Models of helically symmetric binary systems

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Abstract
Results from helically symmetric scalar-field models and first results from a convergent helically symmetric binary neutron-star code are reported here; these are models stationary in the rotating frame of a source with constant angular velocity \(\Omega\). In the scalar-field models and the neutron-star code, helical symmetry leads to a system of mixed elliptic–hyperbolic character. The scalar-field models involve nonlinear terms of the form \(\psi^3\), \((\nabla \psi)^2\) and \(\psi \Box \psi\) that mimic nonlinear terms of the Einstein equation. Convergence is strikingly different for different signs of each nonlinear term; it is typically insensitive to the iterative method used, and it improves with an outer boundary in the near zone. In the neutron-star code, one has no control on the sign of the source, and convergence has been achieved only for an outer boundary less than \(\sim 1\) wavelength from the source or for a code that imposes helical symmetry only inside a near zone of that size. The inaccuracy of helically symmetric solutions with appropriate boundary conditions should be comparable to the inaccuracy of a waveless formalism that neglects gravitational waves, and the (near zone) solutions we obtain for waveless and helically symmetric BNS codes with the same boundary conditions nearly coincide.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Initial data for the inspiral of binary neutron-star (BNS) systems and corresponding quasiequilibrium BNS models have been based either on the initial value equations alone or on the IWM (Isenberg–Wilson–Mathews) spatially conformally flat ansatz. In each case
one solves a truncated version of the Einstein equation for a metric having fewer than the six independent potentials that remain after a choice of gauge. The error of the approximation limits the accuracy of the first orbits of simulations in two ways: the initial data have spurious radiation, and, more importantly, the balance between gravitational attraction and orbital acceleration is not enforced, leading to orbits that are not exactly circular.

One way to go beyond spatial conformal flatness is to construct an analogue in the full theory of the Newtonian binaries that are stationary in a rotating frame. In general relativity, these models are helically symmetric spacetimes [1, 2], with equal amounts of ingoing and outgoing radiation. Binary black holes of this kind were first discussed by Blackburn and Detweiler [3], and models involving nonlinear scalar wave equations have been studied by a group of researchers organized by Price (henceforth the Consortium) [4–9]. Because the power radiated by a helically symmetric binary is constant in time, the spacetime cannot be asymptotically flat. At distances large compared to $1/\Omega$, however, the spacetime of a binary system with a helical Killing vector approximates asymptotic flatness—until, beyond about $10^3 M$ for neutron-star models of mass $M$, the enclosed energy in gravitational waves is comparable to the mass of the binary. The approximation is similar to, and possibly more accurate than, a 3rd post-Newtonian approximation in which the 2 1/2 post-Newtonian radiation is omitted.

In this paper, we report the construction of a convergent neutron-star code in which the full Einstein equation, together with the equation of hydrostatic equilibrium, is solved numerically under the assumption of helical symmetry. Convergence relies on a boundary that is not much farther than a wavelength from the system, and we present results from a number of related nonlinear scalar-field models in which convergence requires either a small coefficient of the nonlinear term or a boundary close to the source. The results of these toy models are surprising in two ways. First, convergence of the scalar-field models is most strongly affected by the sign of the source term, with one choice of sign yielding a convergent solution for remarkably large values of the nonlinear terms we examined. Second, convergence does not depend strongly on the iterative method used to solve the equation, on whether one uses, for example, a Newton–Raphson iteration or an iteration based on a Green function that inverts only a convenient part of the second-order nonlinear operator.

The plan of the paper is as follows. Section 2 introduces the set of toy scalar-field models with nonlinear terms chosen to mimic the nonlinear terms in the dynamical part of the Einstein equation. We describe several iteration methods, one closely related to that of the Uryu code, the others to codes developed by Andrade et al in [7] and further by Bromley, Owen and Price [9]. Section 3 compares the accuracy of codes based on the various iteration methods. Section 4 reports the main results on convergence of the scalar-field models. Finally, section 5 describes Uryu’s neutron-star code, presents its first results, and compares the solution it yields to that of a closely related code based on a waveless approximation.

2. Toy problems

2.1. Helically symmetric, nonlinear scalar wave equations

We consider a scalar field $\psi$ on Minkowski space, satisfying a wave equation with a source $s$ that mimics two objects in circular orbit and with a nonlinear term $\mathcal{N}[\psi]$, whose strength is adjusted by a coefficient $\lambda$:

$$\Box \psi - \lambda \mathcal{N}[\psi] = s.$$  (1)
We use three different nonlinear terms: \( N[\psi] = \psi^3, N[\psi] = |\nabla \psi|^2 \) with \( \nabla \) being the spatial gradient, and \( N[\psi] = \psi \Box \psi \), chosen to represent the types of nonlinear terms that appear in dynamical components of the field equations.

The source \( s \) is a sum of two three-dimensional Gaussian distributions,

\[
s(t, r, \theta, \phi) = \sum_{\pm} \frac{q}{\sqrt{(2\pi)^3}} \exp \left( -\frac{(r \pm R(t))^2}{\sigma^2} \right),
\]

centred about points \( \pm R, R(t) = a[\cos(\Omega t)\hat{x} + \sin(\Omega t)\hat{y}] \), each a distance \( a \) from the origin and each having spread \( \sigma^2 \) and total charge \( q \). The source \( s \) is stationary in a frame moving with angular velocity \( \Omega \); that is, it is Lie-derived by the helical Killing vector

\[
k^{a} = t^{a} + \Omega \phi^{a}
\]

do Minkowski space, where \( t^{a} \) and \( \phi^{a} \) (equivalently \( \partial_{t} \) and \( \partial_{\phi} \)) are generators of time translations and of rotations in the plane of the binary source.

One can regard the scalar-field models as toy models of neutron stars of mass \( M \), if the charge \( q \) of each Gaussian source is identified with \( 4\pi M \). In gravitational units \( (G = c = 1) \), all quantities of a binary star system can be specified in terms of \( M \). In the models presented below, we set \( q = 1, a = 1, \sigma = 0.5 \) and \( \Omega = 0.3 \), corresponding to a binary system of mass \( M \), stellar separation \( 2a = 8\pi M \), stellar radius \( \sigma = 2\pi M \) and velocity \( v = a\Omega = 0.3 \).

A helically symmetric solution, like a genuinely stationary solution, is given by its value \( \psi(t, r, \theta, \phi) \), with angular velocity \( \Omega \), that is, it is Lie-derived by the helical Killing vector \( \psi(t, r, \theta, \phi) = s(t = 0, r, \theta, \phi) = s(t = 0, r, \theta, \phi) \).

The operator \( \mathcal{L} := \nabla^2 - \Omega^2 \partial_{\phi}^2 \) has a mixed character, elliptic inside the light cylinder \( \Omega \sqrt{x^2 + y^2} = 1 \), hyperbolic outside. The difficulties in finding numerical solutions stem from this behaviour. In finding an iterative solution, one inverts the operator \( \mathcal{L} \), but \( \mathcal{L} \) is not negative, and it lacks the contraction property that underlies the convergence of iterative schemes used to invert nonlinear elliptic equations and to prove existence of exact solutions.

### 2.2. Numerical methods

#### 2.2.1. KEH method

In the KEH method one splits off the linear, flat-space operator \( \mathcal{L} \) and inverts it by a choice \( \mathcal{L}^{-1} \) of the Green function. The iterative solution, beginning with \( \Psi_{0} = \mathcal{L}^{-1}S \), is then given by

\[
\Psi_{n+1} = \Psi_{0} + \lambda \mathcal{L}^{-1}N[\Psi_{n}].
\]

Although \( \mathcal{L} \) is not elliptic, the operator associated with each spherical harmonic is the elliptic Helmholtz operator: \( \mathcal{L}[\Psi_{lm}(r)Y_{lm}] = [\nabla^2 + m^2\Omega^2] \Psi_{lm}(r)Y_{lm} \). The corresponding radial operator \( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} = \frac{l(l+1)}{r^2} + m^2\Omega^2 \) has as its eigenfunctions the spherical Bessel functions, from which a Green function is constructed. For definiteness, we choose as \( \mathcal{L}^{-1} \) the form that, for a bounded source, yields the half-advanced + half-retarded solution, namely

\[
\mathcal{L}^{-1}S := \sum_{lm} \int dr' S_{lm}(r') g_{lm}(r', r) Y_{lm}(\theta, \phi),
\]
with
\[ g_{l0} = \frac{1}{2l + 1} \frac{r_<^l}{r_>^l}, \quad g_{lm} = m \Omega j_l(m \Omega r_<) n_l(m \Omega r_>, \quad m \neq 0. \] (7)

Here \( r_\ast = \min(r, r'), r_\ast = \max(r, r') \), and \( j_l(x) \) and \( n_l(x) \) are the spherical Bessel functions of the first and second kinds.

At each iteration, the nonlinear term \( N[\Psi_n] \) serves as an effective source. The polynomial nonlinear function \( \Psi^3 \) is most easily computed by shifting from \( \Psi_{lm}(r) \) back into \( \Psi(r, \theta, \phi) \) and cubing at each point, while the derivative-based nonlinear terms, \( |\nabla \Psi|^2 \) and \( \Psi \Box \Psi \), are calculated using the properties of the spherical harmonics.

Finally, as is usual in codes to solve nonlinear elliptic equations, we use softening and continuation to extend the range of convergence to larger values of \( \lambda \). That is, instead of using \( \Psi_{n+1} \) as defined in equation (5), we can use a softened \( \Psi_{n+1}^\omega \) defined by
\[ \Psi_{n+1}^\omega = \omega \Psi_{n+1} + (1 - \omega) \Psi_n. \] (8)

Given a converged field solution for some nonlinearity with small \( \lambda \), it is sometimes possible to use continuation to obtain a solution for larger \( \lambda \); the converged solution to equation (4) with weak nonlinearity is used as the initial field \( \Psi_0 \) for the iteration of equation (5) for strong nonlinearity. In this way one moves along a sequence of solutions with increasing values of \( \lambda \). The effectiveness of softening and convergence is explored in section 4.

2.2.2. Finite difference and eigenspectral methods. The finite difference code uses an iteration based on the Newton–Raphson method, with numerical approximations that reduce it in part to a secant method. Write the equation to be solved as
\[ F = L\Psi - \lambda N[\Psi] - S = 0. \] (9)

Numerically, \( \Psi \) is given by a set of values \( \Psi_i \) on the three-dimensional spatial slice. Given an initial field \( \Psi_i \), each iterative step generates a modification \( \delta \Psi_i \) by inverting
\[ J_{ij} \delta \Psi_j = -F_i \] (10)

where \( J_{ij} \) is the Jacobian
\[ J_{ij} = \frac{\partial F_i}{\partial \Psi_j}. \] (11)

The Helmholtz operator has the form \( (L\Psi)_i = L_{ij} \Psi_j \) where \( L_{ij} \) is constructed from finite difference operations and incorporates boundary conditions. The corresponding part of \( J_{ij} \) is simple.
\[ J_{ij} = \frac{\partial}{\partial \Psi_j} [L_{ik} \Psi_k - \lambda (N[\Psi])_i - S_i] \] (12)
\[ = L_{ij} - \lambda \frac{\partial N[\Psi]_i}{\partial \Psi_j}. \] (13)

The nonlinear piece of the Jacobian is evaluated numerically by varying local field values.

The eigenspectral code [9] uses the same iterative scheme as the finite difference method, but it employs adapted coordinates and a discretized spectral decomposition. To specify the adapted coordinate system, we begin with comoving Cartesian coordinates \((\tilde{x}, \tilde{y}, \tilde{z})\) where the \( \tilde{z} \)-axis is the axis of rotation. The axes are rotated to a set
\[ \tilde{X} = \tilde{y}, \quad \tilde{Y} = \tilde{z}, \quad \tilde{Z} = \tilde{x} \] (14)
for which the $\hat{Z}$-axis is a line through the centre of each source. The adapted coordinates are chosen to approach spherical polar coordinates, with $\Theta$ measured from the $\hat{Z}$-axis, far from the sources. For each point $(\hat{X}, \hat{Y}, \hat{Z})$, let $r_1$ and $r_2$ be the distances from the source centres, $\theta_1$ and $\theta_2$ corresponding angles from the $\hat{Z}$-axis. Then,

$$ r_1 = \sqrt{(\hat{Z} - a)^2 + \hat{X}^2 + \hat{Y}^2}, $$

$$ r_2 = \sqrt{(\hat{Z} + a)^2 + \hat{X}^2 + \hat{Y}^2}. $$

The adapted coordinates are

$$ \chi = \sqrt{r_1 r_2} $$

$$ \Theta = \frac{1}{2} (\theta_1 + \theta_2) = \frac{1}{2} \tan^{-1} \left( \frac{2 \hat{Z} \sqrt{\hat{X}^2 + \hat{Y}^2}}{\hat{Z}^2 - a^2 - \hat{X}^2 - \hat{Y}^2} \right) $$

$$ \Phi = \tan^{-1}(\hat{X}/\hat{Y}). $$

The spectral decomposition involves the angular Laplacian of $\Theta$ and $\Phi$,

$$ D^2 \Phi := \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial}{\partial \Theta} \right) + \frac{1}{(\sin \Theta)^2} \frac{\partial^2}{\partial \Phi^2}; $$

note that $D^2$ is not the angular part of $\nabla^2$ in adapted coordinates, but it agrees asymptotically with the usual angular Laplacian far from the source. Instead of the spherical harmonics of the continuum Laplacian $D^2$, the eigenspectral code uses the exact eigenvectors of the matrix $L$ obtained by discretizing $D^2$ on the adapted coordinate grid $(\Theta_i, \Phi_j)$. Angular derivatives are represented in $L$ by second-order finite differencing. That is, with $[\sin \Theta D^2 \Phi](\Theta_a, \Phi_b)$ approximated by $\sum_{ij} L_{ab,ij} \Psi(\Theta_i, \Phi_j)$, the normalized eigenvectors $Y_{ij}^{(k)}$ of $L$ satisfy

$$ \sum_{ij} L_{ab,ij} Y_{ij}^{(k)} = -\Lambda^{(k)} \sin \Theta_a Y_{ab}^{(k)}, $$

$$ \sum_{ij} Y_{ij}^{(k)} Y_{ij}^{(k')} \sin \Theta_i \Delta \Theta \Delta \Phi = \delta_{kk'}. $$

With the field expanded in terms of the eigenvectors $Y_{ij}^{(k)}$,

$$ \Psi(\chi, \Theta_i, \Phi_j) = \sum_k a_k^{(k)}(\chi) Y_{ij}^{(k)}, $$

and the operator $L$ written in terms of adapted coordinates (equations (8)–(17) of [9]), $L \Psi$ is expressed in terms of the $Y_{ij}^{(k)}$ (equation (31) of [9]),

$$ \sum_k \left( \alpha_k^{(k)} \frac{d^2 a^{(k)}(\chi)}{d\chi^2} + \beta_k^{(k)} \frac{da^{(k)}(\chi)}{d\chi} + \gamma_k^{(k)} \frac{da^{(k)}(\chi)}{d\chi} \right) = S_k, $$

where the $\alpha_k^{(k)}, \beta_k^{(k)}$ and $\gamma_k^{(k)}$ involve angular derivatives of the $Y^{(k)}$ computed by finite differencing (equations (41)–(43) of [9]). This equation for $a^{(k)}$ is iterated in the same fashion as in the finite difference method to find the solution with nonlinear terms.

If all harmonics were retained, the eigenspectral method would be the equivalent of the finite difference method in adapted coordinates, although in a different basis. The advantage of the adapted coordinates is that the distribution of points encodes most of the physically relevant information in the low-order harmonics. Its disadvantage is that higher order harmonics require an increasingly cumbersome formalism. The code is consequently limited to harmonics $l \leq 2$, saving enough memory to allow high resolution in the radial coordinate.
2.3. Boundary conditions

The KEH method at each iteration finds a solution that has standing wave behaviour for the flat space piece, imposing boundary conditions by the choice of the Green function. There remains freedom to add a homogeneous solution at each iteration, and this has been used to study the sensitivity of convergence to the choice of boundary condition.

The eigenspectral and finite difference methods include boundary conditions in the finite difference matrix for the linear $\square$ operator. Outgoing conditions are imposed on the edges of the grid by enforcing

$$\left( \partial_{\gamma} \Psi - \partial_{\phi} \Psi \right)_{r=r_{\text{max}}} = 0.$$  \hfill (25)

A solution for ingoing radiation can then be generated by a spatial inversion across the plane through the sources and perpendicular to their rotation. At each step, a periodic solution is constructed by superposition of the ingoing and outgoing solutions.

3. Estimating numerical accuracy

3.1. KEH code

Several types of numerical approximation in the KEH code produce effects that can be estimated by convergence testing. Most obvious is the choice of spatial grid in $(r, \theta, \phi)$ on which numerical integration is performed. Very high resolution in $\theta$ and $\phi$ is easily obtained. The number of radial points is more problematic with our simple equispaced grids. A Richardson extrapolation error estimate from varying radial grid spacing shows that precision of $10^{-5}$ is estimated for runs comparing code results. Estimation of the range of $\lambda$ giving convergent solutions is done at lower radial precision.

As the KEH method rests on the decomposition of the field into spherical harmonics, accuracy will depend on the number of harmonics retained in the numerical calculation. Examining the difference between results with an increasing number of harmonics retained shows that a good approximation is to use up to $l = 12$ in the code. The fractional difference between the field calculated with $l_{\text{max}} = 12$ and the field including higher harmonics is less than $10^{-6}$ at each point. To conform to the restriction on paper length, we have relegated to a web version (in [10]) a graphical display of the error as a function of the maximum number of harmonics used.

In the toy models presented below, a solution is computed at each iteration using the half-advanced + half-retarded Green function (6). Because the linear field $\Psi$ of a perpetually radiating source falls off like $r^{-1}$, the nonlinear terms $N = (\nabla \Psi)^2$ and $N = \Psi \square \Psi$ that serve as effective sources for each iteration do not fall off fast enough for the integrals to converge, if the outer boundary extends to infinity. One can, however, pick out a solution that is independent of the outer boundary by fixing the value of $\Psi$ at a finite radius $R$. That is, one can, at each iteration, add the homogeneous solution that maintains the specified value of $\Psi$ at $R$. This has remarkably little effect on the field in the region close to the sources, with less than a 1% change in field strength for points with $r < 6$. This insensitivity of the near-zone field to the amplitude of the waves, when the source dominates the solution, is the reason a helically symmetric solution makes sense as an approximation to an outgoing solution.

3.2. Comparing codes

With different 3D grid patterns for the codes, it was most straightforward to compare results on rays through the volume of interest. We compared the results extrapolated to three orthogonal
Figure 1. The scalar field $\Psi$ as a function of distance $r$ from the origin in units of the orbital radii along the source axis. Each of the four panels corresponds to a unique model, as specified by the form of the nonlinear term, $N[\Psi]$, written above it. In all cases, the angular frequency of rotation is $\Omega = 0.3$ and the source strengths are unity. The plots show results from the KEH method (solid curves), the eigenspectral method (dashed) and the finite difference code (dotted). The insets give more detailed comparisons between these results; the curves are the difference between pairs of solutions relative to the average field. Note that the eigenspectral method gives a periodic modulation relative to the other methods, as a result of the use of only low-order harmonics.

axes: taking the $x$-axis through the centres of the two sources and the $z$-axis perpendicular to the plane of rotation. Both $x$- and $y$-axes show wave behaviour away from the sources; along the $z$-axis, $\psi$ shows only Coulomb-type behaviour, because $Y_{lm}$ vanishes on the axis when $m \neq 0$, and for $m = 0$, $\mathcal{L} \psi = \nabla^2 \psi$.

Sample comparison plots are shown in figure 1. Further comparisons are shown in [10].

The field values on the rays were interpolated for comparison. Differences between fields were divided by the average field value of the three codes to find the relative error, as plotted in the insets of figure 1. We computed the rms of this relative error for the grid points along each ray. These rms values are expressed as percentages in table 1. In the worst case, the rms difference is 3% between codes.

One might note that a smaller rms difference on the rotation axis than on the source and perpendicular axes indicates that discrepancies in the wave region dominate, as in the FD–KEH comparison with $N[\Psi] = \Psi^3$, $\lambda = 100$. In other cases, the error on all three axes is comparable, and there is some shift between codes seen even on the rotation axis, as in the FD–KEH comparison with $N[\Psi] = |\nabla \Psi|^2$, $\lambda = 3$. 
Table 1. A comparison of code output for scalar models. The values in the table give the percent rms difference between of $\Psi$-values from three numerical methods as in figure 1. The rms differences were computed from the points along each principal axis.

| Source axis | Perp. axis | Rotation axis |
|-------------|------------|---------------|
| Linear ($\nabla\Psi = 0$) | | |
| FD to KEH | 0.25 | 0.24 | 0.10 |
| FD to ES | 1.02 | 1.09 | 1.08 |
| ES to KEH | 1.07 | 1.15 | 1.11 |
| $\nabla\Psi = \Psi^3, \lambda = 100$ | | |
| FD to KEH | 3.56 | 3.44 | 0.21 |
| FD to ES | 1.22 | 1.67 | 1.14 |
| ES to KEH | 3.28 | 2.86 | 1.19 |
| $\nabla\Psi = \Psi^3, \lambda = -2.2$ | | |
| FD to KEH | 0.35 | 0.36 | 0.21 |
| FD to ES | 1.96 | 2.02 | 2.08 |
| ES to KEH | 1.77 | 1.99 | 1.92 |
| $\nabla\Psi = |\nabla\Psi|^2, \lambda = 3$ | | |
| FD to KEH | 1.73 | 1.47 | 1.33 |
| FD to ES | 1.39 | 1.80 | 1.44 |
| ES to KEH | 2.69 | 2.76 | 2.77 |
| $\nabla\Psi = |\nabla\Psi|^2, \lambda = -100$ | | |
| ES to KEH | 1.96 | 2.20 | 1.65 |
| $\nabla\Psi = \Psi \nabla\Psi, \lambda = -0.7$ | | |
| FD to KEH | 0.53 | 0.48 | 0.20 |
| FD to ES | 1.06 | 1.13 | 1.13 |
| ES to KEH | 0.99 | 1.23 | 1.00 |

4. Measuring range of convergence

For each code, a series of runs varying the nonlinear amplitude $\lambda$ determines the range of $\lambda$ for which the code gives a convergent solution. The source distribution and boundary location are fixed.

Softening and continuation are used to extend the maximum absolute value of $\lambda$ that allows convergence. For each nonlinear term, strikingly different behaviour is found for opposite signs of $\lambda$. That is, denoting by $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ the largest positive value and the smallest negative value of $\lambda$ for which a code converges, we find for each nonlinear term that $\lambda_{\text{max}}$ and $|\lambda_{\text{min}}|$ differ by at least a factor of 100. In each case, the sign of $\lambda$ that yields greatest convergence is opposite to the sign of the source term, where the source is large. The ‘favourable’ sign damps the effect of the source distribution on the field, while the ‘unfavourable’ sign gives an amplified effective source. For $\nabla\Psi$ and $|\nabla\Psi|^2$, the favourable signs of lambda are $+,-,+$, respectively.

Softening and continuation improved the range of convergence for the favourable sign, but had no effect on the limit of the unfavourably signed $\lambda$. Figure 2 shows the effect of softening, parametrized by $\omega$, and of using continuation on the limiting favourable-sign values of $\lambda$.

Convergence is attained, when softening and continuation are used, for all attempted favourable sign values for $\lambda$, except when using the KEH method with the $\Psi \nabla\Psi$ nonlinear term or when using the finite difference method with the $|\nabla\Psi|^2$ nonlinear term.
The range of convergence results are given in table 2. If softening and continuation maintain convergence to the largest tested value of $\lambda$, the uncertainty in the true maximum value or minimum value of $\lambda$ is indicated by $>$ or $<$. 

4.1. Boundary and convergence

As described in section 5 below, Uryu has obtained a convergent helically symmetric BNS code within a region that extends about one wavelength beyond the centre of the source, by using a waveless formulation outside that radius. Led by this result, we explored the effects of the outer boundary placement on the range of converging $\lambda$ for the fully helical code.

While bringing the boundary in had little effect on the limits of the favourably signed $\lambda$, it did allow a significantly larger magnitude of the unfavourable $\lambda$, as shown in figure 3.

5. Helically symmetric binary neutron-star code

In this section, we report the construction of a first helically symmetric code for binary neutron stars. The code modifies a recently developed numerical method for computing models of compact binary stars in a waveless approximation [10] to produce solutions to the full Einstein–Euler system (the Einstein-perfect fluid equations) with exact helical symmetry. Waveless and helically symmetric solutions are expected to be accurate in the near zone, where the gravitational wave amplitude is small compared to the Coulomb contribution to each metric potential, and, in its present form, the helically symmetric code converges only in
Figure 3. The effect of the outer boundary \( R \) on convergence in the cubic nonlinear model \( \Lambda[\Psi] = \Psi^3 \). The limiting value of \( \lambda \), for which the KEH method converges as a function of outer boundary location, \( R \), in units of orbital radii.

Table 2. The range of nonlinear amplitude \( \lambda \) in scalar models for which codes converged. The first column gives the type of nonlinearity, while the second column indicates the numerical method, as discussed in the text. The third and fourth columns, showing \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \), give the range of \( \lambda \) for which convergence was achieved. Where an inequality is given, no limiting value was found.

| \( \Lambda[\Psi] \) | Code | \( \lambda_{\text{min}} \) | \( \lambda_{\text{max}} \) |
|------------------|------|----------------|----------------|
| \( \Psi^3 \)     | KEH  | -2.3           | 11             |
| \( \Psi^3 \)     | KEH, softened/continued | -2.3           | >1000          |
| \( \Psi^3 \)     | FD   | -2.5           | >1000          |
| \( \Psi^3 \)     | ES   | -2.4           | >1000          |
| \( |\nabla\Psi|^2 \) | KEH  | -30            | 7.7            |
| \( |\nabla\Psi|^2 \) | KEH, softened/continued | <-1000          | 7.7            |
| \( |\nabla\Psi|^2 \) | FD   | -2.0           | 5.6            |
| \( |\nabla\Psi|^2 \) | ES   | -360           | 7.3            |
| \( |\nabla\Psi|^2 \) | ES, continued | <-1000          | 7.3            |
| \( \Psi \Box \Phi \) | KEH  | -0.96          | 1.3            |
| \( \Psi \Box \Phi \) | KEH, softened/continued | -0.96          | 10             |
| \( \Phi \Box \Psi \) | FD   | -1.8           | >1000          |
| \( \Phi \Box \Psi \) | ES   | -1.7           | >1000          |

the near zone. Convergence of the code is achieved by using the waveless formulation outside one wavelength. In effect, the outer waveless solution imposes boundary conditions on the helically symmetric solution at the edge of the near zone.

5.1. Formulation

Our formulation is applied to Einstein’s equation written in \( 3+1 \) form, with spacetime \( \mathcal{M} = \mathbb{R} \times \Sigma \). Let \( n^\alpha \) be the future-pointing unit vector normal to the slices \( \{ t \} \times \Sigma \) of this foliation. The generator of the time coordinate \( t^\alpha \) and the helical symmetry vector \( k^\alpha = t^\alpha + \Omega^\alpha \) are related to \( n^\alpha \) and the shift \( \beta^\alpha \) in the usual way,

\[
t^\alpha = \alpha n^\alpha + \beta^\alpha \quad \text{and} \quad k^\alpha = \alpha n^\alpha + \omega^\alpha,
\]

where \( \omega^\alpha \) is a rotating shift vector \( \omega^\alpha = \beta^\alpha + \Omega \phi^\alpha \).

The projection 4-tensor \( \gamma_{\alpha\beta} \) orthogonal to \( n_\alpha \), defined by

\[
\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta,
\]

(27)
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is associated with the 3-metrics $\gamma_{ab}(t)$ on the spatial slices $\Sigma_t$. (Any spatial tensor on $\Sigma_t$ can be identified with a spacetime tensor orthogonal to $n^a$ on all its indices.) The metric then has the form,

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$  \hfill (28)

The extrinsic curvature of each slice $\Sigma$ is defined by

$$K_{ab} = -\frac{1}{\alpha} \mathcal{L}_n \gamma_{ab},$$  \hfill (29)

where $\mathcal{L}_n$ operating on spatial tensors such as $\gamma_{ab}$ has the meaning

$$\mathcal{L}_n \gamma_{ab} := \frac{1}{\alpha} \partial_t \gamma_{ab} - \frac{1}{\alpha} \mathcal{L}_\beta \gamma_{ab},$$  \hfill (30)

with $\partial_t \gamma_{ab}$ being the pullback of $\mathcal{L}_t \gamma_{ab}$ to $\Sigma$. We introduce a conformal decomposition of the spatial metric, $\gamma_{ab} = \psi^4 \tilde{\gamma}_{ab}$, and a condition $\tilde{\gamma} = f$, where $f$ is the determinant of the flat metric $f_{ab}$ and $\tilde{\gamma}$ the determinant of the conformal metric $\tilde{\gamma}_{ab}$. Further $h_{ab}$ and $h^{ab}$ are introduced by

$$\tilde{\gamma}_{ab} = f_{ab} + h_{ab}, \quad \tilde{\gamma} = f + h^{ab}.$$  \hfill (31)

Helical symmetry, $\mathcal{L}_k \gamma_{ab} = 0$, implies for the 3-metric and extrinsic curvature,

$$\mathcal{L}_k \gamma_{ab} = 0, \quad \mathcal{L}_k K_{ab} = 0.$$  \hfill (32)

Using the relation (26) between $n^a$ and $k^a$, we have

$$\mathcal{L}_n \gamma_{ab} = -\frac{1}{\alpha} \mathcal{L}_n \gamma_{ab}, \quad \text{and} \quad \mathcal{L}_n K_{ab} = -\frac{1}{\alpha} \mathcal{L}_n K_{ab}.$$  \hfill (33)

Projections of the Einstein equation along the normal $n^a$ are the Hamiltonian and momentum constraints,

$$(G_{ab} - 8\pi T_{ab}) n^a n^b = 0,$$  \hfill (34)

$$(G_{ab} - 8\pi T_{ab}) \gamma_{ab} = 0,$$  \hfill (35)

while the spatial projection has a trace and trace-free part

$$(G_{ab} - 8\pi T_{ab}) \gamma_{ab} = 0,$$  \hfill (36)

$$(G_{ab} - 8\pi T_{ab}) \bar{\gamma}_{ab} = 0.$$  \hfill (37)

The above set of equations are solved for the conformal factor $\psi$, the shift $\beta^a$, the lapse $\alpha$ and the deviation of the conformal metric from the flat metric $h_{ab}$, respectively, imposing coordinate conditions, the maximal slicing condition, $K = 0$, and the generalized Dirac condition $D_t \bar{\gamma}_{ab} = 0$, as in the waveless formulation \cite{11}. In the following section, we concentrate on the treatment of the spatially trace-free part of Einstein equation (37), solved for $h_{ab}$, in which the second time derivatives of $h_{ab}$ change the equation from elliptic (in the waveless formulation) to the mixed form that characterizes helically symmetric wave equations. A concrete treatment of the other components of the field equations that yield elliptic equations for $\psi$, $\beta^a$ and $\alpha$, as well as equations of matter source is given in \cite{11}.

The waveless code differs from the helically symmetric code by the requirement that $\tilde{\gamma}_{ab}$ have vanishing derivative along $t^a$ instead of vanishing derivative along $k^a$. The extrinsic curvature and matter variables are still required to be helically symmetric. These requirements result in elliptic equations for the field variables, including the non-conformal part of the 3-metric $h_{ab} = \tilde{\gamma}_{ab} - f_{ab}$.

In the helically symmetric code, with $\partial_t \tilde{\gamma}_{ab}$ non-zero, the elliptic equations are replaced by Helmholtz equations for $h_{ab}$, whose source is almost identical to that of the waveless formulation. We implement a KEH solver for the helical formulation and investigate convergence of the iteration for a compact binary star source.
5.2. Helmholtz equation for $h_{ab}$

Equation (37) has the form

$$ (G_{ab} - 8\pi T_{ab}) \left( \gamma^a \gamma^b - \frac{1}{3} \gamma_{ab} \gamma^{ab} \right) = \mathcal{E}_{ab} - \frac{1}{3} \gamma_{ab} \gamma^{cd} \mathcal{E}_{cd} = 0, \quad (38) $$

where

$$ \mathcal{E}_{ab} := -\xi_{ab} + R_{ab} + K K_{ab} - 2 K_{ac} K^c_{\, ab} - \frac{1}{\alpha} D_a D_b \alpha - 8\pi S_{ab}, \quad (39) $$

with $\bar{R}_{ab}$ the Ricci tensor on $\Sigma$ associated with $\gamma_{ab}$, and $S_{ab}$ the projection of the energy stress tensor, $S_{ab} := T_{ab} \gamma^a \gamma^b$.

By isolating the terms, $\Box h_{ab} := (-\partial^2 + \hat{D} \hat{\Delta}) h_{ab}$, that occur in $\mathcal{E}_{ab}$ in equation (39), one can rewrite equation (38) in the form

$$ \Box h_{ab} = \mathcal{L}_{ab}, \quad (40) $$

where $\Box$ is the flat d’Alembertian operator and $\hat{D} \hat{\Delta} := f^{ab} D_b$. Then, as in the scalar models, helical symmetry of the conformal metric, $\xi \gamma_{ab} = \xi h_{ab} = 0$, results in the operator

$$ \Box h_{ab} = -\partial^2 h_{ab} + \hat{D} \hat{\Delta} h_{ab} = (\hat{\Delta} - \Omega^2 \xi^2) h_{ab}, \quad (41) $$

where the flat Laplacian is defined by $\hat{\Delta} = \hat{D} \hat{\Delta}$.

Even when one uses the Cartesian components $h_{ij}$ of $h_{ab}$, however, $\hat{\Delta} - \Omega^2 \xi^2$ does not coincide with the Helmholtz operator, because $\partial_i h_{ij}$ is a Cartesian component of $\phi \cdot D h_{ab}$, not of $\xi h_{ab}$. To isolate the Helmholtz operator $\mathcal{L} = \hat{\Delta} - \Omega^2 \xi^2 = \hat{\Delta} - \Omega^2 (\phi \cdot D)^2$, we write $\partial_i := \phi \cdot D$ and find

$$ \xi \tilde{\gamma}_{ab} = \partial_i \tilde{\gamma}_{ab} + (\partial_i \tilde{\gamma}_{ac} + \xi \tilde{\gamma}_{ac}) D_b \phi^c + (\partial_i \tilde{\gamma}_{cb} + \xi \tilde{\gamma}_{cb}) D_a \phi^c, \quad (42) $$

where a relation $\phi D_i (D_a \phi^b) = 0$ is used. Moving all terms in equation (42) except $\partial_i \tilde{\gamma}_{ab}$ from the LHS to the RHS of equation (40), we obtain the Helmholtz form

$$ \mathcal{L} h_{ab} := \mathcal{S}_{ab} := 2 \left( \delta_{ab} - \frac{1}{3} \gamma_{ab} \gamma^{cd} \mathcal{E}_{cd} \right) + \frac{1}{\alpha} \gamma_{ab} \hat{D} \hat{\Delta} h_{ab} + \frac{1}{\alpha} \gamma_{ab} \Omega^2 \partial_\rho \gamma^{cd} \partial_\rho h_{cd}, \quad (43) $$

where the barred expression $\bar{\mathcal{E}}_{ab}$ is defined by

$$ \bar{\mathcal{E}}_{ab} := R_{ab}^{N1} + \frac{1}{\alpha} D_a D_b \alpha - 2 \psi^4 \tilde{A}_{ac} \tilde{A}^c_{\, ab} - 8\pi S_{ab} $$

$$ + \frac{1}{2} \left( \frac{\psi^4}{\alpha^2} - 1 \right) \Omega^2 \partial_\phi^2 h_{ab} + \frac{\psi^4}{\alpha^2} \Omega \left( \xi \phi \delta_{ab} + \frac{1}{2} \xi (\partial_i \phi) \tilde{\gamma}_{ab} \right) $$

$$ + \frac{\psi^4}{2\alpha^2} \Omega^2 \left[ (\partial_i \tilde{\gamma}_{ac} + \xi \tilde{\gamma}_{ac}) D_b \phi^c + (\partial_i \tilde{\gamma}_{cb} + \xi \tilde{\gamma}_{cb}) D_a \phi^c \right] $$

$$ + \frac{\psi^4}{2\alpha^2} \xi \phi \delta_{ab} + \frac{\psi^4}{\alpha} \tilde{A}_{ab} \xi \omega \ln \frac{\psi}{\alpha}. \quad (44) $$

In this expression for $\bar{\mathcal{E}}_{ab}$, the coordinate conditions $K = 0$ and $\hat{D} \hat{\gamma}^{ab} = 0$, mentioned above, are imposed, and the trace-free part $A_{ab}$ of the extrinsic curvature, $A_{ab} := K_{ab} - \frac{1}{3} \gamma_{ab} K$, is introduced in the rescaled form $\tilde{A}_{ab} := \psi^{-4} A_{ab}$. Terms $R_{ab}^{N1}$ and $\frac{1}{\alpha} \tilde{R}_{ab}^\phi$ arise from the conformal decomposition of the Ricci tensor [12].

The source (44) can be written concisely without separating the second derivative term $\partial_\phi^2 h_{ab}$ explicitly as above. Applying the helical symmetry condition (33) to equation (39) and
subtracting the $\partial^2 \phi_{hab}$ term from both sides of equation (38), the source term of the exactly helical equation (43) can be rewritten as

$$\mathcal{E}_{ab} := \frac{1}{\alpha} \mathcal{L}_\psi (\psi^4 \tilde{A}_{ab}) - \frac{1}{2} \Omega^2 \partial^2 \phi_{hab} + R_{ab}^{NL} + \frac{3}{2} \partial^2 \phi_{hab} - \frac{1}{\alpha} D_\psi D_\phi \alpha - 2 \psi^4 \tilde{A}_{ac} \tilde{A}_{b}^c - 8\pi S_{ab}. \quad (45)$$

which is equivalent to the above source term (44).

5.3. A different formulation for the use of an elliptic solver, and a comparison with the waveless approximation

Instead of isolating the Helmholtz operator on the LHS for the case of helical symmetry, one can formally isolate an elliptic operator and solve the equation iteratively using an elliptic solver as in the waveless approximation. With this grouping of terms, the helical symmetry condition (33), applied to equation (39), leaves the term $\mathcal{L}_\omega K_{ab}$ as part of the effective source, and the Laplacian of $h_{ab}$ is separated out from $3 \tilde{R}_{ab}$. With gauge conditions $K = 0$ and $\tilde{D}_\psi \tilde{y}^{ab} = 0$, as before, we have

$$\tilde{\Delta} h_{ab} = 2(\tilde{\mathcal{E}}_{ab} - \frac{1}{3} \tilde{\gamma}_{ab} \tilde{\gamma}^{cd} \tilde{\mathcal{E}}_{cd}) - \frac{1}{2} \tilde{\gamma}_{ab} \tilde{D}^e \tilde{D}^f \tilde{h}_{ef}, \quad (46)$$

where $\tilde{\mathcal{E}}_{ab}$ is given by

$$\tilde{\mathcal{E}}_{ab} := \frac{1}{\alpha} \mathcal{L}_\psi (\psi^4 \tilde{A}_{ab}) + R_{ab}^{NL} + \frac{3}{2} \partial^2 \phi_{hab} - \frac{1}{\alpha} D_\psi D_\phi \alpha - 2 \psi^4 \tilde{A}_{ac} \tilde{A}_{b}^c - 8\pi S_{ab}. \quad (47)$$

In equation (47), the first term appears instead of the last three lines in equation (44).

We find that the helically symmetric code does not converge in this method when we solve the above set of equations on $\Sigma$ with a boundary that extends several wavelengths (or more) beyond the source. We were, however, able to obtain a converged solution when exact helical symmetry is imposed only in the near zone, within about a wavelength from the source, and the waveless approximation is used for larger $r$, effectively setting boundary conditions at the boundary of the helically symmetric inner zone. In a waveless approximation [12], the time derivative of the conformal metric, $\partial_t \tilde{\gamma}_{ab}$, is assumed to be zero. As a result the extrinsic curvature is associated with the nonrotating shift $\beta^a$,

$$K_{ab} = -\frac{1}{2} \mathcal{L}_\psi \gamma_{ab} + \frac{1}{2\alpha} \tilde{E}_{ab} \gamma_{ac}, \quad (48)$$

instead of the rotating shift $\omega^a$ as in (33).

In the next section, we compare the near-zone helically symmetric solution to a solution that is everywhere waveless. For the near-zone helically symmetric solution, the change from helical symmetry to the waveless formulation at about one wavelength from the source implies for $K_{ab}$ the condition

$$K_{ab} = \begin{cases} 
\frac{1}{2\alpha} \mathcal{L}_\psi \gamma_{ab} + \frac{1}{2\alpha} (\tilde{E}_{ab} \gamma_{ac} + \Omega \tilde{E}_{ac} \gamma_{ab}), & \text{for } r < \frac{f \pi}{\Omega}, \\
\frac{1}{2\alpha} \mathcal{L}_\psi \gamma_{ab} + \frac{1}{3\alpha} \gamma_{ab} D_c (\Omega \psi^c), & \text{for } r \geq \frac{f \pi}{\Omega},
\end{cases} \quad (49)$$

where $\pi/\Omega$ is the approximate wavelength of the $l = m = 2$ gravitational wave mode. The constant $f$, the coordinate radius of the helically symmetric zone in units of $\pi/\Omega$, is restricted to $f \lesssim 1$ for convergence.
5.4. Numerical code

Our numerical code is based on the finite difference code developed in [11, 13]. The code extends a KEH iteration scheme to the binary neutron-star computation. Cartesian components of the field equations are solved numerically on spherical coordinate grids, $r$, $\theta$, and $\phi$. An equally spaced grid is used from the centre of orbital motion to $5R_0$ where there are 16 or 24 grid points per $R_0$; from $5R_0$ to the outer boundary of computational region a logarithmically spaced grid has 60 or 90, where $R_0$ is the coordinate radius of a compact star along a line passing through the centre of orbit to the centre of a star. Accordingly, for $\theta$ and $\phi$ there are 32 or 48 grid points each from 0 to $\pi/2$ and multipoles are summed up to $l = 32$ [13].

5.5. Numerical solution

In this section we present the results of our code for a binary system modelled by a perfect fluid having polytropic equation of state, $p = K\rho^{\Gamma_1}$, with $\rho$ being the baryon density. We display results for the choices $\Gamma_1 = 2$, appropriate to neutron star matter, for compactness of a star in isolation $(M/R)_\infty = 0.14$, and for half the binary separation $d/R_0 = 1.375$. Solutions with helical symmetry are not uniquely specified by this choice of parameters and equation of state. Because they are stationary solutions with equal amounts of ingoing and outgoing waves, they are not asymptotically flat, and one must find an appropriate choice of boundary conditions. As discussed in the Consortium papers, one seeks conditions that minimize the amplitude of gravitational waves. In the toy models discussed above, conditions are fixed by the choice of a half-advanced + half-retarded Green function at each iteration. In solving the Einstein equation, however, convergence is achieved only in the near zone, and, as we have emphasized, we impose boundary conditions by matching to an exterior waveless solution outside a coordinate radius $f \pi/\Omega$.

For $f \lesssim 1$, the code converges, yielding a helically symmetric solution in the near zone $r \lesssim f \pi/\Omega$. As shown in figure 4, the solution is nearly identical to the waveless solution. The right panel shows a difference larger than 1% only when the metric component itself is smaller than 0.03; as a percentage of $h_{ij}(r = 0)$, the difference is everywhere less than 1%. The threshold of the value of $f$ for convergence is $0.7 \lesssim f \lesssim 1$ depending on the
binary separation, compactness and resolution of finite differencing. We may expect from the result that, with boundary conditions that minimize the amplitude of gravitational waves, the exact helical solution will be close to the waveless solution near the source. This is a hopeful outcome: the waveless and helically symmetric formalisms are each intended to give a solution whose inaccuracy arises from neglecting gravitational waves, and they should give nearly identical results in the near zone where the gravitational wave amplitude is small compared to the Coulomb fields.

In modelling binary neutron stars comparable accuracy is likely to result from codes that match a helically symmetric solution to a waveless solution, from a purely waveless code, and from a helically symmetric code. From a more mathematical perspective, however, finding a solution that has exact helical symmetry on the full spacetime is an appealing goal. In the neutron-star code, we have isolated a set of nonlinear terms that appear responsible for divergence outside the near zone. Improvement of the convergence of the code is required to extend the matching radius beyond a few wavelengths, which may involve a use of metric components in spherical instead of Cartesian coordinates. Further investigation of this alternative is a subject of our future work.

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