BEZOUT’S THEOREM AND COHEN-MACaulAY MODULES

J. MIGLIORE, U. NAGEL, C. PETERSON

Abstract. We define very proper intersections of modules and projective subschemes. It turns out that equidimensional locally Cohen-Macaulay modules intersect very properly if and only if they intersect properly. We prove a Bezout theorem for modules which meet very properly. Furthermore, we show for equidimensional subschemes $X$ and $Y$: If they intersect properly in an arithmetically Cohen-Macaulay subscheme of positive dimension then $X$ and $Y$ are arithmetically Cohen-Macaulay. The module version of this result implies splitting criteria for reflexive sheaves.

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1. Introduction

In this note we present a unified approach in order to study the intersection of projective subschemes and tensor products of reflexive sheaves on projective space over an arbitrary field. A crucial role is played by the new concept of a very proper intersection.

There are two starting points for our considerations. The first is the following result of Huneke and Ulrich.

Proposition 1.1. Let $X \subset \mathbb{P}^n$ denote an equidimensional subscheme having dimension at least 2. Let $H$ be a hyperplane which does not contain any component of $X$. Suppose that $X \cap H$ is arithmetically Cohen-Macaulay. Then $X$ is arithmetically Cohen-Macaulay.

We want to generalize this statement to a result on the intersection of two projective subschemes. Thus we consider the following problem.

Problem: Let $X, Y \subset \mathbb{P}^n$ be equidimensional subschemes such that $\dim X \cap Y \geq 1$. Assume that $X \cap Y$ is arithmetically Cohen-Macaulay. Which “extra conditions” imply that $X$ and $Y$ are arithmetically Cohen-Macaulay?

We assume that the dimension of $X \cap Y$ is at least one because $X \cap Y$ is automatically arithmetically Cohen-Macaulay if it is zero-dimensional. Still, a priori it is not clear at all if the problem above has a reasonable answer. Suppose it has. Then, a comparison
with the result of Huneke and Ulrich suggests to look for some genericity condition which ensures that the subschemes $X$ and $Y$ are in sufficiently general position to each other. Thus, we want to assume that $X$ and $Y$ meet properly. Recall that there is always the following inequality

$$\dim X \cap Y \geq \dim X + \dim Y - n.$$  

If equality holds true then it is said that $X$ and $Y$ intersect properly. However, the following simple example shows that “proper intersection” is not the answer to our problem.

**Example 1.2.** Let $T_1, T_2, T_3 \subset \mathbb{P}^5$ be the three-folds defined by $(x_0, x_1)$, $(x_2, x_3)$ and $(x_1 + x_2, x_0 + x_3)$, respectively. Put $X = T_1 \cup T_2$ and $Y = T_3$. Then $I_X + I_Y$ is a saturated ideal and $X$ and $Y$ intersect properly in an arithmetically Cohen-Macaulay curve. However, $X$ is not arithmetically Cohen-Macaulay.

The analysis of this example lead us to the concept of a “very proper intersection” (cf. Definition [4.1]). Intuitively, two subschemes $X$ and $Y$ intersect very properly if not only $X$ and $Y$ intersect properly but also all their cohomology modules do. We show that the two conditions proper intersection and very proper intersection are equivalent for $X$ and $Y$ if they are both equidimensional and locally Cohen-Macaulay. As one of our goals, we prove that the condition very proper intersection is an answer to our problem above.

**Theorem 1.3.** Let $X, Y \subset \mathbb{P}^n$ denote equidimensional subschemes which meet very properly in an arithmetically Cohen-Macaulay subscheme of positive dimension. Then $X$ and $Y$ are arithmetically Cohen-Macaulay.

It is interesting to see that the example above does not contradict this result. Observe that $X$ is not locally Cohen-Macaulay along the line $T_1 \cap T_2$ which is contained in $Y$. Thus $X$ and $Y$ do not meet very properly.

A natural idea for proving this result is to use Serre’s diagonal trick. This amounts to viewing the intersection of $X$ and $Y$ as consecutive hyperplane sections of their join. Then, one is tempted to apply Proposition [1.1]. Note, however, that this statement assumes that the scheme considered is equidimensional. Furthermore, observe that the occurring hyperplanes are not general because they have to contain the diagonal. In fact, the example above shows that things can go wrong. In order to control these problems we study the cohomology of the subschemes involved. Our methods provide the following version of Bezout’s theorem.

**Theorem 1.4.** Let $X, Y \subset \mathbb{P}^n$ denote equidimensional subschemes which intersect very properly in a non-empty scheme. Then the intersection is equidimensional, and we have

$$\deg X \cap Y = \deg X \cdot \deg Y.$$  

It is well-known that the degree relation is not true for arbitrary proper intersections. Very proper intersections have better properties. The collection of irreducible subschemes of the intersection, which contribute to the Bezout number $\deg X \cdot \deg Y$, corresponds exactly to the whole set of irreducible components of the homogeneous ideal of $X \cap Y$. In particular, $X \cap Y$ has no embedded components.

In the special case of a hypersurface section, the last result gives a condition when the hypersurface section of an equidimensional subscheme remains equidimensional. Thus we may also view Theorem [1.3] as a version of Bertini’s theorem.
Our second starting point for this note were results of Huneke and Wiegand in [7] and [8] on tensor products of maximal modules. Here we think of tensor products as intersections of modules. It turns out that the concept of a very proper intersection can be extended to the case of graded modules and sheaves on projective space. In fact, the results mentioned above are special cases of statements about tensor products of modules. Note also that two reflexive sheaves on projective space always meet properly. Their tensor product is not necessarily reflexive. However, we will show that it is indeed reflexive provided the sheaves intersect very properly.

The paper is organized as follows. In Section 2, we introduce the notion of a slight modification. Then we use cohomological methods in order to relate the unmixedness of a module \( M \) to the unmixedness of \( M/IM \) where \( I \) is a parameter ideal for \( M \). Section 3 is entirely devoted to compute the local cohomology of the join of two modules. The resulting Künneth formulas are essential for the rest of the paper. Very proper intersections are introduced in Section 4. There we also prove our version of Bezout’s theorem for modules. In Section 5, we combine all these techniques in order to relate the Cohen-Macaulay property of the tensor product to its factors. The consequences for reflexive sheaves are drawn in the final section.

Throughout this note, \( R \) denotes a standard graded \( K \)-algebra or a local ring containing a field \( K \). If \( M \) is a module over the graded ring \( R \) then it is understood that \( M \) is a graded \( R \)-module

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## 2. Unmixedness results

In this section we assume that \( R \) is a Gorenstein ring which is either local or a graded \( K \)-algebra where \( K \) is an infinite field. In [14] the unmixedness of an ideal of \( R \) has been characterized by means of its local cohomology modules. First we note that this result can be extended to finitely generated \( R \)-modules. Second we apply it in order to derive sufficient conditions on a subsystem \( \{l_1, \ldots, l_r\} \) of a system of parameters of an unmixed module \( M \) which guarantee that \( M/(l_1, \ldots, l_r)M \) is an unmixed module, too.

Recall that a module is called unmixed if all its associated prime ideals have the same height. This can be characterized cohomologically as follows where we use the convention that a module of negative dimension is the zero module. The result extends [14], Lemma 2.11.

**Lemma 2.1.** Let \( M \) be a finitely generated \( R \)-module of dimension \( d \). Then the following conditions are equivalent:

(a) \( M \) is unmixed.

(b) \( \dim \text{Ext}^i_R(M, R) \leq \dim R - 1 - i \quad \text{if} \quad \dim R - d < i. \)

(c) \( \dim R/\mathfrak{a}_i(M) < i \quad \text{if} \quad i < d \) where \( \mathfrak{a}_i(M) = \text{Ann} H^i_m(M). \)
**Proof.** Choose a complete intersection \( c = (f_1, \ldots, f_{\dim R - d}) \subset \text{Ann} \, M \). Then we can continue as in the proof of [L3], Lemma 4. \( \square \)

We recall the following notion.

**Definition 2.2.** Let \( M \) be a finitely generated \( R \)-module of dimension \( d \). Let \( I \subset R \) be an ideal such that \( \dim M/IM = 0 \). The set \( B = \{a_1, \ldots, a_n\} \subset R \) is said to be an \( M \)-basis of \( I \) if its elements form a minimal basis of \( I \) and every subset of \( B \) consisting of \( d \) elements is a system of parameters of \( M \).

Fortunately such a basis often exists.

**Lemma 2.3.** Let \( I \subset R \) be an ideal and let \( M_1, \ldots, M_s \) be finitely generated \( R \)-modules such that \( \dim M_i/IM_i = 0 \) for all \( i = 1, \ldots, s \). Then it holds:

(a) (local case) The ideal \( I \) has a minimal basis which is an \( M_i \)-basis for all \( i = 1, \ldots, s \).
(b) (graded case) If \( I \) is a homogeneous ideal admitting a generating set consisting of homogeneous elements of degree \( e \) then \( I \) has a minimal basis which is formed by homogeneous elements of degree \( e \) and is an \( M_i \)-basis for all \( i = 1, \ldots, s \).

**Proof.** Claim (a) is Proposition 1.9 and (b) is Proposition 3.3 in [L7]. Note that the latter can fail if \( K \) is a finite field. \( \square \)

Often an ideal \( I \) is called a parameter ideal for \( M \) if \( \dim M/IM = 0 \). We want to extend this notion slightly. We will say that \( I \) is a parameter ideal for \( M \) if \( \dim M/IM = \max\{0, \dim M - \mu(I)\} \) where \( \mu(I) \) denotes the number of minimal generators of \( I \). This means that a parameter ideal is either generated by a subsystem of a system of parameters or it contains a system of parameters for \( M \).

We introduce one more piece of notation. If \( M \) is an \( R \)-module then we call \( M^{sm} = M/H^0_m(M) \) the slight modification of \( M \). If \( M = R/I \) for an ideal \( I \) of \( R \) then \( M^{sm} = R/J \) where \( J \subset R \) is the saturation of the ideal \( I \). More generally, if \( \varphi : F \to M \) is an epimorphism where \( F \) is a free \( R \)-module and we put \( E = \ker \varphi \) then

\[
E^{sat} = \bigcup_k (E : F \, m^k)
\]

is the saturation of \( E \) and \( M^{sm} \cong F/E^{sat} \). Note that \( M \) and \( M^{sm} \) have the same associated sheaf.

The goal of this section is the next result.

**Proposition 2.4.** Let \( \{l_1, \ldots, l_s\} \) be a subsystem of a system of parameters for the finitely generated \( R \)-module \( M \) such that \( I = (l_1, \ldots, l_s)R \) is a parameter ideal for all \( \text{Ext}_R^i(M, R) \) where \( i \neq \dim R - \dim M \). If \( M \) is a graded module we will also assume that all the elements \( l_i \) are homogeneous of degree \( e \). Suppose that \( M^{sm} \) is unmixed. Then \( (M/IM)^{sm} \) is unmixed too. Furthermore, there is a system of generators \( \{h_1, \ldots, h_s\} \) of \( I \), which consists of homogeneous elements of degree \( e \) in the graded case, such that \( (M/(h_1, \ldots, h_i)M)^{sm} \) is unmixed for all \( i = 0, \ldots, s \).

**Proof.** We will show the existence of the elements \( h_i \) successively. Suppose we have found minimal generators \( h_1, \ldots, h_t \) of \( I \) where \( 0 \leq t < s \) such that \( (M/(h_1, \ldots, h_i)M)^{sm} \) is unmixed for all \( i = 0, \ldots, t \). Choose elements \( f_{t+1}, \ldots, f_s \subset I \) (of degree \( e \) in the graded
case) such that \( I = (h_1, \ldots, h_t)R + (f_{t+1}, \ldots, f_s)R \). Put \( N = M/(h_1, \ldots, h_t)M \) and \( J = (f_{t+1}, \ldots, f_s)R \).

Since \( (M/(h_1, \ldots, h_t)M)^{sm} \) is unmixed for all \( i = 0, \ldots, t \), \( \{h_1, \ldots, h_t\} \) is not just a subsystem of parameters for \( M \) but even an \( M \)-filter regular sequence. Hence Theorem 3.3 yields for all integers \( i \)

\[
\mathfrak{a}_i(M) \cdot \ldots \cdot \mathfrak{a}_{i+t}(M) \subset \mathfrak{a}_i(N).
\]

Since \( I \) is a parameter ideal for all \( \operatorname{Ext}^j_R(M, R) \) with \( j \neq \dim R - \dim M \) we conclude by local duality that we have for all \( i < d - t \) where \( d = \dim M \):

\[
\dim R/(\mathfrak{a}_i(N) + J) = \dim R/(\mathfrak{a}_i(N) + I) \\
\leq \max \{ \dim R/(\mathfrak{a}_j(M) + I) \mid i < j < i + t \} \\
= \max \{ 0, \dim R/\mathfrak{a}_j(M) - s \mid i < j < i + t \} \\
\leq \max \{ 0, i + t - 1 - s \}.
\]

Here the latter estimate is due to Lemma 2.1.

Now consider the set

\[ P = \{ i \in \mathbb{Z} \mid i < \dim N = d - t, \ \dim R/\mathfrak{a}_i(N) = i - 1 > 0 \}. \]

We have \( \dim N/JN < \dim N \) and, for all \( i \in P \), \( \dim R/(\mathfrak{a}_i(N) + J) < \dim R/\mathfrak{a}_i(N) \) because \( t < s \). Thus (using arguments as in Lemma 2.3) there is an element \( h \in J \) (homogeneous of degree \( e \) in the graded case) which is a parameter for \( N \) and all \( \operatorname{Ext}^{\dim R - i}_R(N, R) \) with \( i \in P \).

Since \( N^{sm} \) is unmixed the module \( 0 :_N h \) has finite length. Thus the exact sequence

\[ 0 \to 0 :_N h \to N \to N/0 :_N h \to 0 \]

implies

\[ \operatorname{Ext}^j_R(N/0 :_N h, R) \cong \operatorname{Ext}^j_R(N, R) \quad \text{for all } j \neq \dim R. \]

Multiplication by \( h \) provides the exact sequence

\[ 0 \to N/0 :_N h(-e) \to N \to N/hN \to 0. \]

Using the isomorphisms above we see that it induces exact sequences

\[ \operatorname{Ext}^{\dim R - i - 1}_R(N, R)(-e) \xrightarrow{h} \operatorname{Ext}^{\dim R - i - 1}_R(N, R) \to \operatorname{Ext}^{\dim R - i}_R(N/hN, R) \to \operatorname{Ext}^{\dim R - i}_R(N, R) \xrightarrow{h} \operatorname{Ext}^{\dim R - i}_R(N, R). \]

Let \( 0 < i < \dim N - 1 = d - t - 1 \). Since \( N \) is unmixed we have \( \dim \operatorname{Ext}^{\dim R - i}_R(N, R) \leq i - 1 \) and \( \dim \operatorname{Ext}^{\dim R - i - 1}_R(N, R) \leq i \) where for the latter estimate equality holds if and only if \( i + 1 \in P \). But then \( h \) is a parameter for \( \operatorname{Ext}^{\dim R - i - 1}_R(N, R) \) due to our choice of \( h \). Thus in either case we have \( \dim \operatorname{Ext}^{\dim R - i - 1}_R(N, R)/h \operatorname{Ext}^{\dim R - i - 1}_R(N, R) \leq i - 1 \). Therefore, using the exact sequence above we obtain

\[ \dim \operatorname{Ext}^{\dim R - i}_R(N/hN, R) \leq i - 1 \quad \text{for all } i \neq \dim N - 1. \]

Applying the isomorphisms

\[ \operatorname{Ext}^j_R(N/hN, R) \cong \operatorname{Ext}^{j-1}_{R/hR}(N/hN, R/hR)(e) \]
we get that
\[ \dim \Ext_{R/hR}^{\dim R/hR - i}(N/hN, R/hR) \leq \max\{0, i - 1\} \] for all \( i \neq \dim N - 1 \).

According to Lemma 2.1 we conclude that \((N/hN)_{sm}\) is an unmixed module. Thus we may put \( h_{i+1} = h \) and we are done.

\[ \square \]

3. K"unneth formulas

Let \( R \) be a graded \( K \)-algebra where \( K \) is an arbitrary field. Let \( M \) and \( N \) be \( R \)-modules, not necessarily finitely generated. We want to relate the local cohomology modules of \( M \otimes_K N \) to those of \( M \) and \( N \) such that the corresponding homomorphisms are \((R \otimes_K R)\)-linear. Hence we cannot use results of Grothendieck because they would give us only \( K \)-linear homomorphisms. Instead we adapt the method of St"uckrad and Vogel (cf. \[17\]) which they used in order to show a K"unneth formula for the Segre product of \( M \) and \( N \).

We begin with some preliminary results where \( R_1 \) and \( R_2 \) denote graded \( K \)-algebras.

**Lemma 3.1.** Let \( M_i \) and \( N_i \) be graded \( R_i \) modules for \( i = 1, 2 \). Put \( R = R_1 \otimes_K R_2 \). Then there is a natural graded \( R \)-homomorphism
\[ \tau_0: \Hom_{R_1}(N_1, M_1) \otimes_R \Hom_{R_2}(N_2, M_2) \to \Hom_R(N_1 \otimes_K N_2, M_1 \otimes_K M_2). \]

**Proof.** Let \( p_1, p_2 \) be integers and let \( f_i \in [\Hom_{R_i}(N_i, M_i)]_{p_i} \) for \( i = 1, 2 \). We define for homogeneous elements \( n_i \in N_i \) where \( i = 1, 2 \):
\[ \tau_0(f_1 \otimes_K f_2)(n_1 \otimes_K n_2) = f_1(n_1) \otimes_K f_2(n_2). \]

It is clear that this induces a well-defined \( R \)-homomorphism
\[ \tau_0(f_1 \otimes_K f_2): N_1 \otimes_K N_2 \to M_1 \otimes_K M_2 \]
of degree \( p_1 + p_2 \). Hence it provides a well-defined \( K \)-linear map
\[ \tau_0: \Hom_{R_1}(N_1, M_1) \otimes_K \Hom_{R_2}(N_2, M_2) \to \Hom_R(N_1 \otimes_K N_2, M_1 \otimes_K M_2) \]
which preserves degrees. Straightforward computations show that \( \tau_0 \) is even a natural \( R \)-homomorphism.

\[ \square \]

The map of the lemma is often very nice.

**Lemma 3.2.** With the notation and assumptions of Lemma 3.1 suppose additionally that \( N_i \) is a finitely generated \( R_i \)-module for \( i = 1, 2 \). Then the map \( \tau_0 \) is an isomorphism.

**Proof.** First, we assume that the modules \( N_i \) are free. Thus, since the map \( \tau_0 \) is natural we can even assume that \( N_i = R_i(p_i) \) with \( p_i \in \mathbb{Z} \) for \( i = 1, 2 \). Then the claim is easy to check.

Second, we consider the general case. Let
\[ G_i \to F_i \to N_i \to 0 \]
be free graded presentations. Put \( H_i = \Hom_{R_i}(N_i, M_i) \) for \( i = 1, 2 \). We have exact sequences
\[ 0 \to H_i \to \Hom_{R_i}(F_i, M_i) \to \Hom_{R_i}(G_i, M_i) \quad (i = 1, 2) \]
and 
\[(G_1 \otimes_K F_2) \oplus (G_2 \otimes_K F_1) \to F_1 \otimes_K F_2 \to N_1 \otimes_K N_2 \to 0.\]
Since the functor \(- \otimes_K -\) is exact they induce the following commutative diagram with exact rows
\[
\begin{array}{c}
0 \to H_1 \otimes_K H_2 \to \Hom_{R_1}(F_1, M_1) \otimes_K \Hom_{R_2}(F_2, M_2) \\
\downarrow \tau_0 \quad \downarrow \tau_0 \\
0 \to \Hom_R(N_1 \otimes_K N_2, M_1 \otimes_K M_2) \to \Hom_R(F_1 \otimes_K F_2, M_1 \otimes_K M_2)
\end{array}
\]
\[
\begin{array}{l}
\to (\Hom_{R_1}(F_1, M_1) \otimes_K \Hom_{R_2}(G_2, M_2)) \oplus (\Hom_{R_2}(F_2, M_2) \otimes_K \Hom_{R_1}(G_1, M_1)) \\
\downarrow \hat{\tau}_0 \oplus \hat{\tau}_0 \\
\to \Hom_R((G_1 \otimes_K F_2) \oplus (G_2 \otimes_K F_1), M_1 \otimes_K M_2)
\end{array}
\]
where \(\tau_0, \hat{\tau}_0, \hat{\tau}_0\) are the corresponding natural homomorphisms. These maps are even isomorphisms by the first part of the proof. Thus \(\tau_0\) is an isomorphism as well. \(\square\)

For a graded \(R\)-module \(M\) we denote its graded injective hull by \(E(M)\). As the last preparation we need.

**Lemma 3.3.** Let \(I_i\) be graded injective \(R_i\)-modules for \(i = 1, 2\). Let \(\mathfrak{m}\) be the irrelevant maximal ideal of \(R = R_1 \otimes_K R_2\). Then we have
\[H^j_{\mathfrak{m}}(I_1 \otimes_K I_2) = 0\text{ for all } j \geq 1.\]

**Proof.** According to results of Matlis [11] we know that \(I_i\) is a direct sum of injective hulls \(E(R_i/p_i)(p_i)\) where \(p_i \in \Proj(R_i) \cup \{\mathfrak{m}_i\}\) and \(p_i \in \mathbb{Z}\). Since the tensor product and local cohomology commute with direct sums we may assume that \(I_i = E(R_i/p_i)(p_i)\).

If \(p_i = \mathfrak{m}_i\) for \(i = 1, 2\) then we have \(\Supp I_1 \otimes_K I_2 = \{\mathfrak{m}\}\). This implies \(H^j_{\mathfrak{m}}(I_1 \otimes_K I_2) = 0\) for all \(j \geq 1\).

Otherwise we can find homogeneous elements \(y_i \in [R_i]_{p_i}\) such that \(y = y_1 \otimes_K y_2\) has degree \(p > 0\), i.e. \(y \in \mathfrak{m}\). Due to our choice of the elements \(y_i\) the multiplication gives isomorphisms \(I_i \xrightarrow{y_i} I_i(p_i)\). Therefore the multiplication \(I \xrightarrow{y} I(p)\) where \(I = I_1 \otimes_K I_2\) is an isomorphism as well. Thus we obtain for all \(j \geq 1\) isomorphisms \(H^j_{\mathfrak{m}}(I) \xrightarrow{y} H^j_{\mathfrak{m}}(I(p))\).

Since \(\Supp H^j_{\mathfrak{m}}(I) \subset \{\mathfrak{m}\}\) it follows that \(H^j_{\mathfrak{m}}(I) = 0\) for all \(j \geq 0\). \(\square\)

Now we are in the position to state and to prove the main result of this section.

**Theorem 3.4.** Let \(M_i\) and \(N_i\) be graded \(R_i\)-modules and let \(R = R_1 \otimes_K R_2\). Then we have for all \(k \geq 0\):

(a) There are natural graded \(R\)-homomorphisms
\[\tau_k : \bigoplus_{i+j=k} \Ext^i_{R_1}(N_1, M_1) \otimes_K \Ext^j_{R_2}(N_2, M_2) \to \Ext^k_R(N_1 \otimes_K N_2, M_1 \otimes_K M_2).\]

(b) (Künneth formulas) There are natural graded \(R\)-isomorphisms
\[\sigma_k : \bigoplus_{i+j=k} H^i_{\mathfrak{m}_1}(M_1) \otimes_K H^j_{\mathfrak{m}_2}(M_2) \to H^k_{\mathfrak{m}}(M_1 \otimes_K M_2).\]
Lemma 3.5. We state the results in greater generality. First we need.

\[ \bigoplus_{i+j=k} \text{Ext}^i_{R_1}(N_1, M_1) \otimes_K \text{Ext}^j_{R_2}(N_2, M_2) = \bigoplus_{i+j=k} H^i(P_1^\bullet) \otimes_K H^j(P_2^\bullet) \cong H^k(P_1^\bullet \otimes_K P_2^\bullet). \]

According to Lemma 3.1 we have natural homomorphisms of complexes of \( R \)-modules

\[ P_1^\bullet \otimes_K P_2^\bullet = \text{Hom}_{R_1}(N_1, I_1^\bullet) \otimes_K \text{Hom}_{R_2}(N_2, I_2^\bullet) \rightarrow \text{Hom}_R(N_1 \otimes_K N_2, I_1^\bullet \otimes_K I_2^\bullet). \]

Since \( I_1^\bullet \otimes_K I_2^\bullet \) is an acyclic right complex with \( H^0(I_1^\bullet \otimes_K I_2^\bullet) = M_1 \otimes_K M_2 \) there are homomorphisms

\[ H^k(P_1^\bullet \otimes_K P_2^\bullet) \rightarrow H^k(\text{Hom}_R(N_1 \otimes_K N_2, I_1^\bullet \otimes_K I_2^\bullet)) \rightarrow \text{Ext}^k_R(N_1 \otimes_K N_2, M_1 \otimes_K M_2) \]

(cf., for example, [3], Proposition V.1.1a). Now composition of the maps described above gives the desired homomorphisms.

Second we prove (b). Let \( s \) be an integer such that the ideals \( m_1 \) and \( m_2 \) can be generated by less than \( s \) elements. Then we have for all positive integers \( t \) canonical epimorphisms

\[ R/m_1^s \rightarrow R_1/m_1^t \otimes_K R_2/m_2^t \rightarrow R/m^t. \]

They induce the homomorphisms

\[ \text{Ext}^k_R(R/m_1^s, M_1 \otimes_K M_2) \rightarrow \text{Ext}^k_R(R_1/m_1^t \otimes_K R_2/m_2^t, M_1 \otimes_K M_2) \rightarrow \text{Ext}^k_R(R/m^t, M_1 \otimes_K M_2). \]

Taking the direct limit over all \( t \geq 1 \) we obtain

\[ \lim_{\rightarrow} \text{Ext}^k_R(R_1/m_1^t \otimes_K R_2/m_2^t, M_1 \otimes_K M_2) \cong H^k_m(M_1 \otimes_K M_2). \]

Now we use part (a). It provides natural homomorphisms

\[ \text{Ext}^k_{R_1}(R_1/m_1^t, M_1) \otimes_K \text{Ext}^k_{R_2}(R_2/m_2^t, M_2) \rightarrow \text{Ext}^k_R(R_1/m_1^t \otimes_K R_2/m_2^t, M_1 \otimes_K M_2). \]

Taking again the direct limit and using the isomorphism above we get natural homomorphisms

\[ \sigma_k : \bigoplus_{i+j=k} H^i_{m_1}(M_1) \otimes_K H^j_{m_2}(M_2) \rightarrow H^k_m(M_1 \otimes_K M_2). \]

According to [3], Proposition V.4.4, claim (b) holds, if \( \sigma_0 \) is an isomorphism for all modules \( M_1 \) and \( M_2 \) and \( \sigma_k \) is an isomorphism for all injective modules \( M_1 \) and \( M_2 \). The second statement is true by Lemma 3.3. Thus it remains to show that \( \sigma_0 \) is an isomorphism.

Due to Lemma 3.2 the maps

\[ \tau_0 : \text{Hom}_{R_1}(R_1/m_1^t, M_1) \otimes_K \text{Hom}_{R_2}(R_2/m_2^t, M_2) \rightarrow \text{Hom}_R(R_1/m_1^t \otimes_K R_2/m_2^t, M_1 \otimes_K M_2) \]

are isomorphisms. Thus taking the direct limit shows that \( \sigma_0 \) is an isomorphism.

The last result allows us to relate information about two projective subschemes to properties of its embedded join. We state the results in greater generality. First we need.

Lemma 3.5. Let \( M_i \) be graded \( R_i \)-modules for \( i = 1, 2 \). Then we have

(a) \( M_1 \otimes_K M_2 \) is a Cohen-Macaulay module if and only if \( M_1 \) and \( M_2 \) are Cohen-Macaulay.

(b) \( R = R_1 \otimes_K R_2 \) is a Gorenstein algebra if and only if \( R_1 \) and \( R_2 \) are Gorenstein.
Proof. Write $R_i = S_i/I_i$ where $S_i$ is a polynomial ring over $K$ and $I_i \subset S_i$ is a homogeneous ideal. Let $F^{(i)}_\bullet$ denote the minimal free resolution of $M_i$ as $S_i$-module. Then $F^{(1)}_\bullet \otimes_K F^{(2)}_\bullet$ is a minimal free resolution of $M_1 \otimes_K M_2$ as $S_1 \otimes_K S_2$-module. In conjunction with the Auslander-Buchsbaum formula claim (a) follows. Moreover, we see that the Cohen-Macaulay type of $M_1 \otimes_K M_2$ is the product of the Cohen-Macaulay types of $M_1$ and $M_2$. Hence claim (b) follows from the fact that $R$ is Gorenstein if and only if its Cohen-Macaulay type is one.

Now we can show.

Proposition 3.6. Let $M_i$ be graded $R_i$-modules for $i = 1, 2$. Assume that the rings $R_i$ are Gorenstein. Then we have

(a) $M_1 \otimes_K M_2$ is an unmixed $R_1 \otimes_K R_2$-module if and only if $M_1$ and $M_2$ are unmixed.
(b) depth $M_1 \otimes_K M_2 = \text{depth } M_1 + \text{depth } M_2$.
(c) The following conditions are equivalent:
   (i) $M_1 \otimes_K M_2$ is unmixed and locally Cohen-Macaulay.
   (ii) $M_1 \otimes_K M_2$ is Cohen-Macaulay.
   (iii) $M_1$ and $M_2$ are Cohen-Macaulay.

Proof. (a) Since $R = R_1 \otimes_K R_2$ is Gorenstein by the previous lemma we can apply Lemma 2.1. Now suppose that $M_1$ is not unmixed. Then there is an $i > \dim R_1 - \dim M_1$ such that $\dim \text{Ext}^i_{R_1}(M_1, R_1) \geq \dim R_1 - i$. Let $j = \dim R_2 - \dim M_2$. Then we have $\dim \text{Ext}^j_{R_2}(M_2, R_2) = \dim M_2$ and thus

$$\dim \text{Ext}^i_{R_1}(M_1, R_1) \otimes_K \text{Ext}^j_{R_2}(M_2, R_2) \geq \dim R_1 - i + \dim M_2 = \dim R - i - j.$$ 

Hence Theorem 3.4 implies

$$\dim \text{Ext}^{i+j}_{R}(M_1 \otimes_K M_2) \geq \dim R - i - j.$$ 

Since $i + j > \dim R - \dim M_1 \otimes_K M_2$, Lemma 2.1 shows that $M_1 \otimes_K M_2$ is not unmixed.

Conversely, if $M_1$ and $M_2$ are unmixed then $M_1 \otimes_K M_2$ is unmixed too using similar arguments as above.

(b) follows from the cohomological characterization of depth and the Kümeth formula.

(c) According to Lemma 3.3 it suffices to show that (i) implies (iii). Suppose that $M_1$ is not Cohen-Macaulay. Then there is an $i < \dim M_1$ such that $H^i_m(M_1) \neq 0$. Let $j = \dim M_2$. Since $H^j_m(M_1)$ is not finitely generated, the module $H^{i+j}_m(M_1 \otimes_K M_2)$ is not a finitely generated $R$-module and, in particular, is not of finite length. Hence $M_1 \otimes_K M_2$ cannot be equidimensional and locally Cohen-Macaulay which completes the proof.

In the special case where $M_1$ and $M_2$ are $K$-algebras the conclusion from properties of $M_1$ and $M_2$ to properties of $M_1 \otimes_K M_2$ can also be found in [18], Proposition 1.47.

We remark that the results of this section hold also true for modules over local rings containing a field $K$.

4. A Bezout theorem for modules

In this section $R$ will always denote the polynomial ring $K[x_0, \ldots, x_n]$ with its standard grading or a regular local ring containing a field. Moreover, $M$ and $N$ will denote finitely generated graded $R$ modules and $\Delta$ the diagonal ideal of $R \otimes_K R$. 

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We will apply the results of Section 2 where we assumed that the ground field \( K \) is infinite. This assumption is harmless because the Cohen-Macaulay property and Hilbert functions are not effected by field extension. Thus the results of the rest of the paper hold true for arbitrary fields.

Recall that there is an inequality
\[
\dim M \otimes_R N \leq \dim M + \dim N - \dim R.
\]
If equality holds then it is said that \( M \) and \( N \) intersect properly. We need an even stronger condition.

**Definition 4.1.** It is said that \( M \) and \( N \) intersect *very properly* if
\[
\dim M + \dim N \geq \dim R
\]
and
\[
\dim \operatorname{Ext}^i_R(M, R) \otimes_R \operatorname{Ext}^j_R(N, R) = \max\{0, \dim \operatorname{Ext}^i_R(M, R) + \dim \operatorname{Ext}^j_R(N, R) - \dim R\}
\]
for all integers \( i \) and \( j \).

Similarly, we say that subschemes \( X, Y \subset \mathbb{P}^n \) intersect very properly if the corresponding modules \( R/I_X \) and \( R/I_Y \) meet very properly.

We want to compare the two conditions proper and very proper intersection. For this we recall that the canonical module of \( M \) is
\[
K_M = \operatorname{Ext}^{n+1-\dim M}_R(M, R)(-n).
\]
The sets of top-dimensional associated prime ideals of \( M \) and \( K_M \) coincide.

**Lemma 4.2.** It holds:

(a) If \( M \) and \( N \) intersect very properly then they intersect properly.

(b) If \( M \) and \( N \) are unmixed locally Cohen-Macaulay modules which intersect properly then they intersect very properly.

**Proof.** (a) The assumption implies
\[
\dim M + \dim N - (n + 1) = \dim K_M \otimes_R K_N.
\]
Hence \( \dim K_M \otimes_R K_N = \dim M \otimes_R N \) proves the claim.

(b) Since \( M \) and \( M \) meet properly we have
\[
\dim M + \dim N - (n + 1) = \dim M \otimes_R N = \dim K_M \otimes_R K_N \geq 0.
\]
If \((i, j) \neq (n + 1 - \dim M, n + 1 - \dim N)\) then one of the factors of
\[
\operatorname{Ext}^i_R(M, R) \otimes_R \operatorname{Ext}^j_R(N, R)
\]
has finite length because of the assumptions on \( M \) and \( N \). It follows that
\[
\dim \left( \operatorname{Ext}^i_R(M, R) \otimes_R \operatorname{Ext}^j_R(N, R) \right) = 0.
\]
This shows that \( M \) and \( N \) intersect very properly.
We will need a result about the change of the degree under hyperplane section. Let us denote the dimension of the $R$-module $M$ by $e > 0$. Then we can write its Hilbert polynomial as

$$p_M(j) = h_0 \cdot \binom{j+e}{j} + h_1 \cdot \binom{j+e-1}{j} + \ldots + h_e \cdot \binom{t}{t}$$

with integers $h_0, \ldots, h_e$. The degree of $M$ is

$$\deg M = \begin{cases} h_0 & \text{if } \dim M > 0 \\ \text{length}(M) & \text{otherwise.} \end{cases}$$

Since we are not aware of a reference for the following observation in the generality we need, we state it explicitly.

**Lemma 4.3.** Let $f \in R$ be a homogeneous parameter for $M$ of degree $k$. Then we have

$$\deg M/fM = \begin{cases} k \cdot \deg M & \text{if } f \notin p \text{ for all } p \in \text{Ass } M \\ k \cdot \deg M + \deg(0 : M f) & \text{otherwise.} \end{cases}$$

**Proof.** The exact sequence

$$0 \to (0 : M f)(-k) \to M(-k) \xrightarrow{f} M \to M/fM \to 0$$

provides for the Hilbert polynomials

$$p_{M/fM}(j) = p_M(j) - p_M(j - k) + p_{0:Mf}(j - k).$$

Since $\dim M/fM > \dim(0 : M f)$ if and only if $f \notin p$ for all $p \in \text{Ass } M$ where $\dim R/p = \dim M - 1$ the claim follows by comparing the coefficients of the polynomials above.

We will refer to the next result as a Bezout theorem. The name will become clear from the consequences of the result for the intersection of projective schemes. In the proof we will use the isomorphism

$$M \otimes_R N \cong (M \otimes_K N)/\Delta(M \otimes_K N)$$

frequently. Its use is often called diagonal trick or reduction to the diagonal. Analogous to the case of subschemes we call $M \otimes_R N$ the join of $M$ and $N$ and $M \otimes_K N$ their intersection.

**Theorem 4.4.** Suppose that $M$ and $N$ are unmixed modules which intersect very properly. Then $(M \otimes_R N)^{sm}$ is an unmixed module.

Moreover, if $M \otimes_R N$ has positive dimension or depth $M + \text{depth } N \geq n + 1$ then we have

$$\deg M \otimes_R N = \deg M \cdot \deg N.$$  

**Proof.** Let $S = R \otimes_K R$. Then we have by Theorem 3.4 the isomorphisms

$$\bigoplus_{i+j=k} \text{Ext}^i_R(M, R) \otimes_R \text{Ext}^j_R(N, R) \cong \bigoplus_{i+j=k} \text{Ext}^i_R(M, R) \otimes_K \text{Ext}^j_R(N, R) / \Delta(\bigoplus_{i+j=k} \text{Ext}^i_R(M, R) \otimes_K \text{Ext}^j_R(N, R))$$

$$\cong \text{Ext}^k_S(M \otimes_K N, S) / \Delta \text{Ext}^k_S(M \otimes_K N, S).$$
Since $\Delta$ has $n+1$ minimal generators and $M$ and $N$ intersect very properly we see that $\Delta$ is a parameter ideal for all $\text{Ext}_S^k(M \otimes_K N, S)$ and for $M \otimes_K N$. Therefore Proposition 2.4 shows that
\[(M \otimes_K N)/\Delta(M \otimes_K N)^{sm} \cong (M \otimes_R N)^{sm}\]
is an unmixed module proving the first claim.
In order to show the claim on the degrees we show firstly
\[\deg M \otimes_K N = \deg M \cdot \deg N.\]
Let $e$ and $f$ denote the dimension of $M$ and $N$, respectively. For all integers $t \geq 0$ we have the following relation of Hilbert functions:
\[h_{M \otimes_K N}(t) = \sum_{i+j=t} h_M(i) \cdot h_N(j).\]
Let $r$ be an integer such that the Hilbert functions of $M, N$ in degree $i > r$ are given by their respective Hilbert polynomials. Then we obtain for $t \geq 2r$:
\[h_{M \otimes_K N}(t) = \sum_{i=0}^{t} p_M(i) \cdot p_N(t - i) + \sum_{i=0}^{r} [h_M(i) - p_M(i)] \cdot p_N(t - i)\]
\[+ \sum_{i=t-r}^{t} p_M(i) \cdot [h_N(t - i) - p_N(t - i)]\]
\[= \deg M \cdot \deg N \cdot \sum_{i=0}^{t} \binom{i}{e-1} \cdot \binom{t-i}{f-1} + O(t^{e+f-2})\]
\[= \deg M \cdot \deg N \cdot \binom{t}{e+f-1} + O(t^{e+f-2}).\]
It follows that $\deg M \otimes_K N = \deg M \cdot \deg N$ as claimed.
Above we have already applied Proposition 2.4. Now we use its full strength. It provides a minimal basis $\{h_0, \ldots, h_n\}$ of $\Delta$ such that
\[((M \otimes_K N)/(h_0, \ldots, h_i)(M \otimes_K N)^{sm}\]
is an unmixed module for all $i = 0, \ldots, n$. This implies
\[\deg M \otimes_K N = \deg(M \otimes_K N)/(h_0, \ldots, h_n)(M \otimes_K N) = \deg M \otimes_R N\]
according to our assumptions and the previous lemma.

Note that the degree relation in the last statement is very much in the spirit of Bezout's original result on the intersection of plane curves. Nowadays it is well-known that a similar formula cannot be true for arbitrary proper intersections. In general one has to replace the length multiplicity by a suitable intersection multiplicity (cf. [10], [13], [9], Theorem 2.8, [5] and [15]). However, our result gives a condition when the simplest multiplicity works and in that case the extra information that the intersection is unmixed up to an irrelevant component.
Example 4.5. Let $X \subset \mathbb{P}^3$ be the union of two skew lines defined by the ideal $I = (x_0, x_1) \cap (x_2, x_3)$ of the polynomial ring $R = K[x_0, \ldots, x_3]$ and let $Y \subset \mathbb{P}^3$ be the complete intersection defined by $J = (x_1 + x_2, x_0 + x_3)$. Then $M = R/I$ and $N = R/J$ intersect very properly in dimension zero. However, it holds

$$\deg M \otimes_R N = 3 \neq 2 = \deg M \cdot \deg N.$$  

Hence this example shows that in general the degree relation does not hold true in the preceding statement if $\dim M \otimes_R N = 0$.

The previous result implies for subschemes of $\mathbb{P}^n$.

Corollary 4.6. Let $X$ and $Y$ denote equidimensional subschemes of $\mathbb{P}^n$ which intersect very properly. Then $X \cap Y$ is an equidimensional subscheme.

Moreover, if $X \cap Y \neq \emptyset$ then we have

$$\deg X \cap Y = \deg X \cdot \deg Y.$$  

A special case of Bertini’s theorem says that a sufficiently general hyperplane section of an equidimensional subscheme is again equidimensional. The first claim generalizes this statement and makes the condition “sufficiently general” more precise. It implies in particular that $X \cap Y$ does not have embedded components.

Taking Proposition 3.6 into account the proof of Theorem 4.4 shows.

Proposition 4.7. Let $M$ and $N$ be unmixed modules which intersect very properly. Then it holds

$$\text{depth } M \otimes_R N = \max\{0, \text{depth } M + \text{depth } N - (n + 1)\}.$$  

If $\text{depth } M + \text{depth } N \geq n + 1$ the proposition implies

$$\text{depth } M + \text{depth } N = \text{depth } R + \text{depth } M \otimes_R N,$$

i.e., the modules $M$ and $N$ satisfy the depth formula in the sense of Huneke and Wiegand [8].

5. Lifting the Cohen-Macaulay property

Here we look for conditions/results which say essentially that the Cohen-Macaulay property of the very proper intersection of $M$ and $N$ forces $M$ and $N$ to be Cohen-Macaulay modules. We use the same notation as in the previous section.

A first result follows immediately from the diagonal trick. In fact, if $M$ and $N$ intersect properly then any minimal generating set of the diagonal ideal is a subsystem of a system of parameters for $M \otimes_K N$. Thus the isomorphism $M \otimes_R N \cong (M \otimes_K N)/\Delta(M \otimes_K N)$ implies that $M \otimes_K N$ is Cohen-Macaulay. Thus we have seen.

Lemma 5.1. Let $M$ and $N$ denote modules which intersect properly. If $M \otimes_R N$ is Cohen-Macaulay of positive dimension then $M$ and $N$ are Cohen-Macaulay.

Remark 5.2. The result implies for subschemes $X, Y \subset \mathbb{P}^n$: If they meet properly in a non-empty scheme and $R/(I_X + I_Y)$ is Cohen-Macaulay then $X$ and $Y$ are arithmetically Cohen-Macaulay. For a result where we only assume that $X \cap Y$ is arithmetically Cohen-Macaulay, we refer to Corollary 5.3.
Note that in case $R/(I_X + I_Y)$ is Cohen-Macaulay, the scheme $X \cap Y$ is arithmetically Cohen-Macaulay. In general, the converse is not true because the ideal $I_X + I_Y$ is not necessarily saturated.

In order to generalize the statement above we begin with the following observation.

**Lemma 5.3.** Let $M$ denote an unmixed graded $R$-module and let $f \in R$ be a homogeneous $M$-regular element. Suppose that $(M/fM)^{sm}$ has depth $t \geq 2$. Then we have $\text{depth } M = t + 1$ and $M/fM = (M/fM)^{sm}$.

In particular, if $(M/fM)^{sm}$ is a Cohen-Macaulay module of dimension $\geq 2$ then $M$ is Cohen-Macaulay.

**Proof.** In the special case where $M$ is a ring our claim follows from [6], Proposition 2.1. The proof in the general case is similar to those of Huneke and Ulrich. Since it is short we give it for the reader’s convenience.

By the definition of a slight modification we have an exact sequence

$$0 \rightarrow H_m^0(M/fM) \rightarrow M/fM \rightarrow (M/fM)^{sm} \rightarrow 0$$

which implies $H_m^i(M/fM) \cong H_m^i((M/fM)^{sm})$ for all $i \geq 1$. In particular, it follows $H_m^1(M/fM) = 0$.

Let $k$ denote the degree of $f$. Then the exact sequence

$$0 \rightarrow M(-k) \xrightarrow{f} M \rightarrow M/fM \rightarrow 0$$

provides the exact cohomology sequence

$$0 \rightarrow H_m^0(M/fM) \rightarrow H_m^1(M)(-k) \xrightarrow{f} H_m^1(M) \rightarrow H_m^1(M/fM) \rightarrow \ldots .$$

According to Lemma 2.1 the module $H_m^1(M)$ is finitely generated. Since $H_m^1(M/fM) = 0$, Nakayama’s lemma yields $H_m^1(M) = 0$. It follows that $H_m^0(M/fM) = 0$ which implies our claims. 

Now we are ready for the main result of this section. It may be viewed as a vast generalization of the previous lemma. However, the lemma is used in the proof below.

**Theorem 5.4.** Let $M$ and $N$ denote unmixed modules which intersect very properly. Suppose that one of the following conditions is satisfied:

1. $(M \otimes_R N)^{sm}$ is Cohen-Macaulay and has dimension $\geq 2$.
2. $\dim M \otimes_R N = 0$ and $\deg M \otimes_R N = \deg M \cdot \deg N$.

Then $M$ and $N$ are Cohen-Macaulay modules.

**Proof.** Since $M$ and $N$ intersect very properly we can apply Proposition 2.4 as in the proof of Theorem 4.4. Thus there is a minimal basis $\{h_0, \ldots, h_n\}$ of the diagonal ideal $\Delta$ consisting of linear forms such that

$$( (M \otimes_K N)/(h_0, \ldots, h_i)(M \otimes_K N) )^{sm}$$

is an unmixed module for all $i = 0, \ldots, n$. Moreover we have

$$(M \otimes_K N)/(h_0, \ldots, h_n)(M \otimes_K N) \cong M \otimes_R N .$$

Now suppose that assumption (ii) is fulfilled. Then $(M \otimes_K N)/(h_0, \ldots, h_{n-1})(M \otimes_K N)$ is a module of dimension one and degree $\deg M \cdot \deg N$. Since the module $M \otimes_R$
N has the same degree we conclude by Lemma 4.3 that the irrelevant maximal ideal is not an associated prime ideal of \((M \otimes_K N)/(h_0, \ldots, h_{n-1})(M \otimes_K N)\), i.e., \((M \otimes_K N)/(h_0, \ldots, h_{n-1})(M \otimes_K N)\) is Cohen-Macaulay. It follows that \(M \otimes_K N\) is Cohen-Macaulay.

Next, assume that condition (i) is satisfied. Then we can apply Lemma 5.3 successively. It follows that \((M \otimes_K N)_{\text{sm}}\) is Cohen-Macaulay. Due to Proposition 3.5(a) we know that \(M \otimes_K N\) is an unmixed module, thus \((M \otimes_K N)_{\text{sm}} = M \otimes_K N\).

Therefore we have seen that any of the conditions (i) and (ii) implies that \(M \otimes_K N\) is a Cohen-Macaulay module. Thus Lemma 5.3(a) proves our assertion.

The next result was one of the driving forces of this note.

**Corollary 5.5.** Let \(X, Y \subset \mathbb{P}^n\) denote equidimensional subschemes which meet very properly in an arithmetically Cohen-Macaulay subscheme whose dimension is at least one. Then \(X\) and \(Y\) are arithmetically Cohen-Macaulay.

Furthermore, if the Cohen-Macaulay type of \(X \cap Y\) is a prime number then \(X\) or \(Y\) is arithmetically Gorenstein. If \(X \cap Y\) is arithmetically Gorenstein then \(X\) and \(Y\) are arithmetically Gorenstein.

**Proof.** With regard to the previous theorem it suffices to note that the Cohen-Macaulay type of \(X \cap Y\) is the product of the types of \(X\) and \(Y\).

**Remark 5.6.** The theorem and its corollary do not hold true when we replace very proper intersection by the weaker assumption of a proper intersection. Consider the following example: Let \(\hat{X}, \hat{Y} \subset \mathbb{P}^5\) be the cones over the corresponding subschemes \(X, Y \subset \mathbb{P}^3\) in Example 4.3. Then \(\hat{X}\) is locally Cohen-Macaulay at all points except at its vertex. Thus \(\hat{X}\) and \(\hat{Y}\) meet properly but not very properly. The intersection is an arithmetically Cohen-Macaulay curve but \(X\) is not even locally Cohen-Macaulay.

### 6. Some splitting criteria for reflexive sheaves

In this section we apply the methods, which we have developed in the previous sections, in order to study tensor products of reflexive sheaves on projective space.

To start with we note that two maximal \(R\)-modules \(M, N\) intersect always properly. According to Lemma 4.2 \(M\) and \(N\) meet even very properly if \(M\) and \(N\) are locally free.

Recall that the module \(H^0(M)\) is defined as

\[
H^0(M) = \lim_{\rightarrow} \text{Hom}_R(m^t, M).
\]

It is again a graded module which fits into the exact sequence

\[
0 \to H^0_m(M) \to M \to H^0(M) \to H^1_m(M) \to 0.
\]

In particular it holds (cf., for example, [17], Lemma 0.1.8)

\[
H^i_m(H^0(M)) \cong \begin{cases} H^i_m(M) & \text{if } i \geq 2 \\ 0 & \text{if } i \leq 1. \end{cases}
\]

The next result shows essentially that the tensor product has nice properties if the two modules intersect very properly.
Proposition 6.1. Let $M$ and $N$ be maximal $R$-modules which intersect very properly. Then we have:

(a) If $M$ and $N$ are torsion-free then $(M \otimes_R N)^{\text{sm}}$ is torsion-free.
(b) If $M$ and $N$ are reflexive then $H^0(M \otimes_R N)$ is reflexive.

Proof. Claim (a) is a special case of Theorem 4.4 because a maximal $R$-module is torsion-free if and only if it is unmixed.

In order to prove claim (b) we recall that a maximal $R$-module $E$ is reflexive if and only if $\dim R/a_i(E) \leq i - 2$ for all $i < \dim R = n + 1$ (cf. [4], Theorem 3.6). Thus, the Künneth formulas provide that the join $M \otimes_K N$ is a reflexive $(R \otimes_K R)$-module. Since $M$ and $N$ intersect very properly the diagonal $\Delta$ is a parameter ideal for $M \otimes_K N$ and all $\text{Ext}^i_{R \otimes_K R}(M \otimes_K N, R \otimes_K R)$ where $i \neq 0$. Thus, arguing as in the proof of Proposition 3.6 we obtain

$$\dim \text{Ext}^{n+1-i}_R(M \otimes_R N, R) \leq \max\{0, i - 2\} \quad \text{for all } i \neq 0.$$ 

Hence $H^0(M \otimes_R N)$ is a reflexive $R$-module. \qed

In order to discuss the last result we consider some examples where we used the computer algebra program MACAULAY [2] for carrying out the computations.

Example 6.2. Let $I_1$ and $I_2$ denote the homogeneous ideals in $R = K[x_0, \ldots, x_3]$ of two different points in $\mathbb{P}^3$. Let $E_1$ and $E_2$ be the first syzygy modules of $I_1$ and $I_2$, respectively.

Then $I_1 \otimes_R I_1$ is a proper but not a very proper intersection. It turns out that $I_1 \otimes_R I_1 \cong (I_1 \otimes_R I_1)^{\text{sm}}$ is not torsion-free.

Now let us consider $I_1 \otimes_R I_2$. It is a very proper intersection. However, it is not torsion-free. On the other hand $(I_1 \otimes_R I_2)^{\text{sm}}$ is torsion-free, but not reflexive, whereas $H^0(I_1 \otimes_R I_2)$ is even a reflexive $R$-module.

Looking at the reflexive modules $E_1$ and $E_2$ we observe that $E_1 \otimes_R E_1$ is a proper but not a very proper intersection. Furthermore, $E_1 \otimes_R E_1 \cong (E_1 \otimes_R E_1)^{\text{sm}}$ is not reflexive. The situation is different for $E_1 \otimes_R E_2$. It is a very proper intersection and $E_1 \otimes_R E_2 \cong H^0(E_1 \otimes_R E_2)$ is a reflexive $R$-module.

Remark 6.3. The example above shows that the conclusions in Proposition 6.1 are not true if we don’t assume that the intersections are very proper.

Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^n$. As usual we put for all integers $i$

$$H^i_*(\mathcal{F}) = \oplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t)).$$

We say that sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^n$ intersect very properly if the modules $H^0_*(\mathcal{F})$ and $H^0_*(\mathcal{G})$ meet very properly. Then Proposition 6.1 implies.

Corollary 6.4. Let $\mathcal{E}, \mathcal{F}$ denote coherent sheaves on $\mathbb{P}^n$ which intersect very properly. Then we have:

(a) If $\mathcal{F}$ and $\mathcal{G}$ are torsion-free then $\mathcal{F} \otimes \mathcal{G}$ is torsion-free.
(b) If $\mathcal{F}$ and $\mathcal{G}$ are reflexive then $\mathcal{F} \otimes \mathcal{G}$ is reflexive.
Remark 6.5. (i) Again, Example 6.2 shows that the statements are no longer true if the intersections are not very proper.

(ii) Auslander has shown in [1], Lemma 3.1 that the torsion-freeness of $M \otimes_R N$ implies that $M$ and $N$ are torsion-free. Huneke and Wiegand ([7], Theorem 2.7) proved that $M$ and $N$ are reflexive if $M \otimes_R N$ is reflexive.

A corresponding statement does not hold for sheaves. In fact, the example above shows that the reflexivity of $E \otimes F$ does in general not imply that $E$ and $F$ are reflexive too.

As preparation for our last statement we need a result which is analogous to Lemma 5.3.

Lemma 6.6. Let $M$ denote a reflexive $R$-module and let $f \in R$ be a homogeneous $M$-regular element. Suppose that $H^0(M/fM)$ has depth $t \geq 3$. Then $M$ has depth $t + 1$ and $H^0(M/fM) = M/fM$.

Proof. Multiplication by $f$ provides the exact cohomology sequence

$$0 \to H^0_m(M/fM) \to H^1_m(M) \xrightarrow{f} H^1_m(M) \to H^1_m(M/fM) \to H^2_m(M(-k)) \xrightarrow{f} H^2_m(M) \to H^2_m(M/fM) \to \ldots.$$ 

Since we have

$$H^2_m(M/fM) \cong H^2_m(H^0_m(M/fM)) = 0$$

by assumption on the depth $t$ the multiplication by $f$ is surjective on $H^2_m(M)$. But $H^2_m(M)$ is a finitely generated $R$-module because $M$ is reflexive. Hence Nakayama’s lemma implies $H^2_m(M) = 0$. Furthermore, the reflexivity of $M$ provides $H^1_m(M) = 0$. Thus, considering the exact sequence above we conclude that

$$H^0_m(M/fM) = H^1_m(M/fM) = 0.$$

It follows that $H^0_m(M/fM) \cong M/fM$ and we are done.

For short we say that a coherent sheaf $F$ on $\mathbb{P}^n$ splits if it is a direct sum of line bundles. This holds true if and only if $H^0_*(F)$ is a finitely generated free $R$-module.

Proposition 6.7. Let $\mathcal{E}, \mathcal{F}$ denote reflexive sheaves on $\mathbb{P}^n$ $(n \geq 1)$.

(a) If $\mathcal{E} \otimes \mathcal{F}$ splits and $\mathcal{E}$ and $\mathcal{F}$ intersect very properly then $\mathcal{E}$ and $\mathcal{F}$ split.

(b) If $\mathcal{E}$ is a vector bundle such that $H^1_*(\mathcal{E} \otimes \mathcal{E}^*) = 0$ then $\mathcal{E}$ splits.

Proof. (a) The assumption and Proposition 6.1 provide that the tensor product of $E = H^0_*(\mathcal{E})$ and $F = H^0_*(\mathcal{F})$ is a free $R$-module. Therefore, due to Theorem 5.4, $E$ and $F$ are Cohen-Macaulay, thus free.

(b) Put again $E = H^0_*(\mathcal{E})$. Our assumption implies depth $E \otimes_R E^* \geq 3$. Since $E \otimes_R E^* \cong (E \otimes_K E^*)/\Delta(E \otimes_K E^*)$ we conclude by Lemma 6.6 that depth $E \otimes_K E^* \geq n + 4$ and in particular $H^{n+2}_m(E \otimes_K E^*) = 0$. 
Let us denote the $K$-dual $\text{Hom}_K(M, K)$ of a graded $R$-module $M$ by $M^\vee$. The Künneth formulas and local duality provide

\[
H^{n+2}_m(E \otimes_K E^*) = \bigoplus_{i+j = n+2} H^i_m(E) \otimes_K H^j_m(E^*)
\]

\[
= \bigoplus_{i=2}^n H^i_m(E) \otimes_K H^i_m(E^*)^\vee(-n-1).
\]

Since each direct summand must be trivial we conclude that $H^i_m(E) = 0$ for all $i \leq n$, i.e., $E$ is a maximal Cohen-Macaulay module and thus free. \qed

Part (b) of the last statement has been proved over the complex numbers by Luk and Yau in \cite{10} using analytic tools. The general case has been first proved by Huneke and Wiegand (\cite{8}, Theorem 5.2) using their results in \cite{7}.

An analysis of the proof of claim (b) shows that its conclusion follows once we know that $[H^{n+2}_m(E \otimes_K E^*)]_{n+1} = 0$. Thus it seems conceivable that the hypothesis could be weakened by assuming only the vanishing of $H^1(E \otimes E^*(t))$ for a finite number of degrees $t$. We leave this as an open problem.

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: Juan.C.Migliore.1@nd.edu

Fachbereich Mathematik und Informatik, Universität-Gesamthochschule Paderborn, D–33095 Paderborn, Germany
E-mail address: uwen@uni-paderborn.de

Department of Mathematics, Washington University, St. Louis, MO 63130-4899
E-mail address: peterson@math.wustl.edu