Scaling and correlation in financial data

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Abstract

We apply the concept of scaling to the study of some statistical properties of financial data, showing the link between scaling behavior on one hand and correlation properties on the other hand. A non-parametric approach is used to study the scale dependence of the empirical distribution of the price increments \( (x(t + T) - x(t)) \) of S&P Index futures, for time scales ranging from a few minutes to a few days using high-frequency price data. We show that while the variance increases linearly with the timescale, the kurtosis exhibits anomalous scaling properties, indicating a departure from the iid hypothesis. Study of the dependence structure of the increments shows that although the autocorrelation function decays rapidly to zero in a few minutes, the correlation of their squares exhibits a slow power law decay with exponent \( \alpha \simeq 0.37 \), indicating persistence in the scale of fluctuations. We establish a link between the scaling behavior and the dependence structure of the increments: in particular it is shown that the anomalous scaling of kurtosis may be explained by "long memory" properties of the square of the increments.

Nous étudions les propriétés statistiques des fluctuations des prix boursiers en utilisant le concept de lois d’échelle, en mettant en valeur le lien entre lois d’échelle et corrélations temporelles des incréments de prix. Utilisant des données haute-fréquence nous étudions, avec une approche non-paramétrique, la manière dont la distribution \( P_T \) des incréments de prix varie avec l’échelle de temps \( T \) à travers la dépendance de sa variance et sa kurtosis par rapport à l’échelle de temps \( T \), pour \( T \) allant de quelques minutes à plusieurs jours. Alors que la variance croît linéairement avec l’échelle de temps, le kurtosis manifeste une décroissance plus lente que celle d’une marche aléatoire à incréments indépendants, indiquant la présence d’une dépendance non-linéaire dans les incréments, ce que confirme l’étude de fonctions de corrélation non-linéaires, révélant une autocorrélation positive du carré des fluctuations décroissant en loi de puissance avec un exposant \( \alpha \simeq 0.37 \). Nous mettons également en évidence le lien entre cette persistance du carré des incréments d’une part et la décroissance lente de la kurtosis avec l’échelle de temps d’autre part.


1 Motivation

The distributional properties of asset price increments are important both from a theoretical point of view, for understanding market dynamics, and for numerous applications, such as the pricing of derivative products and Value-at-Risk estimations, in which distributional assumptions play a crucial role. These applications, and many others, involve different time horizons, ranging from a few minutes, the typical timescale on which market transactions take place, to several months which is the time horizon portfolio managers are concerned with. This requires knowing the distributional properties of price increments at various time scales.

The problem of comparing the distributions of price changes at various time scales is also interesting from a fundamental point of view. Indeed, as we explain below, whereas studying the distribution of increments on a single timescale cannot distinguish between a process with independent increments and one with dependent increments, studying the deformation of the distribution under a change of timescale provides insight into the dependence structure of the time series. We will further elaborate on this point in section 5.

Many studies have been conducted on the distributional properties of asset prices and returns (for a review see [15]). However most of these studies focus on the distributional properties of returns for a given value of the timescale \( \tau \), using daily, weekly or monthly data for example.

Mandelbrot was the first to emphasize the idea of comparing the distributional properties of price changes at different time scales [9]. The idea behind Mandelbrot’s approach is that of scale invariance: the distribution \( P_T \) of price changes on a time scale \( T \) may be obtained from that of a shorter time scale \( \tau < T \) by an appropriate rescaling of the variable:

\[
P_T(x) = \frac{1}{\lambda} P_\tau\left(\frac{x}{\lambda}\right), \quad \lambda = \left(\frac{T}{\tau}\right)^H
\]

where \( H \) is the self-similarity exponent. The simplest scenario of a scale invariant price process is that of a Lévy flight: a random walk with iid incre-
ments having a stable Lévy distribution[11, 7]. This was indeed the solution proposed by Mandelbrot [9]. In this case, if the increments have a $\mu$-stable Lévy distribution then the process is self-similar with exponent $H = 1/\mu$.

In the recent years the availability of high frequency financial data has rendered possible the study of price dynamics over a wider range of time scales. Several studies have been done in a spirit similar to that of Mandelbrot [9] in the quest of self-similarity in financial time series, on the S&P 500 index [12] and on the CAC40 (Paris stock index) [18]. This scale invariant behavior, observed for short time scales (up to 1000 minutes in [12]), breaks down for longer time scales [2]. Explanations for these observations have been given in terms of truncated Lévy flight models[2, 13].

While inspired by the above approaches, the present study has a different aim: we argue that the existence of non-linear correlations and anomalous scaling indicate that simple random walk models may be insufficient for modeling finer aspects of price fluctuations. Without assuming any scale invariance or self-similarity property, we attempt to characterize the deformation of the probability density function $P_T$ under a change of time scale $T$ by studying the scaling behavior of its variance and its kurtosis, showing that their scaling properties deviate from that of an iid random walk.

Knowledge of the scale dependence of distributional properties of price changes enables us to extract more information on the dynamics of series than studying it on a single time scale, providing information about the dependence structure of the increments: in particular, we show that a link can be established between the scaling behavior of the cumulants and high-order correlation functions of the time series.

2 The data set

The study has been conducted on a high frequency data set of S&P Index future prices between October 1991 and September 1995. There are four maturity dates each year for S&P futures contracts and market activity is most intense for the contract closest to its maturity. We have constituted our data set by recording for each period the price of the S&P futures contract which has the greatest liquidity among all contracts traded in the market, using the above criterion: $x(t)$ is the price of the (unique) futures contract maturing less than three months from $t$. The procedure of sticking together
various price series may artificially introduce price jumps at the maturity
dates. The presence of these jumps may in turn influence various statistical
estimates such as the variance or kurtosis. To avoid this, we have shifted all
prices in the next data set by a constant so as to equalize the last price of a
given set with the first price of the next one, thus eliminating these jumps.
Prices of adjacent contracts are shifted by a constant value such that there is
no jump between adjacent prices at maturity dates. Since our study focuses
on statistical properties of price differences, shifting adjacent prices by the
same constant has no effect whatsoever on our results.

Prices are recorded at 5 minute intervals, the price retained being the
last price recorded during the corresponding 5-minute interval. The data set
thus constituted contains more than 75 000 prices. Throughout the article,
time is given in multiples of \( \tau = 5 \) minutes (\( T = N \tau \)). \( N = 84 \) corresponds
to a working day, \( N = 420 \) to a regular working week.

The size of our data set enables us to study timescales ranging from 5
minutes (\( N = 1 \)) up to a few days (\( N = 200 \)). Estimates for the longest time
scale (\( N = 200 \)) thus correspond to averages over a sample of size \( \geq 3500 \),
enabling large sample approximations to be used.

3 Probability density function of the increments

Let \( K \) be a smooth, positive function with the property

\[
\int_{-\infty}^{+\infty} K(x) dx = 1
\]

If \( (X_i)_{n \geq i \geq 1} \) is a sample realization of a random variable \( X \) the kernel
estimator \( f(x) \) defined by

\[
f(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)
\]

gives a smooth estimator of the probability density function (PDF) of the
variable \( X \). We have calculated a kernel estimator for the PDF of the distribution
of the 5 minute price changes using a Gaussian kernel i.e. \( K(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \)
shown in Figure 1. The non-normal character of the distribution
is better illustrated when it is compared to a Gaussian curve with same mean and variance, also shown in Figure 1.

This impression is confirmed upon calculating the excess kurtosis $\kappa_{5\text{min}}$ of the distribution (see section 4): $\kappa = 15.95$. This leptokurtic character of price changes is a constant feature of high frequency data.

Another feature of high frequency data is that drift effects are negligible compared to fluctuations i.e. the ratio of the mean to the standard deviation is very small. This ratio is found to be $m/\sigma \simeq 3.3 \times 10^{-3}$ in our case: price variations due to drift are roughly a thousand times smaller than those due to 'volatility' - fluctuations around the mean value $m$. Therefore for practical purposes the distribution may be considered as centered around zero as long as the time scales $T$ considered are such that $mN \ll \sigma \sqrt{N}$ i.e. $N \ll N_0 = (\sigma/m)^2 = 9000 \simeq 3$ months.

### 4 Scaling properties

We calculate for each time scale $T = N\tau$ the following quantities:

\[
\begin{align*}
\sigma(T)^2 &= \frac{[x(t+T) - x(t)] - (x(t+T) - x(t))]^2}{\sum_{t=1}^{n} A(t)} \\
\kappa(T) &= \frac{[x(t+T) - x(t)] - (x(t+T) - x(t))]^4}{\sigma(T)^4} - 3
\end{align*}
\]

where the symbol $\overline{A}$ denotes a sample average:

\[\overline{A} = \frac{1}{n} \sum_{t=1}^{n} A(t)\]

$\sigma(T)$ is an estimator of the variance of the distribution $P_T$ and $\kappa(T)$ an estimator of the kurtosis, the kurtosis of a random variable $X$ being defined as:

\[\kappa_X = \frac{E[(X - E(X))^4]}{E[(X - E(X))^2]^2} - 3\]

(where $E$ denotes expectation value) defined such that $\kappa_X = 0$ for a gaussian random variable.
Going from a shorter time scale $\tau$ to a longer time scale $N\tau$ formally corresponds to summing $N$ random variables:

$$x(t + N\tau) - x(t) = \sum_{j=1}^{N} \delta x_j$$

If increments are stationary and independent, the distributions for various timescales are simply related by a convolution relation:

$$P_{N\tau} = P_{\tau} \otimes P_{\tau} \otimes ... \otimes P_{\tau}$$

where $\otimes$ denotes convolution. One can then deduce simply from this relation that for a stochastic process with iid increments with finite variance, the variance of the distribution at scale $T = N\tau$ increases linearly with $N$ and that the kurtosis decreases as $1/N$ [3]. More generally if $c_k(N)$ is the k-th semi-invariant or cumulant [4, 3] of the distribution $P_T = P_{N\tau}$ then the normalized cumulants

$$\lambda_k(N\tau) = \frac{c_k(N\tau)}{\sigma(N\tau)^k} = \frac{\lambda_k(\tau)}{N^{k/2-1}}$$

tend to zero for large $N$, which is another way of seeing the Central Limit Theorem: the cumulants of order $\geq 3$ of the limit distribution are all zero, a property which characterizes the Gaussian distribution. For example the skewness $\lambda_3(N\tau)$ decreases as $1/\sqrt{N}$. Note that $\lambda_4(N\tau)$ is the kurtosis $\kappa(N\tau)$ of the distribution $P_{N\tau}$.

There is no a priori reason to believe that the price of an asset has independent increments. However, it is now known that the independence of increments is not a necessary condition for the Central Limit Theorem to apply [4]: some types of dependence structures still allow for convergence to the Gaussian distribution as $T \to \infty$ in which case the distribution will resemble more closely a Gaussian at longer time scales than at shorter ones and $\lambda_k(T) \to 0$ as $T \to \infty$. However there is no reason anymore for the normalized cumulants $\lambda_k(N\tau)$ to decrease as $1/N^{k/2-1}$ as in the iid case: the presence of ’non-trivial’ scale dependence, which we term anomalous scaling, is the signature of a departure from the iid case.

We have examined the empirical behavior of $\sigma(N\tau)$ and $\kappa(N\tau)$: Figure 2. illustrates the scaling behavior of the variance; it can clearly be seen that the variance is linear in the timescale i.e. $\sigma(N\tau) = N\sigma(\tau)$, as it would be for
independent increments. This property illustrates the additivity of the variances of increments, which is equivalent to the absence of autocorrelation, discussed below.

However if we examine the scaling behavior for higher cumulants of the distribution $P_T$ we notice that their behavior is different than for an iid random walk. Figure 3 shows the behavior of the kurtosis under a change of timescale. As indicated above, in the case of iid increments, the kurtosis decreases by a factor $1/N$ when we multiply the time scale by $N : \kappa(N\tau) = \kappa(\tau)/N$. In Figure 3 we have given a power-law fit of the kurtosis $\kappa(N\tau)$ as a function of the time scale $T = N\tau$. We find that $\kappa(N\tau)$ decreases more slowly than $1/N$:

$$\kappa(N\tau) \simeq \kappa(\tau)/N^\alpha$$

where $\alpha \simeq 0.5$.

In principle one could calculate higher order cumulants of the distribution and examine their scaling properties. However, the error bar for such calculations increases drastically as we go higher in the order of the cumulants.

## 5 Correlation and dependence

It is a well-known fact that price movements in liquid markets do not exhibit any significant autocorrelation: the autocorrelation function of the price changes

$$C(T) = \frac{\delta x_t \delta x_{t+T} - \overline{\delta x_t} \overline{\delta x_{t+T}}}{\text{var}(\delta x_t)}$$

rapidly decays to zero in a few minutes: for $T \geq 15$ minutes it can be safely assumed to be zero within estimation errors\[^1\]. The absence of economically significant linear correlations in price increments and asset returns has been widely documented (see \[^1\] and references within) and often cited as support for the "Efficient Market Hypothesis" \[^5\]. The fast decay of the correlation function implies the additivity of variances: for uncorrelated variables, the variance of the sum is the sum of the variances. The absence of linear correlation is thus consistent with the linear increase of the variance with respect to time scale.
However, the absence of serial correlation does not imply the independence of the increments: it is well known that two random variables may have zero correlation yet not be independent. In the same way, one may give examples of stochastic processes with uncorrelated but not independent increments. One example, given in [3] is the following: let \( X \) be a random variable uniformly distributed on \([0, \pi]\) and define the discrete time stochastic process \( (U_t)_{t \geq 0} \) by the recurrence relation \( U_t = U_{t-1} + \sin(tX) \). A simple calculation then shows that the increments of \( (U_t)_{t \geq 0} \) are uncorrelated; however, they are not independent: indeed, they are deterministic functions of the same random variable, \( X \).

The simplest non-parametric test of the iid hypothesis is the sign test: if the increments have the same median value \( m \) then the number \( T \) of values exceeding \( m \) in a sample of size \( n \) has expected value \( n/2 \) and the statistic \( s = \frac{2T-n}{\sqrt{n}} \) has a standard normal distribution for large samples. The median \( m \) is taken to be the sample median, zero in this case.

We calculate the sign statistic

\[
s = \frac{2T - n}{\sqrt{n}}
\]

for tick data, 5 minute price changes and 30 min price changes. The interval between ticks is irregular and depends on market activity but it is less than one minute. The results are the following:

\[
s_{\text{tick}} = 6.66 \, , \, s_{5\text{min}} = -2.1 \, , \, s_{30\text{min}} = 1.25
\]

Assuming that the sign statistic has a N(0,1) distribution and using a confidence interval of two standard deviations, the results show that although the signs of price movements at very short time resolutions (less than a minute) are significantly correlated, they can be safely considered as independent (as far as the sign test is concerned) for time scales beyond 30 minutes. Even for 5 minutes \( s \) is close to 2 standard deviations.

From a formal point of view, independence is characterized by the fact that for any (measurable) functions \( f, g \) the quantity

\[
C_{f,g}(X,Y) = E[f(X)g(Y)] - E[f(X)]E[g(Y)]
\]

vanishes i.e. the variables \( f(X) \) and \( g(Y) \) are uncorrelated for any non-linear functions \( f \) and \( g \). In particular, \( C_{f,f}(X,Y) = 0 \) : applying any non-linear function to an independent sequence gives an uncorrelated sequence.
In our case this means that increments are stationary and independent if and only if for any function \( f \) the series \( (f(x(t+T) - x(t)))_{t \geq 0} \) is serially uncorrelated. This criterion indicates a simple non-parametric way of testing the hypothesis of iid increments: for a given function \( f \) we calculate

\[
C_f(T) = \frac{\overline{f(\delta x_t)f(\delta x_{t+T})}}{\overline{f(\delta x_t)}f(\delta x_{t+T})}
\]

Formally, \( C_f(T) \) corresponds to a resummation of a certain subclass of correlation functions of the increments \( (\delta x_t)_{t \geq 0} \), with weights corresponding to the Taylor coefficients of \( f \). We use the following non-linear test functions: \( f_1(x) = x^2 \), \( f_2(x) = \cos(x) \) and \( f_3(x) = \ln(1 + x^2) \). In each case we calculate the autocorrelation of the series \( (f(\delta x_t))_{t \geq 0} \). The results are displayed in Figure 5. It can be seen that the series \( (f_1(\delta x_t))_{t \geq 0} \) and \( (f_3(\delta x_t))_{t \geq 0} \) exhibit slowly decaying serial correlations, showing the presence of nonlinear dependence in the data.

To obtain a more detailed picture of the dependence structure of the increments, we proceed to calculate higher order correlation functions. It is well known that the square of the returns -or any other measure of the amplitude of fluctuations for example the absolute value of the increments- exhibits significant autocorrelation.

Figure 6 compares the autocorrelation of the price changes to that of their absolute value. In contrast to the autocorrelation function of the increments which decays rapidly to zero in a few minutes, the autocorrelation of their absolute value decays slowly to zero while staying positive, indicating persistence in the scale of the fluctuations, a phenomenon which can be related to the well known “clustering of volatility”.

Another measure of the scale of the fluctuations is given by the square of the increments; Figure 7 displays the autocorrelation function \( g(T) \) of the square of the increments, defined as:

\[
g(N) = g\left(\frac{T}{\tau}\right) = \frac{\overline{\delta x^2_t\delta x^2_{t+T} - \overline{\delta x^2_t}\overline{\delta x^2_{t+T}}}}{\overline{\delta x^2_t} \overline{\delta x^2_{t+T}}} = \frac{\overline{\delta x^2_t\delta x^2_{t+T} - \overline{\delta x^2_t}\overline{\delta x^2_{t+T}}}}{\mu_4(\tau) - \sigma(\tau)^4} \tag{1}
\]

Fitting \( g(T) \) by an exponential gives very bad results; however the slow decay \( g(T) \) is well represented by a power law:
$$g(k) \simeq \frac{g_0}{k^{\alpha}} \quad \alpha = 0.37 \pm 0.037 \quad g_0 = 0.08$$

the exponent $\alpha$ and the constant $g_0$ being obtained by a regression of $\ln g(T)$ against $T$. We will see in the following section that this approximation is precise enough to finely capture the scaling behavior of the kurtosis.

6 Relation between scaling behavior and correlation structure

The results of the two preceding sections may be blended into a consistent picture of the dynamical properties of the price process by remarking that the scaling behavior of the semi-invariants of the increments is related to their correlation structure.

There is a simple, well-known example of relations between correlation functions and scaling behavior: the relation between the variance of a sum and the covariance of the addends. The variance at time scale $N\tau$ may be expressed as

$$\sigma^2(N\tau) = \sum_{k=1}^{N} \sigma^2(\tau) + 2\sigma^2(\tau) \sum_{k>l} C((k-l)\tau)$$

$$= \sum_{k=1}^{N} \sigma^2(\tau) + 2\sigma^2(\tau) \sum_{k=1}^{N} (N-k)C(k\tau)$$

$$= N\sigma^2(\tau)[1 + 2 \sum_{k=1}^{N} (1 - \frac{k}{N})C(k\tau)]$$

where $C(T)$ is the autocorrelation function of the increments, defined above. therefore implies $\sigma^2(N\tau) = N\sigma^2(\tau)$. The absence of autocorrelation implies linear scaling of the variance.

A similar relation may be derived between the scaling behavior of the kurtosis and the autocorrelation of the square of the increments (the function $g$ defined above). Using the results above, we model the signs of the successive price changes as an independent sequence with zero mean, independent from their absolute value. Using these hypotheses the fourth moment may
be expressed in terms of the moments and correlation functions of shorter
time scale increments $\delta x$; the calculation, details of which are given in the
appendix, gives:

$$
\mu_4(N\tau) = N\mu_4(\tau) + 3(N^2 + N)\sigma(\tau)^4 + 6N(\mu_4 - \sigma(\tau)^4) \sum_{k=1}^{N} g(k) - 6(\mu_4 - \sigma(\tau)^4) \sum_{k=1}^{N} kg(k)
$$

The scaling behavior of the kurtosis is therefore related to the behavior
of the correlation function $g$; if $g$ exhibits a slow power law decay $g(k) \simeq g_0 k^{-\alpha}$
then $\sum_{k=1}^{N} g(k) \simeq g_0/(1 - \alpha)N^{1 - \alpha}$ and $\sum_{k=1}^{N} kg(k) \simeq g_0/(2 - \alpha)N^{2 - \alpha}$ so:

$$
\kappa(N\tau) = \frac{\mu_4(N\tau)}{\sigma^4(N\tau)} - 3 = \frac{\kappa(\tau)}{N} + \frac{6(\kappa(\tau) + 2)}{(2 - \alpha)(1 - \alpha)N^{\alpha}}
$$

Figure 8. shows good agreement for large values of $N$ ($N \geq 50$) between
the right hand side of the above equation and the kurtosis as a function of
time resolution, plotted on the same diagram. We have thus shown that the
anomalous scaling of the kurtosis is well accounted for by the presence of
correlations in the scale of the fluctuations, represented by the square of the
increments.

7 Conclusion

By comparing the statistical properties of price increments of S&P index futures at various time scales, we have shown that studying the resolution dependence or scaling behavior of the variance and the kurtosis of the empirical distribution of price increments enables us to extract much more information than studies on a given time scale. In particular, the scaling properties of the kurtosis of the price changes may be recovered from the correlation function of their squares, which exhibits a slow power law decay with exponent $\alpha = 0.37 \pm 0.037$. Such relations may be generalized to other cumulants and
moments but which are less interesting from an economic point of view.

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Appendix: relation between scaling behavior of kurtosis and correlation of increment squares

Let \( \delta x_k = x(t + k\tau) - x(t + (k-1)\tau) \). We derive here the relation between the kurtosis and the correlation function of the square of the increments \( g(k) \) defined in equation (1). Let us furthermore suppose without loss of generality that \( E[\delta x_k = 0] \) (see also the remark in section 3). The fourth moment \( \mu_4(N\tau) \) is given by

\[
\mu_4(N\tau) = E[((x(t + N\tau) - x(t))^4) ]
= E[(\sum_{k=1}^{N} \delta x_k)^4]
= E[\sum_{i,j,k,l=1}^{N} \delta x_i \delta x_j \delta x_k \delta x_l ]
\]

\[
= \sum_{i=1}^{N} E[\delta x_i^4] + 6 \sum_{i>j} E[\delta x_i^2 \delta x_j^2] + 6 \sum_{i>j>k} E[\delta x_i^2 \delta x_j \delta x_k ]
+ 3 \sum_{i>k} E[\delta x_i^3 \delta x_k] + \sum_{i>j>k} E[\delta x_i \delta x_j \delta x_k \delta x_l ]
\]

We will suppose now that the increments may be written as

\[
\delta x_k = \epsilon_k \gamma_k \quad \epsilon_k \in \{+1,-1\}, \quad \gamma_k > 0
\]

where \((\epsilon_k)\) is a sequence of independent sign variables, but the sequence \((\gamma_k)\) may have an arbitrary dependence structure. Since \( \delta x_k \) is centered
\( E \epsilon_k = 0 \). This hypothesis captures well the behavior of the increments: their magnitudes are correlated ("ARCH effect") but their signs random. We furthermore suppose that \( \gamma_k \) is independent from \( \epsilon_j \). As a result, most of the terms appearing in the above sum vanish:

\[
\begin{align*}
E[\delta x_i^4 \delta x_k] &= E[\epsilon_i \epsilon_k \gamma_i^3 \gamma_k] \\
&= E[\epsilon_i] E[\epsilon_k] E[\gamma_i^3 \gamma_k] = 0 \\
E[\delta x_i \delta x_j \delta x_k \delta x_l] &= E[\epsilon_i] E[\epsilon_j] E[\epsilon_k] E[\epsilon_l] E[\gamma_i \gamma_j \gamma_k \gamma_l] = 0 \\
E[\delta x_i^2 \delta x_j \delta x_k] &= E[\epsilon_j] E[\epsilon_k] E[\gamma_i^2 \gamma_j \gamma_k] = 0
\end{align*}
\]

The expression for the fourth moment therefore reduces to:

\[
\mu_4(N \tau) = \sum_{i=1}^{N} E[\delta x_i^4] + 6 \sum_{i>j} E[\delta x_i^2 \delta x_j^2] \\
= \sum_{j=1}^{N} E[\delta x_j^4] + 6 \sum_{j=1}^{N} (N - j) E[\delta x_j^2 \delta x_{j+j}]
\]

From (1), \( E[\delta x_i^2 \delta x_j^2] \) may now be expressed in terms of the correlation function \( g \):

\[
E[\delta x_i^2 \delta x_j^2] = (\mu_4 - \sigma(\tau)^4) g(|i-j|) + \sigma(\tau)^4
\]

hence, by substitution, the relation between \( \mu_4 \) and \( g \):

\[
\begin{align*}
\mu_4(N \tau) &= N \mu_4(\tau) + 6 \sum_{k=1}^{N} (N - k) [\sigma(\tau)^4 + (\mu_4 - \sigma(\tau)^4) g(k)] \\
&= N \mu_4(\tau) + 3(N^2 - N) \sigma(\tau)^4 + \\
&= 6N(\mu_4(\tau) - \sigma(\tau)^4) \sum_{k=1}^{N} g(k) - 6(\mu_4(\tau) - \sigma(\tau)^4) \sum_{k=1}^{N} k g(k)
\end{align*}
\]

The kurtosis for a time resolution \( T = N \tau \) is therefore given by:
\[
\kappa(N\tau) = \frac{\mu_4(N\tau)}{\sigma(N\tau)^4} - 3
= \frac{N\mu_4(\tau)}{N^2\sigma(\tau)^4} + 3\left(1 - \frac{1}{N}\right) - 3 + 6(\mu_4(\tau) - \sigma(\tau)^4) \sum_{k=1}^{N} \frac{(N - k)g(k)}{N^2\sigma^4(\tau)}
= \frac{\kappa(\tau)}{N} + \frac{6(\kappa(\tau) + 2)}{N} \sum_{k=1}^{N} \left(1 - \frac{k}{N}\right)g(k)
\]

In our case \(g(T) \simeq \frac{g_0}{T^\alpha}\) which implies, for \(\kappa(T)\):

\[
\kappa(N\tau) \simeq \frac{\kappa(\tau)}{N} + \frac{6g_0(\kappa(\tau) + 2)}{N^2} \left(\frac{N^{2-\alpha}}{1-\alpha} - \frac{N^{2-\alpha}}{2-\alpha}\right)
= \frac{\kappa(\tau)}{N} + \frac{6g_0(\kappa(\tau) + 2)}{(2-\alpha)(1-\alpha)N^\alpha}
\]

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FIGURE CAPTIONS:

Figure 1: Probability density of 5 minute increments of S&P 500 index future prices. The lower curve is a gaussian with same mean and variance.

Figure 2: Scaling behavior of the variance of price increments: the variance increases approximately linearly with the time scale, which is consistent with the absence of significant linear correlations.

Figure 3: Scaling behavior of the kurtosis of price increments. The kurtosis decreases much more slowly than $1/N$: a power law fit gives an exponent close to 0.5.

Figure 4: Autocorrelation function of price changes. Time lag is given in multiples of $\tau = 5$ minutes.

Figure 5: Behavior of some non-linear correlation functions of price changes.

Figure 6: Autocorrelation function of absolute price changes: contrary to the price changes, their absolute values display a persistent character.

Figure 7: Autocorrelation function of the square of price changes. Note the slow decay, well represented by a power law with exponent $\alpha = 0.37$.

Figure 8: The scale dependence of the kurtosis (data points) may be reconstructed (solid line) from the autocorrelation function of the square of the price increments, approximated by a power law (Eq.1).
Autocorrelation of price increments vs absolute price increments

S&P futures, 1991-95
Autocorrelation of price changes

S&P 500 index futures 5 min. increments
Probability density of price changes

S&P 500 Futures, 1991-95

- S&P
- Gaussian
N = T/5 minutes

Kurtosis of increments
Theoretical scaling function

Kurtosis

N = T/5 minutes
Autocorrelation of nonlinear functions of price increments

S&P Index futures 1991-95

- \( \cos(x) \)
- \( \ln(1 + x^2) \)
- \( x^2 \)
Scaling behavior of kurtosis

S&P Index futures, 1991-95

Data

Power-law fit \( K(T) = \frac{A}{T^{0.5}} \)
Autocorrelation of square of price changes

S&P 500 Index futures, 1991-95

Power law fit
Time scale $N = T / 5$ minutes