Two quantization approaches to the Bateman oscillator model

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Abstract
We consider two quantization approaches to the Bateman oscillator model. One is Feshbach-Tikochinsky’s quantization approach reformulated concisely without invoking the SU(1,1) Lie algebra, and the other is the imaginary-scaling quantization approach developed originally for the Pais-Uhlenbeck oscillator model. The latter approach overcomes the problem of unbounded-below energy spectrum that is encountered in the former approach. In both the approaches, the positive-definiteness of the squared-norms of the Hamiltonian eigenvectors is ensured. Unlike Feshbach-Tikochinsky’s quantization approach, the imaginary-scaling quantization approach allows to have stable states in addition to decaying and growing states.

Keywords: Bateman oscillator model, Feshbach-Tikochinsky’s approach, Imaginary-scaling quantization

1. Introduction
The Bateman oscillator model\textsuperscript{[1]}, or simply the Bateman model, has repeatedly been investigated as a Lagrangian model for the damped harmonic oscillator since Bateman presented the model about 90 years ago\textsuperscript{[2-12]}. The Bateman Lagrangian, which governs the Bateman model, in actuality describes a doubled system consisting of the (uncoupled) damped and amplified harmonic oscillators. Nevertheless, the Bateman model is widely recognized as a standard model for the damped harmonic oscillator, because the Bateman Lagrangian yields the correct equation of motion of the damped harmonic oscillator and has the desirable property that the Lagrangian itself does not explicitly depend on time.

Canonical quantization of the Bateman model was first performed by Feshbach and Tikochinsky with the aid of the representation theory of the SU(1,1) Lie algebra\textsuperscript{[3,5]}. They obtained the eigenvalues of the Hamiltonian operator and their corresponding eigenvectors. These eigenvalues are necessarily complex numbers, and hence the corresponding eigenstates (in the Schrödinger picture) turn out to be either decaying or growing states. Also, it is seen that the real parts of the Hamiltonian eigenvalues, which can be identified as possible values of energy of the system, are unbounded from below. From a purely dynamical point of view, this will cause the problem of dynamical instability of the system if interactions are turned on. (Applying the framework of thermo field dynamics (TFD)\textsuperscript{[13,14]} to quantizing the Bateman model may bypass this problem\textsuperscript{[15,7]}.) After Feshbach and Tikochinsky performed the canonical quantization of the Bateman model, their results have been re-considered in some different contexts\textsuperscript{[6,7,8,10,11]}. However, it seems that the problem of unbounded-below energy spectrum has not been raised precisely and has not been resolved yet\textsuperscript{[1]}

A similar problem is encountered in the canonical quantization of the Pais-Uhlenbeck oscillator model\textsuperscript{[18]}. Since the Lagrangian of this model contains the second order time-derivative of a coordinate variable in a non-degenerate manner, the corresponding classical Hamiltonian turns out to be unbounded from below in accordance with the Ostrogradsky theorem\textsuperscript{[19,20]}. This undesirable situation is inherited by the standard canonical quantization of the Pais-Uhlenbeck model, leading to the problem of unbounded-below energy spectrum. In order to overcome this problem, with ensuring the positive-definiteness of the squared-norms of the corresponding eigenvectors, Bender and Mannheim proposed an alternative quantization scheme involving the imaginary scaling of position and momentum operators\textsuperscript{[21]}. Subsequently, Mostafazadeh explored mathematical aspects of this quantization scheme and called it the imaginary-scaling quantization\textsuperscript{[22]}. This scheme indeed gives the bounded-below energy spectrum having no corresponding eigenvectors of negative squared-norm.

In this paper, we apply the imaginary-scaling quantization scheme to the Bateman model to obtain the Hamiltonian eigenvalues whose real parts are bounded from below. Of course, the positive-definiteness of the squared-norms of the corresponding eigenvectors is precisely taken into account. Before proceeding to the imaginary-scaling quantization approach to the Bateman model, we first attempt to concisely reformulate Feshbach-Tikochinsky’s quantization approach by exploiting a pseudo

\textsuperscript{1} Recently, quantization of the Bateman model has been studied in connection with a noncommutative space\textsuperscript{[19,15]}. For other recent studies concerning quantization of the Bateman model, see, e.g., Refs.\textsuperscript{[14,15]}. The contents in these studies are not directly related to those treated in the present paper.

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Bogoliubov transformation without invoking the $SU(1,1)$ Lie algebra. After that, we develop the imaginary-scaling quantization approach to the Bateman model by exploiting the combination of an imaginary-scaling transformation and a homogeneous transformation. We will see that the two quantization approaches are realized on an equal footing on the basis of different transformations of the annihilation and creation operators.

This paper is organized as follows: Section 2 provides a brief review of the Bateman model, including a preparation for the two quantization approaches. In Section 3, we reformulate Feshbach-Tikochinsky’s quantization approach in a concise manner, and in Section 4, we study the imaginary-scaling quantization approach to the Bateman model. Section 5 is devoted to a summary and discussion.

2. Bateman model and its canonical quantization

In this section, we briefly review the Bateman model and its quantum-mechanical setup.

The Bateman model is defined by the Lagrangian \[ L = mx\dot{y} + \frac{\gamma}{2}(x\dot{y} - xy) - kxy, \] (2.1)
where $x = x(t)$ and $y = y(t)$ are real coordinate variables, being functions of time $t$, and $m$, $y$, and $k$ are real positive constants. A dot over a variable denotes its derivative with respect to $t$.

From this Lagrangian, the Euler-Lagrange equation for $y$ is derived as

$$m\ddot{x} + \gamma\dot{x} + kx = 0,$$
(2.2)
and similarly, the Euler-Lagrange equation for $x$ is derived as

$$m\ddot{y} - \gamma\dot{y} + ky = 0.$$
(2.3)

Equation (2.2) is precisely the classical equation of motion for the damped harmonic oscillator of mass $m$, spring constant $k$, and damping constant $\gamma$. Equation (2.3) is the classical equation of motion for the amplified harmonic oscillator whose amplitude exponentially grows with time while the amplitude of the damped harmonic oscillator exponentially decays with time. We thus see that the Bateman Lagrangian (2.1) describes a doubled system consisting of the (uncoupled) damped and amplified harmonic oscillators.

Let us introduce the new variables \[ x_1 := \frac{1}{\sqrt{2}}(x + y), \quad x_2 := \frac{1}{\sqrt{2}}(x - y), \] (2.4)
with which the Lagrangian (2.1) can be written as

$$L = \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) - \frac{\gamma}{2}(x_1\dot{x}_2 - \dot{x}_1x_2) - \frac{k}{2}(x_1^2 - x_2^2).$$
(2.5)

The momenta conjugate to $x_1$ and $x_2$ are found to be

$$p_1 := \frac{\partial L}{\partial \dot{x}_1} = mx_1 + \frac{\gamma}{2}x_2, \quad p_2 := \frac{\partial L}{\partial \dot{x}_2} = -mx_2 - \frac{\gamma}{2}x_1.$$ (2.6)

The Hamiltonian is obtained by the Legendre transformation of $L$ as follows:

$$H := p_1\dot{x}_1 + p_2\dot{x}_2 - L = \left(\frac{1}{2m}p_1^2 + \frac{1}{2m}\omega^2x_1^2\right) - \left(\frac{1}{2m}p_2^2 + \frac{1}{2m}\omega^2x_2^2\right) - \frac{\gamma}{2m}(x_1p_2 + x_2p_1),$$
(2.7)
where

$$\omega := \sqrt{k/m - \gamma^2/4m^2}. \tag{2.8}$$

In this paper, we treat only the underdamped-underamplified case by assuming that $\omega$ is real and positive.

Now, regarding the canonical variables $x_i$ and $p_i$ ($i = 1, 2$) as Hermitian operators satisfying $x_i^\dagger = x_i$ and $p_i^\dagger = p_i$, we perform the canonical quantization of the Bateman model by imposing the commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij}\mathbb{I}, \quad (i, j = 1, 2), \quad \text{all others} = 0,$$ (2.9)
where $\mathbb{I}$ denotes the identity operator. In terms of the operators

$$a_i := \frac{\sqrt{\omega}x_i}{\hbar} + i\sqrt{\frac{1}{2m}\omega}p_i,$$ (2.10a)
$$a_i^\dagger := \frac{\sqrt{\omega}x_i}{\hbar} - i\sqrt{\frac{1}{2m}\omega}p_i,$$ (2.10b)
which satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}\mathbb{I}, \quad \text{all others} = 0,$$ (2.11)
the Hamiltonian operator corresponding to the Hamiltonian (2.7) can be expressed as

$$H = H_0 + H_1,$$ (2.12)
with

$$H_0 := \hbar\omega(a_1^\dagger a_1 - a_2^\dagger a_2),$$ (2.13a)
$$H_1 := \frac{1}{2}\sqrt{\frac{\gamma}{m}}(a_1^\dagger a_2 - a_2^\dagger a_1).$$ (2.13b)

As can be readily seen, $H_0$ and $H_1$ are Hermitian (with respect to the $\dagger$-conjugation) and commute. Adopting the naive vacuum state vector $|0\rangle$ specified by

$$a_i|0\rangle = 0,$$ (2.14)
we can construct the Fock basis vectors

$$|n_1, n_2\rangle := \frac{1}{\sqrt{n_1!n_2!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0\rangle \quad (n_i = 0, 1, 2, \ldots).$$ (2.15)
In this case, \( a_i \) and \( a_i^\dagger \) are identified as annihilation and creation operators, respectively. The dual forms of Eqs. (2.14) and (2.15) are given by
\[
\langle 0|a_i^\dagger = 0 , \tag{2.16}
\]
\[
\langle n_1, n_2 | := \frac{1}{\sqrt{n_1!n_2!}} \langle 0|(a_1)^{n_1}(a_2)^{n_2} . \tag{2.17}
\]
Using Eqs. (2.11), (2.14), and (2.16), and imposing the normalization condition \( \langle 0|0 = 1 \), we can show that
\[
\langle m_1, n_2 | n_1, n_2 | = \delta_{m_1n_1} \delta_{m_2n_2} . \tag{2.18}
\]
Hence, it follows that the Fock basis vectors \( |n_1, n_2 \rangle \) have the positive squared-norm 1, and the Fock space spanned by the orthonormal basis \( \{|n_1, n_2 \rangle \} \) is a positive-definite Hilbert space. In this space, the completeness condition of the orthonormal basis reads
\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2 \rangle \langle n_1, n_2 | = 1 . \tag{2.19}
\]
We see that the vectors \( |n_1, n_2 \rangle \) are eigenvectors of \( H_0 \) but not eigenvectors of \( H_1 \), although \( H_0 \) and \( H_1 \) commute. In order to find the simultaneous eigenvectors of \( H_0 \) and \( H_1 \), which are of course eigenvectors of \( H \), we consider invertible transformations in the next two sections.

3. Feshbach-Tikochinsky’s quantization approach revisited

In this section, we perform a reformulation of the canonical quantization approach of Feshbach and Tikochinsky [5] without referring to the \( SU(1, 1) \) Lie algebra.

We first define the operators \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) by
\[
\hat{a}_i := e^{iX} a_i e^{-iX} \quad \hat{a}_i^\dagger := e^{iX} a_i^\dagger e^{-iX} , \tag{3.1}
\]
where \( \theta \) is a complex parameter, and \( X \) is defined by
\[
X := a_1 a_2 + a_1^\dagger a_2^\dagger . \tag{3.2}
\]
It is obvious that \( X^\dagger = X \). The unitarity of \( e^{iX} \) and its associated property \( (\hat{a}_i)^\dagger = \hat{a}_i^\dagger \) hold only when \( \theta \) is purely imaginary. From Eq. (2.11), we see that
\[
| \hat{a}_i, \hat{a}_j^\dagger | = \delta_{i,j} 1 , \quad \text{all others } = 0 . \tag{3.3}
\]
Equation (3.1) can be written as
\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1^\dagger \\
\hat{a}_2^\dagger
\end{pmatrix} , \quad \begin{pmatrix}
\hat{a}_1^\dagger \\
\hat{a}_2^\dagger
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix} . \tag{3.4a}
\]
\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1^\dagger \\
\hat{a}_2^\dagger
\end{pmatrix} , \quad \begin{pmatrix}
\hat{a}_1^\dagger \\
\hat{a}_2^\dagger
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix} . \tag{3.4b}
\]
The transformation \( (a_i, a_i^\dagger) \rightarrow (\hat{a}_i, \hat{a}_i^\dagger) \) looks like a Bogoliubov transformation, but actually it is not the case unless the parameter \( \theta \) is purely imaginary. (If \( \theta \) is purely imaginary, then \( e^{iX} \) is unitary, and the transformation \( (a_i, a_i^\dagger) \rightarrow (\hat{a}_i, \hat{a}_i^\dagger) \) can be said to be a Bogoliubov transformation [23].)

Using Eqs. (3.4) and (3.3), we can express the operators \( H_0 \) and \( H_1 \) as follows:
\[
H_0 = \hbar \omega (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) , \tag{3.5a}
\]
\[
H_1 = \frac{i\hbar y}{2m} \left[ (\hat{a}_1 \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2^\dagger) \cos(2\theta) + (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1) \sin(2\theta) \right] . \tag{3.5b}
\]
Incidentally, \( X \) can be expressed as \( X = \hat{a}_1 \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger \). Since our present purpose is to find the eigenvalues of \( H \), we choose \( \theta \) in such a way that \( H_1 \) takes the form of a linear combination of \( \hat{a}_1^\dagger \hat{a}_1, \hat{a}_2^\dagger \hat{a}_2, \) and \( 1 \). (The operator \( H_0 \) already takes the form of a linear combination of \( \hat{a}_1^\dagger \hat{a}_1 \) and \( \hat{a}_2^\dagger \hat{a}_2 \).) Upon comparison with Feshbach-Tikochinsky’s quantization approach, we set \( \theta = \pm \pi/4 \). Then \( H_1 \) becomes
\[
H_{1(\pm)} := \pm \frac{i\hbar y}{2m} (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1) . \tag{3.6}
\]
The transformation \( (a_i, a_i^\dagger) \rightarrow (\hat{a}_i, \hat{a}_i^\dagger) \) with \( \theta = \pm \pi/4 \) is hereafter referred to as a pseudo Bogoliubov transformation, with the connotation that it is not unitary. Such a non-unitary transformation was also considered in Feshbach-Tikochinsky’s quantization approach based on the \( SU(1, 1) \) Lie algebra. The Hermiticity of \( H_{1(\pm)} \) with respect to the \( \hat{z} \)-conjugation, i.e., \( (H_{1(\pm)})^\dagger = H_{1(\pm)} \), is valid under the conditions
\[
i^\dagger = -i , \quad \gamma^\dagger = -\gamma . \tag{3.7}
\]
Clearly, \( H_0 \) and \( X \) are Hermitian with respect to the \( \hat{z} \)-conjugation.

The Hamiltonian operator (2.12) now reads \( H^{(z)} := H_0^{(z)} + H_{1(\pm)}^{(z)} \). With \( H^{(z)} \), the Heisenberg equation for an implicitly time-dependent operator \( A(t) \) reads \( dA/dt = (ih)^{-1}[A, H^{(z)}] \). Using the commutation relations in Eq. (3.3), we can solve the Heisenberg equations for \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) and obtain
\[
\hat{a}_1(t) = \hat{a}_1(0)e^{-i\lambda \omega t} , \quad \hat{a}_1^\dagger(t) = \hat{a}_1^\dagger(0)e^{-i\lambda \omega t} , \tag{3.8a}
\]
\[
\hat{a}_2(t) = \hat{a}_2(0)e^{i\lambda \omega t} , \quad \hat{a}_2^\dagger(t) = \hat{a}_2^\dagger(0)e^{i\lambda \omega t} , \tag{3.8b}
\]
where \( \lambda := \gamma/2m \). By virtue of the conditions in Eq. (3.7), the \( \hat{z} \)-conjugation relation \( \hat{a}_i(t) = \hat{a}_i^\dagger(t) \) holds at arbitrary time. As can be seen from Eq. (3.3), the \( \hat{z} \)-conjugation involves time reversal. This fact reminds us that in Feshbach-Tikochinsky’s quantization approach, the time reverse, rather than the complex conjugate, is used to define an appropriate normalization integral for a wave function. It is evident that the Hamiltonian operator \( H^{(z)} \) is independent of time.

Next we define the new vectors
\[
|0\rangle := e^{iX}|0\rangle , \quad \langle 0| := \langle 0|e^{-iX} , \tag{3.9}
\]
which satisfy
\[
\hat{a}_i|0\rangle = 0 , \quad \langle 0|\hat{a}_i^\dagger = 0 . \tag{3.10}
\]
owing to Eqs. (2.14) and (2.16). Hence, \( |0\rangle \) and \( \langle 0| \) are established as the vacuum state vectors of the \( \{\hat{a}_i, \hat{a}_i^\dagger\}\) system, and \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) turn out to be annihilation and creation operators, respectively. In this system, we can construct the Fock basis vectors and their dual vectors as follows:

\[
|n_1, n_2\rangle := \frac{1}{\sqrt{n_1! n_2!}} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} |0\rangle, \quad (3.11a)
\]

\[
\langle n_1, n_2| := \frac{1}{\sqrt{n_1! n_2!}} \langle 0| (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2}. \quad (3.11b)
\]

They are related to the old basis vectors in Eqs. (2.15) and (2.17) by

\[
|n_1, n_2\rangle = e^{iX} |n_1, n_2\rangle, \quad \langle n_1, n_2| = \langle n_1, n_2| e^{-iX}. \quad (3.12)
\]

It is easily shown, using Eq. (2.18), that

\[
\langle m_1, m_2| n_1, n_2\rangle = \delta_{m_1 n_1} \delta_{m_2 n_2}. \quad (3.13)
\]

Hence, it follows that the Fock basis vectors \( |n_1, n_2\rangle \) also have the positive squared-norm 1, and the Fock space spanned by the orthonormal basis \( \{|n_1, n_2\rangle\} \) is a positive-definite Hilbert space. The completeness condition (2.19) leads to

\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2\rangle \langle n_1, n_2| = \mathbb{1}. \quad (3.14)
\]

We readily see that the vectors \( |n_1, n_2\rangle \) with \( \theta = \pm \pi/4 \) are simultaneous eigenvectors of \( H_0 \) and \( H_1^{(\pm)} \) and satisfy the Hamiltonian eigenvalue equation

\[
H^{(\pm)} |n_1, n_2\rangle = h^{(\pm)}_{n_1, n_2} |n_1, n_2\rangle \quad (3.15)
\]

with

\[
h^{(\pm)}_{n_1, n_2} := \hbar \omega (n_1 - n_2) \pm i \hbar \lambda (n_1 + n_2 + 1). \quad (3.16)
\]

The Hamiltonian eigenvalues \( h^{(\pm)}_{n_1, n_2} \) are identical to those found earlier by Feshbach and Tikochinsky [5]. In this way, the pseudo Bogoliubov transformation makes it possible to solve the eigenvalue problem of the Hamiltonian operator \( H \) given in Eq. 2.12.

Let us now consider the Schrödinger equation

\[
i \hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle. \quad (3.17)
\]

In order to solve this equation, we expand \( |\psi(t)\rangle \) in terms of the basis \( \{|n_1, n_2\rangle\} \) at \( t = 0 \) rather than the basis \( \{|n_1, n_2\rangle\} \) at \( t = 0 \). Then, using Eq. (3.15), we obtain the particular solutions

\[
|\psi^{(\pm)}_{n_1, n_2}(t)\rangle := \exp (-i h^{(\pm)}_{n_1, n_2} t/\hbar) |n_1, n_2\rangle_{t=0}. \quad (3.18)
\]

which specify the Hamiltonian eigenstates at the time \( t \). The general solution of Eq. (3.17) is given by \( |\psi(t)\rangle = \sum_{n_1, m_1} e^{i m_1 \theta} |\psi^{(\pm)}_{n_1, m_1}(t)\rangle \) with \( c_{n_1, m_1} \) being complex constants. We see that \( |\psi^{(\pm)}_{n_1, m_1}(t)\rangle \) represent decaying states while \( |\psi^{(\pm)}_{n_1, 0}(t)\rangle \) represent growing states, regardless of the possible values of \( n_1 \) and \( n_2 \). This result is due to the presence of the constant term \( \pm i \hbar \lambda \) that remains in \( h^{(\pm)}_{n_1, n_2} \) even when \( n_1 = n_2 = 0 \). Since \( H \) is Hermitian with respect to the \( \dagger \)-conjugation, the dual Schrödinger equation for \( |\psi(t)\rangle \) reads \( d|\psi(t)\rangle/\hbar dt = (-i \hbar)^{-1} (\psi(t)|H. Expanding \( |\psi(t)\rangle \) in terms of the dual basis \( \{\langle n_1, n_2|\} \) at \( t = 0 \), and using the eigenvalue equation \( \langle n_1, n_2| H^{(\pm)} = h^{(\pm)}_{n_1, n_2} \langle n_1, n_2| \) obtained from Eq. (3.11b), we have the particular solutions

\[
|\psi^{(\pm)}_{n_1, n_2}(t)\rangle := \exp \left(i h^{(\pm)}_{n_1, n_2} t/\hbar \right) |n_1, n_2\rangle_{t=0}. \quad (3.19)
\]

It is clear, by taking into account the condition \( h^{(\pm)}_{n_1, n_2} = h^{(\pm)}_{n_2, n_1} \) ensured by Eq. (3.7), that \( |\psi^{(\pm)}_{n_1, n_2}(t)\rangle \) and \( |\psi^{(\pm)}_{n_2, n_1}(t)\rangle \) are related to each other by the \( \dagger \)-conjugation. Equation (3.13) leads to

\[
\langle \psi^{(\pm)}_{n_1, n_2}(t)| \psi^{(\pm)}_{m_1, m_2}(t) \rangle = \delta_{m_1 n_1} \delta_{m_2 n_2}. \]

which demonstrates that the squared-norm of \( |\psi^{(\pm)}_{n_1, n_2}(t)\rangle \) does not change in time. A similar fact was also pointed out by Feshbach and Tikochinsky [5].

Now we briefly mention the divergence to infinity of the standard squared-norm of \( |n_1, n_2\rangle \). We first note that the vectors defined in Eq. (3.9) can be written as

\[
|0\rangle = \frac{1}{\cos \theta} \exp \left( i a_1^\dagger a_2^\dagger \tan \theta \right) |0\rangle, \quad (3.20a)
\]

\[
|0\rangle = \frac{1}{\cos \theta} \exp (-i a_1 a_2 \tan \theta) |0\rangle. \quad (3.20b)
\]

(In actuality, these expressions are well-defined only for \( \theta \) such that \( |\tan \theta| < 1 \). For this reason, the condition \( \theta = \pm \pi/4 \) should here be understood as \( \theta \dagger \pi/4 \) or \( \theta \dagger -\pi/4 \).) When \( \theta \) is not purely imaginary, \( |0\rangle \) is different from the \( \dagger \)-conjugate of \( |0\rangle \), defined by \( |0\rangle := \langle 0| e^{i \theta X} = \langle \cos \theta \dagger |0\rangle e^{i a_1 a_2 \tan \theta} \). The standard squared-norm of \( |0\rangle \) is found to be

\[
\langle 0|0\rangle = \langle 0| e^{i \theta X} \rangle = \frac{1}{|\cos \theta|^2 - |\sin \theta|^2}. \quad (3.21)
\]

If \( \theta \) is purely imaginary, then Eq. (3.21) reduces to \( \langle 0|0\rangle = 1 \) as expected. In contrast, if \( \theta \dagger \pm \pi/4 \), then \( \langle 0|0\rangle \) diverges to infinity. More generally, we can demonstrate that the standard squared-norm of \( |n_1, n_2\rangle \), denoted as \( \langle n_1, n_2| n_1, n_2\rangle \), diverges to infinity when \( \theta = \pi/4 \). Here, \( n_1, n_2 \) denotes the \( \dagger \)-conjugate of \( |n_1, n_2\rangle \) by \( |n_1, n_2\rangle := \langle n_1, n_2| e^{i \theta X} \rangle = \langle n_1, n_2| e^{i \theta X}. Thus, it follows that the ordinary Hilbert space specified by the standard squared-norm is not well-defined in the cases \( \theta = \pm \pi/4 \). Accordingly, it turns out that in the cases \( \theta = \pm \pi/4 \), the Hermiticity of \( H \) does not actually make sense in this Hilbert space and, as a result, \( H \) can possess the purely imaginary eigenvalues \( \pm i \hbar \lambda (n_1 + n_2 + 1) \) [see Eqs. (3.15) and (3.16)]. This situation was also stated by Feshbach and Tikochinsky in a somewhat different context [5]. To avoid the use of the ill-defined Hilbert space mentioned above, we have indeed considered the well-defined Hilbert space specified by the inner product (3.13).

Using Eq. (2.4) at the operator level and Eqs. (2.10), (3.4),
and (3.8), we can obtain for \( \theta = \pi/4, \)
\[
x(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\chi} \left( \tilde{a}_1(0)e^{-i\omega t} + \tilde{a}_2(0)e^{i\omega t} \right),
\]
(3.22a)
\[
y(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{i\chi} \left( \tilde{a}_1(0)e^{-i\omega t} - \tilde{a}_2(0)e^{i\omega t} \right),
\]
(3.22b)
and for \( \theta = -\pi/4, \)
\[
x(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{+i\chi} \left( \tilde{a}_1(0)e^{-i\omega t} + \tilde{a}_2(0)e^{i\omega t} \right),
\]
(3.23a)
\[
y(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\chi} \left( \tilde{a}_1(0)e^{-i\omega t} - \tilde{a}_2(0)e^{i\omega t} \right).
\]
(3.23b)

It can be readily checked that Eqs. (3.22a) and (3.23a) satisfy Eq. (2.2), and Eqs. (3.22b) and (3.23b) satisfy Eq. (2.3). We thus see that Eqs. (2.2) and (2.3) at the operator level are realized in Feshbach-Tikochinsky’s quantization approach reformulated here.

We close this section with a remark on the Hamiltonian operator (2.12). This operator has the same form as one of the Hamiltonian operators argued in TFD [13–14], provided that \( a_2 \) and \( d_2 \) are identified with the so-called tilde conjugates of \( a_1 \) and \( d_1 \), respectively. Noting this fact, Celeghini et al. have investigated a quantum-theoretical aspect of the Bateman model by following the framework of TFD [4, 5]. They claimed the necessity of a field theoretical generalization of the Bateman model. In the thermo field dynamical approach, the minus sign of \( -d_2^\dagger d_2 \) included in \( H_0 \) is essential for describing the thermal degree of freedom. However, from a purely dynamical point of view, this minus sign inevitably causes the problem of dynamical instability of the system if interactions are turned on. In fact, the eigenvalues of \( H_0 \), which are given by \( \text{Re}h_{a_2\bar{a}_2} = \hbar \omega (n_1 - n_2) \) and can be identified as the possible values of energy, are unbounded from below owing to the presence of \( -d_2^\dagger d_2 \). As a result, the dynamical stability of the system is spoiled. This undesirable situation can be overcome by applying the imaginary-scaling quantization scheme [21, 22] to the Bateman model.

4. Imaginary-scaling quantization approach

In this section, we treat the imaginary-scaling quantization of the Bateman model.

First we define the operators \( \tilde{a}_i \) and \( \tilde{a}_i^\dagger \) by
\[
\tilde{a}_i := e^{i\phi} a_i e^{-i\phi}, \quad \tilde{a}_i^\dagger := e^{i\phi} a_i^\dagger e^{-i\phi},
\]
(4.1)
where \( \phi \) is a complex parameter, and \( Y \) is defined by
\[
Y := -\frac{i}{2} \left( a_1^2 - a_2^2 \right).
\]
(4.2)

It is obvious that \( Y^\dagger = Y \). The unitarity of \( e^{i\phi} \) and its associated property \( (\tilde{a}_i)^\dagger = \tilde{a}_i^\dagger \) hold only when \( \phi \) is purely imaginary. We can express \( Y \) as \( Y = -i(\tilde{a}_1^2 - \tilde{a}_2^2)/2 \), from which we see that

\( Y \) is Hermitian with respect to the $\dagger$-conjugation, i.e., \( Y^\dagger = Y \). Equation (2.11) leads to
\[
[\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij} I, \quad \text{all others} = 0.
\]
(4.3)

From the definition of \( Y \), we immediately see that
\[
\tilde{a}_1 = a_1, \quad \tilde{a}_1^\dagger = a_1^\dagger.
\]
(4.4)

Thus it turns out that the transformation \((a_1, a_1^\dagger) \rightarrow (\tilde{a}_1, \tilde{a}_1^\dagger)\) is essentially a squeeze transformation of \((a_2, a_2^\dagger)\), provided that \( \phi \) is purely imaginary and hence \( e^{i\phi} \) is unitary [24, 14]. From now on, we rather choose \( \phi \) to be the real value \( \phi = \pi/2 \). Then, from Eq. (4.1), we have
\[
\tilde{a}_2 = -i\tilde{a}_2^\dagger, \quad \tilde{a}_2^\dagger = -i\tilde{a}_2.
\]
(4.5)

The transformation \((a_2, a_2^\dagger) \rightarrow (\tilde{a}_2, \tilde{a}_2^\dagger) = (-i\tilde{a}_2^\dagger, -i\tilde{a}_2)\) is precisely the imaginary-scaling transformation argued in Refs. 21, 22. Since this transformation can be derived as a non-unitary analog of the squeeze transformation, it can be said to be a pseudo squeeze transformation.

Next we define the operators \( \tilde{a}_i \) and \( \tilde{a}_i^\dagger \) by
\[
\tilde{a}_1 := e^{i\chi} a_1 e^{-i\chi}, \quad \tilde{a}_1^\dagger := e^{i\chi} a_1^\dagger e^{-i\chi},
\]
(4.6)
where \( \chi \) is assumed to be a purely imaginary parameter satisfying \( \chi^\dagger = -\chi \), and \( Z \) is defined by
\[
Z := \tilde{a}_2^\dagger \tilde{a}_2 + \tilde{a}_1^\dagger \tilde{a}_1.
\]
(4.7)

Obviously, \( Z \) is Hermitian with respect to the $\dagger$-conjugation. The unitarity of \( e^{i\chi} \) with respect to the $\dagger$-conjugation and the $\dagger$-conjugation relation \((\tilde{a}_i)^\dagger = \tilde{a}_i^\dagger \) are ensured accordingly. Equation (4.5) leads to
\[
[\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij} I, \quad \text{all others} = 0.
\]
(4.8)

The operators \( \tilde{a}_i \) can be written as linear combinations of \( \tilde{a}_1 \) and \( \tilde{a}_2 \); similarly, the operators \( \tilde{a}_i^\dagger \) can be written as linear combinations of \( \tilde{a}_1^\dagger \) and \( \tilde{a}_2^\dagger \). The transformation \((\tilde{a}_i, \tilde{a}_i^\dagger) \rightarrow (\tilde{a}_j, \tilde{a}_j^\dagger)\) is thus realized as a homogeneous transformation. Combining the expressions of the linear combinations with Eqs. (4.4) and (4.5), we obtain
\[
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_1^\dagger \\
\tilde{a}_2^\dagger
\end{bmatrix} =
\begin{bmatrix}
\cosh \chi & i \sinh \chi \\
-i \sinh \chi & -i \cosh \chi
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_1^\dagger \\
a_2^\dagger
\end{bmatrix},
\]
(4.9a)
\[
\begin{bmatrix}
\tilde{a}_1^\dagger \\
\tilde{a}_2^\dagger \\
\tilde{a}_1 \\
\tilde{a}_2
\end{bmatrix} =
\begin{bmatrix}
\cosh \chi & -i \sinh \chi \\
\sinh \chi & -i \cosh \chi
\end{bmatrix}
\begin{bmatrix}
a_1^\dagger \\
a_2^\dagger \\
a_1 \\
a_2
\end{bmatrix}.
\]
(4.9b)

Now, using Eqs. (4.9) and (4.8), we can express the operators \( H_0 \) and \( H_1 \) defined in Eq. (2.13) as follows:
\[
H_0 = \hbar \omega (\tilde{a}_1^\dagger \tilde{a}_1 + \tilde{a}_2^\dagger \tilde{a}_2 + I),
\]
(4.10a)
\[
H_1 = \frac{\hbar \nu}{2m} \left( (\tilde{a}_1^\dagger \tilde{a}_2 - \tilde{a}_2^\dagger \tilde{a}_1) \cosh(2\chi) + (\tilde{a}_1^\dagger \tilde{a}_1 - \tilde{a}_2^\dagger \tilde{a}_2) \sinh(2\chi) \right).
\]
(4.10b)
Incidentally, \(Z\) can be expressed as \(Z = \hat{a}^\dagger \hat{a}^2 + \hat{a} \hat{a}^2 \hat{a}^\dagger\). Our present purpose is to find the eigenvalues of \(H = H_0 + H_1\) within the framework of imaginary-scaling quantization. To this end, we choose \(\chi\) to be the imaginary value \(\chi = \pm i\pi/4\) so that \(H_1\) can take the form of a linear combination of \(\hat{a}^\dagger \hat{a}^1\) and \(\hat{a}^2 \hat{a}^2\). (The operator \(H_0\) already takes the form of a linear combination of \(\hat{a}^\dagger \hat{a}^1\), \(\hat{a}^2 \hat{a}^2\), and \(1\).) After setting \(\chi = \mp i\pi/4\), the operator \(H_1\) reduces to
\[
\hat{H}_1^{(\chi)} := \frac{-i\hbar y}{2m} \left(\hat{a}^\dagger \hat{a}^1 - \hat{a}^2 \hat{a}^2\right). \tag{4.11}
\]
The Hermiticity of \(\hat{H}_1^{(\chi)}\) with respect to the \(\ddagger\)-conjugation, i.e., \((\hat{H}_1^{(\chi)})^\ddagger = \hat{H}_1^{(\chi)}\) is valid under the conditions
\[
\delta^\ddagger = -i, \quad \gamma^\ddagger = -\gamma. \tag{4.12}
\]
It is obvious that \(H_0\) and \(Z\) are Hermitian with respect to the \(\ddagger\)-conjugation.

The Hamiltonian operator \((4.12)\) now reads \(\hat{H}^{(\chi)} = H_0 + \hat{H}_1^{(\chi)}\). Correspondingly, the Heisenberg equation for an implicitly time-dependent operator \(A(t)\) reads \(dA/dt = (i\hbar)^{-1}[A, \hat{H}^{(\chi)}]\). By using the commutation relations in Eq. \((4.10)\), we can solve the Heisenberg equations for \(\hat{a}_i\) and \(\hat{a}_i^\dagger\), obtaining
\[
\dot{\hat{a}}_1(t) = \hat{a}_1(0)e^{-(i\omega t \ddagger)}, \quad \hat{a}_1^\dagger(t) = \hat{a}_1^\dagger(0)e^{-(i\omega t \ddagger)}, \tag{4.13a}
\]
\[
\dot{\hat{a}}_2(t) = \hat{a}_2(0)e^{-(i\omega t \ddagger)}, \quad \hat{a}_2^\dagger(t) = \hat{a}_2^\dagger(0)e^{-(i\omega t \ddagger)}, \tag{4.13b}
\]
with \(\lambda := \gamma/2m\). By virtue of the conditions in Eq. \((4.12)\), the \(\ddagger\)-conjugation relation \((\hat{a}_i(t))^\ddagger = \hat{a}_i^\dagger(t)\) holds at an arbitrary time. We see from Eq. \((4.13)\) that just like the \(\ddagger\)-conjugation treated in Sec. 3, the \(\ddagger\)-conjugation also involves time reversal. It is evident that the Hamiltonian operator \(\hat{H}^{(\chi)}\) is independent of time.

Let us define the new vectors
\[
|0\rangle := e^{\phi \dagger} |0\rangle, \quad \langle 0| := \langle 0|e^{-\phi \dagger}, \tag{4.14}
\]
which satisfy
\[
\hat{a}_i |0\rangle = 0, \quad \langle 0| \hat{a}_i^\dagger = 0 \tag{4.15}
\]
owing to Eqs. \((2.14)\) and \((2.16)\). Also, using Eqs. \((4.6)\) and \((4.15)\), we can show that
\[
\hat{a}_i |0\rangle = 0, \quad \langle 0| \hat{a}_i^\dagger = 0. \tag{4.16}
\]
Hence, \(|0\rangle\) and \(\langle 0|\) are established as the vacuum state vectors common to both the \((\hat{a}_i, \hat{a}_i^\dagger)\) and \((\hat{a}_i, \hat{a}_i^\dagger)\) systems. From Eqs. \((4.13)\) and \((4.16)\), it turns out that \(\hat{a}_i\) and \(\hat{a}_i^\dagger\) are annihilation operators, while \(\hat{a}_i^\dagger\) and \(\hat{a}_i^\dagger\) are creation operators. In the \((\hat{a}_i, \hat{a}_i^\dagger)\) system, we can construct the Fock basis vectors and their dual vectors as follows:
\[
|n_1, n_2\rangle := \frac{1}{\sqrt{n_1! n_2!}} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} |0\rangle, \tag{4.17a}
\]
\[
\langle n_1, n_2| := \frac{1}{\sqrt{n_1! n_2!}} \langle 0| (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2}. \tag{4.17b}
\]
They are related to the old basis vectors in Eqs. \((2.15)\) and \((2.17)\) by
\[
|n_1, n_2\rangle = e^{\phi \dagger} e^{\phi} |n_1, n_2\rangle, \tag{4.18a}
\]
\[
\langle n_1, n_2| = \langle n_1, n_2| e^{-\phi \dagger} e^{-\phi}. \tag{4.18b}
\]
It is easy to show by using Eq. \((4.18)\) that
\[
\langle n_1, n_2| n_1, n_2\rangle = \delta_{n_1 n_1} \delta_{n_2 n_2}. \tag{4.19}
\]
From this, it follows that the Fock basis vectors \(|n_1, n_2\rangle\) have the positive squared-norm 1, and the Fock space spanned by the orthonormal basis \(|n_1, n_2\rangle\) is a positive-definite Hilbert space. The completeness condition \((4.19)\) now leads to
\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2\rangle \langle n_1, n_2| = 1. \tag{4.20}
\]
We immediately see that the vectors \(|n_1, n_2\rangle\) with \(\phi = \pi/2\) and \(\chi = \pm i\pi/4\) are simultaneous eigenvectors of \(H_0\) and \(\hat{H}_1^{(\chi)}\) and satisfy the Hamiltonian eigenvalue equation
\[
\hat{H}^{(\chi)} |n_1, n_2\rangle = \hat{K}^{(\chi)} |n_1, n_2\rangle, \tag{4.21}
\]
with
\[
\hat{K}^{(\chi)} := \hbar \omega (n_1 + n_2 + 1) \pm i \hbar \lambda (n_1 - n_2). \tag{4.22}
\]
This expression of the Hamiltonian eigenvalues is completely different from the one obtained by Feshbach and Tilokhinsky, namely Eq. \((3.16)\). In fact, the eigenvalues of \(H_0\), which are given at present by \(\text{Re} \hat{K}^{(\chi)} = \hbar \omega (n_1 + n_2 + 1)\), are bounded from below, and therefore the dynamical stability of the system is ensured. Also, \(\text{Re} \hat{K}^{(\chi)}\) include the vacuum state energy \(\hbar \omega\). In this way, the combination of the imaginary-scaling transformation and a homogeneous transformation makes it possible to solve the eigenvalue problem of the Hamiltonian operator \(H\) given in Eq. \((4.12)\), resolving the problem of dynamical instability encountered in Feshbach-Tilokhinsky’s quantization approach.

Now we recall the Schrödinger equation \((3.17)\) and expand the state vector \(|\psi(t)\rangle\) in terms of the basis \(|n_1, n_2\rangle\) at \(t = 0\), instead of the basis \(|n_1, n_2\rangle\) at \(t = 0\). Then, using Eq. \((5.21)\), we obtain the particular solutions of the Schrödinger equation,
\[
|\psi_{n_1, n_2}^{(\chi)}(t)\rangle := \exp \left( -i \hat{K}^{(\chi)} n_1, n_2 / \hbar \right) |n_1, n_2\rangle, \tag{4.23}
\]
which specify the Hamiltonian eigenstates at the time \(t\). The general solution is found to be
\[
|\psi^{(\chi)}(t)\rangle = \sum_{n_1, n_2} \hat{c}_{n_1, n_2} |\psi_{n_1, n_2}^{(\chi)}(t)\rangle.
\]

---

3 If \(|\tan \phi| < 1\), the vectors \(|0\rangle\) and \(\langle 0|\) can be written as
\[
|0\rangle = \frac{1}{\sqrt{\cos \phi}} \exp \left( \frac{1}{2} \tan \phi \right) |0\rangle, \tag{4.15}
\]
\[
\langle 0| = \frac{1}{\sqrt{\cos \phi}} \exp \left( -\frac{1}{2} \tan \phi \right). \tag{4.16}
\]
These expressions cannot be applied to the present case, \(\phi = \pi/2\).
with $\chi_{n_{1},n_{2}}$ being complex constants. We see that both $|\tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle$ and $|\tilde{\psi}^{(a)}_{n_{1},n_{2}}(t)\rangle$ can represent either of decaying or growing states depending on the possible values of $n_{1}$ and $n_{2}$. If $|\tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle$ is the state vector of a decaying (growing) state, then $|\tilde{\psi}^{(a)}_{n_{1},n_{2}}(t)\rangle$ is the state vector of a growing (decaying) state. It is remarkable that the state vectors $|\tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle$ with $n_{1} = n_{2}$ contain no $\gamma$ and represent stable states, because $\text{Im} \tilde{h}^{(s)}_{n_{1},n_{2}}$ vanish when $n_{1} = n_{2}$. Therefore, unlike Feshbach-Tikchonvsky’s quantization approach, the imaginary-scaling quantization approach allows to have stable states in addition to decaying states and growing states. Recall here the dual Schrödinger equation $d|\psi(t)\rangle/ dt = (-i\hbar)^{-1}\langle \psi(t)|H|\psi(t)\rangle$. Expanding $|\psi(t)\rangle$ in terms of the dual basis $|\langle n_{1},n_{2}\rangle\rangle$ at $t = 0$, and using the eigenvalue equation $|\langle n_{1},n_{2}\rangle\rangle = \tilde{h}^{(s)}_{n_{1},n_{2}}|\langle n_{1},n_{2}\rangle\rangle$ obtained from Eq. (4.17b), we have the particular solutions

$$
\langle \tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle := i\tilde{\psi}^{(s)}_{n_{1},n_{2}}(0)\langle n_{1},n_{2}\rangle_{\text{i}t}.
$$

(4.24)

Taking into account the condition $\langle \tilde{h}^{(s)}_{n_{1},n_{2}}\rangle = \tilde{h}^{(a)}_{n_{1},n_{2}}$, ensured by Eq. (4.12), we see that $|\tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle$ and $|\tilde{\psi}^{(a)}_{n_{1},n_{2}}(t)\rangle$ are related to each other by the $s$-conjugation. Equation (4.19) leads to $\langle \tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle = \tilde{h}^{(s)}_{n_{1},n_{2}}|\langle n_{1},n_{2}\rangle\rangle$, which implies that the squared-norm of $|\tilde{\psi}^{(s)}_{n_{1},n_{2}}(t)\rangle$ does not change in time. A similar result was also found in Sec 3.

Using Eq. (2.4) at the operator level and Eqs. (2.10), (4.9), and (4.13), we can obtain for $\chi = i\pi/4$,

$$
x(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t} \left( \tilde{u}_{1}(0)e^{i\omega t} + i\tilde{u}_{2}(0)e^{-i\omega t} \right),
$$

(4.25a)

$$
y(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{i\omega t} \left( \tilde{u}_{1}(0)e^{-i\omega t} - i\tilde{u}_{2}(0)e^{i\omega t} \right),
$$

(4.25b)

and for $\chi = -i\pi/4$,

$$
x(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t} \left( \tilde{u}_{1}(0)e^{-i\omega t} + i\tilde{u}_{2}(0)e^{i\omega t} \right),
$$

(4.26a)

$$
y(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{i\omega t} \left( \tilde{u}_{1}(0)e^{i\omega t} - i\tilde{u}_{2}(0)e^{-i\omega t} \right).
$$

(4.26b)

It can be immediately checked that Eqs. (4.25a) and (4.26a) satisfy Eq. (2.2), and Eqs. (4.25b) and (4.26b) satisfy Eq. (2.3). In this way, it is verified that Eqs. (2.2) and (2.3) at the operator level are realized also in the imaginary-scaling quantization approach. In each of the cases $\chi = i\pi/4$ and $\chi = -i\pi/4$, we observe that $x^{2} = y$ and $y^{2} = x$. From these relations together with $y^{2} = -y$ given in Eq. (4.12), we can recognize that the $s$-conjugation corresponds to the transformation $(x,y,\gamma) \rightarrow (y,x,-\gamma)$, which leaves the Lagrangian (2.1) invariant. On the other hand, it is recognized in Feshbach-Tikchonsky’s quantization approach that the $s$-conjugation corresponds to the transformation $(x,y,\gamma) \rightarrow (\pm x_{i1}/m\omega, \mp x_{i2}/m\omega, \gamma)$, where $x_{i1}$ and $p_{i}$ denote the canonical momenta conjugate to $x$ and $y$, respectively. From this result, we see that unlike the $s$-conjugation, the $s$-conjugation does not have its classical counterpart at the Lagrangian level.

5. Summary and discussion

We have investigated two quantization approaches to the Bateman model. One is Feshbach-Tikchonsky’s quantization approach reformulated concisely without invoking the $SU(1, 1)$ Lie algebra, and the other is the imaginary-scaling quantization approach proposed originally for the Pais-Uhlenbeck model. The former has been developed by applying a pseudo Bogoliubov transformation to the Bateman model, while the latter has been developed by applying the imaginary-scaling transformation and a homogeneous transformation to the Bateman model. The two quantization approaches can thus be realized on an equal footing on the basis of the different transformations of $a_{i}$ and $a_{i}^{†}$. Also, we have pointed out that the imaginary-scaling transformation can be said to be a pseudo squeeze transformation.

We have indeed solved the eigenvalue problem for the Hamiltonian operator $H$ of the Bateman model. By means of the pseudo Bogoliubov transformation, we have simply derived the Hamiltonian eigenvalues $h_{n_{1},n_{2}}^{(s)}$ found earlier by Feshbach and Tikchonsky [5]. In addition, we have derived the alternative Hamiltonian eigenvalues $h_{n_{1},n_{2}}^{(s)}$ by employing the imaginary-scaling quantization scheme [21, 22]. It has been seen that the real part of $h_{n_{1},n_{2}}^{(s)}$ is proportional to $n_{1} - n_{2}$ and the imaginary part is proportional to $n_{1} + n_{2} + 1$. In contrast, the real part of $h_{n_{1},n_{2}}^{(s)}$ is proportional to $n_{1} + n_{2} + 1$ and the imaginary part is proportional to $n_{1} - n_{2}$. As has been clarified above, the eigenvalues $h_{n_{1},n_{2}}^{(s)}$ are desirable than $h_{n_{1},n_{2}}^{(a)}$ from a purely dynamical point of view because Re $h_{n_{1},n_{2}}^{(s)}$ are bounded from below. (By contrast, $h_{n_{1},n_{2}}^{(a)}$ is desirable from the point of view of TFD.) With $h_{n_{1},n_{2}}^{(s)}$, we have obtained the particular solutions of the Schrödinger equation as in Eq. (4.23). Then we have pointed out that the particular solutions with $n_{1} = n_{2}$ represent stable states. Such states do not appear in Feshbach-Tikchonsky’s quantization approach. Also, the stable states cannot be understood at the classical-mechanical level, because all the solutions of Eq. (2.2) represent damped oscillations and all the solutions of Eq. (2.4) represent amplified oscillations, provided that $4mk > \gamma^2$. The emergence of the stable states might be viewed as a stabilization of the Bateman model occurring at the quantum-mechanical level.

We have been able to obtain the two different sets of eigenvalues $h_{n_{1},n_{2}}^{(s)}$ and $h_{n_{1},n_{2}}^{(a)}$ that correspond, respectively, to the two unitary inequivalent basis $|n_{1},n_{2}\rangle$ and $|n_{1},n_{2}\rangle$ determined for the one operator $H$. From this fact, we see that quantum mechanics has, so to speak, flexibility in deriving the set of possible values of a dynamical variable such as $H$. That is, the set of possible values of a dynamical variable is obtained depending on the choice of basis. This flexibility originates in the fact that quantum mechanics is composed of two basic objects – dynamical variables (treated as operators) and state vectors, differently from classical mechanics, which is composed only of dynamical variables.

The Bateman model treats both the damped and amplified harmonic oscillators simultaneously on even ground, and therefore cannot be said to be a model only for the damped harmonic oscillator. To consistently treat only the damped har-
monic oscillator within the framework of analytical mechanics, we need to find a new Lagrangian that, unlike the Caldirola-Kanai Lagrangian, does not explicitly depend on time. This issue should be addressed in the near future.

References

[1] H. Bateman, On dissipative systems and related variational principles, Phys. Rev. 38 (1931) 815.
[2] P. M. Morse, H. Feshbach, Methods of Theoretical Physics, Part I, McGraw-Hill, New York, 1953.
[3] H. Dekker, Classical and quantum mechanics of the damped harmonic oscillator, Phys. Rep. 80 (1981) 1.
[4] M. Razavy, Classical and Quantum Dissipative Systems, 2nd Edition, World Scientific, Singapore, 2017.
[5] H. Feshbach, Y. Titchmarsh, Quantization of the damped harmonic oscillator, Transact. N.Y. Acad. Sci, 38, Ser. II (1977) 44.
[6] E. Celeghini, M. Rasetti, G. Vitello, Quantum dissipation, Ann. Phys. 215 (1992) 156.
[7] Y. N. Srivastava, G. Vitello, A. Widom, Quantum dissipation and quantum noise, Ann. Phys. 238 (1995) 200, arXiv:hep-th/9502044.
[8] M. Blasone, E. Graziano, O. K. Pashaev, G. Vitello, Dissipation and topologically massive gauge theories in the pseudo-euclidean plane, Ann. Phys. 252 (1996) 115, arXiv:hep-th/9603092.
[9] M. Blasone, P. Jiřba, Bateman’s dual system revisited: quantization, geometric phase and relation with the ground-state energy of the linear harmonic oscillator, Ann. Phys. 312 (2004) 354, arXiv:quant-ph/0102128.
[10] D. Chruściński, J. Jurkowski, Quantum damped oscillator I: dissipation and resonances, Ann. Phys. 321 (2006) 854, arXiv:quant-ph/0506007.
[11] D. Chruściński, Quantum damped oscillator II: Bateman’s Hamiltonian vs. 2D parabolic potential barrier, Ann. Phys. 321 (2006) 840, arXiv:quant-ph/0506009.
[12] R. Banerjee, P. Mukherjee, A canonical approach to the quantization of the damped harmonic oscillator, J. Phys. A: Math. Gen. 35 (2002) 5591, arXiv:quant-ph/0108055.
[13] Y. Takahashi, H. Umezawa, Thermo field dynamics, Collect. Phenom. 2 (1975) 55.
[14] H. Umezawa, Advanced Field Theory: Micro, Macro, and Thermal Physics, American Institute of Physics, New York, 1993.
[15] S. K. Pal, P. Nandi, B. Chakraborty, Connecting dissipation and non-commutativity: A Bateman system case study, Phys. Rev. A 97 (2018) 062110, arXiv:1803.03334 [quant-ph].
[16] F. Bagarello, Dissipation evidence for the quantum damped harmonic oscillator via pseudo-bosons, Theor. Math. Phys. 171 (2012) 497, arXiv:1106.4638 [math-ph].
[17] J. Guerrero, F. F. López-Ruiz, V. Aldaya, F. Cossío, Symmetries of the quantum damped harmonic oscillator, J. Phys. A: Math. Theor. 45 (2012) 475303, arXiv:1210.4058 [math-ph].
[18] A. Pais, G. E. Uhlenbeck, On field theories with non-localized action, Phys. Rev. 79 (1950) 145.
[19] M. Ostrogrodzki, Mémoires sur les équations différentielles, relatives au problème des isopérimètres, Mem. Acad. St. Petersbourg VI (1850) 385.
[20] R. P. Woodward, Ostrogrodzki’s theorem on Hamiltonian instability, Scholarpedia 10 (2015) 32243, arXiv:1306.02210 [hep-th].

\[ L_{\text{CK}} = e^{\gamma/m}\left(\frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2\right). \]

which indeed yields Eq. (2.2) and describes only the damped harmonic oscillator. However, it has been pointed out that the canonical quantization based on \(L_{\text{CK}}\) is accompanied by some problems [23, 24]. The Hamiltonian corresponding to \(L_{\text{CK}}\), rather than \(L_{\text{CK}}\) itself, was actually considered by Caldirola and Kanai independently [23, 24]. The doubled Lagrangian \(2L_{\text{CK}}\) was earlier found by Bateman [1] by substituting \(y = e^{\gamma/m}\dot{x}\) into the Bateman Lagrangian \(L_{\text{CK}}\). For this reason, \(L_{\text{CK}}\) is sometimes called the Bateman-Caldirola-Kanai Lagrangian.