < e_{i_k}, \quad (2.5) \]
form a linear basis of the universal enveloping algebra \( \mathfrak{U}(\mathfrak{su}(n)) \) of \( \mathfrak{su}(n) \).

Direct corollary of this theorem is that the symmetrized monomials of degree \( d \) in \((2.4)\) span a linear space \( \mathfrak{U}^d(\mathfrak{su}(n)) \) and the universal enveloping algebra \( \mathfrak{U}(\mathfrak{su}(n)) = \bigoplus_{d=0}^{\infty} \mathfrak{U}^d(\mathfrak{su}(n)) \).

as a linear space is isomorphic to a polynomial algebra in commutative real variables \( \xi_i, \ i = 1, \ldots, n^2 - 1 \).

Furthermore, according to the well-known Gelfand’s theorem \[14\], the description of center, \( \mathcal{Z}(\mathfrak{su}(n)) \), of the enveloping algebra \( \mathfrak{U}(\mathfrak{su}(n)) \) reduces to the study of invariants in commutative symmetrized algebra \( S(\mathfrak{su}(n)) \), which is isomorphic to the algebra of invariant polynomials over \( \mathfrak{su}(n) \). The elements of center \( \mathcal{Z}(\mathfrak{su}(n)) \) are in one to one correspondence with the \( \text{SU}(n) \)-invariant polynomials in \( n^2 - 1 \) real variables, coordinates in \( \mathfrak{su}(n) \). More precisely, the element of \( \mathfrak{U}(\mathfrak{su}(n)) \)
\[ c_r = \sum_{r!} \frac{1}{r!} c_{i_1 \ldots i_r} \sum_{\sigma \in \mathfrak{S}_r} e_{i_{\sigma(1)}} e_{i_{\sigma(2)}} \cdots e_{i_{\sigma(r)}}, \]
where \( \mathfrak{S}_r \) is the group of permutation of \( 1, 2, \ldots, r \), belongs to \( \mathcal{Z}(\mathfrak{su}(n)) \), if and only if \( c_{i_1 \ldots i_r} \) are coefficients of the polynomial in \( \xi_1, \xi_2, \ldots, \xi_r \) variables
\[ \phi(\xi_1, \xi_2, \ldots, \xi_r) = \sum c_{i_1 \ldots i_r} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_r}, \]
which is invariant under the adjoint action
\[ \phi(\xi_1, \xi_2, \ldots, \xi_r) = \phi((\text{Ad} \ g)^T \xi_1, (\text{Ad} \ g)^T \xi_2, \ldots (\text{Ad} \ g)^T \xi_r), \]
with \( (\text{Ad} \ g)^T \) - the matrix of adjoint operator, \( \text{Ad} \ g \), calculated in the basis \( e_{i_1} e_{i_2} \ldots e_{i_r} \).

Therefore, from the algebraic standpoint, the study of the orbit space \( \mathfrak{M}_+ | \text{SU}(n) \) as well as any characteristics of quantum-mechanical observables, invariant under the unitary action \((2.1)\), reduces to the computation of the center \( \mathcal{Z}(\mathfrak{su}(n)) \) of \( \mathfrak{U}(\mathfrak{su}(n)) \).
If the elements $\mathfrak{c}_r$ belong to $\mathcal{Z}$ they are termed as Casimir operators. The number of independent homogeneous Casimir generators for $\text{SU}(n)$ group is equal to rank $\mathfrak{su}(n) = n - 1$.

It is well known, that the quadratic Casimir operator is unique up to the constant factor and is expressible with the aid of the Cartan tensor:

$$C_{ij} = \text{tr}((\text{Ad} \, e_i)(\text{Ad} \, e_j)),$$

Therefore for algebra $\mathfrak{su}(n)$ the quadratic Casimir operator reads

$$\mathfrak{c}_2 = \sum e_i e_i,$$

The higher dimensional Casimirs are expressed via the symmetric structure constants $d_{ijk}$ of $\mathfrak{su}(n)$ algebra [15]. Because further, dealing with the qubit-qutrit system the Casimirs of $\text{SU}(6)$ will be used\(^2\), the expressions for $\mathfrak{c}_i$ are given below:

$$\mathfrak{c}_3 = \sum d_{i_1i_2i_3} e_{i_1} e_{i_2} e_{i_3},$$

$$\mathfrak{c}_4 = \sum d_{j_1j_2j_3} e_{j_1} e_{j_2} e_{j_3} e_{i_1},$$

$$\mathfrak{c}_5 = \sum d_{i_1i_2j_1j_2j_3} e_{i_1} e_{i_2} e_{i_3} e_{i_4} e_{i_5} e_{i_6},$$

$$\mathfrak{c}_6 = \sum d_{i_1i_2j_1j_2j_3k_1k_2} e_{i_1} e_{i_2} e_{i_3} e_{i_4} e_{i_5} e_{i_6}.$$

Now using these operators and decomposition (2.3) based on the isomorphism between center $\mathcal{Z}(\mathfrak{su}(n))$ and $\text{SU}(n)$-invariant polynomials, the following scalars, referred hereafter as Casimir invariants, can be written:

$$\mathfrak{c}_2 = (n - 1) \xi \cdot \xi \quad (2.6)$$

$$\mathfrak{c}_3 = (n - 1) (\xi \lor \xi) \cdot \xi \quad (2.7)$$

$$\mathfrak{c}_4 = (n - 1) (\xi \lor \xi) \cdot (\xi \lor \xi) \quad (2.8)$$

$$\mathfrak{c}_5 = (n - 1) \left((\xi \lor \xi) \lor (\xi \lor \xi)\right) \cdot \xi \quad (2.9)$$

$$\mathfrak{c}_6 = (n - 1) (\xi \lor \xi \lor \xi)^2 \quad (2.10)$$

where

$$(U \lor V)_a := \kappa d_{abc} U_b V_c,$$
with normalization constant $\kappa := \sqrt{n(n-1)/2}$.

Now, using these scalars, the positive semi-definiteness of density matrices for an arbitrary $n$-level quantum system will be formulated.

- **Positivity of density operators**  
  To the best of our knowledge the first analysis of consequences of the constraints on the density operator due to its semi-positive definiteness has been done in the sixtieth of the last century studying the production and decay of resonant states in strong interaction processes [16, 17, 18]. Nowadays, the quantum computing and theory of quantum information reveals the new role of these constraints and recently they have been once more derived [19, 20] 3.

To formulate the semi-definiteness let us choose the Bloch representation for a density operator (2.2) [11]:

$$\rho = \frac{1}{n} \left( I_n + \omega \right), \quad \omega = \kappa \xi \cdot \lambda,$$

(2.11)

where $(n^2 - 1)$-dimensional Bloch vector $\xi \in \mathbb{R}^{n^2-1}$ is contracted with elements $\lambda_i, \ i = 1, \ldots, n^2 - 1$ of the Hermitian basis of $\mathfrak{su}(n)$ Lie algebra. According to [17] 4 a necessary and sufficient condition for the Hermitian matrix to be positive is that the coefficients $S_k$ of its characteristic equation

$$|I - \rho| = x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^n S_n = 0$$

(2.12)

should be non-negative

$$\rho \geq 0 \iff S_k \geq 0 \quad k = 1, \ldots, n.$$  

(2.13)

It is convenient to rewrite these inequalities in terms of normalized coefficients $\bar{S}_k := S_k / \max \{S_k\}$. Since the maximal values of $S_k$ correspond to a maximally degenerate roots; $x_1 = x_2 = \ldots = x_n = 1/n$ of the characteristic equation (2.12) one can express them via the binomial coefficients

$$\max \{S_k\} = \frac{1}{n^k} \binom{n}{n-k},$$

and thus

$$0 \leq \bar{S}_k \leq 1 \quad k = 2, \ldots, n.$$  

(2.14)
Now we are ready to rewrite the constraints (2.14) in terms of the Casimir invariants (2.6)-(2.10). This is possible since, each of three sets, $\mathcal{C}_k$, or $\mathcal{S}_k$, or $t_k = \text{tr}(g^k)$, $k = 2, \ldots, n$, forms the basis of algebraically independent invariants of SU($n$) group (see e.g. [15]). The expressions for the coefficients $S_k$ in terms of $t_m$ are well-known, they are given by determinants:

$$S_k = \frac{1}{k!} \begin{vmatrix}
    t_1 & 1 & 0 & 0 & \cdots & 0 \\
    t_2 & t_1 & 2 & 0 & \cdots & 0 \\
    t_3 & t_2 & t_1 & 3 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    t_{k-1} & t_{k-2} & t_{k-3} & t_{k-4} & \cdots & k-1 \\
    t_k & t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1
\end{vmatrix}.$$ 

Further, $t_m$ can be represented as polynomials in Casimir invariants. Based on the expressions for traces of symmetrized products of Lie algebra basis elements (see Appendix A, cf. also [20]) we have:

$$\text{tr}(\omega^2) = n\mathcal{C}_2,$$
$$\text{tr}(\omega^3) = n\mathcal{C}_3,$$
$$\text{tr}(\omega^4) = n(\mathcal{C}_2^2 + \mathcal{C}_4),$$
$$\text{tr}(\omega^5) = n(2\mathcal{C}_2\mathcal{C}_3 + \mathcal{C}_5),$$
$$\text{tr}(\omega^6) = n(\mathcal{C}_2^3 + 2\mathcal{C}_2\mathcal{C}_4 + \mathcal{C}_3^2 + \mathcal{C}_6).$$

Finally, imposing the following normalization for the Casimir invariants,

$$C_k = \frac{(k - 1)!}{(n - 1)(n - 2) \cdots (n - k + 1)} \mathcal{C}_k,$$

we arrive at a system of inequalities in su(6) Casimir invariants, that defines the positive semi-definiteness of the density matrix of qubit-qutrit pair:

$$0 \leq C_2 \leq 1,$$  \hspace{1cm} (2.15)
$$0 \leq 3C_2 - C_3 \leq 1,$$  \hspace{1cm} (2.16)
$$0 \leq 6C_2 - 5C_2^2 - 4C_3 + C_4 \leq 1,$$  \hspace{1cm} (2.17)
$$0 \leq (1-5C_2)^2 - 30C_2C_3 + 10C_3 - 5C_4 + C_5 \leq 1$$  \hspace{1cm} (2.18)
$$0 \leq (1-5C_2)^3 - 180C_2C_3 + 125C_2C_4 + 20C_3(1+5C_3) - 15C_4 + 6C_5 - C_6 \leq 1.$$  \hspace{1cm} (2.19)
Since the positive semi-definiteness of density matrices plays an exceptional role in the entanglement quantification problem it is reasonable to rewrite the system of inequalities (2.15)-(2.19) in terms of the local SU(2) ⊗ SU(3) invariants.

3 The local unitary invariants

- The local invariance of composite states - When a quantum system is obtained by combining of \( r \)-subsystems with \( n_1, n_2, \ldots, n_r \) levels each, the non-local properties of the composite system are in correspondence with a certain decomposition of the unitary operations (2.1).

In order to discuss this decomposition consider the subgroup of unitary group formed by the local unitary transformations

\[
SU(n_1) \otimes SU(n_2) \otimes \cdots \otimes SU(n_r),
\]

acting independently on the density matrix of each subsystem

\[
g^{(n_i)} \rightarrow g^{(n_i)'} = g^{(n_i)} g^{-1} \quad g \in SU(n_i), \quad i = 1, 2, \ldots, r.
\]

Two states of composite system connected by the local unitary transformations (3.1) have the same non-local properties. The latter can be changed only by the rest of the unitary actions,

\[
SU(n) \quad SU(n_1) \otimes SU(n_2) \otimes \cdots \otimes SU(n_r), \quad n = n_1 n_2 \cdots n_r,
\]

generating the class of non-local transformations.

Having this notions, we are in position to discuss the structure of the corresponding ring of polynomial local invariants, i.e. polynomials in elements of the density matrices, which are scalars under the adjoint local unitary transformations. It is well known that for any reductive linear algebraic group \( G \) and for any finite dimensional \( G \)-module \( V \), the ring \( \mathcal{R}^G \) has the Cohen-Macaulay property [22] and possesses the Hironaka decomposition

\[
\mathcal{R}^G = \bigoplus_{a=0}^{r} J_a \mathbb{C}[K_1, K_2, \ldots, K_s],
\]

8
where $K_b, b = 1, 2, \ldots, s$ are primary, algebraically independent polynomials and $J_a, a = 0, 1, 2, \ldots, r, J_0 = 1$, are secondary, linearly independent invariants respectively. According to that the corresponding Molien function $M_G(q)$ for $\mathcal{R}^G$ [7] can be expressed as follows

$$
M_G(q) = \frac{\sum_{a=0}^{r} q^{\deg J_a}}{\prod_{b=1}^{s} (1 - q^{\deg K_b})}.
$$

In this form it provides us with a certain knowledge on the numbers of algebraically independent polynomials as well as linearly independent invariants.

- **Molien functions for $C[\mathcal{P}^{(2\otimes 2)}]$ and $C[\mathcal{P}^{(2\otimes 3)}]$**

  Let us start with a remark concerning the adjoint action (2.1). Consider a non-degenerate density matrices. In this case using the natural identification of the elements of a linear space spanned by the Hermitian $n \times n$ matrices with the space $\mathbb{R}^{n^2 - 1}$

$$
\rho \rightarrow \rho_{ij}
$$

one can instead of the adjoint action (2.1) consider the linear representation on $\mathbb{R}^{n^2 - 1}$

$$
V_A' = L_{AB} V_B, \quad L_{AB} \in SU(n) \otimes SU(n),
$$

where the line over expression means the complex conjugation.

After this identification in order to get some insight on the structure of the ring of polynomial invariants of linear action of Lie group $G$ on the linear $V$ space we can compute the Molien function

$$
M(\mathbb{C}[V]^G, q) = \int_G \frac{d\mu(g)}{\det(\mathbb{I} - q\pi(g))}, \quad |q| < 1, \quad (3.2)
$$

where $d\mu(g)$ is the Haar measure for Lie group $G$ and $\pi(g)$ is the corresponding representation on $V$. We start with the system of two qubits.

**Two qubits.** In this case the local unitary group is

$$
G = SU(2) \otimes SU(2).
$$

As it is well known for any reductive linear group the integration in (3.2) reduces to the integration over the maximal compact subgroup $K$ of $G$ [4]. In the present case this results in integration over the maximal torus

$$
\pi(g) = \text{diag}(1, 1, z, z^{-1}) \otimes \text{diag}(1, 1, w, w^{-1}),
$$
where $z, w$ - coordinates on one-dimensional tori. Therefore computations reduce to the following two-dimensional integral

$$M_{SU(2) \otimes SU(2)}(q) = \frac{1}{(2\pi i)^2} \int_{|z|=1} \int_{|w|=1} \frac{d\mu}{\Psi(z, w, q)},$$

where

$$d\mu = (1 - z)^2(1 - w)^2 \frac{dz \, dw}{z^2 \, w^2},$$

$$\det(\mathbb{I} - q\pi(g)) = (1 - q) \Psi(z, w, q)$$

$$\Psi(z, w, q) = (1 - q)^3(1 - qz)^2(1 - qw)^2(1 - qz^{-1})^2(1 - qw^{-1})^2$$

$$(1 - qzw)(1 - qz^{-1}w)(1 - qzw^{-1})(1 - qz^{-1}w^{-1}).$$

After integration we get the Molien function

$$M_{SU(2) \otimes SU(2)}(q) = 1 + q^4 + q^5 + 3q^6 + 2q^7 + 2q^8 + 3q^9 + q^{10} + q^{11} + q^{15},$$

which is the palindromic one,

$$M_{SU(2) \otimes SU(2)}(1/q) = -q^{15}M_{SU(2) \otimes SU(2)}(q),$$

with the degree consistent with

$$\dim SU(4) = 15.$$
where

\[
f(x, y, z, q) = \frac{1}{xyz} \frac{(1 - x^{-1})(1 - y^{-1})(1 - z^{-1})(1 - (yz)^{-1})}{\Psi(x, y, z, q)},
\]

\[
det(\mathbb{I} - q\pi(g)) = (1 - q) \Psi(x, y, z, q),
\]

and \(\Psi(x, y, z, q) =\)

\[
(1 - q)^5(1 - qy)^2(1 - qz)^2(1 - qyz)^2(1 - \frac{q}{y})^2(1 - \frac{q}{z})^2(1 - \frac{q}{yz})^2
\]

\[
(1 - qx)^3(1 - qxy)(1 - qxz)(1 - qxyz)(1 - \frac{qy}{x})(1 - \frac{qz}{x})(1 - \frac{qyz}{x})(1 - \frac{q}{xy})(1 - \frac{q}{xz})(1 - \frac{q}{xyz}).
\]

As a result, the Molien function can be represented in the rational form (cf. [12]):

\[
M_{SU(2) \otimes SU(3)}(q) = \frac{N}{D},
\]

where

\[
N = 1 + 4 q + 9 q^2 + 38 q^3 + 69 q^4 + 173 q^5 + 347 q^6 + 733 q^7 + 103 q^8
\]

\[
+ 1403 q^9 + 2796 q^{10} + 5091 q^{11} + 9286 q^{12} + 16058 q^{13} + 27208 q^{14} + 44250 q^{15}
\]

\[
+ 70537 q^{16} + 108430 q^{17} + 163158 q^{18} + 238264 q^{19} + 339974 q^{20} + 472130 q^{21}
\]

\[
+ 641187 q^{22} + 848615 q^{23} + 1098643 q^{24} + 1388741 q^{25} + 1717327 q^{26}
\]

\[
+ 2075836 q^{27} + 2456389 q^{28} + 2843020 q^{29} + 3222408 q^{30} + 3575226 q^{31}
\]

\[
+ 3884797 q^{32} + 4133599 q^{33} + 4398377 q^{34} + 4308636 q^{35} + 4398377 q^{36}
\]

\[
+ 4398377 q^{37} + \ldots + 38 q^{69} + 9 q^{70} + 4 q^{71} + q^{75}
\]

\[
D = (1 - q^2)^3(1 - q^3)^4(1 - q^4)^5(1 - q^5)^4(1 - q^6)^5(1 - q^7)^2(1 - q^8).
\]

This Molien function is the palindromic one

\[
M_{SU(2) \otimes SU(3)}(1/q) = q^{35} M_{SU(2) \otimes SU(3)}(q),
\]

as provided by

\[
\text{dim SU(6)} = 35.
\]

This form of the Molien function serves as source of information on the polynomial ring of SU(2) ⊗ SU(3) invariants. Particularly, one can endeavour
to identify the structure of algebraically independent local unitary scalars. According to (3.4) there are 24 independent scalars in agreement with simple count of \( \dim \left[ \text{SU}(6)/\text{SU}(2) \otimes \text{SU}(3) \right] = 35 - 11 = 24 \). The set of these 24 polynomial invariants may be composed from three invariants of degree 2, four of degree 3, five of degree 4, four of degree 5, five of degree 6, two of degree 7 and one of the degree 8.

Note that the Poincaré series of \( M_{\text{SU}(2) \otimes \text{SU}(3)}(q) \)
\[
M_{\text{SU}(2) \otimes \text{SU}(3)}(q) = \sum_{d=0}^{\infty} \dim \left( \mathcal{P}^{\text{SU}(2) \otimes \text{SU}(3)}_d \right) q^d ,
\]
determines the number of homogeneous polynomial invariants of degree \( d \). According to the calculations of (3.4) the few terms of the Taylor expansion over \( q \) are
\[
M_{\text{SU}(2) \otimes \text{SU}(3)}(q) = 1 + 3q^2 + 4q^3 + 15q^4 + 90q^5 + 25q^6 + 170q^7 + 489q^8 + 1059q^9 + 2600q^{10} + 5641q^{11} + 12872q^{12} + 27099q^{13} + 57990q^{14} + 118254q^{15} + 240187q^{16} + O(q^{17}) .
\]

Now, having in mind the input from the structure of the Molien function (3.4), we attempt to construct the local \( \text{SU}(2) \otimes \text{SU}(3) \) unitary invariants.

- **Constructing \( \text{SU}(2) \otimes \text{SU}(3) \) invariants**
  Let us introduce the decomposition for density matrices well adapted to the case of composite qubit-qutrit system. The space \( \mathfrak{su}(6) \) in (2.2) for \( n=6 \) admits decomposition in the direct sum of three real spaces
  \[
  \mathfrak{su}(6) = \bigoplus_{a=1}^{3} V_a = \mathfrak{su}(2) \otimes \mathbb{I}_3 + \mathbb{I}_2 \otimes \mathfrak{su}(2) + \mathfrak{su}(2) \otimes \mathfrak{su}(3) .
  \]

Using Pauli matrices \( \sigma_i \) as the basis for \( \mathfrak{su}(2) \) and Gell-Mann matrices \( \lambda_a \) as the basis for \( \mathfrak{su}(3) \) (see Appendix A) the density matrix (2.11) for qubit-qutrit system can be written as [9, 10]:
\[
\rho = \frac{1}{6} \left[ \mathbb{I}_6 + \omega \right] , \quad \omega = \alpha + \beta + \gamma ,
\]
where
\[
\alpha := \sum_{i=1}^{3} a_i \sigma_i \otimes \mathbb{I}_3 , \quad \beta := \sum_{a=1}^{8} b_a \mathbb{I}_2 \otimes \lambda_a , \quad \gamma := \sum_{i=1}^{3} \sum_{a=1}^{8} c_{ia} \sigma_i \otimes \lambda_a .
\]
Among the $35=3+8+24$ real parameters $(a_i, b_a, c_{ia})$ the first two sets, $a_i$ and $b_a$, correspond to the Bloch vectors of an individual qubit and qutrit respectively; the evaluation of partial trace yields the reduced matrices for subsystems:

$$
\varrho^{(A)} := \text{tr}_B(\varrho) = \frac{1}{2}(I_2 + \vec{a} \cdot \vec{\sigma}), \quad \varrho^{(B)} := \text{tr}_A(\varrho) = \frac{1}{3}(I_3 + \vec{b} \cdot \vec{\lambda}),
$$

while the variables $c_{ia}$ are entries of the so-called correlation matrix $C = (c_{ia})$.

Now using a trace operation described below we can construct a set of local $SU(2) \otimes SU(3)$ scalars, candidates for the elements of the integrity basis. In analogy with the generators (2.5) of the universal enveloping algebra we consider a set $M$ of non-commutative monomials

$$
M_{i_1...i_d} := X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_d} \in M,
$$

where each of $X_{i_k}, k = 1, \ldots, d$, is one of $\alpha, \beta$, or $\gamma$. To each $M_{i_1...i_d}$ we assign a multidegree $(s, t, q), s + t + q = d$, where $s, t$ and $q$ are degrees of $\alpha, \beta$, and $\gamma$ respectively. The trace operation on monomials $M_{i_1...i_d}$

$$
\text{tr} : M_{i_1...i_d} \to P_{stq}(a_i, b_a, c_{ia}) := \text{tr}(M_{i_1...i_d}) \in P.
$$

defines the map $\text{tr} : M \to P$ of $M$ into the algebra $P$ of homogeneous polynomials in variables $(a_i, b_a, c_{ia})$. A generic term of the polynomial $P_{stq}(a_i, b_a, c_{ia})$ is a convolution of vectors $a_i, b_a$ and matrix $c_{ia}$ with traces

$$
\text{tr}(\sigma_1\sigma_2\cdots\sigma_p \otimes \lambda_1\lambda_2\cdots\lambda_r) = \text{tr}(\sigma_1\sigma_2\cdots\sigma_p) \text{tr}(\lambda_1\lambda_2\cdots\lambda_r),
$$

where $p = s + q$ and $r = t + q$.

Now it is easy to verify that the images of the trace map are local unitary invariants. Indeed, since under the transformation of the form $k_1 \otimes k_2$, where $k_1 \in SU(2)$, and $k_2 \in SU(3)$, the matrices $\sigma$’s and $\lambda$’s in the basis elements of $su(6)$ (see Appendix A) are transformed independently, in adjoint manner

$$
\sigma \to k_1\sigma k_1^{-1}, \quad \lambda \to k_2\lambda k_2^{-1},
$$

the polynomials $\text{tr}(M)$ are invariant against $SU(2) \otimes SU(3)$ action.

Therefore the polynomials $P_{stq}(a_i, b_a, c_{ia})$ are the reserve for the integrity basis of the ring $\mathbb{C}[P]_{SU(2) \otimes SU(3)}$. Now, in contrast to the case of $SU(n)$ Casimir invariants built up with the help of symmetric structure constants
only, dealing with the scalars against the tensor product of groups, the invariants are constructed in terms of the antisymmetric structure constants as well. For example,

\[ \text{tr}(\gamma^3) = c_{ia}c_{jb}c_{kc}\text{tr}(\sigma_i\sigma_j\sigma_k \otimes \lambda_a\lambda_b\lambda_c) = c_{ia}c_{jb}c_{kc}\text{tr}(\sigma_i\sigma_j\sigma_k)\text{tr}(\lambda_a\lambda_b\lambda_c). \]

This quantity being invariant under the SU(2)\(\otimes\)SU(3) action is expressible via totally antisymmetric tensor \(\epsilon_{ijk}\) - structure constants of su(2) algebra and \(f_{abc}\) - structure constants of su(3):

\[ \text{tr}(\gamma^3) = -4\epsilon_{ijk}f_{abc}c_{ia}c_{jb}c_{kc}. \]

Choosing a basis for local invariants, several types of algebraic dependence between the polynomials in \(P\) have to be taken into account. It is worth to consider two illustrative examples. Applying the Hamilton-Cayley theorem for elements \(\alpha, \beta\) and \(\gamma\), considered as Hermitian \(6 \times 6\) matrices, one can determine the algebraic identities for the polynomials of the degree \(d > 7\). Less obvious example of relations between polynomials is due to the identities between the structure constants of the algebra.\(^6\) Let us consider two invariants, both 4-th order in variables \(c_{ia}\) of the correlation matrix \(C\), but one constructed using the invariant symmetric structure constants \(d\) while the second one with the anti-symmetric structure constants \(f\):

\[ \mathcal{I}^{004}(dd) = d_{abc}d_{cpq}(C^TC)_{ab}(C^TC)_{pq}, \]
\[ \mathcal{I}^{004}(ff) = f_{apc}f_{cbq}(C^TC)_{ab}(C^TC)_{pq}. \]

With the aid of identities (A.1) and (A.2) (Appendix A) for the structure constants of su(3) algebra, one can convinced that

\[ \mathcal{I}^{004}(dd) = \frac{2}{3}\mathcal{I}^{004}(ff) - \frac{1}{3}\left[(\text{tr}(C^TC))^2 - 2\text{tr}(C^TC)C^TC)\right]. \]

According to the Poincaré series (3.5) there are 15 homogeneous scalars of order 4, while there are 81 = \(3^4\) monomials in three noncommutative variables. But since the elements \(\alpha\) and \(\beta\) commute this number reduces. Taking into account this commutativity as well as the invariance of trace operation under the cyclic permutations of products, one can find 18 valuable

\(^6\)For the detailed analysis of the relations of that type we refer to [23].
monomials:
\[ \alpha^4, \beta^4, \gamma^4, \alpha^3 \beta, \alpha \beta^3, \alpha^3 \gamma, \alpha \gamma^3, \beta^3 \gamma, \beta \gamma^3, \]
\[ \alpha^2 \beta^2, \alpha^2 \gamma^2, \alpha \gamma \alpha \gamma, \beta^2 \gamma^2, \beta \gamma \beta \gamma, \]
\[ \alpha^2 \beta \gamma, \alpha \beta^2 \gamma, \alpha \beta \gamma^2, \alpha \gamma \beta \gamma. \]

Taking traces of these monomials one can convince that five of them form the kernel of trace map:

\[ \text{tr}(\alpha^3 \beta) = \text{tr}(\alpha \beta^3) = \text{tr}(\alpha^3 \gamma) = \text{tr}(\beta^3 \gamma) = \text{tr}(\alpha^2 \beta \gamma) = 0, \]

and images of last two monomials coincide up to sign

\[ \text{tr}(\alpha \beta \gamma^2) = -\text{tr}(\alpha \gamma \beta \gamma). \]

Therefore the following set of twelve traces

\[
\begin{align*}
\text{tr}(\alpha^4), & \quad \text{tr}(\beta^4), \quad \text{tr}(\alpha^2 \beta^2), \quad \text{tr}(\alpha^2 \gamma^2), \\
\text{tr}(\gamma^4), & \quad \text{tr}(\alpha \gamma^3), \quad \text{tr}(\beta \gamma^3), \quad \text{tr}(\alpha \gamma \alpha \gamma), \\
\text{tr}(\beta^2 \gamma^2), & \quad \text{tr}(\beta \gamma \beta \gamma), \quad \text{tr}(\alpha \beta^2 \gamma), \quad \text{tr}(\alpha \beta \gamma^2),
\end{align*}
\]

plus three 4-th order polynomials constructed as products of second degrees polynomials \( \text{tr}(\alpha^2)\text{tr}(\beta^2), \text{tr}(\alpha^2)\text{tr}(\gamma^2), \text{tr}(\beta^2)\text{tr}(\gamma^2) \), are 15 homogeneous invariant polynomials in accordance with the Poincaré series (3.5).

How difficult is it to extract the independent scalars from this list? It is easy to verify that some traces are expressed in terms polynomials of second order; e.g., \( \text{tr}(\alpha^2 \beta^2) = \frac{1}{6} \text{tr}(\alpha^2)\text{tr}(\beta^2) \). Concerning the remaining monomials one can see that several of them have the same multidegree. Namely, the following “trace” polynomials

1. \( \text{tr}(\alpha^2 \gamma^2) = \frac{1}{6} \text{tr}(\alpha^2)\text{tr}(\gamma^2) \) and \( \text{tr}(\alpha \gamma \alpha \gamma) \),

2. \( \text{tr}(\beta^2)\text{tr}(\gamma^2), \quad \text{tr}(\beta^2 \gamma^2) \) and \( \text{tr}(\beta \gamma \beta \gamma) \),

belong to the same space \( P_{202} \) and \( P_{022} \) respectively. Being linearly independent monomials, they obey the following relations

\[
\begin{align*}
\text{tr}(\alpha^2 \gamma^2) + \text{tr}(\alpha \gamma \alpha \gamma) & = 8 a_{i_1} a_{i_2} c_{i_1 j_1} c_{i_2 j_1}, \\
\text{tr}(\beta^2 \gamma^2) - \frac{1}{6} \text{tr}(\beta^2)\text{tr}(\gamma^2) & = 4 d_{i_1 j_2 k} d_{k j_3 j_4} b_{j_1} b_{j_2} c_{i_1 j_3} c_{i_1 j_4}, \\
\text{tr}(\beta^2 \gamma^2) + \text{tr}(\beta \gamma \beta \gamma) & = 8(\frac{2}{3} b_{j_1} b_{j_2} c_{i_1 j_1} c_{i_1 j_2} + d_{i_1 j_2 k} d_{k j_3 j_4} b_{j_1} b_{j_2} c_{i_1 j_3} c_{i_1 j_4}),
\end{align*}
\]
where summation over all indices is assumed. This circumstance leaves an open question how to build the elements of integrity basis with a certain multidegree using the “trace” polynomials.

We resume our analysis by the following list of linearly independent SU(2)⊗SU(3) scalars which are not products of low orders ones:

- degree 2, three invariants
  \[ \text{tr}(\alpha^2), \text{tr}(\beta^2), \text{tr}(\gamma^2), \]

- degree 3, four invariants
  \[ \text{tr}(\beta^3), \text{tr}(\gamma^3), \text{tr}(\alpha\beta\gamma), \text{tr}(\beta\gamma^2), \]

- degree 4, eight invariants
  \[ \text{tr}(\gamma^4), \text{tr}(\alpha\gamma^3), \text{tr}(\beta\gamma^3), \text{tr}(\alpha\gamma\alpha\gamma), \]
  \[ \text{tr}(\beta^2\gamma^2), \text{tr}(\beta\gamma\beta\gamma), \text{tr}(\alpha\beta^2\gamma), \text{tr}(\alpha\beta\gamma^2). \]

- Casimir invariants decomposition
  - The expansion of the SU(6) Casimir invariants up to the 4-th order (2.6)-(2.8) over the above suggested SU(2)⊗SU(3) “trace” scalars reads:
    \[ 6 \mathcal{C}_2 = \text{tr}(\alpha^2) + \text{tr}(\beta^2) + \text{tr}(\gamma^2), \]
    \[ 6 \mathcal{C}_3 = \text{tr}(\beta^3) + \text{tr}(\gamma^3) + 3 \text{tr}(\beta\gamma^2) + 6 \text{tr}(\alpha\beta\gamma), \]
    \[ 6 \mathcal{C}_4 = \frac{1}{3} \left[ \text{tr}(\alpha^2) \left( 2 \text{tr}(\beta^2) + \text{tr}(\gamma^2) \right) + \frac{1}{4} \text{tr}(\beta^2)^2 - \frac{1}{2} \text{tr}(\gamma^2)^2 - \text{tr}(\beta^2)\text{tr}(\gamma^2) \right] \]
    \[ + 4 \left[ \text{tr}(\alpha\gamma^3) + \text{tr}(\beta\gamma^3) + \text{tr}(\beta^2\gamma^2) + \text{tr}(\alpha\beta^2\gamma) + 3\text{tr}(\alpha\beta\gamma) \right] \]
    \[ + 2 \left[ \text{tr}(\alpha\gamma\alpha\gamma) + \text{tr}(\beta\gamma\beta\gamma) \right] + \text{tr}(\gamma^4). \]

We conclude with the final remark on the applicability of the derived results to the problem of classification of mixed quantum states. Using inequalities (2.15)-(2.19) and results from [21] the well-known Peres—Horodecki criterion for the separability of qubit-qutrit mixed states can be reformulated as a set of inequalities in SU(2)⊗SU(3) scalars.

\[ ^7 \text{Note that 2-nd and 3-d order invariants were proposed in [12].} \]
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A Appendix: Formulas for the $\mathfrak{su}(6)$ algebra

- **The tensorial basis**  For the $\mathfrak{su}(6)$ algebra we use the basis $\{\tau_A\}_{A=1,\ldots,35}$ constructed from the tensor products of the Pauli matrices $\sigma_i \in \mathfrak{su}(2)$:

  \[
  \begin{align*}
  \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
  \end{align*}
  \]

  and eight $\{\lambda_a\}_{a=1,\ldots,8}$ Gell-Mann matrices, forming the $\mathfrak{su}(3)$ basis:

  \[
  \begin{align*}
  \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
  \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
  \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
  \end{align*}
  \]

  The elements $\tau_A$ are enumerated as

  \[
  \begin{align*}
  \tau_i &= \frac{1}{\sqrt{3}} \sigma_i \otimes I_3, \quad \tau_{3+a} = \frac{1}{\sqrt{2}} I_2 \otimes \lambda_a, \\
  \tau_{11+a} &= \frac{1}{\sqrt{2}} \sigma_1 \otimes \lambda_a, \quad \tau_{19+a} = \frac{1}{\sqrt{2}} \sigma_2 \otimes \lambda_a, \quad \tau_{27+a} = \frac{1}{\sqrt{2}} \sigma_3 \otimes \lambda_a.
  \end{align*}
  \]

- **The algebraic structures**  The product of basis elements reads

  \[
  \tau_A \tau_B = \frac{2}{n} \delta_{AB} I + (d_{ABC} + i f_{ABC}) \tau_C,
  \]
The structure constants $d_{ABC}$ and $f_{ABC}$ can be determined via equations

$$d_{ABC} = \frac{1}{4} \text{Tr}([\tau_A, \tau_B] \tau_C), \quad f_{ABC} = -\frac{i}{4} \text{Tr}([\tau_A, \tau_B] \tau_C),$$

where apart from the Lie algebra product, $[\ , \ ]$, the “anti-commutator" of elements, i.e., $\{\tau_A, \tau_B\} = \tau_A \tau_B + \tau_B \tau_A$ has been used.

- **Identities for structure constants** - For the SU($n$) group the the structure constants obey the following identities:

\begin{align*}
  f_{abc} f_{cpq} + f_{bpc} f_{caq} + f_{pac} f_{cbq} &= 0, \\
  d_{abc} f_{cpq} + d_{bpc} f_{caq} + d_{pac} f_{cbq} &= 0, \\
  f_{abc} f_{cpq} &= d_{apc} d_{cbq} - d_{aqc} d_{cbp} + \frac{2}{n} (\delta_{ap} \delta_{bp} - \delta_{aq} \delta_{bp}), \\
  f_{abc} f_{cpq} + f_{aqc} f_{cqb} &= 2 d_{apc} d_{cbq} - d_{abc} d_{cpq} - d_{aqc} d_{cbp} + \frac{2}{n} (2 \delta_{ap} \delta_{bp} - \delta_{ab} \delta_{pq} - \delta_{aq} \delta_{bp}).
\end{align*}

(A.1)

The SU(3) symmetric constants satisfy [24, 25] an important identities

$$d_{abc} d_{cpq} + d_{bpc} d_{caq} + d_{pac} d_{cbq} = \frac{1}{3} (\delta_{ab} \delta_{pq} + \delta_{ap} \delta_{bq} + \delta_{aq} \delta_{bp}).$$

(A.2)

- **The traces** - The traces of symmetrized products of $\mathfrak{su}(n)$ basis elements are

\begin{align*}
  \text{tr} (\tau_a \tau_b) &= 2 \delta_{ab}, \\
  \text{tr} (\tau_a \tau_b \tau_c) &= 2 d_{abc}, \\
  \text{tr} (\tau_a \tau_b \tau_c \tau_d) &= \frac{2^2}{n} \delta_{ab} \delta_{cd} + 2 d_{abc} d_{bcd}, \\
  \text{tr} (\tau_a \tau_b \tau_c \tau_d \tau_e) &= \frac{2^2}{n} (d_{abc} \delta_{de} + \delta_{ab} d_{cde}) + 2 d_{abf} d_{fgy} d_{gde}, \\
  \text{tr} (\tau_a \tau_b \tau_c \tau_d \tau_e \tau_f) &= \frac{2^3}{n^2} \delta_{ab} \delta_{cd} \delta_{ef} + \frac{2^3}{n} (d_{abg} d_{gcd} \delta_{ef} + \delta_{ab} d_{cdg} d_{gef}) + \\
  &\quad + \frac{2^2}{n} d_{abc} d_{def} + 2 d_{abg} d_{gch} d_{hde} d_{vef}.
\end{align*}
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