Equivariant dendroidal sets and simplicial operads

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Abstract

We establish a Quillen equivalence between the homotopy theories of equivariant Segal operads and equivariant simplicial operads with norm maps. Together with previous work, we further conclude that the homotopy coherent nerve is a right-Quillen equivalence from the model category of equivariant simplicial operads with norm maps to the model category structure for equivariant-∞-operads in equivariant dendroidal sets.

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1 Introduction

This paper is the last in a series of five (after [Per18, BP20, BPa, BPb]) and concludes a project to establish a homotopy theoretical equivalence between equivariant colored simplicial operads with norm maps and equivariant-\infty-operads in equivariant dendroidal sets, thus generalizing the analogous Cisinski-Moerdijk project [CM13a, CM13b, CM11] in the non-equivariant setting.

The key novelty (and difficulty) faced in the equivariant setting is that the homotopy theory of operads needs to account for an extra piece of structure, the so called norm maps, which we now briefly recall (for further discussion, see the introductions to any of [Per18],[BP21],[BPa]).

For simplicity, let us focus on the category \( sOp∗ \) of single colored simplicial operads. Letting \( G \) be a fixed finite group, a \( G \)-equivariant (single colored simplicial) operad is a \( G \)-object \( O \in sOp∗_G \). Crucially, note that the \( n \)-th level \( O(n) \) then admits commuting actions by \( \Sigma_n \) and \( G \) or, equivalently, a \( G \times \Sigma_n \) action. One upshot of Blumberg and Hill’s work in [BH15] is that the preferred notion of equivalence in \( sOp∗_G \) is that of graph equivalence, by which we mean maps \( O \rightarrow P \) in \( sOp∗_G \) such that the fixed point maps for graph subgroups

\[
O(n)^\Gamma \sim P(n)^\Gamma \quad \text{for } \Gamma \leq G \times \Sigma_n \text{ such that } \Gamma \cap \Sigma_n = *
\]

are Kan equivalences in \( sSet \). The term graph subgroup is a description of such \( \Gamma \): they are necessarily of the form \( \Gamma = \{(h, \phi(H))| h \in H\} \) for some subgroup \( H \leq G \) and homomorphism \( \phi : H \rightarrow \Sigma_n \). Such fixed points \( O(n)^\Gamma \) are called spaces of norm maps since, for \( X \) an \( O \)-algebra, the algebra multiplication maps on the left below (see, e.g., [BPb, (1.3)])

\[
O(n) \times X^\times n \rightarrow X \quad O(n)^\Gamma \times N_{\Gamma}X \rightarrow X
\]

induce \( H \)-equivariant maps as on the right above, where \( N_{\Gamma} \) is a so called norm object, which denotes \( X^\times n \) together with the \( H \)-action induced by the identification \( H \simeq \Gamma \).

The cornerstone of this project was the discovery by the authors of a category \( \Omega_G \) of \( G \)-trees whose objects encode compositions of norm maps in a \( G \)-operad \( O \) (a detailed discussion can be found in [Per18, §4.3]), and which extends the Moerdijk-Weiss category \( \Omega \) of trees (whose objects encode composition in an operad). This category \( \Omega_G \) then allowed us to build model structures on the categories \( dSet^G = Set^{\Omega_G \times G} \) of equivariant dendroidal sets [Per18, Thm. 2.1] and \( sOp^G \) of equivariant colored simplicial operads [BPb, Thm. A], where in both cases the notion of weak equivalence is determined by (a colored variant of) the norm map data as in (1.1). Our main result in this paper is then the following, generalizing [CM13b, Thm. 8.15].

**Theorem 1.** There is a Quillen equivalence

\[
W_I : dSet^G \rightleftarrows sOp^G : hcN
\]

between equivariant dendroidal sets and equivariant simplicial operads with norm maps.

Here the right adjoint \( hcN \) is a variant of the nerve functor accounting for homotopy information, called the homotopy coherent nerve, while the left adjoint \( W_I \) is a “fattened operadification” which is related to the Boardman-Vogt resolution of operads (see [CM13b, §4] for details).

Writing \( \eta \) for both the terminal category and its nerve, one has natural identifications of slice categories \( dSet^G_{/\eta} \simeq dSet^G \) and \( sOp^G_{/\eta} \simeq sCat^G \), so that by slicing Theorem 1 one recovers Bergner’s result [Ber17] in the \( \infty \)-category context.

However, it is worth nothing that the presence of norm maps make our operadic result far more subtle than the categorical analogue. Indeed, the model structures on \( sSet^G \), \( sCat^G \) used in [Ber17] are built formally from the non-equivariant model structures on \( sSet \), \( sCat \) by using
the abstract framework in [Ste16]. On the other hand, applying that framework to the operadic context results in model structures on $dSet^G$ and $sOp^G$ that only account for the trivial norm maps (i.e. those for which $\Gamma \leq G$ in (1.1)).

The (conclusion to the) proof of Theorem I can be found at the end of §4.4. However, this proof requires some background, which we now recall. Just as in [CM13b], we make use of two additional categories, the category $sdSet^G = \text{Set}^{G \times \Delta^n \times G}$ of equivariant dendroidal simplicial sets and its subcategory $\text{PreOp}^G$ of equivariant preoperads, which fit into a diagram

$$
\begin{array}{ccc}
\text{PreOp}^G & \xleftarrow{\gamma^*} & \text{sOp}^G \\
N & \searrow & \downarrow \text{hcN} \\
\text{sdSet}^G & \xleftarrow{\gamma_*} & \text{dSet}^G
\end{array}
$$

where $N$ is the nerve functor and $\gamma^*, \gamma_*$ are the natural inclusions.

The model structures on the categories featured in (1.3) were built in previous work. More specifically, [Per18, Thm. 2.1] provides the model structure on $dSet^G$, [BPb, Thm. 1] provides the model structure on $sOp^G$, [BP20, Def. 4.22] gives the model structure on $sdSet^G$, and [BP20, Thm. 4.39] provides the model structure on $\text{PreOp}^G$. These model structures generalize those in the work of Cisinski-Moerdijk in the non-equivariant operadic context, which in turn generalize corresponding model structures in the categorical context. The following table, first appearing in [BP20], summarizes the relevant model structures, along with the nomenclature for the fibrant objects.

| “categories up to htpy” | “operads up to htpy” | “equivariant operads up to htpy” |
|------------------------|----------------------|-------------------------------|
| $\text{simplicial sets } sSet$ | $\text{dendroidal sets } dSet$ | $\text{equivariant dendroidal sets } dSet^G$ |
| Joyal model structure | model str. from [CM11] | model structure from [Per18] |
| $\infty$-categories | $\infty$-operads | $G$-$\infty$-operads |
| $\text{bisimplicial sets } ssSet$ | $\text{simplicial dendroidal sets } sdSet$ | $\text{equiv. simplicial dendroidal sets } sdSet^G$ |
| Rezk model structure | model str. from [CM13a] | model structure from [BP20] |
| complete Segal spaces | complete dend. Segal spaces | complete equiv. dend. Segal spaces |
| $\text{Segal precategories } \text{SeCat}$ | $\text{Segal preoperads } \text{PreOp}$ | $\text{equiv. Segal preoperads } \text{PreOp}^G$ |
| $\text{Hirschowitz-Simpson}$ | model str. from [CM13a] | model structure from [BP20] |
| Segal categories | Segal operads | equiv. Segal operads |
| $\text{simplicial categories } sCat$ | $\text{simplicial operads } sOp$ | $\text{equiv. simplicial operads } sOp^G$ |
| Bergner model structure | model str. from [CM13b] | model structure from [BPb] |

Table 1: A summary of models for $\infty$-categories, $\infty$-operads, and $G$-$\infty$-operads.

Considering now the functors in (1.3), we have previously established that $\gamma$ and $\gamma^*$ are both left-Quillen equivalences [BP20, Thms. 4.30 and 4.41].

The proof strategy for establishing Theorem I can then be summarized as follows.

First, the $(W, \text{hcN})$ adjunction is shown to be Quillen (Proposition 4.55).

Second, the square (1.3) is shown to commute at the level of homotopy categories (this is shown at the end of §4.4, by establishing the zigzag of weak equivalences in (4.60)).

Third and last, it thus suffices to show that the top horizontal functor $N$ in (1.3) induces an equivalence of homotopy categories. As in [CM13b], this last step requires some care. The functor $N$ preserves all weak equivalences (cf. the proof of Theorem 4.48) and is thus already a derived functor, but it is not quite right Quillen due to $\text{PreOp}^G$ not having enough fibrant objects or, dually, having too many cofibrant objects. To address this, we show in §3.4 that $\text{PreOp}^G$
admits an alternative model structure, called the \textit{tame model structure} (Theorem 3.47), with the same weak equivalences but less cofibrant objects. Using this alternative model structure, it can then be shown (Theorem 4.48) that \( N \) becomes a right-Quillen equivalence, concluding the argument.

### 1.1 Outline

First, in §2 we mostly recall some notions from previous work that are used throughout. Namely, §2.1 and §2.2 recall the necessary properties of the categories \( \Omega \) of trees and \( \Omega_G \) of \( G \)-trees, while §2.3 recalls the model structure on the category \( \mathsf{dSet}^G \) of equivariant dendroidal sets.

The overall goal of §3 is to establish the existence of the alternative \textit{tame model structure} on \( \mathsf{PreOp}^G \). The content of §3.1 and §3.2 is again expository in nature, recalling the model structures on \( \mathsf{sdSet}^G \), \( \mathsf{PreOp}^G \). In §3.3 we introduce a somewhat novel construction, called the \textit{fibered tensor product} \( \otimes_C \), which is used in §3.4 to describe and build the tame model structure on \( \mathsf{PreOp}^G \). The use of \( \otimes_C \) is motivated by the observation that the model structure on \( \mathsf{sOp}^G \) can be described using an analogous tensor product, thus simplifying the task of showing that \( \tau: \mathsf{PreOp}^G \to \mathsf{sOp}^G : N \) is a Quillen adjunction (cf. Theorem 4.48).

The goal of §4 is to prove Theorem I (up to Lemma 4.37, whose proof is postponed to §5). First, §4.1 recalls the model structure on \( \mathsf{sOp}^G \). Then, in §4.3 we establish Theorem 4.48, showing that the top functor \( N \) in (1.3) induces an equivalence of homotopy categories. Lastly, §4.4 concludes the proof of Theorem I by showing that (1.3) commutes in a homotopical sense.

Our last main section §5 is dedicated to the rather technical proof of Lemma 4.37, which examines the homotopical properties of certain pushouts in \( \mathsf{Op}^G \) after applying the nerve functor \( N: \mathsf{Op}^G \to \mathsf{dSet}^G \), and is at the core of the proof of Theorem 4.48 and thus also of Theorem I.

Lastly, in Appendix A we give an explicit description of the discretization of a \( G \)-\( \infty \)-operad \( X \in \mathsf{dSet}^G \), adapting the similar non-equivariant description in [MW09, §6]. This then allows us to show that, for a fibrant operad \( \mathcal{O} \in \mathsf{sOp}^G \), the natural discretizations of \( hcN(\mathcal{O}) \in \mathsf{dSet}^G \) and \( N(\mathcal{O}) \in \mathsf{sdSet}^G \) coincide (Proposition A.22), thus generalizing the non-equivariant analogue result [CM13b, Prop. 4.8].

Remark 1.4. This paper utilizes and compares several model structures on related categories. We list them below, along with internal references for their definitions.

- model structure on \( \mathsf{dSet}^G \), Theorem 2.37.
- joint model structure on \( \mathsf{sdSet}^G \), Theorem 3.3.
- normal model structure on \( \mathsf{PreOp}^G \), Theorems 3.15 and 3.33.
- tame model structure on \( \mathsf{PreOp}^G \), Theorem 3.47.
- model structure on \( \mathsf{sOp}^G \), Theorem 4.9.

### 2 Equivariant trees and dendroidal sets

In this mostly expository section, we recall the categories of trees, as well as the associated presheaf categories, which will be needed throughout the paper. A more detailed discussion can be found in [Per18, §§5,§6], [BP20, §2].
2.1 Trees and forests

We start by recalling the Moerdijk-Weiss category $\Omega$ of trees [MW07]. First, each object of $\Omega$ can be encoded by a (rooted) tree diagram $T$ as below.

![Tree Diagram](image)

(2.1)

Edges with no vertices $\circ$ above them are called leaves, the unique bottom edge is called the root, and edges that are neither are called inner edges. In the example above, $a$, $b$ and $d$ are leaves, $r$ is the root, and $c$ and $e$ are inner edges. The sets of edges, inner edges, and vertices of a tree $T$ are denoted $E(T)$, $E_i(T)$, and $V(T)$, respectively.

Describing the maps in $\Omega$ requires some care. To do so, we recall the algebraic notion of a broad poset, originally due to Weiss [Wei12] and further developed in [Per18]. For each edge $t$ in a tree topped by a vertex $\circ$, we write $t^\uparrow$ for the tuple of edges immediately above $t$. In (2.1) one has $r^\uparrow = cde$, $c^\uparrow = ab$, and $e^\uparrow = \epsilon$, where $\epsilon$ denotes the empty tuple. We then encode each vertex symbolically as $t^\uparrow \leq t$, which we call a generating broad relation. This notation is motivated by a form of transitivity. For example, in (2.1) the relations $cde \leq r$ and $ab \leq c$ generate, under broad transitivity, the relation $abde \leq r$, and one may similarly obtain relations $cd \leq r$ and $abd \leq r$.

These relations, together with identity relations $t \leq t$, then form the broad poset associated with $T$ (alternatively, this broad poset data is essentially equivalent to the data of the colored operad $\Omega(T)$ associated to $T$, cf. [MW07, §3], [Per18, Rem. 4.4] or (4.17)).

A map of trees $\varphi : S \to T$ in $\Omega$ is then an underlying map of edge sets $\varphi : E(S) \to E(T)$ which preserves broad relations.

If an edge $t$ is pictorially above (or equal to) an edge $s$, we write $t \preceq d s$. Equivalently, $t \preceq d s$ if there exists a broad relation $s^1 \ldots s^n \preceq s$ such that $t = s_i$ for some $i$.

Moreover, our discussion will be simplified by assuming that $\Omega$ has exactly one representative of each planarized tree, by which we mean a tree together with a planar representation as in (2.1) (alternatively, planarizations can be formalized as suitable extensions of $\preceq d$ to a total order [BP21, §3.1]). Importantly, this implies that each map $\varphi : S \to T$ in $\Omega$ has a strictly unique factorization $S \to S' \to T$ as an isomorphism followed by a planar map [BP21, Prop. 3.24]. Informally, $S'$ is obtained by giving $S$ the planarization “pulled back” from $T$. Note that, in particular, the subcategory of planar maps is skeletal, i.e. the only planar isomorphisms are the identities.

**Notation 2.2.** We write $\eta$ for the stick tree, the unique tree with a single edge and no vertices.

**Example 2.3.** The edge labels in each tree $S_i$ below determine maps $E(S_i) \to E(T)$, where $T$ is as in (2.1). For $i \leq 4$ this encodes maps $S_i \to T$ in $\Omega$, but not for $i = 5$.

![Tree Diagrams](image)

$^1$That is, this factorization is not simply unique up to unique isomorphism.
A map of trees $\varphi:S \to T$ is called:

- a **tall map** if $\varphi(l_S) = l_T$ and $\varphi(r_S) = r_T$, with $l(\cdot)$ and $r(\cdot)$ denoting the tuple of leaf edges and the root edge;
- a **face map** if it is injective on edges; an **inner face** if it is also tall; and an **outer face** if, for any factorization $\varphi \simeq \varphi_1 \varphi_2$ with $\varphi_1, \varphi_2$ face maps and $\varphi_2$ inner, $\varphi_2$ is an isomorphism;
- a **degeneracy** if it is surjective on edges and preserves leaves (and is thus tall).

Pictorially, inner face maps $S \to T$ remove some edges in $T$ (and merge the vertices adjacent to those edges), outer face maps remove some vertices of $T$, and degeneracies collapse some of the unary vertices of $S$.

**Example 2.4.** In Example 2.3, $S_1 \to T$ is an inner face, $S_2 \to T$ is an outer face, $S_3 \to T$ is a face that is neither inner nor outer, and $S_4 \to T$ is a degeneracy.

**Notation 2.5.** In the remainder of §2 we will label a map in $\Omega$ by the letters d/i/o/t/f/p to indicate that the map is a degeneracy/inner face/outer face/tall/face/planar.

**Proposition 2.6 ([BP20, Prop. 2.2]).** A map of trees $\varphi:S \to T$ has a factorization, unique up to unique isomorphisms,

$$S \xrightarrow{d} S' \xrightarrow{i} S'' \xrightarrow{o} T \tag{2.7}$$

as a degeneracy followed by an inner face followed by an outer face.

**Remark 2.8.** A map $\varphi:S \to T$ is tall (resp. a face) iff in the decomposition (2.7) the component labeled o (resp. d) is an isomorphism. As such, by combining the first two (resp. last two) maps in (2.7) one recovers the “tall-outer face” (resp. “degeneracy-face”) factorization of the map $\varphi$ [BP21, Prop. 3.36]; [Per18, Prop. 5.37].

**Remark 2.9.** Following the previous remark, it is natural to consider the class of maps $\varphi:S \to T$ such that the inner face factor in (2.7) is an isomorphism. We call these maps convex, since they are readily seen to be characterized by the following property: if $e <_d e' <_d e''$ in $T$ and $e, e''$ are in the image of $\varphi$ then so is $e'$. Notably, it follows from this characterization that convex maps are also closed under composition.

Equivalently, $\varphi$ is convex precisely if the “tall-outer face” and “degeneracy-face” factorizations coincide. In particular, outer faces are characterized as the convex faces.

When accounting for planar structures, one has the following refinement of Proposition 2.6.

**Proposition 2.10** (cf. [BP20, Prop. 2.2]). A map of trees $\varphi:S \to T$ has a strictly unique factorization

$$S \xrightarrow{d} S_p \xrightarrow{pd} \varphi S \xrightarrow{pi} \overline{\varphi S} \xrightarrow{po} T \tag{2.11}$$

as an isomorphism followed by a planar degeneracy, a planar inner face, and a planar outer face.

**Remark 2.12.** The notation $\varphi S$ is motivated by the fact that this tree has edge set $E(\varphi S) = \varphi(E(S))$, while the notation $\overline{\varphi S}$ is an instance of the outer closure of an inner face notation in [BP20, Not. 2.14].

**Remark 2.13.** Generalizing Remarks 2.8 and 2.9, one has that, for any subset $S \subseteq \{\simeq, pd, pi, po\}$ of the arrow labels in (2.11), the type of maps whose factors labeled by $S$ are identities is closed under composition.

For example, the maps such that the factors labeled $\simeq$ and $pi$ are identities are the planar convex maps, while those maps such that the factors labeled $pd$ and $po$ are identities are the (possibly not planar) inner face maps. Both of these kinds of maps are closed under composition.
A corolla is a tree with a single vertex. We note that the subcategory of $\Omega$ spanned by corollas and isomorphisms is naturally identified with to the category $\Sigma$ of standard finite ordered sets \{1,2,\ldots,n\} and (non-ordered) isomorphisms.

**Notation 2.14.** For $T \in \Omega$ and $v \in V(T)$, we write $T_v \hookrightarrow T$ for the subcorolla whose vertex is $v$.

Next, we recall the categories of (colored) forests used in [BPa, Def. 3.21].

**Definition 2.15.** The category $\Phi$ of forests is the coproduct completion of the category $\Omega$ of trees: objects are formal coproducts $F = \cup_{i \in I} F_i$ with $F_i \in \Omega$, and an arrow $\varphi : \cup_{i \in I} F_i \rightarrow \cup_{j \in J} F'_j$ is given by a map of indexing sets $\varphi : I \rightarrow J$ and maps $\varphi_i : F_i \rightarrow F'_\varphi(i)$ in $\Omega$ for each $i \in I$.

The sets of edges, inner edges, vertices of a forest $F = \cup_i F_i$ are defined in the natural way as

$$E(F) = \cup_i E(F_i), \quad E^i(F) = \cup_i E^i(F_i), \quad V(F) = \cup_i V(F_i).$$

**Remark 2.16.** As with trees $T \in \Omega$, we assume that each forest $F = \cup_{i \in I} F_i \in \Phi$ is planarized [BP21, Def. 3.2 and Rem. 3.15], which is equivalent to choosing a total order of the indexing set $I$ and a planarization of each $F_i$. Moreover, we similarly assume that $\Phi$ contains exactly one representative of each planarization, so that the only planar isomorphisms are again the identities.

However, we caution that, in order to pullback a planarization along $\varphi : F \rightarrow \tilde{F}$, one needs to assume $\varphi$ sends roots of $F$ to $\leq_d$-incomparable edges of $\tilde{F}$, i.e. that $\varphi$ is an independent map [Per18, Def. 5.28]. As such, in the context of forests our definition of planar map requires that the map is independent [BP21, Prop. 3.19]. In particular, the factorization $F \xrightarrow{\tilde{\varphi}} F' \rightarrow \tilde{F}$ of a map $\varphi$ as an isomorphism followed by a planar map exists only if $\varphi$ is independent [BP21, Prop. 3.24].

**Definition 2.17.** Let $\mathcal{C}$ be a set of colors. The category $\Phi_\mathcal{C}$ of $\mathcal{C}$-forests has:

- objects pairs $\tilde{F} = (F, c)$ with $F \in \Phi$ a forest and $c : E(F) \rightarrow \mathcal{C}$ a coloring of its edges;
- arrows $\tilde{F} = (F, c) \rightarrow (F', c') = \tilde{F}'$ maps $\varphi : F \rightarrow F'$ in $\Phi$ such that $c = c' \varphi$.

Lastly, we write $\Omega_\mathcal{C} \subset \Phi_\mathcal{C}$, which we call the category of $\mathcal{C}$-trees, for the full subcategory spanned by the objects whose underlying forest is a tree, and $\Sigma_\mathcal{C} \subset \Omega_\mathcal{C}$, which we call the category of $\mathcal{C}$-corollas, for the further subcategory of objects whose underlying tree is a corolla and whose maps are isomorphisms. Note that a change of color map $f : \mathcal{C} \rightarrow \mathcal{D}$ induces a map $f : \Phi_\mathcal{C} \rightarrow \Phi_\mathcal{D}$ via $\tilde{F} = (F, c) \mapsto (f F, f c)$. Also, $\tilde{F} = (F, c) \mapsto (f F, f c)$. 

### 2.2 Equivariant trees

We next recall the category $\Omega_G$ of equivariant trees, which encodes the combinatorics of compositions of norm maps. A thorough discussion can be found in [Per18, §5] or [BP20, §2].

We begin with an example. Let $G = \langle \rho, \sigma | \rho^4 = 1, \sigma^2 = 1, \sigma \rho = \rho^3 \sigma \rangle = \{1, \rho, \rho^2, \rho^3, \sigma, \rho \sigma, \rho^2 \sigma, \rho^3 \sigma \}$ be the dihedral group with 8 elements, and $L \leq K \leq H \leq G$ denote the subgroups $H = \langle \rho^2, \sigma \rangle$, $K = \langle \rho^2 \rangle$, $L = \{1\}$. There is then a $G$-tree $T \in \Omega_G$ with expanded representation given by the two
trees on the left below, and orbital representation given by the single tree on the right.

\[
\begin{array}{cccccc}
  & a & b & \rho^2a & \sigma a & \rho a \\
  & c & \sigma c & \rho a & \rho^2\sigma a & \rho^3\sigma a \\
 & d & \sigma d & \rho c & \rho^2\sigma c & \rho^3\sigma a \\
\end{array}
\]

edge labels in the expanded representation encode a $G$-action, so that the edges labeled $a, b, c, d$ have isotropies $L, K, K, H$, respectively. On the other hand, the orbital representation displays the edge orbits in the expanded representation, labeled by the associated transitive $G$-set.

Formally, $\Omega_G$ is defined as follows. Write $\Phi^G$ for the category of $G$-forests, i.e. $G$-objects in the category $\Phi$ of forests in Definition 2.15. We then define $\Omega_G \subseteq \Phi^G$ as the full subcategory consisting of the (non-empty) $G$-forests such that the $G$-action is transitive on tree components.

For $T \in \Omega_G$, we write $E_G(T) = E(T)/G$, $E'_G(T) = E'(T)/G$, $V_G(T) = V(T)/G$ for the sets of edge orbits, inner edge orbits, and $G$-vertices, respectively.

**Remark 2.19.** We caution that there is a proper inclusion, $\Omega^G \subsetneq \Omega_G$, where $\Omega^G$ denotes $G$-objects in $\Omega$.

**Remark 2.20.** We find it convenient to consider multiple descriptions of a $G$-tree $T \in \Omega_G$.

First, we have the forest description $T = \cup_{i \in I} T_i$, where $I$ is a transitive $G$-set. Second, given a tree component $T_*$ with stabilizer $H \leq G$, one has a decomposition $T \simeq G \cdot H T_*$. Third, writing $\Gamma \leq G \times \text{Aut}(T_*)$ for the graph subgroup (cf. (1.1)) associated to the homomorphism $\varphi : H \to \text{Aut}(T_*)$ that encodes the $H$-action on $T_*$, one has a quotient description $T \simeq (G \cdot T_*)/\Gamma$.

**Remark 2.21.** The two representations in (2.18) play complementary roles in the discussion of $G$-trees. On the one hand, maps in $\Omega_G$ are best understood via expanded representations, which encode the relevant broad poset structures. On the other hand, orbital representations pictorially encode composition data for norm maps of operads (see e.g. [Per18, Ex. 4.9], [BP20, Rem. 3.39]).

Following the discussion at the end of §2.1, each $G$-tree $T = \cup_{i \in I} T_i$ is planarized, so that each tree component $T_i$ is planarized and the set $I$ of components is totally ordered.

As was the case in $\Omega$, maps $\varphi : S \to T$ in $\Omega_G$ are likewise built from a few basic types of maps.

Recall that a map $\varphi : \cup_{i \in I} S_i \to \cup_{j \in J} S_j$ in $\Omega_G$ is described by a map of sets $\varphi : I \to J$ and maps of trees $S_i \to T_{\varphi(i)}$ for $i \in I$. A map $\varphi$ is called a quotient map if all maps $S_i \to T_{\varphi(i)}, i \in I$ are isomorphisms in $\Omega$ and called a sorted map if $\varphi : I \to J$ is an ordered isomorphism of sets. Further, a sorted map is called a sorted degeneracy/tall map/face/inner face/outer face if each of the component maps is a degeneracy/tall map/face/inner face/outer face in $\Omega$.

**Definition 2.22.** We write $\Omega^G_0 \subseteq \Omega_G$ for the wide subcategory of $G$-trees and quotient maps. Further, we write $\Sigma_G \subseteq \Omega^G_0$ for the full subcategory spanned by $G$-corollas and quotient maps, where $C \in \Omega_G$ is a $G$-corolla if $|V_G(C)| = 1$ or, equivalently, if its trees components are corollas.

**Example 2.23.** Let $G = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ be the cyclic group with two elements. In the diagram
below, the assignment \( \alpha \mapsto a, \bar{\alpha} \mapsto -a, \beta \mapsto b, \gamma \mapsto c \) determines a quotient map \( S \to T \).

\[
\begin{array}{cccc}
\alpha & \beta & \gamma & G + \gamma \\
\bar{\alpha} & -\bar{\alpha} & -\gamma & G + \beta \\
S & T & c & G + a \\
& a & -a & G/G + c \\
& & & G/G + b
\end{array}
\]

**Notation 2.24.** Following Notation 2.14, for \( T \in \Omega_G \) and a \( G \)-vertex \( v \in V_G(T) \) given as a \( G \)-orbit \( [v_i] \) with \( v_i \in V(T) \), we write \( T_v = u_{[v_i]} T_{v_i} \). Informally, \( T_v \) is the \( G \)-corolla formed by the associated “\( G \)-orbit of corollas” \( T_{v_i} \).

**Corollary 2.25** (cf. Prop. 2.6, [Per18, Rem. 5.49]). A map \( \varphi: S \to T \) in \( \Omega_G \) has a factorization

\[
S \xrightarrow{s} S' \xrightarrow{si} S'' \xrightarrow{sp} T^* \xrightarrow{\varphi} T,
\]

unique up to unique sorted isomorphisms, as a sorted degeneracy followed by a sorted inner face, a sorted outer face, and a quotient map.

**Remark 2.26.** All sorted maps \( \varphi: S \to T \) in \( \Omega_G \) are independent maps, so one may ask if such a map is a planar map, which is then equivalent to each component map \( S_i \to T_{v(i)} \) being planar.

On the other hand, a quotient map \( S \to T \) in \( \Omega_G \) is independent iff it is an isomorphism, so a quotient is planar iff it is an identity. More generally, asking for a quotient map to be planar involves componentwise planar maps, i.e. for the maps \( S_i \to T_{v(i)} \) to be planar, is the same as requiring the maps \( S_i \to T_{v^i} \) to be identities (since we already know they are isomorphisms). A quotient map with this additional property of being componentwise planar is called a pullback map (cf. [BP21, Ex. 3.27]).

**Corollary 2.27** (cf. Prop. 2.10, [Per18, Rem. 5.49]). A map \( \varphi: S \to T \) in \( \Omega_G \) has a strictly unique factorization

\[
S \xrightarrow{s} S \xrightarrow{sp} S' \xrightarrow{sp} S'' \xrightarrow{s''} \varphi^* T \to T,
\]

as a sorted isomorphism followed by a sorted planar degeneracy, a sorted planar inner face, a sorted planar outer face, and a pullback map.

Lastly, in contrast to the notion of sorted face map above, the description of the model structure on equivariant dendroidal sets \( \mathbf{dSet}^G \) will require an additional notion of face.

**Definition 2.28.** Let \( T = u_i T_i \) be a \( G \)-tree. An (outer) face of \( T \) is an (outer) face map \( U \to T_i \) from some \( U \in \Omega \) to a component \( T_i \) of \( T \). A face of \( T \) is called planar if the map \( U \to T_i \) is a planar map.

We write \( \text{Face}(T) \) for the \( G \)-poset of planar faces of \( T \), and let \( \text{Face}_{\leq}(T) \subseteq \text{Face}(T) \) denote the subposet spanned by the planar outer faces of \( T \) with no inner edges (these are the faces determined by either a single edge or a single vertex of \( T \)).

### 2.3 Equivariant dendroidal sets

Recall that the category of *dendroidal sets* [MW07] is the presheaf category \( \mathbf{dSet} = \mathbf{Set}^{\Omega^{op}} \). For a (finite) group \( G \), the category of \( G \)-dendroidal sets is then the \( G \)-object category \( \mathbf{dSet}^G = \mathbf{Set}^{G \times \Omega^{op}} \).
One key subtlety when working with \( \text{dSet}^G \) is that for each equivariant dendroidal set \( X \in \text{dSet}^G \) its levels \( X(U) \) are indexed by non-equivariant trees \( U \in \Omega \), while the key classes of maps in \( \text{dSet}^G \) are defined in terms of G-trees \( T \in \Omega_G \).

To describe these maps, we first extend the Yoneda embedding \( \Omega[-]: \Omega \to \text{dSet} \) notation for the representable functors \( \Omega[U](V) = \Omega(V, U) \) to obtain extended Yoneda embeddings (here the right embedding is simply obtained from the left embedding by taking \( G \)-objects)

\[
\begin{align*}
\Phi & \quad \Omega[-] \quad \text{dSet} \\
\Phi^G & \quad \Omega[-] \quad \text{dSet}^G
\end{align*}
\]

(2.29)

Note that, since \( G \)-trees \( \Omega_G \) are defined as a subcategory of \( G \)-forests \( \Phi^G \), the right side of (2.29) defines representables \( \Omega[T] \in \text{dSet}^G \) for \( T \in \Omega_G \). These representable presheaves \( \Omega[T] \) then allow us to generalize the key presheaves in dendroidal sets \( \text{dSet} \) (cf. \([CM13a, \S 2]\)) to equivariant dendroidal sets \( \text{dSet}^G \) (cf. \([Per18, \S 6]\) or \([BP20, \S 2.3]\)), as follows.

**Definition 2.30.** Let \( T = u_i T_i \) be \( G \)-tree. The boundary \( \partial \Omega[T] \subseteq \Omega[T] \) is defined by

\[
\partial \Omega[T] = \bigcup_i \partial \Omega[T_i] = \colim_{U \in \text{Face}(T), U \subseteq T_i} \Omega[U] = \bigcup_{U \in \text{Face}(T), U \subseteq T_i} \Omega[U].
\]

Next, for \( \emptyset \neq E \subseteq E'(T) \) a non-empty \( G \)-subset of inner edges and writing \( E_i = E \cap E(T_i) \), the \( G \)-inner horn \( \Lambda^E[T] \subseteq \Omega[T] \) is defined by

\[
\Lambda^E[T] = \bigcup_i \Lambda^{E_i}[T_i] = \colim_{U \in \text{Face}(T), (T_i - E_i) \cup U} \Omega[U] = \bigcup_{U \in \text{Face}(T), (T_i - E_i) \cup U} \Omega[U].
\]

(2.31)

Lastly, the Segal core \( \text{Sc}[T] \subseteq \Omega[T] \) is defined by

\[
\text{Sc}[T] = \bigcup_i \text{Sc}[T_i] = \colim_{U \in \text{Face}_c(T)} \Omega[U] = \bigcup_{U \in \text{Face}_c(T)} \Omega[U].
\]

**Remark 2.32.** For \( T \in \Omega_G \), a decomposition \( \Omega \simeq G \cdot H \cdot T \) with \( T \in \Omega^H \) yields identifications

\[
\Omega[T] \simeq G \cdot H \cdot \Omega[T], \quad \partial \Omega[T] \simeq G \cdot H \cdot \partial \Omega[T], \quad \Lambda^E[T] \simeq G \cdot H \cdot \Lambda^E[T], \quad \text{Sc}[T] \simeq G \cdot H \cdot \text{Sc}[T],
\]

where \( E_i = E \cap E(T_i) \).

Adapting the non-equivariant story, the maps in Definition 2.30 are then the basis for a model structure on \( \text{dSet}^G \). In the following, recall that a class of maps in a category is called saturated if it is closed under pushouts, retracts, and transfinite compositions.

**Definition 2.33.** The class of \( G \)-normal monomorphisms in \( \text{dSet}^G \) is the saturation of the boundary inclusions \( \partial \Omega[T] \to \Omega[T] \) for \( T \in \Omega_G \).

The class of \( G \)-inner anodyne extensions in \( \text{dSet}^G \) is the saturation of the \( G \)-inner horn inclusions \( \Lambda^E[T] \to \Omega[T] \) for \( T \in \Omega_G \) and \( \emptyset \neq E \subseteq E'(T) \) a non-empty \( G \)-subset.

**Definition 2.34.** A map \( X \to Y \) in \( \text{dSet}^G \) is called a \( G \)-inner fibration if it has the right lifting property with respect to all \( G \)-inner horn inclusions \( \Lambda^E[T] \to \Omega[T] \).

Moreover, if \( X \to * \) is a \( G \)-inner fibration then \( X \in \text{dSet}^G \) is called a \( G \)-\( \infty \)-operad.

Informally, one may view \( G \)-\( \infty \)-operads as "operads with weak composition laws for norm maps".
To recall the model structure on $\text{dSet}^G$, we need two more ingredients. First, we write
\[ \iota_!: \text{sSet} \rightleftarrows \text{dSet} : \iota^* \]
for the adjunctions where the left adjoints $\iota_!$ are the natural inclusions (given by “extension by $\emptyset$”). Note that the leftmost adjunction is induced by the natural inclusion $\iota : \Delta \to \Omega$ as the linear trees. Second, we write
\[ \tau : \text{sSet} \rightleftarrows \text{Cat} : \tau_* \]
\[ \tau_! : \text{dSet} \rightleftarrows \text{Op} : \tau^* \]
for the adjunctions where the right adjoints are the nerve functors given by $\tau_!(\text{NC})(n) = \text{Cat}([n], \mathcal{C})$ for $\mathcal{C} \in \text{Cat}$ and $\tau_!(\text{NO})(T) = \text{Op}(\Omega(T), \mathcal{O})$ for $\mathcal{O} \in \text{Op}$ and $\Omega$ the colored operad (of sets) generated by $T$ (cf. [MW07, §3], [Per18, Rem. 4.4, Ex. 4.6] or (4.17)).

Recall [MW09, Prop. 5.3 and Thm. 6.1] that the nerve functors $\tau$ are fully faithful, with their essential image characterized as those presheaves that satisfy a Segal condition.

The following theorem synthesizes Theorem 2.1, Proposition 8.8, and Theorem 8.22 of [Per18].

**Theorem 2.37 ([Per18]).** There exists a left proper model structure on $\text{dSet}^G$ such that:

- the cofibrations are the $G$-normal monomorphisms;
- the fibrant objects are the $G$-$\infty$-operads;
- the fibrations between $G$-$\infty$-operads are the $G$-inner fibrations $X \to Y$ such that the induced maps on fixed-point homotopy categories $\tau_*^\times(X^H \to Y^H)$ are isofibrations of categories for all $H \leq G$;
- the weak equivalences are the smallest class of maps closed under 2-out-of-3 which contains the $G$-inner anodyne extensions and the trivial fibrations (i.e. those maps with the right lifting property against the $G$-normal monomorphisms).

In addition to the category $\text{dSet}^G = \text{Set}^{G^{\text{op}}}$ of equivariant dendroidal sets, there is also a category $\text{dSet}_G = \text{Set}^{G^{\text{op}}}$. We call the category of genuine dendroidal sets. As it turns out, several natural constructions in the non-equivariant setting generalize to produce objects in $\text{dSet}_G$ rather than in $\text{dSet}^G$ (e.g. $\text{dSet}_G$ is essential to establishing the characterization of the fibrant objects in $\text{dSet}^G$, cf. [Per18, §8.2]), so we next recall the connection between the two categories.

Let $\mathcal{O}_G$ denote the orbit category of the group $G$, i.e. the category of transitive $G$-sets $G/H$ for $H \leq G$ and $G$-set maps. Regarding the group $G$ as a single object category, one then has a fully faithful inclusion $\iota_! : G^{\text{op}} \to \mathcal{O}_G$ sending the object of $G$ to the free $G$-orbit $G/e$. In addition, there is a fully faithful inclusion $\Omega \times \mathcal{O}_G \to \Omega_G$ given by $(T, G/H) \mapsto G/H \cdot T$. Altogether, one obtains a commutative diagram of fully faithful inclusions as follows.

\[
\begin{array}{ccc}
\Delta \times G^{\text{op}} & \xrightarrow{\iota_!} & \Omega \times G^{\text{op}} \\
\downarrow^\iota \quad & & \downarrow^\iota \\
\Delta \times \mathcal{O}_G & \xrightarrow{\iota_!} & \Omega \times \mathcal{O}_G \\
\end{array}
\]

The connection between $\text{dSet}^G$ and $\text{dSet}_G$ is then given by the rightmost adjunction
\[ \iota^* : \text{sSet}^{G^{\text{op}}} \rightleftarrows \text{dSet}_G : \iota_* \]
\[ \iota_! : \text{dSet}^G \rightleftarrows \text{dSet}_G : \iota_*, \]
where we note that the right adjoints $v_\ast, v_{G, \ast}$ are fully faithful inclusions. Explicitly, one has $v_\ast X(G/H) = X^H$ and $v_{G, \ast} X(T) = dSet^G(\Omega[T], X) \simeq X(T_x)^H$ for $T = G \cdot H$, with $T_x \in \Omega^H$.

The fully faithful functors appearing in (2.35), (2.36), (2.39) then fit into commutative diagrams as below, where $Op_G$ is the category of genuine equivariant operads (discussed below).

$$\begin{align*}
\text{Cat}^G & \xrightarrow{\iota_1} \text{Op}^G & \xrightarrow{\iota_{G, \ast}} & \text{Op}_G \\
\text{sSet}^G & \xrightarrow{\tau} \text{dSet}^G & \xrightarrow{\tau_{G, \ast}} & \text{dSet}_G \\
N & \downarrow & \downarrow & \downarrow N_G \\
\text{Cat}^G & \xrightarrow{\iota_1} \text{Op}^G & \xrightarrow{\iota_{G, \ast}} & \text{Op}_G \\
\text{sSet}^G & \xrightarrow{\tau} \text{dSet}^G & \xrightarrow{\tau_{G, \ast}} & \text{dSet}_G
\end{align*}$$

(2.40)

Here, genuine equivariant operads are an extension of the notion of operad which in the single colored context was first defined in [BP21] via algebraic means. However, to sidestep the technical work needed to extend the definition in [BP21] to the colored context, here we follow the approach in [BP20, Def. 3.35] and regard $Op_G$ simply as the full subcategory of $dSet_G$ of those objects satisfying the strict Segal condition below. As such, the fact that $N_G$ in (2.40) is fully faithful is tautological (so that the top $v_{G, \ast}$ functor is just a restriction of the lower $v_{G, \ast}$ functor).

**Definition 2.41.** A presheaf $Z \in dSet_G$ is called a genuine equivariant operad if it satisfies the strict right lifting condition against the Segal core inclusions $v_{G, \ast} (Sc[T] \to \Omega[T])$ for all $T \in \Omega_G$.

By taking left adjoints of the vertical nerve functors in (2.40) we obtain the following diagram where those squares that feature natural transformations do not commute. Here the existence of the dashed left adjoint $\tau_G$ to $N_G$ requires justification, with a full discussion of $\tau_G$ being the objective of Appendix A (see also the discussion in [BPc, Rem. 4.26]).

$$\begin{align*}
\text{sSet}^G & \xrightarrow{\iota_1} \text{dSet}^G & \xrightarrow{\iota_{G, \ast}} & \text{dSet}_G \\
\text{Cat}^G & \xrightarrow{\iota_1} \text{Op}^G & \xrightarrow{\iota_{G, \ast}} & \text{Op}_G \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{Cat}^G & \xrightarrow{v_\ast} \text{Op}^G & \xrightarrow{v_{G, \ast}} & \text{Op}_G \\
\text{sSet}^G & \xrightarrow{\tau} \text{dSet}^G & \xrightarrow{\tau_{G, \ast}} & \text{dSet}_G
\end{align*}$$

(2.42)

### 3 The tame model structure on preoperads

Our goal in this section is to build the alternative tame model structure $\text{PreOp}^G_{tame}$ on preoperads that is needed for the nerve functor $N: sOp^G \to \text{PreOp}^G$ in (1.3) to be right Quillen.

First, §3.1, §3.2 recall the model structures on simplicial dendroidal sets $\text{sdSet}^G$ and preoperads $\text{PreOp}^G$ built in [BP20]. Then, §3.3 builds an auxiliary construction, the fibered simplicial tensoring $\otimes_{\mathbb{E}_\ast}$, which has a key role in our description of the tame model structure in §3.4.

### 3.1 Equivariant dendroidal Segal spaces

We recall the several model structures on the category of equivariant simplicial dendroidal sets $\text{sdSet}^G = \text{Set}^{\Delta^{op} \times \Omega^{op} \times G}$ introduced in [BP20]. We first recall some notation.

**Notation 3.1.** For $X \in \text{sdSet}^G$, we write $X_n(U)$ for the evaluation at $n \in \Delta$ and $U \in \Omega$, and refer to $n$ and $U$ as the simplicial and dendroidal directions. More generally, we write

$$X(-): \text{sSet} \to \text{dSet}^G, \quad X(-): \text{dSet}^G \to \text{sSet}$$

(3.2)
for the colimit-preserving functors such that \(X_{\Delta[n]} = X_n\) and \(X(G \cdot \Omega[U]) = X(U)\) for \(n \geq 0\), \(U \in \Omega\). Explicitly, \(X_K(U) = s\text{Set}(K, X(U))\) and \(X_A = d\text{Set}^G(A, X_n)\).

Additionally, we have a natural fully-faithful inclusion
\[
\mathcal{C}: d\text{Set}^G \rightarrow s\text{Set}^G
\]
as presheaves that are constant along the simplicial direction (that is, \(X(T) \in s\text{Set}^G\) is discrete for all \(T \in \Omega\)), and we will often refer to presheaves with their images under \(\mathcal{C}\).

Lastly, for \(A \in d\text{Set}^G\) and \(K \in s\text{Set}\) we write \(A \times K\) for the presheaf \((A \times K)_n(U) = A(U) \times K_n\); more generally, for maps \(A \rightarrow B\) in \(d\text{Set}^G\) and \(K \rightarrow L\) in \(s\text{Set}\), we write \((A \rightarrow B) \odot (K \rightarrow L)\) for the pushout product map with target \(B \times L\) (see, for example, [Rie14, 11.1.7]).

The various model structures on \(s\text{Set}^G\) arise from the theory of (generalized) Reedy categories.

First, since \(\Delta^\text{op}\) is a Reedy category, the identification \(s\text{Set}^G = (d\text{Set}^G)^{\Delta^\text{op}}\) together with the model structure on \(d\text{Set}^G\) from [Per18] (cf. Theorem 2.37) yields the simplicial Reedy model structure on \(s\text{Set}^G\). Note that the weak equivalences in this model structure are the dendroidal equivalences, i.e. maps \(f: X \rightarrow Y\) such that \(X_n \rightarrow Y_n\) is a weak equivalence in \(d\text{Set}^G\) for all \(n \geq 0\).

Second, as discussed in [BP20, Ex. A.7], \(\Omega^\text{op} \times G\) is a generalized Reedy category and the family of graph subgroups \(\{K \leq \text{Aut}(T)^{\text{op}} \times G \mid \Gamma \cap \text{Aut}(T)^{\text{op}} = \ast\}_{T \in \Gamma}\)
(see (1.1)) is Reedy admissible in the sense of [BP20, Ex. A.2]. Hence, by [BP20, Thm. A.8], the identification \(s\text{Set}^G = s\text{Set}^{\Omega^\text{op} \times G}\) together with the Kan model structure on \(s\text{Set}\) yields the (equivariant) dendroidal Reedy model structure on \(s\text{Set}^G\). The weak equivalences in this model structure are the simplicial equivalences, i.e. maps \(f: X \rightarrow Y\) such that \(X(\Omega[T]) \rightarrow Y(\Omega[T])\) are Kan equivalences in \(s\text{Set}\) for all \(T \in \Omega_G\).

Third, as the simplicial and dendroidal Reedy model structures in \(s\text{Set}^G\) above have the same cofibrations, the joint left Bousfield localization framework in [BP20, §4.1] yields the following.

**Theorem 3.3.** The simplicial and dendroidal Reedy model structures on \(s\text{Set}^G\) have a smallest\(^2\) common left Bousfield localization, which we call the joint model structure. Moreover:

(i) the joint model structure is left proper;

(ii) both the dendroidal and simplicial equivalences in \(s\text{Set}^G\) are also joint equivalences;

(iii) \(X\) is joint fibrant iff \(X\) is both simplicial and dendroidal Reedy fibrant;

(iv) if \(X, Y\) are joint fibrant then a map \(X \rightarrow Y\) is a joint equivalence iff it is a simplicial equivalence iff it is a dendroidal equivalence;

(v) if \(X, Y\) are dendroidal fibrant then a map \(X \rightarrow Y\) is a joint equivalence iff it is a dendroidal equivalence iff it is an equivalence in \(d\text{Set}^G\);

(vi) if \(X \rightarrow Y\) is a joint (co)fibration, the level maps \(X_n \rightarrow Y_n\) are (co)fibrations in \(d\text{Set}^G\) and the maps \(X(\Omega[T]) \rightarrow Y(\Omega[T])\), \(T \in \Omega_G\) are (co)fibrations in \(s\text{Set}\).

In particular, if \(X\) is joint fibrant then \(X_n \in d\text{Set}^G\) and \(X(\Omega[T]) \in s\text{Set}\) are fibrant.

**Proof.** The existence of a smallest common left Bousfield localization is an application of [BP20, Prop. 4.1], with the hypothesis that the dendroidal/simplicial Reedy model structures admit localizations being guaranteed by Hirschhorn’s existence result [Hir03, Thm. 4.1.1] (in particular, both Reedy model structures are left proper and cellular by [Hir03, Theorems 15.3.4, 15.7.6]

---

\(^2\)Here "smallest" means that the class of weak equivalences is as small as possible.
respectively, the proofs of which do not depend on the Reedy category being strict). (i) then follows from [Hir03, Thm. 4.1.1(3)]. (ii) holds by definition. (iii) and (iv) are [BP20, Prop. 4.1(i)(ii)]. (v) is [BP20, Cor. 4.29(iii)]. Lastly, (vi) follows from [BP20, Lemmas A.27(i), A.29(i)].

Remark 3.4. For \( A \to B \) a normal monomorphism in \( \text{dSet}^G \) and \( K \to L \) a monomorphism in \( \text{sSet} \), the map \( (A \to B) \Box (K \to L) \) is a cofibration in any of the model structures on \( \text{sdSet}^G \). Moreover, if \( A \to B \) (resp. \( K \to L \)) is a weak equivalence in \( \text{dSet}^G \) (resp. \( \text{sSet} \)), then \( (A \to B) \Box (K \to L) \) is a dendroidal (resp. simplicial) equivalence in \( \text{sdSet}^G \).

Fourth (and last), one has the (equivariant) dendroidal Segal space model structure on \( \text{sdSet}^G \), which is the left Bousfield localization of the dendroidal Reedy model structure with respect to the Segal core inclusions

\[
S \varepsilon[T] \to \Omega[T], \quad T \in \Omega_G.
\]

Remark 3.5. The joint model structure on \( \text{sdSet}^G \) can also be described as a further localization of the dendroidal Segal space model structure imposing a "completion condition" [BP20, Rem. 4.27], and is thus also called the complete (equivariant) dendroidal Segal space model structure with complete equivalences.

Lemma 3.6. The weak equivalences in the dendroidal Reedy, dendroidal Segal space, and joint Reedy model structures on \( \text{sdSet}^G \) are closed under filtered colimits.

Remark 3.7. More explicitly, weak equivalences are closed under filtered colimits if a map of filtered colimits \( \text{colim}_i C_i \to \text{colim}_i D_i \) is a weak equivalence whenever all \( C_i \to D_i \) are. Notably, this is equivalent to the claim that filtered colimits \( \text{colim}_i C_i \) are homotopy colimits.

Proof. Weak equivalences in the dendroidal Reedy model structure are simplicial equivalences, so in that case the result is inherited from the analogous claim for \( \text{sSet} \). The result for the latter two model structures follows since they are left Bousfield localizations of the dendroidal Reedy model structure (as the alternative condition in Remark 3.7 is clearly preserved under localization).

In what follows we will often make reference to the 0-(co)skeleton of some \( X \in \text{sdSet}^G \) in the dendroidal Reedy structure. To avoid confusion with the 0-(co)skeleton for the simplicial Reedy structure, and noting that \( \eta \) is the only tree in \( \Omega \) of degree 0, we introduce the following notation.

Notation 3.8. Let \( X \in \text{sdSet}^G \). We write \( \text{sk}_\eta X, \text{csk}_\eta X \in \text{sdSet}^G \) for the (co)skeleta described by

\[
(\text{sk}_\eta X)(U) = \bigcup_{E(U)} X(\eta), \quad (\text{csk}_\eta X)(U) = \prod_{E(U)} X(\eta).
\]

3.2 Segal preoperads

We next recall the normal model structure on preoperads \( \text{PreOp}^G \) studied in [BP20, §4, §5].

Definition 3.9. The category of (equivariant) preoperads \( \text{PreOp}^G \) is the full subcategory of \( \text{sdSet}^G \) spanned by those \( X \) such that \( X(\eta) \) is a discrete simplicial set.

Definition 3.10. Given \( X \in \text{PreOp}^G \) we call \( X(\eta) \) the color set of \( X \) and denote the associated color set functor as follows.

\[
\text{PreOp}^G \xrightarrow{\varepsilon \bullet} \text{Set}^G \xrightarrow{\varepsilon_X = X(\eta)} \text{Set}^G
\]
Further, for each fixed $C \in \text{Set}^G$ we write $\text{PreOp}^G_C \subset \text{PreOp}^G$ for the fiber subcategory of (3.11) over $C$, consisting of those $X$ such that $X(\eta) = C$ and maps that are the identity on colors.

Lastly, for $f : C \to D$ a map of $G$-sets of colors we define adjoint functors

$$f^* : \text{PreOp}^G_C \rightleftarrows \text{PreOp}^G_D : f_*$$

via the pushout and pullback squares below (note that $sk_\eta f_* A = \coprod C \Omega[\eta]$ depends only on $C$ while $csk_\eta f^* X = \prod D \Omega[\eta]$ depends only on $D$)

$$\begin{array}{c}
sk_\eta A \rightarrow sk_\eta f_* A \\
\downarrow \hspace{2cm} \downarrow \\
f_* A \rightarrow f_* A
\end{array} \quad \begin{array}{c}
f^* X \rightarrow X \\
\downarrow \hspace{2cm} \downarrow \\
csk_\eta f^* X \rightarrow csk_\eta X
\end{array}$$

Remark 3.13. The color functor (3.11) is both a Grothendieck opfibration and fibration (cf. [BP, §2.1]), with cocartesian (resp. cartesian) arrows the natural maps $A \to f_* A$ (resp. $f^* X \to X$). In particular, maps $f_* A \to B$ over a map of colors $D \to E$ are in bijection with maps $A \to B$ over the over the composite map of colors $C \to D \to E$, and dually for $f^* X$.

The inclusion $\gamma^* : \text{PreOp}^G \to \text{sdSet}^G$ admits both a left adjoint $\gamma_!$ and a right adjoint $\gamma^*$

$$\begin{array}{c}
\text{PreOp}^G \\
\gamma_! \leftarrow \gamma^* \rightarrow \text{sdSet}^G
\end{array}$$

described by the following pushout and pullback squares.

$$\begin{array}{c}
sk_\eta X \rightarrow \pi_0 sk_\eta X \\
\downarrow \hspace{2cm} \downarrow \\
X \rightarrow \gamma_! X
\end{array} \quad \begin{array}{c}
\gamma_* X \rightarrow X \\
\downarrow \hspace{2cm} \downarrow \\
csk_\eta X \rightarrow csk_\eta X
\end{array}$$

More explicitly: $\gamma_! X(U) = X(U)$ for non-linear trees $U \in \Omega \setminus \Delta$, while $\gamma_* X([n])$ for $[n] \in \Delta$ linear is given by the pushout on the left below; $\gamma_* X(U)$ is given by the pullback on the right below.

$$\begin{array}{c}
X(\eta) \rightarrow \pi_0 X(\eta) \\
\downarrow \hspace{2cm} \downarrow \\
X([n]) \rightarrow \gamma_* X([n])
\end{array} \quad \begin{array}{c}
\gamma_* X(U) \rightarrow X(U) \\
\downarrow \hspace{2cm} \downarrow \\
\Pi_{E(U)} X_0(\eta) \rightarrow \Pi_{E(U)} X(\eta)
\end{array}$$

By largely formal arguments, the joint model structure on $\text{sdSet}^G$ in Theorem 3.3 induces a model structure on $\text{PreOp}^G$, as follows. We say a map $X \to Y$ in $\text{PreOp}^G$ is a joint equivalence (resp. normal monomorphism) if $\gamma^* X \to \gamma^* Y$ is a joint equivalence (resp. normal monomorphism) in $\text{sdSet}^G$. The following is then [BP20, Thms. 4.39 and 4.42], with the additional “moreover” claims inherited from the analogous conditions in $\text{sdSet}^G$ in Theorem 3.3(i) and Lemma 3.6.

Theorem 3.15. There is a model structure on $\text{PreOp}^G$, called the normal model structure, such that weak equivalences (resp. cofibrations) are the joint equivalences (normal monomorphisms).

Moreover, this model structure is left proper and weak equivalences are closed under filtered colimits.

Lastly, the adjunction $\gamma^* : \text{PreOp}^G \rightleftarrows \text{sdSet}^G : \gamma_*$ is a Quillen equivalence between the normal model structure and the joint model structure on $\text{sdSet}^G$. 

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For our purposes, we also need to recall a convenient “Dwyer-Kan description” of the joint equivalences between fibrant objects in \( \text{PreOp}^G \). For this purpose, we first introduce the following new notation, which extends notation in [BP20, Def. 5.7] and will simplify our discussion of the nerve functor (see, e.g. (4.21)).

**Notation 3.16.** Let \( X \in \text{PreOp}^G \), \( A \in \text{dSet}^G \), and \( c : A(\eta) \to X(\eta) \) be a \( G \)-equivariant map.

We define \( X_c(A) \in \text{sSet} \) as the pullback below (here the two squares are identical, providing only different descriptions of the bottom-right corner).

\[
\begin{array}{ccc}
X_c(A) & \rightarrow & X(A) \\
\downarrow & & \downarrow \\
* & \rightarrow & X(\sk_1 A)
\end{array}
\quad
\begin{array}{ccc}
X_c(A) & \rightarrow & X(A) \\
\downarrow & & \downarrow \\
* & \rightarrow & \left( \prod_{A(\eta)} X(\eta) \right)^G
\end{array}
\]

Further, when \( A = G : \Omega[U] \) for \( U \in \Omega \), we abbreviate \( X_c(G : \Omega[U]) \) as \( X_c(U) \).

Note that one thus has a coproduct decomposition

\[
X(A) \simeq \bigsqcup_{c : A(\eta) \to X(\eta)} X_c(A).
\]

**Remark 3.18.** Our primary instances of Notation 3.16 occur when \( A = \Omega[T] \) for \( T \in \Omega_G \), in which case \( \Omega[T](\eta) = E(T) \), so that \( c : \Omega[T](\eta) \to X(\eta) \) can be regarded as a coloring of the edges \( E(T) \) of \( T \) by the colors \( X(\eta) \) of the preoperad \( X \).

**Remark 3.19.** Specifying Remark 3.18 to the case of \( T = C \) a \( G \)-corolla, the coloring \( c : \Omega[C](\eta) \to X(\eta) \) is tantamount to a \( G \)-equivariant map \( c : \partial \Omega[C] \to X \), i.e. to a \( G \)-profile in the sense of [BP20, Def. 5.6]. Further, since there is an identification \( X(\partial \Omega[C]) \simeq \prod_{[e_i] \in E_G(C)} X(\eta)^{H_i} \) with \( H_i \leq G \) the isotropy of the edge \( e_i \), the data of the coloring \( c \) is equivalent to a choice of \( x_i \in X(\eta)^{H_i} \) for each \( [e_i] \in E_G(C) \). As such, one has an identification

\[
X_c(\Omega[C]) = X(x_1, \ldots, x_n; x_0)
\]

where the mapping space \( X(x_1, \ldots, x_n; x_0) \) is as defined in [BP20, Def. 5.7]. Further, the decomposition in (3.17) then extends the decomposition in [BP20, Rem. 5.14].

**Remark 3.20.** Fix \( C \in \text{Set}^G \) and consider the fiber subcategory \( \text{PreOp}^G_C \subset \text{PreOp}^G \).

For \( U \in \Omega \), the decomposition \( X(U) = \prod_{C \in \Omega(U) = c} X_c(U) \) in (3.17) (note that we are using the abbreviated notation at the end of Notation 3.16) then induces an equivalence of categories

\[
\begin{array}{ccc}
\text{PreOp}^G_C & \xrightarrow{\simeq} & \text{Fun}_*(G \times \Omega_C^{op}, \text{sSet}) \\
(U \mapsto X(U)) & \mapsto & ((U, c) \mapsto X_c(U))
\end{array}
\]

where \( \Omega_C \) denotes the category of \( C \)-colored trees of Definition 2.17, \( G \times \Omega_C^{op} \), is an instance of [BP20, Ex. 2.14] extending the category \( \Omega_C^{op} \) by adding \( G \)-action arrows \( \bar{U} = (U, c) \to (U, gc) = g\bar{U} \) for \( g \in G \), and \( \text{Fun}_*(G \times \Omega_C^{op}, \text{sSet}) \subset \text{Fun}(G \times \Omega_C^{op}, \text{sSet}) \) is the subcategory of pointed functors, i.e. functors \( Y \) such that \( Y(\eta_c) = * \), where \( c \in C \) is a color and \( \eta_c \) denotes the stick tree colored by \( c \).

**Remark 3.22.** Given the equivalence (3.21) and the alternative notation \( \bar{U} = (U, c) \) for \( C \)-trees in \( \Omega_C \), it seems natural to abbreviate \( X_c(U) \) as \( X(\bar{U}) \). However, we will make significant use of the notation \( X_c(\Omega[T]) \) in Remarks 3.18, 3.19, and this latter notation is not readily recovered from (3.21). As such, when dealing with preoperads we work only with the \( X_c(A), X_c(U) \) notations, reserving the \( \mathcal{O}(\bar{C}) \) style notation for the context of operads \( \mathcal{O} \in \text{sOp}^G \) (see §4.1).
Remark 3.23. Let $X \in \text{PreOp}^G$. For any $G$-tree $T \in \Omega_G$ and coloring $\epsilon : E(T) \to X(\eta)$ one has

$$X_\epsilon (\text{Sc}[T]) \simeq \prod_{v \in V_G(T)} X_{\epsilon_v} (\Omega[T_v])$$

where $\epsilon_v$ denotes the restricted coloring given by the composite $E(T_v) \to E(T) \xrightarrow{\epsilon} X(\eta)$, and $T_v$ is as in Notation 2.24.

We can now recall the notion of Segal operad, cf. [CM13b, Def. 5.5], [BP20, Def. 4.40].

Definition 3.24. A preoperad $X \in \text{PreOp}^G$ is called a (equivariant) Segal operad if $X(\Omega[T]) \to X(\text{Sc}[T])$ is a Kan equivalence for each $T \in \Omega_G$. A Segal operad $X$ is further called a Reedy fibrant Segal operad if $\gamma^* X$ is dendroidal Reedy fibrant in $\text{sdSet}^G$.

Remark 3.25. By (3.17) and Remark 3.23, $X \in \text{PreOp}^G$ is a Segal operad iff the natural maps

$$X_\epsilon (\Omega[T]) \to \prod_{v \in V_G(T)} X_{\epsilon_v} (\Omega[T_v])$$

are Kan equivalences for all $T \in \Omega_G$ and $G$-equivariant colorings $\epsilon : E(T) \to \mathbf{C}_X = X(\eta)$.

Additionally, by [BP20, Rem. 4.41] a Segal operad $X$ is a Reedy fibrant Segal operad iff $\gamma^* X$ is fibrant in the dendroidal Segal space model structure on $\text{sdSet}^G$.

Remark 3.26. Since, for any preoperad $X \in \text{PreOp}^G$ the discrete simplicial sets $X(\eta)$ are Kan complexes, one can form a dendroidal fibrant replacement $X \to \tilde{X}$ in $\text{sdSet}^G$ such that $X(\eta) \simeq \tilde{X}(\eta)$, so that $\tilde{X}$ is again a preoperad.

Moreover, since the maps $X(\Omega[T]) \to \tilde{X}(\Omega[T])$ for $T \in \Omega_G$ are Kan equivalences, (3.17) implies that so are the maps $X_\epsilon (\Omega[T]) \to \tilde{X}_\epsilon (\Omega[T])$ for any coloring $\epsilon : E(T) \to X(\eta)$.

Note that Remark 3.25 then implies that $X$ is a Segal operad iff $\tilde{X}$ is a Reedy fibrant Segal operad.

We will show that joint equivalences between Segal operads admit a Dwyer-Kan type description in terms of fully faithfulness and essential surjectivity conditions (cf. Theorem 4.9). To describe essential surjectivity, we need to recall a discrete algebraic structure associated to a Segal preoperad. In the following, we make use of the category $\text{dSet}_G = \text{Set}^G$ of genuine dendroidal sets discussed in §2.3, as well as its natural simplicial generalization $\text{sdSet}_G = \text{ssSet}^G$.

Definition 3.27. Given a Segal operad $X \in \text{PreOp}^G$, we define its homotopy genuine operad $\text{ho}(X) \in \text{sdSet}_G$ by

$$\text{ho}(X) = \pi_0 (v_* \gamma^* X) \quad (3.28)$$

with $v_* : \text{sdSet}_G \to \text{sdSet}_G$ (cf. (2.39)) and $\pi_0 : \text{sdSet}_G \to \text{dSet}_G$ defined in the natural way.

Remark 3.29. When $G = \ast$, (3.28) reduces to $\text{ho}(X) = \pi_0 (\gamma^* X)$ and the Segal condition for $X \in \text{PreOp}$ induces a strict Segal condition on $\text{ho}(X) \in \text{dSet}$, i.e. $\text{ho}(X)$ is the nerve of an operad, cf. (2.36). Moreover, if $X \in \text{PreCat} \subseteq \text{PreOp}$ is a precategory, i.e. $X \in \text{ssSet} \subseteq \text{sdSet}$ with $X(\eta)$ a discrete simplicial set, then $\text{ho}(X) \in \text{dSet}$ is the nerve of a category.

Remark 3.30. The “genuine operad” moniker for $\text{ho}(X) \in \text{dSet}_G$ refers to the fact that this presheaf satisfies a certain strict Segal condition, as shown in [BP20, Prop. 5.9] (technically the cited result only covers the case of $X$ Reedy fibrant, but it is immediate that for $X, \tilde{X}$ as in Remark 3.26 there is a natural identification $\text{ho}(X) \simeq \text{ho}(\tilde{X})$).

However, for our present purposes we will not need the full strength of this statement, but only a more familiar consequence. Recalling (cf. (2.38)) the inclusion $\iota_G : \Delta \times \text{O}_G \to \Omega_G$ given
by $([n], G/H) \mapsto G/H \cdot [n]$, one has that $\iota^*_G h_0(X) \in \text{sSet}^{G^p}$ is given by $(\iota^*_G h_0(X))(G/H) = \iota^* h_0(X^H) = h_0(\iota^* X^H)$, where $\iota^*, h_0$ are as in (2.35), Remark 3.29. As such, the Segal condition for $h_0(X)$ implies that $\iota^*_G h_0(X)$ is a coefficient system of nerves of categories [BP20, Rem. 5.11].

**Definition 3.31.** A map $f: X \to Y$ of Segal operads in $\text{PreOp}^G$ is called:

(i) **fully-faithful** if, for all $G$-corollas $C \in \Sigma_G$ and $G$-equivariant colorings $\mathbf{c}: E(C) \to \mathbf{c}_X = X(\eta)$, the induced map

$$X_t(\Omega[C]) \to Y_f(\Omega[C])$$

is a Kan equivalence in $\text{sSet}$;

(ii) **essentially surjective** if the map $\iota^*_G h_0(X) \to \iota^*_G h_0(Y)$ of $G$-coefficient systems of categories is levelwise essentially surjective;

(iii) a **Dwyer-Kan equivalence** if it is both fully-faithful and essentially surjective.

**Notation 3.32.** Arrows in the category (with nerve) $(\iota^*_G h_0(X))(G/H) = h_0(\iota^* X^H)$ are encoded by maps $\Omega[1] \to X^H$. If $\Omega[1] \to X^H$ encodes an isomorphism, we call $\Omega[1] \to X^H$ a $H$-equivalence.

The following summarizes [BP20, Thms. 5.51 and 5.48], with the additional fact that the “further” claim holds for all Segal operads, rather than just the Reedy fibrant ones, following from Remark 3.26.

**Theorem 3.33.** The fibrant objects in $\text{PreOp}^G_{\text{normal}}$ are precisely the Reedy fibrant Segal operads, i.e. the preoperads $X \in \text{PreOp}^G$ such that $\gamma^* X \in \text{sSet}^G$ is a Segal space.

Further, a map between Segal operads is a joint equivalence iff it is a Dwyer-Kan equivalence.

### 3.3 Fibered simplicial (co)tensor on preoperads

In this section we introduce an auxiliary simplicial tensoring on $\text{PreOp}^G$ that will play a key role in our definition of the tame model structure $\text{PreOp}^G_{\text{tame}}$ in §3.4, as well as streamline the comparison between $\text{PreOp}^G_{\text{tame}}$ and $\text{sOp}^G$ in §4.3.

We first define the adjoint simplicial cotensoring, which admits a very simple description in terms of the $X_t(U)$ construction introduced in Notation 3.16 (and the identification (3.21)).

**Definition 3.34.** Given $X \in \text{PreOp}^G$ and $K \in \text{sSet}$ we define their fiber cotensor $\{K, X\}_{\mathbf{c}_*} \in \text{PreOp}^G_{\mathbf{c}}$ by

$$\{K, X\}_{\mathbf{c}_*}(U) = X_{\mathbf{c}}(U)^K.$$ (3.35)

for $U \in \Omega$ a tree and $\mathbf{c}: E(T) \to \mathbf{c} = X(\eta)$ a coloring.

Alternatively, $\{K, X\}_{\mathbf{c}_*}$ is given by the pullback in $\text{sSet}^G$ (where the bottom map is induced by the map $K \to \ast$, and the left square simply evaluates the right square at $U \in \Omega$)

$$\begin{array}{ccc}
\{K, X\}_{\mathbf{c}_*}(U) & \longrightarrow & X(U)^K \\
\downarrow & & \downarrow \\
\Pi_{E(U)} X(\eta) & \longrightarrow & (\Pi_{E(U)} X(\eta))^K
\end{array}$$

$$\begin{array}{ccc}
\Pi_{E(U)} X(\eta) & \longrightarrow & (\Pi_{E(U)} X(\eta))^K \\
\downarrow & & \downarrow \\
\{\text{csk}_\eta X & \longrightarrow & (\text{csk}_\eta X)^K
\end{array}$$

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Definition 3.36. Given $X \in \text{PreOp}_G^\mathcal{C}$ and $K \in \text{sSet}$ we define their fiber tensor $X \otimes_{\mathcal{C}} K \in \text{PreOp}_G^\mathcal{C}$ by the pushout in $\text{sdSet}_G^{\mathcal{C}}$

$$(sk_\eta X) \times K \to X \otimes_{\mathcal{C}} K,$$ (3.37)

More explicitly, one has $(X \otimes_{\mathcal{C}} K)(U) = X(U) \times K$ whenever $U$ is a non-linear tree (equivalently, $\Omega(U, \eta) = \emptyset$) and that $(X \otimes_{\mathcal{C}} K)([n])$ is given by the following pushout when $U = [n]$ is linear.

$$(X(\eta) \times K) \to X(\eta)$$

We have an analogous fibered pushout product map we’ll denote by $(X \to Y) \square_{\mathcal{C}} (K \to L)$.

Remark 3.38. In [CM13b, (7.1.5)] the objects $\Omega[T] \otimes_{\mathcal{C}} K$ were denoted $\Omega[K,T]$ and built by hand.

Remark 3.39. If $K \in \text{sSet}$ is connected, comparing the left square in (3.14) with the square (3.37) yields an identification $\gamma_!(X \times K) \simeq X \otimes_{\mathcal{C}} K$.

Remark 3.40. For each fixed $K \in \text{sSet}$ the fiber tensor and cotensor determine an adjunction as on the left below.

$$\text{PreOp}_G^{\mathcal{C}} \xrightarrow{(-) \otimes_{\mathcal{C}} K^{\mathcal{C}}_{\cdot}} \text{PreOp}_G^{\mathcal{C}} \xleftarrow{(K,-)_{\mathcal{C}}^{\mathcal{C}}_{\cdot}} \text{PreOp}_G^{\mathcal{C}}$$

Moreover, this adjunction is a fibered adjunction over the color set functor $\mathcal{C} \cdot \text{PreOp}_G^{\mathcal{C}} \to \text{Set}_G^{\mathcal{C}}$, cf. [BPa, Def. 2.22]. In particular, for each fixed $G$-set of colors $\mathcal{C}$ one has a restricted adjunction as on the right above. Moreover, by [BPa, Prop. 2.24] and its dual, these functors are compatible with (co)coterminal arrows. I.e., cf. Remark 3.13, one has $(f \cdot A) \otimes_{\mathcal{C}} K \simeq f(A \otimes_{\mathcal{C}} K)$ and $(K, f^*X)_{\mathcal{C}}^{\mathcal{C}} = f^*(\{K,X\}_{\mathcal{C}}^{\mathcal{C}})$.

Remark 3.41. Remark 3.40 implies that the fiber cotensor $X \otimes_{\mathcal{C}} K$ preserves colimits on the $X$ variable. However, some care is needed when dealing with the $K$ variable. For each fixed color $G$-set $\mathcal{C}$, one has that the functor

$$\text{PreOp}_G^{\mathcal{C}} \times \text{sSet} \xrightarrow{(-) \otimes_{\mathcal{C}} (-)} \text{PreOp}_G^{\mathcal{C}}$$

is part of a two-variable adjunction, which in particular means that $X \otimes_{\mathcal{C}} \cdot (-) : \text{sSet} \to \text{PreOp}_G^{\mathcal{C}}$ (where $\mathcal{C} = X(\eta)$) preserves colimits. On the other hand, this means that $X \otimes_{\mathcal{C}} (-) : \text{sSet} \to \text{PreOp}_G^{\mathcal{C}}$ only preserves those colimits which coincide in $\text{PreOp}_G^{\mathcal{C}}$ and $\text{PreOp}_G^{\mathcal{C}}$, namely the connected colimits. On the other hand, for coproducts one instead has that the canonical map

$$\cup_i X \otimes_{\mathcal{C}} K_i \to X \otimes_{\mathcal{C}} (\cup_i K_i)$$

is a coterminal arrow over the fold map $\nabla : \cup_i \mathcal{C} \to \mathcal{C}$, i.e. $\nabla_!(\cup_i X \otimes_{\mathcal{C}} K_i) \simeq X \otimes_{\mathcal{C}} (\cup_i K_i)$. 

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Remark 3.42. Let \( K \in \sSet \), and \( X \to Y \) any map in \( \PreOp^G \) which is the identity on colors and

Then the top horizontal maps in (3.37) for \( X, Y \) coincide, and likewise for the left square in (3.14) for \( X \times K, Y \times K \). It thus follows that the squares below are pushout squares in \( \sdSet^G \).

\[
\begin{array}{ccc}
X \times K & \xrightarrow{\gamma} & X \otimes \epsilon_* K \\
\downarrow & & \downarrow \\
Y \times K & \xrightarrow{\gamma} & Y \otimes \epsilon_* K
\end{array}
\]

Lemma 3.43. Let \( f: X \to Y \) be a map in \( \PreOp^G \) and \( k: K \to L \) be a map in \( \sSet \).

Then \( \gamma^*(f \square \epsilon_* k) \) is a pushout in \( \sdSet^G \) of the map

\( (f: X \to Y) \square (K \to L) \).

Proof. Since the left square below is a pushout square (this follows by noting that the horizontal arrows are cocartesian, as per Remark 3.40; see the universal property in Remark 3.13)

\[
\begin{array}{ccc}
X \otimes \epsilon_* K & \xrightarrow{f} & f_* X \otimes \epsilon_* K \\
\downarrow & & \downarrow \\
X \otimes \epsilon_* L & \xrightarrow{f} & f_* X \otimes \epsilon_* L
\end{array}
\]

one has \( (X \to Y) \square \epsilon_* k \simeq (f_* X \to Y) \square \epsilon_* k \). Since \( f_* X \to Y \) is the identity on colors, Remark 3.42 then says that the two middle squares below are pushout squares.

\[
\begin{array}{ccc}
f_* X \times K & \xrightarrow{f} & f_* X \times L \\
\downarrow & & \downarrow \\
Y \times K & \xrightarrow{f} & Y \otimes \epsilon_* L
\end{array}
\]

A standard argument (e.g. [RV14, Obs. 5.1]) then shows that the pushout map for the rightmost square above is a pushout of the pushout map for the leftmost square, finishing the proof. \( \square \)

3.4 Definition and existence of the tame model structure

The following adapts [CM13b, §7.7], repackaged using the fibered cotensoring \( \otimes \epsilon_* \).

Definition 3.44. The tame cofibrations in \( \PreOp^G \) are the saturation of the following maps

(1) \( G/H \cdot (\varnothing \to \Omega[\eta]) \) for \( H \leq G \);

(2) \( \Omega[C] \otimes \epsilon_* (\partial \Delta[n] \to \Delta[n]) \) for \( G \)-corollas \( C \in \Sigma_G, n \geq 0 \);

(3) \( (\Sc[T] \to \Omega[T]) \square \epsilon_* (\partial \Delta[n] \to \Delta[n]) \) for \( G \)-trees \( T \in \Omega_G, n \geq 0 \).

Definition 3.45. A precategory \( I \in \PreCat \simeq \PreOp \downarrow \Omega[\eta] \) is called a pseudo-interval if \( I(\eta) = \{0,1\} \), the map \( \Omega[\eta] \sqcup \Omega[\eta] = \sk_0 I \to I \) is a tame cofibration, and the map \( I \to \Omega[\eta] \) is a weak equivalence.

Definition 3.46. The tame anodyne cofibrations in \( \PreOp^G \) are the saturation of the following maps
(TA1) \( G/H \cdot (\Omega[n] \to I) \) for \( H \leq G \) and \( I \in \text{PreCat} \) a countable pseudo-interval;

(TA2) \( \Omega[C] \otimes e, (A^i[n] \to \Delta[n]) \) for \( C \in \Sigma_G \), \( 0 \leq i \leq n, n \geq 1 \);

(TA3) \((\text{Sc}[T] \to \Omega[T]) \triangleleft e, (\partial \Delta[n] \to \Delta[n]) \) for \( T \in \Omega_G \), \( n \geq 0 \).

We can now state the main result of this section.

**Theorem 3.47** (cf. [CM13b, Thm. 7.19]). There is a left proper model structure on \( \text{PreOp}^G \), called the tame model structure, such that:

- weak equivalences are the joint equivalences (i.e. detected by inclusion into \( \text{sdSet}^G \));
- the generating cofibrations are the maps \((TC1),(TC2),(TC3); \)
- \( X \in \text{PreOp}^G \) is fibrant iff \( X \to * \) has the right lifting property against \((TA1),(TA2),(TA3); \)
- a map \( X \to Y \) between fibrant objects is a fibration iff it has the right lifting property against \((TA1),(TA2),(TA3); \).

Moreover, the identity adjunction \( \text{PreOp}^G_{tame} \rightleftarrows \text{PreOp}^G_{normal} \) is a Quillen equivalence.

Before proving Theorem 3.47, we collect a few lemmas.

**Lemma 3.48.** Tame cofibrations (resp. tame anodyne cofibrations) are cofibrations (resp. trivial cofibrations) in the normal model structure on \( \text{PreOp}^G \).

**Proof.** It suffices to check the given claims for the generating maps in Definitions 3.44, 3.46.

The (TC1) case is immediate. For (TC2),(TC3),(TA2),(TA3) we apply Lemma 3.43 (note that for (TC2),(TA2) the map \( f : X \to Y \) is \( \emptyset \to \Omega[C] \), so that \( f_iX \to Y \) is the inclusion \( \partial \Omega[C] \to \Omega[C] \)) and in all such cases it is clear by Remark 3.4 that the corresponding map \( (f_iX \to Y) \triangleleft k \) is a (trivial) cofibration in \( \text{sdSet}^G \). (TA1) follows by definition of pseudo-interval and the (TC1),(TC2),(TC3) cases.

**Lemma 3.49.** Any map \( X \to Y \) which has the right lifting property against \((TC1),(TC2),(TC3)\) is a joint equivalence in \( \text{PreOp}^G \).

**Proof.** Writing \( f : C \to D \) for the underlying map of colors, consider the factorization \( X \to f^*Y \to Y \). Noting that lifting problems against (TC1) depend only on objects and both of (TC2) and (TC3) consist of maps which are identities on objects, we see that \( X \to Y \) has the right lifting property against (TC1) iff \( f^*Y \to Y \) does and the right lifting property against (TC2),(TC3) iff \( X \to f^*Y \) does. We argue separately that \( f^*Y \to Y \) and \( X \to f^*Y \) are joint equivalences.

Consider first the map \( f^*Y \to Y \). Note now that \( f^*Y \to Y \) has the right lifting proper against all maps \( \partial \Omega[T] \to \Omega[T] \times \Delta[n] \). Indeed, if \( T = G/H \eta \) is a stick \( G \)-tree, this is precisely the lifting condition against (TC1), and otherwise it follows automatically since \( \partial \Omega[T] \to \Omega[T] \times \Delta[n] \) is the identity on objects. Therefore, the levels \( (f^*Y)_n \to Y_n \) are trivial fibrations in \( \text{dSet}^G \), showing that \( f^*Y \to Y \) is a dendroidal equivalence, and thus a joint equivalence, cf. Theorem 3.3(ii).

Consider now the map \( X \to f^*Y \). The lifting property against (TC2) together with the decompositions in (3.17) then say that the maps \( X_\iota(\Omega[C]) \to (f^*Y)_\iota(\Omega[C]) \) are trivial \( G \)-fibrations for all \( G \)-corollas \( C \in \Sigma_G \) and colorings \( c : E(C) \to \mathcal{C} \). Now let \( T \in \Omega_G \) be a \( G \)-tree and \( c : E(T) \to \mathcal{C} \) be a coloring, and consider the following diagram.

\[
\begin{array}{ccc}
X_\iota(\Omega[T]) & \longrightarrow & X_\iota(\text{Sc}[T]) & \longrightarrow & \Pi_{v \in \mathcal{V}_{\mathcal{G}}(T)} X_{\iota}(\Omega[T_v]) \\
\downarrow & & \downarrow & & \downarrow \\
(f^*Y)_\iota(\Omega[T]) & \longrightarrow & (f^*Y)_\iota(\text{Sc}[T]) & \longrightarrow & \Pi_{v \in \mathcal{V}_{\mathcal{G}}(T)} (f^*Y)_\iota(\Omega[T_v])
\end{array}
\]
By the above, the right vertical map is a trivial Kan fibration, and thus so is the isomorphic map \( X_*(Sc[T]) \to (f^*Y)_*(Sc[T]) \), and likewise for the total map \( X(Sc[T]) \to f^*Y(Sc[T]) \). As the lifting property against (TC3) yields that \( X(\Omega[T]) \to f^*Y(\Omega[T]) \times_{f^*Y(Sc[T])} X(Sc[T]) \) is a trivial Kan fibration for all \( G \)-trees, so is \( X(\Omega[T]) \to f^*Y(\Omega[T]) \), showing that \( X \to f^*Y \) is a simplicial equivalence, and thus a joint equivalence, cf. Theorem 3.3(ii).

**Remark 3.50.** Tame cofibrant replacement in \( \text{PreOp}^G \) can be performed without changing objects. Indeed, given any \( A \in \text{PreOp}^G \), one has that \( \sk_n A = \bigsqcup_{A(\eta)} \Omega[\eta] \) is tame cofibrant by (TC1). Thus, the small object argument for (TC2), (TC3) applied to the map \( \sk_n A \to A \) gives a factorization \( \sk_n A \to \bar{A} \to A \) where: \( \sk_n A \to \bar{A} \) is in the saturation of (TC2), (TC3), so that \( \bar{A} \) is tame cofibrant; \( \bar{A} \to A \) has the right lifting property against (TC2), (TC3) by construction and against (TC1) since it is the identity on objects, and is thus a joint equivalence by Lemma 3.49.

**Lemma 3.51.** Let \( X \in \text{PreOp}^G \) be a Segal operad, and \( \Omega[1] \to X^H \) be a H-equivalence (Notation 3.32) for some \( H \leq G \). Then there exists a countable pseudo-interval \( I \) and factorization \( \Omega[1] \to I \to X^H \).

**Proof.** Recall (Remark 3.26) that one can find a simplicial equivalence \( X \xrightarrow{\sim} \tilde{X} \) with \( \tilde{X} \) fibrant in the normal model structure on \( \text{PreOp}^G \). Since, cf. Theorem 3.33, Reedy fibrant Segal preoperads are in particular Segal spaces, it is well known that, for \( J \) the nerve of the contractible groupoid with two objects, one has a dashed arrow as on the left below (this is originally due to Rezk [Rez01, Thm. 6.2], applied to \( \tau^* \tilde{X}^H \); alternatively, see [BP20, Prop. 5.26(iv)]).

\[
\begin{array}{cc}
\Omega[1] & \longrightarrow X^H \\
\downarrow & \\
J & \longrightarrow \tilde{X}^H
\end{array}
\]

Writing \( J \xrightarrow{\sim} \bar{J} \to \tilde{X}^H \) for the “trivial cofibration followed by fibration” factorization in the normal model structure on \( \text{PreOp} \), we now form the right diagram above, where the square is a pullback. Here we note that \( X^H \to \tilde{X}^H \) is a simplicial equivalence, i.e. the maps \( X^H(T) \to \tilde{X}^H(T) \) are Kan equivalences for each \( T \in \Omega \), while the maps \( \bar{J}(T) \to \tilde{X}^H(T) \) are Kan fibrations. Hence, since \( s\text{Set} \) is (right) proper, \( J' \to \bar{J} \) is again a simplicial equivalence.

By construction, the canonical map \( J' \to \Omega[\eta] \) is a simplicial equivalence, but to obtain the required countability and the tame cofibrancy condition for \( J' \) to be a pseudo-interval (Definition 3.45), we will need to replace \( J' \). Firstly, a countable replacement can be obtained by adapting either the argument between Lemmas 4.2 and 4.3 of [Ber07a] or the more refined argument in the proof of [HSS00, Lemma 5.1.7]. Briefly, since the spaces \( J'(\bar{T}) \) are all contractible, one may build nested countable subpresheaves \( I'^{\lambda} \subseteq J' \) as in

\[
\Omega[1] = I'^0 \subseteq I'^1 \subseteq I'^2 \subseteq \ldots \subseteq J'
\]

such that all maps \( I'^{\lambda}(\bar{T}) \to I'^{\lambda+1}(\bar{T}) \) are nullhomotopic (informally, and given a countable \( I'^{\lambda} \), one needs only countably many simplices of \( J' \) to kill of the homotopy groups of \( I'^{\lambda} \); hence by adding those simplices and closing under the presheaf operations one obtains \( I'^{\lambda+1} \)). Thus, setting \( I' = \bigcup_{\lambda} I'^{\lambda} \) we still have that \( I' \to \Omega[\eta] \) is a simplicial equivalence, but \( I' \) is now countable.

Lastly, the small object argument for (TC2), (TC3) applied to \( \Omega[1] \to I' \) gives a factorization \( \Omega[1] \to I \to I' \) where \( I \to I' \) is a joint equivalence (see the argument in Remark 3.50), and the countable preoperad \( I \) now has the tame cofibrancy property required to be a pseudo-interval. □
Proof of Theorem 3.47. Note that, assuming the existence of the tame model structure, the “moreover” and “left proper” claims follow from Lemma 3.48, saying that tame cofibrations are normal cofibrations, and the fact that $\text{PreOp}^G_{tame}$, $\text{PreOp}^G_{normal}$ both have joint equivalences as weak equivalences.

To show the existence claims, we will verify conditions C1,C2,C3,C4,C5 in [Sta14, Prop. 2.3], which is a variation of J. Smith’s theorem [Bek00, Thm. 1.7] that includes a further criterion for detecting fibrations between fibrant objects. In the remainder of the proof, we write $\mathcal{I}$ (resp. $\mathcal{J}$) for the union of the sets (TC1),(TC2),(TC3) (resp. (TA1),(TA2),(TA3)), and $\mathcal{W}$ for the class of weak equivalences, so that notions such as $\mathcal{I}$-cof, $\mathcal{J}$-fibrant have the same meaning as in [Sta14].

$\text{PreOp}^G$ is certainly locally presentable, as it is a presheaf category. That the weak equivalences in $\text{PreOp}^G$ are accessible follows since they are the preimage by $\gamma^*$ of the weak equivalences in $\text{sdSet}^G$ (see [Lur09, Cor. A.2.6.5] and [Lur09, Cor. A.2.6.6]).

Conditions C1 and C3 therein are equivalent to the 2-out-of-6 condition for weak equivalences, and are thus inherited from $\text{sdSet}^G$. Moreover, C2 has already been verified in Lemma 3.49.

We next check C4. Note first that the maps in $\mathcal{I}$-cof $\cap \mathcal{W}$ are closed under pushout and transfinite composition, as they are trivial cofibrations in the normal model structure in $\text{PreOp}^G$, so that C4 needs only be checked for the maps in (TA1),(TA2),(TA3) themselves, rather than their saturation. The case of maps in (TA1) is tautological, by definition of pseudo-interval. The fact that the maps in (TA2),(TA3) are in the saturation of (TC2),(TC3) is clear, and the fact that these maps are weak equivalences follows from Lemma 3.48.

Lastly, we check C5. The lifting condition against (TA3) says that $\mathcal{J}$-fibrant objects are such that the maps $X(\Omega[T]) \to X(\text{Sc}[T])$ are trivial Kan fibrations, and thus that such $X$ are Segal operads. Therefore, by the “further” statement in Theorem 3.33 it suffices to check that $\mathcal{J}$-fibrations between Segal operads which are also DK equivalences have the right lifting property against the maps in (TC1),(TC2),(TC3). Given $X \to Y$ a $\mathcal{J}$-fibration with $\mathcal{J}$-fibrant $Y$, the lifting property against (TC3) is tautological since (TC3) equals (TA3). Next, the lifting property against (TA2) says that the maps $X(\Omega[T]) \to f^*Y(\Omega[T])$ are Kan fibrations, and the DK condition says that these are Kan equivalences, so that such maps have the right lifting property against (TC2). Lastly, given any lifting problem against a map in (TC1), essential surjectivity (Definition 3.31(ii)) and Lemma 3.51 produce a lifting problem against a map in (TA1) which has a solution, providing a solution to the original problem. This finishes the proof.

For later use, we record the following.

Lemma 3.52. For all $T \in \Omega$, the objects $\text{Sc}[T], \Omega[T]$ are tame cofibrant in $\text{PreOp}^G$.

Proof. The case of $\text{Sc}[T]$ follows from the pushout below, where $\coprod_{E(T)} \Omega[\eta]$ is tame cofibrant by (TC1) and the left vertical map is a tame cofibration by (TC2) with $n = 0$.

$$
\begin{align*}
\coprod_{v \in V_G(T)} \partial \Omega[T_v] & \longrightarrow \prod_{E(T)} \Omega[\eta] \\
\downarrow & \\
\coprod_{v \in V_G(T)} \Omega[T_v] & \longrightarrow \text{Sc}[T]
\end{align*}
$$

The case of $\Omega[T]$ follows since (TC3) with $n = 0$ says that $\text{Sc}[T] \to \Omega[T]$ is a tame cofibration.
4 The Quillen equivalences

This section establishes our main result, Theorem 1, up to the key Lemma 4.37, to be shown in §5.

We do so in two steps. After recalling the Dwyer-Kan model structure on operads $sOp^G$ in §4.1, §4.2 we show in Theorem 4.48 in §4.3, that the nerve functor $N: sOp^G \to PreOp^G_{tame}$ is a right Quillen equivalence, where $PreOp^G_{tame}$ has the tame model structure from §3.4. Using (1.3), this yields two zigzags of Quillen adjunctions between $sOp^G$ and the joint model structure on $sdSet^G$, and the proof concludes in §4.4 by showing that the two associated derived composites agree up to joint equivalence, cf. (4.60).

4.1 Equivariant simplicial operads

We write $sOp^G = Op^G(sSet)$ for the category of $G$-equivariant colored simplicial operads. This category admits several descriptions: as the algebras for a composition product $\circ$; as the algebras for a free operad monad $F$; as the subcategory of preoperads satisfying a strict Segal condition.

Our primary goal in this section and the following is to recall (and repack) the model structure on $sOp^G$ built in [BPb, Thm. A]. The work in loc. cit. is based on the free operad monad perspective on $sOp^G$, which is technically involved, but this paper needs only a brief overview of that perspective. Instead, and in preparation for the proof of the Quillen equivalence $PreOp^G \leftrightarrow sOp^G$ in §4.3, we will find it useful in this section to also use the strict Segal condition perspective, which also plays a key role in Appendix A and the side paper [BPC].

We start by recalling the category $\Sigma_C$ of $C$-corollas (cf. Definition 2.17). An object of $\Sigma_C$ is given by a $C$-colored corolla as on the left below. Then letting $n$ be the number of leaves of $\vdash C$, which we call the arity of $\vdash C$, and $\sigma \in \Sigma_n$, the picture below depicts a generic map in $\Sigma_C$.

$$\begin{array}{ccc}
\vdash C & \xrightarrow{\sigma} & \vdash C_{\sigma}^{-1} \\
\vdash C & \xrightarrow{\sigma} & \vdash C_{\sigma}^{-1}
\end{array}$$

(4.1)

Alternatively, $C$-corollas can be represented simply as strings in $C$, which we call $C$-profiles (3 cf. Remark 3.19). In profile notation, (4.1) then becomes

$$\begin{array}{ccc}
\vdash C = (c_1, \ldots, c_n; c_0) & \xrightarrow{\sigma} & (c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c_0) = \vdash C_{\sigma}^{-1}.
\end{array}$$

For this reason, we also refer to $\Sigma_C$ as the $C$-symmetric category.

Lastly, we note that the notation $\vdash C_{\sigma}^{-1}$ used above comes from the natural right action of $\Sigma_n$ on $C$-profiles of arity $n$ via $(c_1, \ldots, c_n; c_0) \sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c_0)$.

**Definition 4.2.** The category $sSym$ of simplicial symmetric sequences has:

- objects given by pairs $(\mathcal{C}, X)$ with $\mathcal{C} \in \mathcal{C}$ a set of colors and $\Sigma^{op}_\mathcal{C} \xrightarrow{X} sSet$ a functor;
- maps $(\mathcal{C}, X) \to (\mathcal{D}, Y)$ given by maps of colors $f: \mathcal{C} \to \mathcal{D}$ and $\Phi$ as below.

3These were called $\mathcal{C}$-signatures in [BPb, BPa].
More explicitly, by specifying (4.1) to a map \( \overline{\mathcal{C}} \sigma \to \overline{\mathcal{C}} \) in \( \Sigma \varepsilon \) (note that \( \overline{\mathcal{C}} = (\mathcal{C}\sigma)\sigma^{-1} \)), one has that a symmetric sequence \( X: \Sigma^\text{op} \to \text{SSet} \) has structure maps

\[
X(\overline{\mathcal{C}}) = X(\epsilon_1, \ldots, \epsilon_n; \sigma_0) \xrightarrow{\sigma} X(\epsilon(\sigma(1)), \ldots, \epsilon(\sigma(n)); \sigma_0) = X(\mathcal{C}\sigma)
\]

for \( \overline{\mathcal{C}} \) of arity \( n \) and \( \sigma \in \Sigma_n \).

We next describe operads. We write \( \mathcal{F} \leftrightarrow \mathcal{G} \) given by the left Kan extension below (the description of the monad structure is found in [BPa, Defs. 3.44, A.23, A.32]).

\[
\begin{array}{ccc}
\Omega \varepsilon \to \Sigma \varepsilon & \xrightarrow{\mathcal{T} \mapsto \Pi_{v \in V(T)} X(T_v)} & \text{SSet} \\
\xrightarrow{\mathcal{L} \mathcal{A} \mathcal{n} \mathcal{E} \mathcal{N}} \downarrow & & \downarrow \\
\Sigma^\text{op} \varepsilon & \xrightarrow{\mathcal{S} \mathcal{O} \mathcal{P}} & \text{SSet}
\end{array}
\]

The category \( \mathcal{S} \mathcal{O} \mathcal{P} \) of colored simplicial operads can then be described as the category of fiber \( \mathcal{F} \)-algebras on \( \mathcal{S} \mathcal{Y} \mathcal{M} \) (where fiber algebras are those algebras for which the multiplication \( \mathcal{F} \mathcal{O} \to \mathcal{O} \) preserves colors, cf. [BPa, Def. 2.27]). For \( G \) a finite group, we then write \( \mathcal{S} \mathcal{O} \mathcal{P}^G \) (resp. \( \mathcal{S} \mathcal{Y} \mathcal{M}^G \)) for the category of \( G \)-objects on \( \mathcal{S} \mathcal{O} \mathcal{P} \) (resp. \( \mathcal{S} \mathcal{Y} \mathcal{M} \)) which we call the category of \( G \)-equivariant colored simplicial operads (symmetric sequences). Note that, by an abstract argument [BPa, Prop. 2.35], \( \mathcal{F} \) induces a monad on \( \mathcal{S} \mathcal{Y} \mathcal{M}^G \) whose category of fiber algebras is \( \mathcal{S} \mathcal{O} \mathcal{P}^G \).

Mirroring (3.11), we then have color set functors

\[
\begin{align*}
\mathcal{S} \mathcal{Y} \mathcal{M} \xrightarrow{\varepsilon \varepsilon} \text{Set} & \quad \mathcal{S} \mathcal{O} \mathcal{P} \xrightarrow{\varepsilon \varepsilon} \text{Set} \\
\mathcal{S} \mathcal{Y} \mathcal{M}^G \xrightarrow{\varepsilon \varepsilon} \text{Set}^G & \quad \mathcal{S} \mathcal{O} \mathcal{P}^G \xrightarrow{\varepsilon \varepsilon} \text{Set}^G
\end{align*}
\]

**Remark 4.5.** Replacing \( \text{SSet} \) with \( \text{Set} \) in Definition 4.2 and (4.4) one recovers the analogue non-simplicial categories \( \mathcal{S} \mathcal{Y} \mathcal{M}, \mathcal{O} \mathcal{P}, \mathcal{S} \mathcal{Y} \mathcal{M}^G, \mathcal{O} \mathcal{P}^G \). As is well known, there is then a fully faithful inclusion \( \mathcal{S} \mathcal{O} \mathcal{P} \subset \mathcal{O} \mathcal{P}^\Sigma^{\mathcal{E} \mathcal{N}} \) as those simplicial objects with a constant set of colors.

The color set functors above are all Grothendieck fibrations and, moreover, the monad \( \mathcal{F} \) is suitably compatible with these fibrations. In [BPa, §3] the Grothendieck fibration perspective is used to describe the fibers \( \mathcal{S} \mathcal{Y} \mathcal{M}^G, \mathcal{S} \mathcal{O} \mathcal{P}^G \) of those objects with a fixed \( G \)-color set \( \mathcal{E} \) and maps which are the identity on colors. However, here we will be able to take a more direct approach.

If \( \mathcal{E} \) is a \( G \)-set of colors, one has a left \( G \)-action on the set of \( \mathcal{E} \)-profiles. Thus, if \( X \in \mathcal{S} \mathcal{Y} \mathcal{M}^G \) has color set \( \mathcal{E}_X = \mathcal{E} \), one has, generalizing (4.3), that \( X \) has structure maps

\[
X(\overline{\mathcal{C}}) = X(\epsilon_1, \ldots, \epsilon_n; \sigma_0) \xrightarrow{\sigma} X(g\epsilon(\sigma(1)), \ldots, g\epsilon(\sigma(n)); g\sigma_0) = X(g\mathcal{C}\sigma)
\]

for \( \overline{\mathcal{C}} \) a \( \mathcal{E} \)-profile of arity \( n \) and \( (g, \sigma) \in G \times \Sigma_n^G \).

Note that, implicit in the \( g\mathcal{C}\sigma \) notation in (4.6) is the fact that \( G \times \Sigma_n^G \) has a left action on \( \mathcal{E} \)-profiles of arity \( n \). As such, given a subgroup \( \Gamma \leq G \times \Sigma_n^G \) and \( \overline{\mathcal{C}} \) of arity \( n \), we say that \( \Gamma \) stabilizes \( \overline{\mathcal{C}} \) if \( g\mathcal{C}\sigma = \overline{\mathcal{C}} \) for all \( (g, \sigma) \in \Gamma \). In particular, (4.6) then implies that, if \( \Gamma \) stabilizes \( \overline{\mathcal{C}} \) and for \( X \in \mathcal{S} \mathcal{Y} \mathcal{M}^G \) (resp. \( \mathcal{O} \in \mathcal{S} \mathcal{O} \mathcal{P}^G \)) with \( G \)-color set \( \mathcal{E} \), the level \( X(\overline{\mathcal{C}}) \) (resp. \( \mathcal{O}(\overline{\mathcal{C}}) \)) has an action by \( \Gamma \).

**Notation 4.7.** For \( x \in X(\overline{\mathcal{C}}) \) with \( \overline{\mathcal{C}} \) of arity \( n \) and \( (g, \sigma) \in G \times \Sigma_n^G \) we write \( gx\sigma \in X(g\mathcal{C}\sigma) \) for the image of \( x \) under (4.6). Note that this defines an action of \( G \times \Sigma_n^G \) on \( \bigsqcup_{\overline{\mathcal{C}}} X(\overline{\mathcal{C}}) \) of arity \( n \).

Moreover, if \( x\sigma = x \) only when \( \sigma = id \) we say that \( x \) is \( \Sigma \)-free.
Remark 4.8. Let \( G_\mathcal{C} \) denote the groupoid with objects the \( \mathcal{C} \)-profiles and arrows \( \mathcal{C} \to g\mathcal{C} \sigma \) for \( \mathcal{C} \) of arity \( n \) and \((g, \sigma) \in G \times \Sigma_n^{op} \). In other words, \( G_\mathcal{C} \) is the coproduct over \( n \geq 0 \) of the action groupoids for the actions of \( G \times \Sigma_n^{op} \) on \( n \)-ary profiles\(^4\). Equation (4.6) then identifies \( s\text{Sym}_\mathcal{C} \cong s\text{Set}^{G_\mathcal{C}} \).

Before describing the model structure on \( s\text{Op}^G \), we need to recall two more ingredients.

First, a subgroup \( \Gamma \leq G \times \Sigma_n^{op} \) is called a \( G \)-graph subgroup if \( \Gamma \cap \Sigma_n^{op} = \{*\} \). Equivalently, one can readily show that we must have \( \Gamma = \{(h, \phi(h)^{-1}) \mid h \in H\} \) for some subgroup \( H \leq G \) and homomorphism \( \phi : H \to \Sigma_n \).

Second, one has functors (compare with Definition 3.27 and the subsequent remarks)

\[
\text{sOp} \xrightarrow{\pi_0} \text{Op} \xrightarrow{\iota^*} \text{Cat}
\]

where \( \pi_0 \) is computed levelwise, i.e. \( (\pi_0\mathcal{O})(\mathcal{C}) = \pi_0(\mathcal{O}(\mathcal{C})) \) and \( \iota^* \) forgets non-unary operations.

Generalizing [Ber07b, CM13b], we showed in [BPb] that \( s\text{Op}^G \) has a Dwyer-Kan style model structure. More precisely, we have the following result, which is [BPb, Thm. A], with the alternative characterization of fibrations provided by [BPb, Prop. 3.79].

Theorem 4.9. The category \( s\text{Op}^G \) has a cofibrantly generated model structure with weak equivalences (resp. fibrations) those maps \( F : \mathcal{O} \to \mathcal{P} \) such that:

- \( F \) is fully faithful (resp. a local fibration), i.e. the induced maps
  \[
  \mathcal{O}(\mathcal{C})^\Gamma \to \mathcal{P}(F\mathcal{C})^\Gamma
  \]
  are Kan equivalences (resp. Kan fibrations) in \( s\text{Set} \) for all \( \mathcal{C}_\sigma \)-profiles \( \mathcal{C} = (c_1, \ldots, c_n; \sigma_0) \) and all graph subgroups \( \Gamma \leq G \times \Sigma_n^{op} \) which stabilize \( \mathcal{C} \);

- \( F \) is essentially surjective (resp. an isofibration), i.e. the induced maps of usual categories
  \[
  \iota^* \pi_0\mathcal{O}^H \to \iota^* \pi_0\mathcal{P}^H
  \]
  are essentially surjective (resp. isofibrations) for all \( H \leq G \).

The sets of generating (trivial) cofibrations in \( s\text{Op}^G \) are recalled in Remark 4.35 in §4.2, after discussing the fibered simplicial cotensoring in \( s\text{Op}^G \).

For a fixed \( G \)-set of colors \( \mathcal{C} \), the category of \( \mathcal{C} \)-colored symmetric sequences \( s\text{Sym}_\mathcal{C} \) admits an auxiliary model structure defined analogously to (4.10).

Proposition 4.12. For all \( G \)-sets \( \mathcal{C} \), \( s\text{Sym}_\mathcal{C} \) has a model structure where \( X \to Y \) is a (trivial) fibration (resp. weak equivalence) iff \( X(\mathcal{C})^\Gamma \to Y(\mathcal{C})^\Gamma \) is a (trivial) Kan fibration (resp. weak equivalence) for all \( \mathcal{C}_\sigma \)-profiles \( \mathcal{C} \) and all graph subgroups \( \Gamma \leq G \times \Sigma_n^{op} \) stabilizing \( \mathcal{C} \).

Proof. Using the identification \( s\text{Sym}_\mathcal{C} \cong s\text{Set}^{G_\mathcal{C}} \) in Remark 4.8, this is the \( F^\mathcal{C}_\mathcal{C} \)-model structure from [BPa, Defn. 4.13], where \( F^\mathcal{C}_\mathcal{C} \) is the family of subgroups of \( G_\mathcal{C} \) ([BPa, Defn. 4.11]) defined by \( F^\mathcal{C}_\mathcal{C} = \pi_0 F^\mathcal{C} \) for \( \pi_0 : G \times \Sigma_n^{op} \to G \times \Sigma_n^{op} \) ([BPa, Defn. 5.1, Remark 5.2]); explicitly, for an \( n \)-ary \( \mathcal{C} \)-profile \( \mathcal{C} \), \( (F^\mathcal{C}_\mathcal{C})_{\mathcal{C}} \) is the collection of graph subgroups \( \Gamma \leq G \times \Sigma_n^{op} \) which stabilize \( \mathcal{C} \).

As the genuine model structure on \( s\text{Set}^G \) exists by e.g. [BPa, Prop. 4.10], this \( F^\mathcal{C}_\mathcal{C} \)-model structure exists by [BPa, Prop. 4.17].

\(^4\)In [BPa, Prop. 3.17], we use the alternative description \( G_\mathcal{C} = G \times \Sigma_n^{op} \).
Proposition 4.13. Let \( f: A \to B \) be a map in \( s\text{Sym}_c^G \cong s\text{Set}^{G_c} \). The following are equivalent:

(i) \( f \) is a cofibration;

(ii) \( f \) is a monomorphism and the stabilizer of every \( x \in B \setminus f(A) \) is a graph subgroup;

(iii) \( f \) is a monomorphism and every \( x \in B \setminus f(A) \) is \( \Sigma \)-free (cf. Notation 4.7).

Proof. By [BPa, Rem. 4.14] the generating cofibrations in \( s\text{Sym}_c^G \) then have the form \( G_c(\tilde{C}, -)/\Gamma \cdot (\partial \Delta[k] \to \Delta[k]) \) with \( \Gamma \in \mathcal{F}_{c}^\partial, k \geq 0 \), so that (i) \( \iff \) (ii) follows by adapting [Ste16, Prop. 2.16] or [Per18, Prop. 6.5]. (ii) \( \iff \) (iii) is straightforward. \( \square \)

In light of (iii) in the previous result, a color fixed map of operads \( O \to P \) such that the underlying map of symmetric sequences is a cofibration is called a \( \Sigma \)-cofibration. Combining Proposition 4.13 with [BPa, Prop. 5.26] (also, see [BPb, Prop. 3.11(ii)]) yields the following.

Proposition 4.14. If \( f: O \to P \) is a color fixed cofibration in \( s\text{Op}^G \) and \( O \) is \( \Sigma \)-cofibrant then \( f \) is a \( \Sigma \)-cofibration. In particular, cofibrant operads are \( \Sigma \)-cofibrant.

Remark 4.15 (cf. [BPa, Rem. 3.51]). As in (3.12), for \( f: \mathbb{C} \to \mathbb{D} \) a map of \( G \)-sets of colors one has adjunctions

\[
\begin{align*}
&\mathcal{f}_! : s\text{Sym}_c^G \cong s\text{Sym}_c^G : \mathcal{f}_* \quad \mathcal{f}_! : s\text{Op}_c^G \cong s\text{Op}_c^G : \mathcal{f}_*.
\end{align*}
\]

The right adjoints, both denoted \( \mathcal{f}_* \), are given by \( \mathcal{f}_* Y(\tilde{C}) = Y(\mathcal{f}(\tilde{C})) \) (cf. Definition 2.17). The left adjoint \( \mathcal{f}_! \) on symmetric sequences is given by \( \mathcal{f}_! X = \text{Lan}_{\Sigma^c_\mathbb{C} \to \Sigma^c_\mathbb{D}} X \). The left adjoint \( \mathcal{f}_! \) on operads is given on free operads by \( \mathcal{f}_! CO = \mathcal{F} \mathcal{f}_! O \) and thus (since any operad \( O \) is a coequalizer \( \text{coeq}(\mathcal{F} \mathcal{f}_! O \rightrightarrows \mathcal{F} O) \)) on general operads \( O \) by

\[
\text{coeq}(\mathcal{F}(\mathcal{f}_! O) \rightrightarrows \mathcal{F} \mathcal{f}_! O).
\]

(4.16)

4.2 The nerve and the fibered simplicial (co)tensor on operads

This section discusses two necessary constructions: the nerve functor \( N: s\text{Op} \to \text{PreOp} \) and, in analogy with the fibered tensoring \( \otimes_{\mathbb{E}_*} \) in \( \text{PreOp} \) from §3.3, a similar tensoring \( \otimes_{\mathbb{E}_*} \) on \( s\text{Op} \).

We first recall and extend the operadification-nerve functor adjunction in (2.36) to

\[
\tau: d\text{Set} \rightrightarrows \text{Op}: N \quad \tau: \text{PreOp} \rightrightarrows s\text{Op}: N
\]

where the rightmost adjunction simply applies the leftmost adjunction to each simplicial level (indeed, by Definition 3.9, Remark 4.5 both \( \text{PreOp} \) and \( s\text{Op} \) are characterized by demanding that the color sets are constant in the simplicial direction). One then has the formulas

\[
\tau X = \text{colim}_{[U] \to \ast} \Omega(U), \quad (N\mathcal{O})(U) = \text{Op}(\Omega(U), O), \quad U \in \Omega
\]

for \( X \in d\text{Set}, O \in \text{Op} \), and with \( \Omega(U) \) the free operad determined by a tree \( U \in \Omega \), that we now recall. In fact, we define \( \Omega(-) \) slightly more generally. For \( F \in \Phi \) a forest, \( \Omega(F) \) is the \( \mathcal{E}(F) \)-colored operad which, evaluated at a \( \mathcal{E}(F) \)-colored corolla \( \tilde{C} = (C, \mathcal{c} : \mathcal{E}(C) \to \mathcal{E}(F)) \), is given by

\[
\Omega(F)(\tilde{C}) = \begin{cases} \ast & \text{if } \mathcal{c} : \mathcal{E}(C) \to \mathcal{E}(F) \text{ defines a map } C \to F \text{ in } \Phi \\ \emptyset & \text{otherwise.} \end{cases}
\]

(4.18)

Note that, as the levels of \( \Omega(F) \) are \( \ast \) or \( \emptyset \), one has at most one way to compose operations, and thus at most one possible operad structure on \( \Omega(F) \). That this structure exists (i.e. that composition is well defined) follows since, for a tree \( U \in \Omega \), a coloring \( \mathcal{c} : \mathcal{E}(U) \to \mathcal{E}(F) \) defines a map \( U \to F \) in \( \Phi \) iff the restrictions \( \mathcal{c}_v : \mathcal{E}(U_v) \to \mathcal{E}(F), v \in \mathcal{V}(U) \) define maps \( U_v \to F \) in \( \Phi \).
Remark 4.19. Recalling the $X_ε(U)$ notation in Notation 3.16, one has that $(NΩ(F))_ε(U)$ is given exactly as in (4.18) with the corolla $C$ replaced by a general tree $U$, so that $NΩ(F) = Ω[F]$.

We will also make use of an alternative description of $Ω(F)$, as follows.

First, if $C ∈ Σε$ is a $C$-corolla, we denote its representable functor in $Sym_ε = \text{Set}^{s操}$ by $Σ_ε[C] = Σ_ε^o(C, -)$. Second, for $F ∈ Φ_ε$ a $C$-forest, we extend the $Σ_ε[-]$ notation via

$$Σ_ε[F] = \prod_{v ∈ V(F)} Σ_ε[F_v],$$

where $u^C$ is the coproduct in $Sym_ε$ (rather than in the larger category $Sym$). Third, for $F ∈ Φ$ an (uncolored) forest we write $F^r$ for $F$ with its tautological $E(F)$-coloring, i.e. $F^r = (F, τ: E(F) → E(F))$, and abbreviate $Στ[F] = Σ_{E(F)}[F^r]$. All together, one then has an identification

$$Ω(F) = FΣτ[F] \quad (4.20)$$

which, informally, says that “$Ω(F)$ is freely generated by the vertices of $F$”.

We now recall [MW09, Prop. 5.3 and Thm. 6.1] that the nerve $N: Op → dSet$ is then a fully faithful inclusion whose (essential) image can be characterized as those dendroidal sets $X ∈ dSet$ with the strict right lifting property against inner horn inclusions $Λ^ε[U] → Ω[T]$ for $U ∈ Ω, ε ∈ E(U)$. Next, following either [CM13a, Prop. 2.5 and Cor. 2.6] or [BP20, Props. 3.22 and 3.31], this is in turn equivalent to the strict right lifting property of $X$ against Segal core inclusions $Sc[U] → Ω[T]$ for $U ∈ Ω$, which is in turn equivalent to the strict Segal conditions (cf. Definition 3.24) below, demanding that the maps

$$X(U) \xrightarrow{=} X(Sc[U]), \quad U ∈ Ω \quad X_ε(U) \xrightarrow{=} \prod_{v ∈ V(U)} X_ε(U_v), \quad U ∈ Ω, c: E(U) → X(η) \quad \quad (4.21)$$

are all isomorphisms. Moreover, one then has the following alternate formula for the nerve $NO$ evaluated at $U ∈ Ω, c: E(U) → Op$.

$$(NO)_ε(U) = \prod_{v ∈ E(U)} O(U_v, ε_v) = \prod_{v ∈ E(U)} O(Uh_v). \quad (4.22)$$

Remark 4.23. Setting $X = NO$ and unpacking (3.2), the left isomorphisms in (4.21) become $dSet(Ω[U], NO) \xrightarrow{=} dSet(Sc[U], NO)$, yielding isomorphisms $τ(Sc[U]) \xrightarrow{=} τ(Ω[U]) = Ω(U)$.

Mirroring §3.3, we next describe the fibered simplicial (co)-tensoring $⊗_ε$ on $sOp_ε^{C}$. First, for $O ∈ sOp_ε^{C}$ and $K ∈ sSet$ we define the fiber cotensor $\{K, O\}_ε$, via the pointwise simplicial cotensor, i.e.

$$\{K, O\}_ε(C) = O(C)^K. \quad (4.24)$$

Remark 4.25. The fact that $\{K, O\}_ε$ as described above has an operad structure can be seen by considering nerves. Indeed, one readily checks that the strict Segal condition (4.21) for $NO$ implies the same condition for the preoperad $\{K, NO\}_ε$ (defined as in (3.35)). Thus, (4.24) describes the levels of the unique (up to isomorphism) operad $\{K, O\}_ε$, such that $N\{K, O\}_ε = \{K, NO\}_ε$.

We now turn to the fiber tensor $(-) ⊗_ε K$, left adjoint to (4.24). First, note that (4.24) still makes sense at the level of symmetric sequences, i.e. with $O ∈ sSym_ε^{C}$ replaced with $X ∈ sSym_ε^{C}$. Then, at the level of symmetric sequences, the left adjoint construction $X × K ∈ sSym_ε^{C}$ is simply
given pointwise by $(X \times K)(\tilde{C}) = X(\tilde{C}) \times K$. It is now formal that, on a free operad $\mathcal{F}X$, the tensor $\mathcal{F}X \otimes_{\mathcal{E}_{*}} K$ adjoint to (4.24) is given by $(\mathcal{F}X) \otimes_{\mathcal{E}_{*}} K = \mathcal{F}(X \times K)$ so that, for a general $O \in \mathcal{sOp}^{G}$ (which has a description $O \simeq \text{coeq}(\mathcal{F}O \Rightarrow \mathcal{F}O)$ as a coequalizer of free algebras) it is given by (cf. (4.16))

$$O \otimes_{\mathcal{E}_{*}} K \simeq \text{coeq}(\mathcal{F}(\mathcal{F}O \times K) \Rightarrow \mathcal{F}(O \times K)).$$

(4.26)

**Remark 4.27.** In [CM13b, §7.1], the objects $\Omega(T) \otimes_{\mathcal{E}_{*}} K$ were denoted $T[K]$ and built by hand.

**Remark 4.28.** The analogues of Remarks 3.40,3.41 apply mutatis mutandis to the operadic fiber tensor. In particular, one has that the canonical map

$$u_{i}O \otimes_{\mathcal{E}_{*}} K_{i} \to O \otimes_{\mathcal{E}_{*}} (u_{i}K_{i})$$

is a cocartesian arrow over the fold map $\nabla: u_{i}C \to C$, i.e. $\nabla (u_{i}O \otimes_{\mathcal{E}_{*}} K_{i}) = O \otimes_{\mathcal{E}_{*}} (u_{i}K_{i})$.

**Proposition 4.30.** For all $X \in \text{PreOp}^{G}$ and $K \in \mathcal{sSet}$, one has a natural identification

$$\tau(X \otimes_{\mathcal{E}_{*}} K) = \tau(X) \otimes_{\mathcal{E}_{*}} K$$

with the first (resp. second) $\otimes_{\mathcal{E}_{*}}$ is the fiber simplicial tensoring of $\text{PreOp}^{G}$ (resp. $\mathcal{sOp}^{G}$).

**Proof.** This is equivalent to the already established (cf. Remark 4.25) adjoint identification $N\{K,O\}_{\mathcal{E}_{*}} \simeq \{K,N\mathcal{O}\}_{\mathcal{E}_{*}}$ for $O \in \mathcal{sOp}^{G}$. \square

**Remark 4.31.** Proposition 4.30 is a slight generalization of [CM13b, Prop. 7.2], which establishes the case $X = \Omega[U]$, $U \in \Omega$ by direct inspection (cf. Remarks 3.38,4.27).

**Remark 4.32.** Let $\Gamma \subseteq G \times \Sigma_{n}^{op}$ be the graph subgroup given by $\Gamma = \{(h,\phi(h)^{-1})|h \in H\}$ for $H \leq G$, $\phi:H \to \Sigma_{n}$. Writing $C_{n}$ for the $n$-corolla, $\phi$ defines a left $H$-action on $C_{n}$, so that one obtains an associated $G$-corolla $C = \Sigma_{G}[C_{n}]$. It is straightforward to check that there are natural identifications (here we view the natural left $G^{op} \times \Sigma_{n}$-action on $G \cdot C_{n}$ as a right $G \times \Sigma_{n}^{op}$-action)

$$\left(\left[G \cdot C_{n}\right]/\Gamma \simeq \Gamma \cdot H \cdot C_{n} = C \right) \quad \Sigma_{\tau}[G \cdot C_{n}]/\Gamma \simeq \Sigma_{\tau}[G \cdot H \cdot C_{n}] = \Sigma_{\tau}[C]$$

in $\Phi^{G}$ and $\text{Sym}^{G}$, respectively.

We end the section by discussing the generating sets for the model category $\mathcal{sOp}^{G}$ in Theorem 4.9, as given in [BPb, Def. 3.19]. We need one last ingredient, cf. [BPb, Def. 3.4], Definition 3.45.

**Definition 4.34.** Let $[1]$ denote the free isomorphism category, i.e. the contractible groupoid with two objects 0, 1. An interval is a cofibrant simplicial category $I \in \mathcal{sCat}_{(0,1)}$ equivalent to $[1]$.

**Remark 4.35.** Specifying [BPb, Def. 3.19] for the category $\mathcal{sSet}$, the notation therein becomes

$$\mathcal{F}(\Sigma_{\tau}[G \cdot C_{n}]/\Gamma \cdot f) \simeq \mathcal{F}(\Sigma_{\tau}[C]/f) \simeq \mathcal{F}(\Sigma_{\tau}[C]) \otimes_{\mathcal{E}_{*},f} \mathcal{F}(\Omega(C) \otimes_{\mathcal{E}_{*},f})$$

where the first identification is (4.33), the second follows by definition of $\otimes_{\mathcal{E}_{*}}$, and the third is (4.20). We thus have that the generating cofibrations in $\mathcal{sOp}^{G}$ are the maps

(C1) $\varnothing \to G/H \cdot \Omega(\eta)$ for $H \leq G$;

(C2) $\Omega(C) \otimes_{\mathcal{E}_{*}} (\partial \Delta[n] \to \Delta[n])$ for $C \in \Sigma_{G}$, $n \geq 0$.

while the generating trivial cofibrations are (for the countability condition, see [BPb, Rem. 3.17])
(A1) \( G/H \cdot (n \to \mathbb{G}) \) for \( H \leq G \) and \( \mathbb{G} \) an interval with countably many simplices;

(A2) \( \Omega(C) \otimes_{\varepsilon_*} (A_i[n] \to \Delta[n]) \) for \( C \in \Sigma_G, 0 \leq i \leq n, 1 \leq n. \)

In (C2),(A2) above the group \( G \) acts only on \( \Omega(C) \) and not on the featured simplicial sets. The following lemma considers the case where the simplicial sets also have a \( G \)-action (for a discussion of the genuine model structure on \( s\text{Set}^G \), see [BPa, Def. 4.1]).

**Lemma 4.36.** For \( C \in \Sigma_G \) and \( A \to B \) a genuine (trivial) cofibration in \( s\text{Set}^G \), \( \Omega(C) \otimes_{\varepsilon_*} (A \to B) \) is a (trivial) cofibration in \( s\text{Op}^G \).

**Proof.** Since \( \Omega(C) \otimes (-) : s\text{Set}^G \to s\text{Op}^G \) preserves colimits, by [BPa, (4.4)] it suffices to consider the case \( (A \to B) = G/H \cdot (K \to L) \) for \( K \to L \) a generating (trivial) cofibration in \( s\text{Set} \). Now consider the following diagram.

\[
\begin{array}{ccc}
(G/H \cdot \Omega(C)) \otimes_{\varepsilon_*} K & \to & \Omega(C) \otimes_{\varepsilon_*} (G/H \cdot K) \\
\downarrow & & \downarrow \\
(G/H \cdot \Omega(C)) \otimes_{\varepsilon_*} L & \to & \Omega(C) \otimes_{\varepsilon_*} (G/H \cdot L)
\end{array}
\]

By (4.29) the horizontal arrows are cocartesian, while the vertical arrows fix colors, so this is a pushout square. The result now follows since \( G/H \cdot \Omega(C) \) decomposes as a coproduct \( \cup_i \Omega(C_i) \) with \( C_i \in \Sigma_G \), so that the left vertical map above is a coproduct of maps in (C2). \( \square \)

### 4.3 Equivalence between preoperads and operads

Our goal in this subsection is to prove Theorem 4.48, establishing the Quillen equivalence between preoperads \( \text{PreOp}^G \) and operads \( \text{Op}^G \). The key to proving this result is given by Lemma 4.42 and the subsequent Corollary 4.45, which allow us to understand the counit of the adjunction. These latter results in turn depend on the following key result, whose proof is deferred to §5.

**Lemma 4.37.** Suppose that \( \mathcal{O} \in \text{Op}^G \) is \( \Sigma \)-cofibrant. Further, let \( C \in \Sigma_G \) be any \( G \)-corolla, \( r \geq 1 \) a positive integer and consider a pushout in \( \text{Op}^G \) of the form below.

\[
\begin{array}{ccc}
\partial \Omega(C)^{ur} & \to & \mathcal{O} \\
\downarrow & & \downarrow \\
\Omega(C)^{ur} & \to & \mathcal{P}
\end{array}
\] (4.38)

Then the induced map

\[
\Omega[C]^{ur} \cup_{\partial \Omega(C)^{ur}} \mathcal{NO} \to \mathcal{NP} \] (4.39)

is \( G \)-inner anodyne.

**Remark 4.40.** Both (4.38) and (4.39) are unchanged if the copowers \( (-)^{ur} \) in \( \text{Op}^G \), \( \text{dSet}^G \) are replaced with fibered copowers \( (-)^{ur}_{\varepsilon_*} = (-) \otimes_{\varepsilon_*} \{1, \ldots, r\} \) in \( \text{Op}_{E(1)}^G \), \( \text{dSet}_{E(1)}^G \).

In addition, since \( \partial \Omega(C) = \Omega(C) \otimes_{\varepsilon_*} \emptyset \), one is moreover free to replace the left vertical map in (4.38) with \( \Omega(C) \otimes_{\varepsilon_*} K \to \Omega(C) \otimes_{\varepsilon_*} L \) for \( K \to L \) any inclusion of sets.

**Remark 4.41.** The integer \( r \geq 1 \) in Lemma 4.37 is included to match our required application in Lemma 4.42. However, it readily follows by induction on \( r \) that one needs only prove the \( r = 1 \)
the functor

\[
\Omega \left[ \mathcal{C} \right]^{ur} \overset{u_{\partial \mathcal{C}^{ur}}} \longrightarrow NO_r \\
\downarrow \quad \downarrow \\
\Omega \left[ \mathcal{C} \right]^{ur+1} \overset{u_{\partial \mathcal{C}^{ur+1}}} \longrightarrow \Omega \left[ \mathcal{C} \right] \overset{u_{\partial \mathcal{C}}} \longrightarrow NO_r \longrightarrow NO_{r+1}
\]

where the square is a pushout so that induction on \( r \) and the \( r = 1 \) case yield that all horizontal maps are \( G \)-inner anodyne. The proof of the interesting \( r = 1 \) case will occupy the entirety of \( \S \).

**Lemma 4.42** (cf. [CM13b, Lemma 8.2]). Let \( A \to B \) be a tame cofibration in \( \text{PreOp}^G \), \( O \in \text{sOp}^G \) a \( \Sigma \)-cofibrant \( G \)-operad, and consider a pushout diagram in \( \text{sOp}^G \) of the form below.

\[
\tau A \longrightarrow O \\
\downarrow \quad \downarrow \\
\tau B \longrightarrow P
\]

(4.43)

Then \( O \to P \) is a \( \Sigma \)-cofibration and

\[ B \cup_A NO \to NP \]  

(4.44)

is a weak equivalence.

Setting \( A = \emptyset, O = \emptyset \) in the previous result yields the following.

**Corollary 4.45.** If \( B \in \text{PreOp}^G \) is tame cofibrant, then \( B \sim N \tau B \) is a weak equivalence.

**Proof of Lemma 4.42.** We first consider the case where \( A \to B \) is in one of (TC1),(TC2),(TC3).

The (TC1) case is immediate, since then \( O \to P \) is the \( \Sigma \)-cofibration \( O \to O \cup G/H \cdot \Omega(\eta) \) and (4.44) is the isomorphism \( NO \cup G/H \cdot \Omega(\eta) \sim N(\bigcup G/H \cdot \Omega(\eta)) \).

The (TC3) case is also straightforward: since \( \tau A \to \tau B \) is an isomorphism (Remark 4.23), one can take \( O = P \), so that (4.44) becomes a section of the map \( NO \to B \cup_A NO \), which is a trivial cofibration (as it is a pushout of \( A \to B \)), and 2-out-of-3 thus implies that (4.44) is a weak equivalence.

The most interesting case is then (TC2). Firstly, by Proposition 4.30 the functor \( \tau \) sends maps in (TC2) to maps in (C2), so \( O \to P \) is indeed a \( \Sigma \)-cofibration by [BPa, Prop. 5.26]. Next, fixing a simplicial level \( m \geq 0 \), \( A_m \to B_m \) then has the form \( \Omega \left[ \mathcal{C} \right] \otimes_{\mathcal{E}^m} (\partial \Delta[n]_m \to \Delta[n]_m) \) so that \( \tau A_m \to \tau B_m \) has the form \( \Omega \left[ \mathcal{C} \right] \otimes_{\mathcal{E}^m} (\partial \Delta[n]_m \to \Delta[n]_m) \). But then (following the discussion in Remark 4.40) Lemma 4.37 yields that all levels \( B \cup_A NO_m \to NP_m \) for \( m \geq 0 \) are equivalences in \( \text{dSet}^G \), showing that \( B \cup_A NO \to NP \) is indeed a joint equivalence in \( \text{PreOp}^G \).

We now turn to the case of \( A \to B \) a general tame cofibration. As usual, \( A \to B \) is a retract of a transfinite composition of pushouts of generating cofibrations. Since the conclusions of the result are invariant under retracts, we are free to assume that \( A \to B \) is a transfinite composite

\[ A = A_0 \to A_1 \to A_2 \to \cdots \to \text{colim}_{\beta < \kappa} A_\beta = B. \]

where each map \( A_\beta \to A_{\beta+1} \) for \( \beta < \kappa \) is a pushout of a map in one of (TC1),(TC2),(TC3).

Defining \( O_\beta \) by replacing \( A \to B \) with \( A \to A_\beta \) in the pushout (4.43), \( O \to P \) becomes the transfinite composite of the maps \( O_\beta \to O_{\beta+1} \) and (4.44) becomes \( \text{colim}_{\beta < \kappa} (A_\beta \cup_A NO \to NO_\beta) \).

It thus suffices to show, by induction on \( \beta < \kappa \), that the maps \( O_\beta \to O_{\beta+1} \) are \( \Sigma \)-cofibrations and that the maps \( A_\beta \cup_A NO \to NO_\beta \) are weak equivalences (sufficiency of the latter condition

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uses the fact that filtered colimits of weak equivalences in \( \text{PreOp}^G \) are weak equivalences, cf. Theorem 3.15). Consider now the following diagrams.

\[
\begin{align*}
\tau A & \rightarrow O \\
\tau A & \rightarrow O_eta \\
\tau A_{\beta+1} & \rightarrow O_{\beta+1} \\
A_\beta \cup_A N O & \rightarrow A_{\beta+1} \cup_A N O \\
N O_\beta & \rightarrow A_{\beta+1} \cup_A N O_\beta \\
N O_{\beta+1} &
\end{align*}
\]

The induction hypothesis states that \( O \rightarrow O_\beta \) is a \( \Sigma \)-cofibration and that the map \( A_\beta \cup_A N O \rightarrow N O_\beta \) is a weak equivalence. Therefore, \( O_\beta \) is \( \Sigma \)-cofibrant and both vertical maps marked \( \sim \) in the rightmost diagram above are weak equivalences (by left properness of \( \text{PreOp}_{\text{tame}}^G \)) and thus the induction step will follow provided that the result holds for \( A_\beta \rightarrow A_{\beta+1} \) and \( O_\beta \). But \( A_\beta \rightarrow A_{\beta+1} \) is assumed to be a pushout of a map in (TC1),(TC2),(TC3), in which case the result is already known, and thus noting that the result is invariant under pushouts finishes the proof.

Before proving Theorem 4.48, we recall the two following results, which are adapted from [JT07] (see Proposition 7.15 therein).

**Proposition 4.46.** A cofibration \( A \rightarrow B \) in a model category is a weak equivalence if it has the left lifting property against all fibrations between fibrant objects.

**Corollary 4.47.** An adjunction \( F : C \rightleftarrows D : G \) between model categories is a Quillen adjunction provided that \( F \) preserves cofibrations and \( G \) preserves fibrations between fibrant objects.

**Theorem 4.48.** The adjunction

\[
\tau : \text{PreOp}^G_{\text{tame}} \rightleftarrows \text{sOp}^G : N
\]

is a Quillen equivalence.

**Proof.** We first show that \( N \) preserves and detects weak equivalences. To see this, note first that all objects in the image of \( N \) are Segal operads (cf. Remark 3.25, (4.21)) so that, by Theorem 3.33, a map in the image of \( N \) is a weak equivalence if it is a Dwyer-Kan equivalence. But it is clear that \( N \) preserves and reflects fully-faithful maps, and since \( N (\tau O)^H = \tau^* ((NO)^H) \) for \( H \leq G \) one likewise has that \( N \) preserves and reflects essentially surjective maps.

Next, we use Corollary 4.47 to show that (4.49) is a Quillen adjunction. First, \( \tau \) preserves cofibrations since, by Proposition 4.30, \( \tau \) sends maps in (TC1),(TC2) to maps in (C1),(C2) and maps in (TC3) to isomorphisms (Remark 4.23). Second, to show that \( N \) preserves fibrations between fibrant objects, by using the characterization in Theorem 3.47 it suffices, thanks to an adjunction argument, to show that \( \tau \) sends the maps in (TA1),(TA2),(TA3) to trivial cofibrations. Moreover, as we already know that \( \tau \) preserves cofibrations, we need only show that \( \tau \) sends (TA1),(TA2),(TA3) to weak equivalences. The cases (TA2),(TA3) are again immediate from Proposition 4.30, but (TA1) requires a different argument (which could also be used for (TA2),(TA3)). Writing \( A \rightarrow B \) for a map in (TA1), one always has that \( A, B \) are tame cofibrant, so that Corollary 4.45 and 2-out-of-3 imply that \( N \tau A \rightarrow N \tau B \) is a weak equivalence and thus, since \( N \) reflects weak equivalences, that \( \tau A \rightarrow \tau B \) itself is a weak equivalence. This shows that (4.49) is a Quillen adjunction.
Lastly, for the Quillen equivalence claim, let \( B \in \text{PreOp}_G \) be tame cofibrant and \( O \in \text{Op}_G \) be fibrant. We must show the left map below is a weak equivalence iff the right composite is.

\[
\tau B \to O, \quad B \cong N\tau B \to NO
\]

This now follows from Corollary 4.45 and the fact that \( N \) preserves and detects weak equivalences. \(\square\)

4.4 The homotopy coherent nerve and the proof of the main result

This section proves Theorem I. We first recall the \( W : d\text{Set}^G \rightleftarrows \text{Op}_G : hcN \) adjunction in (1.2).

In the categorical setting, the left adjoint \( W_! \) admits an explicit description, due to Dugger and Spivak [DS11], in terms of so called necklaces, which we extend to the operadic setting in [BPc]. We now summarize the results therein we will need.

**Definition 4.50.** For a tree \( U \in \Omega \) there is a simplicial operad \( W(U) \in \text{Op} \) with set of colors \( E(U) \) and whose \( n \)-simplices evaluated at a \( E(U) \)-corolla \( C = (C,E) \) are

\[
W(U)_n(C) = \begin{cases}
\{\text{factorizations } C \xrightarrow{i} F_0 \xrightarrow{i_0} \cdots \xrightarrow{i_{n-1}} F_n \xrightarrow{i_n} U\} & \text{if } E(C) \xrightarrow{i} E(U) \text{ gives a map in } \Omega \\
\emptyset & \text{otherwise.}
\end{cases}
\]

where we label maps in \( \Omega \) as \( t/i/f/p \) to indicate they are tall/inner faces/faces/planar (cf. §2.1).

**Remark 4.51.** The factorization description in Definition 4.50 reflects our approach in [BPc], which makes heavy use of the factorizations in Proposition 2.10. However, there is a simpler and more familiar description of \( W(U)(C) \). If one lets \( C \xrightarrow{t} U \xrightarrow{\alpha} U \) denote the unique “tall map followed by planar outer face” factorization, repeated use of Proposition 2.10 shows that the \( F_0 \to \cdots \to F_n \) strings in Definition 4.50 are precisely the strings of planar inner faces of \( U_G \). And since the latter are in bijection with strings of subsets of inner edges \( E(U_G) \), we have

\[
W(U)(C) = \begin{cases}
\Delta[1]^\ast E(U_G) & \text{if } E(C) \xrightarrow{i} E(U) \text{ gives a map in } \Omega \\
\emptyset & \text{otherwise},
\end{cases}
\]

which recovers the description in [CM13b, §4].

**Remark 4.52.** One neat feature of the description in Definition 4.50 is that the nerve \( NW(U) \) can be defined identically (cf. [BPc, Def. 4.1]; compare with Remark 4.19). As an aside, one can verify directly that [BPc, Def. 4.1] defines a preoperad satisfying the strict Segal condition [BPc, Rem. 4.6], thereby inducing the operad structure on \( W(U) \).

The adjunction

\[
W : d\text{Set} \rightleftarrows \text{Op} : hcN
\]

is then defined by

\[
W_! X = \text{colim}_{U[U] \to X} W(U) \quad \text{hcN}(U) = \text{Op}(W(U), O)
\]

with the analogous equivariant adjunction (1.2) obtained by taking \( G \)-objects.

For a \( G \)-tree \( T = \bigcup T_i = G \cdot_H T_\ast \) in \( \Omega_G \) we abbreviate \( W(T) = W_!(\Omega[T]) \). Note that, since the \( T_i \) have disjoint edge sets, we thus have

\[
W(T) = \bigcup W(T_i) = G \cdot_H W(T_\ast).
\]

The following formalizes some key observations in the proof of [CM13b, Prop. 4.5].
Lemma 4.53. For \( \eta \neq T \in \Omega^G \) a tree with a \( G \)-action and \( G \)-subset \( \emptyset \neq E \subseteq \ell(T) \), one has pushout diagrams in \( s\text{Op}^G \) (for \( C = \text{tr}(T) \) the corolla with the same number of leaves as \( T \))

\[
\begin{array}{ccc}
\Omega(C) \otimes \epsilon \rightarrow W(\Omega[T]) & \xrightarrow{\Omega(C) \otimes \epsilon \rightarrow \Lambda^E[\Delta[1]^\times E(T)])} W(\Lambda^E[T]) \\
\downarrow & \downarrow & \downarrow \\
\Omega(C) \otimes \epsilon \rightarrow W(T) & \xrightarrow{\Omega(C) \otimes \epsilon \rightarrow \Delta[1]^\times E(T)} W(T)
\end{array}
\]

(4.54)

where \( \partial \left( \Delta[1]^\times E(T) \right) \rightarrow \Delta[1]^\times E(T) \) and \( \lambda^E \left( \Delta[1]^\times E(T) \right) \rightarrow \Delta[1]^\times E(T) \) are the pushout products

\[
(\partial \Delta[1] \rightarrow \Delta[1]) \rangle_{\Delta^E(T)} \ , \ (\partial \Delta[1] \rightarrow \Delta[1]) \rangle_{\Delta^E(T)} \nobla (\{1\} \rightarrow \Delta[1]) \rangle_{\Delta^E(T)}
\]

with \( G \)-action induced by the action on \( \ell(T) \).

Proof. Note first that, by (4.20) and the discussion above (4.26),

\[
\Omega(C) \otimes \epsilon, K \simeq (\mathbb{F}\Sigma_r[C]) \otimes \epsilon, K \simeq \mathbb{F}(\Sigma_r[C] \times K).
\]

Let us write \( \overline{C} = (C, c) \) for the corollas colored by either \( E(C) \) or \( E(T) \) via the (planar) coloring sending leaves to leaves and the root to the root. By definition of \( \mathbb{F} \), cf. (4.4), one has

\[
(\mathbb{F}(\Sigma_r[C] \times K))(\overline{C}) = (\Sigma_r[C] \times K)(\overline{C}) = K,
\]

so that the horizontal maps in (4.54) are the unique maps given by the identity at the \( \overline{C} \) level, as per the calculations of \( W(\Omega[T]) \rightarrow W(\Lambda^E[T]) \) in [BPc, Examples 4.41,4.42].

Lastly, to see that the squares in (4.54) are pushout squares note that, after taking nerves, it is clear that the left vertical inclusions attach precisely those dendrices missing from the right vertical inclusions. I.e., upon applying the nerve functor, (4.54) induces pushouts in \( d\text{Set}^G \). The result now follows since the nerve reflects colimits, due to being a fully faithful right adjoint.

Proposition 4.55 (cf. [CM13b, Prop. 4.9]). \( W: d\text{Set}^G \rightleftarrows s\text{Op}^G:hcN \) is a Quillen adjunction.

Proof. We verify the conditions in Corollary 4.47. Combining the pushouts in Lemmas 4.53 and 4.36 yields that \( W \) preserves cofibrations and sends \( G \)-inner anodyne extensions to trivial cofibrations. By adjunction, the latter claim implies that, if \( f: \mathcal{O} \rightarrow \mathcal{P} \) is a fibration between fibrant objects, then \( hcN(f):hcN\mathcal{O} \rightarrow hcN\mathcal{P} \) is a \( G \)-inner fibration between \( G \)-\( \infty \)-operads. Thus, the remaining claim that \( hcN \) preserves fibrations between fibrant objects reduces to showing that the maps \( \tau_* \left( \tau \left( hcN(f)^H \right) \right) \) is \( \tau \left( hcN_g(f)^H \right) \) for \( H \leq G \) are isofibrations of (usual) categories (cf. Theorem 2.37). But since by definition of fibration in \( s\text{Op}^G \) the maps \( \tau_* \pi_0 f^H \) in (4.11) are isofibrations, the result follows by the identification \( \pi_0 Q \simeq \tau \left( hcN(Q) \right) \) for fibrant operads \( Q \in s\text{Op} \), cf. [CM13b, Prop. 4.8].

Remark 4.56. The identification \( \pi_0 Q \simeq \tau \left( hcN(Q) \right) \) in [CM13b, Prop. 4.8] used above identifies two procedures for discretizing a simplicial operad \( Q \in s\text{Op} \) to obtain its homotopy operad \( \pi_0 Q \in \text{Op} \).

Notably, however, neither Proposition 4.55 nor the original [CM13b, Prop. 4.9] require the full strength of [CM13b, Prop. 4.8], as essential surjectivity depends only on the the categories of unary operations within operads. Nonetheless, and in light of the fully faithful inclusions in (2.40), it is natural to ask if [CM13b, Prop. 4.8] extends to the context of genuine operads. The answer to this question is affirmative, and is given by Proposition A.22 in Appendix A.
We now turn to the proof of our main result, Theorem 1.

Recall that, given an object $X$ in a model category $\mathcal{M}$, a simplicial frame for $X$ is a fibrant replacement $\alpha(X) \to \tilde{X}(\bullet)$ of the constant simplicial object $\alpha(X)$ in the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$. Moreover, if $X$ was already fibrant, one is free to assume that $\tilde{X}(0) = X$.

In addition, we will need some variants of [BP20, Prop. 4.5], regarding the categories $\mathsf{sSet}^{\Delta^{op}}$ and $(\mathsf{sdSet}^{G})^{\Delta^{op}}$. We set some notation: for $X$ in one of these categories, $X(m)$ (resp. $X_n$) denotes a level obtained by fixing the $(-)^{\Delta^{op}}$ simplicial coordinate (resp. a simplicial coordinate in $\mathsf{sSet}$, $\mathsf{sdSet}^{G}$). Further, $\delta^{*}X$ denotes the object $X_n(n), n \geq 0$ given by the diagonal in the simplicial coordinates.

**Remark 4.57.** The proof of [BP20, Prop. 4.5(ii)] (or, alternatively, adapting [BP20, Prop. 4.24(ii)]) shows that a Reedy fibrant $X(\bullet) \in \mathsf{sSet}^{\Delta^{op}}$ is joint fibrant (i.e. its transpose swapping the two simplicial directions is also Reedy fibrant) iff the vertex maps $X(m) \to X(0)$ (induced by the vertices $[0] \to [m]$) are Kan equivalences.

**Lemma 4.58.** If $X \in (\mathsf{sdSet}^{G})^{\Delta^{op}}$ is Reedy fibrant over the dendroidal Reedy model structure on $\mathsf{sdSet}^{G}$ and the vertex maps $X(m) \to X(0)$ are simplicial equivalences in $\mathsf{sdSet}^{G}$ then the two maps

$$X(0) \to \delta^{*}X \leftarrow X_0$$

are also simplicial equivalences in $\mathsf{sdSet}^{G}$.

**Proof.** By definition, we need to show that, for each $T \in \Omega_G$, the maps

$$X(0)(\Omega[T]) \to \delta^{*}X(\Omega[T]) \leftarrow X_0(\Omega[T])$$

are Kan equivalences in $\mathsf{sSet}$. Both of these equivalences will follow from [BP20, Prop. 4.5(iv)] provided we show that $X(\Omega[T])$ is a joint fibrant object in $\mathsf{sSet}$. And since the vertex maps $X(\Omega[T])(m) \to X(\Omega[T])(0)$ are Kan equivalences by assumption on $X$, by Remark 4.57 it remains only to check that $X(\Omega[T])$ is Reedy fibrant in $\mathsf{sSet}^{\Delta^{op}}$. For this last claim, note first that the Reedy fibrancy assumption on $X$ is that the matching maps (see, e.g., [BP20, Thm. A.8]) $X(m) \to M_mX(\bullet)$ are dendroidal fibrations in $\mathsf{sdSet}^{G}$. Unpacking definitions, this means that, for every normal monomorphism $A \to B$ in $\mathsf{dSet}^{G}$, the maps

$$X(m)(B) \to X(m)(A) \times_{M_mX(\bullet)(A)} M_mX(\bullet)(B)$$

are Kan fibrations. But setting $A \to B$ to be $\emptyset \to \Omega[T]$ we get that the maps $X(\Omega[T])(m) \to M_mX(\Omega[T])(\bullet)$ are Kan fibrations, i.e. that $X(\Omega[T])$ is Reedy fibrant in $\mathsf{sSet}^{\Delta^{op}}$. \(\square\)

**Proof of Theorem 1.** Consider the square of adjunctions on the left below (where we depict only the right adjoints). We already know that all four adjunctions therein are Quillen, and that those adjunctions other than the $(\mathcal{W}_1, hcN)$ adjunction are Quillen equivalences ([BP20, Thms. 4.30 and 4.41] and Theorem 4.48). Next, we consider the induced diagram of homotopy categories and derived functors on the right. Crucially, note that while the right Quillen functors $N$ and $hcN$ must be right derived, the left Quillen functors $\gamma^*$ and $c_!$ do not, since they preserve all weak equivalences (this holds for $\gamma^*$ by definition, cf. Theorem 3.47, and for $c_!$ since it sends weak equivalences to dendroidal equivalences, cf. Theorem 3.3(ii)).

$$
\begin{array}{ccc}
\mathsf{PreOp}^{G}_{\text{tame}} & \leftarrow & \mathsf{sOp}^{G} \\
\gamma^* & \downarrow & c_! \\
\mathsf{sdSet}^{G} & \leftarrow & \mathsf{dSet}^{G}
\end{array}
\sim
\begin{array}{ccc}
\mathsf{HoPreOp}^{G} & \leftarrow & \mathsf{HosOp}^{G} \\
\gamma^! & \downarrow & c_! \\
\mathsf{HosdSet}^{G} & \leftarrow & \mathsf{HodSet}^{G}
\end{array}
$$

(4.59)
Recalling that a Quillen adjunction is a Quillen equivalence iff the induced adjunction of homotopy categories is an equivalence adjunction, the desired claim that $(W, hcN)$ is a Quillen equivalence will thus follow provided we show that the right square in (4.59) commutes up to natural isomorphism. In other words, we reduce to showing that for fibrant operads $O \in sOp^G$ there is a natural zigzag of joint equivalences between $\gamma^* NO$ and $hcN NO$.

We now discuss this zigzag. Assume $O \in sOp^G$ is fibrant. First, choose a (functorial) fibrant simplicial frame $\overline{O}(\bullet) \in (sOp^G)_{\Delta^{op}}$, where we assume $\overline{O}(0) = O$. Next, let $\gamma^* N\overline{O}(\bullet) \rightarrow \overline{Q}(\bullet)$ be a Reedy fibrant replacement in $(sdSet^G)_{\Delta^{op}}$. We note that both $\overline{O}$ and $\overline{Q}$ have two simplicial directions: the frame direction, whose levels are written as $\overline{O}(m), \overline{Q}(m)$, and an internal direction (determined by the simplicial levels in $sOp^G, sdSet^G$), whose levels are written $\overline{O}_n, \overline{Q}_n$. Our desired zigzag of joint equivalences in $sdSet^G$ will have the form below.

\[
\gamma^* NO = \gamma^* N\overline{O}(0) \xrightarrow{(a)} \overline{Q}(0) \xrightarrow{(b)} \delta^* \overline{Q} \xrightarrow{(c)} \overline{Q}_0 \xrightarrow{(d)} (\gamma^* N\overline{O})_0 \xrightarrow{(e)} hcN\overline{O} \xrightarrow{(f)} hcN NO
\] (4.60)

Firstly, the map $(a)$ is a joint equivalence by definition of $\overline{Q}$.

Next, the maps $(b), (c)$ are joint equivalences (in fact, simplicial equivalences) by Lemma 4.58. Here, we note that though, a priori, the maps $\overline{Q}(m) \rightarrow \overline{Q}(0)$ are only joint equivalences, they are in fact simplicial equivalences, since the levels $\overline{Q}(m)$ are joint fibrant, cf. Theorem 3.3(iv), [BP20, Lemma A.29(i)].

To see that $(d)$ is a joint equivalence, note that one has identifications ($\bullet$ tracks the simplicial index)

\[
\overline{Q}_0(\Omega[T])(\bullet) \xrightarrow{\text{sdSet}^G(\Omega[T], \overline{Q}(\bullet))} \text{PreOp}^G(\Omega[T], \gamma_* \overline{Q}(\bullet))
\]

\[
(\gamma^* N\overline{O})_0(\Omega[T])(\bullet) = \text{sdSet}^G(\Omega[T], \gamma^* N\overline{O}(\bullet)) = \text{PreOp}^G(\Omega[T], N\overline{O}(\bullet))
\] (4.61)

in $sSet$ for each $T \in \Omega_G$. Next, since the counit maps $\gamma^* \gamma_* \overline{Q}(\bullet) \rightarrow \overline{Q}(\bullet)$ are joint equivalences (due to $\gamma^* \gamma_* \overline{Q}(\bullet)$ already computing a derived functor) in $sdSet^G$, one has that the maps $N\overline{O}(\bullet) \rightarrow \gamma_* \overline{Q}(\bullet)$ are joint equivalences in $\text{PreOp}^G$. Therefore, and since $\gamma_*$ is right Quillen, both $N\overline{O}$ and $\gamma_* \overline{Q}$ are simplicial frames for $NO$ in the tame model structure $\text{PreOp}^G_{tame}$. Thus, the fact that $(d)$ is a joint equivalence follows since both halves of (4.61) compute the mapping space from $\Omega[T]$ to $NO$ in the tame model structure (this uses the observation that $\Omega[T]$ is tame cofibrant, cf. Lemma 3.52).

For $(e)$, in which case we need also describe the map, we consider the identifications

\[
(\gamma^* N\overline{O})_0(\Omega[T])(\bullet) \xrightarrow{\text{PreOp}^G(\Omega[T], N\overline{O}(\bullet))} sOp^G(\Omega[T], \overline{O}(\bullet))
\]

\[
(hcN\overline{O})(\Omega[T])(\bullet) \xrightarrow{sOp^G(W(T), \overline{O}(\bullet))} sOp^G(W(T), \overline{O}(\bullet))
\] (4.62)

in $sSet$ for each $T \in \Omega_G$. The map $(e)$ is then induced by the unique color preserving map $W(T) \rightarrow \Omega(T)$ (recall that the mapping spaces of $\Omega(T)$ are either $\ast$ or $\emptyset$). Thus, since $W(T) \rightarrow \Omega(T)$ is a weak equivalence of cofibrant operads in $sOp^G$ and $\overline{O}$ is a simplicial frame for $O$, it follows that (4.62) likewise computes mapping spaces, and thus $(e)$ is indeed a weak equivalence.

Lastly, that $(f)$ is a weak equivalence follows from the identification $hcN \overline{O} = hcN \overline{O}$, together with the fact that $hcO(\bullet) \rightarrow \overline{O}(\bullet)$ is a levelwise equivalence of levelwise fibrant operads (where “levelwise” means “for fixed $\bullet$”) and $hcN: sOp^G \rightarrow dSet^G$ being right Quillen.

**Remark 4.63.** The previous proof is a close variation of the proof of [CM13b, Thm. 8.14], although the equivariant context forces us to use a more formal argument.
More precisely, the given proof of [CM13b, Thm. 8.14] relies on [CM13b, Thm 5.9(v)], which states that a preoperad \( X \in \text{PreOp} \) is equivalent in \( \text{sdSet} \) to the presheaf \( T \mapsto \text{Map}(\Omega[T], X) \) for \( T \in \Omega \) (here \( \text{Map}(\cdot, \cdot) \) denotes the homotopy space of maps). However, in the equivariant context the assignment \( T \mapsto \text{Map}(\Omega[T], X) \) for \( T \in \Omega_G \) does not produce a presheaf in \( \text{sdSet}^G \) (since the levels of such presheaves are indexed by \( U \in \Omega \)) but rather a presheaf in the category \( \text{sdSet}_G \) of simplicial objects on \( \text{dSet}_G \) (cf. (2.39)), which does not appear in (4.59).

As such, rather than attempt to formulate and use an analogue of [CM13b, Thm 5.9(v)], our proof replaces the role of that result with an explicit analysis of the simplicial framings needed to define the homotopy mapping spaces appearing in [CM13b, Thm 5.9(v)].

**Remark 4.64.** There is a natural way to attempt to simplify the zigzag (4.60) in the previous proof. Namely, one may attempt to replace the first four maps therein with the simpler two map zigzag

\[
\gamma^* N \tilde{O}(0) \to \delta^* \gamma^* N \tilde{O} \leftarrow (\gamma^* N \tilde{O})_0. \tag{4.65}
\]

As it turns out, it can be shown that (4.65) consists of weak equivalences, but our argument for this is substantially more involved than the argument given for (4.60).

Briefly, if \( \tilde{X} \) in \( (\text{PreOp}_{\text{frame}})^{\Delta^\op} \) is a simplicial frame for \( X \) in \( \text{PreOp}^G \), one can find a levelwise simplicial equivalence \( \tilde{X} \to \tilde{Y} \) with \( \tilde{Y} \) a simplicial frame in \( (\text{PreOp}_{\text{normal}})^{\Delta^\op} \). One then has that \( \tilde{X}(0) \to \delta^* \tilde{X} \leftarrow \tilde{X}_0 \) consists of weak equivalences iff \( \tilde{Y}(0) \to \delta^* \tilde{Y} \leftarrow \tilde{Y}_0 \) does, and the latter can be shown by following the (rather involved) proof of [BP20, Prop. 5.41] with \( \tilde{Y} \) taking the role of \( X^\wedge \) therein.

As a side note, [BP20, Prop. 5.41] is one of the keys to the proof of [BP20, Thm. 5.48], which establishes the DK description of weak equivalences between fibrant objects in \( \text{PreOp}^G, \text{sdSet}^G \), so our proof of Theorem I does indirectly rely on [BP20, Prop. 5.41].

### 5 nerves of free extensions are homotopy pushouts

This section will prove the following key lemma, which is the equivariant analogue of [CM13b, Prop. 3.2]. For a comparison with the work in [CM13b, §3], see Remarks 5.40, 5.39.

**Lemma 5.1.** Suppose that \( O \in \text{Op}^G \) is \( \Sigma \)-cofibrant. Further, let \( C \in \Sigma_G \) be any \( G \)-corolla and consider a pushout in \( \text{Op}^G = \text{Op}^G(\text{Set}) \) of the form below.

\[
\begin{align*}
\partial \Omega(C) & \to O \\
\Omega(C) & \to \mathcal{P}
\end{align*}
\]

Then the induced map

\[
\Omega[C] \cup_{\partial \Omega(C)} NO \to NP
\]

is \( G \)-inner anodyne.

#### 5.1 The characteristic edge lemma

The proof of Lemma 5.1 will use the characteristic edge lemma [BP20, Lemma 3.4], repackaged below as Lemma 5.11 in a form better suited for our application. We start with some notation. For a discussion of (non-)degenerate dendrices in a dendroidal set, see [Per18, Prop. 5.62].
**Notation 5.4.** Let $Y \in \text{dSet}^G$ be a $G$-dendroidal set and $y: \Omega[U^y] \to Y$ a dendrex, $U^y \in \Omega$.

We write $(y) = y(\Omega[U^y])$ and refer to $(y) \subseteq Y$ as the principal subpresheaf generated by $y$.

Moreover, if some (and thus any) non-degenerate representative $y$ is free with respect to the $\text{Aut}(U^y)$-action (via precomposition), we say $y$ and $(y)$ are $\Sigma$-free. If all dendrices $y$ are $\Sigma$-free, we say $Y$ itself is $\Sigma$-free.

Given a map of trees $V \to U^y$ we write $\partial_V y$ for the composite $\Omega[V] \to \Omega[U^y] \xrightarrow{\partial} Y$.

**Remark 5.5.** Note that $(y) = (\bar{y})$ if $y, \bar{y}$ are both degeneracies of a common non-degenerate dendrex. In particular, if the chosen representatives $y, \bar{y}$ are both nondegenerate, there must exist an isomorphism $\varphi: U^y \xrightarrow{\sim} U^{\bar{y}}$ (which is unique if $(y)$ is $\Sigma$-free) such that $y = \bar{y} \circ \varphi$.

**Notation 5.6.** Given a $\Sigma$-free $(y)$, a coherent inner edge set $E^{(y)}$ for $(y)$ is a collection of subsets $E^y \subseteq E'(U^y)$ for each non-degenerate representative $y$ of $(y)$, and such that $E^y = \varphi(E'(U^y))$ for the unique $\varphi$ with $y = \bar{y} \circ \varphi$. Note that $E^{(y)} = \{E^y\}$ is entirely determined by any of the $E^y$.

**Remark 5.7.** For $Y \in \text{dSet}^G$, the group $G$ acts on dendrices by postcomposition, i.e. $g y$ is the composite $\Omega[U^y] \xrightarrow{\sim} Y \xrightarrow{\sim} Y$. In particular, note that $U^{g y} = U^y$. Moreover, this action extends to principal subpresheaves as $g(y) = (gy)$.

As such, if $(y)$ is $\Sigma$-free, a coherent inner edge set $E^{(y)} = \{E^y \subseteq E'(U^y)\}$ for $(y)$ gives rise to a coherent inner edge set $g E^{(y)} = \{E^y \subseteq E'(U^y)\}$ for $(g y)$ with the same edge sets $E^y$.

The following essentially replicates [BP20, Def. 3.1] as generalized in [BP20, Rem. 3.7], except with dendrices $y: \Omega[U^y] \to Y$ mostly replaced with the principal presheaves $(y) \subseteq Y$. The reformulation of (Ch0.2) and the descending chain condition are discussed in Remarks 5.9, 5.10.

**Definition 5.8.** Let $f: X \to Y$ be a monomorphism in $\text{dSet}^G$ and $(\{y\})$ a set of $\Sigma$-free principal subpresheaves of $Y$. Suppose further that $(\{y\})$ is equipped with a poset structure compatible with the $G$-action in Remark 5.7 and which satisfies the descending chain condition. For each $(y)$ denote

$$X_{\leq (y)} = X \cup \bigcup_{(z) \leq (y)} \{\bar{z}\}$$

Given a coherent inner edge set $\Xi^{(y)} = \{\Xi^y \subseteq E'(U^y)\}$, non-degenerate representative $y: \Omega[U^y] \to Y$, and a subface $V \hookrightarrow U^y$, we write $\Xi^V_y = \Xi^y \cap E'(V)$.

We say $(\{\Xi^{(y)}\})$ is a characteristic inner edge collection of $(\{y\})$ with respect to $X$ if, for some (and thus any) choice of non-degenerate representatives $y: \Omega[U^y] \to Y$, one has that:

(Ch0.1) $y: \Omega[U^y] \to Y$ is injective away from $y^{-1}(X_{\leq (y)})$;

(Ch0.2) $(\{y\})$ and $(\{\Xi^{(y)}\})$ are $G$-equivariant, in the sense that $g(y) \in (\{y\})$ and $g \Xi^{(y)} = \Xi^{g(y)}$, i.e. $\Xi^y = \Xi^{g(y)}$;

(Ch1) if $V \hookrightarrow U^y$ is an outer face and $\Xi^V_y = \emptyset$, then $(\partial_V y) \subseteq X_{\leq (y)}$;

(Ch2) if $V \hookrightarrow U^y$ is any face and $(\partial_V - \Xi^V_y (y)) \subseteq X$, then $(\partial_V y) \subseteq X_{\leq (y)}$;

(Ch3) if $(\bar{y}) \notin (\{y\})$, $V \hookrightarrow U^y$, and $(\partial_V - \Xi^V_y (y)) \subseteq (\bar{y})$, then $(\partial_V y) \subseteq X_{\leq (y)}$.

**Remark 5.9.** In [BP20, Rem. 3.7] the role of each presheaf $(y)$ is played by a special chosen representative, which we here denote by $y^\text{pl} \in (y)$. The motivation for this is that in some key examples, such as in [BP20, Ex. 3.9], one can choose preferred “planar representatives for principal presheaves”, allowing for a pictorial depiction of the dendrices and poset as in [BP20, Fig. 3.1].
There is then a bijection $\{\langle y \rangle \} = \{g y^{pl}\}$ between principal presheaves and the set of representatives, but while the former has a $G$-action the latter a priori does not (as $g y^{pl}$ may not be planar). Translating the $G$-action along this bijection one has that the action of $g$ on $y^{pl}$ gives $(g y^{pl})^{pl}$ and (ii),(iii) in (Ch0.2) of [BP20, Rem. 3.7] precisely encode this action on planar representatives.

**Remark 5.10.** Recall that a poset satisfies the descending chain condition.

As such, while [BP20, Lemma 3.4] assumed that the poset $\{\langle y \rangle \}$ was finite, since its proof follows by iteratively adding elements to $G$-equivariant convex subsets of the poset (cf. the last paragraph of the proof in loc. cit.), the argument generalizes to any poset satisfying the descending chain condition.

**Lemma 5.11** (cf. [BP20, Lemma 3.4]). If $\{\langle y \rangle \}$ is a characteristic inner edge collection of $\{\langle y \rangle \}$ with respect to $X$ then

$$X \to X \cup \bigcup_{\langle y \rangle} \langle y \rangle$$

(5.12)

is $G$-inner anodyne.

### 5.2 Proof of the key lemma

This section will prove Lemma 5.1 by applying the characteristic edge lemma, Lemma 5.11. This will require a fair bit of preparation, starting with a description of the pushout operad $P$ in (5.2).

First, we let $\mathcal{E} = \mathcal{E}_G$ and write $\bar{\mathcal{E}} = (C, \mathcal{E})$ for $c : E(C) \to \mathcal{E}$ the map of colors induced by the top horizontal map in (5.2). One then has identifications (cf. (4.20) and Remark 4.15)

$$\bar{c} : \Omega(C) = \bar{c} (F \Sigma_r[C]) \simeq F (c \Sigma_r[C]) \simeq F \left( \Sigma_{c}[\bar{C}] \right)$$

$$\bar{c} : \Omega(C) = \bar{c} (F \emptyset [E(C)]) \simeq F (c \emptyset [E(C)]) \simeq F (\emptyset [E])$$

(for $\emptyset [E]$ the initial object in $\Sym_{\mathcal{E}}^G$) allowing us to rewrite (5.2) as the following pushout in $\Op^G_{\mathcal{E}}$

$$\begin{array}{ccc}
F(\emptyset [E]) & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
F (\Sigma_{c}[\bar{C}]) & \longrightarrow & \mathcal{P}.
\end{array}$$

By [BPa, Lemma 5.8] with $\varphi : X \to Y$ the map $\emptyset [E] \to \Sigma_{c}[\bar{C}]$ (and as further detailed in [BPa, Rem. A.53]) one then has, for each $\mathcal{E}$-corolla $\bar{D} \in \Sigma_{c}$, a formula

$$\mathcal{P} (\bar{D}) \simeq \bigcup_{[T] \in \boldsymbol{\Omega}_{\mathcal{E}} (\bar{D})} \left( \prod_{v \in V^{ct}(T)} \mathcal{O}(\bar{T}_v) \times \prod_{v \in V^{in}(T)} \Sigma_{c}[\bar{C}](\bar{T}_v) \right)^{\cdot \Aut_{\mathcal{E}}(\bar{T}) \Aut_{\mathcal{E}}(\bar{D})}$$

(5.13)

(39)

(where, since $X = \emptyset [E]$, it is likewise always $Q^G_{\mathcal{P}}(\varphi) = \emptyset [E]$ in [BPa, Lemma 5.8]).

To explain the notation in (5.13), we need to recall the category $\Omega^p$ of alternating trees [BP21, Def. 5.52]. First, we consider trees whose vertices are partitioned $V(T) = V^{ct}(T) \cup V^{in}(T)$ into $\bullet$-labeled vertices, called active, and $\circ$-labeled vertices, called inert, as in Example 5.14. Such trees are called alternating if each edge is adjacent to exactly one $\bullet$-labeled vertex (i.e. if adjacent vertices have different labels $\bullet$, $\circ$ and vertices adjacent to the root and leaves are $\bullet$-labeled). $\Omega^p_{\mathcal{E}}$ is then defined by coloring edges, cf. Definition 2.17. Further, as in (4.4), one has a leaf-root functor $\text{lr} : \Omega^p_{\mathcal{E}} \to \Sigma_{c}$ given by $\bar{T} = (T, c) \mapsto (c(l_1), \ldots, c(l_n); c(r))$ for $l_i$ the leaves and $r$ the root, which gives rise to the undercategory $\bar{D} \downarrow \Omega^p_{\mathcal{E}}$ appearing in (5.13).
Example 5.14. In the following, $T$ is alternating while $S$ is not.

Before proceeding, we recall two key properties of the map $\varphi$ above that will be used in what follows. First, for a vertex $T_v$ of $T$, write $S_v$ for the outer face of $S$ whose outer edges are the image of $T_v$ (alternatively, $T_v \to S_v \to S$ is the “tall-outer factorization” of $T_v \to T \to S$). Then, $\varphi$ is a label map, meaning that if $T_v$ has a ●-label (resp. ○-label) then so do all vertices in $S_v$. Moreover, $\varphi$ is ○-inert, meaning that if $T_v$ has a ○-label then $S_v$ is a corolla (on a terminological note, ○-labeled nodes are called inert precisely because we work only with ○-inert maps).

The decomposition (5.13) will be the key to verifying the characteristic edge conditions in Definition 5.8. To do so, we will first find it useful to discuss a number of special types of dendrices and principal subpresheaves of $NP$, suggested by (5.13). Recall that, by the strict Segal condition characterization of nerves [CM13a, Cor. 2.6], a dendrex $p: \Omega[U] \to NP$ is uniquely specified by the tree $U \in \Omega$ together with a coloring $E(U) \to C$ and a choice of operations $\{p_v \in P(\overrightarrow{U}_v)\}_{v \in V(U)}$. Moreover, for $\overrightarrow{D} = (D, d)$ a $C$-corolla, we will throughout make use of the decomposition

$$\left(\Omega[C] \cup \partial \Omega[C]\right) \varnothing_h \overrightarrow{D} = \Sigma_{\varnothing}[\overrightarrow{C}](\overrightarrow{D}) \cup \varnothing(\overrightarrow{D}) \quad (5.15)$$

where we recall that the left side is an instance of the $X_{\varnothing}(U)$ notation in Notation 3.16.

Definition 5.16. A dendrex $p: \Omega[U] \to NP$ is called:

- elementary if for each vertex $U_v \to U$, one has $\partial_{U_v} \in \varnothing \cup \Sigma_{\varnothing}[\overrightarrow{C}]$, cf. (5.15);
- alternating if $U \in \Omega^a$ is an alternating tree and for each active (resp. inert) vertex $U_v \to U$ one has $\partial_{U_v} \in \varnothing$ (resp. $\partial_{U_v} \in \Sigma_{\varnothing}[\overrightarrow{C}]$);
- canonical if it is non-degenerate and has a degeneracy which is alternating.

Definition 5.17. Let $(p) \subseteq NP$ be a principal subpresheaf. We say $(p)$ is:

- unital if there is a representative $p: \Omega[U] \to NP$ with $U = \eta$ the stick tree;
- reduced if there is a representative $p: \Omega[U] \to NP$ with $U \in \Sigma$, i.e. with $U$ a corolla;
- elementary if there is an elementary representative $p: \Omega[U] \to NP$;
- canonical if there is a canonical (equivalently, alternating) representative $p: \Omega[U] \to NP$.

Remark 5.18. A dendrex is elementary iff its degeneracies are elementary, so the definition of elementary subpresheaf does not depend on the choice of representative.
Notation 5.19. Recalling that any tree $U$ has an associated corolla $\text{lr}(U)$ (counting leaves of $U$) together with a map $\text{lr}(U) \to U$, we abbreviate $\partial_r \circ p = \partial_{\text{lr}(U)} \circ p$ and call $(\partial_r \circ p)$ the \textit{reduction} of $(p)$.

Remark 5.20. A reduced principal subpresheaf $(r) \subseteq \mathcal{NP}$ is simply the data of a single operation in $\mathcal{P}$, and so the coproduct decomposition of (5.13) implies that for each such $(r)$, there exists an alternating dendrex $\alpha \in \Omega(T) \to \mathcal{NP}$, unique up to isomorphism, such that $(\partial_r \alpha) = (r)$. Moreover, $(a)$ is thus the only canonical subpresheaf with reduction $(r)$, and we write $(r) = (a)$ to denote this.

Lastly, note that one thus has that $(p)$ is canonical iff $(p) = (\partial_r \circ p)\chi$.

Remark 5.21. A reduced subpresheaf $(r)$ is unital iff $(r)\chi = \text{unit}$, in which case $(r) = (\text{unit})\chi$.

Remark 5.22. If $e : \Omega[U^e] \to \mathcal{NP}$ is an elementary dendrex, the tree $U^e$ can be naturally regarded as an $(\mathcal{O}, \bar{C})$-labeled tree by labeling each vertex $U^e_v \to U^e$ according to whether $\partial_{U^e} e \in \mathcal{O}$ or $\partial_{U^e} \bar{e} \in \Sigma \bar{C}$.

By the alternating tree analogue of [BP21, Prop. 5.57], there is hence a unique alternating tree $U^a$ together with a tall planar $\bar{C}$-inert label map $U^a \to U^e$, and it then follows that $\partial_{U^a} e$ is an alternating dendrex so that $(\partial_{U^a} e) = (\partial_r(e))\chi$. In particular, this shows that $(\partial_r(e))\chi \subseteq \langle e \rangle$.

Example 5.23. Should $U^e$ in the previous remark be the tree $S$ in Example 5.14, then $U^a \to U^e$ is the map $\varphi : T \to S$ depicted therein.

Definition 5.24. Let $e : \Omega[U^e] \to \mathcal{NP}$ be a non-degenerate elementary dendrex. We write $\Xi^e \subseteq E^e(U^e)$ for the subset of inner edges that are adjacent to at least one $\bar{C}$-labeled vertex.

Remark 5.25. Let $e : \Omega[U^e] \to \mathcal{NP}$ be a non-degenerate elementary dendrex, $U^a \to U^e$ be as in Remark 5.22, and write $a = \partial_{U^a} e$. Since $U^a$ is alternating, all of its inner edges are adjacent to a $\bar{C}$-labeled vertex. Therefore, the fact that $U^a \to U^e$ is a tall $\bar{C}$-inert label map implies that $\Xi^e$ consists of those inner edges which are in the image of $U^a$.

Proposition 5.26. Let $e : \Omega[U^e] \to \mathcal{NP}$ be a non-degenerate elementary dendrex. Then $(e)$ is canonical iff $\Xi^e = E^e(U)$.

Proof. We use the notation in Remark 5.25. Since $(a) = (\partial_r(e))\chi$, $(e)$ is canonical iff $(a) = (e)$, i.e. iff $U^a \to U^e$ is a degeneracy. But since a map of trees is a degeneracy iff it is tall and surjective, it follows that $U^a \to U^e$ is a degeneracy iff its image includes all inner edges, i.e. iff $\Xi^e = E^e(U)$.

Remark 5.27. If $U \neq \eta$ is not the stick tree, then $\text{lr}(U) \simeq U - E^e(U)$ is the inner face removing all inner edges.

Lemma 5.28. For any principal subpresheaf $(p) \subseteq \mathcal{NP}$ there exists an elementary subpresheaf $(e) \subseteq \mathcal{NP}$, non-degenerate representative $e : \Omega[U^e] \to \mathcal{NP}$, and a subset $E \subseteq \Xi^e$ such that $\partial_{U^e} - E(e)$ is non-degenerate and $(p) = (\partial_{U^e}-E(e))\chi$. In particular, $(p) \subseteq \langle e \rangle$.

Proof. Let $p : \Omega(U) \to \mathcal{NP}$ be a non-degenerate representative. We first build $e$.

For each vertex $U_v \to U$, write $p_v = \partial_{U_v} p$ and, noting that $(p_v)$ is reduced, we further write $c_v : \Omega[U^c_v] \to \mathcal{NP}$ for some canonical representative of $(p_v)r$ (cf. Remark 5.20). The identity $(p_v) = (\partial_r e_v)$ implies that $U_v \simeq \text{lr}(U^c_v)$, so that by choosing tall maps $U_v \to U^c_v$, one obtains a $U$-substitution datum [BP21, Def. 3.43] which by [BP21, Prop. 3.46] can be assembled into a tree $U^c$ together with a tall map $U \to U^c$ such that for every vertex $U_v$, the “tall map followed by outer face” factorization of the composite $U_v \to U \to U^c$ is given by $U_v \to U^c_v \to U^c$.

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Since each vertex of $U^c$ is in exactly one of the outer trees $U^c_v$, we define $e : \Omega[U^c] \rightarrow NP$ as the unique dendrex such that $\partial_v e = e_v$. Note that since the $e_v$ are non-degenerate then so is $e$.

Since $\partial_U c = p$ was chosen to be non-degenerate, it remains to show that $U \rightarrow U^c$ identifies $U^c = U^c - E$ for some $E \subseteq \Xi_c$. Since $(p_v), (p_h)_{\chi}$ are non-unital (by assumption on $p$ and Remark 5.21), the $U^c_v$ are not stick trees, and Remark 5.27 implies $U \rightarrow U^c - E$ for $E = u_{\in V(U)} E(U^c)$. That $E \subseteq \Xi_c$, i.e. that any edge in $E$ is adjacent to a $C$-labeled vertex, follows from Proposition 5.26 applied to the $(e_v)$. □

**Lemma 5.29.** Suppose $c : \Omega[U^c] \rightarrow NP$ is elementary and $(\partial_v c)$ is not unital. Then there exists an inner face map $U^c \rightarrow U^c$ such that $\partial_U c$ is canonical.

**Proof.** Let $U^a \rightarrow U^c$ be as in Remark 5.22.

Writing $a = \partial_U a$, $r = \partial_v r$, and letting $c : \Omega[U^c] \rightarrow NP$ be a canonical representative of $(a) = (r)_{\chi}$, one has that $a$ is a degeneracy of $c$, i.e. there is a degeneracy map $U^a \rightarrow U^c$ such that $\partial_U a c = a$. Since $(r) = (\partial_v r)$ is not unital, neither is $(r)_{\chi}$ (cf. Remark 5.21), so that $U^c$ can not be the stick tree $\eta$, and thus $U^a \rightarrow U^c$ has a section which is an inner face (this follows from [Per18, Cor. 5.38] since no edge of $U^c$ is both a root and a leaf). But then the composite $U^c \rightarrow U^a \rightarrow U$ must be a face (or else $c = \partial_U a$ would be degenerate) and is tall, and is hence an inner face. □

**Lemma 5.30.** Let $c : \Omega[U^c] \rightarrow NP$ be a non-unital canonical dendrex, $e : \Omega[U^c] \rightarrow NP$ an elementary dendrex, and $\text{lr}(U^c) \rightarrow U^c$ a tall map.

Then, if the solid diagram below commutes, there exists a tall dashed map making the diagram commute.

$$
\begin{array}{c}
\Omega[\text{lr}(U^c)] \\
\downarrow \\
\Omega[U^c] \\
\downarrow \\
\Omega[U^c] \\
\end{array}
\xymatrix{\Omega[\text{lr}(U^c)] \ar[r]^-c & \Omega[U^c] \ar[r]^e & NP}
$$

(5.31)

**Remark 5.32.** The requirement that $(c)$ is non-unital is essential, as there may exist non-unital $(c) \subseteq NC$ such that $(\partial, c) = (\partial, c)$ is unital, in which case no dashed arrow as in (5.31) can exist.

**Proof.** Since commutativity of (5.31) implies $(\partial, c) = (\partial, c)$, which is not unital by assumption, by Lemma 5.29 there is an inner face $U^c \rightarrow U^c$ such that $c = \partial_U c$ is canonical.

By definition of canonical dendrex, there are degeneracies $U^a \rightarrow U^c$, $U^a \rightarrow U^c$ with $U^a, U^a$ alternating trees and such that the composites $\Omega[U^a] \rightarrow \Omega[U^c] \xrightarrow{c} NP$, $\Omega[U^a] \rightarrow \Omega[U^c] \xrightarrow{e} NP$ are alternating dendrices. And since $\text{lr}$ sends tall maps to isomorphisms, we can form the diagram

$$
\begin{array}{c}
\Omega[\text{lr}(U^c)] \\
\downarrow \\
\Omega[U^a] \\
\downarrow \\
\Omega[U^c] \\
\end{array}
\xymatrix{\Omega[\text{lr}(U^c)] \ar[r]^-c & \Omega[U^a] \ar[r]^e & \Omega[U^c]}
$$

We will argue that all dashed vertical isomorphisms exist. That the first vertical isomorphism exists is trivial. The existence of the second vertical isomorphism follows from (5.13) which implies that, for $a, \bar{a}$ alternating dendrices, all isomorphisms $\partial a \simeq \partial \bar{a}$ are induced from an isomorphism $\bar{a} \simeq \bar{a}$. Lastly, the existence of the third isomorphism follows from the fact that the factorization of degenerate dendrices through non-degenerate dendrices is unique up to (unique) isomorphism [Per18, Prop. 5.62]. □

**Lemma 5.33.** Let $e : \Omega[U^c] \rightarrow NP$ be a non-degenerate elementary dendrex, $\bar{e} : \Omega[U^c] \rightarrow NP$ be an elementary dendrex, and $U^c - E \rightarrow U^c$ be a map with $E \subseteq \Xi_c$. 42
Then, if the solid diagram below commutes, there exists a dashed map making the diagram commute.

\[
\begin{array}{ccc}
\Omega[U^c-E] & \rightarrow & \Omega[U^c] \\
\downarrow & & \downarrow \\
\Omega[U^c] & \rightarrow & NP
\end{array}
\] (5.34)

**Proof.** We abbreviate \(U' = U^c - E\). Note first that, by applying the “tall map followed by outer face” factorization to \(U' \rightarrow U''\) to obtain \(U' \rightarrow \tilde{U} \rightarrow U''\), the dendrex \(\partial_{\tilde{U}}\bar{e}\) is still elementary (being an outer face of an elementary dendrex), so we reduce to the case where \(U' \rightarrow U''\) is a tall map.

For each vertex \(U'_v \rightarrow U''\) we apply the “inner face followed by outer face factorization” to the composites \(U'_v \rightarrow U''\), \(U'_v \rightarrow U''\) to get \(U'_v \rightarrow U'' \rightarrow U''\) and, further writing \(e_v = \partial_{U'_v}^c\bar{e}, \bar{e}_v = \partial_{U'_v}\bar{e}\), we obtain solid diagrams

\[
\begin{array}{ccc}
\Omega[U'_v] & \rightarrow & \Omega[U'_{v\bar{e}}] \\
\downarrow & & \downarrow \\
\Omega[U'_{v\bar{e}}] & \rightarrow & NP
\end{array}
\] (5.35)

We now claim that \(e_v\) is canonical. Indeed, \(e_v\) is non-degenerate elementary since it is an outer face of \(e_v\), which is also non-degenerate elementary. And, since all inner edges of \(U'_v\) are in \(E \subseteq \Xi^c\), they are all adjacent to \(\tilde{C}\)-labeled vertices, so \(e_v\) is indeed canonical by Proposition 5.26.

Since \(\langle e_v \rangle\) is non-unital (or \(U'_v \rightarrow U'_{v\bar{e}}\) would be a degeneracy), by Lemma 5.30 there is a tall dashed arrow in (5.35) for each \(v \in V(U')\), i.e. a \(U^c\)-substitution datum. Thus by [BP21, Prop. 3.46] we obtain the desired dashed arrow in (5.34).

**Corollary 5.36.** If \(e: \Omega[U^c] \rightarrow NP\) is a non-degenerate elementary dendrex and \(E \subseteq \Xi^c\), then \((e)\) is the smallest elementary subpresheaf containing \((\partial_{U^c-E}(e))\), i.e. if \((\partial_{U^c-E}(e)) \subseteq (\bar{e})\) with \(\bar{e}\) elementary then \((e) \subseteq (\bar{e})\).

**Proof.** \((\partial_{U^c-E}(e)) \subseteq (\bar{e})\) yields the diagram (5.34) and the dashed arrow therein shows \((e) \subseteq (\bar{e})\).

**Corollary 5.37.** An elementary subpresheaf \((e)\) is canonical iff it is the smallest elementary subpresheaf containing \((\partial_{e})\).

**Proof.** As noted in Remark 5.20, \((e)\) is canonical iff \((e) = (\partial_{e})_\chi\). If \((\partial_{e})\) is unital, then \((\partial_{e})_\chi = (\partial_{e})_\lambda\), which is elementary (cf. Remark 5.21), so the claim is clear. Otherwise, letting \(c: \Omega[U^c] \rightarrow NP\) be a canonical representative of \((\partial_{e})_\lambda\), one has \(\Xi_e = E'(U'^c)\) and \((\partial_{e})_\chi = (\partial_{U'^c-E'(e)}\bar{e})\) so, by Corollary 5.36, \((\partial_{e})_\chi\) is the smallest elementary containing \((\partial_{e})\).

**Corollary 5.38.** Suppose \(NP\) is \(\Sigma\)-free, and let \(e: \Omega[U^c] \rightarrow NP\) be an elementary non-degenerate dendrex and \(E, E' \subseteq \Xi^c\). If \((\partial_{U^c-E}(e)) = (\partial_{U^c-E'}(e))\), then in fact \(E = E'\).

**Proof.** If \((\partial_{U^c-E}(e)) = (\partial_{U^c-E'}(e))\) then one can find a solid diagram as below

\[
\begin{array}{ccc}
\Omega[U^c-E] & \rightarrow & \Omega[U^c] \\
\downarrow & & \downarrow \\
\Omega[U^c-E'] \rightarrow & \Omega[U^c]
\end{array}
\]
and thus by Lemma 5.33 one can also find the vertical dashed arrow. But since $NP$ is $\Sigma$-free by assumption, the dashed arrow must be the identity, so that $U^e = E = U^e - E'$ and hence $E = E'$.

**Proof of Lemma 5.1.** We will verify the characteristic edge conditions in Definition 5.8. Note that, as $\partial\Omega[T] \to \Omega[T]$ is a generating cofibration in $s\text{Op}^\Sigma$, $O \to \mathcal{P}$ is a cofibration with $\Sigma$-cofibrant source and hence, by Proposition 4.14, $\mathcal{P}$ is $\Sigma$-cofibrant. One thus has that $NP$ is $\Sigma$-free (cf. [BM03, Lemma 5.9]) so that the $\Sigma$-freeness conditions in both Definition 5.8 and Corollary 5.38 are satisfied.

We set $X = \Omega[C] \cup_{\partial\Omega[C]} NO$, with the $G$-poset of principal subpresheaves formed by the elementary subpresheaves $(e)$ under inclusion, and the characteristic inner edge sets $\Xi(e) = \{\Xi^e\}$ given as in Definition 5.24. Note that by Lemma 5.28 every dendrex of $NP$ is in some $(e)$, so that the map (5.12) is indeed $\Omega[C] \cup_{\partial\Omega[C]} NO \to NP$.

Let $e: \Omega[U^e] \to NP$ be a non-degenerate elementary dendrex. We note the following:

(a) if $\Xi^e = \emptyset$ then either all vertices of $U^e$ are $\mathcal{O}$-labeled, i.e. $e \in NO$, or $U^e$ is a $\mathcal{C}$-labeled corolla, i.e. $e \in \Sigma_e[\mathcal{C}]$. In other words, $\Xi^e = \emptyset$ iff $(e) \subseteq X \subseteq NO \cup_{\partial\Omega[C]} \Omega[C]$.

(b) any outer face of $e$ is again elementary, as is any inner face $\partial_{U^e - f}(e)$ such that $f \notin \Xi^e$ (since then both vertices adjacent to $f$ are $\mathcal{O}$-labeled). Therefore, by Corollary 5.36, we see that a face of $e$ is *not* in some elementary $(e) \varsubsetneq (e)$ iff it is of the form $\partial_{U^e - f}(e)$ for some $E \subseteq \Xi^e$.

We now check the characteristic edge conditions. (Ch0.2) is clear.

For (Ch1), by (b) any proper outer face of $e$ is in $X_{e(e)}$, so we need only consider the case of $V = U^e$ with $\Xi^e = \emptyset$, in which case $(e) \subseteq X \subseteq X_{e(e)}$ by (a).

For (Ch2),(Ch3), by (a) and the first half of (b) one needs only consider the case of $V = U^e - E$ where $E \subseteq \Xi^e$ and $\Xi^e \neq \emptyset$. But then $V - \Xi^e = U^e - \Xi^e$ and, by the second half of (b), one has $(\partial_{U^e - \Xi^e}(e)) \subseteq X_{e(e)}$ iff $(e) \subseteq X_{e(e)}$ iff $\partial_{U^e - E}(e) \subseteq X_{e(e)}$, so (Ch2),(Ch3) follow.

Lastly, we address (Ch0.1). By (a) we need only consider the case of $\Xi^e \neq \emptyset$, so that by (b) the complement of the preimage $e^{-1}(X_{e(e)})$ consists of the faces isomorphic to $U^e - E$ for $E \subseteq \Xi^e$. Injectivity of $e$ within each isomorphism class of the faces away from $e^{-1}(X_{e(e)})$ follows from $NP$ being $\Sigma$-free, while injectivity across distinct isomorphism classes of faces is Corollary 5.38.

**Remark 5.39.** Condition (Ch0.1) is, by some margin, the subtlest condition in the previous proof, and the main reason for the chosen formulations of Lemmas 5.30, 5.33. In particular, we note that injectivity of $e: \Omega[U^e] \to NP$ will in general fail away from $e^{-1}(X_{e(e)})$. For example, two edges/vertices of $U^e$ may be assigned the same color/operation, and similarly for larger outer faces. In fact, injectivity may even fail on inner faces $T^e - E$ where $E \notin \Xi^e$.

**Remark 5.40.** The proof of the characteristic edge lemma [BP20, Lemma 3.4] gives a filtration of $\Omega[C] \cup_{\partial\Omega[C]} NO \to NP$ by anodyne maps which, for $G = \ast$, matches the filtration in [CM13b, §3].

As such, our arguments largely adapt [CM13b], though we also patch an apparent gap in the proof therein. Namely, we believe that the appeal to [CM13b, Lemma 3.4] in the proof of [CM13b, Lemma 3.5] is incorrect (namely, in the terminology of Definition 5.16, the assumption in [CM13b, Lemma 3.4] only holds for *elementary* dendrices). To fix this, we replace the role of [CM13b, Lemma 3.4] with [BP20, Lemma 3.4]. From this point of view, [CM13b, Lemma 3.5] amounts to verifying (Ch1),(Ch2),(Ch3), though the tricky condition (Ch0.1) appears left unaddressed.
6 Indexing system analogue results

As in [BP20, §6] and [Per18, §9], we dedicate our final main section to discussing the variants of our main results for the indexing systems of Blumberg-Hill [BH15] or, more precisely, the slightly more general weak indexing systems introduced by the authors [Per18, §9], [BP21, §4.4] (and independently identified by Gutierrez-White [GW18]).

Weak indexing systems can regarded as the operadic generalization of the notion of a family of subgroups (cf. Remark 6.3). Recall that families \( \mathcal{F} \) of a subgroup \( G \) are in bijection with sieves \( \mathcal{O}_\mathcal{F} \subseteq \mathcal{O}_G \) of the orbit category of \( G \).

**Definition 6.1.** A sieve of a category \( \mathcal{C} \) is a full subcategory \( \mathcal{S} \subseteq \mathcal{C} \) such that, for all arrows \( A \to B \) in \( \mathcal{C} \), \( B \in \mathcal{S} \) implies that \( A \) and \( A \to B \) are also in \( \mathcal{S} \).

**Definition 6.2.** A subcategory \( \Omega_\mathcal{F} \subseteq \Omega_G \) is called a weak indexing system if:

(i) \( \Omega_\mathcal{F} \) is a sieve in \( \Omega_G \) and;

(ii) for \( T \in \Omega_G \), one has \( T \in \Omega_\mathcal{F} \) iff \( T_v \in \Omega_\mathcal{F} \) for all \( v \in V_G(T) \).

**Remark 6.3.** The \( \mathcal{F} \) in \( \Omega_\mathcal{F} \) refers to an equivalent presentation of a weak indexing system. Fixing \( \Omega_\mathcal{F} \), let \( \mathcal{F}_n \) for \( n \geq 0 \) denote the family of those graph subgroups \( \Gamma \leq G \times \Sigma_n \) (cf. (1.1)) such that, equipping the \( n \)-corolla \( C_n \) with the \( H \)-action given by the associated homomorphism \( \phi_H : H \to \Sigma_n \), the induced \( G \)-corolla \( G \cdot H \cdot C_n \) is in \( \Omega_\mathcal{F} \). The symbol \( \mathcal{F} \) then denotes the collection of families \( \{ \mathcal{F}_n \}_{n \geq 0} \), whose data is equivalent to \( \Omega_\mathcal{F} \).

For later reference, replacing \( C_n \) above with a tree \( T \in \Omega \) and \( \Sigma_n \) with Aut\((T)\), we write \( \mathcal{F}_T \) for the family of graph subgroups \( \Gamma \leq G \times \text{Aut}(T) \) for which the similarly built \( G \cdot H \cdot T \) is in \( \Omega_\mathcal{F} \).

We note that the fact that each \( \mathcal{F}_n \) is a family is consequence of \( \Omega_\mathcal{F} \) being a sieve. More precisely, this follows from the sieve condition for the quotient maps in \( \Omega_G \). However, the \( \mathcal{F}_n \) must satisfy additional operadic closure properties (resulting from the sieve condition for face maps in \( \Omega_G \)), which are made explicit in [BH15, Def. 3.22], [Rub, Def. 2.12].

**Remark 6.4.** By vacuousness, Definition 6.2(ii) implies that any weak indexing system \( \Omega_\mathcal{F} \) contains the equivariant stick trees \( G/H \cdot \eta \). As such, by the sieve condition, all equivariant linear trees \( G/H \cdot [n] \) for \( [n] \in \Delta \) are likewise in \( \Omega_\mathcal{F} \). As such, \( \mathcal{F}_1 \) always contains all subgroups \( H \leq G \approx G \times \Sigma_1 \).

For both of the categories appearing in Theorem 1, each weak indexing system \( \mathcal{F} \) gives rise to model structures (on the same underlying categories) dSet\( \mathcal{F} \), sOp\( \mathcal{F} \) which, loosely speaking, are built by replacing the role of \( \Omega_G \) (or the families of graph subgroups \( \mathcal{F}_n \)) with \( \Omega_\mathcal{F} \) (the families \( \mathcal{F}_n \)) throughout. With minor changes to our proofs, discussed in the remainder of the section, one then obtains an \( \mathcal{F} \)-variant of Theorem 1, stated below as Theorem 6.8.

First, we recall that \( \mathcal{F} \)-variant model structures on all the categories in the square (1.3) have already been built, with some of the adjunctions therein already known to be Quillen equivalences.

- Replacing \( \Omega_G \) with \( \Omega_\mathcal{F} \) in Definitions 2.33, 2.34 one obtains the classes of \( \mathcal{F} \)-normal monomorphisms, \( \mathcal{F} \)-inner anodyne maps, and \( \mathcal{F} \)-inner fibrations in dSet\( \mathcal{F} \). Using these alternative classes, the \( \mathcal{F} \)-model structure dSet\( \mathcal{F} \) [Per18, Thm. 2.2] is described as in Theorem 2.37.

- As in the discussion before Theorem 3.3, since \( \Delta^{op} \) is a Reedy category, the identification sdSet\( \mathcal{F} \) = (dSet\( \mathcal{F} \))\( \Delta^{op} \) yields a \( \mathcal{F} \)-simplicial Reedy model structure on sdSet\( \mathcal{F} \) with weak equivalences the \( \mathcal{F} \)-endodendroidal equivalences, i.e. maps \( X \rightarrow Y \) for which the maps \( X_n \rightarrow Y_n, n \geq 0 \) are weak equivalences in dSet\( \mathcal{F} \).
Similarly, as $\Omega^p \times G$ is generalized Reedy and the collection of families $\{F_T\}_{T \in \Omega}$ is Reedy admissible, the identification $\text{sdSet}^G = (\text{sdSet})^{\Omega^p \times G}$ yields a $F$-dendroidal Reedy model structure on $\text{sdSet}^G$ with weak equivalences the $F$-simplicial equivalences, i.e. maps $X \to Y$ for which the maps $X(\Omega[T]) \to Y(\Omega[T])$ for $T \in \Omega_F$ are Kan equivalences in $\text{sdSet}$.

The $F$-joint/complete/Rezk model structure on $\text{sdSet}^G$ (cf. the discussion above [BP20, Thm. 6.7]), denoted $\text{sdSet}^{\underline{G}}_F$, then combines the $F$-simplicial and $F$-dendroidal Reedy model structures as in Theorem 3.3.

- The $F$-normal model structure on $\text{PreOp}^G$, denoted $\text{PreOp}^G_{\text{normal},F}$ (cf. the discussion above [BP20, Thm. 6.8]), is obtained from the $F$-joint model structure $\text{sdSet}^{\underline{G}}_F$ as in Theorem 3.15.

- The adjunctions $F: \text{sdSet}^{\underline{G}}_F \rightleftarrows \text{sdSet}^{\underline{G}}_F; c^\ast$ and $\gamma^\ast: \text{PreOp}^G_{\text{normal},F} \rightleftarrows \text{sdSet}^{\underline{G}}_F; \gamma_\ast$ are Quillen equivalences [BP20, Thms. 6.7 and 6.8].

- The $F$-model structure on $\text{sOp}^G$, denoted $\text{sOp}^G_F$, is an instance of [BPb, Thm. A] with $F_\alpha$ therein as discussed in Remark 6.3. This model structure is described as in Theorem 4.9 by restricting the $\Gamma \subseteq G \times \Sigma^p_0$ to $F_\alpha$. Likewise, the generating sets are as in Remark 4.35 with $C \in \Sigma_G$ restricted to $C \in \Sigma_F = \Sigma_G \cap \Omega_F$.

Next, the notions of $F$-Segal operad and $F$-Dwyer Kan equivalence are obtained by restricting $T \in \Omega_G$, $C \in \Sigma_G$ in Definitions 3.24,3.31 to $T \in \Omega_F$, $C \in \Sigma_F$. As in Theorem 3.33, one has the following, which is left unstated in [BP20, §6], but follows from [BP20, Thm. 6.9] and the $F$-version of [BP20, Cor. 5.51] (in fact, we need only the easier “only if” half of the latter, which follows by the same proof, restricting $T$ therein to $\Omega_F$).

**Remark 6.5.** A map between $F$-Segal operads is a $F$-joint equivalence iff it is an $F$-Dwyer-Kan equivalence.

Adapting the work in §3.4, and writing $(FTC2),(FTC3),(FTA2),(FTA3)$ for the maps in Definitions 3.44,3.46 restricted to $C \in \Sigma_F$, $T \in \Omega_F$ we now get the following extension of Theorem 3.47. Note that, by Remark 6.4, there is no need to alter (TC1),(TA1).

**Theorem 6.6.** There is a left proper model structure on $\text{PreOp}^G$, called the $F$-tame model structure and denoted $\text{PreOp}^G_{\text{tame},F}$, such that:

- weak equivalences are the $F$-joint equivalences (detected by inclusion into $\text{sdSet}^G_F$);

- the generating cofibrations are the maps $(TC1),(FTC2),(FTC3)$;

- $X \in \text{PreOp}^G_{\text{tame},F}$ is fibrant iff $X \to \ast$ has the right lifting property against $(TA1),(FTA2),(FTA3)$;

- a map $X \to Y$ between fibrant objects is a fibration iff it has the right lifting property against $(TA1),(FTA2),(FTA3)$.

Moreover, the identity adjunction $\text{PreOp}^G_{\text{tame},F} \rightleftarrows \text{PreOp}^G_{\text{normal},F}$ is a Quillen equivalence.

**Proof.** All the arguments in the proof of Theorem 3.47, as well as in the proofs of necessary Lemmas 3.48,3.49,3.51 carry through, with the biggest differences being that Lemma 3.48 uses the obvious $F$-variant of Remark 3.4 and that the verification of $C_5$ in the proof of Theorem 3.47 replaces the appeal to Theorem 3.33 with an appeal to Remark 6.5. □
We now turn to the $\mathcal{F}$-variants of the Quillen equivalences established in §4. The $\Sigma$-cofibrations $\mathcal{O} \to \mathcal{P}$ in $sOp^G$ (or $Op^G$) discussed after Proposition 4.13 generalize to the notion of $\Sigma_\mathcal{F}$-cofibration, demanding that any point $x \in \mathcal{P} \setminus \mathcal{O}$ at a profile $\tilde{G}$ of arity $n$ must have isotropy in $\mathcal{F}_n$. We note that this is connected to the characterization, given in the discussion after [Per18, Def. 9.8], of the $\mathcal{F}$-normal monomorphisms $X \to Y$ in $dSet^G$ as the maps such that points $x \in Y \setminus X$ at a tree $T$ have isotropy in $\mathcal{F}_T$. Indeed, the nerve formula (4.22) and Definition 6.2(ii) imply that, for $\mathcal{O} \to \mathcal{P}$ a $\Sigma_\mathcal{F}$-cofibration between $\Sigma_\mathcal{F}$-cofibrant objects in $Op^G$, the nerve map $NO \to NP$ is an $\mathcal{F}$-normal monomorphism between $\mathcal{F}$-normal objects in $dSet^G$.

The following is the extension of Theorem 4.48.

**Theorem 6.7.** The following adjunction is a Quillen equivalence.

$$\tau: \text{PreOp}_{\text{tame}, \mathcal{F}}^G \xrightarrow{\simeq} sOp^G_{\Sigma_\mathcal{F}}: \text{N}$$

**Proof.** Most of the proof of Theorem 4.48 extends, with the role of Theorem 3.33 replaced with Remark 6.5. The trickiest part concern the appeals to Corollary 4.45. To obtain an $\mathcal{F}$-version of the latter, we need an $\mathcal{F}$-version of Lemma 4.42. Analyzing its proof, this comes down to having $\mathcal{F}$-variants of Lemma 3.6 and of Lemma 5.1. For Lemma 3.6, its proof immediately generalizes.

As for Lemma 5.1, [BPa, Prop. 5.26] ensures that the map $\mathcal{O} \to \mathcal{P}$ in (5.2) is a $\Sigma_\mathcal{F}$-cofibration between $\Sigma_\mathcal{F}$-cofibrant objects. Thus, by the discussion above, the map $\Omega[C \cup_{\partial[C]} NO \to NP]$ in (5.3) has $\mathcal{F}$-normal target. Hence, since this map is shown to be $G$-anodyne via the characteristic edge lemma [BP20, Lemma 3.4], whose proof follows by attaching $G$-inner horns determined by $NP$, the fact that $NP$ is $\mathcal{F}$-normal implies that (5.3) must be built by attaching only $\mathcal{F}$-inner horns, and thus be an $\mathcal{F}$-anodyne map (this is identical to the argument before [BP20, Thm. 6.7]).

Lastly, we have the following, which is the $\mathcal{F}$-variant of the main result, Theorem 1.

**Theorem 6.8.** Let $\mathcal{F}$ be a weak indexing system. One then has a Quillen equivalence

$$W_i: dSet^G \leftrightarrow sOp^G: \text{hcN}. \quad (6.9)$$

**Proof.** First, the fact that (6.9) is a Quillen adjunction follows as in the proof of Proposition 4.55 where we note that, in the appeal to Lemma 4.36, the fact that for $C \in \Sigma_\mathcal{F}$ the decomposition $G/H \cdot \Omega(C) \simeq \sqcup_i \Omega(C_i)$ satisfies $C_i \in \Sigma_\mathcal{F}$ follows from the sieve condition in Definition 6.2(6).

The remainder of the Quillen equivalence proof follows as written, using the same zigzag (4.60) (the $\mathcal{F}$ version of Lemma 4.58 works as expected, using $\mathcal{F}$-simplicial equivalences) with the only notable point being that, for the map (d) therein, one needs to know that $\Omega[T]$ for $T \in \Omega_\mathcal{F}$ is $\mathcal{F}$-tame fibrant, which is clear from the proof of Lemma 3.52.

### A The homotopy genuine equivariant operad

The goal of this appendix is to establish Proposition A.22, which compares two procedures of discretizing an equivariant operad $\mathcal{O} \in sOp^G$, and is the full equivariant generalization of [CM13b, Prop. 4.8], cf. Remark 4.56.

Most of the work will be spent describing the "genuine operadification" functor $\tau_G: dSet_G \to Op_G$ first mentioned in (2.42). In general, $\tau_G Z$ for some $Z \in dSet_G$ is given by the formula (A.5) below, though this formula is cumbersome in practice (and included mostly for completeness). Instead, our focus will be on the special case of $Z \in dSet_G$ a genuine $G$-$\infty$-operad (Definition A.6), for which $\tau_G Z$ admits a simpler description as a *homotopy operad*, which we denote $\text{ho}(Z)$.
(Definition A.19). This alternative ho(Z) construction mimics the similar description of \( \tau X \) for \( X \in \text{dSet} \) an \( \infty \)-operad given in [MW09, §6], so that the majority of this appendix is a fairly direct generalization of the work, although with one interesting nuance. Namely, [MW09, §6] makes heavy use of tree diagrams, and associated faces/horns, whose equivariant generalization turns out to involve orbital representations of G-trees rather than expanded representations (cf. (2.18)). As such, we will need to recall the notions of orbital faces (Def. A.1) and orbital horns (Eq. (A.16)) introduced in [BP20, §2.2, §2.3], which are distinct from any of the notions recalled in §2.3.

We first recall the notion of orbital face.

**Definition A.1.** A map \( S \to T \) of G-trees that is injective on edges is called an orbital face map.

**Example A.2.** Let \( T \) be the G-tree in (2.18), whose edge orbits we abbreviate as \( Ga, Gb, Gc, Gd \).

\( T \) has orbital faces \( T \to Gd = T \to \{d, \rho d\} \) and \( T \to Gc = T \to \{c, \sigma c, \rho c, \rho \sigma c\} \), where we note that the expanded representation of \( T \to Gd \) has four tree components (in contrast to \( T \to Gc \), whose expanded representations have only two components). The term “orbital” refers to the fact that orbital faces correspond to (usual) faces of the underlying tree in the orbital representation.

As one would expect, an orbital outer face \( S \to T \) is called inner if it is given by removing inner edges (e.g. \( T \to Gc \) in Example A.2). Orbital faces allow for a general description of \( \tau_G(Z) \) for any \( Z \in \text{dSet}_G \), cf. (A.5) below. This description makes use of the following, which is the key definition in [BPc], and is connected to the description of the \( W \)-construction, Definition 4.50, built therein.

**Definition A.3.** The category \( \text{Nec}^c_G \) of G-dendroidal necklaces and tall maps has:

(i) objects planar orbital inner faces \( n: J \to T \), called necklaces;

(ii) maps from \( n: J \to T \) to \( n^*: J^* \to T^* \) given by tall maps \( \varphi: T \to T^* \) such that \( \varphi(J) \supseteq J^* \).

The category \( \text{Nec}^c_G \) above is part of a larger category \( \text{Nec}_{\Omega G} \) where maps need not be tall, though defining the latter requires more care (see [BPc, Def. 3.3 and Prop. 3.14]). Nonetheless, by restricting to \( \text{Nec}^c_G \) one has a leaf-root functor \( \text{lr}_{\text{Nec}^c_G} \to \Sigma_G \) given by \( (J \to T) \mapsto \text{lr}(T) \) where (cf. (4.4), (5.13), Notation 5.19) the G-corolla \( \text{lr}(T) \) replaces each tree component of \( T \) with the corolla with the same number of leaves (e.g. in Example (A.2) one has \( \text{lr}(T) = T \to Gc \)). [BPc, Rem. 4.26] can then be restated as follows.

**Remark A.4.** Let \( Z \in \text{dSet}_G \). Then, at a G-corolla \( C \) one has the formula

\[
\tau_G Z(C) = \text{colim}_{(C \to \text{lr}(J \to T)) \in (C; \text{Nec}^c_G)^{op}} \prod_{v \in \Omega_G(J)} Z(T_v). \tag{A.5}
\]

Noting that any necklace \( (J \to T) \) receives a map \( (T \to T) \to (J \to T) \) in \( \text{Nec}^c_G \), one has that every element in the colimit (A.5) has a representative in \( \prod_{v \in \Omega_G(J)} Z(T_v) \) for some G-tree \( T \in \Omega_G \) (though general necklaces are needed to encode all the relations).

We now turn our attention to the special case of G-\( \infty \)-operads \( Z \in \text{dSet}_G \), defined as follows, and for which all elements of \( \tau_G Z(C) \) in (A.5) will be represented by elements of \( Z(C) \) itself.

**Definition A.6.** \( Z \in \text{dSet}_G \) is a genuine G-\( \infty \)-operad if it has the right lifting property against all maps \( v_G, (A^E[T] \to \Omega[T]) \) for \( T \in \Omega_G \), G-subset \( E \subseteq E^G(T) \), and \( v_{G, *} \) as defined in (2.39).

**Remark A.7.** Genuine G-operads (Definition 2.41) can also be defined via a strict lifting property against the maps \( v_{G, *} (A^E[T] \to \Omega[T]) \), cf. [BP20, Def. 3.35]. Thus, genuine G-\( \infty \)-operads can be viewed as “weak” genuine G-operads, much as the relation between quasicategories and categories.

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Remark A.8. Since $\nu_{G,*}$ is fully faithful one has that $X \in \text{dSet}^G$ is a $G$-$\infty$-operad (cf. Definition 2.34) iff $\nu_{G,*} X \in \text{dSet}^G$ is a genuine $G$-$\infty$-operad.

We now turn to the task of describing $\tau_G Z$ for $Z \in \text{dSet}_G$ a genuine $G$-$\infty$-operad.

We start with some notation. Given a multiset $I$ of edges of a tree $T \in \Omega$ (formally, a function $I : E(T) \to \mathbb{N}_0$), we write $\sigma^I T \in \Omega$ for the tree obtained by degenerating $T$ once for each edge in $I$. More explicitly, $\sigma^I T$ is the unique tree such that there is a planar degeneracy $\pi : \sigma^I T \to T$ with $|\pi^{-1}(e)| = I(e) + 1$. Moreover, note that, if $T \in \Omega_G$ is a $G$-tree, then $\sigma^I T \in \Omega_G$ can be defined if $I$ is $G$-equivariant (formally, this means that the multiset $I$ factors as $I : E(T) \to E_G(T) \to \mathbb{N}_0$; i.e. if $I(e) = n$ then so is $I(g e) = n$ for all $g \in G$).

Our main interest will be in degeneracies of $G$-corollas. Up to isomorphism, a $G$-corolla $C \in \Sigma_G$ is determined by the number $0 \leq k$ of leaf orbits and isotropy subgroups $H_i \leq H_0 \leq G$ for $0 \leq i \leq k$, where $H_0$ is the isotropy of a (chosen) root edge. Pictorially, such a $G$-corolla has the orbital representation (cf. (2.18)) given on the left below, but in this section we will find it more convenient to label edge orbits using coset notation as on the right below, so that $[e_i] = G e_i$ denotes the $G$-orbit of $e_i$.

\[
\begin{array}{c}
G/H_1 \quad \cdots \quad G/H_k \\
\downarrow \quad \updownarrow \\
G/H_0 \\
\downarrow \\
C
\end{array}
\quad \begin{array}{c}
[e_1] \quad \cdots \quad [e_k] \\
\downarrow \quad \updownarrow \\
[e_0] \\
\downarrow \\
C
\end{array}
\]  

(A.9)

We will then abbreviate $\sigma^I C = \sigma^{[e_i]} C$, and write $e_i$, $e'_i$ for the two edges of $\sigma^I C$ that degenerate the edge $e_i$ of $C$, with $e_i$ denoting the inner edge and $e'_i$ the outer edge.

\[
\begin{array}{c}
[e_1] \quad \cdots \quad [e_k] \\
\downarrow \\
[e_0] \\
\downarrow \\
[e'_0] \\
\downarrow \\
\sigma^I C
\end{array}
\quad \begin{array}{c}
[e'_1] \\
\downarrow \\
[e_0] \\
\downarrow \\
[e_1] \\
\downarrow \\
[e'_k] \\
\downarrow \\
\sigma^I C
\end{array}
\]  

(A.10)

The $G$-tree $\sigma^I C$ then has an orbital inner face $\sigma^I C - [e_i]$ obtained by removing $[e_i]$ as well as an orbital outer face obtained by removing $e'_i$, which we denote $\sigma^I C - [e'_i]$. Moreover, note that we have natural identifications $C \simeq \sigma^I C - [e_i]$, $C \simeq \sigma^I C - [e'_i]$.

In what follows, we will find it convenient to simplify notation by denoting maps $\nu_{G,*} \Omega[T] \to Z$, where $T \in \Omega_G$ and $Z \in \text{dSet}_G$, simply as $T \to Z$.

Definition A.11. Let $Z \in \text{dSet}_G$ be a genuine $G$-$\infty$-operad and $C$ a $G$-corolla with edge orbits $[e_0], \ldots, [e_k]$. Given two operations $f, g : C \to Z$, we write $f \sim g$ if there exists a map $H : \sigma^I C \to Z$ such that

- $f$ equals the restriction $H|_{\sigma^I C - [e_i]}$;
- $g$ equals the restriction $H|_{\sigma^I C - [e'_i]}$;
- the restriction $H|_{\sigma^I [e_i]}$ is the degeneracy $\sigma^I [e_i] \to [e_i] \to C \to Z$.

Remark A.12. Note that, if $f \sim g$, then it must be that $f|_{\partial C} = g|_{\partial C}$.
Example A.13. Let $G = \mathbb{Z}/2 = \{ \pm 1 \}$ and consider the $G$-corolla with orbital and expanded representations as given on the left below.

\[
\begin{array}{c}
G \cdot e \\
\downarrow e \\
\downarrow -e
\end{array}
\quad
\begin{array}{c}
G \cdot e' \\
\downarrow e' \\
\downarrow -e'
\end{array}
\quad
\begin{array}{c}
G \cdot e \\
\downarrow e \\
\downarrow -e
\end{array}
\quad
\begin{array}{c}
G \cdot e' \\
\downarrow e' \\
\downarrow -e'
\end{array}
\]

$C$ then has a single $G$-leaf orbit $[e] = G \cdot e$ so that, for $f, g : C \to Z$, one has $f \sim g$ if there exists a dendrex $H : \sigma^{-e} C \to Z$ such that

\[
f = H|_{\sigma^{-e} C \setminus \{ e' \}} \quad g = H|_{\sigma^{-e} C \setminus \{ e \}} \quad H_{\sigma^{-e} e}, H_{\sigma^{-e} e'} \text{ are degenerate.} \quad (A.14)
\]

It is worthwhile to compare this equivariant relation $f \sim g$ with the relations obtained if one forgets the $G$-actions. Indeed, while (A.14) implicitly assumes that all of $f, g, H$ are $G$-equivariant, by omitting that assumption one can reinterpret (A.14) as defining a relation $f \sim_c g$ between not necessarily $G$-equivariant maps $f, g : C \to Z$.

A priori, the $\sim_c$ relation differs from the non-equivariant $\sim_e$ and $\sim_{-e}$ relations obtained forgetting the $G$-actions and regarding $C$ as a non-equivariant corolla. However, for $f, g, H$ as in (A.14) one has

\[
f = H|_{\sigma^{-e} C \setminus \{ e' \}} \sim_c H|_{\sigma^{-e} C \setminus \{ e \}} \sim_e H|_{\sigma^{-e} C \setminus \{ e, e' \}} = g \quad (A.15)
\]

so that, by Lemma A.18(b) below, one has that $f \sim_c g$ in fact implies $f \sim_e g$. Moreover, the converse statement follows immediately by using degeneracies.

More generally, similar considerations show that the $\sim$ relations are compatible with restricting the $G$-actions to $H$-actions for some $H \leq G$.

Our next task is to establish the key properties of the $\sim_i$ relations. This will adapt a slew of arguments in [MW09, §6] that make use of lifting properties against horn inclusions. However, as $\sim_i$ is motivated by the orbital representations (A.9),(A.10), we will need a notion of horn that is likewise motivated by such representations. This is the notion of orbital horn, introduced in [BP20, §2.3], and defined as follows. Letting $T \in \Omega_G$ be a $G$-tree, $E \subseteq E'(T)$ be a $G$-subset, and writing $\text{Face}_o(T)$ for the poset of planar orbital faces, the orbital $G$-inner horn $\Lambda^E_o[T] \in dSet^G$ is given by

\[
\Lambda^E_o[T] = \colim_{S \in \text{Face}_o(T), (T-E)+S} \Omega[S] = \bigcup_{S \in \text{Face}_o(T), (T-E)+S} \Omega[S]. \quad (A.16)
\]

We caution that this differs from the $\Lambda^E[T]$ horns in (2.31) and Definition A.6, as in general one has $\Lambda^E_o[T] \nsubseteq \Lambda^E[T]$ (see [BP20, Ex. 2.34] for a detailed example). Nonetheless, as orbital $G$-inner horn inclusions $\Lambda^E_o[T] \to \Omega[T]$ are $G$-inner anodyne [BP20, Prop. 3.13], the genuine $G$-operads in Definition A.6 also have the right lifting property against the maps $v_{G,*}(\Lambda^E_o[T] \to \Omega[T])$.

Example A.17. For the trees $\sigma^i C$ in (A.10), the orbital inner horn $\Lambda^{[e_i]}_o[\sigma^i C]$ is the union of the orbital faces $\sigma^i C - [e_i']$ and $\sigma^i [e_i]$ (as expected by treating the orbital picture as a usual tree).

Lemma A.18 (cf. [MW09, Prop. 6.3 and Lemma 6.4]). Let $Z \in dSet^G$ be a genuine $G$-oo-operad and $C$ a $G$-corolla with edge orbits $[e_0], \cdots, [e_k]$. Then:
(a) each of the relations \( \sim_i \) in Definition A.11 is an equivalence relation;
(b) all the equivalence relations \( \sim_i \) coincide.

Proof. We first address (a).

For reflexivity \( f \sim_i f \), we take the exhibiting homotopy \( H \) to be the degeneracy \( \sigma^i C \xrightarrow{\sigma^i} C \xrightarrow{f} Z \).

Both symmetry and transitivity will use the tree \( \sigma^{ii} C \) below, which degenerates \([e_i]\) twice.

For symmetry, suppose \( f \sim_i g \) with \( H: \sigma^i C \to Z \) the exhibiting homotopy. Define a map
\[
\tilde{H}: \Lambda^0_{[e_i]} [\sigma^{ii} C] \to Z \text{ via }
\]
\[
\tilde{H}|_{\sigma^{ii} C-[e_i]} = H, \quad \tilde{H}|_{\sigma^{ii} C-[e_i^j]} = f \circ \sigma^i, \quad \tilde{H}|_{\sigma^{ii} [e_i]} = f \circ \sigma^{ii}. 
\]
Since the orbital inner horn inclusion \( \tilde{H}: \Lambda^0_{[e_i]} [\sigma^{ii} C] \to \Omega[C] \) is \( G \)-inner anodyne by [BP20, Prop. 3.13], \( \tilde{H} \) admits an extension \( \tilde{H}: \sigma^{ii} C \to Z \). The restriction \( \tilde{H}|_{\sigma^{ii} C-[e_i]} \) then provides the homotopy exhibiting \( g \sim_i f \), and symmetry of \( \sim_i \) follows.

Next, suppose \( f \sim_i g \) and \( g \sim_i h \), and let \( H: \sigma^i C \to Z, \ K: \sigma^i C \to Z \) be the exhibiting homotopies. Define a map \( \tilde{H}: \Lambda^0_{[e_i]} [\sigma^{ii} C] \to Z \) by
\[
\tilde{H}|_{\sigma^{ii} C-[e_i]} = H, \quad \tilde{H}|_{\sigma^{ii} C-[e_i]} = K, \quad \tilde{H}|_{\sigma^{ii} [e_i]} = f \circ \sigma^{ii} = g \circ \sigma^{ii} = h \circ \sigma^{ii}.
\]
\( \tilde{H} \) again admits an extension \( \tilde{H}: \sigma^{ii} C \to Z \), and the restriction \( \tilde{H}|_{\sigma^{ii} C-[e_i^j]} \) provides the homotopy exhibiting \( f \sim_i g \), and transitivity of \( \sim_i \) follows.

We next turn to (b). Consider the tree \( \sigma^{ij} C \) which degenerates \( C \) once along each of \([e_i]\) and \([e_j]\).

Suppose \( f \sim_i g \) with \( H: \sigma^i C \to Z \) the associated homotopy. Define a map \( \tilde{H}: \Lambda^0_{[e_i]} [\sigma^{ij} C] \to Z \) by
\[
\tilde{H}|_{\sigma^{ij} C-[e_i^j]} = H, \quad \tilde{H}|_{\sigma^{ij} C-[e_i]} = f \circ \sigma^j, \quad \tilde{H}|_{\sigma^{ij} C-[e_i]} = f \circ \sigma^j.
\]
Yet again, \( \tilde{H} \) admits an extension \( \tilde{H}: \sigma^{ij} C \to Z \), and the restriction \( \tilde{H}|_{\sigma^{ij} C-[e_i]} \) provides a homotopy exhibiting \( g \sim_j f \). (b) now follows. \( \Box \)
In light of Lemma A.18, given operations \( f, g: C \to Z \) with \( C \) a \( G \)-corolla and \( Z \) a genuine \( G\)-\( \infty \)-operad, we will henceforth write \( f \sim g \) whenever \( f \sim_i g \) for some (and thus all) \( i \). We now extend the \( \sim \) relation, allowing us to define \( \text{ho}(Z) \) for \( Z \) a genuine \( G\)-\( \infty \)-operad.

**Definition A.19.** Let \( T \in \Omega_G \) be a \( G \)-tree and \( Z \in dSet_G \) be a genuine \( G\)-\( \infty \)-operad.

Given dendrices \( x, y: T \to Z \) we write \( x \sim y \) if there are equivalences of restrictions \( x|_{T_v} \sim y|_{T_v} \) for all \( G \)-vertices \( v \in V_G(T) \).

Further, we define \( \text{ho}(Z)(T) = Z(T)/\sim \).

**Proposition A.20.** Let \( Z \in dSet_G \) be a genuine \( G\)-\( \infty \)-operad. Then the assignment \( T \mapsto \text{ho}(Z)(T) \) is a contravariant functor on \( T \in \Omega_G \), i.e. \( \text{ho}(Z) \in dSet_G \).

**Proof.** It suffices to show that the \( \sim \) equivalence relations are compatible with the generating classes of maps in \( \Omega_G \), namely (cf. Proposition 2.10) (planar) degeneracies, inner faces, outer faces, and quotient maps.

The cases of degeneracies and outer faces are obvious. For quotients, since any quotient \( T \to T' \) of \( G \)-trees induces quotients on \( G \)-vertices, it suffices to consider the case of a quotient \( C \xrightarrow{\pi} C \) of \( G \)-corollas. But it is straightforward to check that a homotopy exhibiting \( f \sim_0 g \) also induces a homotopy exhibiting \( f \circ \pi \sim_0 g \circ \pi \) (notably, this needs not hold for the relations \( f \sim_i g \) for \( 0 < i \), since then the exhibiting homotopy may instead exhibit a string of relations \( f \circ \pi \sim \cdots \sim g \circ \pi \) as in (A.15), due to quotient maps possibly sending two leaf orbits to the same leaf orbit).

It remains to address the most interesting case, that of inner faces. Since inner faces can be factored as composites of inner faces that each collapse a single inner edge orbit, it suffices to consider the case of faces \( D \to T \), where \( T \) has a single inner edge orbit. That is, we can assume that there are \( G \)-corollas \( C_1, C_2 \) such that \( T = C_1 \cup [e_i] C_2 \) and \( D = T - [e_i] \), as illustrated below.

![Diagram](https://via.placeholder.com/150)

The claim is now that, if \( x, y: T \to Z \) are such that \( x|_{C_1} \sim y|_{C_1} \) and \( x|_{C_2} \sim y|_{C_2} \), then one must have \( x|_D \sim y|_D \). This will follow from the following two claims:

(i) if \( x, y: T \to Z \) are such that \( x|_{C_1} = y|_{C_1} \) and \( x|_{C_2} = y|_{C_2} \) then \( x|_D = y|_D \);

(ii) given \( x: T \to Z, f: C_1 \to Z \) and \( g: C_2 \to Z \) such that \( f \sim x|_{C_1}, g \sim x|_{C_2} \), there exists \( y: T \to Z \) such that \( y|_{C_1} = f, y|_{C_2} = g \) and \( y|_D = x|_D \).
To show (i) and (ii), consider the degeneracies \( \sigma^0T \) and \( \sigma^1T \) pictured below.

Given \( x, y \) as in (i), one can now build a map \( H: \Lambda_{\sigma_0}^{|c|} [\sigma^0T] \to Z \) by

\[
H|_{\sigma^0T-\{e_0\}} = x, \quad H|_{\sigma^0T-\{e_0'\}} = y, \quad H|_{\sigma^1C_1} = x|_{C_1} \circ \sigma^0 = y|_{C_1} \circ \sigma^0.
\]

Letting \( \bar{H}: \sigma^0T \to Z \) be an extension of \( H \), the restriction \( \bar{H}|_{\sigma^0T-\{e_1\}} \) provides the desired homotopy \( x|_D \sim y|_D \), showing (i).

Lastly, let \( x, f, g \) be as in (ii), and let \( K: \sigma^1C_1 \to Z \) exhibit the relation \( f \sim_i x|_{C_1} \) and \( \bar{K}: \sigma^1C_2 \to Z \) exhibit the relation \( x|_{C_2} \sim_i g \) (note the reversed order). Now build the map \( H: \Lambda_{\sigma_0}^{|c'|} [\sigma^1T] \to Z \) by

\[
H|_{\sigma^1T-\{e_1\}} = x, \quad H|_{\sigma^1C_1} = K, \quad H|_{\sigma^1C_2} = \bar{K}.
\]

Again letting \( \bar{H}: \sigma^1T \to Z \) be a lift, the restriction \( \bar{H}|_{\sigma^1T-\{e_0'\}} \) provides the desired \( y:T \to Z \) in (ii), finishing the proof.

**Corollary A.21.** Let \( Z \in \mathbf{dSet}_G \) be a genuine \( G \)-\( \infty \)-operad. Then:

(a) \( \text{ho}(Z) \in \mathbf{dSet}_G \) is a genuine equivariant operad (Definition 2.41).

(b) the quotient map \( Z \to \text{ho}(Z) \) is the universal map from \( Z \) to a genuine equivariant operad.

In particular, (a) and (b) yield a natural identification \( \text{ho}(Z) \simeq \tau_G(Z) \).

**Proof.** Note first that, since by Remark A.12 the relation \( \sim \) preserves edge colorings, Definition A.19 implies that any map \( v_{\sigma^1} \circ S_0[T] \to \text{ho}(Z) \) admits a factorization \( v_{\sigma^1} \circ S_0[T] \to Z \overset{q}{\to} \text{ho}(Z) \).

The right lifting property for \( \tau_G(Z) \) against the maps \( v_{\sigma^1} \circ (S_0[T] \to \Omega[T]) \) is then automatic from the lifting property for \( Z \).

For strictness, note that Definition A.19 can be reinterpreted as saying that a pair of dendrices \( \Omega[T] \Rightarrow Z \) give rise to the same point of \( \text{ho}(Z) \), i.e. the composites \( \Omega[T] \Rightarrow Z \overset{q}{\to} \text{ho}(Z) \) coincide, iff the composites \( S_0[T] \Rightarrow \Omega[T] \Rightarrow Z \overset{q}{\to} \text{ho}(Z) \) coincide, showing strictness, and establishing (a).

For (b), since \( \text{ho}(Z) \) is a quotient of \( Z \), it suffices to show that any map of the form \( F: Z \to Y \) with \( Y \) a genuine equivariant operad must also enforce the \( \sim \) relation. For a \( G \)-corolla \( C \) and \( f, g: C \Rightarrow Z \) such that \( H: \sigma^1C \to Z \) exhibits \( f \sim_i g \), the strict lifting condition for \( Y \) shows that the maps \( F \circ H: \sigma^1C \to Y \), \( F(f) \circ \sigma^1: \sigma^1C \to Y \) must coincide, and hence \( F(f) = F(g) \). The claim that \( F \) respects equivalences of general pairs of dendrices \( T \Rightarrow Z \) is now clear from Definition A.19.

The following is the equivariant analogue of [CM13b, Prop. 4.8], as discussed in Remark 4.56.
Proposition A.22. Let \( O \in sOp^G \) be a fibrant operad. Then there is a natural isomorphism of genuine equivariant operads

\[
\tau_G(hcN(O)) \cong \pi_0(v_G, N\mathcal{O}).
\]  \hfill (A.23)

Proof. To ease notation, we abbreviate \( v_G, \ast \) as \( v_\ast \) throughout the proof.

By [BP20, Prop. 5.9], \( \pi_0(v_\ast N\mathcal{O}) \) is a genuine equivariant operad, and the existence of the map in (A.23) will be an application of Corollary A.21(b).

Firstly, note that we have the following identifications, naturally on \( T \in \Omega_G \).

\[
v_\ast hcN(O)(T) \cong sOp^G(W;\Omega[T], O) \cong sdSet^G(NW;\Omega[T], N\mathcal{O}) \cong sdSet_G(v_\ast NW;\Omega[T], v_\ast N\mathcal{O})
\]

where the second and third identifications use the fact that \( N:Op \to dSet \) and \( v_\ast: dSet \to dSet_G \) are fully faithful inclusions. One now has a map

\[
sdSet_G(v_\ast NW;\Omega[T], v_\ast N\mathcal{O}) \to sdSet_G(v_\ast NW;\Omega[T], \pi_0 v_\ast N\mathcal{O})
\]

\[
\cong dSet_G(\pi_0 v_\ast NW;\Omega[T], \pi_0 v_\ast N\mathcal{O})
\]

\[
\cong dSet_G(v_\ast \Omega[T], \pi_0 v_\ast N\mathcal{O})
\]

\[
= (\pi_0 v_\ast N\mathcal{O})(T)
\]

so altogether we obtain a map \( v_\ast hcN(O) \to \pi_0 v_\ast N\mathcal{O} \) and hence, by Corollary A.21, the desired map

\[
\tau_G(hcN(O)) \to \pi_0 v_\ast N\mathcal{O}.
\]

Moreover, both of these are quotients of \( v_\ast hcN(O) \), so to prove that this map is an isomorphism one needs only show that any two operations \( f, g: C \to hcN(O) \) of \( v_\ast hcN(O) \) that are identified in \( \pi_0 v_\ast N\mathcal{O} \) were already identified in \( ho(hcN(O)) \). But this now follows from the pushout below, cf. Lemma 4.53,

\[
\begin{array}{ccc}
\Omega(C) \otimes_{\mathcal{E}} \partial\Delta[1] & \longrightarrow & W_i(\partial\Delta[\sigma^0 C]) \\
\downarrow & & \downarrow \\
\Omega(C) \otimes_{\mathcal{E}} \Delta[1] & \longrightarrow & W(\sigma^0 C)
\end{array}
\]

since mapping out of the bottom left (resp. right) corner encodes homotopy in \( \pi_0 v_\ast N\mathcal{O} \) (resp. \( ho(hcN(O)) \)). \( \square \)

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