Entropy production and Onsager symmetry in neoclassical transport processes of toroidal plasmas

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(Received 23 June 1995; accepted 25 September 1995)

Entropy production and Onsager symmetry in neoclassical transport processes of magnetically confined plasmas are studied in detail for general toroidal systems, including nonaxisymmetric configurations. It is found that the flux surface average of the entropy production defined from the linearized collision operator and the gyroangle-averaged distribution function coincides with the sum of the inner products of the thermodynamic forces and the conjugate fluxes consisting of the Pfirsch–Schlüter, banana-plateau, nonaxisymmetric parts of the neoclassical radial fluxes and the parallel current. It is proved from the self-adjointness of the linearized collision operator that the Onsager symmetry is robustly valid for the neoclassical transport equations in the cases of general toroidal plasmas consisting of electrons and multi-species ions with arbitrary collision frequencies. It is shown that the Onsager symmetry holds whether or not the ambipolarity condition is used to reduce the number of the conjugate pairs of the transport fluxes and the thermodynamic forces. The full transport coefficients for the banana-plateau and nonaxisymmetric parts are separately derived, and their symmetry properties are investigated. The nonaxisymmetric transport equations are obtained for arbitrary collision frequencies in the Pfirsch–Schlüter and plateau regimes, and it is directly confirmed that the total banana-plateau and nonaxisymmetric transport equations satisfy the Onsager symmetry.

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I. INTRODUCTION

Transport processes caused by binary Coulomb collisions between charged particles in toroidal magnetic configurations are described by the neoclassical transport theory. Particle and heat transport fluxes observed in most fusion devices exceed the predictions of the neoclassical transport theory, and thus are called anomalous transport. The anomalous transport is considered to result from the turbulent fluctuations driven by various plasma instabilities, which are not taken into account by the neoclassical theory. However, the neoclassical theory is regarded as a standard model with a well-established framework, which gives lower limits of the transport fluxes in quiescent plasmas close to thermal equilibrium, and it also has practical use for predicting the parallel transport such as a bootstrap current with more accuracy than the predictions for the radial transport. The characteristics of the neoclassical theory lie in the inclusion of the effects of the global magnetic field geometry due to the long mean free path of the hot particles. In less collisional regimes such as plateau and banana regimes, the processes in deriving the neoclassical transport equations, which relate the neoclassical transport fluxes to the thermodynamic forces, are more complicated than the derivation of the classical transport equations in more collisional regimes. Since the neoclassical transport coefficients contain the parameters relating to both the collisionality and the magnetic geometry, it is less trivial than in the case of the classical transport to show the validity of the Onsager symmetry for the neoclassical transport. In the cases of axisymmetric systems, the analytical expressions of the full neoclassical transport coefficients for all collisionality regimes have been obtained, and it is well-known that the Onsager symmetry holds for the axisymmetric neoclassical transport matrix.

The neoclassical transport equations for nonaxisymmetric systems are more complicated due to the geometrical complexities, and they involve the nonambipolar parts. The absence of symmetry may also cause the breakup of magnetic surfaces into islands and ergodic regions, although such problems are beyond the scope of this work. Here we assume the existence of toroidal nested magnetic surfaces as in other previous works. Balescu and Fantechi derive the full neoclassical transport coefficients for the nonaxisymmetric plasma in the plateau regime, and claim that the Onsager symmetry partly breaks down in that case. Contrary to Ref. 11, we show in the present work that the Onsager symmetry is robustly valid for the neoclassical transport equations in the cases of general toroidal plasmas consisting of electrons and multi-species ions with arbitrary collision frequencies.

Concerning the Onsager relation, it is important to discuss the entropy production resulting from the transport processes since the Onsager relation, if it holds, is satisfied only by the transport matrix connecting the conjugate pairs of transport fluxes and thermodynamic forces, which should be specified through the entropy production. According to the terminology in Ref. 3, we expect that the kinetic form of the entropy production defined from the collision operator should coincide with its thermodynamic form, in which the entropy production is expressed as the sum of the products of the conjugate pairs of the fluxes and forces. In Chap. 17 of Ref. 3, Balescu presented detailed analyses on the kinetic and thermodynamic forms of the entropy production in the classical and neoclassical transport processes for the axisymmetric case. Using the Hermitian moment representation, he confirmed that the kinetic form of the entropy production includes the thermodynamic form given by the products of the thermodynamic forces and their conjugate classical and
neoclassical Pfirsch–Schlüter fluxes, although he did not identify the thermodynamic form corresponding to the neoclassical banana-plateau transport. In this work, before the proof of the Onsager symmetry, we show for general toroidal geometry the complete coincidence between the kinetic form of the entropy production and its full thermodynamic form, including all contributions from the classical and neoclassical (Pfirsch–Schlüter, banana-plateau, nonaxisymmetric) fluxes. The main difference between our treatment and that in Ref. 3 is that we use directly the distribution function and the drift kinetic equations instead of the Hermitian moment expansion.

The proof of the Onsager symmetry for the neoclassical transport equations in general toroidal configurations is given in the similar manner to that in the Appendix of Ref. 2. The proof uses the self-adjointness of the linearized collision operator and the formal solution of the linearized drift kinetic equation, although neither axisymmetry of the magnetic configuration nor any condition for collisionality is required. We also derive the full neoclassical transport coefficients in the nonaxisymmetric system for collision frequencies in the Pfirsch–Schlüter and plateau regimes, from which the Onsager symmetry of the full neoclassical transport matrix is directly confirmed.

In the axisymmetric configurations, the ambipolarity is automatically satisfied by the neoclassical transport and the radial electric field does not affect the transport fluxes. On the other hand, in the nonaxisymmetric configurations, the radial electric field is determined through the particle transport equations if the ambipolarity condition is imposed. In both cases with and without the ambipolarity condition, we give the neoclassical transport equations and check their Onsager symmetry.

This work is organized as follows. In Sec. II, the entropy production defined from the collision operator is divided into the two parts, which are derived from the gyroangle-averaged and gyroangle-dependent parts of the distribution function. The entropy production from the gyroangle-dependent distribution function is given by the sum of the inner products of the classical radial particle and heat fluxes and the radial gradient thermodynamic forces. The transport matrix relating these classical fluxes and forces is shown to satisfy the Onsager symmetry. We find that the entropy production from the gyroangle-averaged distribution function is written as the sum of the inner products of the thermodynamic forces and the corresponding conjugate fluxes which consist of the Pfirsch–Schlüter, banana-plateau, nonaxisymmetric parts of the neoclassical radial fluxes and the parallel current. In Sec. III, using the formal solution of the linearized drift kinetic equation and the self-adjointness of the linearized collision operator, we prove that the Onsager symmetry is satisfied by the neoclassical transport equations for arbitrary collision frequencies in general toroidal systems, including nonaxisymmetric cases. The effects of the ambipolarity on the neoclassical transport coefficients are examined for both axisymmetric and nonaxisymmetric cases to show the robust validity of the Onsager symmetry independent of the use of the ambipolarity condition. In Sec. IV, the full transport coefficients are derived for the banana-plateau and nonaxisymmetric parts, separately, and their symmetry properties are investigated. We derive the nonaxisymmetric transport coefficients for arbitrary collision frequencies in the Pfirsch–Schlüter and plateau regimes, and directly confirm that the total banana-plateau and nonaxisymmetric transport equations satisfy the Onsager symmetry. Finally, conclusions and discussions are given in Sec. V. There, we discuss the reason why our two main results, i.e., the complete correspondence between the kinetic and thermodynamic forms of the entropy production, and the Onsager symmetry in the nonaxisymmetric case, were not confirmed in Ref. 3 and in Ref. 11, respectively.

II. ENTROPY PRODUCTION IN CLASSICAL AND NEOClassICAL TRANSPORT PROCESSES

Here, we show that the thermodynamic form of the entropy production is equivalent to its kinetic form defined from the collision operator for the classical and neoclassical transport in the general case of magnetically confined plasma with arbitrary toroidal geometry. For that purpose, we first describe several properties of the collision operator which is denoted for species $a$ by

$$C_a = \sum_b C_{ab}(f_a f_b),$$  \hspace{1cm} (1)

where $f_a$ ($f_b$) is the distribution function of the species $a$ ($b$) and $C_{ab}$ represents the contribution from the collision between the particles $a$ and $b$.

The collision operator conserves the particles’ number, momentum, and kinetic energy, which is written as

$$\mathbb{E} \ d^3v C_{ab} = 0,$$
$$\mathbb{E} \ d^3vm_a v C_{ab} + \mathbb{E} \ d^3vm_b v C_{ba} = 0,$$ \hspace{1cm} (2)
$$\mathbb{E} \ d^3v \frac{1}{2} m_a v^2 C_{ab} + \mathbb{E} \ d^3v \frac{1}{2} m_b v^2 C_{ba} = 0.$$

Furthermore, the collision operator is invariant under arbitrary translational and rotational transform of the velocity variable $v$ of distribution functions, which is expressed by

$$\mathcal{T} C_{ab}(f_a f_b) = C_{ab} (\mathcal{T} f_a \cdot \mathcal{T} f_b),$$
$$\mathcal{R} C_{ab}(f_a f_b) = C_{ab} (\mathcal{R} f_a \cdot \mathcal{R} f_b),$$ \hspace{1cm} (3)

where $\mathcal{T}$ and $\mathcal{R}$ denote functions $f$ with arbitrary translational and rotational transform of the velocity variable $v$, respectively.

The entropy production for the species $a$ is defined from the collision operator by

$$\dot{S}_a = \sum_b \dot{S}_{ab} = - \sum_b \mathbb{E} \ d^3v (\ln f_a) C_{ab}(f_a f_b).$$ \hspace{1cm} (4)

The second law of the thermodynamics or the positive definiteness of the entropy production is given by

$$\dot{S}_{ab} + \dot{S}_{ba} \geq 0,$$
$$\sum_a \dot{S}_a \geq 0,$$ \hspace{1cm} (5)
where the total entropy production \( \Sigma_a \dot{S}_a \) vanishes if and only if the distribution functions for all species are the Maxwellian with the same temperature and the same mean velocity. However, even if two particle species with much different masses (such as for electrons and ions) have the Maxwellian distributions with different temperatures, the collisional heat exchange between the two species is negligibly slow and the entropy production due to their collisions is so small that we hereafter neglect it based on the small mass ratio ordering.

For magnetically confined plasmas, the lowest-order distribution function for the species \( a \) with respect to the drift ordering is given by the Maxwellian with no mean velocity as

\[
f_a = f_{a0} + f_{a1} + \mathcal{O}(\delta^2),
\]

where \( v_{\text{ta}}^2 = 2T_a/m_a \) and \( x_a = v/v_{\text{ta}} \) are defined from the temperature \( T_a \). The distribution functions and the collision operator are perturbatively expanded in the drift ordering parameter \( \delta = p/L \) (\( p \): the thermal gyroradius, \( L \): the equilibrium scale length) as

\[
f_a = f_{a0} + f_{a1} + \mathcal{O}(\delta^2),
\]

where the linearized collision operator \( C_{ab}^L \) is defined by

\[
C_{ab}^L(f_{a0} f_{b0}) = C_{ab}(f_{a0} f_{b0}) + \mathcal{O}(\delta^2).
\]

The linearized collision operator also has the conservation and symmetry properties, which are expressed by Eqs. (2) and (3) with \( C_{ab} \) replaced by \( C_{ab}^L \). The self-adjointness of the linearized collision operator is written as

\[
T_a \mathcal{E} d^3 v \frac{g_a}{f_{a0}} C_{ab}^L(h_{a1} h_{b1}) + T_b \mathcal{E} d^3 v \frac{g_b}{f_{b0}} C_{ba}^L(h_{b1} h_{a1}) = 0,
\]

which is exactly valid for \( T_a = T_b \) and is approximately satisfied for \( T_a \neq T_b \) when \( m_a/m_b \ll 1 \) or \( m_b/m_a \ll 1 \). For example, in the case of collisions between ions (\( a = i \)) and electrons (\( b = e \)) with \( T_i \neq T_e \), \( C_{ie}^L \) contains a part which breaks the complete self-adjointness although it is neglected to the lowest order in \((m_i/m_e)^2\) (see Sec. IV of Ref. 13).

Concerning the positive definiteness of the entropy production described in Eq. (5), we find the positive definiteness of the quadratic form associated with the linearized collision operator as

\[
\ln f_a = \ln f_{a0} + f_{a1} + \mathcal{O}(\delta^2),
\]

the entropy production \( \dot{S}_a \) up to \( \mathcal{O}(\delta^2) \) is given by

\[
\sigma_a = \sum_b \sigma_{ab} = -\sum_b \mathcal{E} d^3 v \frac{f_{a1} f_{b0}}{f_{a0}} C_{ab}(f_{a0} f_{b1}).
\]

Instead of Eq. (5), we have from Eq. (10)

\[
T_a \sigma_{ab} + T_b \sigma_{ba} \geq 0, \quad \sum_a T_a \sigma_a = 0.
\]

Now, let us divide the first-order distribution function \( f_{a1} \) into the gyroangle-averaged part \( f_{a1} \) and the gyroangle-dependent part \( \tilde{f}_{a1} \) as

\[
f_{a1} = f_{a1} + \tilde{f}_{a1}.
\]

Due to the rotational symmetry of the collision operator in the velocity space shown in Eq. (3), the entropy production \( \sigma_a \) separates into the corresponding parts as

\[
\sigma_a = \tilde{\sigma}_a + \tilde{\sigma}_a,
\]

where

\[
\tilde{\sigma}_a = -\sum_b \mathcal{E} d^3 v \frac{f_{a1} f_{b0}}{f_{a0}} C_{ab}(\tilde{f}_{a1} \tilde{f}_{b1}),
\]

\[
\tilde{\sigma}_a = -\sum_b \mathcal{E} d^3 v \frac{f_{a1} f_{b0}}{f_{a0}} C_{ab}(\tilde{f}_{a1} \tilde{f}_{b1}).
\]

First, we consider the entropy production \( \tilde{\sigma}_a \) due to the gyroangle-dependent part of the distribution function. The gyroangle-dependent part \( \tilde{f}_{a1} \) is given from the lowest-order distribution function \( f_{a0} \) by

\[
\tilde{f}_{a1} = \frac{\mathbf{v} \times \mathbf{n}}{\Omega_a} \cdot \nabla f_{a0} - \frac{\mathbf{v} \times \mathbf{n} \cdot \nabla \mathbf{v}}{\Omega_a} \cdot \mathbf{v} = \frac{m_a}{\Omega_a} \mathbf{u}_{a0} \cdot \nabla \mathbf{f}_{a0} + \mathbf{u}_{a0} \cdot \mathbf{q}_{a0} \nabla \Sigma_a = \frac{m_a}{\Omega_a} \mathbf{u}_{a0} \cdot \nabla \mathbf{f}_{a0} + \mathbf{u}_{a0} \cdot \mathbf{q}_{a0} \nabla \Sigma_a = \frac{m_a}{\Omega_a} \mathbf{u}_{a0} \cdot \nabla \mathbf{f}_{a0} + \mathbf{u}_{a0} \cdot \mathbf{q}_{a0} \nabla \Sigma_a,
\]

which is valid for \( T_a \neq T_b \) to the lowest order of the small mass ratio \( m_a/m_b \ll 1 \) (or \( m_b/m_a \ll 1 \)) as Eq. (9). Using in Eq. (4)

\[
\ln f_{a1} = \ln f_{a0} + f_{a1} + \mathcal{O}(\delta^2),
\]

the thermodynamic forces \( X_{a1} \) and \( X_{a2} \) are defined from the radial gradients of the pressure \( p_a \), the electrostatic potential \( \Phi \), and the temperature \( T_a \) as

\[
X_{a1} = -\frac{\rho'_a}{n_a} - e_a \Phi', \quad X_{a2} = -T_a \frac{\partial}{\partial T_a}.
\]

In terms of which the perpendicular components of the fluid velocity and the heat flow are
\[ u_{a} = \frac{1}{n_{a}} \int d^3v f_{a \perp} v_{\perp} = \frac{e X_{a}}{e_{a} B} \nabla V \cdot \mathbf{n}, \]
\[ q_{a} = \frac{1}{n_{a}} \int d^3v f_{a \perp} v_{\perp} x_{a}^2 - \frac{5}{2} = \frac{5 c X_{a}}{2 e_{a} B} \nabla V \cdot \mathbf{n}. \]  
(20)

Substituting Eq. (17) into Eq. (16), we obtain
\[ T_{a} \bar{\sigma}_{a} = J_{a1}^{cl} X_{a1} + J_{a2}^{cl} X_{a2}, \]  
(21)

where
\[ J_{a1}^{cl} = \Gamma_{a}^{cl} \nabla V, \quad J_{a2}^{cl} = \frac{1}{T_{a}} q_{a}^{cl} \nabla V \]  
(22)
are the radial components of the classical particle and heat fluxes defined by
\[ \Gamma_{a}^{cl} = \frac{c}{e_{a} B} F_{a1} \cdot \mathbf{n}, \quad \frac{1}{T_{a}} q_{a}^{cl} = \frac{c}{e_{a} B} F_{a2} \cdot \mathbf{n}. \]  
(23)

Here the friction forces \( F_{a1} \) and \( F_{a2} \) are given by
\[ F_{a1} = \sum_{b} \int d^3v m_{a} v_{\perp} C_{ab}^{L} (f_{a1} \cdot f_{b1}), \]
\[ F_{a2} = \sum_{b} \int d^3v m_{a} v_{\perp} x_{a}^2 - \frac{5}{2} C_{ab}^{L} (f_{a1} \cdot f_{b1}). \]  
(24)

It should be noted that, from the rotational symmetry of the collision operator, that \( \tilde{F}_{a1} \) does not contribute to the perpendicular friction forces \( F_{a1} \), while \( \tilde{F}_{a1} \) does not contribute to the parallel friction forces \( F_{a2} \). Equation (21) shows that the entropy production \( \bar{\sigma}_{a} \) defined from the gyro-angulare-dependent part of the distribution function is caused by the classical particle and heat transport, and that the classical fluxes \( J_{a1}^{cl} \) and \( J_{a2}^{cl} \) are conjugate to the thermodynamic forces \( X_{a1} \) and \( X_{a2} \), respectively. The momentum conservation due to the collision operator gives
\[ \sum_{a} F_{a1} = 0, \]  
(25)
which in turn causes the classical fluxes to satisfy the ambipolarity as
\[ \sum_{a} e_{a} F_{a}^{cl} = 0. \]  
(26)

We have the relations between the perpendicular friction forces and flows from Eqs. (17) and (24) as
\[ \int F_{a1} \cdot \mathbf{F}_{a2} = \sum_{b} \int l_{11}^{ab} l_{22}^{ab} \mathbf{u}_{\perp}^{\perp} - \frac{2}{5} q_{\perp}^{ab}, \]  
(27)
where the coefficients \( l_{jk}^{ab} \) are the same ones defined in Ref. 2 and are given by
\[ l_{jk}^{ab} = \delta_{jk} \frac{m_{a}^2}{T_{a}} \sum_{a^{'}} \int d^3v l_{1j-1}^{(3/2)}(x_{a^{'}}) C_{aa^{'}} [v_{i}], \]
\[ \times L_{k-1}^{(3/2)}(x_{a}^{'}) f_{a^{'}} f_{a}, \quad \frac{m_{a} m_{b}}{T_{b}} \int d^3v \]
\[ \times L_{j-1}^{(3/2)}(x_{a}^{'}) C_{ab} [f_{a}, v_{i} L_{k-1}^{(3/2)}(x_{a}^{'}) f_{b}]. \]  
(28)

Here, \( L_{n}^{(3/2)}(x^2) = 1, \) \( L_{1}^{(3/2)}(x^2) = \frac{1}{2} x^2, \) \( \cdots, \) are the Laguerre polynomials of order \( \frac{3}{2} \). In Eqs. (27) and (28), the rotational symmetry of the linearized collision operator is used. From the self-adjointness of the linearized collision operator shown in Eq. (9), the coefficients \( l_{jk}^{ab} \) have the following symmetry:
\[ l_{jk}^{ab} = 0. \]  
(29)

The momentum conservation property described in Eq. (25) imposes another constraint on the coefficients \( l_{jk}^{ab} \):
\[ \sum_{a} l_{jk}^{ab} = 0. \]  
(30)

From Eq. (21) and the ambipolarity condition \( \sum_{a} e_{a} F_{a}^{cl} = 0 \) given by Eq. (26), we obtain
\[ \sum_{a} T_{a} \bar{\sigma}_{a} = \sum_{a} \left[ J_{a1}^{cl} X_{a1} + J_{a2}^{cl} X_{a2} \right] = \sum_{a \neq 1} J_{a1}^{cl} X_{a1} + \sum_{a} J_{a2}^{cl} X_{a2} \]  
(31)
where \( X_{a1}^{*} (a \neq 1) \) is defined by
\[ X_{a1}^{*} = X_{a1} - \frac{e_{a}}{e_{b}} X_{b1} = - \frac{p_{a}'}{n_{a}} + \frac{e_{a} p_{b}'}{e_{b} n_{b}} (a \neq 1). \]  
(32)

Here we have chosen a certain particle species denoted by \( I \). We hereafter regard \( I \) as the ion species with the smallest particle number density. If a plasma consists of electrons and a single ion species \( i \), we take \( I = i \).

Equation (31) shows that the number of the conjugate pairs of the classical fluxes and thermodynamic forces is reduced by employing the new pairs \( (J_{a1}^{*} X_{a1}), J_{a2}^{*} X_{a2} \) instead of \( (J_{a1}, X_{a1}), (J_{a2}, X_{a2}) \). We also find that the radial electric field does not appear in the new set of the thermodynamic forces. The transport equations which relate the classical fluxes \( (J_{a1}, J_{a2}) \) to the thermodynamic forces \( (X_{a1}, X_{a2}) \) are obtained from Eqs. (20), (22), (23), and (27) as
\[ \left( L_{a}^{cl} \right)^{ab} = \sum_{b} \int \left( L_{b}^{cl} \right)^{ab} \frac{(L_{a}^{cl})^{ab}}{T_{a}} X_{b1}, \]  
(33)
where the classical transport coefficients \( (L_{a}^{cl})^{jk}_{ab} \) are given by
\[ (L_{a}^{cl})^{jk}_{ab} = (-1)^{j+k-1} \frac{c_{a} e_{a} B}{e_{b} e_{b}} l_{jk}^{ab} (j, k = 1, 2). \]  
(34)

From Eq. (29), we have the Onsager symmetry for the classical transport as
\[ (L_{a}^{cl})^{jk}_{ab} = (L_{a}^{cl})^{kj}_{ab} (j, k = 1, 2). \]  
(35)

Equation (30) yields
\[ \sum_{a} e_{a} (L_{a}^{cl})^{ab}_{1k} = 0 \quad (k = 1, 2). \]  
(36)

We easily find from Eqs. (32), (33), (35), and (36) that the classical transport equations for the pairs \( (J_{a1}^{*}, X_{a1}^{*}), (J_{a2}^{*}, X_{a2}^{*}) \) are given by.

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\[ j_{a_1 (a + r)}^{f_1} = \sum_{b=1}^{n} (L^{j kl}_{cb}) X_{a_1 b 1} + \sum_{b=1}^{n} (L^{j kl}_{cb}) X_{a_2 b 2} , \]
\[ j_{a_2}^{f_1} = \sum_{b=1}^{n} (L^{j kl}_{cb}) X_{a_1 b 1} + \sum_{b=1}^{n} (L^{j kl}_{cb}) X_{a_2 b 2} , \]

which shows that the transport coefficients \((L^{j kl}_{cb})\) are the same as in Eq. (33), except for the limitation \((a,j), (b,k) \neq (1,1)\), and that the Onsager symmetry is valid for both of the conjugate pairs. We should note that the ambipolarity condition (36) reduces the number of the thermodynamic forces required for determining the classical fluxes by one, and that the radial electric field does not enter the reduced set of the thermodynamic forces \((X^{a_1 (a + r)}_{a_1}) - X_{a_2} \).

Next, let us consider the entropy production \(\bar{\sigma}_a\) due to the first-order gyro-angle-averaged distribution function \(f_{a_1}\), which satisfies the linearized drift kinetic equation:1–3,7–11,14,15

\[ v_{\perp} n \cdot \nabla f_{a_1} + v_{da} \cdot \nabla f_{a_0} = - \frac{e_a}{T_a} v_{\parallel} E_{\perp} f_{a_1} = C_{a}^b (\nabla f_{a_1}) , \]

where \(v_{\perp} n\) is the sum of the \(\text{E} \times \text{B}, \nabla B\) and curvature drift velocities, and \(C_{a}^b (\nabla f_{a_1}) = \sum_{b} C_{a}^b (\nabla f_{a_1}) \). Here, it should be noted that the electric drift term \(v_{\parallel} \nabla f_{a_1}\) is sometimes retained in the linearized drift kinetic equation16 which causes the nonlinear radial electric field dependence of the neoclassical transport coefficients and of the ambipolarity condition for a nonaxisymmetric system. There are two main reasons why the electric drift term \(v_{\parallel} \nabla f_{a_1}\) is not included in Eq. (38). One is due to the drift ordering \(v_{\perp} n \cdot \nabla f_{a_1}\), from which \(v_{\perp} n \cdot \nabla f_{a_1}\) should be neglected compared with \(v_{\parallel} n \cdot \nabla f_{a_1}\) in the linearized drift kinetic equation. Equation (38) is the standard linearized drift kinetic equation widely accepted in literatures (see Refs. 1–3 and Refs. 7–11). (However, if the radial electric field much larger than

\[ \left\langle \int d^3 v f_{a_1} v_{\perp} \cdot \nabla = \left( \frac{\nabla V}{m_a \Omega_a} \cdot \nabla \right) \cdot \nabla p_{a_1} + \nabla \cdot \pi_a \right\rangle \]

\[ = - \frac{c}{e_a B^2} \left( \frac{B}{B} \right) \nabla \cdot \nabla p_{a_1} + \nabla \cdot \pi_a )
\]

\[ = - \frac{c}{e_a B^2} \left( \frac{B}{B} \right) \nabla \cdot \nabla p_{a_1} + \nabla \cdot \pi_a ) + \frac{c}{e_a B^2} \nabla \cdot \nabla p_{a_1} - \frac{c}{e_a B^2} \left( \frac{1}{B} \nabla \cdot \nabla p_{a_1} + \nabla \cdot \pi_a \right) \left( \frac{B}{B^2} \right) \left( \frac{B}{B} \right) \left( \frac{B}{B^2} \right) \right) \]

\[ = J_{a_1}^{\text{pol}} - \frac{n_a c}{B^2} \left( \frac{E}{B} \right) \left( \frac{B}{B^2} \right) \left( \frac{B}{B^2} \right) \left( \frac{B}{B^2} \right) \right) \]

\[ \left( \int d^3 v f_{a_1} \left( \frac{x_3^2}{2} \right) v_{\perp} \cdot \nabla \right) \]

\[ = - \frac{c}{e_a B^2} \left( \frac{B}{B} \right) \nabla \cdot \nabla p_{a_1} + \nabla \cdot \pi_a \right) \]

\[ = - \frac{c}{e_a B^2} \left( \frac{B}{B} \right) \nabla \cdot \nabla p_{a_1} + \nabla \cdot \pi_a \right) + \frac{c}{e_a B^2} \nabla \cdot \nabla p_{a_1} \left( \frac{B}{B^2} \right) \left( \frac{B}{B^2} \right) \right) \]

\[ = J_{a_2}^{\text{pol}} , \]
where the Hamada coordinates \((V, \theta, \zeta)\) are employed to represent the poloidal and toroidal components of the magnetic fields (see Appendix A) and Eq. (A2) is used. Here, we have used the definitions

\[
\rho = \frac{1}{f_0} \int draft^{\frac{1}{2}} m_a v_i^2 f_{ai}, \quad \theta = \frac{1}{f_0} \int d draft^{\frac{1}{2}} m_a v_i^2 \times m_a v_i^2 (x_i^2 - \frac{1}{2}) f_{ai}, \quad \pi_a = (\rho - \rho_0)(\eta_i - \frac{1}{2}) = \int d draft^{\frac{1}{2}} m_a (v_i^2 - \frac{1}{2} v_i^2 f_{ai})(\eta_i - \frac{1}{2}) = \int d draft^{\frac{1}{2}} m_a (\frac{1}{2} v_i^2 f_{ai})(\eta_i - \frac{1}{2}).
\]

The neoclassical particle and heat fluxes are given by

\[
J_{a1}^{\text{nc1}} = \left\langle \rho a \bar{\nabla} V \right\rangle^{\text{nc1}} = J_{a1}^{\text{PS}} + J_{a1}^{\text{pp}} + J_{a1}^{\text{ma}},
\]

\[
J_{a2}^{\text{nc2}} = \frac{1}{T_a} \left\langle \rho a \bar{\nabla} V \right\rangle^{\text{nc2}} = J_{a2}^{\text{PS}} + J_{a2}^{\text{pp}} + J_{a2}^{\text{ma}},
\]

which consist of the Pfirsch–Schlüter \((J_{a1}^{\text{PS}})\), the banana-plateau \((J_{a1}^{\text{pp}})\), and nonaxisymmetric \((J_{a1}^{\text{ma}})\) parts defined by

\[
J_{a1}^{\text{PS}} = \left\langle \rho a \bar{\nabla} V \right\rangle^{\text{PS}} = \frac{c}{e_B a B^2} \left\langle B \bar{\nabla} \phi_B^2 + B \bar{\nabla} \pi_B \right\rangle,
\]

\[
J_{a1}^{\text{pp}} = \left\langle \rho a \bar{\nabla} V \right\rangle^{\text{pp}} = \frac{c}{e_B a B^2} \left\langle B \bar{\nabla} \pi_B \right\rangle,
\]

\[
J_{a1}^{\text{ma}} = \left\langle \rho a \bar{\nabla} V \right\rangle^{\text{ma}} = \frac{c}{e_B a B^2} \left\langle B \bar{\nabla} \pi_B \right\rangle.
\]

Here the Pfirsch–Schlüter fluxes are written as parts of the neoclassical fluxes as in Refs. 1–3 where the term “neoclassical” is used for the transport due to guiding center motions in a toroidal magnetic configuration affected by collisions, which is a contrast to the “classical” transport caused by particle gyro-motions with collisions. (However, the term “neoclassical” is sometimes used in a narrower sense for referring to the transport fluxes due to particles with long mean free paths such as the banana-plateau fluxes, which exclude the Pfirsch–Schlüter fluxes.)

Then, we find that the flux surface average of the second term in the right-hand side of Eq. (39) is given by the products of the neoclassical radial fluxes and the thermodynamic forces as

\[
\frac{n_e c}{T_a} \int_{a1} J_{a1}^{\text{nc1}} \frac{E_{a1}(B E_a)}{T_a} + \frac{n_e c}{T_a B^2} \frac{E_{a1} B \bar{\nabla} \left\langle B^2 \right\rangle}{B^2} X_{a1},
\]

where Eq. (A7) is used. Then, we finally obtain from Eqs. (39), (44), and (45) the thermodynamic form of the flux-surface-averaged entropy production \(\bar{\sigma}_a\) as

\[
T_a \bar{\sigma}_a = J_{a1}^{\text{nc1}} X_{a1},
\]

where the parallel flux \(J_{a3}\) and the parallel force \(X_{a3}\) are defined by

\[
J_{a3} = \frac{n_e (B u_{ia})}{(B E_a)^{3/2}}, \quad X_{a3} = \frac{\left\langle B E_a \right\rangle}{(B E_a)^{3/2}}.
\]

Thus the flux surface average of the entropy production due to the gyroangle-averaged distribution functions is given in the thermodynamic form, in which the neoclassical radial fluxes \(J_{a1}^{\text{nc1}}\), \(J_{a2}^{\text{nc2}}\) and the parallel current \(J_{a3}\) are conjugate to the radial gradient forces \(X_{a1}\), \(X_{a2}\) and the parallel electric field \(X_{a3}\), respectively. It should be noted that the neoclassical thermodynamic form of the entropy production can be obtained only through the magnetic surface average as in Eq. (48), which is a remarkable contrast to the classical thermodynamic form (21) defined locally in the configuration space.

Now, let us consider the ambipolarity condition for the neoclassical particle fluxes. Using the momentum conservation (25) by collisions and the charge neutrality condition \(\Sigma_a n_a e_a = 0\), we obtain the flux surface average of the total parallel momentum balance equation as \(\Sigma_a \left\langle B \bar{\nabla} \pi_B \right\rangle = 0\). Then, we find from the definitions in Eq. (43) that the intrinsic ambipolarity holds for both of the Pfirsch–Schlüter and banana-plateau particle fluxes in the same way as for the classical particle fluxes, which implies that the ambipolar conditions,

\[
\sum_a e_a J_{a1}^{\text{ps}} = \sum_a e_a J_{a1}^{\text{pp}} = 0,
\]

are valid for arbitrary values of the thermodynamic forces \((X_{a1}, X_{a2}, X_{a3})\). On the other hand, the nonaxisymmetric particle fluxes \(J_{a1}^{\text{ma}}\) and accordingly the total neoclassical particle fluxes \(J_{a1}^{\text{nc1}}\) are not ambipolar generally. If the ambipolarity condition for the total neoclassical particle fluxes,

\[
\sum_a e_a J_{a1}^{\text{nc1}} = 0,
\]

is imposed, we find in the similar way as in Eq. (31) that Eq. (48) is rewritten as
\[ \sum_a T(a) \sigma_a = \sum_{a1} J^{ncl,1} + \sum_a J^{ncl,2} + J_{E} X_{E}. \]  

In axisymmetric toroidal systems, the nonaxisymmetric fluxes \( J^{ncl,a}_a \) (\( j = 1,2 \)) vanish and therefore the neoclassical particle fluxes \( J^{ncl}_a \) are intrinsically ambipolar. As discussed in the next section, in nonaxisymmetric systems, the ambipolarity condition (51) combined with the neoclassical transport equations gives a constraint on the thermodynamic forces \((X_{a1}, X_{a2}, X_{E}\)) from which the radial electric field \(-\Phi'\) is expressed by a linear form in the pressure and temperature gradients and the parallel electric field. Then, independent thermodynamic forces for nonaxisymmetric systems are given not by the set \((X_{a1}, X_{a2}, X_{E})\) but by the reduced one \((X^*_a(X_{a1}+X_{a2}), X_{E})\). In the present work, we show the neoclassical transport equations for both cases with \((X_{a1}, X_{a2}, X_{E})\) and with \((X^*_a(X_{a1}+X_{a2}), X_{E})\) used as the forces, in order to elucidate the relation of the ambipolarity to the axisymmetry and to the Onsager symmetry.

### III. ONSAGER SYMMETRY OF NEOCLASSICAL TRANSPORT EQUATIONS FOR GENERAL TOROIDAL SYSTEMS

In this section, it is proved that the Onsager symmetry is satisfied by the neoclassical transport equations for general toroidal systems, including nonaxisymmetric cases. For that, it is convenient to define the distribution function \( g_a \) by

\[ g_a = \tilde{f}_a - e \frac{E}{B} \int \frac{d^3 l}{B} [B(E) - \frac{B^2}{(B^2)^{1/2}}(B(E))]. \]

where \( f^{'d}l \) denotes the integral along the magnetic field line. Then, Eq. (38) is rewritten as

\[ g_a \cdot \nabla g_a - C_{L}^{(a)} g_a = \frac{1}{\tau_a} f_{a0}(S_{a1} X_{a1} + S_{a2} X_{a2} + S_{a3} X_{a3}), \]

where the functions \( S_{aj} \) (\( j = 1,2,3 \)) are defined by

\[ S_{aj} = \left( \frac{x_a^2}{2} - \frac{x_a}{2} \right) \frac{1}{df_{a0}} \left( S_{a1} X_{a1} + S_{a2} X_{a2} + S_{a3} X_{a3} \right), \]

Here, it is worthwhile making some remarks on the case where, just as in Ref. 16, the electric drift term \(-E \tilde{g}_a \cdot \nabla \tilde{g}_a\) is added in the left-hand side of Eq. (54). Here, in \( v_{E} \nabla \tilde{g}_a \), the spatial derivative \( \nabla \) is taken with \( (x, v, \alpha) \) as independent phase space variables defined in Ref. 16, and the electric drift velocity is given by \( v_{E} = c \mathbf{E} \mathbf{X} / (B^2) \) to satisfy an important phase space conservative property (see Ref. 16). In that case, nonlinear neoclassical transport equations are derived due to the nonlinear dependence on the radial electric field as mentioned after Eq. (38). Here, let us artificially regard the radial electric field \( E_v = -\partial \Phi / \partial V \) added in the left-hand side of Eq. (54) as an independent parameter, although \( E_v \) is already contained as a part of the thermodynamic forces in the right-hand side of Eq. (54). By doing this, the resulting transport equations are written in the apparently linear form \( \mathbf{J} = \mathbf{L}(E_v) \cdot \mathbf{X} \) with \( E_v \) as a parameter in the transport matrix \( \mathbf{L} \). Then, it is shown that the proof of the Onsager symmetry for \( \mathbf{L}(E_v) \) in this section is still valid even if \( \mathbf{A}_E \tilde{g}_a \) is retained. This follows from the fact that the operator \( \mathbf{A}_E \) has the same properties as \( v_{E} \cdot \mathbf{n} \cdot \nabla \) which are given by \( \mathbf{A}_E f_{a0} = 0 \) and \( (\delta d^3 v, \mathbf{A}_E F) = 0 \) for an arbitrary function \( F(x) \):

\[ \nabla g_a \cdot \nabla g_a - C_{L}^{(a)} g_a = \frac{1}{\tau_a} f_{a0}(S_{a1} X_{a1} + S_{a2} X_{a2} + S_{a3} X_{a3}), \]

where the have used the following formula for an arbitrary function \( F(x) \):

\[ \nabla g_a \cdot \nabla g_a - C_{L}^{(a)} g_a = \frac{1}{\tau_a} f_{a0}(S_{a1} X_{a1} + S_{a2} X_{a2} + S_{a3} X_{a3}), \]

Noting in Eq. (54) that the left-hand side is linear with respect to \( \tilde{g}_a \) and that \( X_{aj} \) (\( j = 1,2,3 \)) occur in the right-hand side as parameters, we find that the solution \( \tilde{g}_a \) of Eq. (54) is given by

\[ g_a = \sum_{a} (G_{ab1} X_{b1} + G_{ab2} X_{b2} + G_{ab3} X_{b3}), \]

where \( G_{abj} \) (\( j = 1,2,3 \)) are defined as the solutions of

\[ \nabla G_{abj} - \sum_{a'} C_{aa'}^{L}(G_{a'b}, G_{a'j}) = 0, \]

Then, using Eqs. (56) and (58), we obtain the transport equations relating \( J^{ncl}_a \) (\( j = 1,2 \)) and \( J_{a3} \) to \( X_{aj} \) (\( j = 1,2,3 \)) as

\[ J^{ncl}_a = \sum_{b} \sum_{k=1,2} L_{jk}^{ab} X_{bk} + L_{kJ}^{a} X_{E}, \]

where the transport coefficients \( L_{jk}^{ab} \) are given by

\[ L_{jk}^{ab} = \int d^3 v S_{aj} G_{abk}, \]

Equation (60) is rewritten as the transport equations relating \( J^{ncl}_a \) (\( j = 1,2 \)) and \( J_{E} \) to \( X_{aj} \) (\( j = 1,2 \)) and \( X_{E} \):

\[ J^{ncl}_a = \sum_{k=1,2} \sum_{b=0} L_{jk}^{ab} X_{bk} + L_{kJ}^{a} X_{E}, \]

where the coefficients \( L_{jk}^{a} \) (\( j = 1,2,3 \)) and \( L_{E} \) are given by

\[ L_{jk}^{a} = \int d^3 v S_{aj} G_{abk}, \]

\[ L_{E} = \int d^3 v S_{aj} G_{abk} X_{E}, \]

\[ L_{jk}^{a} = \int d^3 v S_{aj} G_{abk} X_{E}, \]
In order to show the Onsager symmetry of the transport coefficients, it is useful to separate Eq. (59) into even and odd parts with respect to \( v \) as

\[
u \cdot \nabla G_{abj} = - \sum_a C_{a0}^L (G_{abj}^+ + G_{abj}^-) = \delta_{ab} \frac{1}{T} f_{b0} S_{bj},
\]

(63)

where the superscripts \(+\) and \(-\) denote the even and odd parts, respectively. Noting that \( S_{aj} \) are even for \( j = 1, 2 \) and odd for \( j = 3 \), and using Eqs. (61) and (64), we obtain

\[
\sum_A T_A E^{2} v \frac{1}{f_{a0}} G_{Akb} v \cdot \nabla G_{Aaj} - \sum_{A, B} T_A E^{2} v \frac{1}{f_{a0}} G_{Akb} C_{AB}^L (G_{Aaj}^+, G_{Baj}^-) = \delta_{j1} \delta_{j2} L_{jk}^{ab} + \delta_{k3} L_{jk}^{ba},
\]

(65)

from which we have

\[
(\delta_{j1} + \delta_{j2}) L_{jk}^{ab} + \delta_{k3} L_{jk}^{ba} = - \sum_{A, B} T_A E^{2} v \frac{1}{f_{a0}} G_{Akb} C_{AB}^L (G_{Aaj}^+, G_{Baj}^-) + G_{Aaj}^- C_{AB}^L (G_{Akb}^+, G_{Bbk}^-).
\]

(66)

We find from the self-adjointness of the linearized collision operator given by Eq. (9) that the right-hand side of Eq. (66) is invariant under the permutation of the subscripts \((a, j) \rightarrow (b, k)\) and that

\[(\delta_{j1} + \delta_{j2}) L_{jk}^{ab} + \delta_{k3} L_{jk}^{ba} = (\delta_{k1} + \delta_{k2} - \delta_{k3}) L_{jk}^{ab}.
\]

Thus, we obtain

\[(\delta_{j1} + \delta_{j2} - \delta_{j3}) L_{jk}^{ab} = (\delta_{k1} + \delta_{k2} - \delta_{k3}) L_{jk}^{ab},
\]

which is rewritten as the well-known Onsager relations:

\[
L_{jk}^{ab} = L_{jk}^{ba} \quad (j, k = 1, 2),
\]

\[
L_{j3}^{ab} = - L_{j3}^{ba} \quad (j = 1, 2),
\]

\[
L_{33}^{ab} = L_{33}^{ba}.
\]

(67)

We see from Eqs. (63) and (67) that

\[
L_{JE}^{ab} = - L_{JE}^{ba} \quad (j = 1, 2),
\]

(68)

Equations (67) and (68) show that the Onsager symmetry is satisfied by the transport matrix which combines the conjugate pairs of the fluxes \((J_{a1}^{nc}, J_{a2}^{nc}), J_{E}^{nc}\) and the forces \((X_{a1}^a, X_{a3}^a, X_E)\) for general toroidal systems.

Here, let us discuss the relation between the transport equations and the ambipolarity condition. In axisymmetric systems, intrinsic ambipolarity holds for the neoclassical particle fluxes and is expressed in terms of the relation between the transport coefficients as

\[
\sum_a e_a L_{1j}^{ab} = \sum_a e_a L_{1E}^{ab} = 0 \quad (j = 1, 2, 3).
\]

(69)

Then, the number of the conjugate pairs of fluxes and forces is reduced by one as shown in Eq. (52) using \(X_{a1}^a(\alpha = \beta)\) instead of \(X_{a1}^a\). We find from Eqs. (67), (68), and (69) that the transport equations relating the fluxes \((J_{a1}^{nc}, J_{a2}^{nc}), J_{E}^{nc}\) to \((X_{a1}^a, X_{a2}^a, X_E)\) in the axisymmetric case are given by

\[
J_{11}^{nc} = \sum_{b \neq l} L_{1l}^{ab} X_{b1}^a + \sum_b L_{12}^{ab} X_{b2}^a + L_{1E}^{ab} X_E,
\]

(70)

\[
J_{a2}^{nc} = \sum_{b \neq l} L_{2l}^{ab} X_{b1}^a + \sum_b L_{22}^{ab} X_{b2}^a + L_{2E}^{ab} X_E,
\]

\[
J_E = \sum_b L_{E1}^{ab} X_{b1}^a + \sum_b L_{E2}^{ab} X_{b2}^a + L_{EE}^{ab} X_E.
\]

In the transport equations (70), the transport coefficients \(L_{jk}^{ab}\) are the same as in Eq. (62) except for the limitation \((a, j) \neq (I, 1)\), and the Onsager symmetry is still valid. In nonaxisymmetric systems, the ambipolarity condition (51) gives a relation between the thermodynamic forces which is used to express one thermodynamic force \(X_I\) in terms of the other thermodynamic forces \((X_{a1}^a, X_{a2}^a, X_E)\) as

\[
X_I = - e_1^n \sum_{a, b} e_a e_b L_{11}^{ab} + \sum_a e_a \times \sum_{b \neq l} L_{1l}^{ab} X_{b1}^a + \sum_b L_{12}^{ab} X_{b2}^a + L_{1E}^{ab} X_E.
\]

(71)

Equation (71) determines the radial electric field \(-\Phi' = X_I / e_1 + P' J_s / (n_e J_e)\) as a linear combination of the pressure and temperature gradients and the parallel electric field.

Then, in the nonaxisymmetric case with the ambipolarity condition (71), the transport equations relating \((J_{a1}^{nc}, J_{a2}^{nc}), J_{E}^{nc}\) to \((X_{a1}^a, X_{a2}^a, X_E)\) are given by

\[
J_{11}^{nc} = \sum_{b \neq l} \mathcal{Q}_{11}^{nc} X_{b1}^a + \sum_b \mathcal{Q}_{12}^{nc} X_{b2}^a + \mathcal{Q}_{1E}^{nc} X_E,
\]

(72)

\[
J_{a2}^{nc} = \sum_{b \neq l} \mathcal{Q}_{21}^{nc} X_{b1}^a + \sum_b \mathcal{Q}_{22}^{nc} X_{b2}^a + \mathcal{Q}_{2E}^{nc} X_E,
\]

\[
J_E = \sum_b \mathcal{Q}_{E1}^{nc} X_{b1}^a + \sum_b \mathcal{Q}_{E2}^{nc} X_{b2}^a + \mathcal{Q}_{EE}^{nc} X_E.
\]

where the transport coefficients \(\mathcal{Q}_{jk}^{nc}, \mathcal{Q}_{jE}^{nc}, \mathcal{Q}_{Ej}^{nc}\), and \(\mathcal{Q}_{EE}^{nc}\) are
The parallel momentum balance equations combined with the friction-flow relations are given in the 13M approximation by

\[ \langle \mathbf{B} \cdot \nabla \mathbf{\tau} \rangle + n_e \langle E_1 B_2 \rangle \]

\[ \langle \mathbf{B} \cdot \nabla \mathbf{\Theta} \rangle \]

\[ = - \frac{n_a m_a}{\tau_{e1}} \langle \mathbf{j}_1 \rangle - \frac{n_c}{2 \rho_i} \langle B q_{i1} \rangle, \]

\[ \langle \mathbf{B} \cdot \nabla \mathbf{\Theta} \rangle = \langle B q_{i2} \rangle \]

(75)

where the dimensionless friction coefficients \( \langle \mathbf{j}_1 \rangle = (\tau_{a1} / n_a m_a) \langle \mathbf{j}_1 \rangle \) are given by \( \langle \mathbf{j}_1 \rangle = Z_i, \langle \mathbf{j}_1 \rangle = \frac{3}{2} Z_i, \langle \mathbf{j}_1 \rangle = \sqrt{\frac{1}{2}} Z_i \), and \( \langle \mathbf{j}_1 \rangle = \sqrt{\frac{3}{2}} \) with the ion charge number \( Z_i \). The species summation of the parallel momentum balances reduces to

\[ \langle \mathbf{B} \cdot \nabla \mathbf{\tau} \rangle + \langle \mathbf{B} \cdot \nabla \mathbf{\tau} \rangle = 0, \]

(77)

where the momentum conservation (25) in collisions and the charge neutrality condition \( \Sigma_a e_a n_a = 0 \) are used.

Solving the linearized drift kinetic equation gives the equations for the parallel viscosities, which have the following form for all collision frequencies in the Pfirsch–Schlüter, plateau, and banana regimes:

\[ \langle \mathbf{B} \cdot \nabla \mathbf{\tau} \rangle = - \frac{n_a m_a}{\tau_{a1}} \frac{c_{a1}}{c_{a2}} \langle \mathbf{j}_1 \rangle - \frac{\mu_{a1}}{\mu_{a2}} \langle \mathbf{j}_1 \rangle \]

\[ \times \left( \frac{2}{5 \rho_c} \langle q_{i1} \rangle \right) \left( - \langle G_a \rangle / e_a \right) X_{i2} \]

(78)

where \( c_{a} \) and \( \hat{\mu}_{a} \) (\( j = 1,2,3 \)) are the dimensionless parameters for the viscosity coefficients, and \( \langle G_a \rangle \) represents the geometrical factor which measures the deviation from the axisymmetric configuration. These parameters \( c_{a} \) and \( \hat{\mu}_{a} \) (\( j = 1,2,3 \)) and \( \langle G_a \rangle \) are given in Appendix C and all of them are generally dependent on the collision frequencies although, in the axisymmetric case, the geometrical factor is given by \( \langle G_a \rangle = (\mathbf{B}) / B_j \) for both species and for all the collision frequencies.

Using Eqs. (76)–(78), the ion parallel flows and viscosity are written as

\[ \langle B q_{i1} \rangle = (\mathbf{G}_i) c_{e1} X_{i1} + \hat{\mu}_{i1} / \mu_{i1} \]

\[ - \frac{1}{c_{i1} \hat{\mu}_{i1}} \frac{\tau_{i1}}{n_i m_i} \langle \mathbf{B} \cdot \nabla \mathbf{\tau} \rangle, \]

(79)

\[ 2 \left( \frac{\hat{\mu}_{i1}}{\hat{\mu}_{i2}} \right) \langle B q_{i1} \rangle = \frac{2}{5 \rho_i} \left( \frac{c_{e1} \hat{\mu}_{i1} \hat{\mu}_{i3} + (\hat{\mu}_{i2})^2}{\hat{\mu}_{i1} (c_{i1} \hat{\mu}_{i3} + \hat{\mu}_{i2})} \right) \langle \mathbf{G}_i \rangle c_{e1} X_{i2} \]

\[ + \frac{\hat{\mu}_{i2}}{\hat{\mu}_{i1}} \frac{\tau_{i1}}{n_i m_i} \langle \mathbf{B} \cdot \nabla \mathbf{\tau} \rangle, \]

(80)

IV. BANANA-PLATEAU AND NONAXISYMMETRIC TRANSPORT COEFFICIENTS

In the previous section, we have shown the Onsager symmetry for the transport coefficients relating the neoclassical radial fluxes and the parallel current \( (J_{a1}^{\text{nc}}, J_{a2}^{\text{nc}}) \) to the radial gradient forces and the parallel electric field \( (X_{a1}, X_{a2}, X_E) \) in general toroidal systems. The neoclassical radial particle and heat fluxes \( (J_{a1}^{\text{nc}}, J_{a2}^{\text{nc}}) \) consist of the Pfirsch–Schlüter, banana-plateau, and nonaxisymmetric parts while, as shown in Appendix B, it is well known that the transport equations for the Pfirsch–Schlüter fluxes \( (J_{a1}^{\text{ps}}, J_{a2}^{\text{ps}}) \) and the radial gradient forces \( X_{a1}, X_{a2} \) satisfy the Onsager symmetry. Thus, it is clear that the sum of the banana-plateau and nonaxisymmetric radial fluxes, and the parallel current \( (J_{a1}^{\text{bp}} + J_{a1}^{\text{nc}} + J_{a2}^{\text{nc}} + J_{a2}^{\text{nc}}) \) are related to the forces \( (X_{a1}, X_{a2}, X_E) \) by the transport coefficients with the Onsager symmetry. In this section, using the 13 moment (13M) approximation,1 we derive the transport equations for the banana-plateau flows and those for the nonaxisymmetric fluxes separately, in the case of general toroidal plasma consisting of electrons and one ion species. Then, the symmetry properties are investigated for each of the transport equations, and the Onsager symmetry for the total transport is directly confirmed.
\[
\langle B \cdot \nabla \Theta \rangle = - \frac{n_m c_1 \mu_{12}^i \mu_{13} - (\mu_{12})^2}{\mu_{11}^i (c_1 \mu_{13} + \tilde{f}_{22})} \left( \frac{G_i}{e_t} \right) \frac{X_{i2}}{e_t} - \frac{\mu_{12}^i}{\mu_{11}^i} \langle B \cdot \nabla \pi_e \rangle, \tag{81}
\]

where we have defined \( \mu_{11}^i = \mu_{11} - \gamma \mu_{12} \) and \( \mu_{12}^i = \mu_{12} - \gamma \mu_{13} \) with \( \gamma = c_1 \mu_{12}/(c_1 \mu_{13} + \tilde{f}_{22}) \). Equations relating the electron parallel viscosities \( \langle B \cdot \nabla \pi_e \rangle \) and \( \langle B \cdot \nabla \Theta \rangle \) to the thermodynamic forces \( (X_{e1}, X_{e2}, X_{i1}, X_{i2}, X_E) \) are obtained from Eqs. (75), (78), and (79) as

\[
\begin{align*}
\frac{\tilde{f}_{11}}{c_1} &= \frac{1}{\mu_{12}^i} \left[ \frac{1}{\mu_{11}^i} \frac{n_m c_1}{\tau_{ee}} \right] \frac{X_{e1} + \mu_{12}^i}{X_{e2}} + \frac{1}{\mu_{11}^i} \frac{X_{i1} + \mu_{12}^i}{X_{i2}} \\
- \frac{\tilde{f}_{22}}{c_1} &= \frac{1}{\mu_{12}^i} \left[ \frac{1}{\mu_{11}^i} \frac{n_m c_1}{\tau_{ee}} \right] \frac{X_{e2}}{X_{e1}} - \frac{1}{\mu_{11}^i} \frac{X_{i2}}{X_{i1}} + \frac{n_m c_1}{\tau_{ee}} \frac{X_{i1}}{X_{i2}}
\end{align*}
\]

(82)

We also find from Eq. (75) that the parallel current is divided into the classical part \( J_{E1} \) and the neoclassical part \( J_{E2} \) due to the electron parallel viscosities as

\[
J_E = \langle B J_i \rangle / (B^2)^{1/2} = J_{E1} + J_{E2},
\]

(83)

\[
J_{E1} = \sigma_s X_{e1}, \quad J_{E2} = \frac{\sigma_s}{n_e} \left( \frac{B^2}{\tau_{ee}} \right)^{1/2} \langle B \cdot \nabla \pi_e \rangle.
\]

(84)

where \( \sigma_s = (n_e e \tau_{ee}) / m_e \tilde{f}_{11} \tilde{f}_{22} / (\tilde{f}_{11}^2 - (\tilde{f}_{12}^2)^2) \) is the classical Spitzer conductivity.

Finally, we obtain from Eqs. (77) and (81)–(83) the banana-plateau transport equations which relate \( \langle j_{e1}, j_{i1}, j_{E1}, j_{i2}, j_{E2} \rangle \) to \( (X_{e1}, X_{e2}, X_{i1}, X_{i2}, X_E) \) as

\[
\begin{align*}
\langle j_{E1} \rangle &= \frac{1}{\mu_{12}^i} \left[ \frac{1}{\mu_{11}^i} \frac{n_m c_1}{\tau_{ee}} \right] \frac{X_{e1}}{X_{e2}} + \frac{1}{\mu_{11}^i} \frac{X_{i1}}{X_{i2}} + \frac{n_m c_1}{\tau_{ee}} \\
\langle j_{i1} \rangle &= \frac{1}{\mu_{12}^i} \left[ \frac{1}{\mu_{11}^i} \frac{n_m c_1}{\tau_{ee}} \right] \frac{X_{e2}}{X_{e1}} - \frac{1}{\mu_{11}^i} \frac{X_{i2}}{X_{i1}} + \frac{n_m c_1}{\tau_{ee}} \frac{X_{i1}}{X_{i2}} \\
\langle j_{E2} \rangle &= \frac{1}{\mu_{12}^i} \left[ \frac{1}{\mu_{11}^i} \frac{n_m c_1}{\tau_{ee}} \right] \frac{X_{i1}}{X_{i2}} - \frac{1}{\mu_{11}^i} \frac{X_{i2}}{X_{i1}} + \frac{n_m c_1}{\tau_{ee}} \frac{X_{i1}}{X_{i2}}.
\end{align*}
\]

(84)

The banana-plateau transport coefficients given in Eqs. (85)–(93) satisfy the following symmetry properties:

\[
\begin{align*}
\langle L^{b^p}_{EE} \rangle &= \frac{B^q}{B^q} \langle L^{b^p}_{EE} \rangle, & \langle L^{b^p}_{E1} \rangle &= \frac{B^q}{B^q} \langle L^{b^p}_{E1} \rangle, \\
\langle L^{b^p}_{i1} \rangle &= \frac{B^q}{B^q} \langle L^{b^p}_{i2} \rangle, & \langle L^{b^p}_{ij} \rangle &= \frac{B^q}{B^q} \langle L^{b^p}_{ij} \rangle, & (j,k) &= (1,2), \\
\langle L^{b^p}_{a} \rangle &= \frac{B^q}{B^q} \langle L^{b^p}_{a} \rangle, & (a,e,i,j) &= (1,2).
\end{align*}
\]

(84)

Thus, we confirm that the banana-plateau transport equations have the complete Onsager symmetry in the axisymmetric case where \( \langle G_i \rangle = \langle G_e \rangle = \langle B \rangle / B^q \). The intrinsic ambipolarity of the banana-plateau particle fluxes is expressed by
Here, the force and the radial electric field of forces \((E_r, F_r)\) depend on the radial electric field \(E_r\) in the banana plateau radial particle and heat fluxes, plateau transport. However, if the electrons and the ions be-

Using the above relations, the fluxes \(J_{e1}^i, J_{r1}^i, J_{e2}^i, J_{r2}^i, \ldots\) are related to the new reduced set of forces \(X_{e1}^*, X_{e2}^*, X_{e3}^*, X_{r1}^*, \ldots\) by the transport equations

\[
\begin{align*}
L_{e1}^i & = \frac{\langle G_i \rangle}{Z \langle G_e \rangle} (L_{e1}^i)_{e1}^i \quad (a = e, i; j = 1, 2), \\
L_{r1}^i & = \frac{\langle G_i \rangle}{Z \langle G_e \rangle} (L_{r1}^i)_{e1}^i, \\
L_{e2}^i & = \frac{\langle G_i \rangle}{Z \langle G_e \rangle} (L_{e2}^i)_{e1}^i, \\
L_{r2}^i & = \frac{\langle G_i \rangle}{Z \langle G_e \rangle} (L_{r2}^i)_{e1}^i, \\
L_{e3}^i & = \frac{\langle G_i \rangle}{Z \langle G_e \rangle} (L_{e3}^i)_{e1}^i, \\
L_{r3}^i & = \frac{\langle G_i \rangle}{Z \langle G_e \rangle} (L_{r3}^i)_{e1}^i.
\end{align*}
\]

where \(c_{ia} \) and \( \langle G_{ia} \rangle \) are given in Appendix C and \( \hat{\mu}_{ai} \) \((j = 1, 2, 3)\) are the same as in Eq. (78). Here and hereafter, we consider the toroidal viscosities and the nonaxisymmetric fluxes only for the Pfirsch–Schluter and plateau regimes since the expressions similar to Eq. (99) have not been obtained yet for the banana regime.

Appendix C shows that the ratio between the toroidal and parallel viscosities \(c_{ta}/c_a\) is related to the geometrical factor \(\langle G_a \rangle\) in Eq. (78) by the following equation:

\[
1 - \frac{c_{ta}}{c_a} \frac{\langle B^2 \rangle}{\langle B^2 \rangle} = \frac{B^2 \langle G_a \rangle}{B^2 \langle B^2 \rangle},
\]

which is an essential relation for showing the Onsager symmetry of the total neoclassical transport.

Using Eqs. (78) and (99), the toroidal viscosities are written in terms of the parallel viscosities and the thermodynamic forces as

\[
\frac{\langle B_r \cdot \nabla \tau_{ci} \rangle}{\langle B_r \cdot \nabla \tau_{ci} \rangle} = \frac{c_{ia}}{c_a} \frac{\langle B_r \cdot \nabla \tau_{ci} \rangle}{\langle B_r \cdot \nabla \tau_{ci} \rangle} + \frac{n_a m_a}{\tau_a} \hat{\mu}_{ai} \frac{G_{ia}}{G_a} - \frac{\langle G_{ia} \rangle}{\langle G_a \rangle} - \frac{\langle G_{ia} \rangle}{\langle G_a \rangle} - \frac{\langle G_{ia} \rangle}{\langle G_a \rangle} - \frac{\langle G_{ia} \rangle}{\langle G_a \rangle} - \frac{\langle G_{ia} \rangle}{\langle G_a \rangle}.
\]

Then, from Eqs. (43), (84), and (101), we obtain the nonaxisymmetric transport equations relating \(J_{e1}^a, J_{r1}^a, J_{e2}^a, J_{r2}^a, \ldots\) to \(X_{e1}, X_{e2}, X_{e3}, X_{r1}, \ldots\):
It is found that \( (L_{mn}^{ia})_{jk} \) is always valid although \( (L_{mn}^{ia})_{ji} \) is satisfied only when \( c_{ie}/c_{ei} = c_{ii}/c_{ii} \). We see from Eq. (100) that the condition \( c_{ie}/c_{ei} = c_{ii}/c_{ij} \) is equivalent to \( \langle G_i \rangle = \langle G_i \rangle \), which holds if the electrons and the ions belong to the same collisionality regime.

Finally, combining the banana-plateau transport equations (84) with the nonaxisymmetric transport equations (102), and using the relation (100), the transport equations for the total of the banana-plateau and nonaxisymmetric fluxes \( \left( J_{mn}^{ia}, J_{mn}^{a1}, J_{mn}^{aj}, J_{mn}^{bi} \right) \) are obtained as

\[
\begin{align*}
J_{mn}^{ia} & = J_{mp}^{ia} + J_{mn}^{aj}, \\
J_{mn}^{a1} & = J_{mp}^{a1} + J_{mn}^{aj}, \\
J_{mn}^{aj} & = J_{mp}^{aj} + J_{mn}^{aj}, \\
J_{mn}^{bi} & = J_{mp}^{bi} + J_{mn}^{be},
\end{align*}
\]

where the total transport coefficients are given by

\[
\begin{align*}
\left( L_{mn}^{ia} \right)_{jk} & = \frac{B^6 \langle G_i \rangle}{B^6 \langle G_i \rangle} \left( L_{mn}^{ia} \right)_{ji} \left( L_{mn}^{a1} \right)_{ji} \left( L_{mn}^{aj} \right)_{ji} \left( L_{mn}^{bi} \right)_{ji} \\
& \quad + \frac{n_e m_e c^2}{\tau_{ie}} \left( \mu_{c1} \hat{\mu}_{c2} \hat{\mu}_{c3} \right), \\
\left( L_{mn}^{a1} \right)_{jk} & = \frac{B^6 \langle G_i \rangle}{B^6 \langle G_i \rangle} \left( L_{mn}^{a1} \right)_{ji} \left( L_{mn}^{a1} \right)_{ji} \left( L_{mn}^{aj} \right)_{ji} \left( L_{mn}^{bi} \right)_{ji} \\
& \quad + \frac{n_e m_e c^2}{\tau_{ie}} \left( \mu_{c1} \hat{\mu}_{c2} \hat{\mu}_{c3} \right), \\
\left( L_{mn}^{aj} \right)_{jk} & = \frac{B^6 \langle G_i \rangle}{B^6 \langle G_i \rangle} \left( L_{mn}^{aj} \right)_{ji} \left( L_{mn}^{a1} \right)_{ji} \left( L_{mn}^{aj} \right)_{ji} \left( L_{mn}^{bi} \right)_{ji} \\
& \quad + \frac{n_e m_e c^2}{\tau_{ie}} \left( \mu_{c1} \hat{\mu}_{c2} \hat{\mu}_{c3} \right), \\
\left( L_{mn}^{bi} \right)_{jk} & = \frac{B^6 \langle G_i \rangle}{B^6 \langle G_i \rangle} \left( L_{mn}^{bi} \right)_{ji} \left( L_{mn}^{a1} \right)_{ji} \left( L_{mn}^{aj} \right)_{ji} \left( L_{mn}^{bi} \right)_{ji} \\
& \quad + \frac{n_e m_e c^2}{\tau_{ie}} \left( \mu_{c1} \hat{\mu}_{c2} \hat{\mu}_{c3} \right).
\end{align*}
\]

Thus, from the above definitions and the symmetry properties given by Eq. (94), we can directly confirm that the total banana-plateau and nonaxisymmetric transport coefficients satisfy the following Onsager symmetry:

\[
\left( L_{mn}^{ia} \right)_{jk} = \left( L_{mn}^{ja} \right)_{kj} = \left( L_{mn}^{ja} \right)_{jk} = \left( L_{mn}^{ja} \right)_{jk},
\]

as seen from Eqs. (85)–(93) and Eqs. (106)–(108), the ion–ion coefficients \( (L_{mn}^{ia})_{jk} \) are \( \langle (m_e/m_i)^{1/2} \rangle \) larger than the other coefficients.

When the ambipolarity condition is imposed in the nonaxisymmetric case, we obtain, in the same way as in Eqs. (71)–(73), the radial electric field:

\[
-\Phi' = \frac{p_i'}{n \mu e_i} + \frac{X_{11}}{e_i},
\]

(109)
V. CONCLUSIONS AND DISCUSSION

In this work, we have investigated the entropy production, the full transport equations, and their Onsager symmetry for the neoclassical transport processes in magnetically confined plasmas with general toroidal configurations. It was clearly shown that, for both the classical and neoclassical transport processes, the kinetic form of the entropy production defined from the linearized collision operator is equivalent to its thermodynamic form written as the inner product of the thermodynamic forces and their conjugate transport fluxes. The entropy production from the gyroangle-dependent distribution function corresponds to the sum of the products of the classical radial particle and heat fluxes and the radial gradient thermodynamic forces, while the magnetic surface average of the entropy production from the gyroangle-averaged distribution function is given by the sum of the products of the thermodynamic forces and their conjugate fluxes which consist of the Pfirsch–Schlüter, banana-plateau, nonaxisymmetric parts of the neoclassical radial fluxes and the parallel current. This equivalence between the kinetic and thermodynamic forms of the entropy production for the full neoclassical transport fluxes were not confirmed by Balescu in Chap. 17 of Ref. 3. The reason is now discussed.

In deriving the thermodynamic form of the entropy production, we used the linearized drift kinetic equation without employing the Hermitian moment expansion of the distribution function. Balescu expressed the kinetic form of the entropy production as the quadratic form of only the vector Hermitian moment part of the distribution function which corresponds to the l = 1 part of the Legendre polynomial expansion. However, since the neoclassical banana-plateau and nonaxisymmetric fluxes are caused by the parallel and toroidal viscosities, the tensor Hermitian moment part (or the l = 2 part of the Legendre polynomial expansion) needs to be included for deriving the neoclassical thermodynamic form of the entropy production. Furthermore, as shown in Appendix D, we find that it is necessary to include all the tensor moments including higher order parts with l = 3, 4, 5, . . . , in the kinetic form of the entropy production to derive the neoclassical thermodynamic form in the banana and plateau regimes. This is intuitively understandable by considering the resonant particles responsible for the neoclassical fluxes in the plateau regime. The resonant particle distribution is highly anisotropic in velocity space and is approximated by a delta function in pitch angle so that the all Hermitian moments of the tensor Hermitian moment part of the velocity distribution function which corresponds to the l = 1 part of the Legendre polynomial expansion. However, since the neoclassical banana-plateau and nonaxisymmetric fluxes are caused by the parallel and toroidal viscosities, the tensor Hermitian moment part (or the l = 2 part of the Legendre polynomial expansion) needs to be included for deriving the neoclassical thermodynamic form of the entropy production. Furthermore, as shown in Appendix D, we find that it is necessary to include all the tensor moments including higher order parts with l = 3, 4, 5, . . . , in the kinetic form of the entropy production to derive the neoclassical thermodynamic form in the banana and plateau regimes. This is intuitively understandable by considering the resonant particles responsible for the neoclassical fluxes in the plateau regime. The resonant particle distribution is highly anisotropic in velocity space and is approximated by a delta function in pitch angle so that the all Hermitian moments (or all lth-order Legendre polynomials) are required. As shown in Appendix D, the operator $v_i \cdot \nabla$ in the drift kinetic equation (38) introduces the anisotropic distribution in the velocity space and it connects the lth-order moment with $(l \pm 1)$th-order moments in contrast with the linearized collision operator which is isotropic in the velocity space and connects the lth-order moment with the same lth-order moment alone. In the Pfirsch–Schlüter regime, the collision operator dominates $v_i \cdot \nabla$ and the distribution function has small contributions from higher-order moments representing the anisotropy. Then, in the Pfirsch–Schlüter regime, the l = 1 vector moment is enough to express the entropy production as in Ref. 3, while the negligibly small viscosity-
induced neoclassical fluxes are included in the \( l = 2 \) tensor moment part. On the other hand, as the collision frequency decreases, the operator \( \psi \cdot n \cdot \nabla \) is comparable to, and then dominates, the collision operator and all \( l \)-order moments are required in the kinetic entropy production functional to obtain its neoclassical thermodynamic form in the banana and plateau regimes.

We also proved from the formal solution of the linearized drift kinetic equation with the self-adjoint linearized collision operator that the Onsager symmetry is robustly valid for the neoclassical transport equations for general toroidal plasmas consisting of electrons and multi-species ions with arbitrary collision frequencies. Furthermore, we derived in Sec. IV, in the case of a single ion species, the full banana-plateau transport coefficients for all collisionality regimes and the full nonaxisymmetric transport coefficients for the Pfirsch–Schlüter and plateau regimes. The symmetry properties of these transport matrices were separately examined and the Onsager symmetry for their total transport equations was confirmed. We discussed the effects of the ambipolarity condition on the transport equations in detail for both axisymmetric and nonaxisymmetric configurations. The ambipolarity condition reduces by one the number of the conjugate pairs of the transport fluxes and the thermodynamic forces. In the axisymmetric case, the intrinsic ambipolarity holds and the radial electric field does not affect the transport. On the other hand, for broken toroidal symmetry, a radial current is a function of the thermodynamic forces \((X_{aj}, X_{E})\) in which the radial electric field is included. When the ambipolarity condition is imposed in the nonaxisymmetric case, the radial electric field is given by a linear combination of the other thermodynamic forces. We showed that the Onsager symmetry is satisfied whether the conjugate pairs of the fluxes and forces are reduced by the ambipolarity condition or not.

Balescu and Fantechi derived the full neoclassical transport coefficients for the plateau regime in the nonaxisymmetric configuration and claimed that the Onsager symmetry is slightly broken by the nonaxisymmetry. They showed the transport equations only for the reduced pairs of the fluxes and forces, in which the radial electric field is eliminated by the ambipolar condition. There, terms of \( \partial / \partial (m_{e}/m_{i})^{1/2} \) were neglected in expressing the radial electric field in terms of the other forces as in Eq. (114). Then, the resultant transport coefficients in Ref. 11 deviate from those in Eq. (112) and the Onsager symmetry is broken in them since part of \( \partial / \partial (m_{e}/m_{i})^{1/2} \) terms necessary for the symmetry are dropped.

The banana-plateau and nonaxisymmetric transport equations obtained here are valid whether electrons and ions belong to the same collisionality regime or not. When both species are in the same collisionality regime, we find that the thermodynamic forces for the banana-plateau radial particle and heat fluxes and the bootstrap current in the nonaxisymmetric systems.

Since we proved the robust validity of the Onsager symmetry for the neoclassical transport equations, even in the nonaxisymmetric cases, this symmetry property can be utilized for the calculation of the nonaxisymmetric transport coefficients in the banana regime which were not given in Sec. IV. For example, from the banana-plateau transport coefficients \((L_{aj}^{bp})_{E} \) and \((L_{aj}^{bp})_{E} \) \((a = e, i; j = 1, 2)\) for the banana regime given in Sec. IV, we can immediately obtain part of the nonaxisymmetric transport coefficients for the banana regime as \((L_{aj}^{na})_{E} \) \(:= (L_{aj}^{bp})_{E} - (L_{aj}^{bp})_{E} \) \((a = e, i; j = 1, 2)\).

In the previous work, we investigated the neoclassical and anomalous transport in weakly turbulent plasma and described the entropy production and the Onsager symmetry in electrostatic turbulence.\(^{17}\) We are also investigating a unified description of the transport equations, the entropy production, and the Onsager symmetry for neoclassical and turbulent processes with both electrostatic and electromagnetic fluctuations, which we will report on in a future work.

ACKNOWLEDGMENTS

The author (HS) thanks Professor M. Okamoto for his encouragement of this work, and Dr. N. Nakajima for useful discussions.

This work is supported in part by the Grant-in-Aid from the Japanese Ministry of Education, Science, and Culture, and in part by U.S. Department of Energy Grant No. DE-FG05-80ET-53088.

APPENDIX A: HAMADA COORDINATES AND INCOMPRESSIBLE FLOWS

In general toroidal configurations, the magnetic field is written in the contravariant form:

\[
B = B^{\theta} \frac{\partial x}{\partial \theta} + B^{\phi} \frac{\partial x}{\partial \phi},
\]

where \( \theta \) and \( \phi \) are the poloidal and toroidal angle variables, respectively, and corresponding basis vectors for the contravariant representation are given by \( \partial x / \partial \theta \) and \( \partial x / \partial \phi \).

Throughout this paper, we employ the Hamada coordinates \((V, \theta, \phi)\), where the flux label \( V \) represents the volume enclosed by the flux surface, and the periods of the angle variables are normalized as \( 2 \pi = t d \theta = t d \phi = 1 \). The Jacobian is unity \( \nabla V \cdot \nabla \theta \times \nabla \phi = 1 \), so that the flux surface average is simply written as \( \langle \cdot \rangle = t d \theta d \phi \langle \cdot \rangle \). The poloidal and toroidal fields are given by \( B_{e} = C_{e} \partial x / \partial \theta = C_{e} \partial V / \partial \theta \times \nabla V \) and \( B_{i} = C_{i} \partial x / \partial \phi = C_{i} \partial V / \partial \phi \times \nabla \phi \). The contravariant poloidal and toroidal components are the surface flux quantities \( B_{e}^{p} = C_{e} \partial V / \partial \theta \times \nabla \phi \) and \( B_{i}^{p} = C_{i} \partial V / \partial \phi \times \nabla \theta \). The vector product of \( B \) and \( \nabla V \) is given by the linear combination of \( B \) and \( \partial x / \partial \phi \) as

\[
B \times \nabla V = B_{\phi} \frac{\partial x}{B^{\phi}} - B^{\phi} \frac{\partial x}{B^{\phi}},
\]

(22)
where \( \mathbf{B}_c = \mathbf{B} \cdot \partial \mathbf{x} / \partial \zeta \).

When a solenoidal vector field \( \mathbf{U} (\nabla \cdot \mathbf{U} = 0) \) is tangential to magnetic surfaces \( \nabla \cdot \mathbf{U} = 0 \) and satisfies \( \mathbf{U} \times \mathbf{B} = \nabla K(V) \) with some flux quantity \( K(V) \), it is written in the Hamada coordinates as

\[
\mathbf{U} = U^\theta \frac{\partial \mathbf{x}}{\partial \theta} + U^\zeta \frac{\partial \mathbf{x}}{\partial \zeta},
\]

where both the contravariant components \( U^\theta = \mathbf{U} \cdot \nabla \theta \) and \( U^\zeta = \mathbf{U} \cdot \nabla \zeta \) are flux surface quantities. Since both of the flow velocity \( \mathbf{u}_a \) and the heat flow \( \mathbf{q}_a \) are incompressible \( \nabla \cdot \mathbf{u}_a = \nabla \cdot \mathbf{q}_a = 0 \) [see Eq. (D12)] and satisfies the above conditions [see Eq. (20)] to the lowest order in \( \delta \), all of their contravariant components \( u^\theta_a = \mathbf{u}_a \cdot \nabla \theta \), \( u^\zeta_a = \mathbf{u}_a \cdot \nabla \zeta \), \( q^\theta_a = \mathbf{q}_a \cdot \nabla \theta \), and \( q^\zeta_a = \mathbf{q}_a \cdot \nabla \zeta \) are flux surface quantities. Also, \( \mathbf{u}_a \) and \( \mathbf{q}_a \) are separated into the parallel and perpendicular components as

\[
\mathbf{u}_a = u_{ia} \mathbf{n} + u_{\perp a}, \quad \mathbf{q}_a = q_{ia} \mathbf{n} + q_{\perp a},
\]

where the perpendicular components \( u_{\perp a} \) and \( q_{\perp a} \) are given by the thermodynamic forces \( X_{a1} \) and \( X_{a2} \) as in Eq. (20). Then, the contravariant poloidal and toroidal flow components are given by the linear combinations of the parallel flow components and the thermodynamic forces as

\[
\begin{align*}
\frac{u^\theta_a}{B} &= B q^\theta_a \quad \frac{\partial X_{a1}}{\partial a}, \\
\frac{u^\zeta_a}{B} &= B q^\zeta_a \quad \frac{\partial X_{a2}}{\partial a}, \\
\frac{2}{5} \frac{q^\theta_a}{p_a} &= \frac{B^2}{5} \frac{q_{ia}}{p_a} \quad \frac{\partial X_{a2}}{\partial a}, \\
\frac{2}{5} \frac{q^\zeta_a}{p_a} &= \frac{B^2}{5} \frac{q_{ia}}{p_a} + \frac{B \theta}{B^2} \frac{\partial X_{a2}}{\partial a},
\end{align*}
\]

where \( B = \mathbf{B} \cdot \partial \mathbf{x} / \partial \theta \). We find from Eq. (A5) that

\[
\begin{align*}
\langle B^2 \rangle u^\theta_a &= \langle u_{ia} B \rangle - \frac{\partial \langle B \zeta \rangle}{\partial a} \frac{\partial X_{a1}}{\partial a}, \\
\langle B^2 \rangle u^\zeta_a &= \langle u_{ia} B \rangle + \frac{\partial \langle B \theta \rangle}{\partial a} \frac{\partial X_{a1}}{\partial a}, \\
\langle B^2 \rangle \frac{2}{5} \frac{q^\theta_a}{p_a} &= \frac{B^2}{5} \frac{q_{ia}}{p_a} - \frac{\partial \langle B \zeta \rangle}{\partial a} \frac{\partial X_{a2}}{\partial a}, \\
\langle B^2 \rangle \frac{2}{5} \frac{q^\zeta_a}{p_a} &= \frac{B^2}{5} \frac{q_{ia}}{p_a} + \frac{\partial \langle B \theta \rangle}{\partial a} \frac{\partial X_{a2}}{\partial a}.
\end{align*}
\]

The following formulas are obtained from Eqs. (A5) and (A6):
The conjugate pairs.

It is shown that the toroidal viscosities are given in the following form for both of the Pfirsch–Schlichting, plateau, and banana regimes are given in the following form:

\[ \frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\pi}_a \rangle}{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Theta}_a \rangle} = \frac{n_a m_a}{\tau_{aa}} \sum \frac{\mu_{a1}}{\mu_{a1}} \frac{\mu_{a2}}{\mu_{a2}} \frac{\mu_{a3}}{\mu_{a3}} \left( \frac{u_{ia} B}{2} \right) \frac{G_{ia}}{e_a} + \frac{X_{ia}}{X_{ia}} . \]

(C1)

where \( c_a \) and \( \tilde{\mu}_{aj} (j=1,2,3) \) are the dimensionless parameters for the viscosity coefficients and \( G_{ia} \) represents the geometrical factor. The collision frequency \( \tau_{aa} \) is defined in Ref. 2. The dimensionless viscosity parameters \( \tilde{\mu}_{aj} (j=1,2,3) \) are written as

\[ \tilde{\mu}_{aj} = \frac{1}{2} (v_T \tau_{aa})^2 \frac{\langle (\mathbf{B} \cdot \nabla)(\mathbf{B} \cdot \nabla) \rangle}{\langle \mathbf{B}^2 \rangle} \]  

(C2)

It is shown from the solution of the linearized drift kinetic equation that the parallel viscosities in general toroidal configurations for all collision frequencies in the Pfirsch–Schlichter, plateau, and banana regimes are given in the following form:

\[ \frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\pi}_a \rangle}{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Theta}_a \rangle} = \frac{n_a m_a}{\tau_{aa}} \sum \frac{\mu_{a1}}{\mu_{a1}} \frac{\mu_{a2}}{\mu_{a2}} \frac{\mu_{a3}}{\mu_{a3}} \left( \frac{u_{ia} B}{2} \right) \frac{G_{ia}}{e_a} + \frac{X_{ia}}{X_{ia}} . \]

(C1)

where \( c_a \) and \( \tilde{\mu}_{aj} (j=1,2,3) \) are the dimensionless parameters for the viscosity coefficients and \( G_{ia} \) represents the geometrical factor. The collision frequency \( \tau_{aa} \) is defined in Ref. 2. The dimensionless viscosity parameters \( \tilde{\mu}_{aj} (j=1,2,3) \) are written as

\[ \tilde{\mu}_{aj} = \frac{1}{2} (v_T \tau_{aa})^2 \frac{\langle (\mathbf{B} \cdot \nabla)(\mathbf{B} \cdot \nabla) \rangle}{\langle \mathbf{B}^2 \rangle} \]  

(C2)

It is shown from the solution of the linearized drift kinetic equation that the parallel viscosities in general toroidal configurations for all collision frequencies in the Pfirsch–Schlichter, plateau, and banana regimes are given in the following form:

\[ \frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\pi}_a \rangle}{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Theta}_a \rangle} = \frac{n_a m_a}{\tau_{aa}} \sum \frac{\mu_{a1}}{\mu_{a1}} \frac{\mu_{a2}}{\mu_{a2}} \frac{\mu_{a3}}{\mu_{a3}} \left( \frac{u_{ia} B}{2} \right) \frac{G_{ia}}{e_a} + \frac{X_{ia}}{X_{ia}} . \]

(C1)

where \( c_a \) and \( \tilde{\mu}_{aj} (j=1,2,3) \) are the dimensionless parameters for the viscosity coefficients and \( G_{ia} \) represents the geometrical factor. The collision frequency \( \tau_{aa} \) is defined in Ref. 2. The dimensionless viscosity parameters \( \tilde{\mu}_{aj} (j=1,2,3) \) are written as

\[ \tilde{\mu}_{aj} = \frac{1}{2} (v_T \tau_{aa})^2 \frac{\langle (\mathbf{B} \cdot \nabla)(\mathbf{B} \cdot \nabla) \rangle}{\langle \mathbf{B}^2 \rangle} \]  

(C2)

It is shown from the solution of the linearized drift kinetic equation that the parallel viscosities in general toroidal configurations for all collision frequencies in the Pfirsch–Schlichter, plateau, and banana regimes are given in the following form:

\[ \frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\pi}_a \rangle}{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Theta}_a \rangle} = \frac{n_a m_a}{\tau_{aa}} \sum \frac{\mu_{a1}}{\mu_{a1}} \frac{\mu_{a2}}{\mu_{a2}} \frac{\mu_{a3}}{\mu_{a3}} \left( \frac{u_{ia} B}{2} \right) \frac{G_{ia}}{e_a} + \frac{X_{ia}}{X_{ia}} . \]

(C1)

where \( c_a \) and \( \tilde{\mu}_{aj} (j=1,2,3) \) are the dimensionless parameters for the viscosity coefficients and \( G_{ia} \) represents the geometrical factor. The collision frequency \( \tau_{aa} \) is defined in Ref. 2. The dimensionless viscosity parameters \( \tilde{\mu}_{aj} (j=1,2,3) \) are written as

\[ \tilde{\mu}_{aj} = \frac{1}{2} (v_T \tau_{aa})^2 \frac{\langle (\mathbf{B} \cdot \nabla)(\mathbf{B} \cdot \nabla) \rangle}{\langle \mathbf{B}^2 \rangle} \]  

(C2)

It is shown from the solution of the linearized drift kinetic equation that the parallel viscosities in general toroidal configurations for all collision frequencies in the Pfirsch–Schlichter, plateau, and banana regimes are given in the following form:

\[ \frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\pi}_a \rangle}{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Theta}_a \rangle} = \frac{n_a m_a}{\tau_{aa}} \sum \frac{\mu_{a1}}{\mu_{a1}} \frac{\mu_{a2}}{\mu_{a2}} \frac{\mu_{a3}}{\mu_{a3}} \left( \frac{u_{ia} B}{2} \right) \frac{G_{ia}}{e_a} + \frac{X_{ia}}{X_{ia}} . \]

(C1)
The expressions similar to Eq. (C5) have not been obtained yet for the banana regime. We find from Eqs. (C3), (C4), and (C6) that the ratio between the toroidal and parallel viscosities $c_{a}/c_{a}$ is related to the geometrical factor $\langle G_{a} \rangle$ by the following equation:

$$1 - \frac{c_{a}}{c_{a}} \frac{\langle B^{2} \rangle}{\langle B^{2} \rangle} = B_{\theta}^{\theta}(G_{a}) \langle B_{\theta} \rangle$$

which is an essential relation for showing the Onsager symmetry of the total neoclassical transport. We find from Eqs. (C1) and (C5) that the toroidal viscosities are written in terms of the parallel viscosities and the thermodynamic forces as

$$\left\{ \begin{array}{l}
\frac{\langle B_{\theta} \cdot \nabla \cdot \phi \rangle}{\langle B_{\theta} \rangle} = \frac{c_{a}}{c_{a}} \frac{\langle B_{\theta} \cdot \nabla \cdot \phi \rangle}{\langle B_{\theta} \rangle} + \frac{m_{a} n_{a}}{\tau_{a}} c_{a} \langle G_{a} \rangle \\
- \langle G_{a} \rangle = \frac{e_{a}}{e_{a}} \langle \dot{\varphi}_{a} \rangle, \\
\langle \dot{\varphi}_{a} \rangle - \langle \dot{\varphi}_{a} \rangle = \langle X_{a} \rangle 
\end{array} \right.$$  \hspace{1cm} (C9)

**APPENDIX D: DRIFT KINETIC EQUATION AND LEGENDE POLYNOMIAL EXPANSION**

The linearized drift kinetic equation is written as

$$\nu_{a}(x) \cdot \nabla \partial \dot{f}_{a} + v_{da} \cdot \nabla f_{a} - \frac{e_{a}}{f_{a}} \nu_{a} E_{a} f_{a} = C_{a}^{L}(\tilde{f}_{a})$$  \hspace{1cm} (D1)

where $\tilde{f}_{a}$ and $f_{a}$ are regarded as functions of the phase space variables $(x, E, \mu)$ ($\tilde{E} = \frac{1}{2} m_{a} \dot{x}^{2} + e_{a} \Phi$: the particle’s energy; $\mu = m_{a} \dot{x}^{2}/(2B)$: the magnetic moment). Using $(x, v, \xi)$ ($\xi = \nu_{a} / v}$ as the phase space variables instead of $(x, E, \mu)$, we consider the expansion by the Legendre polynomials $P_{l}(\xi)$ ($P_{0}(\xi) = 1, P_{1}(\xi) = \xi, P_{2}(\xi) = \frac{3}{2} \xi^{2} - \frac{1}{2}, \cdots$) for an arbitrary function $F(x, v, \xi)$ as

$$F(x, v, \xi) = \sum_{l=0}^{\infty} F^{(l)}(x, v, \xi),$$

$$F^{(l)}(x, v, \xi) = P_{l}(\xi) \frac{2l+1}{2} \int_{-1}^{1} d \eta P_{l}(\eta) F(x, v, \eta).$$

The $l$th Legendre component corresponds to the $l$-order tensor Hermitian moment part in the Hermitean moment representation employed by Balescu in Ref. 3. The first term in the left-hand side of Eq. (D1) is rewritten in the new phase space variables as

$$\nu_{a}(x) \cdot \nabla \partial \tilde{f}_{a} - \nabla \cdot \nabla B - \frac{e_{a}}{f_{a}} \nu_{a} E_{a} f_{a} = \mathcal{A} \tilde{f}_{a},$$  \hspace{1cm} (D3)

We have the following formulas for the Legendre polynomials:

$$\langle \mathcal{A} \tilde{f}_{a} \rangle = \langle \tilde{f}_{a} \rangle - \frac{e_{a}}{f_{a}} \nu_{a} E_{a} f_{a}.$$  \hspace{1cm} (D4)

We find from Eqs. (D3) and (D4) that the operator $\mathcal{A}$ with $(x, E, \mu)$ as the phase space variables in the drift kinetic equation (D1) transforms the $l$th Legendre component to the linear combination of the $(l \pm 1)$th components:

$$\mathcal{A} F^{(l)} = (\mathcal{A} F^{(l-1)})^{(l+1)} + (\mathcal{A} F^{(l+1)})^{(l-1)}.$$

Contrastively, from the velocity space isotropy of the collision operator described in Eq. (3), the operator $C_{a}^{L}$ transforms the $l$th component to only the $l$th one. The second and third terms in the right-hand side of Eq. (D1) are rewritten from Eqs. (18) and (20) as

$$\left[ \frac{2}{3} + \frac{1}{3} P_{2}(\xi) \right] \int_{-1}^{1} \frac{1}{2} \int_{-1}^{1} \frac{2}{5p_{a}} \nabla \cdot \mathbf{q}_{a} f_{a}$$

and

$$- \frac{e_{a}}{f_{a}} \nu_{a} E_{a} f_{a},$$

respectively. The former is proportional to the divergence of the diamagnetic flows and contains only the zeroth and second Legendre polynomial components while the latter is proportional to the parallel electric field and contains only the first Legendre component.

Now, let us write the drift kinetic equation (D1) by each Legendre component separately. The zeroth Legendre component (or the scalar moment part) of Eq. (D1) is given by

$$\langle \mathcal{A} \tilde{f}_{a}^{(l=0)} \rangle = \frac{2}{3} \int_{-1}^{1} \frac{1}{2} \int_{-1}^{1} \frac{2}{5p_{a}} \nabla \cdot \mathbf{q}_{a} f_{a}$$

Here, the $l=1$ Legendre component $\tilde{f}_{a}^{(l=1)}$ of the distribution function is expanded by the Laguerre polynomials $L_{2}^{(3/2)}(\chi_{a}^{2})$, $L_{2}^{(5/2)}(\chi_{a}^{2})$ as

$$\tilde{f}_{a}^{(l=1)} = \frac{2}{v_{ta}} \xi \mathbf{f}_{a} + \mathbf{q}_{a} \cdot \mathbf{v}_{a} - \frac{e_{a}}{v_{ta}} \nu_{a} E_{a} f_{a}$$

where $\tilde{f}_{a}^{(l=1, j \geq 2)}$ denotes the sum of the $j$th Laguerre polynomial components with $j \geq 2$, which is neglected in the 13M approximation. The first term of Eq. (D8) is rewritten as
\[
\begin{aligned}
\mathcal{f}_{a_{-1}}^{j(1-1)}(t=0) &= \frac{2}{3} \mathbf{x}_{a}^{T} \mathbf{v} \cdot (u_{ia} \mathbf{n}) + \frac{1}{2} \mathbf{x}_{a}^{2} - \frac{5}{2} \\
 &\times \frac{2}{5p_{a}} \mathbf{v} \cdot (q_{ia} \mathbf{n}) f_{a0} \\
 &+ (\mathcal{A}_{a_{-1}}^{j(1-1,j \geq 2)}(t=0)).
\end{aligned}
\]  

(D10)

Then, Eq. (D8) reduces to
\[
\begin{aligned}
\mathfrak{a}_{a}^{l(0)} &= C_{a}^{l(0)} \mathfrak{a}_{a},
\end{aligned}
\]  

(D11)

Integrating Eq. (D11) multiplied by 1 and \(x_{a}^{2}\) over the velocity space, we obtain the incompressibility of \(u_{a}\) and \(q_{a}\):
\[
\nabla \cdot u_{a} = \nabla \cdot q_{a} = 0,
\]  

(D12)

where we used the particle number and energy conservation by the collision operator described in Eq. (2). [Exactly speaking, \(\nabla \cdot q_{a} = 0\) is valid to the lowest order of the small mass ratio as in Eq. (9).] Thus, we have
\[
\begin{aligned}
\mathcal{f}_{a_{-1}}^{j(0)}(t=0) &= \mathfrak{C}_{a}^{l(0)}(\mathfrak{a}_{a}),
\end{aligned}
\]  

(D13)

from which the contribution from the scalar moment part \(\mathfrak{a}_{a}^{l(0)}\) of the distribution function to the kinetic form of the entropy production is given by
\[
\begin{aligned}
\mathfrak{a}_{a}^{l(0)} &= - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathcal{f}_{a_{-1}}^{j(0)} \mathfrak{C}_{a}^{l(0)}(\mathfrak{a}_{a}) \\
&= - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(0)}(\mathfrak{a}_{a}) .
\end{aligned}
\]  

(D14)

In the 13M approximation, \(\mathfrak{a}_{a}^{l(0)}\) vanishes.

Taking the first Legendre component (or the vector moment part) of Eq. (D1), we have
\[
\begin{aligned}
\mathcal{A}_{a_{-1}}^{j(1-1,j \geq 2)}(t=0) &= - \frac{e_{a} E_{ia}}{T_{a}} \mathfrak{u} f_{a0} = C_{a_{-1}}^{l(1-1,j \geq 2)}(\mathfrak{a}_{a}),
\end{aligned}
\]  

(D15)

We obtain the following parallel momentum balance equations from velocity integration of Eq. (D15) multiplied by \(m_{a}v_{\xi}\) and \(m_{a}v_{\xi}(x_{a}^{2} \frac{\xi}{2})^{j_{2}}\):
\[
\begin{aligned}
\mathbf{n} \cdot (\nabla p_{a} + \nabla \cdot \mathbf{u}_{a}) - n_{a} e_{a} &E_{i1} = F_{i1a}, \\
\mathbf{n} \cdot (\nabla \theta_{a} + \nabla \cdot \Theta_{a}) = F_{i2a}.
\end{aligned}
\]  

(D16)

Using Eqs. (D9) and (D15), the contribution from the vector moment part \(\mathfrak{a}_{a}^{l(1-1)}\) to the kinetic form of the entropy production is given by
\[
\begin{aligned}
\mathfrak{a}_{a}^{l(1-1)} &= - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathcal{f}_{a_{-1}}^{j(1-1)} C_{a}^{l(1-1)}(\mathfrak{a}_{a}) \\
&= \frac{1}{T_{a}} n_{a} e_{a} u_{ia} E_{i1} - \frac{1}{T_{a}} u_{ia} \mathfrak{a}_{a}^{l(0)}(\mathfrak{a}_{a}) \\
&= \frac{1}{T_{a}} n_{a} e_{a} u_{ia} E_{i1} - \frac{1}{T_{a}} u_{ia} \left[ \mathfrak{u}_{ia} \mathfrak{a}_{a}^{l(1-1)} + \bar{\mathfrak{a}}_{a}^{l(2-1)}(\mathfrak{a}_{a}) \right] \\
&+ \frac{2}{5p_{a}} q_{ia} \mathbf{n} \cdot (\nabla \theta_{a1} + \nabla \cdot \Theta_{a}) \\
&+ \frac{2}{5p_{a}} \bar{\mathfrak{a}}_{a}^{l(1-1,j \geq 2)}(\mathfrak{a}_{a}) \\
&= \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(1-1)}(\mathfrak{a}_{a}) .
\end{aligned}
\]  

(D17)

where the last integral vanishes in the 13M approximation.

The second Legendre component (or the tensor moment part) of Eq. (D1) is given by
\[
\begin{aligned}
\mathcal{A}_{a_{-1}}^{j(1-1)+j(1-3)}(t=0) &= \frac{1}{2} P_{2}(\xi) x_{a}^{2} \\
&\times \left[ \mathfrak{u}_{ia} \mathfrak{a}_{a}^{l(0)} + \mathfrak{a}_{a}^{l(1-1),j \geq 2} \right] f_{a0} \\
&+ C_{a}^{l(1-1)}(\mathfrak{a}_{a}),
\end{aligned}
\]  

(D18)

which is rewritten by Eqs. (D9) and (D12) as
\[
\begin{aligned}
\mathfrak{a}_{a}^{l(2-1)}(t=0) &= \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(1-1)}(\mathfrak{a}_{a}) \\
+ f_{a}^{l(1-3)}(t=0) &= C_{a}^{l(2-1)}(\mathfrak{a}_{a}).
\end{aligned}
\]  

(D19)

The contribution from the tensor moment part \(\mathfrak{a}_{a}^{l(2-2)}(t=0)\) to the kinetic form of the entropy production is given by
\[
\begin{aligned}
\mathfrak{a}_{a}^{l(2-2)} &= - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(2-2)} C_{a}^{l(2-2)}(\mathfrak{a}_{a}) \\
&= - \frac{1}{T_{a}} \left[ (p_{ia} - p_{a}) u_{ia} + (\Theta_{ia} - \Theta_{a}) \right] \frac{2}{5p_{a}} q_{ia} \\
&\cdot \nabla \ln B - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(1-1,j \geq 2)} + f_{a}^{l(1-3)}(t=0).
\end{aligned}
\]  

(D20)

Similarly, for higher-order tensor moment parts with \(l \geq 3\), we have
\[
\mathcal{A}_{a_{-1}}^{j(1-1)+j(1-3)}(t=0) = C_{a}^{l(1-3)}(\mathfrak{a}_{a}) \quad (for \ l \geq 3),
\]  

(D21)

from which we obtain
\[
\begin{aligned}
\mathfrak{a}_{a}^{l(1)} &= - \mathfrak{E} d^{3} \mathfrak{u}_{a} C_{a}^{l(1)}(\mathfrak{a}_{a}) \\
&= - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(1)}(\mathfrak{a}_{a}) \quad (for \ l \geq 3).
\end{aligned}
\]  

(D22)

Taking the magnetic surface average of each order moment part of the entropy production in Eqs. (D14), (D17), (D20), and (D22) gives
\[
\begin{aligned}
\langle \mathfrak{a}_{a}^{l(0)} \rangle &= - \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(0)}(\mathfrak{a}_{a}) \langle \mathfrak{a}_{a}^{l(1-1,j \geq 2)} \rangle \\
\langle \mathfrak{a}_{a}^{l(1)} \rangle &= \frac{1}{T_{a}} n_{a} e_{a} (u_{ia} E_{i1}) - \frac{1}{T_{a}} u_{ia} \mathfrak{n} \cdot (\nabla p_{a} + \nabla \cdot \mathbf{n}_{a}) \\
&+ \frac{2}{5p_{a}} \mathfrak{n} \cdot (\nabla \theta_{a} + \nabla \cdot \Theta_{a}) \\
&- \mathfrak{E} d^{3} \mathfrak{u}_{a} \mathfrak{a}_{a}^{l(1-1,j \geq 2)}(\mathfrak{a}_{a}) \langle \mathfrak{a}_{a}^{l(1-1)} \rangle \langle \mathfrak{a}_{a}^{l(1-2)} \rangle \\
&= \langle \mathfrak{a}_{a}^{l(0)} \rangle \langle \mathfrak{a}_{a}^{l(1-1)} \rangle \langle \mathfrak{a}_{a}^{l(1-2)} \rangle \\
&= \langle \mathfrak{a}_{a}^{l(0)} \rangle \langle \mathfrak{a}_{a}^{l(1-1)} \rangle \langle \mathfrak{a}_{a}^{l(1-2)} \rangle.
\end{aligned}
\]  

(D23)
Here, as is understandable from the form of $\tilde{f}_{a1}$ shown in Eq. (17), the contribution of the vector Hermitian moments contains $\tilde{\sigma}_a$ which is defined by Eq. (16) and written in the classical thermodynamic form of Eq. (21). In the Pfirsch–Schlüter regime, the higher-order moments representing higher anisotropy become small so that the $l=2$ tensor moments corresponding to the viscosities (and therefore to the banana-plateau and nonaxisymmetric transport fluxes) are smaller than the $l=1$ vector moment corresponding to the classical and Pfirsch–Schlüter fluxes by a small factor $v_{\text{Tsa}}\tau_{\text{sa}}/L$ ($L$: the scale length of the magnetic configuration). In this case, the $l=3$ moments are further smaller and the total entropy production is approximated in the lowest order of $v_{\text{Tsa}}\tau_{\text{sa}}/L$ with the 13M approximation by Eq. (D.29) as

$$\sum_a T_a(\langle\tilde{\sigma}_a\rangle + \langle\tilde{\sigma}_a^{(l=1)}\rangle).$$

(D.30)

which takes the thermodynamic form consisting of the products of the thermodynamic forces and their conjugate classical and Pfirsch–Schlüter fluxes. [Here, it is noted that the classical fluxes $J_{a1}^c$ ($j=1,2$) is defined not by the magnetic surface average but by the local quantity in Eq. (22).] However, for small collision frequencies as in the plateau and banana regimes, contributions from higher-order tensor moments with $l\geq 2$ are comparable to the $l=1$ vector moment and are indispensable in order to derive the complete neoclassical thermodynamic form including the banana-plateau and nonaxisymmetric transport fluxes.

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