How to make a tiny black hole?

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Abstract

This is a brief review of critical phenomena in gravitational collapse. The conceptual issues are emphasized and some directions for future research are suggested. The paper is not addressed to the experts in the field – for them little will be new. It is rather meant to introduce others into one of the most rapidly developing areas of research in general relativity with the hope of attracting them into the subject.

The gravitational collapse of matter leading to a formation of a black hole is probably the most fascinating prediction of general relativity. On the one hand, this phenomenon has significant astrophysical consequences since it is believed to be the end-point of evolution of massive stars. On the other hand, the understanding of the dynamics of gravitational collapse is a major theoretical challenge in general relativity and a subject of intensive studies. One of the main goals of these studies is to confirm or falsify the cosmic censorship hypothesis, which, loosely speaking, says that a physically realistic gravitational collapse cannot result in a naked singularity, i.e. a singularity visible to a distant observer. Although it would be most interesting to assert the validity of the cosmic censorship hypothesis for the vacuum Einstein equations, this problem is too hard because, at present, only the spherically symmetric equations are tractable by analytical techniques, and, unfortunately, in this case the vacuum Einstein equations become trivial. Thus, to have nontrivial spherically symmetric
dynamics it is indispensable to couple matter fields.

From this theoretical perspective, the physical relevance of a matter model is of secondary importance – the priorities are simplicity and generality. The simplest choice is a minimally coupled linear massless scalar field – this system reduced to spherical symmetry was studied by Christodoulou in a series of papers (see [1,2] and references therein). Analyzing the evolution of asymptotically flat regular initial data, he showed that "weak" (in the sense of a certain function norm physically corresponding to a measure of energy concentration) initial data posses global time evolution which asymptotes the flat spacetime, whereas "strong" initial data collapse to a black hole. These results suggested that for a one-parameter family of initial data interpolating between "weak" and "strong" regime, there would be a critical parameter value corresponding to the threshold of black hole formation. If so, there arises a natural question: what is the mass of the black hole at the threshold? Is it infinitesimal, or is there a finite mass gap in the spectrum of black hole masses? I will refer to these two possibilities as to type II and type I behaviours, respectively.

The program of studying this question numerically was initiated and carried out with remarkable success by Choptuik [3]. His results can be summarized as follows. Consider a one-parameter interpolating family of initial data for the spherically symmetric Einstein-massless scalar field equations. A typical initial profile for the scalar field is an ingoing Gaussian wave with adjustable amplitude. With such a profile one can associate two length scales: the physical extent (the width), $L$, and the Schwarzschild radius $R_S$. The dimensionless parameter, $p = R_S/L$, which is monotonically related to the amplitude of the Gaussian, characterizes the degree of energy concentration. In agreement with Christodoulou’s results, for weakly coupled data, $L >> R_S$, the ingoing wave implodes through the center and then

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1This terminology originates from the formal analogy with first and second order phase transitions. Notice that type I behaviour is typical in the astrophysical context with the mass gap being of the order of the Chandrasekhar mass for fermionic matter.
escapes to infinity leaving behind the flat spacetime. When $L \approx R_S$, the gravitational interaction becomes important, the energy of the imploding wave is partially trapped and a black hole forms. Let us denote by $p^*$ a critical value of the parameter which separates black-hole-spacetimes ($p > p^*$) from no-black-hole ones ($p < p^*$). Given two values $p_{\text{weak}}$ and $p_{\text{strong}}$, it is (in principle) straightforward to find $p^*$ by bisection.

This problem looks so natural that one might wonder why it had not been studied earlier. Actually it had, but the results were inconclusive because of insufficient numerical accuracy. The point is that the features of near-critical solutions are exponentially sensitive to the distance from the threshold $|p - p^*|$ and, as this distance tends to zero, there appears an oscillating structure on progressively smaller spatio-temporal scales. To probe this structure it was instrumental to use a sophisticated numerical code. Choptuik used an adaptive mesh-refinement algorithm which allowed him to approach the threshold almost down to the machine precision $|p - p^*|/p^* \approx 10^{-15}$. The effort Choptuik invested in implementing his algorithm payed with interest – having a high resolution code, he was not only able to resolve the mass-gap question, but also, as a premium, found unexpected and intriguing phenomena at the threshold of black hole formation.

The most important observation was the universality of critical behaviour. Here universality refers to the fact that, in the so called intermediate asymptotics (i.e. before a solution "decides" whether to collapse or to disperse), all near-critical solutions have the same (i.e. family-independent) shape in the strong-field region (i.e. near the center of implosion). In other words, in this asymptotic regime the details of initial data are washed out. The precisely critical solution ($p = p^*$), called a choptuon, has an unusual symmetry of discrete self-similarity, that is it reproduces itself (echoes) on progressively finer scales: $r \to re^{-\Delta}$, $t^* - t \to (t^* - t)e^{-\Delta}$, where $t^*$ is the accumulation time of successive echoes and a constant $\Delta \approx 3.44$.

The second important result was the resolution of the mass gap problem. Choptuik has found that near-super-critical data form black holes with masses satisfying the power-law
\[ M_{BH} \simeq C(p - p^*)^\gamma, \]  

where the proportionality constant \( C \) is family-dependent, but the critical exponent \( \gamma \approx 0.37 \) is again universal. Thus, by fine-tuning the parameter \( p \), one can make a black-hole of arbitrarily small mass\(^2\). This gives the answer to the question posed in the title. At this point I should warn the reader, who might be anxious to produce a tiny black hole in this manner in his lab, that the experiment is dangerous – the smaller the black hole, the stronger the gravitational field at the horizon!

The choptuon is a limiting, sort of zero-mass black hole. In physical terms, it can be viewed as a collapsing radiating ball of field energy for which the rate of collapse is exactly balanced by the rate of energy loss by radiation, so that when the ball shrinks to zero radius all of its energy is radiated away\(^4\). Due to the accumulation of echoes, the curvature of the critical solution diverges at the origin as \( t \to t^* \). This singularity is visible from null infinity, as Hamadé and Stewart have demonstrated\(^5\), so, strictly speaking, the choptuon constitutes a counterexample to the cosmic censorship hypothesis. On the one hand, this counterexample is disturbing because it shows that the evolution of perfectly regular initial data for the realistic matter may lead to the formation of regions of spacetime with arbitrarily large curvature which are not surrounded by a event horizon. On the other hand, the phenomenon is not generic – the naked singularity can be destroyed by an arbitrarily small perturbation.

Soon after Choptuik’s discovery a similar critical behaviour has been observed in several other models of gravitational collapse with different sources\(^6,7\) and even, in one notable case, without spherical symmetry\(^8\). I shall not go into the details of these models – let me only note that the overall picture of criticality is qualitatively the same as in the scalar field

\(^2\)Perhaps it is worth stressing that the absence of a mass gap for black holes in the Einstein-massless scalar field system, is not a trivial consequence of the scale invariance of the model – the latter implies only that a mass gap could not be universal.
collapse, possibly with one difference: in certain models the critical solution is not discretely, but *continuously* self-similar. These studies have lent support to the conjecture that such features like universality, black-hole mass scaling, and self-similarity (discrete or continuous) are the robust properties of type II gravitational collapse\(^3\).

Having emphasized the genericity of type II critical collapse, it is appropriate to point out that the above mentioned models share a common, rather restrictive, property: they do not have regular stationary solutions. This fact reduces the long-time outcomes of evolution basically to two possibilities: collapse or dispersal\(^4\). Clearly, in a model having a *stable* stationary solution, there is an additional possibility that this solution will be the end-point of evolution of some (presumably large) set of initial configurations – after all this is how the stars have been formed. An even more interesting situation arises when a model admits an *unstable* stationary solution with exactly one unstable mode. Such a solution can play a role of a critical configuration separating collapse from dispersion, thereby giving rise to type I critical behaviour. This phenomenon has been found very recently in the Einstein-Yang-Mills model\(^5\), where depending on a class of initial data, both types, I and II, of critical behaviour are present. An interesting implication of that is the existence of a critical line in the parameter space which interpolates between theses two types. The transition point lying on this line, corresponding to a sort of superposition of both critical solutions, can be located by fine-tuning the parameters of certain *two*-parameter families of initial data. Thus,

\(^3\)Numerical values of a critical exponent \(\gamma\) and a periodicity scale \(\Delta\) do depend on a model. Actually, in the first three analyzed models (scalar field, axisymmetric gravity, radiation fluid) the critical exponent \(\gamma\) had the same value (up to numerical errors), which suggested the universality in the broader sense of model-independence. More accurate recent computations strongly suggest that this fact was just a misleading numerical coincidence.

\(^4\)A priori there is also a possibility of chaotic evolution, however, to my knowledge, it has not been observed in this context.
the presence of stationary solutions in a model enriches the possible long-time asymptotics and may lead to complex phase diagrams. I am confident that the analysis of [9], which has opened this issue, will be followed by further activity on related models.

The analytical understanding of the numerical phenomenology of critical behaviour described above is a great theoretical challenge. Although at present there are no rigorous results in this area, a substantial progress has been made on a heuristic level. First of all, we have a fairly convincing picture of the origin of universality. The main assumption of the mechanism explaining universality is that the critical solution has exactly one unstable mode [10]. Then the stable manifold $W_S$ of the critical solution has codimension one and separates (at least locally) the phase space of a model into spacetimes containing a black hole and spacetimes that do not. An interpolating one-parameter family of initial data is a curve in the phase space which intersects the stable manifold $W_S$ at the critical parameter value $p = p^*$. The critical data are attracted along $W_S$ towards the critical solution (for this reason the critical solution is sometimes called a codimension one attractor). The near-critical data, by continuity, initially remain close to $W_S$ and approach the critical solution (intermediate asymptotics), but ultimately are repelled from $W_S$ by the growing unstable mode.

A sketch of the near-critical evolution.
The fact that the same unstable mode dominates the long-time behaviour of all near-critical initial data is the origin of universality. It should be stressed that this mechanism explains the critical dynamics controlled by a given codimension one attractor. If there are multiple attractors in the phase space, there arise crossover phenomena associated with transitions between different basins of attraction. For example, this happens in the Einstein-Yang-Mills model mentioned above, where the transition point representing the coexistence of two types of critical behaviour is nothing else but the codimension two intersection of the stable manifolds of two critical solutions.

This framework gives also a natural explanation of the power-law (1), and, more importantly, provides a method of computing the critical exponent $\gamma$. For continuously self-similar critical solutions the argument goes as follows. As I wrote above, the evolution of a small perturbation around the critical solution is dominated by the single unstable mode. The amplitude of this mode is proportional to $|p - p^*|(t^* - t)^{-\lambda}$, where the Lyapunov exponent $\lambda$ is positive. Assuming that this perturbation leads to collapse ($p > p^*$), it follows from dimensional analysis that the scale of mass of a resulting black hole is set by the time in which the perturbation grows to a finite size: $(p - p^*)(t^* - t)^{-\lambda} \sim O(1)$. Hence, $M_{BH} \sim (p - p^*)^{1/\lambda}$, and the comparison with (1) yields $\gamma = 1/\lambda$. Thus, to compute $\gamma$ it suffices to find a critical solution and then, using linear stability analysis, calculate the Lyapunov exponent of the unstable mode. The critical exponents have been computed in this manner in several models. The fact that these calculations have reproduced the values obtained from dynamical simulations, confirms the correctness of the overall picture.

In the case of a discretely self-similar critical solution the calculation is technically much more complicated – here one obtains a sort of nonlinear hyperbolic eigenvalue problem for the periodicity scale $\Delta$.

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5 A critical solution can be computed by inserting the self-similarity ansatz into the field equations. In the case of continuous self-similarity this reduces the problem to solving a system of ODE’s. The case of discrete self-similarity is more complicated – here one obtains a sort of nonlinear hyperbolic eigenvalue problem for the periodicity scale $\Delta$. 

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more involved but the basic idea remains the same [12].

The mechanism described above is analogous to the standard renormalization group (RG) description of second order phase transitions in statistical physics. Actually, it is more than analogy – the time evolution can be viewed as the flow in the phase space generated by the iteration of RG transformation [14,15] (which amounts to a suitable rescaling of variables). The critical solution corresponds to a fixed point (continuous self-similarity) or a limit cycle (discrete self-similarity) of the RG transformation [16]. This approach is very useful in determining the so called irrelevant terms in the evolution equations. For example, it has been observed by Choptuik that the inclusion of a mass or self-interaction term to the Einstein-massless scalar field equations does not affect quantitatively the critical behaviour. In other words, all models of the minimally coupled gravitating scalar field with different self-interaction potentials belong to the same universality class (here universality refers to model-independence). This follows immediately from the RG transformation for the scalar field which has the form

$$\phi_L(r, t^* - t) \equiv R_L \phi(r, t^* - t) = \phi(r/L, (t^* - t)/L), \quad L > 1. \quad (2)$$

It is easy to check that the "potential" terms in the field equations for $\phi_L$ are multiplied by $L^{-2}$ (as compared to the "kinetic" terms) and therefore after many iterations these terms become negligible, ergo their presence does not change the long-time asymptotics.

Although the ideas borrowed from the theory of phase transitions in statistical physics have been heuristically very helpful in understanding the critical behaviour in gravitational collapse, the speculations about possible deep physical connection between these phenomena are not, in my opinion, justified and belong rather to an art of writing introductions in the Physical Review Letters. This remark applies in particular to the interpretation of the black hole mass as an order parameter, which is based on the striking similarity of the power-law (1) with the analogous formula for the spontaneous magnetisation in the ferromagnetic/paramagnetic phase transition. It seems to me that instead of taking such analogies too literally, it would be more fruitful to search insight in toy-models exhibiting
dynamical phase transitions. Let me illustrate what I mean with the following example.

Consider a bistable dissipative gradient system

$$\frac{\partial f}{\partial t} = -\nabla E(f).$$

(3)

Here the energy functional $E(f)$, whose gradient flow generates the evolution, is assumed to be bounded from below and to have exactly two minima $f_{\pm}$. If, in addition, the space of finite energy functions is connected, then one can show (modulo technicalities) by minimax argument that there exists a saddle point of energy (denote it by $f^*$). These three extrema of energy are the stationary solutions of eq.(3), two stable ones and one unstable. It follows from (3) that the energy decreases in time, so it is natural to expect (and in fact can be proven in many cases) that these stationary solutions are the only possible end-points of the evolution (assuming that global evolution exists). Of course, generic initial data will flow to one of the stable solutions $f_{\pm}$. By definition such data comprise respective basins of attraction, $W_{\pm}^S$, of these solutions. A codimension one boundary between $W_{-}^S$ and $W_{+}^S$ is a stable manifold, $W_{S}^\ast$, of the saddle point $f^\ast$. Now, consider, as above, a one-parameter family $f_p$ of initial data interpolating between $W_{-}^S$ and $W_{+}^S$. As usual, let $p^\ast$ be the critical parameter value for which $f_p$ intersects $W_{S}^\ast$. In the intermediate asymptotic regime the evolution of near-critical initial data is dominated by the single unstable mode of $f^\ast$

$$f_p(t) \simeq f^\ast + C(p - p^\ast) e^{\lambda t} \xi,$$

(4)

where $\xi$ is the eigenmode associated with the positive eigenvalue $\lambda$. Here the amplitude $C$ is the only vestige of initial data. Depending on the sign of $C$, the solution $f_p(t)$ will ultimately evolve towards $f_+$ or $f_-$. The ”lifetime” $T$ of the near-critical solution staying in the vicinity of $f^\ast$ is determined by the condition $|p - p^\ast| e^{\lambda T} \sim O(1)$ which gives $T \sim (-1/\lambda) \ln |p - p^\ast|$.}

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6For the sake of brevity I am cavalier about the mathematical ”details” (such as an important issue of compactness). I simply assume tacitly that the functional $E(f)$, when defined in a suitable function space, has all needed properties.
This model reflects quite well certain features of type I critical collapse observed in the Einstein-Yang-Mills model. Of course there are substantial differences, but one cannot expect too much from such a simple model. It would be most interesting to construct a toy-model of type II critical behaviour. This could give insight into the origin of discrete self-similarity, which is widely considered as the most mysterious feature of critical collapse. Let me end with the speculation that, in this respect, there might be a remote mathematical connection between echoing and the dynamical formation of fine structure in certain material phase transformations, where the flow of energy to higher and higher wavenumbers can be understood in terms of models described by eq.(3) with the energy functional which does not attain a minimum but has minimizing sequences with finer and finer structure [17].

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