L\textsuperscript{p}-NORM INEQUALITY USING Q-MOMENT AND ITS APPLICATIONS

TOMOHIRO NISHIYAMA

Abstract. For a measurable function on a set which has a finite measure, an inequality holds between two L\textsuperscript{p}-norms. In this paper, we show similar inequalities for the Euclidean space and the Lebesgue measure by using a q-moment which is a moment of an escort distribution. As applications of these inequalities, we first derive upper bounds for the Renyi and the Tsallis entropies with given q-moment and derive an inequality between two Renyi entropies. Second, we derive an upper bound for the probability of a subset in the Euclidean space with given L\textsuperscript{p}-norm on the same set.

Keywords: L\textsuperscript{p}-norm, q-moment, q-expectation value, Tsallis entropy, Renyi entropy, maximum entropy, escort distribution.

1. INTRODUCTION

We consider the measure space (\(\mathbb{R}^n, \mathcal{M}, \mu\)), where \(\mathcal{M}\) is a \(\sigma\)-algebra and \(\mu\) is the Lebesgue measure. When \(S \in \mathcal{M}\) has a finite measure, for \(0 < q < r\) and measurable function \(f\) on \(S\), the following inequality holds.

\[
\|f\|_q \leq \mu(S)^{\frac{1}{r} - \frac{1}{q}} \|f\|_r, \tag{1}
\]

where \(\|f\|_p\) is a \(L^p\)-norm [7] defined as follows.

Definition 1. Let \(S \in \mathcal{M}\). For \(0 < p < \infty\),

\[
\|f\|_p \overset{\text{def}}{=} \left( \int_S |f(x)|^p d^n x \right)^{\frac{1}{p}}.
\]

For \(p = \infty\),

\[
\|f\|_\infty \overset{\text{def}}{=} \text{ess sup}_{x \in S} |f|.
\]

In this paper, we show a similar inequality between two \(L^p\)-norms by using a q-moment when \(S = \mathbb{R}^n\) (See Theorem 1 and 2 in section 2).

For \(0 < q < r \leq \infty\) and \(n = 1\),

\[
\|f\|_q \leq (C \mu_{q, \alpha})^{\frac{1}{r} - \frac{1}{q}} \|f\|_r, \tag{2}
\]

where \(C\) is a constant and \(\mu_{q, \alpha}\) is the \(\alpha\)-th order q-moment defined as follows.

Definition 2. Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a measurable function which satisfies \(0 < \|f\|_q < \infty\). Let \(X \in \mathbb{R}^n\). We define a q-expectation value [1][9].

\[
E_q[X] \overset{\text{def}}{=} \int_{\mathbb{R}^n} x |f(x)|^q d^n x
\]

Especially, when \(q = 1\), we write \(E_1[X]\) as \(E[X]\).

\(E_q[X]\) is also an expected value of a escort distribution \(\frac{|f(x)|^q}{\int_{\mathbb{R}^n} |f(x)|^q d^n x}\) [2].
Definition 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function and \( X \in \mathbb{R} \). For \( b \in \mathbb{R} \) and \( \alpha > 0 \), we define the \( \alpha \)-th order \( q \)-moment as follows.
\[
\mu_{q,\alpha} \triangleq E_q\lfloor X - b \rfloor^\alpha
\]
When \( b = E_q|X| \), \( \mu_{q,\alpha} \) is the central \( q \)-moment.

\( \mu_{q,\alpha} \) is also the \( \alpha \)-th order moment of a escort distribution.

In (2), \( \mu_{q,\alpha}^{\frac{1}{q}} \) corresponds to \( \mu(S) \) in (1) and we can interpret \( \mu_{q,\alpha}^{\frac{1}{q}} \) as the “range” of the region function \( f \) spreads.

As applications of the inequality (2) and the multivariate version of (2), we derive an inequality between two Rényi entropies and derive upper bounds for the Rényi and the Tsallis entropies with given \( q \)-moment. We also obtain an upper bound for the Shannon differential entropy as a limit of the Rényi entropy.

Furthermore, we derive an upper bound for the probability of a subset in \( \mathbb{R}^n \) with given \( L^p \)-norm on the same set. This is a generalization of the result in [5].

2. Main Results

Theorem 1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function with finite \( q \)-moment. Let \( \|f\|_q, \|f\|_r < \infty \).

For \( 0 < q < r \leq \infty \),
\[
\|f\|_q \leq \left(C \mu_{q,\alpha}\right)^{\frac{1}{r} - \frac{1}{q}} \|f\|_r,
\]
where \( C \) is a constant which only depends on \( \alpha \). The example value of \( C \) is \( \frac{2}{\Gamma\left(\frac{1}{\alpha}\right)}(\alpha e)^{\frac{1}{\alpha}} \).

Proof. For a non-negative convex function \( \phi_t \), we consider the following value.
\[
V = \int_{\mathbb{R}} |f(x)|^q \phi_t(\mu_{q,\alpha}^{-1}|x - b|^\alpha)dx,
\]
where the function \( \phi_t(x) \) satisfies \( \phi_t(1) = 1 \). We transform this equation as follows.
\[
V = \int_{\mathbb{R}} |f(x)|^q dx \times E_q[\phi_t(\mu_{q,\alpha}^{-1}|X - b|^\alpha)] = \|f\|_q^2 E_q[\phi_t(\mu_{q,\alpha}^{-1}|X - b|^\alpha)]
\]
Applying the Jensen’s inequality to this equation and using Definition 3 give
\[
V \geq \|f\|_q^2 \phi_t(\mu_{q,\alpha}^{-1} E_q[|X - b|^\alpha]) = \|f\|_q^2 \phi_t(1) = \|f\|_q^2
\]
Furthermore, for \( 1 \leq s,t \leq \infty \), by applying the Hölder’s inequality to \( \phi_t \), we have
\[
V \leq \|f\|_q^s \left(\int_{\mathbb{R}} \phi_t(\mu_{q,\alpha}^{-1}|x - b|^\alpha)^t dx\right)^{\frac{1}{t}} ,
\]
where \( \frac{1}{s} + \frac{1}{t} = 1 \). By assumption \( q < r \), we can put \( r = qs \) and \( 1 \leq t < \infty \). Then, we have
\[
V \leq \|f\|_q^s \left(\int_{\mathbb{R}} \phi_t(\mu_{q,\alpha}^{-1}|x - b|^\alpha)^t dx\right)^{\frac{1}{t}} ,
\]
where we use \( \|f\|_q^s = \|f\|_q^s = \|f\|_q^s \).

Since \( \phi_t(x) = \exp\left(-\frac{\beta}{2}(x - 1)\right) \) is a convex function and satisfies \( \phi_t(1) = 1 \), substituting \( \phi_t(x) = \exp\left(-\frac{\beta}{2}(x - 1)\right) \) into RHS of this inequality, we have
\[
\int_{\mathbb{R}} \phi_t(\mu_{q,\alpha}^{-1}|x - b|^\alpha)^t dx = \int_{\mathbb{R}} \exp\left(-\beta(\mu_{q,\alpha}^{-1}|x - b|^\alpha - 1)\right) dx ,
\]
where $\beta > 0$. Changing from the variable $x$ to $y = \mu_{q,a}^{-\frac{1}{\alpha}}(x - b)$ gives
\[
\int_\mathbb{R} \phi_t(\mu_{q,a}^{-1}|x-b|^\alpha)^\gamma dx = 2 \exp(\beta \mu_{q,a}^{-\frac{1}{\alpha}}) \int_0^\infty \exp(-\beta y^\alpha)dy = \frac{2}{\alpha} \Gamma(\frac{1}{\alpha}) \mu_{q,a}^{-\frac{1}{\alpha}} \exp(\beta - \frac{1}{\alpha}).
\] (10)

$\exp(\beta - \frac{1}{\alpha})$ has a minimum value at $\beta = \frac{1}{\alpha}$. Substituting this condition into (10) and (8) gives
\[
V \leq \|f\|_q^q (C \mu_{q,a}^{-\frac{1}{\alpha}})^{\frac{1}{p}}.
\] (11)

where $C = \frac{2}{\alpha} \Gamma(\frac{1}{\alpha})(ae)^{\frac{1}{p}}$. Combining (3) and (11) gives
\[
\|f\|_q \leq \|f\|_r (C \mu_{q,a}^{-\frac{1}{\alpha}})^{\frac{1}{p}}.
\] (12)

Combining $\frac{1}{p} + \frac{1}{q} = 1$ and $r = qs$, we have $\frac{1}{p} = \frac{1}{q} - \frac{1}{r}$. Substituting this equation into (12), the result follows.

Next, we prove the multivariate version.

**Definition 4.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function.

For $b \in \mathbb{R}^n$ and $X \in \mathbb{R}^n$, we define a multivariate $q$-moment as follows.
\[
\Sigma_{q,b} \overset{\text{def}}{=} E_q[(X - b)(X - b)^T],
\]

where $T$ denotes the transpose of a vector.

When $b = E_q[X]$, we write $\Sigma_{q,b}$ as $\Sigma_q$ and $\Sigma_q$ is equal to a $q$-covariance matrix.

When $q = 1$, we write $\Sigma_{q,b}$ as $\Sigma_b$ and $\Sigma$ denotes a covariance matrix.

**Theorem 2.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function with finite multivariate $q$-moment. Let $\|f\|_q$, $\|f\|_r < \infty$.

For $0 < q < r \leq \infty$ and $n \geq 1$
\[
\|f\|_q \leq (C (\det \Sigma_{q,b})^{\frac{1}{2}})^{\frac{1}{\frac{1}{p} - \frac{1}{r}}} \|f\|_r.
\] (13)

$C$ is a constant which only depends on $n$. The example value of $C$ is $C = (2\pi e)^{\frac{1}{p}}$.

**Proof.** We can prove this theorem in the same way as the theorem 1.

First, we consider the following value.
\[
V = \int_{\mathbb{R}^n} |f(x)|^q \phi_t((x - b)^T \Sigma_{q,b}^{-1}(x - b)) d^n x,
\] (14)

where $\phi_t$ is a non-negative convex function which satisfies $\phi_t(0) = 1$. By applying the Jensen’s inequality to this equation in the same way as Theorem 1 we get
\[
V \geq \|f\|_q^q \phi_t(n) = \|f\|_q^q.
\] (15)

Next, for $1 \leq s, t \leq \infty$, by applying the Hölder’s inequality to (14) and putting $r = qs$, we have
\[
V \leq \|f\|_q^q \left( \int_{\mathbb{R}^n} \phi_t((x - b)^T \Sigma_{q,b}^{-1}(x - b))^\gamma d^n x \right)^{\frac{1}{\gamma}}.
\] (16)

Substituting $\phi_t(x) = \exp(-\frac{\beta}{2}(x - n))$ into this inequality gives
\[
\int_{\mathbb{R}^n} \phi_t((x - b)^T \Sigma_{q,b}^{-1}(x - b))^\gamma d^n x = \exp(\beta n) \int_{\mathbb{R}^n} \exp(-\beta(x - b)^T \Sigma_{q,b}^{-1}(x - b)) d^n x.
\] (17)

Changing the variable from $x$ to $y = \Sigma_{q,b}^{-\frac{1}{2}}(x - b)$ gives
\[
\exp(\beta n) \int_{\mathbb{R}^n} \exp(-\beta(x - b)^T \Sigma_{q,b}^{-1}(x - b)) d^n x = \exp(\beta n)(\det \Sigma_{q,b})^{\frac{1}{2}} \left( \frac{\pi}{\beta} \right)^{\frac{n}{2}}.
\] (18)
exp(\beta)\beta^{-\frac{1}{2}} has a minimum value at \beta = \frac{1}{2}. Substituting this condition into (15) and combining (16) and (17) give

\[ V \leq \|f\|^q (C(\det \Sigma_q, b) \beta) \frac{1}{2}, \]

where \( C = (2\pi e)^{\frac{n}{2}} \). Combining (15) and (19) gives

\[ \|f\|_q \leq \|f\|_r (C(\det \Sigma_q, b) \beta) \frac{1}{2t}, \quad (20) \]

Combining \( s + t = 1 \) and \( r = qs \), we have

\[ \frac{1}{qt} = \frac{1}{q} - \frac{1}{r}. \]

Substituting this equation into (20), the result follows.

### 3. Applications

In this section, \( C \) denotes a constant which only depends on the dimension \( n \). The example value of \( C \) is \((2\pi e)^{\frac{n}{2}}\).

#### 3.1. Application for the Rényi and the Tsallis entropies

We derive upper bounds for the Rényi and the Tsallis entropies with given \( q \)-covariance matrix and derive an inequality between two Rényi entropies by using Theorem 2.

**Corollary 1.** Let \( f \) be a probability density function on \( \mathbb{R}^n \) with finite \( q \)-covariance matrix and \( h_p(X) \overset{\text{def}}{=} \frac{1}{p} - p \log \|f\|_p \) be the Rényi entropy.

For \( p > 1 \),

\[ h_p(X) \leq \frac{1}{2} \log (\det \Sigma) + \log C. \quad (21) \]

For \( 0 < p < 1 \),

\[ h_p(X) \leq \frac{1}{2} \log (\det \Sigma_p) + \log C. \quad (22) \]

**Proof.** For \( p > 1 \), by putting \( q = 1, r = p, b = E_q[X] \) and using \( \|f\|_1 = 1 \) in Theorem 2, the result follows.

For \( 0 < p < 1 \), by putting \( r = 1, q = p, b = E_q[X] \) and using \( \|f\|_1 = 1 \) in Theorem 2, the result follows.

We can derive the optimal constant \( C \) by using the distribution that maximizes the Rényi entropy [4].

In the limit \( p \to 1 \), the Rényi entropy is the Shannon differential entropy and we have

\[ h(X) \leq \frac{1}{2} \log (\det \Sigma) + \log C, \quad (23) \]

where \( h(X) \) is the Shannon differential entropy. When \( C = (2\pi e)^{\frac{n}{2}} \), this inequality is consistent with the well-known upper bound of the Shannon entropy.

**Corollary 2.** Let \( f \) be a probability density function on \( \mathbb{R}^n \) with finite \( q \)-covariance matrix and \( h_p(X) \overset{\text{def}}{=} \frac{1}{p} - p \log \|f\|_p \) be the Rényi entropy.

For \( 0 < q < r \leq \infty \),

\[ \frac{1-q}{q} h_q(X) \leq \frac{1-r}{r} h_r(X) + \left( \frac{1}{q} - \frac{1}{r} \right) \left( \frac{1}{2} \log (\det \Sigma_q) + \log C \right). \quad (24) \]

**Proof.** Taking the logarithm of Theorem 2, the result follows.

**Corollary 3.** Let \( f \) be a probability density function on \( \mathbb{R}^n \) with finite \( q \)-covariance matrix and \( S_q(X) \overset{\text{def}}{=} \frac{1}{q-1} (1 - \|f\|_q^q) \) be the Tsallis entropy [8].

For \( q > 1 \),

\[ \exp_q(S_q(X)) \leq C(\det \Sigma)^{\frac{1}{2}}, \quad (25) \]
where \( \exp_q(x) \defeq [1 + (1 - q)x]^{1/\tilde{q}} \).

For \( 0 < q < 1 \),

\[
\exp_q(S_q(X)) \leq C(\det \Sigma_q)^{\frac{1}{2}}.
\]

\[\text{(26)}\]

**Proof.** From the definition of the Tsallis entropy and \( \exp_q(x) \), we have \( \exp_q(S_q(X)) = \|f\|_q^{1/q} \). The rest part of the proof is almost the same as Corollary \[\](\[\]).

### 3.2. Application for the upper bound of a probability

We derive an upper bound for the probability of a subset with given \( L^p \)-norm on the same set. Since we use some functions in this subsection, we use the following notation.

**Notation.**

- For a non-negative measurable function \( F \),
  \[
  E_F[X] \defeq \frac{\int_{\mathbb{R}^n} xF(x)dx}{\int_{\mathbb{R}^n} F(x)dx}.
  \]
- \( \Sigma_{F,b} \defeq E_F[(X - b)(X - b)^T] \).

  When \( b = E_q[X] \), we write \( \Sigma_{F,b} \) as \( \Sigma_F \).

- For \( \Omega \subseteq \mathbb{R}^n \),
  \[
  I_\Omega(x) = \begin{cases} 
  1 & (x \in \Omega) \\
  0 & (x \notin \Omega)
  \end{cases}
  \]

**Proposition 1.** Let \( f \) be a probability density function on \( \mathbb{R}^n \) with finite covariance matrix \( \Sigma_f \). Let \( \Omega \subseteq \mathbb{R}^n \) and \( P(\Omega) \) be a probability of \( \Omega \).

For \( r > 1 \) and \( n \geq 1 \),

\[
P(\Omega)^{1 + \frac{n}{2} + \frac{n}{r}} \leq (C(\det \Sigma_f)^{\frac{1}{2}})^{1 - \frac{1}{r}} \|fI_\Omega\|_r.
\]

\[\text{(27)}\]

**Proof.** First, we put

\[
\psi(x) \defeq (\Sigma_f)^{-\frac{1}{2}}(x - E_f[X])
\]

\[
y = \psi(x)
\]

\[
\Omega = \psi(\Omega)
\]

\[
\hat{f}(y) \defeq (\det \Sigma_f)^{\frac{1}{2}}f(x)
\]

\[
g(x) \defeq f(x)I_\Omega(x)
\]

\[
\hat{g}(y) \defeq \hat{f}(y)I_\Omega(y).
\]

Since \( (\det \Sigma_f)^{\frac{1}{2}}dx = dx \), we have

\[
\int_{\mathbb{R}^n} \hat{g}(y)^r dx = \int_{\Omega} \hat{f}(y)^r dx
\]

\[
= (\det \Sigma_f)^{\frac{r-1}{2}} \int_{\Omega} f(x)^r dx
\]

\[
= (\det \Sigma_f)^{\frac{r-1}{2}} \int_{\mathbb{R}^n} g(x)^r dx.
\]

From this equation, we have

\[
\|\hat{g}\|_r = (\det \Sigma_f)^{\frac{r-1}{2}} \|g\|_r = (\det \Sigma_f)^{\frac{1}{2}(1 - \frac{1}{r})} \|g\|_r.
\]

\[\text{(30)}\]

Especially, when \( r = 1 \), \( \|\hat{g}\|_1 = \|g\|_1 = P(\Omega) \).
Furthermore, using the GM-AM inequality \((\det A)^{\frac{1}{n}} \leq \frac{1}{n} \text{Tr} A\), we have

\[
\det \Sigma \hat{g},0 \leq \left(\frac{1}{n} \text{Tr} \Sigma \hat{g},0\right)^n = \frac{1}{\|\hat{g}\|_1^n} \left(\frac{1}{n} \sum_i \int_{\mathbb{R}^n} y_i^2 \hat{g}(y) \, dy\right)^n \tag{31}
\]

and

\[
\sum_i \int_{\mathbb{R}^n} y_i^2 \hat{g}(y) \, dy \leq \sum_i \int_{\mathbb{R}^n} y_i^2 \hat{f}(y) \, dy = E_f[\text{Tr}(YY^T)]
\]

\[
= E_f[\text{Tr}(\Sigma_f^{-1}(X - E_f[X])(X - E_f[X])^T)]
\]

\[
= \text{Tr}(\Sigma_f^{-1}E_f[(X - E_f[X])(X - E_f[X])^T]) = n.
\]

(31) and (32) give

\[
\det \Sigma \hat{g},0 \leq \frac{1}{\|\hat{g}\|_1^n}. \tag{33}
\]

Applying Theorem 2 for \(\hat{g}, q = 1\) and \(b = 0\) gives

\[
\|\hat{g}\|_1 \leq (C(\det \Sigma \hat{g},0)^{\frac{1}{2}})^{1+\frac{1}{r}} \|\hat{g}\|_r. \tag{34}
\]

By using (30) and (33), we have

\[
\|g\|_1 \leq (C(\det \Sigma \hat{g},0)^{\frac{1}{2}})^{1+\frac{1}{r}} \|\hat{g}\|_r \leq \frac{1}{\|g\|_1^{\frac{1}{2}(1+\frac{1}{r})}}(C(\det \Sigma_f)^{\frac{1}{2}})^{1+\frac{1}{r}} \|g\|_r. \tag{35}
\]

By transforming this equation and using \(P(\Omega) = \|g\|_1\), the result follows.

**Corollary 4.** Let \(f\) be a probability density function on \(\mathbb{R}^n\) with finite covariance matrix \(\Sigma_f\). Let \(\Omega \subseteq \mathbb{R}^n\) and \(P(\Omega)\) be a probability of \(\Omega\).

For \(n \geq 1\),

\[
P(\Omega) \leq (C(\det \Sigma_f)^{\frac{1}{2}} \|fI_{\Omega}\|_{\infty})^{\frac{n+1}{n+2}}. \tag{36}
\]

**Proof.** By substituting \(r = \infty\) into (24), the result follows.

Since \(\|fI_{\Omega}\|_{\infty} = \text{ess sup}_{x \in \Omega} f(x)\), when the supremum of \(f(x)\) in \(\Omega\) is given, we can derive the probability upper bound by using Corollary 4.

4. Conclusion

In the first half, we have shown inequalities between two \(L^p\)-norms by using the \(q\)-moment for the Euclidean space and the Lebesgue measure.

In the latter half, by applying these inequalities to probability theory, we have derived the inequality that holds between two Rényi entropies, and have derived upper bounds for the Rényi and the Tsallis entropies with given \(q\)-moment. In particular, by using the result of the Rényi entropy, we have shown an upper bound for the Shannon entropy in the limit \(q \to 1\).

Furthermore, we have derived the upper bound for the probability of the subset in \(\mathbb{R}^n\) with given \(L^p\)-norm on the same set.

We hope we will find the optimal constants \(C\) for each inequality.
References

[1] Sumiyoshi Abe and GB Bagci. Necessity of q-expectation value in nonextensive statistical mechanics. *Physical Review E*, 71(1):016139, 2005.
[2] Christian Beck and Friedrich Schögl. *Thermodynamics of chaotic systems: an introduction*. Number 4. Cambridge University Press, 1995.
[3] Keith Conrad. Probability distributions and maximum entropy. *Entropy*, 6(452):10, 2004.
[4] Oliver Johnson and Christophe Vignat. Some results concerning maximum rényi entropy distributions. In *Annales de l’Institut Henri Poincaré (B) Probability and Statistics*, volume 43, pages 339–351. No longer published by Elsevier, 2007.
[5] Tomohiro Nishiyama. Improved chebyshev inequality: new probability bounds with known supremum of pdf. *arXiv preprint arXiv:1808.10770*, 2018.
[6] Alfred Rényi. On measures of entropy and information. Technical report, HUNGARIAN ACADEMY OF SCIENCES Budapest Hungary, 1961.
[7] Walter Rudin. *Real and complex analysis*. Tata McGraw-Hill Education, 2006.
[8] Constantino Tsallis. Possible generalization of boltzmann-gibbs statistics. *Journal of statistical physics*, 52(1-2):479–487, 1988.
[9] Constantino Tsallis, RenioS Mendes, and Anel R Plastino. The role of constraints within generalized nonextensive statistics. *Physica A: Statistical Mechanics and its Applications*, 261(3-4):534–554, 1998.