Synchronization and separation in the Johnson schemes

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(joint with Mohammed Aljohani and John Bamberg)
Synchronization

You are in a dungeon consisting of a number of rooms. Passages are marked with coloured arrows. Each room contains a special door; in one room, the door leads to freedom, but in all the others, to death. You have a map of the dungeon, but you do not know where you are.
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You can check that (Blue, Red, Blue) takes you to room 1 no matter where you start.
Automata

The diagram on the last page shows a finite-state deterministic automaton. This is a machine with a finite set of states, and a finite set of transitions, each transition being a map from the set of states to itself. The machine starts in an arbitrary state, and reads a word over an alphabet consisting of labels for the transitions (Red and Blue in the example); each time it reads a letter, it undergoes the corresponding transition.
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A reset word is a word with the property that, if the automaton reads this word, it arrives at the same state, independent of its start state. An automaton which possesses a reset word is called synchronizing.
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Not every finite automaton has a reset word. The Černý conjecture asserts that, if an $n$-state automaton is synchronizing, then it has a reset word of length at most $(n - 1)^2$. (If true, this would be best possible.) The conjecture is still open after half a century, and has motivated a lot of work on synchronization.
Automata and transformation semigroups

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Algebraically, if $\Omega = \{1, \ldots, n\}$ is the set of states, then any transition is a map from $\Omega$ to itself. Reading a word composes the corresponding maps, so the set of maps corresponding to all words is a transformation semigroup on $\Omega$. So an automaton is a transformation semigroup with a distinguished generating set.

An automaton is synchronizing if and only if the transformation semigroup contains a map of rank 1, that is, whose image has cardinality 1.
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Let $\Gamma$ be a simple (undirected) graph on the vertex set $\Omega$. An **endomorphism** of $\Gamma$ is a map on $\Omega$ which takes edges of $\Gamma$ to edges of $\Gamma$; there is no restriction to what it does to a non-edge (which can map to a non-edge, or an edge, or collapse to a vertex).
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The endomorphisms of $\Gamma$ form a monoid, the **endomorphism monoid** of $\Gamma$, denoted by $\text{End}(\Gamma)$.
The obstruction to synchronization

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**Theorem**

* A transformation semigroup $S$ on $\Omega$ fails to be synchronizing if and only if there is a non-null graph $\Gamma$ on $\Omega$ with clique number equal to chromatic number (that is, with core a complete graph) such that $S \leq \text{End}(\Gamma)$.  

Proof of the theorem

It is clear that, if $S \leq \text{End}(\Gamma)$, where $\Gamma$ is a non-null graph, then $S$ is not synchronizing, since no edge of $\Gamma$ can be collapsed by an endomorphism.
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In the other direction, suppose that $S$ is a transformation semigroup on $\Omega$. Define a graph $\Gamma$ with vertex set $\Omega$ by joining $v$ to $w$ if and only if there is no element $s \in S$ for which $vs = ws$. ▶
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- \( S \) is synchronizing if and only if \( \Gamma \) is a null graph. (For if any two vertices can be mapped to the same place, then after \( n - 1 \) such steps we can map the whole of \( \Omega \) to a single point.)
$S \leq \text{End } \Gamma$. (Suppose that $v, w$ is an edge of $\Gamma$. By definition, there is no $s \in S$ with $vs = ws$. Could there exist $s \in S$ such that $\{vs, ws\}$ is a non-edge? If so then there exists $t \in S$ with $vst = wst$, a contradiction.)
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\[ \Gamma \text{ has clique number equal to chromatic number.} \] (Take an element \( s \in S \) of smallest possible rank. The image of \( s \) cannot be further collapsed, and so is a clique in \( \Gamma \); and the map \( s \) is then a colouring of \( \Gamma \) with the points of the image as colours.)
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This is abuse of language since a permutation group, regarded as a semigroup, can never be synchronizing!
From the semigroup result, we immediately obtain the following theorem.

**Theorem**

*The permutation group $G$ is non-synchronizing if and only if there is a non-trivial $G$-invariant graph $\Gamma$ with clique number equal to chromatic number.*
The obstruction for permutation groups

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Here a graph is *non-trivial* if it is neither complete nor null. Note that, in the previous proof, the graph associated with a semigroup is complete if and only if the semigroup is a permutation group, since any non-permutation will collapse some pair.
There is a closely related property, which can be phrased in terms of graphs as follows (this was not the original form). The transitive permutation group $G$ is non-separating if there is a non-trivial $G$-invariant graph for which the product of clique number and independence number is equal to the number of vertices. (For a vertex-transitive graph, the product of clique number and independence number cannot exceed the number of vertices.)
Separation

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If no such graph exists, then $G$ is **separating**.

Note that, if a vertex-transitive graph has clique number equal to chromatic number, then all the colour classes in a minimal colouring have the same size, so the product of clique and independence numbers is equal to the number of vertices. Thus,

*A separating permutation group is synchronizing.*
We now have the following hierarchy for permutation groups:

**Theorem**

\[ 2\text{-homogeneous} \Rightarrow \text{separating} \Rightarrow \text{synchronizing} \Rightarrow \text{primitive}. \]
The big problem

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**Question**

Determine the synchronizing (or separating) permutation groups.
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For more details about the status of this problem, and related concepts on permutation groups derived from semigroups, see J. Araújo, P. J. Cameron and B. Steinberg, “Between primitive and 2-transitive: synchronization and its friends”, *European Math. Soc. Surveys* 4 (2017), 101–184.
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$S_n$ on $k$-sets: the Johnson scheme
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So any $G$-invariant graph is defined by a subset $I$ of $\{0, \ldots, k - 1\}$, with two $k$-sets joined if their intersection belongs to $I$. Let us call this graph $\Gamma_I(n,k)$. We have to decide whether such graphs can have clique number equal to chromatic number, or product of clique number and independence number equal to $\binom{n}{k}$. 
The Erdős–Ko–Rado theorem (proved 1938, published 1961) says that, for $n$ sufficiently large in terms of $k$ and $t$, the largest size of a family of $t$-intersecting $k$-subsets of $\{1, \ldots, n\}$ is $\binom{n-t}{k-t}$, and is realised by the family of $k$-sets containing a fixed $t$-set.
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Steiner systems

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The $k$-sets containing a fixed $t$-set form a clique in this graph, (of Erdős–Ko–Rado type, or EKR type). A short calculation shows that and the product of the sizes of these sets is $\binom{n}{k}$.
Thus:

Proposition

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In particular, if $k | n$, then a partition of \( \{1, \ldots, n\} \) into $k$-sets is a Steiner system $S(1, k, n)$, and $S_n$ on $k$-sets is not separating.
The conjecture

Conjecture

There is a function $F$ such that, if $n > F(k)$, then $S_n$ acting on $k$-sets is non-separating if and only if a Steiner system $S(t,k,n)$ exists for some $t$ with $0 < t < k$. 
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In other words, out of all the graphs $\Gamma_I(n, k)$, the only ones that matter for large $n$ are those with $I = \{0, \ldots, t - 1\}$ or $I = \{t, \ldots, k - 1\}$. 
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So the conjecture can be re-phrased: for $n > G(k)$, $S_n$ on $k$-sets is non-separating if and only if the divisibility conditions hold for some $t$ with $0 < t < k$. 

And what about synchronizing?

There is a similar conjecture. A large set of Steiner systems $S(t, k, n)$ is a partition of the set of $k$-subsets of an $n$-set into Steiner systems. If a large set exists, then $S_n$ on $k$-sets is not synchronizing.
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There is a function $H$ such that, for $n > H(k)$, $S_n$ acting on $k$-sets is non-synchronizing if and only if a large set of Steiner systems $S(t, k, n)$ exists for some $t$ with $0 < t < k$. 
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Less is known about the existence of large sets, and we do not feel confident enough to conjecture an analogue of Keevash’s theorem for them.
Large sets

Two results on large sets, which show that $S_n$ on $k$-sets is non-synchronizing for these values:

Theorem

- (Baranyai) If $k$ divides $n$, then there is a large set of $S(1, k, n)$ systems (a 1-factorisation of the complete $k$-uniform hypergraph).
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- (Lu–Teirlinck) For $n \equiv 1$ or $3 \pmod{6}$, $n > 7$, there is a large set of Steiner triple systems ($S(2, 3, n)$ systems).
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- *(Lu–Teirlinck)* For $n \equiv 1$ or $3 \pmod{6}$, $n > 7$, there is a large set of *Steiner triple systems* ($S(2, 3, n)$ systems).

Note that $n \equiv 1$ or $3 \pmod{6}$ are necessary and sufficient conditions, due to Kirkman, for the existence of $S(2, 3, n)$. The nonexistence of a large set for $n = 7$ was shown by Cayley (indeed there do not exist more than two disjoint $S(2, 3, 7)$s).
Results

The separation conjecture is true for $k \leq 5$. Here are precise results for $k \leq 4$; the case $k = 5$ has been completed by Mohammed Aljohani but not yet published.

Theorem

- For $n \geq 5$, $S_n$ acting on 2-sets is synchronizing if and only if it is separating; this occurs if and only if $n$ is odd.

- For $n \geq 7$, $S_n$ on 3-sets is synchronizing if and only if it is separating; this occurs if and only if $n \equiv 2, 4, 5 \pmod{6}$ and $n > 8$.

- For $n \geq 9$, $S_n$ on 4-sets is synchronizing if and only if it is separating; this occurs if and only if $n \equiv 3, 5, 6, 7, 9, 11 \pmod{12}$ and $n > 9$. 
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- For $n \geq 9$, $S_n$ on 4-sets is synchronizing if and only if it is separating; this occurs if and only if $n \equiv 3, 5, 6, 7, 9 \text{ or } 11 \pmod{12}$ and $n > 9$. 
These agree with the conjecture. e.g. for \( k = 4 \), by results of Hanani, the necessary and sufficient conditions for the existence of \( S(t, 4, n) \) are \( n \equiv 0 \pmod{4} \) for \( t = 1 \), \( n \equiv 1 \) or \( 4 \pmod{12} \) for \( t = 2 \), and \( n \equiv 2 \) or \( 4 \pmod{6} \) for \( t = 3 \).
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The lines of the Fano plane form a 7-clique in \( \Gamma_{\{1\}}(3, 7) \) (any two meet in a point). For each line \( L \), the set of five 3-sets consisting of \( L \) and the four 3-sets disjoint from it form a coclique in this graph, and the seven such cocliques as \( L \) ranges over the lines is a 7-colouring of the graph.
Proof tools

The main tool is a theorem of Delsarte:

**Theorem**

*Let* $A$ *be an association scheme on* $v$ *vertices and let* $\Gamma$ *be the union of some of the graphs in the scheme. If* $C$ *is a clique and* $S$ *is an coclique in* $\Gamma$, *then* $|C| \cdot |S| \leq v$. *If equality holds and* $x$ *and* $y$ *are the respective characteristic vectors of* $C$ *and* $S$, *then*

\[(x E_j x^\top)(y E_j y^\top) = 0 \text{ for all } j > 0,\] *where* $E_0, E_1, \ldots$ *are the minimal idempotents in the Bose–Mesner algebra of the scheme.*
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In order to apply this, we need expressions for the minimal idempotents in terms of the basis matrices of the algebra. These are given by the **Q-matrix** of the scheme.
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In order to apply this, we need expressions for the minimal idempotents in terms of the basis matrices of the algebra. These are given by the **Q-matrix** of the scheme. For the Johnson scheme, the entries of the Q-matrix are expressed in terms of the **Eberlein polynomials**. This can also be found in Delsarte’s thesis.
Exceptions

We saw in the theorem earlier that there are exceptions for $k = 3, n = 7, 8$, and for $k = 4, n = 9$. We saw the case $k = 3, n = 7$ earlier.
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For $k = 3, n = 8,$ the Fano plane $S(2, 3, 7)$ is a 7-clique in the graph corresponding to intersection 1. A 7-colouring of this graph is given by the extension $S(3, 4, 8)$: for each of the 7 parallel classes of blocks, give a colour to the 3-subsets of the two blocks in this class.
For $k = 4, n = 9$, there is an overlarge set of $S(3, 4, 8)$ systems on 9 points, a partition of the 4-sets into 9 such systems, each omitting one point, found by Breach and Street. This is a 9-colouring of the graph on 4-sets corresponding to intersections 1 and 3. It is straightforward to find a 9-clique in this graph.
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Breach and Street found computationally that there are just two overlarge sets up to isomorphism, each admitting a doubly transitive automorphism group. Praeger and I found a more conceptual proof, using the geometric phenomenon of triality on the hyperbolic quadric in PG(7, 2).
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So small exceptions are often very beautiful configurations! The pattern continues, since examples arising from $S(4, 5, 11)$ give exceptions to the conjecture in the case $k = 5, n = 11$ or $n = 12$. 
Other association schemes

**Hamming schemes:** Since the Hamming graph $H(n, q)$ has clique number and chromatic number $q$, it is non-synchronizing. (Indeed, primitive groups which are non-basic (that is, contained in a wreath product with the product action) are non-synchronizing.)
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**$q$-Johnson schemes:** Here similar considerations apply to the Johnson schemes. But the theory of Steiner systems in $q$-Johnson schemes is in its infancy, the first nontrivial example having been found by Michael Braun, Tuvi Etzion, Patric R. J. Östergård and Alexander Vardy in 2016. There are hard open problems here!
Polar spaces: Only the case of points has been studied. A classical polar space is non-synchronizing if and only if it has either an ovoid and a spread, or a partition into ovoids; it is non-separating if it has an ovoid. The complete solution to which polar spaces have these properties is not yet known despite many decades of research.
Polar spaces: Only the case of points has been studied. A classical polar space is non-synchronizing if and only if it has either an ovoid and a spread, or a partition into ovoids; it is non-separating if it has an ovoid. The complete solution to which polar spaces have these properties is not yet known despite many decades of research. However, it is known that for the parabolic quadric in \(\text{PG}(4, q)\) for odd \(q\) (corresponding to the orthogonal group \(O_5(q)\)), there exist ovoids but no spreads and no partitions into ovoids. This gives an infinite family of groups which are synchronizing but not separating – the only such examples known apart from two exceptions, \(G_2(2)\) and its derived group.
**Diagonal groups:** The diagonal groups (one of the cases in the O’Nan–Scott theorem) with more than two factors are non-synchronizing; the proof of this is elementary for \( n \) even, but for \( n \) odd it uses the **Hall–Paige conjecture** (proved fairly recently using the Classification of Finite Simple Groups).

For two factors, the property depends on factorisation properties of finite simple groups. Mohammed Aljohani is working on this, but currently we do not have an example of such a group which is synchronizing.

Others: Pick your favourite family of permutation groups or association schemes. Which ones are synchronizing, which are separating? The answer is probably not known! (The synchronization property is closed under coarsening, and a synchronizing scheme is primitive – that is, all relations are connected.)
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