TIME-CONSISTENT MULTIPERIOD MEAN SEMIVARIANCE PORTFOLIO SELECTION WITH THE REAL CONSTRAINTS

PENG ZHANG*
School of Economics and Management
South China Normal University
Guangzhou 510006, China

YONGQUAN ZENG AND GUOTAI CHI
College of Humanities and Social sciences
Zhongkai University of Agriculture and Engineering
Guangzhou 510225, China
Faculty of Management and Economics
Dalian University of Technology
Dalian 116024, China

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Abstract. In this paper, a new multiperiod mean semivariance portfolio selection with the transaction costs, borrowing constraints, threshold constraints and cardinality constraints is proposed. In the model, the return and risk of assets are characterized by mean value and semivariance, respectively. Because the semivariance operator is not separable, the optimal solution of the model is not time-consistent. The time-consistent strategy for this model can be obtained by using game approach. The time-consistent strategy, which is a mix integer dynamic optimization problem with path dependence, is approximately turned into a dynamic programming problem by approximate dynamic programming method. A novel discrete approximate iteration method is designed to obtain the optimal time-consistent strategy, and is proved linearly convergent. Finally, the comparison analysis of trade-off parameters is given to illustrate the idea of our model and the effectiveness of the designed algorithm.

1. Introduction. How to allocate wealth among investment opportunities is a central problem in financial economics. The first mathematical formulation of selecting a portfolio was provided by Markowitz(1952)[28], who combined probability theory and optimization theory to model the behavior of the investor. Though variance has been a rather popular measure of risk in portfolio selection, it has limitations (Markowitz(1959)[27], Simkowitz and Beedles(1978)[34]. One distinguished limitation is that variance considers high returns as equally undesirable as low returns because high returns will also contribute to the extreme of variance. To overcome the limitation of the mean variance models, downside risk measure is directly used...
to replace variance. Semivariance that was first introduced in Markowitz (1959) is one of the best-known downside risk measures.

The Markowitz’s mean-variance model has been extended to multiperiod setting by many scholars. Among others, the representative works include Li and Ng (2000), who extend the originally mean variance problem to multiperiod setting and embed the original problem into a tractable auxiliary problem, then derive the analytical portfolio strategy and the efficient frontier. The further study about multiperiod portfolio selection model includes, e.g., Zhu et al. (2004), Gao et al. (2015) and Cong and Oosterlee (2016). Yan and Li (2009) substituted variance with semi-variance as the risk measure to deal with the multiperiod portfolio selection problem. Pınar (2007) used the downside-risk measure to study the multiperiod portfolio selection problem. However, the optimal strategy of the above multiperiod mean variance or semivariance portfolio selection is under pre-commitment strategy, which is optimal at time zero even though it is not optimal later on.

It is necessary for a rational investor to find an optimal strategy with time-consistency. Cui et al. (2012) propose a weak time consistency. Wang and Forsyth (2011) compare the efficient frontiers obtained from the time consistent optimal policy and the pre-commitment optimal policy, when the bankruptcy and no short-selling constraints are added to the problem. Björk and Murgoci (2010) studied the problem under a general Markovian framework in both discrete-time and continuous-time portfolio selection. Basak and Chabakauri (2010) provided a time-consistent strategy of multiperiod portfolio selection by applying the game theory. Czichowsky (2013) generalized the formulations of Basak and Chabakauri (2010) and justified that the discrete-time time-consistent strategy converged to that of the continuous-time setting. Wei et al. (2013) provided the time-consistent solution to the multiperiod mean-variance asset–liability management problem under the regime switching market. Björk et al. (2014) discussed a more realistic multiperiod mean-variance model in which the risk aversion is dependent on the current total wealth. Bensoussan et al. (2014) extended the work of Björk et al. (2014) to a multiperiod portfolio optimization problem without short selling constraints. Wu and Zeng (2015) discussed the time-consistent strategy for a generalized multiperiod mean variance criterion for defined-contribution pension schemes with mortality risk. Readers may refer to Lioui (2013), Chen et al. (2014), Birgit et al. (2014), Zhou et al. (2016), Wu and Chen (2015), Cui et al. (2014), Cui et al. (2017) and Liu and Chen (2018) for the discussion of the time-consistent strategies for different multiperiod portfolio optimization models.

Because of its practical relevance, this cardinality constrained Markowitz model has been intensively studied in the last decade. Especially from the computational viewpoint, some researchers proposed exact solution methods, i.e., Bienstock (1996) proposed a branch-and-cut algorithm; Bertsimas and Shioda (2009) extended the algorithm of Bienstock (1996) presenting a tailored procedure, based on Lemke’s pivoting algorithm; Li et al. (2006) proposed a convergent Lagrangian method. Another Lagrangian relaxation method was proposed in Shaw et al. (2008) with application to some undisclosed real-life problems with up to 500 assets; Cesaroni et al. (2013) proposed increasing set algorithm; Cui et al. (2013), and Sun et al. (2013) used the Lagrangian decomposition technique to construct tight convex relaxations to solve the cardinality constrained model; Murray and Shek (2012)
presented a local relaxation method; Le Thi et al. (2009, 2014) [21, 20] presented convex functions algorithms. Since exact solution methods are able to solve only a fraction of cardinality constrained Markowitz models, many heuristic algorithms have also been proposed, i.e., Anagnostopoulos and Mamanis (2011) [1] proposed multiobjective evolutionary algorithm; Fernández and Gómez (2007) [17] presented neural networks algorithm; Ruiz-Torrubiano and Suarez (2010) [32] provided solutions to the cardinality model using different heuristics algorithms, i.e., genetic algorithm, simulated annealing and various estimation of distribution algorithms; Woodside-Oriakhi et al. (2011) [40] examined the application of genetic algorithm, tabu search and simulated annealing for finding the cardinality constrained efficient frontier; Deng et al. (2012) [16] proposed improved particle swarm algorithm; Soleimani et al. (2009) [35] presented genetic algorithm; Vercher and BernUdez (2013) [37] proposed multi-objective evolutionary algorithm. In these studies, it appears that the computational complexity for the solution of the cardinality constrained model is much greater than the one required by the classical Markowitz model or by several other of its refinements. Indeed, the standard Markowitz model is a convex quadratic programming problem, while the cardinality constrained Markowitz model is a mixed integer quadratic programming problem that falls into the NP-hard problems. The contribution of this paper is as follows: To the best of our knowledge, no author study the time-consistent strategy for the multiperiod portfolio selection model with the transaction costs, borrowing constraints, threshold constraints and cardinality constraints. In this paper, we propose a new multiperiod mean semivariance portfolio selection model with these real constraints. Because of the transaction costs and cardinality constraints, the proposed model is a mix integer dynamic optimization problem with path dependence. A novel discrete approximate iteration method is designed to seek the optimal time-consistent strategy.

The rest of this paper is organized as follows. In Section 2, we introduce the problem description and notations, some properties and the frictional market constraints of the multiperiod portfolio. A new multiperiod mean semivariance portfolio selection model with the transaction costs, borrowing constraints, threshold constraints and cardinality constraints is proposed. In Section 3, the time-consistent strategy for the proposed model can be derived by using the game approach. In Section 4, the model, which is a mix integer dynamic optimization problem with path dependence, is transformed into a dynamic programming problem by using the approximate dynamic programming method. A novel discrete approximate iteration method is proposed to seek the optimal time-consistent strategy and is proved linearly convergent. In Section 5, we give the comparison analysis of trade-off parameters to illustrate the idea of our model and the effectiveness of the designed algorithm. Finally, some conclusions are given in Section 6. Please use this AIMS template to prepare your tex file after the paper is accepted by an AIMS journal.

2. The formulation of multiperiod portfolio selection problem. In this section, we introduce the transaction costs, borrowing constraints, threshold constraints and cardinality constraints and present a multiperiod portfolio selection with real constraints.

2.1. Real constraints for multiperiod portfolio selection. We consider a multiperiod portfolio optimization problem, where an investor is going to invest in \( n \) risky assets and a risk-free asset with a positive initial wealth of \( X_0 \). The investment will be made at the beginning of the period \( t \) of a \( T \) period portfolio selection
Let the definite. Assume that the whole investment process is self-financing, that is, the total wealth of the investor at the end of period is expressed as the corresponding mean return vector and the semivariance–covariance matrix of \( X_t \) at period \( t \) can be expressed as \( \text{svar}(X_t) \), \( u_{it} \) be the investment amount of risk-free asset at period \( t \), \( u_t = (u_{1t}, u_{2t}, \ldots, u_{nt}, u_{ft})' \), \( l_{it} \) and \( p_{it} \) be respectively the lower and upper bound constraints of risky assets \( i \) at period \( t \). Furthermore, for any \( 1 \leq t \leq T \), let \( r_t = E(R_t) \) be the corresponding mean return vector and the semivariance–covariance matrix be denoted by \( H_t' = (\text{Cov}(R_{it}, R_{jt}))_{n \times n} \), where \( H_t' \) is assumed to be semi-positive definite. Assume that the whole investment process is self-financing, that is, the investor does not invest the additional capital during the portfolio selection.

The mean value of the portfolio \( u_t = (u_{1t}, u_{2t}, \ldots, u_{nt}, u_{ft})' \) at period \( t \) can be expressed as

\[
r_{pt} = \sum_{i=1}^{n} r_{it} u_{it} + r_{ft}(X_{t-1} - \sum_{i=1}^{n} u_{it}) - X_{t-1} = (r_{ft} - 1)X_{t-1} + \sum_{i=1}^{n}(r_{it} - r_{ft})u_{it},
\]

\( t = 1, \ldots, T \)

(1)

In this paper, the transaction cost is assumed a \( V \)-shaped function of differences between the \( t \)th period portfolio \( u_t = (u_{1t}, u_{2t}, \ldots, u_{nt}, u_{ft}) \) and the \( t \)-th period portfolio \( u_{t-1} = (u_{1(t-1)}, u_{2(t-1)}, \ldots, u_{n(t-1)}, u_{f(t-1)}) \). That’s to say, the transaction cost for asset \( i \) at period \( t \) is \( c_{it}(|u_{it} - u_{i(t-1)}|) \). Hence, the total transaction cost of the portfolio \( u_t = (u_{1t}, u_{2t}, \ldots, u_{nt}, u_{ft}) \) at period \( t \) can be expressed as

\[
C_t = \sum_{i=1}^{n} c_{it}(|u_{it} - u_{i(t-1)}|), t = 1, \ldots, T
\]

(2)

Thus, the net return rate of the portfolio \( u_t \) at period \( t \) can be denoted as

\[
r_{Nt} = (r_{ft} - 1)X_{t-1} + \sum_{i=1}^{n}(r_{it} - r_{ft})u_{it} - \sum_{i=1}^{n} c_{it}(|u_{it} - u_{i(t-1)}|), t = 1, \ldots, T
\]

(3)

According to the above descriptions of the financial market, the multiperiod wealth process can be formulated as

\[
X_t = r_{Nt} + X_{t-1} = r_{ft}X_{t-1} + \sum_{i=1}^{n}(r_{it} - r_{ft})u_{it} - \sum_{i=1}^{n} c_{it}(|u_{it} - u_{i(t-1)}|), t = 1, \ldots, T
\]

(4)

**Definition 2.1.** Let \( \xi \) be a random variable with finite expected value \( e \). Then the semi-variance of \( \xi \) is defined by \( \text{svar}([\xi]) = E[(\xi - e)^-]^2 \), where

\[
(\xi - e)^- = \begin{cases} 
\xi - e & \text{if } \xi \leq e \\
0 & \text{if } \xi > e
\end{cases}
\]

(5)

According to **Definition 2.1**, the semi-variance of the portfolio \( x_t \) can be expressed as

\[
\text{svar}_t(x_t) = \text{svar}_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \ldots + r_{nt}x_{nt}) = x_t' H_t' x_t
\]

(6)
where
\[ H'_t = (\sigma_{ijt})_{n \times n}, (\sigma_{ijt})^- = E[(R_{it} - r_{it})^-(R_{jt} - r_{jt})^-] \quad (7) \]
The threshold constraints of multiperiod portfolio selection can be expressed as
\[ l_{it} \leq u_{it} \leq p_{it} \quad (8) \]
where \( l_{it} \) and \( p_{it} \) are the lower and upper bounds constraints of \( x_{it} \), respectively.

Let the preset value be \( u_{ft} \), where \( u_{ft} > 0 \), the borrowing constraint of risk-free asset at period \( t \) is
\[ u_{ft} = X_{t-1} - \sum_{i=1}^{n} u_{it} \geq u_{ft}^b \quad (9) \]

To calculate cardinality constraints for the multiperiod portfolio model, zero-one decision variables are added as follows:
\[ z_{it} = \begin{cases} 1 & \text{if any of asset } i \text{ of period } t(i = 1, \ldots, n; t = 1, \ldots, T) \text{ is held} \\ 0 & \text{otherwise} \end{cases} \quad (10) \]
where \( \sum_{i=1}^{n} z_{it} \leq K, K \) is positive integer.

2.2. The multiperiod portfolio selection problem. The traditional multiperiod mean-semivariance portfolio optimization model only considers the expectation and semivariance of terminal wealth. However, in real world, the investor not only cares for the expectation and semivariance of terminal wealth, but also concerns about intermediate expected values and intermediate semivariances. We consider a set of numbers \( w_t > 0, \eta_t > 0 \) for \( t=1, \ldots, T \). A multiperiod portfolio selection with the transaction costs, borrowing constraints, threshold constraints and cardinality constraints is as follows:
\[
\begin{align*}
\max & \sum_{t=1}^{T} w_t (E(X_t) - \eta_t \text{var}(X_t)) \\
\text{s.t.} & \\
& X_{t} = r_{ft}X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft})u_{it} - \sum_{i=1}^{n} c_{it}(u_{it} - u_{it(t-1)}), t = 1, \ldots, T \quad (a) \\
& X_{t-1} - \sum_{i=1}^{n} u_{it} \geq u_{ft}^b \quad (b) \\
& \sum_{i=1}^{n} z_{it} \leq K, z_{it} \in (0, 1) \quad (c) \\
& l_{it}z_{it} \leq u_{it} \leq p_{it}z_{it}, i = 1, \ldots, n; t = 1, \ldots, T \quad (d)
\end{align*}
\]
where \( u_{ft}^b \) is the preset value, and \( u_{ft}^b \leq 0 \). The Model 11 consists of an objective, namely, the maximization of the investors’ cumulative utility. The details of the problem are shown in Model 11, where constraint \( (a) \) denotes the wealth accumulation constraint; constraint \( (b) \) indicates that the investment amount of risk-free asset at period \( t \) must exceed the given lower bound; constraint \( (c) \) represents the desired number of assets in the portfolio must not exceed the given value \( K \); constraint \( (d) \) states threshold constraints of \( x_{it} \).

3. Time-consistent strategy for the multiperiod portfolio model. No researcher has studied the time-consistent strategy for the multiperiod mean semivariance model with transaction costs, borrowing constraints, threshold constraints and cardinality constraints. In this paper, we reformulate the problem into a game problem (Björk and Murgoci (2010)[6]). In this section, we will develop the
time-consistent strategy for the generalized multiperiod mean-semivariance portfolio model with the real constraints.

Let $u_{k-1}(X_{k-1}, u) = \sum_{t=k}^{T} u_t(E_{k-1}(X_t) - \eta_t \text{var}_{k-1}(X_t))$

According to the Definition 2.2 in Björk and Murgoci (2010)[6], the time-consistent strategy for Model 11 can be defined as follows.

**Definition 3.1.** Considering a fixed control law $u^{TC}(k-1)$. Let $u(k-1) = (u_{k-1}^{TC}, u_{k-1}^{TC}, \ldots, u_{T-1}^{TC})$, where $u_{k-1}$ is arbitrarily control variable. Then $u^{TC}(k-1)$ is said to be a time-consistent strategy if for all $k = 1, 2, \ldots, T$, the following conditions hold

$$\max_{u_{k-1}} u_{k-1}(X_{k-1}, u(k-1)) = u_{k-1}(X_{k-1}, u^{TC}(k-1))$$

where $u^{TC}(k-1) = (u_{k-1}^{TC}, u_{k-1}^{TC}, \ldots, u_{T-1}^{TC})$.

Björk and Murgoci (2010)[6] view the above setup as a noncooperative game. The above time-consistent strategy $u^{TC}(k-1)$ is said to be a sub-game perfect Nash equilibrium strategy. Under the framework of non-cooperative game, for an arbitrary initial point $(k-1, X_{k-1})$, suppose that there is one player, we refer to this as “player number $k$ at each time period $k-1$, and the rule presented in Definition 2 is that player number $k$ can only choose the strategy $u_{k-1}$ to maximize $u_{k-1}(X_{k-1}, u(k-1))$ given that his/her successors choose the equilibrium strategy $(u_{k-1}^{TC}, \ldots, u_{T-1}^{TC})$.

From the above definition, Model 11 can be reformulated as the following game models

$$(MV_k^{TC}) : \max \sum_{t=k}^{T} w_t(E_b(X_t) - \eta_t \text{var}_b(X_t))$$

subject to

$$\begin{cases}
X_j = r_{fj}X_{j-1} + \sum_{i=1}^{n} (r_{ij} - r_{fj})u_{ij} - \sum_{i=1}^{n} c_{ij} | u_{ij} - u_{i(j-1)} |, j = k \\
X_{j-1} - \sum_{i=1}^{n} u_{ij} \geq u_{fj} \\
\sum_{i=1}^{n} z_{ij} \leq K, z_{ij} \in (0,1) \\
l_{ij}z_{ij} \leq u_{ij} \leq p_{ij}z_{ij} \\
X_t = r_{ft}X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft})u_{it} - \sum_{i=1}^{n} c_{it} | u_{it} - u_{i(t-1)} |, t = 1, \ldots, T \\
X_{t-1} - \sum_{i=1}^{n} u_{it} \geq u_{ft} \\
\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in (0,1) \\
l_{it}z_{it} \leq u_{it} \leq p_{it}z_{it} \\
u_{it}^{TC} \text{solves}(MV_{it}^{TC}), t = k + 1, \ldots, T
\end{cases}
$$

The problem in the period $T$ is given as

$$(MV_T^{TC}) : \max w_T(E_{T-1}(X_T) - \eta_T \text{var}_{T-1}(X_T))$$

subject to

$$\begin{cases}
X_T = r_{fT}X_{T-1} + \sum_{i=1}^{n} (r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n} c_{iT} | u_{iT} - u_{i(T-1)} | \\
X_{T-1} - \sum_{i=1}^{n} u_{iT} \geq u_{fT} \\
\sum_{i=1}^{n} z_{iT} \leq K, z_{iT} \in (0,1) \\
l_{iT}z_{iT} \leq u_{iT} \leq p_{iT}z_{iT}
\end{cases}
$$
where \( E_k(X_T) \) and \( \text{svar}_k(X_T) \) respectively denote the conditional expectation and variance based on the information of the time period \( k, k = 1, \ldots, T \). Björk and Murgoci (2010)[6] showed that the so-called time consistent strategy means that optimal strategy obtained at time period \( m_1 \) agrees with that derived at time period \( m_2 \) where \( m_1 < m_2 \). It is easy to find that the strategies derived by Models 16 and 17 satisfy the time-consistency characteristics.

\[
\begin{align*}
&\ w_T[E_{T-1}(X_T) - \eta_T \text{svar}_{T-1}(X_T)] = \\
&\ w_T[E_{T-1}(r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n}c_{iT} \mid u_{iT} - u_{i(T-1)} \mid)] - \eta_T \text{svar}_{T-1}(r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n}c_{iT} \mid u_{iT} - u_{i(T-1)} \mid)] \\
&\ = w_T[(r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n}c_{iT} \mid u_{iT} - u_{i(T-1)} \mid)] - \eta_T \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijT}u_{iT}u_{jT} \\
\end{align*}
\]

According to Equation 14, the Model 13 can be turned into the follows:

\[
\begin{align*}
\text{max } r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n}c_{iT} \mid u_{iT} - u_{i(T-1)} \mid - \eta_T \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijT}u_{iT}u_{jT} \\
\text{s.t.} \\
X_T = r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n}c_{iT} \mid u_{iT} - u_{i(T-1)} \mid \\
X_{T-1} - \sum_{i=1}^{n} u_{iT} \geq u^b_{fT} \\
\sum_{i=1}^{n} z_{iT} \leq K, z_{iT} \in (0, 1) \\
l_{iT}z_{iT} \leq u_{iT} \leq p_{iT}z_{iT} \\
\end{align*}
\]

According to Björk and Murgoci (2010)[6] and Model 15, the Model 12 and Model 13 can be turned into as follows:

\[
\begin{align*}
\text{max } \frac{1}{T} \sum_{t=1}^{T} w_t[r_{fT}X_{t-1} + \sum_{i=1}^{n}(r_{it} - r_{fT})u_{it} - \sum_{i=1}^{n}c_{it} \mid u_{it} - u_{i(t-1)} \mid - \eta_T \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijT}u_{iT}u_{jT}] \\
\text{s.t.} \\
X_t = r_{fT}X_{t-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n}c_{iT} \mid u_{iT} - u_{i(T-1)} \mid \\
X_{t-1} - \sum_{i=1}^{n} u_{it} \geq u^b_{fT} \\
\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in (0, 1) \\
l_{it}z_{it} \leq u_{it} \leq p_{it}z_{it}, \quad i = 1, \ldots, n; t = 1, \ldots, T \\
\end{align*}
\]

The Model 16, which is a mix integer dynamic optimization with path dependence, is difficult to solve directly.

4. **Solution algorithm.** In this section, the Model 16 will be approximated into a dynamic programming problem. A novel discrete iteration method will be proposed to solve the problem. The linear convergence of the method will be proved.
4.1. The proposed model approximated to dynamic programming problem. The local/single-period optimization problem of Model 16 at period \( t \) is as follows:

\[
\max r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid - \eta_t \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} u_{it} u_{jt}
\]

s.t.

\[
\begin{align*}
X_t &= r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft} \\
\sum_{i=1}^{n} z_{it} &\leq K, z_{it} \in (0, 1) \\
l_{it} z_{it} &\leq u_{it} \leq p_{it} z_{it}, \quad i = 1, \ldots, n; t = 1, \ldots, T
\end{align*}
\]

(17)

Let \( u_{i(t-1)} = \bar{u}_{i(t-1)} \), where \( \bar{u}_{i(t-1)} \) is preset value, \( \sum_{i=1}^{n} \bar{u}_{i(t-1)} + f_{(t-1)} = X_{(t-2)} \), the Model 16 can be approximated into the following model:

\[
\max \sum_{t=1}^{T} w_t [r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid - \eta_t \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} u_{it} u_{jt}]
\]

s.t.

\[
\begin{align*}
X_t &= r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft} \\
\sum_{i=1}^{n} z_{it} &\leq K, z_{it} \in (0, 1) \\
l_{it} z_{it} &\leq u_{it} \leq p_{it} z_{it}, \quad i = 1, \ldots, n; t = 1, \ldots, T
\end{align*}
\]

(18)

The local/single-period optimization problem of Model 18 at period \( t \) is as follows:

\[
\max r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid - \eta_t \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} u_{it} u_{jt}
\]

s.t.

\[
\begin{align*}
X_t &= r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft} \\
\sum_{i=1}^{n} z_{it} &\leq K, z_{it} \in (0, 1) \\
l_{it} z_{it} &\leq u_{it} \leq p_{it} z_{it}, \quad i = 1, \ldots, n; t = 1, \ldots, T
\end{align*}
\]

(19)

\[\text{Theorem 4.1.} \quad \text{Let the optimal solution and objective function value of Model 17 and Model 19 respectively be } u_1^{*}, u_2^{*} \text{ and } g(u_1^{*}), f(u_2^{*}). \text{Then } g(u_1^{*}) - g(u_2^{*}) \leq 2 \sum_{i=1}^{n} c_{it} \mid u_{it} - \bar{u}_{i(t-1)} \mid.\]

\[\text{Proof.} \quad \text{Because the feasible solution set of the Model 17 is same as Model 19, } u_1^{*} \text{ and } u_2^{*} \text{ are, respectively the feasible solutions of Model 17 and Model 19. Then,}\]

\[g(u_1^{*}) \geq g(u_2^{*}) \text{ and } f(u_2^{*}) \geq f(u_1^{*}) \text{ that is } g(u_1^{*}) + f(u_1^{*}) \geq g(u_2^{*}) + f(u_2^{*}) \]

(20)

\[g(u_1^{*}) - g(u_2^{*}) + f(u_2^{*}) - f(u_1^{*}) \geq 0 \]

(21)
The left-hand side of Eq. 21 is
\[ g(u_1^*) - g(u_2^*) + f(u_1^*) - f(u_2^*) = \sum_{i=1}^{n} c_i |u_i^* - u_{i(t-1)}| - \sum_{i=1}^{n} c_i |u_i - u_{i(t-1)}| \]
\[ + \sum_{i=1}^{n} c_i |u_i^{2*} - u_{i(t-1)}| - \sum_{i=1}^{n} c_i |u_i^{2*} - u_{i(t-1)}| \leq 2 \sum_{i=1}^{n} c_i |u_{i(t-1)} - u_{i(t-1)}| \]
(22)

Which ends the proof.

Because \( c_{it} \ll r_{it} \), where asset \( i \in \text{efficient asset set of portfolio,} \)
\[ 2 \sum_{i=1}^{n} c_i |u_{i(t-1)} - u_{i(t-1)}| \] is small, that \( g(u_1^*) - g(u_2^*) \) is also small.

4.2. The smallest and biggest value of state variable at every period. In Model 19, investors can choose \( X_t \) between \( X_t^{\min} \) and \( X_t^{\max} \). \( X_t^{\min} \) and \( X_t^{\max} \) can be respectively obtained as follows:

The investor considers to maximize the expected return of the portfolio at period \( t \).
\[
\max r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_i |u_{it} - u_{i(t-1)}| \\
\begin{align*}
X_t &= r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_i |u_{it} - u_{i(t-1)}| \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft}^b \\
\sum_{i=1}^{n} z_{it} &\leq K, z_{it} \in (0, 1) \\
l_{it} z_{it} &\leq u_{it} \leq p_{it} z_{it}, \quad i = 1, \ldots, n; t = 1, \ldots, T
\end{align*}
\]
(23)

Let \( y_{it} = |u_{it} - u_{i(t-1)}| \). Then the Model 23 can be turned into as follows.
\[
\max r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_i y_{it} \\
\begin{align*}
X_t &= r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_i y_{it} \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft}^b \\
y_{it} &\geq u_{it} - u_{i(t-1)} \\
y_{it} &\geq -(u_{it} - u_{i(t-1)}) \\
\sum_{i=1}^{n} z_{it} &\leq K, z_{it} \in (0, 1) \\
l_{it} z_{it} &\leq u_{it} \leq p_{it} z_{it}
\end{align*}
\]
(24)

\( u_t^* \) (the optimal solution \( u_t = (u_{ft}, u_{1t}, u_{2t}, \ldots, u_{nt})' \) ) can be obtained solving Model 24 by the CPLEX method. Then, \( X_t^{\max} \) can be obtained as follows:
\[
X_t^{\max} = r_{ft} X_{t-1} + \sum_{i=1}^{n} r_{it} u_{it} - \sum_{i=1}^{n} c_i y_{it}, t = 1, \ldots, T
\]
where \( X_0 \) is initial wealth, which is preset value. The investor only considers to minimize the variance of the portfolio at period \( t \), that is, the \( X_{t-1} \) can be invested in risk-free asset.
4.3. The discrete iteration method. In this section, a novel discrete iteration method is proposed to solve the Model 18. The method is as follows: Firstly, according to the network approach, discretizes the state variables and transforms the model into multiperiod weighted digraph. Secondly, uses the max-plus algebra (Heidergott et al. (2006)) [19] to solve the largest path that is the admissible solution. Thirdly, based on the admissible solution, continues iterating until the two admissible solutions are real near. Finally, the method is proved linearly convergent.

The state variable $X_t$ of the period $t$ is discretized into four intervals of same widths from the smallest value to the biggest one. It means that there are five discrete values for the state variable in every period. In this way, Model 18 is transformed into a multiperiod weighted digraph as shown in Fig. 1. The investment period, the value of the objective function of the period $t$ and a discrete value of the state variable are respectively represented by the stage, the weight of the period $t$ and the point of the multiperiod weighted digraph.

Figure 1. The multiperiod weighted digraph

In this section, a discrete iteration method will be proposed to solve the Model 18.

**Step 1.** The discrete state variables at period $t (t = 1, \ldots, T)$ can be obtained by discretizing the interval value of $X_{t}^{\text{max}} - X_{t}^{\text{min}}$ into four equalities. That is

$$X_{it} = X_{t}^{\text{min}} + \frac{X_{t}^{\text{max}} - X_{t}^{\text{min}}}{4} i, i = 0, \ldots, 4$$

**Step 2.** The weight of the arcs in Figure 1 can be obtained as follows:

When $X_{jt}$ and $X_{k(t-1)}$ are known the sub-problem at period $t$ of the Model 18 can be turned into

$$\max r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} | u_{it} - u_{it(t-1)} | - \eta t \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijt} u_{it} w_{jt}$$

$$s.t.$$ 

$$X_{it} = r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} \sum_{i=1}^{n} c_{it} | u_{it} - u_{it(t-1)} |$$

$$X_{t-1} = \sum_{i=1}^{n} u_{it} \geq u_{ft}$$

$$\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in (0, 1)$$

$$l_{it} z_{it} \leq u_{it} \leq p_{it} z_{it}$$
Let $y_{it} = |u_{it} - u_{i(t-1)}|$. Then the Model 25 can be turned into as follows.

$$
\max r_{ft}X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft})u_{it} - \sum_{i=1}^{n} c_{it}y_{it} - \eta_t \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijt}u_{it}u_{jt}
$$

subject to:

$$
\begin{align*}
X_t &= r_{ft}X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft})u_{it} - \sum_{i=1}^{n} c_{it}y_{it} \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft}^b \\
y_{it} &\geq u_{it} - u_{i(t-1)} \\
X_{t-1} - \sum_{i=1}^{n} u_{it} &\geq u_{ft}^b \\
\sum_{i=1}^{n} z_{it} &\leq K, z_{it} \in (0, 1) \\
l_{it}z_{it} &\leq u_{it} \leq p_{it}z_{it} \\
y_{it} &\geq u_{it} - u_{i(t-1)} \\
y_{it} &\geq -(u_{it} - u_{i(t-1)})
\end{align*}
$$

(26)

$u^*_t$ (the optimal solution $u_t = (u_{ft}, u_{1t}, u_{2t}, \ldots, u_{nt})$) can be obtained solving Model 26 by the CPLEX method. Simultaneously, the objective function value $F_t(j, k)$ can be obtained as follows:

$$
F_{ft}X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft})u_{it}^* - \sum_{i=1}^{n} c_{it}y_{it}^* - \eta_t \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijt}u_{it}^*u_{jt}^*
$$

Step 3. Calculation of the longest path of the multiperiod weighted digraph

According to the max-plus algebra (Heidergott et al., 2006)[19], Zhang and Zhang (2014)[44], the longest path $F(1)$ of the multiperiod weighted digraph of the first iteration can be obtained as follows:

$$
F(1) = F(1) \otimes F(2) \otimes \ldots \otimes F(T)
$$

where $F(1) = (F^{(1)}(1, j))_{1 \times 5}, F(2) = (F^{(2)}(1, i, j))_{5 \times 5}, \ldots, F(T) = (F^{(T)}(1, j))_{5 \times 5}.$

Step 4. The discrete iteration method $k + 1$ th iteration can be described as follows:

Let the longest path of the $k$ th iteration be $X_0 \rightarrow X^{(k)}_{i1} \rightarrow X^{(k)}_{i2} \rightarrow \ldots \rightarrow X^{(k)}_{iT}$.

The optimal solutions of the longest path of Figure 1 are also feasible solutions of the multiperiod mean-absolute deviation portfolio selection model. Based on the $X_0, X^{(k)}_{i1}, X^{(k)}_{i2}, \ldots, X^{(k)}_{iT}$, the state variables from period 1 to period $T$ are discretized into four intervals as following three steps.

Step 4.1. $X_{2\text{min}}^{(k)}$ and $X_{2\text{max}}^{(k)}$ are discretized into two same internals, respectively. The five discrete points of $S_2$, i.e., $X^{(k)}_{2\text{min}}, X^{(k)}_{2\text{max}}, X^{(k)}_{2\text{mid}}, X^{(k)}_{2\text{mid}}, X^{(k)}_{2\text{max}}$ can be obtained.

Step 4.2. Based on $(X^{(k)}_{i3}, \ldots, X^{(k)}_{iT+1})$, using the same method, the state variables from period 3 to period $T + 1$ are respectively discretized into the five points. The utility of period $t$ can also be obtained.

Step 4.3. The longest path of the $k + 1$ th iteration $F^{(k+1)}$ and another feasible solution can be obtained as follows:

$$
F^{(k+1)} = F^{(k+1)} \otimes F^{(k+1)} \otimes \ldots \otimes F^{(k+1)}
$$

(28)
where $F^{(k+1)}_1 = (F^{(k+1)}_1(i,j))_{1 \times 5}, F^{(k+1)}_2 = (F^{(k+1)}_2(i,j))_{5 \times 5}, \ldots, F^{(k+1)}_T = (F^{(k+1)}_T(i,j))_{5 \times 5}.

If $|F^{(k+1)} - F^{(k)}| \leq 10^{-6}$, then the optimal solution of the longest path $F^{(k+1)}$ also is the approximately optimal solution of the Model 22. Otherwise $k = k + 1$ turn Step 4.1.

4.4. Convergence property of the discrete iteration method.

Theorem 4.2. Let $H_t^+$ be semi-positive definite, the discrete iteration method is linearly convergent.

Proof. Let the longest path in period 1 be $F^{\text{max}}_1(0,j_1)$, and the longest path in period $t$ be $F^{\text{max}}_t(i_{t-1}, j_t) t = 2, \ldots, T$. Then the upper bound of the solution of Model 18 is $F^{\text{max}}_1(0, j_1) \times F^{\text{max}}_2(i_1, j_1) \times \ldots \times F^{\text{max}}_T(i_{T-1}, j_T)$.

The longest path of the multiperiod weighted digraph of the $k$th iteration $F^{(k)}$ obtained as follows:

$$F^{(k)} = F^{(k)}_1 \otimes F^{(k)}_2 \otimes \ldots \otimes F^{(k)}_T$$

where $F^{(k)}_1 = (F^{(k)}_1(0,j))_{1 \times 5}, F^{(k)}_2 = (F^{(k)}_2(i,j))_{5 \times 5}, \ldots, F^{(k)}_T = (F^{(k)}_T(i,j))_{5 \times 5}.$

Let the longest path of the $k$th iteration be $X_0 \rightarrow X^{(k)}_{i_1} \rightarrow X^{(k)}_{i_2} \ldots \rightarrow X^{(k)}_{i_T}$.

Using the Step 4, the multiperiod weighted digraph of the $k + 1$th iteration can be obtained. The solution is becoming bigger and bigger. Because the solutions of Model 18 are increasing sets and there is an upper bound of the solution of Model 18. Then, the discrete iteration method is convergent.

Let the optimal value of period $t$ of Model 18 be $F^{*}_t, F^{(k)}_t$ be the optimal solution of the $k$th iteration at period $t$.

Because the objective function of Model 18 is concave, then $F^{(k+1)}_t \geq F^{*}_t \geq F^{(k)}_t$.

Because $F^{*}_t \geq F^{(k+1)}_t, F^{*}_t \geq F^{(k)}_t$, then $F^{*}_t - F^{(k+1)}_t \leq F^{*}_t - F^{(k)}_t$. So, $|F^{(k+1)}_t - F^{*}_t| \leq 1$. So the discrete iteration method is linearly convergent.

Thus, the proof of Theorem 4.2 is ended. □

5. Numerical examples. In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses thirty stocks from NASDAQ for his/her investment. The stocks codes are respectively S1 (AABA.O), S2 (AAAE.O), S3 (AAPPL.O), S4 (AON.O), S5 (AWWW.O), S6 (AAXN.O), S7 (CAMT.O), S8 (NTES.O), S9 (ABMD.O), S10 (ACAD.O), S11 (CGNX.O), S12 (ACHN.O), S13 (CHNA.O), S14 (ACI.W.O), S15 (CBK.A.O), S16 (INTC.O), S17 (ADBE.O), S18 (ADP.O), S19 (ADS.K.O), S20 (COKE.O), S21 (CTRP.O), S22 (GOOG.L.O), S23 (VRTX.O), S24 (MSFT.O), S25 (AGNC.O), S26 (AIMC.O), S27 (AMKR.O), S28 (SBUK.K.O), S29 (NDAQ.O), S30 (AMGN.O). We collect historical data of them from January 2009 to December 2018 and set every month as a period to handle the historical data. Investors intend to make five periods of investment with initial wealth $X_0 = 1,000,000$ dollars and their wealth can be adjusted at the beginning of each period. Suppose that the transaction costs of assets of the two periods investment take the same value $c_{it} = 0.003(i = 1, \ldots, 30, t =$
1, . . . , 5), $r_{ft} = 1.003$, the lower $l_{it} = 0$ and upper bound constraints $u_{it} = 200000$ dollars ($i = 1, \ldots , 30; t = 1, \ldots , 5$).

In order to examine the impact of the risk aversion coefficient $\eta$, the desired number of assets $K$ and the preset investment amount of risk-free asset $u_{ft}^b$, we compute the optimal solution, or terminal wealth of the following five problems.

(1) When $K = 8$, $u_{ft}^b = -500000$ dollars, the biggest and smallest terminal wealth ($X_t^{\max}$ and $X_t^{\min}$) can be respectively obtained as follows:

(2) When $K = 8$, $u_{ft}^b = -500000$ dollars, the preference coefficients $\eta = 0.000001$, the optimal solution of Model 18 can be obtained as follows:

(3) When $K = 6$ and $K = 8$, $u_{ft}^b = -500000$ dollars, $\eta = 0, 0.000001, \ldots , 0.08$, the terminal wealth of Model 18 can be respectively obtained as follows.

(4) When $K = 8$, $u_{ft}^b = -500000$ dollars and $u_{ft}^b = -100000$ dollars, $\eta = 0, 0.000001, \ldots , 0.000009$, the terminal wealth of Model (18) can be obtained as follows.

(5) When $K = 0, 1, \ldots , 13$, $u_{ft}^b = -100000$ dollars, $\eta = 0.000001$, the terminal wealth of Model 18 can be obtained as follows.

The expected return rate of asset $i$, $i = 1, \ldots , 30$, at period $t$, $t = 1, \ldots , 5$ can be obtained by the moving average method. The covariance matrix $\sigma_i^2 = (\sigma_{ijt})_{n \times n}$ can be obtained as follows:

$$\sigma_{ijt} = \frac{1}{m} \sum_{k=1}^{m} (R_{ik} - r_{it}) (R_{jk} - r_{jt})$$

(1) When $K = 8$, $u_{ft}^b = -500000$ dollars, $X_t^{\max}$ can be obtained as follows by using the CPLEX to solve the Model 24:

$$X_1^{\max} = 1046100, X_2^{\max} = 1093162, X_3^{\max} = 1142504, X_4^{\max} = 1191512, X_5^{\max} = 1238747.$$  

When investor put all money in the risk-free asset, the risk of portfolio is the smallest, i.e., the expected wealth is also the smallest.

$$X_1^{\min} = 1003000, X_2^{\min} = 1006009, X_3^{\min} = 1009027, X_4^{\min} = 1012054, X_5^{\min} = 1015090.$$  

Because $X_t$ is the amount of wealth at period $t$,

$$r_{ft} X_{t-1} + \sum_{i=1}^{n} (r_{it} - r_{ft}) u_{it} - \sum_{i=1}^{n} c_{it} \left| u_{it} - \overline{u}_{it(t-1)} \right| = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijt} u_{it} u_{jt}.$$  

(2) When $K = 8$, $u_{ft}^b = -500000$ dollars, $\eta = 0.000001$, we use the discrete approximate iteration method to solve the Model 18. We can obtain the corresponding results as follows.

When $K = 8$, $u_{ft}^b = -500000$ dollars, $\eta = 0.000001$, the optimal investment strategy at period 1 is $x_{31} = 200000.0, x_{61} = 200000.0, x_{71} = 200000.0, x_{81} = 200000.0, x_{91} = 200000.0, x_{101} = 200000.0, x_{111} = 100000.0, x_{281} = 200000.0$ and other risk asset 0, which means investor should allocate his initial wealth on asset 3, asset 6, asset 7, asset 8, asset 9, asset 10, asset 11, asset 28, and other risk asset by the amount of 200000.0, 200000.0, 200000.0, 200000.0, 200000.0, 200000.0, 200000.0, 100000.0, 200000.0 among the thirty stocks, respectively. From the Table 1, the optimal investment strategy for period 2, 3, 4 and 5 can be obtained. In this case, the available terminal wealth is 1235206.

(3) When $K = 6$, $K = 8$, $\eta = 0, 0.000001, \ldots , 0.08$, we use the discrete approximate iteration method to solve the Model 18. We can obtain the corresponding results as follows.

Where $X_5$ and $X_5^t$ are the terminal wealth of $K = 6$ and $K = 8$, respectively.

In Table 2, the experiments in this paper correspond to the values of $K = 6$, and $K = 8$, $u_{ft}^b = -500000$ dollars, $\eta = 0, 0.000001, \ldots , 0.08$. It can be seen that the
The optimal investment proportions

| t  | Asset 1 | Asset 2 | Asset 3 | Asset 4 | Asset 5 | Asset 6 | Asset 7 | Asset 8 | Asset 9 | Asset 10 | X1 |
|----|---------|---------|---------|---------|---------|---------|---------|---------|---------|----------|----|
| 1  | 100000  | 200000  | 0       | 200000  | 0       | 0       | 0       | 200000  | 0       | 200000   | 1044290|
| 2  | 100000  | 200000  | 0       | 200000  | 0       | 0       | 0       | 200000  | 0       | 200000   | 1901960|
| 3  | 100000  | 200000  | 0       | 200000  | 0       | 0       | 0       | 200000  | 0       | 200000   | 1140219|
| 4  | 100000  | 200000  | 0       | 200000  | 0       | 0       | 0       | 200000  | 0       | 200000   | 1188760|
| 5  | 100000  | 200000  | 0       | 200000  | 0       | 0       | 0       | 200000  | 0       | 200000   | 1235266|

Table 1. The optimal solutions when $K = 8$, $u_{ft}^b=-500000$ dollars, $\eta_t=0.000001$

| $\eta_t$ | 0 | 0.000001 | 0.000002 | 0.000003 | 0.000004 | 0.000005 | 0.000006 | 0.000007 | 0.000008 |
|----------|---|----------|----------|----------|----------|----------|----------|----------|----------|
| X5       | 1200337 | 1198835  | 1192648  | 1187153  | 1181143  | 1175134  | 1169125  | 1163116  |
| X6       | 1237168 | 1236862  | 1236556  | 1236250  | 1235944  | 1235638  | 1235332  | 1235026  |
| X7       | 1119367 | 1118019  | 1065707  | 1048836  | 1046399  | 1035338  | 1019634  | 1015952  | 1027745 |
| X8       | 1127173 | 1126253  | 1067599  | 1050035  | 1041300  | 1036057  | 1032563  | 1030067  | 1028194 |
| X9       | 1026339 | 1025214  | 1020151  | 1018366  | 1017621  | 1017139  | 1016767  | 1016526  | 1016347 |
| X10      | 1027090 | 1025575  | 1023031  | 1018585  | 1017711  | 1017871  | 1016837  | 1016588  | 1016400 |
| $\eta_t$ | 0.0009 | 0.001    | 0.002    | 0.003    | 0.004    | 0.005    | 0.006    | 0.007    | 0.008    |
| X11      | 1016208 | 1016095  | 1015893  | 1015425  | 1015033  | 1015281  | 1015747  | 1015354  | 1015216 |
| X12      | 1016256 | 1016139  | 1015941  | 1015441  | 1015052  | 1015600  | 1015256  | 1015211  | 1015221 |
| X13      | 0.049   | 0.04     | 0.03     | 0.04     | 0.05     | 0.06     | 0.07     | 0.08     | 0.09     |
| X14      | 1015281 | 1015195  | 1015141  | 1015123  | 1015113  | 1015113  | 1015106  | 1015106  | 1015106 |
| X15      | 1015207 | 1015195  | 1015142  | 1015126  | 1015117  | 1015117  | 1015106  | 1015106  | 1015106 |

Table 2. The terminal wealth when $K=6$, and $K = 8$, $u_{ft}^b=-500000$ dollars, $\eta_t=0, 0.000001, \ldots, 0.08$

terminal wealth becomes smaller, when the preset the preference coefficients $\eta_t$ in the interval $[0, 0.06]$ become larger, which reflects the influence of the parameter $h_t$ on portfolio selection; the terminal wealth is same, when the preset the preference coefficients $\eta_t \geq 0.06$. It can be seen that when the preference coefficients $\eta_t$ in the interval $[0, 0.05]$, the terminal wealth of $K = 8$ is more than $K = 6$; when the preference coefficients $\eta_t \geq 0.06$, the terminal wealth of $K = 6$ and $K = 8$ is same.

(4) When $K = 8$, $u_{ft}^b=-500000$ and $u_{ft}^b=-100000$, $\eta_t=0, 0.000001, \ldots, 0.000009$, we use the discrete approximate iteration method to solve the Model 18. We can obtain the corresponding results as follows.

Where $X_5$ and $X'_5$ denote the terminal wealth of $u_{ft}^b=-500000$ and $u_{ft}^b=-1000000$.

In Table 3, the experiments in this paper correspond to the terminal wealth, When $K = 8$, $u_{ft}^b=-500000$ and $u_{ft}^b=-1000000$, $\eta_t=0, 0.000001, \ldots, 0.000009$. It can be seen that the terminal wealth of $u_{ft}^b=-1000000$ is bigger than the terminal wealth of $u_{ft}^b=-500000$, when $\eta_t=0, 0.000001, \ldots, 0.000009$, which reflects the influence of the parameter $\eta_t$ on portfolio selection.
Table 3. The terminal wealth when $K = 8$, $u_{ft}^b = -500000$ and $u_{ft}^b = -1000000$, $u_t = 0, 0.000001, \ldots, 0.000009$

| $\eta$ | 0 | 0.0000001 | 0.0000002 | 0.00000003 | 0.0000004 |
|--------|---|----------|----------|------------|----------|
| $X_2$  | 1237168 | 1237167 | 1237031 | 1236828 | 1235819 |
| $X_4$  | 1240294 | 240294  | 1240154 | 1240154 | 1239692 |
| $\eta$ | 0.0000005 | 0.0000006 | 0.0000007 | 0.0000008 | 0.0000009 |
| $X_5$  | 1235560 | 1234737 | 1234664 | 1234642 | 1233682 |
| $X_5$  | 1239692 | 1237130 | 1237130 | 1237130 | 1237130 |

(5) When $K = 0, 1, \ldots, 13$, $u_{ft}^b = -500000$ dollars, $\eta = 0.0000001$, we use the discrete approximate iteration method to solve the Model 18. We can obtain the corresponding results as follows.

Table 4. The terminal wealth when $K = 0, 1, \ldots, 13$, $u_{ft}^b = -500000$ dollars, $\eta = 0.0000001$

| $K$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| $X_5$ | 1015090 | 1065669 | 1102726 | 1136562 | 1160553 | 1181499 | 1202122 |
| $K$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $X_5$ | 1221479 | 1240294 | 1258686 | 1275590 | 1284427 | 1284833 | 1284833 |

In the used data sets, the same optimal solutions with the $K=12$ is suitable for $K \geq 12$. The experiments in this paper correspond to the values of $K$ in the interval [0,12]. It can be seen that, as will be seen in Table 4, the terminal wealth also becomes larger, when $K$ in the portfolio become larger, which reflects the influence of $K$ on portfolio selection.

6. Conclusions. In this paper, a multiperiod mean semivariance portfolio selection model is proposed. The proposed model covered transaction costs, borrowing constraints, threshold constraints and cardinality constraints. Because of the transaction costs, the model is a dynamic optimization problem with path dependence. The optimal strategy does not satisfy time consistency, because the semivariance operator is not separable. In order to obtain the time-consistent optimal strategy, the proposed model is approximately turned into a dynamic programming problem by applying the approximate dynamic programming method and the game theory. A novel discrete approximate iteration method is designed to obtain the optimal time-consistent portfolio strategy, and is proved convergent. These results can unify many existing research on multiperiod mean semivariance portfolio problems. Several comparison analysis of trade-off parameters are given to illustrate the idea of our model and the effectiveness of the designed algorithm.

Our work can be extended to several directions. For example, we can further consider the time-varying risk aversion of multiperiod portfolio selection. We can also use VaR, CVaR, et al. to measure the risk of multiperiod portfolio selection. An interesting topic is to consider our model under different transaction costs. So the multiperiod portfolio models based on the means, other risk, the time-varying risk aversion and different transaction costs will be some future directions on the proposed approach in solving real life problems.
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E-mail address: zhangpeng300478@aliyun.com
E-mail address: 151499768@qq.com
E-mail address: chigt@dlut.edu.cn