An Example of Poincaré Symmetry with a Central Charge

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Abstract

We discuss a simple system which has a central charge in its Poincaré algebra. We show that this system is exactly solvable after quantization and that the algebra holds without anomalies.

I. INTRODUCTION

Central charges in symmetry algebras arise in two ways. They can be anomalies of a quantized theory, like (in two space-time dimensions) the Virasoro anomaly (triple derivative Schwinger term) in the diffeomorphism algebra of a diffeomorphism invariant theory [1] [2] or in the (infinite) conformal algebra of a conformally invariant theory [3] [4]. However, they can also arise classically, as for example in the conformally invariant Liouville model, where the (infinite) conformal algebra possesses a center obtained already by canonical (non-quantal) Poisson brackets [5] or as in the asymptotic symmetry group of anti-de Sitter space in 2 + 1 dimensions [6]. Another instance arises in the non-relativistic field theoretic realization of the Galileo group.

With the appearance of a number of systems with central charges in their symmetry algebras at the classical level, it is useful to study a simple model with this behavior. We examine a charged, scalar field in 1+1 dimensions, interacting with a constant external electric field. The Poincaré algebra of this system has a central charge appearing already at the classical level. As a check on Poincaré invariance, we verify the Dirac-Schwinger relation. This leads to a modified energy-momentum tensor. Next, we take the massless case and show that we can quantize the system and solve it exactly. Finally, we look at the quantized algebra of the massless case and show that it holds without anomalies, with the electric charge operator functioning as the central charge.

II. THE CLASSICAL SYMMETRIES

We begin with the Lagrangian in 1+1 dimensions for a complex scalar field of charge $e$ interacting with a constant external electromagnetic field, described by a position-dependent vector potential.

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\[ L = (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi \]  

(1)

with

\[ D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi \]  

(2)

\[ A_\mu = -\frac{1}{2} \epsilon_{\mu\nu} F x_\nu, \]  

(3)

where we use a flat metric of signature (+−) and define \( \epsilon^{01} = -\epsilon_{01} = +1 \). Because of the position dependence in the externally presented vector potential, the theory does not have manifest translation symmetry. However, since physical motion is manifestly translation invariant, we recover invariance of the action by adding a connection term to the transformation law of the field \( \phi \) under translation.

\[ \delta_\mu \phi = (\partial^\mu + \frac{1}{2} ie \epsilon^{\mu\nu} x_\nu F) \phi \]

\[ = (D^\mu + ie \epsilon^{\mu\nu} x_\nu F) \phi \]  

(4)

\[ \delta_\mu L = \partial^\mu \left[ (D^\nu \phi)^* D_\nu \phi - m^2 \phi^* \phi \right] \]  

(5)

The associated Noether current \( T^{\mu\nu}_C \) is the energy-momentum tensor for this theory

\[ T^{\mu\nu}_C = \theta^{\mu\nu} + 2 A^\nu J^\mu, \]  

(6)

\[ \theta^{\mu\nu} = (D^\mu \phi)^* (D^\nu \phi) + (D^\nu \phi)^* (D^\mu \phi) - \eta^{\mu\nu} \left( (D^\alpha \phi)^* (D_\alpha \phi) - m^2 \phi^* \phi \right), \]  

(7)

\[ J^\mu = ie [\phi^* (D^\mu \phi) - (D^\mu \phi)^* \phi]. \]  

(8)

Using the field equations of motion,

\[ (D^\mu D_\mu + m^2) \phi = 0, \]  

(9)

we see that our currents obey

\[ \partial_\mu J^\mu = 0 \]  

(10)

\[ \partial_\mu \theta^{\mu\nu} = \partial_\mu \theta^{\nu\mu} = \epsilon^{\mu\alpha} F J_\alpha \]  

(11)

\[ \partial_\mu T^{\mu\nu}_C = 0. \]  

(12)

Note that \( T^{\mu\nu}_C \) is not symmetric: it is conserved in the first index \( \mu \), while the second index \( \nu \) denotes the direction of the translation. Thus we arrive at our first unconventional result: Even though the fields are spinless, their canonical energy-momentum tensor is not symmetric.

In addition, the Lagrangian has the usual Lorentz symmetry

\[ \delta_L \phi = \epsilon^{\alpha\beta} x_\alpha \partial_\beta \phi \]  

(13)

\[ \delta_L L = \partial_\beta \left( \epsilon^{\alpha\beta} x_\alpha L \right) \]  

(14)

with the associated conserved current

\[ M^\mu = \epsilon_{\alpha\beta} x^\alpha (\theta^{\mu\beta} + A^{\beta} J^\mu) \]  

(15)

\[ \partial_\mu M^\mu = 0 \]  

(16)
We can define operators on linear functionals $G$ of $\phi$

\[
P^\mu (G[\phi]) = G[\delta^\mu_\nu \phi]
\]

\[
M(G[\phi]) = G[\delta_L \phi]
\]  

In this way, $P^\mu$ generates translations, and $M$ generates Lorentz transformations. Defining a bracket to be the commutator of these operators on linear functionals of $\phi$,

\[
[M, P^\mu] (G[\phi]) = G[\delta_\mu^\nu \delta_L \phi - \delta_L \delta_\nu^\mu \phi]
\]

\[
= G\left[\epsilon^{\mu\nu}(\partial_\nu + \frac{1}{2}ie\epsilon_{\nu\alpha} x^\alpha F)\phi\right]
\]

\[
= \epsilon^{\mu\nu} P_\nu(G[\phi])
\]  

\[
[P^\mu, P^\nu] (G[\phi]) = G[\delta_\nu^\mu \delta_L^\nu \phi - \delta_L \delta_\nu^\mu \phi]
\]

\[
= G[ie\epsilon^{\mu\nu} F \phi]
\]

\[
= -\epsilon^{\mu\nu} F Q (G[\phi]),
\]  

where $Q$ is the operator that multiplies by $-ie$, so it simply commutes with all other operators that act on linear functionals of $\phi$.

\[
Q(G[\phi]) = G[-ie\phi] = -ie G[\phi]
\]  

Thus, we have a Poincaré algebra in 1 + 1 dimensions with a central charge $\mathbb{1}$.

\[
[M, P^\mu] = \epsilon^{\mu\nu} P_\nu
\]

\[
[P^\mu, P^\nu] = -\epsilon^{\mu\nu} F Q
\]  

This algebra can be realized using Poisson brackets with the charges

\[
P^\mu = \int dx T_0^{0\mu}(t, x)
\]

\[
M = \int dx M^0(t, x)
\]  

\[
Q = \int dx J^0(t, x),
\]  

Since the charges are the spatial integrals of the time components of conserved currents, the charges are time-independent, assuming that the field $\phi$ dies off sufficiently rapidly. (We shall later show that the quantized versions of these operators are explicitly time-independent in the massless case.) We calculate $\pi$, the momentum conjugate to $\phi$, to be $\pi = (D^0 \phi)^*$. Similarly, the momentum conjugate to $\phi^*$ is $\pi^* = (D^0 \phi)$. Writing $J^\mu$, $T_0^{0\mu}$, and $M^0$ in terms of these quantities,

\[
J^0(t, x) = ie(\phi^* \pi^* - \phi \pi)
\]

\[
J^1(t, x) = ie((D_1 \phi)^* \phi - \phi^* (D_1 \phi))
\]  

\[
T_0^{00}(t, x) = \pi^* \pi + (D_1 \phi)^* (D_1 \phi) + m^2 \phi^* \phi + x F J^0
\]

\[
T_0^{01}(t, x) = -\pi (D_1 \phi) - (D_1 \phi)^* \pi^* + t F J^0
\]

\[
M^0(t, x) = x \left(\pi^* \pi + (D_1 \phi)^* (D_1 \phi) + m^2 \phi^* \phi\right)
\]

\[
+ t \left(\pi (D_1 \phi) + (D_1 \phi)^* \pi^*\right) + \frac{1}{2}(x^2 - t^2) F J^0
\]  

(31)
we can now calculate the equal-time Poisson brackets we need.

\[
\begin{align*}
\left[ J^0(x), J^0(y) \right] &= 0 \quad (32) \\
\left[ J^0(x), T^0_\mu(y) \right] &= -\epsilon^{\mu\nu} J^0(y) \delta'(x - y) \quad (33) \\
\left[ M^0(x), J^0(y) \right] &= \left( xJ^1(x) - tJ^0(x) \right) \delta'(x - y) \quad (34) \\
\left[ T^0_0(x), T^0_0(y) \right] &= \left( T^0_0(x) + T^0_0(y) \right) \delta'(x - y) \quad (35) \\
\left[ T^0_0(x), T^0_1(y) \right] &= \left( T^1_0(x) + T^0_0(y) + F(x - y) J^0(y) \right) \delta'(x - y) \quad (36) \\
\left[ M^0(x), T^0_\mu(y) \right] &= \left( -x_\nu \left( T^0_\nu(x) + T^\mu_\nu(y) \right) + x^\mu T^\nu_\nu(y) + \frac{1}{2} \left( x_\lambda x^\lambda \right) \epsilon^{\mu\nu} F J^\nu(y) \right. \\
&\quad \left. -\epsilon^{\mu\nu} F(x_\nu - y_\nu)x^\lambda J^\lambda(x) + y_\lambda \left( x^\mu \epsilon^{\nu\lambda} - x^\nu \epsilon^{\mu\lambda} \right) F J^\nu(y) \right) \delta'(x - y) \quad (37)
\end{align*}
\]

(where the common time argument has been suppressed). Thus, the charges satisfy

\[
\begin{align*}
[Q, M] &= 0 \quad (38) \\
[Q, P^\mu] &= 0 \quad (39) \\
[M, P^\mu] &= \epsilon^{\mu\nu} P_\nu \quad (40) \\
[P^\mu, P^\nu] &= -\epsilon^{\mu\nu} F Q \quad (41)
\end{align*}
\]

This is the same algebra that we obtained before in (22-23).

III. THE DIRAC-SCHWINGER RELATION

The Dirac-Schwinger relation is a method of proving Lorentz invariance. A system is Lorentz invariant if the energy-momentum tensor obeys the following condition (for Poisson brackets):

\[
\left[ T^0_0(x), T^0_0(y) \right] = \left( T^0_1(x) + T^0_1(y) \right) \delta'(x - y) \quad (42)
\]

The energy-momentum tensor we obtained in (3) obeys (42) with the indices reversed:

\[
\left[ T^0_0(x), T^0_0(y) \right] = \left( T^1_0(x) + T^0_1(y) \right) \delta'(x - y). \quad (43)
\]

Unfortunately, $T^0_1 \neq T^1_0$ is not symmetric in its indices, so the Dirac-Schwinger condition fails.

However, if we modify the energy-momentum tensor to make it symmetric, the condition then holds. By adding a superpotential $\epsilon^{\mu\beta} \partial_\beta V^\nu$, we obtain a new energy-momentum tensor

\[
T^{\mu\nu} = T^{\mu\nu}_C + \epsilon^{\mu\beta} \partial_\beta V^\nu = \theta^{\mu\nu} + \epsilon^{\alpha\nu} x_\alpha F J^\mu + \epsilon^{\mu\beta} \partial_\beta V^\nu. \quad (44)
\]

Requiring $T^{\mu\nu}$ to be symmetric, we obtain

\[
\partial_\mu V^\mu = -F x_\mu J^\mu. \quad (45)
\]
Since \( J^\mu \) is conserved, we can define a new variable \( h \) by

\[
J^\mu = \epsilon^{\mu\nu} \partial_\nu h
\]  

(46)

With this new expression for \( J^\mu \), we obtain

\[
\partial_\mu V^\mu = \partial_\mu (Fx_\nu \epsilon^{\mu\nu} h)
\]  

(47)

For solutions to (47) and (46), we can take

\[
V^\mu = Fx_\nu \epsilon^{\mu\nu} h
\]  

(48)

\[
h(t, x) = \int_{-\infty}^{\infty} dy \frac{1}{2} \epsilon(x - y) J^0(t, y),
\]  

(49)

where \( \epsilon(x - y) \) denotes the sign function. For these solutions, \( T^{\mu\nu} \) simplifies to

\[
T^{\mu\nu} = \theta^{\mu\nu} - \eta^{\mu\nu} hF
\]  

(50)

Checking our commutation relations, we see that

\[
[T^{00}(x), T^{00}(y)] = (T^{01}(x) + T^{01}(y)) \delta'(x - y),
\]  

(51)

so the Dirac-Schwinger condition indeed holds for \( T^{\mu\nu} \).

Alternately, we can derive the energy-momentum tensor (50) by varying the metric in a generally covariant version of the Lagrangian (1). We start by writing the generally covariant Lagrangian

\[
L = \sqrt{-g} (F^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) - m^2 \phi^* \phi)
\]  

(52)

with \( A_\mu \) no longer an external quantity, but a functional of the metric satisfying the relation

\[
\partial_\mu A_\nu - \partial_\nu A_\mu = \epsilon^{\mu\nu} F \sqrt{-g}
\]  

(53)

Varying the metric, we get

\[
\delta L = \frac{1}{2} \sqrt{-g} \partial_{\mu\nu} \delta g^{\mu\nu} - \sqrt{-g} J^\mu \delta A_\mu
\]  

(54)

Owing to the covariant conservation of \( J^\mu \), we can write

\[
\sqrt{-g} J^\mu = \epsilon^{\mu\nu} \partial_\nu h
\]  

(55)

as a defining relation for \( h \). Substituting (53) in (54) in the variation of the action and integrating by parts, we get

\[
\delta S = \int d^2x (\frac{1}{2} \sqrt{-g} \partial_{\mu\nu} \delta g^{\mu\nu} - \epsilon^{\mu\nu} \partial_\mu \delta A_\nu h).
\]  

(56)

Using our relation (53) and simplifying, we end up with our result

5
\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial L}{\partial g_{\mu\nu}} = \delta_{\mu\nu} - g_{\mu\nu} hF \]  

(57)

This lends credence that this is the correct energy-momentum tensor to use for the Dirac-Schwinger relation.

Unfortunately, the momentum associated with \( T_{\mu\nu} \) differs from the momentum associated with \( T_{\mu\nu}^C \).

\[
\int dx \left( T^{01} - T_{C}^{01} \right) = \int dx \partial_1 V^1 = V^1 \big|_{x=\infty} = -tQ
\]

(58)

Furthermore, by taking the time derivative of this difference, we see that both momenta cannot be conserved simultaneously, except possibly in the uncharged sector. (We shall later calculate the charges associated with our original energy-momentum tensor \( T_{\mu\nu}^C \) in the zero-mass case and see explicitly that they are time-independent. Therefore, the momentum associated with \( T_{\mu\nu} \) is not conserved, except in the uncharged sector.)

IV. SOLUTION OF THE EQUATIONS OF MOTION (FOR ZERO MASS)

For the remainder of this paper, we shall take our field \( \phi \) to be massless by setting \( m = 0 \). We shall also absorb \( e \) into \( F \), replacing \( eF \) by \( F \). Then, we shall use the equations of motion and cannonical commutation relations to quantize the system.

Putting \( \phi \) in the form

\[
\phi = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{i(k - \frac{1}{2} tF)x} \left( f(T)a_k + f(T)^* b_{-k}^\dagger \right) |F|^{-1/4},
\]

(59)

\[
T = \frac{\epsilon(F)}{|F|^{1/2}} (Ft - k)
\]

(60)

and setting the mass to zero, our equations of motion \([5]\) reduce to

\[
\left( \frac{d}{dT} \right)^2 + T^2) f(T) = 0
\]

(61)

which has a closed form solution in terms of Bessel functions

\[
f(T) = A(k)f_1(T) + B(k)f_2(T),
\]

(62)

\[
f_1(T) = |T|^{1/2} J_{-1/4}(\frac{1}{2} T^2),
\]

(63)

\[
f_2(T) = \epsilon(T)|T|^{1/2} J_{1/4}(\frac{1}{2} T^2).
\]

(64)

\( A(k) \) and \( B(k) \) are arbitrary functions that parameterize the creation and annihilation operators, \( a_k \) and \( b_k^\dagger \), which satisfy the usual commutation relations.

We now impose canonical quantization relations. Our form for \( \phi \) \([5]\) automatically satisfies \([\phi, \phi^*] = [\pi, \pi^*] = 0 \). The condition \([\phi, \pi] = [\phi^*, \pi^*] = i\delta(x - y) \) require

\[
f(T)f'(T)^* - f'(T)f(T)^* = i,
\]

(65)
which with the help of the Bessel function identity
\[ J_{-3/4}(x)J_{-1/4}(x) + J_{3/4}(x)J_{1/4}(x) = \frac{\sqrt{2}}{\pi x} \] (66)
implies
\[ A(k)B^*(k) - B(k)A^*(k) = \frac{\sqrt{2}}{4}\pi i. \] (67)

Apart from this constraint (which can be seen as a normalization condition), the choice of these functions remains arbitrary below. If one insists on choosing a parameterization, two useful parameterizations are the constant parameterization and the parameterization which reduces to the standard free-field expression in the limit where \( F \) goes to zero.

\[ A(k) = \frac{\pi}{4} F^{-1/2}|k| \left( J_{-3/4}(k^2/2|F|) - i\epsilon(Fk)J_{1/4}(k^2/2|F|) \right) \] (68)
\[ B(k) = -i\frac{\pi}{4} F^{-1/2}|k| \left( J_{-1/4}(k^2/2|F|) - i\epsilon(Fk)J_{3/4}(k^2/2|F|) \right). \] (69)

This second parameterization satisfies the requirement (67) by the Bessel function identity (66).

V. QUANTIZATION OF THE ALGEBRA

In this section, we shall use the original energy-momentum tensor \( T_{\mu\nu}^{(C)} \) and verify that its associated charges are time-independent. Inserting the solution for \( \phi \) (59, 62) into the classical expressions for the charges (24-26), and normal ordering, we get expressions for the quantized charges in terms of creation and annihilation operators.

\[ Q = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e \left[ a_k^\dagger a_k - b_{-k}^\dagger b_{-k} \right] \] (70)
\[ M = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ \left( -k^2/2F - \frac{\sqrt{2}}{\pi} iF(\partial A\partial B^* - \partial B\partial A^*) \right) \left( a_k^\dagger a_k - b_{-k}^\dagger b_{-k} \right) 
- \frac{1}{\sqrt{2}\pi} iF (A^*\partial B - B\partial A^* + A\partial B^* - B^*\partial A) \left( a_k^\dagger(\partial a)_k - (\partial b^\dagger)_{-k}b_{-k} - (\partial a^\dagger)_{k}a_{k} + b_{-k}^\dagger (\partial b)_k \right) 
- \frac{1}{8} F \left( a_k^\dagger(\partial^2 a)_k - (\partial^2 b^\dagger)_{-k}b_{-k} - 2(\partial a^\dagger)_{k}(\partial a)_{k} 
+ 2(\partial b^\dagger)_{-k}(\partial b)_{-k} + (\partial^2 a^\dagger)_{k}a_{k} - b_{-k}^\dagger (\partial^2 b)_k \right) 
+ \frac{\sqrt{2}}{\pi} iF (\partial AB - A\partial B) ((\partial a)_k b_{-k} + a_k (\partial b)_{-k}) 
+ \frac{\sqrt{2}}{\pi} iF (\partial AB - A\partial B)^* \left( -a_k^\dagger(\partial b^\dagger)_{-k} - (\partial a^\dagger)_{k}b_{-k}^\dagger \right) \right] \] (71)
\[ P^0 = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ -\sqrt{2} \pi F(A^* \partial B - B \partial A^* + A \partial B^* - B^* \partial A) \left( a_k^\dagger a_k + b_{-k}^\dagger b_{-k} \right) + \frac{1}{2} i F \left( a_k^\dagger (\partial a)_k + (\partial b^\dagger)_{-k}b_{-k} - (\partial a^\dagger)_k a_k - b_{-k}^\dagger (\partial b)_{-k} \right) + \frac{2\sqrt{2} \pi}{F} (\partial AB - A \partial B) a_k b_{-k} + \frac{2\sqrt{2} \pi}{F} (\partial AB - A \partial B^* a_k^\dagger b_{-k}^\dagger) \right] \]

\[ P^1 = \int_{-\infty}^{\infty} \frac{dk}{2\pi} k [a_k^\dagger a_k - b_{-k}^\dagger b_{-k}] \]

We note that these charges are manifestly time-independent, as claimed earlier. These charges satisfy the algebra

\[ [Q, P^u] = [Q, M] = 0 \]
\[ [M, P^u] = -ie^{\mu\nu} P_\nu \]
\[ [P^u, P^v] = ie^{\mu\nu} F Q \]

with no anomalies.

As a note of caution, we formed the charges first and then calculated the commutators, rather than calculating the commutators of the local currents and then integrating over space. In addition to the Virasoro anomaly, the currents have other anomalies. These other anomalies depend on time as well as the particular parameterization chosen for \( A(k) \) and \( B(k) \) in (62). Furthermore, when these anomalies are integrated over space, the results are ill-defined and depend on how the spatial integration is evaluated. As an illustrative example, let us sketch the calculation of the commutator between the quantized charges \( Q \) and \( M \) by integrating the commutator between \( :J^0: \) and \( :M^0: \).

\[ [Q, M] = \int dx dy \left[ : J^0(t, x) :, : M^0(t, y) : \right] \]

\[ :M^0(t, x) : = x : \theta^{00}(t, x) : - t : \theta^{01}(t, x) : + \frac{1}{2} (x^2 - t^2) F : J^0(t, x) : \]

\[ :J^0(t, x) : = \int \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{i(k-k')x} \left( \partial_{T_k} - \partial_{T_k'} \right) S(k, k', T_k, T_k') \]

\[ :\theta^{00}(t, x) : = \int \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{i(k-k')x} |F|^{1/2} \left( \partial_{T_k} \partial_{T_{k'}} + T_k T_{k'} \right) S(k, k', T_k, T_k') \]

\[ :\theta^{01}(t, x) : = \int \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{i(k-k')x} \epsilon(F) |F|^{1/2} \left( i\partial_{T_{k'}} T_k - i\partial_{T_{k}} T_k \right) S(k, k', T_k, T_{k'}) \]

\[ S(k, k', T_k, T_{k'}) = f_k f_{k'} a_{k'}^\dagger a_k + f_k^* f_{k'}^* b_{-k'}^\dagger b_{-k} + f_k f_{k'} a_k b_{-k'} + f_k^* f_{k'}^* b_{-k}^\dagger a_{k'} \]

where \( T_k \) is the expression (51) for \( T \), and the added subscript allows us to denote the same expression with the momentum appearing in the subscript substituted for \( k \). Similarly, \( f_k \) denotes \( f(k, T_k) \). It must also be noted that the independent variables with respect to the integrals are the momenta \( (k, k', \text{etc.}), x, \) and \( t \); however, the independent variables with respect to the derivatives are the momenta, the \( T \) variables with respect to each of the momenta \( (T_k, T_{k'}, \text{etc.}), \) and \( x \). The operator \( S \) satisfies
\[ [S(k, k', T_k, T_{k'}), S(p, p', T_p, T_{p'})] = 2\pi\delta(k - p') \left( f_k \tilde{f}_{k'}^* - f_k^* \tilde{f}_{k'} \right) S(p, k', T_p, T_{k'}) \]
\[ -2\pi\delta(p - k') \left( f_p \tilde{f}_{k'}^* - f_p^* \tilde{f}_{k'} \right) S(k, p', T_k, T_{p'}) \]
\[ +2\pi\delta(p - k')2\pi\delta(k - p') \left( f_k \tilde{f}_{k'}^* f_p^* f_{p'} - f_k^* f_{k'} f_p f_{p'} \right), \] (83)

where the delta functions are with respect to the same independent variables as the integration (momenta, \( t \), and \( x \)). As a consequence of the different collections of independent variables, derivatives of the \( S \) commutator (82) may not vanish, even though (83) vanishes identically.

We can now calculate the other commutators. The source of the inconsistency will be most evident if we do not evaluate the integrals over the delta functions in the anomalous term yet.

\[ [J^0(t, x), J^0(t, y)] = 0 \] (84)
\[ [J^0(t, x), \theta^{00}(t, y)] = -iJ^1(t, y)\delta'(x - y) \] (85)
\[ [J^0(t, x), M^0(t, y)] = -i \left( yJ^1(t, y) - tJ^0(t, y) \right)\delta'(x - y) \] (86)

\[ +y \int \frac{dk \, dk' \, dp \, dp'}{2\pi \, 2\pi \, 2\pi \, 2\pi} e^{i(k - k')x + i(p - p')y} ie \left( \partial_{T_k} - \partial_{T_{k'}} \right) \]
\[ |F|^{1/2} \left( \partial_{T_k} \partial_{T_{k'}} + T_p T_{p'} \right) \]
\[ 2\pi\delta(p - k')2\pi\delta(k - p') \left( f_k \tilde{f}_{k'}^* f_p^* f_{p'} - f_k^* f_{k'} f_p f_{p'} \right) \] (87)

\[ [Q, M] = \int dx dy \frac{dk \, dk' \, dp \, dp'}{2\pi \, 2\pi \, 2\pi \, 2\pi} y e^{i(k - k')x + i(p - p')y} ie \left( \partial_{T_k} - \partial_{T_{k'}} \right) \]
\[ |F|^{1/2} \left( \partial_{T_k} \partial_{T_{k'}} + T_p T_{p'} \right) \]
\[ 2\pi\delta(p - k')2\pi\delta(k - p') \left( f_k \tilde{f}_{k'}^* f_p^* f_{p'} - f_k^* f_{k'} f_p f_{p'} \right) \] (88)

\[ = \int \frac{dk \, dk' \, dp \, dp'}{2\pi \, 2\pi \, 2\pi \, 2\pi} \]
\[ (-i)2\pi\delta(k - k')2\pi\delta'(p - p')2\pi\delta(p - k')2\pi\delta(k - p') \]
\[ ie \left( \partial_{T_k} - \partial_{T_{k'}} \right) |F|^{1/2} \left( \partial_{T_p} \partial_{T_{p'}} + T_p T_{p'} \right) \]
\[ \left( f_k \tilde{f}_{k'}^* f_p^* f_{p'} - f_k^* f_{k'} f_p f_{p'} \right) \] (89)

This final form (89) illustrates the problem with evaluation – there are four delta functions of three independent quantities (four independent momenta, but three independent differences of momenta). Evaluating the charges first is equivalent to evaluating the first two delta functions first. In this order of evaluation, the term vanishes. Evaluating the current commutator first amounts to evaluating the last two delta functions first. In this case,
the value of the term depends on the order in which we evaluate the remaining two delta functions. This is equivalent to the value of the term depending on how the $x$ and $y$ integrations are evaluated. Since we are interested in the commutators of the charges, it seems more reasonable to evaluate them completely, including the spatial integration, before introducing any commutators. Fortunately, the term whose commutator gives an anomaly has a vanishing coefficient after the spatial integration. In this way we can see that what seems to be an ill-defined anomalous term actually vanishes.

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REFERENCES

[1] D. Cangemi, R. Jackiw, Phys. Lett. B337 (1994), 271
[2] C. Teitelboim, Phys. Lett. B126, (1983), 41
[3] S. Fubini, A.J. Hanson, R. Jackiw, Phys. Rev. D7 (1973), 1732
[4] Conformal Algebra in Space-Time, S. Ferrara, R. Gatto, A.F. Grillo (Springer-Verlag; Berlin, Heidelberg, New York; 1973)
[5] E. D’Hoker, R. Jackiw, Phys. Rev. D26 (1982), 3517
[6] J.D. Brown, M. Henneaux, Commun. Math. Phys. 104 (1986), 207
[7] A discussion of the extended Poincaré group can be found in D. Cangemi, R. Jackiw, Annals Phys. 225 (1993), 229
[8] D. Cangemi, R. Jackiw, Phys. Lett. B299 (1993), 24