Qudit hypergraph states

F. E. S. Steinhoff,1,2,3 C. Ritz,1 N. I. Miklin,1 and O. Gühne1

1Naturwissenschaftlich Technische Fakultät, Universität Siegen, Walter-Flex-Str. 3, D-57068 Siegen, Germany
2Instituto de Física, Universidade Federal de Goiás, 74001-970, Goiânia, Goiás, Brazil
3Instituto de Engenharia, Universidade Federal de Mato Grosso, 78060-900 Várzea Grande, Mato Grosso, Brazil

(Dated: January 23, 2017)

We generalize the class of hypergraph states to multipartite systems of qudits, by means of constructions based on the $d$-dimensional Pauli group and its normalizer. For simple hypergraphs, the different equivalence classes under local operations are shown to be governed by a greatest common divisor hierarchy. Moreover, the special cases of three qutrits and three ququarts is analysed in detail.

PACS numbers: 03.65.Ud, 03.67.Mn

I. INTRODUCTION

The physical properties of multipartite systems are highly relevant for practical applications as well as foundational aspects. Despite their importance, multipartite systems are in general very complex to describe and little analytical knowledge is available in the literature. Well-known examples in many-body physics are the various spin models, which are simple to write down, but where typically not all properties can be determined analytically. The entanglement properties of multipartite systems are no exception and already for pure states it is known that a complete characterization is, in general, not a feasible task [1, 2]. This motivates the adoption of simplifications that enable analytical results or at least to infer properties in a numerically efficient way.

One approach in this direction with broad impact in the literature is that based on a graph state encoding [3]. Mathematically, a graph consists of a set of vertices and a set of edges connecting the vertices. Graph states are a class of genuinely multipartite entangled states that are represented by graphs. This class contains as a special case the whole class of cluster states, which are the key ingredients in paradigms of quantum computing, e.g., the one-way quantum computer [4] and quantum error correction [5] or for the derivation of Bell inequalities [6].

Recently, there has been an interest in the generalization of graph states to a broader class of states known as hypergraph states [9]. In a hypergraph an edge can connect more than two vertices, so hypergraph states are associated with many-body interactions beyond the usual two-body ones. Interestingly, the mathematical description of hypergraph states is still very simple and elegant and in Ref. [11] a full classification of the local unitary equivalence classes of hypergraphs states up to four qubits was obtained. Also, in Refs. [11, 12] Bell and Kochen-Specker inequalities have been derived and it has been shown that some hypergraph states violate local realism in a way that is exponentially increasing with the number of qubits. Finally, recent studies in condensed matter theory showed that this class of states occur naturally in physical systems associated with topological phases [10]. Originally, hypergraph states were defined as members of an even broader class of states known as locally maximally entangleable (LME) states [13], which are associated to applications such as quantum fingerprinting protocols [14]. Hypergraph states are then known as $\pi$-LME states and display the main important features of the general class of LME states.

Up to now, hypergraph states were defined only in the multi-qubit setting, while graph states can be defined in systems with arbitrary dimensions. In higher dimensions graph states have many interesting properties not present in the two-dimensional setting. For example, there are considerable differences between systems where the underlying local dimensions are a prime or non-prime [15]. Another difference is the construction of Bell-like arguments for higher-dimensional systems [16].

In the present work, we extend the definition of hypergraph states to multipartite systems of arbitrary dimensions (qudits) and analyse their entanglement properties. Especially, we focus on the equivalence relations under local unitary (LU) operations or under stochastic local operations assisted by classical communication (SLOCC). In particular, the possible local inter-conversions between different entangled hypergraph states are governed by a greatest-common-divisor hierarchy. Note that the whole class of qudit graph states is a special case of our formulation.
Figure 1. Example of a graph state represented by the multi-graph \( G = (V, E) \), where \( V = \{1, 2, 3, 4\} \) and \( E = \{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{1, 4\}, \{1, 4\}, \{2\}, \{2\}, \{3\}\}. The graph state in this case is \( |G\rangle = Z_{12}Z_{14}Z_{16}Z_{3456}|\rangle^\dagger \).

The paper is organized as follows: In Section II we start by giving a brief review of the concepts and results that are at the basis of our formulation. This includes a description of the Pauli and Clifford groups in a \( d \)-dimensional system, as well as a general look on qudit graph states. In Section III we introduce the definitions associated with qudit hypergraph states. Section IV presents some properties of the stabilizer formalism used for qudit hypergraph states. Section V introduces the problem of classifying the SLOCC and LU classes of hypergraph states, first describing the different techniques employed and then proving a series of results on this classification. Finally, we present some concrete examples in low dimensional tripartite systems, where already the main differences between systems of prime and non-prime dimensions become apparent. We reserve to the Appendices the related subjects of a phase-space description and local complementation of qudit graphs.

II. BACKGROUND AND BASIC DEFINITIONS

We consider an \( N \)-partite system \( \mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i \), where the subsystems \( \mathcal{H}_i \) have the same dimension \( d \). A graph is a pair \( G = (V, E) \), where \( V \) is the set of vertices and \( E \) is a set comprised of 2-element subsets of \( V \) called edges. Likewise, a hypergraph is a pair \( H = (V, E) \), where \( V \) are the vertices and \( E \) is a set comprised of subsets of \( V \) with arbitrary number of elements; a \( n \)-element \( e \) in \( E \) is called an \( n \)-hyperedge. In some sense, a hyperedge is an edge that can connect more than two vertices. A multi-(hyper)graph is a set where the (hyper)edges are allowed to appear repeated. An example of a multi-graph can be found in Fig. 1, while one of a multi-hypergraph can be found in Fig. 2. Given two integers \( m \) and \( n \), their greatest common divisor will be denoted by \( \text{gcd}(m, n) \). The integers modulo \( n \) will be denoted as \( \mathbb{Z}_n \).

A. The Pauli group and its normalizer

Taking inspiration in the formulation of qubit hypergraph states, we adopt here the description based on the Pauli and Clifford groups in finite dimensions. In a \( d \)-dimensional system with computational basis \( \{|q\rangle\}_{q=0}^{d-1} \),

let us consider the unitary operators given by

\[ Z = \sum_{q=0}^{d-1} \omega^q |q\rangle\langle q|; \quad X = \sum_{q=0}^{d-1} |q \oplus 1\rangle\langle q| \]  \hspace{1cm} (1)

with the properties \( X^d = Z^d = I \) and \( X^nZ^n = e^{-\frac{\pi i}{d}} \), \( X = e^{\frac{\pi i}{d}} \), \( Q = \sum_{q=0}^{d-1} |q\rangle\langle q| \) and \( P = \sum_{q=0}^{d-1} |p_q\rangle\langle p_q| \) are discrete versions of the position and momentum operators; here \( |p_q\rangle = F|q\rangle \) and \( F = d^{-1/2} \sum_{q=0}^{d-1} \omega^q |q\rangle\langle q| \) is the discrete Fourier transform. Thus, \( X \) performs displacements in the computational (position) basis, while \( Z \) performs displacements in its Fourier transformed (momentum) basis.

Another set of important operators are the so-called Clifford or symplectic operators, defined as

\[ S(\xi, 0, 0) = \sum_{q=0}^{d-1} |\xi q\rangle\langle q|; \hspace{1cm} (2) \]

\[ S(1, \xi, 0) = \sum_{q=0}^{d-1} \omega^{\xi q^2 q^{-1}} |q\rangle\langle q|; \hspace{1cm} (3) \]

\[ S(1, 0, \xi) = \sum_{q=0}^{d-1} \omega^{-\xi q^2 q^{-1}} |p_q\rangle\langle p_q| \]. \hspace{1cm} (4)

These operators are invertible and unitary whenever the values of \( \xi \) and \( d \) are coprime (see proof of Lemma 2) and generate the normalizer of the Pauli group, which is usually referred as the Clifford group. The interested reader can check a more broad formulation in terms of a discrete phase-space in the Appendix A or in the Ref. [15].
B. Qudit graph states

We briefly review the theory of the so-called qudit graph states, which is well established in the literature [17]. The mathematical object used is a multi-graph $G = (V, E)$; we call $m_e \in \mathbb{Z}_d$ the multiplicity of the edge $e$, i.e., the number of times the edge appears. Given a multi-graph $G = (V, E)$, we associate a quantum state $\ket{G}$ in a $d$-dimensional system in the following way:

- To each vertex $i \in V$ we associate a local state $\ket{+} = \ket{\psi_0} = d^{-1/2} \sum_{q=0}^{d-1} \ket{q}$.
- For each edge $e = \{i, j\}$ and multiplicity $m_e$ we apply the unitary
  \[
  Z_e^{m_e} = \sum_{q=0}^{d-1} \ket{q_i} \otimes (Z_j^{m_e})^q
  \] (5)
  on the state $\ket{+}^V = \bigotimes_{i \in V} \ket{+}_i$. Thus, the graph state is defined as
  \[
  \ket{G} = \prod_{e \in E} Z_e^{m_e} \ket{+}^V.
  \] (6)

We allow among the edges $e \in E$ the presence of “loops”, i.e., an edge that contains only a single vertex. A loop of multiplicity $m$ on vertex $k$ means here that a local gate $(Z_k)^m$ is applied to the graph state. An example of a qudit graph state in a system with dimension $d > 3$ is shown in Fig. 1.

An equivalent way of defining a qudit graph state is via the stabilizer formalism [17]. Given a multi-graph $G = (V, E)$, define for each vertex $i \in V$ the operator $K_i = X_i \prod_{e \in E} Z_e^{m_e(i)}$. The set $K_i$ generates an abelian group known as the stabilizer. The unique $+1$ common eigenstate of these operators is precisely the state $\ket{G}$ associated to the multi-graph $G$. Moreover, the set of common eigenstates of these operators form a basis of the global state space, the so-called graph state basis.

The local action of the generalized Pauli group on a graph state is easy to picture and clearly preserves the graph state structure. As already said, the action of $Z_i^m$ corresponds to a loop of multiplicity $m$ on the qubit $i$, while the action of $X_i^m$ corresponds to loops of multiplicity $m$ on the qubits in the neighbourhood of the qubit $m$; this last observation is a corollary of Lemma 1.

The action of the local Clifford group is richer and enables the conversion between different multi-graphs in a simple fashion. For prime dimensions, the action of the gate $S_e(\xi, 0, 0)$ enables the conversion between edges of different multiplicities, while the gate $S_e(1, -1, 0)$ is associated to the operation known as local complementation - see Appendix B. Moreover, in non-prime dimensions the possible conversions between edges are governed by a greatest common divisor hierarchy, as show in more generality ahead - see Proposition 2 and Theorem 4.

III. QUDIT HYPERGRAPH STATES

We introduce now the class of hypergraph states in a system with underlying finite dimension $d$. Before proceeding, we need first the concept of a controlled operation on a multipartite system. From a given local operation $M$ one can define a controlled operation $M_{ij}$ between qudits $i$ and $j$ as

\[
M_{ij} = \sum_{q=0}^{d-1} \ket{q_i} \bra{q_i} \otimes M_j^q
\] (7)

Likewise, a controlled operation between three qudits, $i$, $j$ and $k$ is defined recursively as

\[
M_{ijk} = \sum_{q=0}^{d-1} \ket{q_i} \bra{q_i} \otimes M_{jk}^q
\] (8)

and in general the controlled operation between $n$ qudits labeled by $I = \{i_1, i_2, \ldots, i_n\}$ is given by

\[
M_I = M_{i_1i_2\ldots i_n} = \sum_{q=0}^{d-1} \ket{q_{i_1}} \bra{q_{i_1}} \otimes M_{i_2\ldots i_n}^q
\] (9)

A prominent example is the CNOT operation, which is simply the bipartite controlled operation generated by the $X$ gate - $\text{CNOT} = \sum q |q\rangle \langle q| \otimes X^q$. Although our formulation can be done in terms of this gate and its multipartite versions, it is preferable to use an equivalent formulation in terms of controlled-phase gates $Z_I$, since these gates are mutually commuting and thus are easy to handle. Explicitly, the controlled phase gate on $n$ particles is given by

\[
Z_e = \sum_{q_1=0}^{d} \ldots \sum_{q_{n-1}=0}^{d} |q_1 \ldots q_{n-1}\rangle \langle q_1 \ldots q_{n-1}| Z^{q_1\ldots q_{n-1}}
\]

\[
= \sum_{q_1=0}^{d} \ldots \sum_{q_{n}=0}^{d} |q_1 \ldots q_n\rangle \langle q_1 \ldots q_n| \omega^{q_1\ldots q_n}
\] (10)

The mathematical object used here to represent a given state is a multi-hypergraph $G = (V, E)$; as usual, we call $m_e \in \mathbb{Z}_d$ the multiplicity of the hyperedge $e$, i.e., the number of times the hyperedge appears. Given a multi-hypergraph $H = (V, E)$, we associate a quantum state $\ket{H}$ in a $d$-dimensional system in the following way:

- To each vertex $i \in V$ we associate a local state $\ket{+} = d^{-1/2} \sum_{q=0}^{d-1} \ket{q}$.
- For each hyperedge $e \in E$ with multiplicity $m_e$ we apply the controlled-unitary $Z_e^{m_e}$ on the state $\ket{+}^V = \bigotimes_{i \in V} \ket{+}_i$. Thus, the hypergraph state is defined as
  \[
  \ket{H} = \prod_e Z_e^{m_e} \ket{+}^V.
  \] (11)
We allow among the hyperedges $e \in E$ the presence of “loops”, i.e., an edge that contains only a single vertex. Also empty edges are allowed, they correspond to a global sign. A loop of multiplicity $m$ on vertex $k$ means here that a local gate $(Z_k)^m$ is applied to the graph state. An example of hypergraph state is illustrated in Fig. 2.

Equivalently, one can define a hypergraph state as the unique $+1$ eigenstate of a maximal set of commuting stabilizer operators $K_I = X_I \prod_{e \in E} Z_{e \setminus \{k\}}^i$; we explain this approach after the proof of Lemma 1 below.

An important special class of hypergraph states are the so-called $n$-elementary hypergraph states, which are those constituted of a single hyperedge $e$ between $n$ qudits, i.e., the state has the simple form $|H\rangle = Z_{e}^{m_e} |+\rangle^V$. For this subclass, the main entanglement properties depend on the multiplicity $m_e$ of the hyperedge, as shown in the next sections.

For completeness, we cite alternative formulations of hypergraph states that are potentially useful in other scenarios. First, we notice that the multiplicities of the hyperedges can be encoded as well in the adjacency matrix of the theory of graphs. For graph states, for example, the $\Gamma$ tensor is a matrix, the hyperedges can be encoded as well in the adjacency matrix of the theory of graphs. Many local quantum operations, especially those coming from the local Clifford group, are elegantly described as matrix operations over the adjacency matrix of the multi-graph $G$.

One can also work in the Schrödinger picture: the form of the state in the computational basis is given by:

$$|H\rangle = \sum_{q=0}^{d-1} \omega^f(q) |q\rangle,$$  \hspace{1cm} (12)

where $q \equiv (q_1, q_2, ..., q_n)$ and $f$ is a function from $\mathbb{Z}_d$ to $\mathbb{Z}_d$ defined by $f(q) = \bigoplus_{q_1, ..., q_n} \in E \cref k q_k$. For qubits, for example, the function $f$ is a Boolean function and this encoding is behind applications such as Deutsch-Jozsa and Grover’s algorithms [9].

\section{Properties of Hypergraph States and the Stabilizer Formalism}

In the following sections, we derive some properties of hypergraph states, the controlled $Z$ operation on many qudits and the stabilizer formalism. These tools will later be used for the SLOCC and LU classification.

\subsection{Local action of Pauli and Clifford groups}

We now consider the effect of unitaries from the Pauli and Clifford groups on a hypergraph state. First, we need some simple relations:

\textbf{Lemma 1.} The following relations hold:

$$X^i_k Z^j_k X_k = Z_{I \setminus \{k\}} Z_I \hspace{1cm} (13)$$

$$Z_I X_k Z^j_k = X_k Z_{I \setminus \{k\}} \hspace{1cm} (14)$$

\textbf{Proof.} We prove by induction on the cardinality $n$ of the index set $I$, i.e., on the number of qudits. For $n = 2$, remembering the relation $X^i_k Z X = \omega Z$, we see that

$$X^i_k Z_j k X_k = X^i_k \left( \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes Z^q_k \right) X_k$$

$$= \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes (X^i_k Z^q_k X_k)$$

$$= \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes (\omega^q Z^q_k)$$

$$= \left( \sum_{q=0}^{d-1} \omega^q |q_j\rangle \langle q_j| \otimes I_k \right) \left( \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes Z^q_k \right)$$

$$= Z_j Z_k$$

and the relations are valid. Now, let us consider the set $I$ having cardinality $n$ and the set $I' = I \cup \{j\}$, with $j \neq k$. Then we have

$$X^i_k Z_{I'} X_k = X^i_k \left( \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes Z^q_{I'} \right) X_k$$

$$= \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes (X^i_k Z^q_{I'} X_k)$$

$$= \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes (Z_{I \setminus \{k\}} Z_I Z^q_{I \setminus \{k\}})$$

$$= \left( \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes Z^q_{I \setminus \{k\}} \right) \left( \sum_{q=0}^{d-1} |q_j\rangle \langle q_j| \otimes Z^q_{I} \right)$$

$$= Z_{I \setminus \{k\}} Z_{I'}$$

i.e., if the relations are valid for $n$, then they are also valid for $n + 1$. \hfill \qed

The effect of applying the gate $X^i_k$ on an elementary hypergraph is then to create a hyperedge of same multiplicity on the neighbourhood of the qudit $k$:

$$X^i_k |H\rangle = X^i_k Z_{m_e}^e |+\rangle^V = Z_{m_e}^{m_e \setminus \{k\}} Z_{m_e} X_k^i |+\rangle^V = Z_{m_e}^{m_e} |H\rangle$$

It is important to notice that the hyperedges induced on the neighbourhood of $k$ by repeated applications of $X^i_k$ have multiplicities that are divisible by $m_e$. Depending on the primality of the underlying dimension $d$, there are restricted possibilities of inducing hyperedges on neighbouring qudits via this procedure, a difference in relation to the qubit case.

Moreover, since the local $Z_k$ gate commutes with any $Z_I$, it can always be locally removed by applying $Z_k^\dagger$. As
explained previously, we adopt the convention of representing any $Z_k$ acting on a hypergraph state as a loop - a 1-hyperedge - around the vertex $k$; higher potencies $(Z_k)^m$ are represented by $m$ loops around $k$. Thus, the action of the local Pauli group constituted of the unitaries $X_k^m Z_k^n$ is to create $n$ loops on the vertex $k$ and $m$-hyperedges on the neighbourhood of $k$.

Let us turn now to the action of the local Clifford group. The local gate from Eq. \([2]\) performs permutations on the computational basis, via the mapping $S(\xi^{-1},0,0)ZS^\dagger(\xi^{-1},0,0) = Z^\xi$, where $Z$ acts on a single particle. Based on this, we can derive the action on multiparticle $Z_e$ gates, corresponding to a hyperedge $e$. It turns out that for $d$ prime, it is always possible to convert a $k$-hyperedge of multiplicity $m$ ($m \neq 0 \mod d$) to another $k$-hyperedge of multiplicity $m'$ ($m' \neq 0 \mod d$). For non-prime $d$, the $k$-hyperedges that are connectable via permutations are those whose multiplicities have an equal greatest common divisor with the dimension $d$. In detail, we can formulate

**Proposition 2.** Let $k, k' \in \mathbb{Z}_d$ be such that $\gcd(d,k) = \gcd(d,k') = g$. Then there exists a Clifford operator $S$ such that $S(Z_e)^k S^\dagger = (Z_e)^{k'}$.

**Proof.** One can find the proof of this proposition in Ref. [18].

For a detailed discussion on the Clifford group see Ref. [19]. For completeness, we provide in Appendix C alternative proofs of this Proposition. The local Clifford gates in Eq. \([3]\) and Eq. \([4]\) are associated with the local complementation of qudit graphs, which is explained in detail in Appendix B.

### B. Stabilizer formalism

From relation \([14]\) we can construct the stabilizer operator on a vertex $i$:

$$|H\rangle = \prod_{e \in E} Z_e |+\rangle^V = \prod_{e \in E} Z_e X_i Z_e^\dagger Z_e |+\rangle^V$$

$$ = X_i \prod_{e \in E^*} Z_{e \backslash \{i\}} |H\rangle = K_i |H\rangle$$  \(15\)  

with $K_i = X_i \prod_{e \in E^*} Z_{e \backslash \{i\}}$ and where $E^*$ denotes all edges containing $i$. Hence, the operators $K_i$ stabilize the hypergraph state $|H\rangle$. An equivalent way is expressing the stabilizer operator in the compact way $K_i = X_i Z_{N_i}$, where $Z_{N_i} = \prod_{j \in N_i} Z_j$, when $N_i$ is the neighbourhood of $i$. Moreover, these operators are mutually commuting:

$$K_i K_j = (X_i \prod_{e \in E} Z_{e \backslash \{i\}}) (X_j \prod_{e \in E} Z_{e \backslash \{j\}})$$

$$= (\prod_{e \in E} Z_e X_i \prod_{e \in E} Z_{e \backslash \{i\}} Z_e^\dagger) (\prod_{e \in E} Z_e X_j \prod_{e \in E} Z_{e \backslash \{j\}} Z_e^\dagger)$$  \(17\)

$$= \prod_{e \in E} Z_e X_i X_j \prod_{e \in E} Z_e^\dagger$$

$$= \prod_{e \in E} Z_e X_j X_i \prod_{e \in E} Z_e^\dagger = K_j K_i$$  \(18\)

Indeed, these operators generate a maximal abelian group on the number $n$ of qudits. The group properties of closure and associativity are straightforward, while the identity element comes from $K_i^d = I$ and the inverse of $K_i$ being simply $K_i^\dagger$. Each operator in this group has eigenvalues $1, \omega, \omega^2, \ldots, \omega^{d-1}$ and their $d^n$ common eigenvectors form an orthonormal basis of the total Hilbert space, the hypergraph basis with elements given by

$$|H_{k_1,k_2,\ldots,k_N}\rangle = Z_1^{-k_1} Z_2^{-k_2} \ldots Z_N^{-k_N} |H\rangle$$  \(19\)

where the $k_i$s assume values in $\mathbb{Z}_d$. Notice also that

$$|H\rangle \langle H| = \frac{1}{d^n} \prod_{i \in V} (I + K_i + K_i^2 + \ldots + K_i^{d-1})$$  \(20\)

In the qubit case, these non-local stabilizers are observables and were used for the development of novel non-contextuality and locality inequalities [11,12]. For $d > 2$, these operators are no longer self-adjoint in general, but we believe techniques similar to Ref. [26] could be used to extend the results of the qubit case to arbitrary dimensions.

### C. Local measurements in $Z$ basis and ranks of the reduced states

It is possible to give a graphical description of the measurement of a non-degenerate observable $M = \sum_q m_q |q\rangle \langle q|$ on a hypergraph state in terms of hypergraph operations. Obtaining outcome $m_q$ when measuring $M$ on the qudit $k$ of the hypergraph state $|H\rangle = \prod_e Z_e^{m_e} |+\rangle^V$ amounts to performing the projection $P_q^{(k)} |H\rangle$, where $P_q^{(k)} = |q_k\rangle \langle q_k|$. The state after the measurement is then $|q_k\rangle \otimes |H'\rangle$, where $|H'\rangle = \prod_e Z_e^{m_e-q} |+\rangle^V \backslash \{k\}$.

Moreover, the calculation of ranks of reduced states can be done graphically. We show now a lemma that will be important in future derivations:

**Lemma 3.** For a $d$-dimensional system, the rank of any reduced state of an $n$-elementary hypergraph state is $d/\gcd(d,m_e)$ where $m_e$ is the multiplicity of the hyperedge.
that the operators $Z_{\alpha}$ where $|\alpha\rangle$ is one vertex, we see that $\sum_{q=0}^{d-1} |q\rangle\langle q| \otimes (Z_{\alpha}^{qm_e})^{q}|+\rangle V = \sum_{q=0}^{d-1} |q\rangle\langle q| \otimes (Z_{\alpha}^{qm_e})^{q}|+\rangle V$, (23)

where $e = \{12\ldots n\}$ is the $n$-hyperedge. Let us first consider the case where only a single system is traced out. Tracing out subsystem 1, we arrive at

$$\rho_{e\setminus\{1\}} = Tr_1(|H\rangle\langle H|) = \sum_{q=0}^{d-1} |q\rangle\langle q| \otimes (Z_{\alpha}^{qm_e})^{q}|+\rangle V$$

where

$$|H_q^{(1)}\rangle = (Z_{\alpha}^{qm_e})^{q}|+\rangle V^{\setminus\{1\}}.$$  

The number of different values (modulo $d$) of the product $qm_e$, with $q = 0,1,2,\ldots,d-1$ is $d/gcd(d,m_e)$, since in $Z_d$ one has $m_e q = m_e q'$ iff $q = q'$ modulo $d/gcd(d,m_e)$.

Moreover, the $d/gcd(d,m_e)$ different vectors $|H_q^{(1)}\rangle$ are linearly independent. We prove this via induction on the number of vertices of $V \setminus \{1\}$. For $V \setminus \{1\}$ composed of one vertex, we see that $|H_q^{(1)}\rangle = Z_{\alpha}^{qm_e}|+\rangle_2$ and

$$\sum_q \alpha_q |H_q^{(1)}\rangle = (\sum_q \alpha_q Z_{\alpha}^{qm_e})|+\rangle_2$$

where $\alpha_q$ are arbitrary complex numbers. Then $\sum_q \alpha_q |H_q^{(1)}\rangle = 0$ iff $(\sum_q \alpha_q Z_{\alpha}^{qm_e}) = 0$. To see this, not that the operators $Z_{\alpha}^{qm_e}$ are diagonal in the computational basis and the vector $|+\rangle_2$ is an equal superposition of all basis elements. Therefore, $(\sum_q \alpha_q Z_{\alpha}^{qm_e})|+\rangle_2 = 0$ implies already $(\sum_q \alpha_q Z_{\alpha}^{qm_e}) = 0$.

Since the Pauli operators form a basis of the Lie algebra $sl(d(C))$ and are thus linearly independent and given that the operators $Z_{\alpha}^{qm_e}$ are Pauli operators we have that $(\sum_q \alpha_q Z_{\alpha}^{qm_e}) = 0$ implies that $\alpha_q = 0$ for all $q$, i.e., the vectors $|H_q^{(1)}\rangle$ are linearly independent.

For $V \setminus \{1\}$ composed of two vertices, $|H_q^{(1)}\rangle = Z_{\alpha}^{qm_e}|+\rangle_2$. By the same argument above, $|H_q^{(1)}\rangle$ are linearly independent iff $Z_{\alpha}^{qm_e}$ are linearly independent operators.

We have

$$\sum_q \alpha_q Z_{\alpha}^{qm_e} = \sum_q |q\rangle\langle q| \otimes (\sum_q \alpha_q Z_{\alpha}^{qm_e})$$

$$= \sum_q |q\rangle\langle q| \otimes (\sum_q \alpha_q Z_{\alpha}^{q^2 m_e})$$

$$|0\rangle\langle 0| \otimes (\sum_q \alpha_q I_3)$$

$$+ |12\rangle\langle 12| \otimes (\sum_q \alpha_q Z_{\alpha}^{2}) + \ldots$$

We see then that $\sum_q \alpha_q Z_{\alpha}^{qm_e} = 0$ can be satisfied only if $\alpha_q = 0$ for all $q$, since this is the only way to have a null term $|12\rangle\langle 12| \otimes (\sum_q \alpha_q Z_{\alpha}^{q^2 m_e})$, given that the $Z_{\alpha}^{qm_e}$ operators are linearly independent. Thus $|H_q^{(1)}\rangle$ are linearly independent for $V \setminus \{1\}$ composed of two vertices as well. The general induction argument is now clear and it is obvious that the vectors $|H_q^{(1)}\rangle$ are linearly independent in general. Hence the rank of $\rho_{e\setminus\{1\}}$ is $d/gcd(d,m_e)$.

One can directly check using the representation in Eq. (10) that if more than two qudits are traced out the same arguments apply. Thus, the rank of any reduced state is $d/gcd(d,m_e).

$$\square$$

V. SLOCC AND LU CLASSES OF HYPERGRAPHS

A. SLOCC and LU transformations

The phenomenon of entanglement is a consequence of the physical restriction to local operations by agents separated by space-like distances. It is thus important to identify when it is possible to inter-convert two different quantum states by means of local operations or, more specifically, characterize their equivalence under SLOCC or LU operations. Finding the SLOCC/LU classes to which a given hypergraph state belong is in general a cumbersome task even in the qubit case, but in the following we will present several results and ideas that can be used for tackling this task.

Let first define the basic notation. Two pure $n$-partite states $|\phi\rangle$ and $|\psi\rangle$ are equivalent under local unitaries if one has a relation like

$$|\phi\rangle = \bigotimes_{i=1}^{n} U_i |\psi\rangle,$$  

\text{V. SLOCC AND LU CLASSES OF HYPERGRAPHS}

\text{A. SLOCC and LU transformations}

The phenomenon of entanglement is a consequence of the physical restriction to local operations by agents separated by space-like distances. It is thus important to identify when it is possible to inter-convert two different quantum states by means of local operations or, more specifically, characterize their equivalence under SLOCC or LU operations. Finding the SLOCC/LU classes to which a given hypergraph state belong is in general a cumbersome task even in the qubit case, but in the following we will present several results and ideas that can be used for tackling this task.

Let first define the basic notation. Two pure $n$-partite states $|\phi\rangle$ and $|\psi\rangle$ are equivalent under local unitaries if one has a relation like

$$|\phi\rangle = \bigotimes_{i=1}^{n} U_i |\psi\rangle,$$  

where $\alpha_q$ are arbitrary complex numbers. Then $\sum_q \alpha_q |H_q^{(1)}\rangle = 0$ iff $(\sum_q \alpha_q Z_{\alpha}^{qm_e}) = 0$. To see this, not that the operators $Z_{\alpha}^{qm_e}$ are diagonal in the computational basis and the vector $|+\rangle_2$ is an equal superposition of all basis elements. Therefore, $(\sum_q \alpha_q Z_{\alpha}^{qm_e})|+\rangle_2 = 0$ implies already $(\sum_q \alpha_q Z_{\alpha}^{qm_e}) = 0$.
where the \( U_i \) are unitary matrices, acting on the \( i \)-th particle. The question whether two multiqubit states are LU equivalent or not can be decided by bringing the states into a normal form under LU transformations \( 23 \).

More generally, the states are equivalent under stochastic local operations and classical communication (SLOCC) iff there exist invertible local operators (ILOs) \( A_i \) such that

\[
|\phi\rangle = \bigotimes_{i=1}^{n} A_i |\psi\rangle.
\]

(33)

Physically, this means that \( |\phi\rangle \) can be reached starting from \( |\psi\rangle \) by local operations and classical communication with a non-zero probability.

Although general criteria for SLOCC-equivalence of multipartite states do not exist, it is possible to find general necessary conditions that are useful as exclusive constraints. For instance, SLOCC transformations can clearly not change the rank of a reduced state \( \rho_i \). Moreover, for special classes of states sufficient conditions for SLOCC-equivalence can be found.

**B. Elementary hypergraphs**

We now address the problem of SLOCC classification of \( n \)-elementary hypergraph states. The classification depends on the greatest common divisor between the underlying dimension and the hyperedge multiplicity, as show in the following theorem:

**Theorem 4.** For a \( d \)-dimensional \( n \)-partite system, two \( n \)-elementary hypergraph states with hyperedge multiplicities \( k \) and \( k' \) are equivalent under LU, and hence also under SLOCC if \( \gcd(d, k) = \gcd(d, k') \). For the case that \( \gcd(d, k) \neq \gcd(d, k') \) the states are inequivalent under SLOCC.

**Proof.** If \( \gcd(d, k) = \gcd(d, k') \) we can, according to Proposition 1 find a local Clifford transformation with \( SZ^kS^k = Z^{k'} \). So we have \( S |H\rangle = SZ^k |+\rangle^n = Z^{k'} S |+\rangle^n = |H'\rangle \), since \( S |+\rangle^n = |+\rangle^n \). For the other implication, note that if \( \gcd(d, k) \neq \gcd(d, k') \), the single-system reduced states have different ranks by Lemma 3 and thus the states are not SLOCC equivalent.

\[ \square \]

In other words, the number of different elementary hypergraph SLOCC classes is the number of different values (modulo \( d \)) of \( \gcd(d, k) \), which is obviously the number of divisors of \( d \). It is remarkable that in this case SLOCC equivalence is the same as equivalence under local Clifford operations, by Proposition 2. For \( d \) prime, all values \( k \in \mathbb{Z}_d \) are obviously coprime with \( d \) and hence the following implication is straightforward:

**Corollary 5.** For \( d \) prime, all \( n \)-elementary hypergraph states are equivalent under SLOCC.

Finally, we note that also some other hypergraphs states are LU equivalent to elementary hypergraph states. From Lemma 1 and the discussion that followed, one sees that the action of the local gate \( X^i \) on an \( n \)-elementary hypergraph state creates a \( n - 1 \)-hyperedge on the neighbourhood of \( i \) with equal multiplicity \( m_e \) of the \( n \)-hyperedge \( e \). Acting \( k \) times with this local gate, i.e., application of \( (X^i)^k \) results in inducing, in the neighbourhood of the qudit \( i \), a \( n - 1 \)-hyperedge of multiplicity \( km_e \). As shown in the proof of Lemma 3 the number of different values of the product \( km_e \) is given by \( d/\gcd(d, m_e) \). Thus, the higher the value \( \gcd(d, m_e) \), the smaller the possible \( n - 1 \)-hyperedges that can be created (or erased) within an elementary hypergraph state.

**C. Tools for SLOCC classification**

In this section we will explain some more refined criteria for proving or disproving SLOCC equivalence. As already mentioned, the rank of the reduced states, \( r(\rho_i) = Tr_{S_3}(\rho_S) \) (where \( S \) denotes the set of all subsystems), as a simple way of identifying inequivalences.

To find a finer distinction we employ a method based on Ref. \[23\] that uses a \( 1|23\ldots n \) split of the system to identify types of inequivalent bases of the \( (2,3,\ldots,n) \)-subspace, which results in a lower bound on the number of actual SLOCC-classes. As we want to infer for a given state its SLOCC-class, there remains the following problem to be solved: identifying the basis which has minimal entanglement in its basis vectors. Accordingly, we refer to this tool as minimally entangled basis (MEB) criterion. A major disadvantage of this method is that with growing number \( n \) of subsystems, the entanglement structure within the bases becomes more complex, as it rises recursively from the total number of SLOCC-classes of the \( (n-1) \)-partite systems.

The MEB of an \( n \)-partite quantum state is defined as follows:

**Definition 6.** Consider a state \( |\psi_{12\ldots n}\rangle = \sum_{a_1,a_2,\ldots,a_n=0}^{d-1,d-1,\ldots,d-1} c_{a_1,a_2,\ldots,a_n} |a_1,a_2,\ldots,a_n\rangle \) in a \( d^2 \times n \) system. According to Ref. \[23\], we define the \( d \times (d^{n-1}) \) coefficient-matrix \( C_{1|2\ldots n} \) in the canonical basis \( \{e_i\} \) as follows:

\[
C_{1|2\ldots n} = \sum_{a_1,a_2,\ldots,a_n=0}^{d-1,d-1,\ldots,d-1} c_{a_1,a_2,\ldots,a_n} e_1^T \otimes \cdots \otimes e_n^T
\]

This matrix holds all information about the total state. From the singular value decomposition (SVD) of this matrix, \( C_{1|2\ldots n} = U_1 D V_{2\ldots n}^\dagger \), we can identify a basis \( \{v_k\} \) of the right subspace \( (2,\ldots,n) \), where individual basis vectors \( |v_k\rangle \) might be entangled.
Within this framework, we define a minimally entangled basis (MEB) \( \{ v_k \}_{\text{MEB}} \) of \( |\psi_{12...n}\rangle \) as the one within which the number of full product vectors is maximal under the condition that it spans the same subspace as \( \{ v_k \} \).

With this definition we can state:

**Lemma 7.** Two n-partite quantum states \( |\phi\rangle, |\psi\rangle \) of the same subsystem-dimensionality and equal reduced ranks are SLOCC-inequivalent, if their MEBs have a different number of product vectors.

**Proof.** The action of the ILOs \( A_i \), where \( i = 1, 2, \ldots n \), on \( C_{12\ldots n} \) in its SVD is identified to be

\[
A_i U_i D_i [(A_2 \otimes \ldots \otimes A_n)V_{2\ldots n}]^\dagger.
\]

We analyse the basis \( \{ v_k \} \) of the right subspace. The Schmidt rank of each basis vector can be changed by \( A_1 \) exclusively, which corresponds to a basis transformation of the subspace. If the states \( |\phi\rangle, |\psi\rangle \) are SLOCC equivalent, by definition there exists ILOs \( A_i \) which map \( |\phi\rangle \) into \( |\psi\rangle \) and thus map the basis of the right subspace of \( |\phi\rangle \) into the basis of the right subspace of \( |\psi\rangle \). The MEB of \( |\psi\rangle \) will be then a valid MEB for \( |\phi\rangle \), implying that the number of product vectors is the same. \( \square \)

In the above Lemma we consider states \( |\phi\rangle, |\psi\rangle \) that have equal reduced ranks, because otherwise these states are automatically SLOCC-inequivalent and there is no point in calculating their MEBs.

Notice that inequivalent MEBs can exclude SLOCC equivalence, but an equivalence of MEBs does not, in general, guarantee SLOCC-equivalence. An exception is the case where the right subspace is spanned by a complete product basis. The reason is that in this case they are SLOCC equivalent to a generalized GHZ state:

**Lemma 8.** Two genuine n-partite entangled quantum states \( |\phi\rangle, |\psi\rangle \) of the same subsystem-dimensionality and equal reduced single-particle ranks are SLOCC-equivalent, if their MEBs are complete product bases.

**Proof.** We show that the existence of a complete product basis within the right subspace is sufficient to find ILOs that transform \( |\psi\rangle \) (and \( |\phi\rangle \)) to the GHZ type state \( |\psi_{\text{GHZ}}\rangle \sim \sum_{k=0}^{r} \bigotimes_{i=1}^{n} |k_i\rangle \), where \( r \) is the rank of the reduced single-particle states.

Let us assume that the basis vectors \( |v_k\rangle \) are all product vectors. Therefore, they can be written as

\[
|v_k\rangle = \bigotimes_{i=2}^{n} |\phi_i^{(k)}\rangle
\]

In order to map this onto the GHZ state, we only have to find ILO \( A_i^{(k)} \) on the particles \( i = 2, \ldots n \) such that for any particle the set of states \( \{ |\phi_i^{(k)}\rangle \} \) is mapped onto the states \( \{ |k_i\rangle \} \). This is clearly possible: since the reduced state ranks are all \( r \), the set \( \{ |\phi_i^{(k)}\rangle \} \) consists of \( r \) linearly independent vectors. Finally, on the first particle we have to consider the left basis \( |u_k\rangle \). These vectors are orthogonal, and we can find a unitary transformation that maps it to \( \{ |k_1\rangle \} \).

Based on the Lemmata presented in this subsection we wrote computer programs which we regard as tools which we use later for classification of tripartite hypergraphs of dimension 3 and 4.

1. **Tool # 1**

The first program checks whether there exist a state \( \rho \) in the subspace spanned by a given set of pure states \( \{ |v_i\rangle \} \) for \( i = 1, \ldots, K, K \leq d \) that has a positive partial transpose (PPT) with respect to any bipartition \( [28] \). This problem can be formulated as following semidefinite program (SDP)

\[
\begin{align*}
\min_{\lambda} & \quad 0 \\
\text{subject to} & \quad \rho = \sum_{ij} \lambda_{ij} |v_i\rangle\langle v_j|, \\
& \quad \rho \geq 0, \\
& \quad \forall \text{ bipartitions } M|M, \rho^T_M \geq 0, \\
& \quad \lambda^T = \lambda, \text{ Tr}(\lambda) = 1,
\end{align*}
\]

where last condition means \( \lambda \) is a hermitian \( K \times K \) matrix with trace 1, and \( \rho^T_M \) denotes partial transpose of matrix \( \rho \) with respect to the subsystem \( M \).

This tool can be used to prove that there is no product vector in the right subspace \( (2, \ldots, n) \) of an n-partite state \( |\phi\rangle \), where \( K \) is the number of basis vectors in the right subspace. If the above SDP is infeasible it implies that there is no separable state in the subspace \( (2, \ldots, n) \), which in turns implies that there is no product vector. If for some other \( n \)-partite state \( |\psi\rangle \) there is a product vector in the right subspace \( (2, \ldots, n) \) the two states \( |\phi\rangle \) and \( |\psi\rangle \) are SLOCC-inequivalent according to Lemma 7.

2. **Tool # 2**

The second program is a slight modification of the first one and it checks whether there exist a PPT state of rank \( K \) in the subspace spanned by \( K \) linearly independent vectors \( \{ |v_i\rangle \}, i = 1, \ldots, K, K \leq d \). If the optimal value \( \epsilon \)
of the following semidefinite program
\[
\begin{align*}
\min & \quad \epsilon \\
\text{subject to} & \quad \varrho = \sum_{ij} \lambda_{ij} |v_i\rangle\langle v_j|, \\
& \quad \varrho \geq 0, \\
& \quad \forall \text{bipartitions } M|\overline{M}, \quad \varrho^{TM} \geq 0, \\
& \quad \varrho \geq \epsilon \left( \sum_i |v_i\rangle\langle v_i| \right), \\
& \quad \lambda^I = \lambda, \quad \text{Tr}(\lambda) = 1,
\end{align*}
\]

is greater than 0, and if the found PPT state \( \varrho \) can be proven to be (fully) separable, then by the range criterion (see Ref. [29]) it means that in the subspace spanned by \( |v_i\rangle \) there are \( K \) product states which span the same subspace. This program can be used to prove SLOCC-equivalence of states \( |\phi\rangle \) and \( |\psi\rangle \) according to Lemma [8] if for both states the above conditions are satisfied for their right subspace of at least one bipartition.

3. Tool # 3

Finally, it is convenient to perform a numerical optimization in order to find product states \( |v^p_i\rangle \), \( i = 1, \ldots, K' \), \( K' \leq K \) in the subspace spanned by the given set of vectors \( |v_i\rangle \) for \( i = 1, \ldots, K \), \( K \leq d \). This can be done by, let us say, maximizing the purity of the reduced states \( |\overline{M}\rangle \), \( \text{Tr}(\varrho^2_M) \), where \( \varrho_M = \text{Tr}_M(\varrho) \) is the reduced state of the subsystem \( M \) for each bipartition and minimizing the scalar product \( \langle |v^p_i\rangle|v^p_j\rangle \rangle^2 \) between each pair of product vectors for each unique pair \( \{i, j\} \), \( i, j \in \{1, \ldots, K'\} \). Minimizing the scalar products makes the program to look for linearly independent vectors which in the best case are orthogonal.

As we will see in the next section, for most of the tripartite hypergraph states of dimension 3 and 4 numerical optimization, described above, gives explicit form of product states in the right subspace if they exist. Moreover, knowing the exact form of product states for the case where a full product basis exists for both states \( |\phi\rangle \) and \( |\psi\rangle \) allows us to find an explicit SLOCC transformation between these states.

D. Tools for LU-classification

Let us now discuss tools how LU equivalence can be characterized. In principle, this question can be decided using the methods of Ref. [27], but for the examples in the next section some other methods turn out to be useful.

If LU equivalence should be proven, an obvious possibility is to find directly the corresponding LU transformation. This has been used in Theorem [4]. For proving non-equivalence, one can use entanglement measures such as geometric measure [20], since such measures are invariant under LU transformations. Another possibility is the white-noise tolerance of witnessing entanglement [21]. The latter method works as follows: For an entangled state which is detected by a witness one can assign an upper limit of white noise which can be added to the state, such that the state can still be detected by that witness. Clearly, if two states are equivalent under LU, they have the same level of white-noise tolerance of entanglement detection. Now if one considers a class of decomposable witnesses the estimation of this level for a given state can be accomplished effectively by means of semidefinite programming [22]. Below, we use a method described in Ref. [21] to witness genuine multipartite entanglement of hypergraph states and to determine the corresponding white-noise tolerance of that witnessing.

Using the tools described above we present a classification in terms of SLOCC- and LU-equivalence of tripartite hypergraph states for dimensions 3 and 4 in the next section.

VI. EXAMPLES

We now consider the special cases of a tripartite system with prime dimension 3 and a tripartite system with the smallest non-prime dimension 4 as examples.

A. Classification of \( 3 \otimes 3 \otimes 3 \)

In the case of a tripartite system of qutrits, there is only one SLOCC equivalence class of hypergraph states and two LU equivalence classes: the GHZ state and the 3-elementary hypergraph state. These two states are inequivalent by LU for presenting different values of geometric measure of entanglement and white noise tolerance in Table [1].

This classes can be derived as follows: Let us first consider the GHZ state. From Appendix B it follows that the GHZ state can be converted to the graph state represented by the local complementation of the GHZ graph via local symplectic unitaries. From Proposition [2] we see that a hyperedge of arbitrary multiplicity can be converted to an hyperedge of any other multiplicity via local symplectic permutations. Thus, the tripartite GHZ state is equivalent to any other tripartite graph state via local symplectic unitaries.

If we consider the elementary hypergraph state, a 3-hyperedge can be converted to another 3-hyperedge of arbitrary multiplicity via symplectic permutations. In addition, the 3-elementary hypergraph is equivalent to any other 3-hypergraph, since edges (2-hyperedges) of arbitrary multiplicities can be created via repeated application of the \( X^\dagger \) gate in a neighbouring qutrit.

Finally, in order to show the SLOCC equivalence, local invertible operations connecting these two LU subclasses can be achieved by applying \( A_1 \) to one of the qutrits of
the graph state and $A_{2,3}$ to the other two, where
\begin{align}
A_1 &= \frac{1}{4\sqrt{3}} \begin{pmatrix}
-2\sqrt{3} - 2i & 4i & 4i \\
-4\sqrt{3} + 4i & \sqrt{3} + i & -5\sqrt{3} + 7i \\
-6\sqrt{3} - 2i & -\sqrt{3} - 5i & -\sqrt{3} + 7i \\
\end{pmatrix}, \\
A_{2,3} &= \frac{1}{3} \begin{pmatrix}
\frac{\sqrt{3}e^{i\pi/6}}{e^{i2\pi/3}} & 1 & \frac{1}{2} + \frac{2}{\sqrt{3}} \\
\frac{\sqrt{3}e^{i\pi/6}}{e^{i2\pi/3}} & \frac{\sqrt{3}e^{i\pi/6}}{e^{i2\pi/3}} & \frac{1}{2} + \frac{2}{\sqrt{3}} \\
\end{pmatrix}. 
\end{align}

This local operations were found with the help of the tool# 3 (numeric optimization program described in the previous section), which gives in this case full product basis for right subspaces of all states from Table I.

### B. Classification of $4 \otimes 4 \otimes 4$

In the case of a tripartite system of ququarts, there are five SLOCC and six LU equivalence classes of hypergraph states. All possible states with respect to permutations and equivalence of edge multiplicities (local Clifford permutation $S$ converts the multiplicity of the 3-hyperedge from 1 to 3 (since $3 = 3^{-1}$ modulo 4, see Proposition 2, see also Fig. 3.b.), are shown in the Table II and the interconversion between representatives within the same class are explained in detail in what follows.

#### 1. Class 1

Class 1 contains hypergraph states with at least two edges of multiplicity 1 and with either no hyperedges, or with hyperedge of multiplicity 2. All these states belong to the same LU-equivalence class.

LU-equivalence among first three states of class 1 (see Table III) is governed by standard local complementation operations, which can be used to create a new edge of multiplicity 1 in the neighborhood of qudit 2, while applying these operations once more generates an edge of multiplicity 2 in the neighborhood of qudit 2 (see Appendix B for more details). The same local complementation is responsible for LU equivalence among last three states the first LU class.

To prove LU equivalence between these two subgroups of states (with no hyperedge and with 2-hyperedge) we find with the Tool# 3 explicit form of their MEBs, which appear to consist of product vectors. It can be shown then that local transformation between these states is unitary. Here we present such local unitary for transformation from the Figure 4.

\begin{align}
U_{1,2,3} &= \frac{1}{2} \begin{pmatrix}
1 + i & 0 & 1 - i & 0 \\
0 & 0 & 0 & -2 \\
-1 - i & 0 & 1 + i & 0 \\
0 & -2 & 0 & 0 \\
\end{pmatrix}.
\end{align}

#### 2. Class 1$'$

Class 1$'$ contains all hypergraph states which have 3-hyperedge of multiplicity 1. LU equivalence of the states within this class is governed by the unitary $(X^\dagger)^m$, which, when allied to some qudit, generates edges of multiplicity $m$ on the neighbourhood of the qudit (see Lemma 1).

#### 3. Class 2

Class 2 consists of two LU equivalence classes. The representative of the first LU-equivalence class are the graph states composed of two and three edges of multiplicity 2, though the representatives of the second LU class are the hypergraph state with a 3-hyperedge of multiplicity 2 with possible edges of multiplicity 2.

We can perform some form of “local complementation” between two states from the first LU class by applying the following unitaries in the basis $\{|p_0\rangle, |p_1\rangle, |p_2\rangle, |p_3\rangle\}$:

\begin{align}
U_{1,3} &= \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & i & 0 \\
0 & \sqrt{2} & 0 & 0 \\
-i & 0 & -1 & 0 \\
0 & 0 & 0 & \sqrt{2} \\
\end{pmatrix} ;
U_2 &= \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\end{align}

Applying the local $(X^\dagger)^m$ unitary to some qudit of the states from the second LU class generates edges of multiplicity $2m$ (i.e., 0 or 2) on the neighborhood of that qudit.

Using the Tool #3 one can find the local operation corresponding to SLOCC equivalence between these LU classes. For the representatives shown on the Fig. 4 the corresponding LO is

\begin{align}
A_{1,2,3} &= \frac{1}{2} \begin{pmatrix}
-i(1 + \sqrt{3}) & 0 & (1 - \sqrt{3}) & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 \\
\end{pmatrix}.
\end{align}

One can easily check that $A_{1,2,3}$ is invertible, but not unitary. To show that there is no local unitary transformation possible, one can look at the entanglement measures for these LU classes (see Table III).

#### 4. Class 3

The representatives of class 3 are the elementary hypergraph states with a 3-hyperedge of multiplicity 2, one edge of multiplicity 1 and possible edges of multiplicity 2. These three states are in the same LU class and the local transformation between them is $(X^\dagger)$ applied on one of the qudits.

#### 5. Class 4

The representatives of class 4 are graph states composed of one or two edges of multiplicity 2 and one
| Class | Schmidt ranks | Representatives | Geom. measure/ w-noise tolerance |
|-------|---------------|-----------------|---------------------------------|
| 1     | 1|23 3 2|13 3 3|1|2 | 2 | 3 | 3 | 3 | 0.66 | ∼ 62.5% |
|       | 3|12 3 3 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 0.53 | ∼ 76.0% |

Table I. Table of one SLOCC class and two LU classes of 3-qutrit hypergraph multipartite entangled states.

| Class | Schmidt ranks | Representatives | Geom. measure/ w-noise tolerance |
|-------|---------------|-----------------|---------------------------------|
| 1     | 1|23 4 2|13 4 3|2|12 4 2 | 2 | 3 | 3 | 3 | 3 | 3 | 0.75 | ∼ 84.2% |
|       | 3|12 4 3 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 0.58 | ∼ 87.1% |
| 2     | 1|23 2 2|13 2 3 3|1|2 | 2 | 3 | 3 | 3 | 3 | 3 | 0.50 | ∼ 91.4% |
|       | 3|12 2 4 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 0.32 | ∼ 88.7% |
| 3     | 1|23 4 2|13 4 3 3|1|2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 0.75 | ∼ 86.1% |
|       | 3|12 4 3 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 0.75 | ∼ 88.8% |

Table II. Table of SLOCC classes and LU classes of 3-ququart hypergraph multipartite entangled states.
edge of multiplicity 1. Applying the local unitaries $U_1 = (S(1,1,0))^4$, $U_2 = S(1,0,1)$, $U_3 = S(1,1,0)$ to the first state creates an edge of multiplicity 2 between qudits 2 and 3.

6. SLOCC-inequivalence of Classes 1-4,1'

To prove the SLOCC-inequivalence of states of most of the classes it is sufficient to look at their Schmidt ranks for each bipartition (see Table II). Exceptions are pairs of classes 1,1' and 3,4.

To prove that there is no SLOCC transformation between states from classes 3 and 4 let us consider the vectors from the right subspace for bipartition 2|13 for two representatives from each class. From the Schmidt decomposition of the state from class 3 one finds directly that there is at least one product vectors in the right subspace of parties 13, i.e. MEB contains at least one product vector. For the state from class 4 we can prove that in the corresponding subspace there are no product vectors in the MEB using the Tool #1. Thus from Lemma 7 it follows that these states belong to different SLOCC classes.

Unfortunately, we were not able to prove SLOCC-inequivalence of states from classes 1 and 1' using the tools presented above. In fact, using the Tool #3 we found that the states from class 1 have a full product basis in their right subspace for each bipartition and the Tool #2 showed that for the states from class 1' there are states with PPT and full rank in their right subspace. However, the optimal value $\epsilon$ of the SDP of the Tool #2 for the states in class 1' was in order of $10^{-5}$. Besides, the direct numerical search for SLOCC transformation bringing a states in class 1 to some state in class 1' returned states of fidelity of almost 1, though the numerical search for SLOCC transformation in the opposite direction, from a state in 1' to some state in 1, succeed in returning states of fidelity of only 0.875. This difference in fidelities of local transformations in different directions is typical for the three-qubit states of GHZ and W classes, which suggests that classes 1 and 1' are inequivalent.

VII. CONCLUSIONS

In this work we generalized the class of hypergraph states to systems of arbitrary finite dimensions. For the special class of elementary hypergraph states we obtained the full SLOCC and LU classification in terms of the greatest common divisor, which also governs other properties such as the ranks of reduced states. For tripartite systems of local dimensions 3 and 4, we obtained all SLOCC and LU classes by developing new theoretical and numerical methods based on the original concept of MEBs.

Some open questions are worth to mention. In the multiqubit case, hypergraph states are a special case of LME states; it would be interesting to generalize the class of LME states to arbitrary dimensions and see if a similar relation holds. Nonlocal properties of qudit hypergraph states were not a part of this work and deserve a separate consideration. Finally, possible applications of these states as a resource for quantum computing should be investigated.

VIII. ACKNOWLEDGMENTS

FESS is thankful to M. Hofmann for suggesting the problem of extending hypergraph states to qudits. For discussions and comments, we acknowledge A. Asadian, C. Budroni, M. Gachechiladze, M. Huber, M. Grassl and T.M. Kraft.

The work has been supported by the program Science without Borders from the Brazilian agency CAPES, by CNPq, through the program PVE/PPD (Pesquisador Visitante Especial/Pós-Doutorado Junior), Project No. 401230/ 2014-7, Grant No. 150733/2015-1, the DAAD, the EU (Marie Curie CIG 293993/ENFOQI), the ERC (Consolidator Grant 683107/TempoQ), the FQXi Fund (Silicon Valley Community Foundation) and the DFG.
IX. APPENDIX

A. Phase-space picture

Infinite-dimensional systems are often described through position and momentum operators $Q$ and $P$ in a phase-space picture. Displacements in this quantum phase-space are performed by unitaries

$$D(q, p) = e^{i(pQ - qP)}$$

(42)

where $q$ and $p$ are real numbers. These unitaries satisfy $D(q, p) D(q', p') = e^{-i(p-q') - p'} D(q', p') D(q, p)$, characterizing a faithful representation of the Heisenberg-Weyl Lie group. Performing the transformations

$$Q \rightarrow \kappa Q + \lambda P;$$
$$P \rightarrow \mu Q + \nu P,$$

(43)

(44)

subjected to the condition $\kappa \nu - \lambda \mu = 1$, will bring the Heisenberg-Weyl group into itself. In other words, unitaries that perform these transformations will generate the normalizer of the Heisenberg-Weyl group. This group constitutes the so-called symplectic group in continuous variables and is related to important concepts in quantum optics such as squeezing.

In finite-dimensional systems it is possible to give an analogous description in terms of a discrete phase-space, whenever the Hilbert space dimension is a power of a prime number [15]. A general displacement in this discrete phase-space is then performed by an operator $D(m, n) = \omega^{mn} X^n Z^m$; the set of these displacement operators form an unitary representation of the discrete Heisenberg-Weyl group through the multiplication rule

$$D(m, n) D(m', n') = \omega^{(m'n - mn')/2} D(m + m', n + n'),$$

Symplectic transformations in a discrete-phase space act over the Pauli operators in the following fashion

$$SX^i S^\dagger = \omega^{\kappa\lambda i} X^i Z^\lambda,$$

$$SZS^\dagger = \omega^{\mu\nu i} X^i Z^\nu,$$

(45)

(46)

subjected to the condition $\kappa \nu - \lambda \mu = 1 \pmod d$. An arbitrary symplectic operator $S(\kappa, \lambda, \mu)$ can be decomposed as

$$S(\kappa, \lambda, \mu) = S(1, \xi_1) S(1, \xi_2, 0) S(\xi_3, 0),$$

(47)

where the operators in the right-hand side are given in (2), (3), (4) and

$$\xi_1 = \mu \kappa (1 + \lambda \mu)^{-1};$$

$$\xi_2 = \mu \kappa^{-1} (1 + \lambda \mu);$$

$$\xi_3 = \kappa (1 + \lambda \mu)^{-1}.$$

(48)

(49)

(50)

The actions of gates (3) and (4) are respectively given by

$$S(1, \xi, 0) X S(1, -\xi, 0) = \omega^{\xi^2} X Z \xi;$$

$$S(1, \xi, 0) Z S(1, -\xi, 0) = Z;$$

$$S(1, 0, \xi) Z S(1, 0, -\xi) = \omega^{-\xi^2} Z X \xi;$$

$$S(1, 0, \xi) X S(1, 0, -\xi) = X.$$

(51)

(52)

(53)

(54)

B. Local complementation of qudit graphs

The graph operation known as local complementation of a graph $G = (V, E)$ at the vertex $a \in V$ consists of the following mapping:

$$G \rightarrow G' = (V, E \cup E_{N_a})$$

(55)

where $E_{N_a}$ are the edges in the neighbourhood of $a$ and $\cup$ denotes the set operation of symmetric sum, i.e., $A \cup B = \{A \cup B \} \setminus \{A \cap B \}$. The implementation of such operation for qudit graph states is known in the literature [17] and is restricted to prime-dimensional systems. Here we give a simpler derivation of this implementation, which is also valid for some special cases in non-prime dimensional systems. Note that we consider here only graphs with edges of multiplicity one, which are equivalent to graphs with edges of multiplicity coprime with the underlying dimension $d$.

From the section on stabilizers of a hypergraph state, we get as a special case that the operators

$$K_i = X_i \prod_{e \in E^*} Z_{c\setminus\{i\}}^c = X_i Z_{N_i}, \quad i \in V$$

(56)

generate the stabilizer group of the graph state $|G\rangle$ represented by the graph $G = (V, E)$. The graph state $|G\rangle$ is thus the unique $+1$ eigenstate of the operators $K_i$.

Theorem 9. Given a graph state $|G\rangle$ composed of edges with multiplicity one, let $U_a = S_a(1,0,-1) S_{N_a} (1, -1, 0)$. Then, $U_a |G\rangle = |G'\rangle$, where $G'$ is the local complementation of $G$ at the vertex $a \in V$.

Proof. Let $\{K_i\}_{i \in V}$ denote the set of stabilizer operators of $G$ and let $S$ be the stabilizer group generated by them. It is clear that $U_a K_i U_a^\dagger = K_i$ if $i$ is not in $N_a$, while for $c \in N_a$ we have $U_a K_c U_a^\dagger = K_c^{-1} K_c'$, where $K_c'$ is the stabilizer operator for the vertex $c$ of $G'$. We have then that $U_a S U_a^\dagger = S'$, where $S'$ is the group generated by the stabilizer operators of $G'$.

Another way of proving this result is to consider the action of $S_a(1,0,-1)$, which is simply

$$S_a(1, 0, -1) |G\rangle = S_{N_a}(1,1,0) |G'\rangle$$

(57)

and thus applying $S_{N_a}(1, -1, 0)$ on the state above will map $G$ into its local complementation $G'$.

C. Proofs of Proposition 2

Proof. We define $k = 0g$ and $k' = 0g$ and consider first a single particle gate $Z$. Looking at the action of $S$, $S^\dagger$ and $Z$ on a basis vector $|x\rangle$ one sees that a corresponding $S$ can be found, if we can find an $\xi$ such that

$$\frac{kx}{\xi} = k'x \pmod d$$

(58)
holds for any $x$. Dividing by $g$, this is equivalent to $\alpha x = \beta x \xi \mod (d/g)$. The value of $\xi$ is found by considering first $\xi' = \alpha/\beta \mod (d/g)$. This is well defined, since $\beta$ and $d/g$ are coprime. It remains to construct a final $\xi$ that is coprime with $d$. The $\xi'$ fulfills $\beta \xi' = \alpha + y(d/g)$ for some $y$. It follows that $\xi'$ does not have any prime factors already contained in $(d/g)$ since $\alpha$ and $d/g$ are coprime, but $\xi'$ may still have prime factors present in $g$ (but absent in $d/g$). If this is the case, we choose $\xi = \xi' + d/g$. This is allowed, since $\xi'$ was defined $\mod (d/g)$. Now, $\xi$ has no prime factors contained in $g$ (but absent in $d/g$), and still no prime factors contained in $d/g$. So, it is coprime to $d$, and $S$ is unitary.

Finally, if a multiparticle gate $Z_k$ is considered, the proof is the same, starting from the representation in Eq. \[10\].

Below is an alternative proof of Proposition \[2\].

**Proof.** For $\gcd(d,k) = 1$, i.e., $k$ is coprime with $d$, there exists $k^{-1}$ such that $kk^{-1} = 1$; this multiplicative inverse is given by $k^{-1} = k^{\lambda(d) - 1}$, where $\lambda(d)$ is the Carmichael function \[30\]. The function $f : \mathbb{Z}_d \to \mathbb{Z}_d$ given by $f(q) = qk$ is injective \[31\], since $qk = q'k$ if $q = q'$. But it is also surjective since it is a function from $\mathbb{Z}_d$ to itself. Hence, $f$ is a bijection, the unitary $S_k = \sum_{q=0}^{d-1} |q\rangle \langle qk|$ is well-defined and $S_k Z S_k^\dagger = Z^k$; notice that this corresponds to the Clifford gate \[3\]. Defining the Clifford operator $S = S_{k-1} S_{k-1}^\dagger$, it follows that $SZ^k S^\dagger = S_{k-1}(S_{k-1}^\dagger Z^k S_{k-1}) S_{k-1}^\dagger = S_{k-1} Z S_{k-1}^\dagger = Z^k$.

If $\gcd(d,k) > 1$, then there exists $c$ coprime with $d$ such that $k = gc$. In order to prove this, let us take the prime decomposition of $d$, i.e., $d = p_1^{n_1} p_2^{n_2} \ldots p_N^{n_N}$. Let us consider the decomposition $\prod_{i=1}^N Z_{d_i}$, where each component $c_i$ is coprime with $d_i$. Indeed, these values $c$ are formed by all possible different combinations of the component values $c_i$. The only values $k = gc$ which are not already in the form $k = gc$ are only for which the fixed $j$ and $\alpha$ are coprime, for fixed $1 \leq n \leq n_j$. The decomposition of $k = gc$ is simply $k = (k_1, k_2, \ldots, k_0, \ldots, k_N)$ and the non-zero values $k_i$, $i \neq j$, are coprime with $d_i$. Thus, the values $c$ coprime with $d$ for which $c_i = k_i$ for all $i \neq j$ yields $gc = gc_k$.

Let $\gcd(d,k) = \gcd(d,k') = g$; then there exists $c$, $c'$ coprime with $d$ such that $k = gc$ and $k' = gc'$, by the discussion above. Let $S$ be the Clifford operator such that $SZ^c S^\dagger = Z^c$. Then $SZ^k S^\dagger = S(Z^c)^g S^\dagger = (Z^c)^g = Z^{k'}$. \[\square\]

[1] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[2] O. Gühne and G. Tóth, Phys. Rev. 474, 1 (2009).
[3] M. Hein, J. Eisert, and H.J. Briegel, Phys. Rev. A 69, 062311 (2004); M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. van den Nest and H.-J. Briegel, in Quantum Computers, Algorithms and Chaos, edited by G. Casati, D.L. Shepelyansky, P. Zoller, and G. Benenti (IOS Press, Amsterdam, 2006), quant-ph/0602096.
[4] R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A 68, 022312 (2003).
[5] D. Gottesman, Stabilizer Codes and Quantum Error Correction, PhD thesis, CalTech, Pasadena (1997); D. Schlingemann, Quantum Inf. Comput. 2, 307 (2002).
[6] D. Greenberger, M. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. 58, 1131 (1990); N. D. Mermin, Rev. Mod. Phys. 65, 803 (1993); D.P. DiVincenzo and A. Peres, Phys. Rev. A 55, 4089 (1997); O. Gühne and A. Cabello, Phys. Rev. A 77, 032108 (2008).
[7] B. Jungnitsch, T. Moroder and O. Gühne, Phys. Rev. A 84, 032310 (2011).
[8] A. Kitaev, Ann. Phys. 321, 2 (2006).
[9] R. Qu, J. Wang, Z. Li, and Y. Bao, Phys. Rev. A 87, 022311 (2013); M. Rossi, M. Huber, D. Bruß, and C. Macchiavello, New J. Phys. 15, 113022 (2013).
[10] B. Yoshida, Phys. Rev. B 93, 155131 (2016); J. Miller and A. Miyake, npj Quantum Information 2, 16036 (2016).
[11] O. Gühne, M. Cuquet, F. E. S. Steinhoff, T. Moroder, M. Rossi, D. Bruß, B. Kraus, and C. Macchiavello J. Phys. A: Math. Theor. 47, 335303 (2014).
[12] M. Gachchiladze, C. Budroni, and O. Gühne, Phys. Rev. Lett. 116, 070401 (2016).
[13] T. Carle, B. Kraus, W. Dür, and J. I. de Vicente, Phys. Rev. A 87, 012328 (2013); C. Krenzyska and B. Kraus, Phys. Rev. A 79, 052304 (2009).
[14] C.E. Mora, H.J. Briegel, and B. Kraus, Int. J. Quant. Inf. 5, 729 (2007).
[15] A. Vourdas, Rep. Prog. Phys. 67, 267 (2004).
[16] W. Tang, S. Yu, and C.H. Oh, Phys. Rev. Lett. 110, 100403 (2013).
[17] D. Schlingemann, Quantum Inf. Comput. 4, 287 (2004); E. Hostens, J. Dehaene, and B. de Moor, Phys. Rev. A 71, 042315 (2005); M. Bahramgiri, and S. Beigi, arxiv:quant-ph/0610267; S. Y. Looi, L. Yu, V. Gheorghiu, and R. B. Griffiths, Phys. Rev. A 78, 042303 (2008); A. Keet, B. Fortescue, D. Markham, and B. C. Sanders, Phys. Rev. A 82, 062315 (2010).
[18] M. A. Nielsen, M. J. Bremner, J. L. Dodd, A. M. Childs, and C. M. Dawson Phys. Rev. A 66, 022317 (2002).
[19] D. M. Appleby, J. Math. Phys. 46, 052107 (2005).
[20] T.-C. Wei and P. M. Goldbart, Phys. Rev. A 68, 042307 (2003).
[21] B. Jungnitsch, T. Moroder and O. Gühne, Phys. Rev. Lett. 106, 190502 (2011).
[22] L. Vandenbergh and S. Boyd, SIAM Review 38, 49 (1996).
[23] L. Lamata J. Leon, D. Salgado, and E. Solano, Phys. Rev. A 74, 052336 (2006).
[24] G. H. Hardy, E. M. Wright, An introduction to the theory of numbers, Oxford: Clarendon Press (1938).
[25] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An introduction to the theory of numbers*, Wiley (1991).

[26] W. Son, C. Brukner, and M. S. Kim, Phys. Rev. Lett. **97**, 110401 (2006); A. Asadian, C. Budroni, F. E. S. Steinhoff, P. Rabl, and O. Gühne, Phys. Rev. Lett. **114**, 250403 (2015).

[27] B. Kraus, Phys. Rev. Lett. **104**, 020504 (2010); B. Liu, J.-L. Li, X. Li, and C.-F. Qiao, Phys. Rev. Lett. **108**, 050501 (2012); T.-G. Zhang, M.-J. Zhao, M. Li, S.-M. Fei, and X. Li-Jost, Phys. Rev. A **88**, 042304 (2013).

[28] A. Peres, Phys. Rev. Lett. **77** 1413 (1996) M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, (1996).

[29] P. Horodecki, Phys. Lett. A **232** 333 (1997).

[30] R. D. Carmichael, Am. Math. Mont. **19** 22 (1917); P. Erdős, C. Pomerance, and E. Schmutz, Acta Arithmetica **58**, 363 (1991).

[31] A more elementary way of proving is noticing that $qk = q'k$ iff $q = q' \mod (d/gcd(d, k))$. 