THE RATIO OF HOMOLOGY RANK TO HYPERBOLIC VOLUME, I

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Abstract. We show that for every finite-volume hyperbolic 3-manifold $M$ and every prime $p$ we have $\dim H_1(M; \mathbb{F}_p) < 168.602 \cdot \text{vol}(M)$. There are slightly stronger estimates if $p = 2$ or if $M$ is non-compact. This improves on a result proved by Agol, Leininger and Margalit, which gave the same inequality with a coefficient of 334.08 in place of 168.602. It also improves on the analogous result with a coefficient of about 260, which could have been obtained by combining the arguments due to Agol, Leininger and Margalit with a result due to Böröczky. Our inequality involving homology rank is deduced from a result about the rank of the fundamental group: if $M$ is a finite-volume orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 2-semifree, then $\text{rank} \pi_1(M) < 1 + \lambda_0 \cdot \text{vol} M$, where $\lambda_0$ is a certain constant less than 167.79.

1. Introduction

It is a standard consequence of the Margulis Lemma that for every $n \geq 2$ there exists a constant $\lambda > 0$ such that for every finite-volume hyperbolic $n$-manifold $M$ we have $\text{rank} \pi_1(M) \leq \lambda \cdot \text{vol} M$ (where vol denotes hyperbolic volume). In particular, there is a constant $\lambda > 0$ such that for every finite-volume orientable hyperbolic 3-manifold and every prime $p$ we have

$$\text{dim } H_1(M; \mathbb{F}_p) \leq \lambda \cdot \text{vol } M.$$  \hfill (1.0.1)

According to [4, Proposition 2.2], (1.0.1) always holds with $\lambda = 334.08$. The authors of [4] have informed us that they were aware that the result could be improved using the sphere-packing results of [6]; a straightforward application of these results shows that (1.0.1) holds with a value of $\lambda$ that is approximately equal to 260.

Theorem 5.4 of this paper includes the assertion that (1.0.1) always holds with a value of $\lambda$ just under 168.602.

Thus we have:

**Theorem.** For any finite-volume orientable hyperbolic 3-manifold $M$, and any prime $p$, we have

$$\text{dim } H_1(M; \mathbb{F}_p) < 168.602 \cdot \text{vol}(M).$$

Theorem 5.4 also asserts that if one takes $p = 2$, or restricts attention to the case where $M$ is non-compact, then the constant in the conclusion of the theorem stated above can be improved in the first place to the right of the decimal point.
Theorem 5.4 is deduced from Proposition 5.2, which gives a bound on the rank of a fundamental group under a certain condition. Recall that a group \( G \) is said to be \( k \)-semifree for a given positive integer \( k \) if each subgroup of \( G \) having rank at most \( k \) is a free product of free abelian groups. Proposition 5.2 asserts that if \( M \) is a finite-volume orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is 2-semifree, then \( \text{rank} \ \pi_1(M) < 1 + \lambda_0 \cdot \text{vol} \ M \), where \( \lambda_0 \) is a certain constant less than 167.79.

In the sequels to this paper, by combining the methods of the present paper with a variety of deep results and techniques (including the Four Color Theorem and the desingularization techniques of [3] and [8]), we will prove results that significantly improve the estimates given by Proposition 5.2 and Theorem 5.4. While some of these results involve mild additional topological hypotheses, there does appear to be a strict improvement of Theorem 5.4 for the case where \( p = 2 \) and \( M \) is closed.

The proof of Proposition 5.2 is a refinement of the corresponding result in [4] (see the final displayed formula in the proof of [4, Proposition 2.2]). The refinement uses a crucial new idea to improve the estimate. In order to describe this improvement, we shall begin by reviewing the relevant argument given in [4], using some terms that are explained in the body of the present paper.

According to [2, Corollary 4.2], the hypothesis that the finite-volume orientable 3-manifold \( M \) has a 2-semifree fundamental group implies that \( \varepsilon := \log 3 \) is a Margulis number (see Definitions 4.4 below) for \( M \). One fixes a finite subset \( S \) of \( M \) which is a maximal \( \varepsilon \)-discrete set contained in the \( \varepsilon \)-thick part of \( M \) (see Definitions 4.2 and 4.3). By a Voronoi-Dirichlet construction, which we explain in considerable detail in Section 2 below, one associates with the set \( S \) a “cell complex” \( K_S \) whose underlying space is \( M \). (When \( M \) is closed, \( K_S \) is a certain kind of CW complex.) Each (open) 3-cell of \( K_S \) is the homeomorphic image under a locally isometric covering map \( q : \mathbb{H}^3 \to M \) of the interior of a “Voronoi region,” which is a convex polyhedron \( X \) associated with a point \( P \) of \( q^{-1}(S) \), and having \( P \) as an interior point. We think of \( P \) as the “center” of the Voronoi region \( X \).

Now \( \pi_1(M) \) is carried by a connected graph \( \mathcal{G} \) which has \( S \) as its vertex set, and has one edge “dual” to each 2-cell of \( K_S \) meeting the \( \varepsilon \)-thick part of \( M \). To bound the rank of \( \pi_1(M) \) as a multiple of \( \text{vol} \ M \), it therefore suffices to bound the first betti number of \( \mathcal{G} \); this is done by bounding the number of vertices of \( \mathcal{G} \) as a multiple of \( \text{vol} \ M \), and bounding the valence of an arbitrary vertex. In order to bound the number of vertices of \( \mathcal{G} \) (or equivalently the cardinality of \( S \)) as a multiple of \( \text{vol} \ M \), it suffices to give a lower bound for the volume of an arbitrary 3-cell in \( K_S \). In [4] this is done by observing that each 3-cell contains an isometric copy of a ball of radius \( \varepsilon/2 \) in \( \mathbb{H}^3 \), and using the volume of this ball as a lower bound; as mentioned above, this can be improved using the sphere-packing results of [6], and we have incorporated this in our estimates.

Bounding the valence of a vertex of \( \mathcal{G} \) amounts to bounding, for a given Voronoi region \( X \), the number of two-dimensional faces of \( X \) which meet the pre-image under \( q \) of the \( \varepsilon \)-thick part of \( M \). In the present sketch we shall refer to such faces as “thick faces.” Any thick face \( F \) determines a Voronoi region \( Y \) which “neighbors” \( X \) along \( F \) in the sense that \( F = X \cap Y \).
The neighboring Voronoi region $Y$ contains the ball $E$ of radius $\varepsilon/2$ about the “center” of $Y$; furthermore, the “centers” of $X$ and $Y$ are separated by a distance of at most $2\varepsilon$, which implies that $E$ is contained in a ball $N_0$ of radius $5\varepsilon/2$ about the “center” $P$ of $X$. This implies an upper bound of $[B(5\varepsilon/2)/B(\varepsilon/2) - 1] = 493$ for the valence of a vertex of $\mathcal{G}$, where $B(r)$ denotes the volume of a ball of radius $r$ in $\mathbf{H}^3$. This is the valence bound used in [4].

The main new idea in the proof of Proposition 4.10 involves a somewhat different approach to bounding the valence of a vertex. We fix a constant $R$ between $2\varepsilon$ and $5\varepsilon/2$, and denote by $N$ the closed ball of radius $R$ about the point $P$. The strategy is to find a relatively large constant $c > 0$ such that, if $F$ is a thick face of $X$, and $Y$ is the neighboring Voronoi region determined by $F$, the volume of $Y \cap N$ is bounded below by $c$. This will give an upper bound slightly less than $B(R)/c$ for the valence of a vertex of $\mathcal{G}$.

To carry out this strategy, we choose a point $T$ that lies in the intersection of a given thick face $F$ with the preimage under $q$ of the $\varepsilon$-thick part of $M$. The definition of a Voronoi region implies that the distance $D$ from $T$ to $P$ is equal to the distance from $T$ to the “center” $Q$ of the neighboring Voronoi region $Y$. Then $Y$ contains the set $Z$ (an “ice cream cone”) which is defined as the convex hull of the union of $\{T\}$ with the closed ball of radius $\varepsilon/2$ about $Q$, while $N$ contains the closed ball of radius $\rho := R - D$ centered at $T$. In the notation that is formalized later in this introduction, this closed ball is denoted by $\text{nbhd}_\rho(T)$. Since $Y \supset Z$ and $N \supset \text{nbhd}_\rho(T)$, the set $Y \cap N$ contains $Z \cap \text{nbhd}_\rho(T)$.

One can use elementary hyperbolic geometry to calculate the volume of $Z \cap \text{nbhd}_\rho(T)$ in terms of $R, \varepsilon$ and $D$. If we recall that $\varepsilon = \log 3$, and take $R = 2 \log 3 + 0.15$ (a choice of $R$ that will turn out to minimize our valence bound), the expression for the volume of $Z \cap \text{nbhd}_\rho(T)$ becomes a function of the parameter $D$, which can take any value in the interval $[\varepsilon/2, \varepsilon]$. We show that $c := 0.496$ is a lower bound for this function on the subinterval $[R/2 - \varepsilon/4, \varepsilon]$. When $\varepsilon/2 \leq D \leq R/2 - \varepsilon/4$, it turns out that in the geometric situation described above, $Y \cap N$ contains the ball of radius $\varepsilon/2$ about $Q$; this gives a lower bound of $B(\varepsilon/2) > c$ for the volume of $Y \cap N$. Thus $c$ is a lower bound for this volume in all cases.

This provides an improved upper bound of 314 for the valence of a vertex of $\mathcal{G}$. This improved valence bound is the crucial new ingredient in the proof of Proposition 5.2.

Theorem 5.4 is deduced from Proposition 5.2 by the same method as is used for the corresponding step in [4]. The constants that appear in Theorem 5.4 incorporate small improvements that are provided by results from [2], [11] and [13].

The definition and properties of the complex $K_S$ are presented in Section 2 of the present paper. This section contains a good deal of expository material about convex polyhedra and Voronoi decompositions, since the foundations of the theories in question do not seem to be extensively documented in the literature; our main references are to [12] and [10]. In Section 3 we develop the quantitative hyperbolic geometry needed to compute the volume of the set which is denoted $Z \cap \text{nbhd}_\rho(T)$ in the above discussion. In Section 4 we give a general version, for any Margulis number $\varepsilon$, of the argument sketched above for the case $\varepsilon = \log 3$;
the result is embodied in Corollary 4.11. We regard this corollary, and Proposition 4.10 from which it is deduced, as the central results of this paper, although their statements are more technical than that of Theorem 5.4. In Section 5, we do the numerical calculations needed to pass from Corollary 4.11 to Proposition 5.2 and Theorem 5.4.

We summarize here some conventions that will be used in the body of the paper.

If $X$ is a group, we will write $Y \leq X$ to mean that $Y$ is a subgroup of $X$.

The symbol “dist” will denote the distance in a metric space when it is clear from the context which metric space is involved. If $p$ is a point of a metric space, and $r$ is a real number, we will denote by $\text{nbhd}_r(p)$ the set of all $x \in X$ such that $\text{dist}(x, p) < r$. Thus $\text{nbhd}_r(p)$ is a neighborhood of $p$ if $r > 0$, and is empty if $r \leq 0$.

The isometry group of a metric space $X$ is denoted $\text{Isom}(X)$. If $X$ is an orientable Riemannian manifold, $\text{Isom}_+(X)$ will denote the orientation-preserving subgroup of $\text{Isom}(X)$.

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2. The Voronoi complex

In this section, after a brief review of facts about convex polyhedra in $\mathbb{H}^n$, we introduce the notion of a polyhedral complex in $\mathbb{H}^n$, and show how a discrete set in $\mathbb{H}^n$ defines a polyhedral complex, its Voronoi complex. We then show how, if $S$ is a finite subset of a hyperbolic $n$-manifold $M$, the Voronoi complex defined by the pre-image of $S$ under a locally isometric covering map $\mathbb{H}^n \to M$ induces a “generalized polyhedral complex” with $M$ as its underlying space, and an associated dual graph.

Definitions, Notation and Remarks 2.1. Let $n$ be a positive integer. A closed convex set in $\mathbb{E}^n$ or $\mathbb{H}^n$ may be defined to be a non-empty set which is the intersection of a collection of closed half-spaces, or to be a non-empty closed set which contains the line segment joining any two of its points; in the hyperbolic case, the equivalence of the two definitions follows from [10, Lemma 1.5]. If $X$ is any closed convex set in $\mathbb{E}^n$ or $\mathbb{H}^n$, and if $W$ denotes the smallest totally geodesic subspace of $\mathbb{E}^n$ or $\mathbb{H}^n$ containing $X$, we define the dimension of $X$ to be the dimension of $W$, and we define the interior of $X$, denoted $\text{int } X$, to be its topological interior relative to $W$. The boundary of $X$ is $\partial X := X - \text{int } X$.

A support hyperplane for the closed convex set $X$ is a hyperplane which (i) has non-empty intersection with $X$, and (ii) is the boundary of a closed half-space containing $X$. A face of $X$ is a subset $F$ of $\mathbb{E}^n$ or $\mathbb{H}^n$ which either is equal to $X$, or is the intersection of $X$ with a support hyperplane for $X$. (In the latter case we say that $F$ is a proper face of $X$.) Note that a proper face of $X$ has strictly lower dimension than $X$.

We define a convex polyhedron in $\mathbb{H}^n$ or $\mathbb{E}^n$ to be a non-empty set which is the intersection of a family of closed half-spaces whose bounding hyperplanes form a locally finite family. Note that a face of a convex polyhedron $X$ is itself a convex polyhedron and is contained in $X$.

We define a facet of a convex polyhedron $X$ to be a maximal proper face of $X$. 
In this paper we will use a number of facts about closed convex sets and convex polyhedra in $\mathbb{H}^n$ that are the counterparts of well-known facts about closed convex sets and convex polyhedra in $\mathbb{E}^n$. We will list several such facts here and then give very brief hints about how to prove them.

2.1.1. Every $d$-dimensional closed convex set $X$ in $\mathbb{H}^n$ is homeomorphic to a set of the form $D^d - \Xi$, where $D^d$ denotes the closed unit ball in $\mathbb{R}^d$ and $\Xi$ is some subset of $S^{d-1} = \partial D^d$. Furthermore, under a homeomorphism between $X$ and $D^d - \Xi$, the interior and boundary of $X$, defined as above, correspond respectively to $\text{int} D^d$ and $S^{d-1} - \Xi$.

2.1.2. The boundary of any closed convex set is the union of its proper faces.

2.1.3. If $X$ is a closed convex set, any non-empty set which is a finite intersection of faces of $X$ is a face of $X$.

2.1.4. If $X$ is a convex polyhedron, any face of a face of $X$ is a face of $X$.

2.1.5. Every face of a convex polyhedron $X$ is a finite intersection of codimension-1 faces of $X$.

2.1.6. The faces of a convex polyhedron form a locally finite family of sets.

Assertion 2.1.1 is proved in the same way as its Euclidean counterpart (which is a standard exercise).

Assertion 2.1.3, which is the hyperbolic analogue of [12, Section 2.4, Assertion 10], is easily proved using the hyperboloid model of $\mathbb{H}^n$. If $F_1, \ldots, F_m$ are faces of the closed convex set $X \subset \mathbb{H}^n$, we may write $F_i = \Pi_i \cap X$ for $i = 1, \ldots, m$, where each $\Pi_i$ is a support hyperplane for $X$. If we choose a point $x_0 \in F_1 \cap \cdots \cap F_m$, and use the conventions of [10, Section 1], we may write each $\Pi_i$ as the intersection of $\mathbb{H}^n$ with a time-like hyperplane $B_i = \{ x \in \mathbb{R}^{n+1} : u_i \circ (x - x_0) = 0 \}$, where $u_i \in \mathbb{R}^{n+1}$ is a vector such that $u_i \circ (x - x_0) \geq 0$ for every $x \in X$. If we then set $u = u_1 + \cdots + u_m$, then $B := \{ x \in \mathbb{R}^{n+1} : u \circ (x - x_0) = 0 \}$ is time-like and $\Pi := B \cap \mathbb{H}^n$ is a support hyperplane for $X$, so that $F := F_1 \cap \cdots \cap F_m = \Pi \cap X$ is a face of $X$.

In the special case of a convex polyhedron $X$ which is a finite intersection of closed half-spaces, the proofs of Assertions 2.1.2, 2.1.4, 2.1.5, and 2.1.6 above are almost identical to those of their respective Euclidean analogues, [12, Section 2.2, Assertion 4], [12, Section 2.6, Assertion 1], [12, Section 2.6, Assertion 5], and [12, Section 2.6, Assertion 6]. The latter results concern closed convex sets and so-called “polyhedral sets” in $\mathbb{E}^n$; in [12], a “polyhedral set” is defined to be a subset of $\mathbb{E}^n$ which is a finite intersection of closed half-spaces.

The conclusions of the results cited from [12] are the same as those of their hyperbolic analogues, except that the conclusion of [12, Section 2.6, Assertion 6] gives finiteness, rather than just local finiteness, of the set of faces. In the proofs of the hyperbolic versions, one uses Assertion 2.1.3 above in place of [12, Section 2.4, Assertion 10]. Some trivial adjustments are needed, because the definitions given in [12] allow a “polyhedral set” or a “face” to be empty. Also, in adapting the proof of [12, Section 2.6, Assertion 5] to give a proof of Assertion 2.1.5
above, one should replace the term “facet” by “codimension-1 face”; the logic of the proof goes through equally well after this change. (We shall see in 2.1.9 below that the notions of facet and of codimension-1 face are a posteriori equivalent.)

It is a straightforward exercise to extend the proofs of Assertions 2.1.2, 2.1.4, and 2.1.6 to the case of an arbitrary convex polyhedron in \( H^n \) from the special case of a finite intersection of closed half-spaces. To extend 2.1.5 to the general case, one needs the following fact, which is a more interesting exercise. Let us say that a family \( \mathcal{H} \) of closed half-spaces in \( H^n \) is irredundant if \( \mathcal{H} \) has no proper subfamily \( \mathcal{H}' \) such that \( \bigcap_{H \in \mathcal{H}} H = \bigcap_{H' \in \mathcal{H}'} H \).

2.1.7. If \( \mathcal{H} \) is a family of closed half-spaces in \( H^n \) whose bounding hyperplanes form a locally finite family, then \( \mathcal{H} \) has an irredundant subfamily \( \mathcal{H}' \) such that \( \bigcap_{H \in \mathcal{H}} H = \bigcap_{H' \in \mathcal{H}'} H \).

In particular, every convex polyhedron in \( H^n \) is the intersection of an irredundant family of closed half-spaces whose bounding hyperplanes form a locally finite family.

(In [12], the case of the first assertion of 2.1.7 in which the family \( \mathcal{H} \) is finite was left implicit, as this case is trivial.)

The following facts will be useful.

2.1.8. The boundary of any convex polyhedron is the set-theoretical disjoint union of the interiors of all its proper faces.

To prove 2.1.8, first note that an induction on the dimension of a convex polyhedron \( X \), using 2.1.2 and 2.1.4, shows that \( X \) is the union of the interiors of its faces. To prove disjointness, suppose that \( F_1 \) and \( F_2 \) are distinct faces of \( X \). By 2.1.3, \( G := F_1 \cap F_2 \) is a face of \( X \), and the definition of a face then implies that \( G \) is a face of \( F_i \) for \( i = 1, 2 \). Since \( F_1 \neq F_2 \), there is an index \( i_0 \in \{1, 2\} \) such that \( G \) is a proper face of \( F_{i_0} \). We therefore have \( G \subset \partial F_{i_0} \), so that \( \text{int} F_1 \cap \text{int} F_2 = \emptyset \), and 2.1.8 is proved.

2.1.9. A face of a convex polyhedron \( X \) is a facet of \( X \) if and only if it has codimension 1.

The “only if” part of 2.1.9 follows immediately from 2.1.5. To prove the “if” part, suppose that \( E \) is a face of \( X \) which is not a facet. Then \( X \) has a proper face \( F \) with \( E \subsetneq F \). This implies that \( E \) is a proper face of \( F \), and hence \( \dim E < \dim F < \dim X \); thus \( E \) cannot have codimension 1 in \( X \).

Definitions, Notation and Remarks 2.2. A polyhedral complex in \( H^n \) is a locally finite collection \( \mathcal{X} \) of convex polyhedra such that (i) for each \( X \in \mathcal{X} \), all the faces of \( X \) belong to \( \mathcal{X} \), and (ii) for all \( X, Y \in \mathcal{X} \), the set \( X \cap Y \) either is a common face of \( X \) and \( Y \) or is empty. We denote by \( |\mathcal{X}| \) the union of all the convex polyhedra that belong to \( \mathcal{X} \).

The open cells of a polyhedral complex \( \mathcal{X} \) are the sets of the form \( \text{int} X \) for \( X \in \mathcal{X} \). Note that by 2.1.1, each open cell of \( \mathcal{X} \) is topologically an open ball of some dimension.

It follows from 2.1.8 and the definition of a polyhedral complex that \( |\mathcal{X}| \) is set-theoretically the disjoint union of the open cells of \( \mathcal{X} \).
Definition, Notation and Remarks 2.3. Let $\mathcal{S}$ be a locally finite (i.e. discrete and closed) subset of $\mathbb{H}^n$. For each point $P \in \mathcal{S}$, we set

$$X_{P,\mathcal{S}} = \{W \in \mathbb{H}^n : \text{dist}(W, P) \leq \text{dist}(W, Q) \text{ for every } Q \in \mathcal{S}\}.$$  

We shall sometimes write $X_P$ in place of $X_{P, \mathcal{S}}$ in situations where it is understood which set $S$ is involved.

A set having the form $X_{P, \mathcal{S}}$ for some $P \in \mathcal{S}$ will be called a Voronoi region for the locally finite set $\mathcal{S}$.

Proposition 2.4. Let $\mathcal{S}$ be a locally finite subset of $\mathbb{H}^n$. Then:

1. each Voronoi region for $\mathcal{S}$ is a convex polyhedron;
2. if $X_0, \ldots, X_m$ are Voronoi regions then $X_0 \cap \cdots \cap X_m$ is a face of $X_i$ for $i = 0, \ldots, m$;
3. if $X$ is a Voronoi region then every codimension-1 face of $X$ has the form $X \cap Y$ for some Voronoi region $Y$; and
4. if $X_0$ is a Voronoi region then every face of $X_0$ has the form $X_0 \cap X_1 \cap \cdots \cap X_m$ for some Voronoi regions $X_1, \ldots, X_m$.

Proof. Assertion (1) is included in [10, Lemma 5.2]. (The conventions of [10] regarding convex polyhedra are the same as those of the present paper, except that the term “face” is defined in [10] to be what we call a proper face.)

Lemma 5.2 of [10] also asserts that if $m > 0$, and if $X_0, \ldots, X_m$ are distinct Voronoi regions for a locally finite set $\mathcal{S} \subset \mathbb{H}^n$, then $X_0 \cap \cdots \cap X_m$ is a proper face of $X_i$ for $i = 0, \ldots, m$. This implies (2).

To prove (3), let $F$ be any codimension-1 face of the Voronoi region $X$. Choose a point $U \in \text{int} F$. Since $U$ lies on the boundary of $X$, it follows from the definition of a Voronoi region (see 2.3) that there is some Voronoi region $Y \neq X$ such that $U \in Y$. Set $G = X \cap Y$; it follows from Assertion (2) of the present lemma that $G$ is a face of $X$; it is a proper face since $Y \neq X$. By 2.1.3, $L := F \cap G$ is a face of $X$; since $L$ is contained in the face $F$ of $X$, it follows from the definition that $L$ is a face of $F$. But $L$ contains the interior point $U$ of $F$, and hence $L = F$. This means that $F \subset G$. But according to 2.1.9, the codimension-1 face $F$ of $X$ is a facet, i.e. a maximal proper face, of $X$; since $G$ is a proper face of $X$, it now follows that $F = G$, i.e. $F = X \cap Y$.

To prove (4), suppose that $E$ is a face of a Voronoi region $X$. According to 2.1.5, we may write the face $E$ of $X$ as a finite intersection $F_1 \cap \cdots \cap F_m$ of codimension-1 faces of $X$. By Assertion (3), we may write $F_i = X \cap Y_i$ for $i = 1, \ldots, m$, where $Y_1, \ldots, Y_m$ are Voronoi regions. Then we have $E = X \cap Y_1 \cap \cdots \cap Y_m$, and the proof of (4) is complete. □

Notation 2.5. Let $\mathcal{S}$ be a locally finite subset of $\mathbb{H}^n$. We denote by $\mathcal{X}_\mathcal{S}$ the set of all convex polyhedra which are faces of Voronoi regions for $\mathcal{S}$.

Proposition 2.6. Let $\mathcal{S}$ be a locally finite subset of $\mathbb{H}^n$. Then $\mathcal{X}_\mathcal{S}$ is a polyhedral complex.
Proof. By Assertion (1) of Proposition 2.4, each Voronoi region is a convex polyhedron, and hence so are its faces. Thus the elements of $X_\mathcal{F}$ are convex polyhedra.

To prove that the family $X_\mathcal{F}$ is locally finite, it suffices—in view of 2.1.6—to prove that the Voronoi regions for $\mathcal{F}$ form a locally finite family. Thus for each $Q \in H^n$, we must show that the set $\mathcal{T}_Q$ of all points $P \in X_\mathcal{F}$ such that $X_{P,\mathcal{F}} \cap \text{nbhd}_1(Q) \neq \emptyset$ is finite. For this purpose, fix a point $P_0 \in X_\mathcal{F}$. Given any point $P \in \mathcal{T}_Q$, we may choose a point $R \in X_{P,\mathcal{F}} \cap \text{nbhd}_1(Q)$. The definition of $X_{P,\mathcal{F}}$ gives $\text{dist}(R, P) \leq \text{dist}(R, P_0)$. By the triangle inequality we then have $\text{dist}(Q, P) \leq \text{dist}(Q, P_0) + 2 \text{dist}(R, Q) < \text{dist}(Q, P_0) + 2$. As the latter inequality holds for every $P \in \mathcal{T}_Q$, the set $\mathcal{T}_Q$ is bounded, and is therefore finite since $\mathcal{F}$ is locally finite. This establishes the local finiteness of $X_\mathcal{F}$.

It follows from 2.1.4 that any face of a polyhedron in $X_\mathcal{F}$ is a polyhedron in $X_\mathcal{F}$. Now suppose that $E$ and $E'$ are elements of $X_\mathcal{F}$. By Assertion (4) of Proposition 2.4, we may write $E = X_0 \cap \cdots \cap X_m$ and $E' = X'_0 \cap \cdots \cap X'_{m'}$, where $X_0, \ldots, X_m$ and $X'_0, \ldots, X'_{m'}$ are Voronoi regions for $\mathcal{F}$. According to Assertion (2) of Proposition 2.4, $E \cap E' = X_0 \cap \cdots \cap X_m \cap X'_0 \cap \cdots \cap X'_{m'}$ is a face of $X_0$. Since the face $E \cap E'$ of $X_0$ is contained in the face $E$ of $X_0$, it follows from the definitions that $E \cap E'$ is a face of $E$. Similarly it is a face of $E'$. This shows that $X_\mathcal{F}$ is a polyhedral complex.

\[ \square \]

Remarks 2.7. If $\mathcal{F}$ is a locally finite subset of $H^n$, then the $n$-dimensional open cells of $X_\mathcal{F}$ are precisely the interiors of the Voronoi regions for $\mathcal{F}$. It now follows from the definitions that for each $P \in \mathcal{F}$, the open $n$-cell $\text{int} X_P$ is the unique open cell of $X_\mathcal{F}$ containing $P$, and that $P$ is the unique point of $\mathcal{F}$ lying in $\text{int} X_P$. Thus each point of $\mathcal{F}$ lies in a unique open $n$-cell of $X_\mathcal{F}$, and each open $n$-cell contains a unique point of $\mathcal{F}$.

Definition 2.8. We define a **generalized polyhedral complex** to be an ordered quadruple $K = (\mathcal{M}, \mathcal{A}, (\mathcal{D}_H)_{H \in \mathcal{A}}, (\Phi_H)_{H \in \mathcal{A}})$, where

1. $\mathcal{M}$ is a Hausdorff space;
2. $\mathcal{A}$ is a locally finite collection of pairwise disjoint subsets of $\mathcal{M}$ whose union is $\mathcal{M}$;
3. $(\mathcal{D}_H)_{H \in \mathcal{A}}$ is an indexed family of convex polyhedra each contained in a hyperbolic space of dimension at least 2;
4. $\Phi_H$ is a continuous map of $\mathcal{D}_H$ into $\mathcal{M}$ for each $H \in \mathcal{A}$;
5. $\Phi_H | \text{int} \mathcal{D}_H$ maps $\text{int} \mathcal{D}_H$ homeomorphically onto $H$ for each $H \in \mathcal{A}$; and
6. for each $H \in \mathcal{A}$ and for each face $F$ of $\mathcal{D}_H$, there exist an element $C$ of $\mathcal{A}$ and an isometry $\iota : F \to \mathcal{D}_C$ such that $\Phi_H | F = \Phi_C | \iota$.

Given a generalized polyhedral complex $K = (\mathcal{M}, \mathcal{A}, (\mathcal{D}_H)_{H \in \mathcal{A}}, (\Phi_H)_{H \in \mathcal{A}})$, we will call $\mathcal{M}$ the **underlying space** of $K$ and will denote it by $|K|$. The set $\mathcal{A}$ will be denoted $\mathcal{A}^K$, and its elements will be called **cells** of $K$. According to 2.1.1, each cell is topologically an open ball of some dimension. For each $d \geq 0$, the set of all $d$-cells, i.e. cells of dimension $d$, will be denoted by $\mathcal{A}^{d,K}$, and the union of all $d$-cells will be denoted by $|\mathcal{A}^{d,K}|$. For each cell $H$, the convex polyhedron $\mathcal{D}_H$ and the map $\Phi_H$ will be denoted respectively by $\mathcal{D}^H$ and $\Phi^H$. 

Note that in the special case where $D^K_H$ is compact for every $H \in A^K$, the collection $A^K$ is in particular a CW complex with underlying space $|K|$. Indeed, in this case, according to 2.1.1, for each cell $H \in A^K$, the polyhedron $D^K_H$ is topologically a closed ball whose interior is mapped homeomorphically onto $H$ by $\Phi^K_H$; the map $\Phi^K_H$ therefore gives rise to a characteristic map for $H$.

2.9. If $K$ is a generalized polyhedral complex, it follows from Condition (6) of Definition 2.8 that for each $H \in A^K$ and for each face $F$ of $D^K_H$, the interior of $F$ is mapped homeomorphically by $\Phi^K_H$ onto a cell of $K$. Given any cell $C$ of $K$, we define the star of $C$ in $K$ to be the set of all ordered pairs $(H, F)$ such that $H$ is a cell of $K$, $F$ is a face of $D^K_H$, and $\Phi^K_H(\text{int } F) = C$. We define the dimension of an element $(H, F)$ of the star of $C$ to be the dimension of $H$.

Note that if $K$ is a generalized polyhedral complex such that $|K|$ is an $n$-manifold without boundary, the star of each $(n-1)$-cell of $K$ contains exactly two elements of dimension $n$.

Remarks and Notation 2.10. Let $n \geq 2$ be an integer, and let $M$ be a hyperbolic $n$-manifold. Let us write $M = H^n/\Gamma$, where $\Gamma \leq \text{Isom}(H^n)$ is discrete and torsion-free, and let $q : H^n \to \Gamma$ denote the quotient map. Suppose that $\mathcal{X}$ is a polyhedral complex with $|\mathcal{X}| = H^n$, such that

1. $\mathcal{X}$ is invariant under $\Gamma$ in the sense that $\gamma \cdot X \in \mathcal{X}$ whenever $\gamma \in \Gamma$ and $X \in \mathcal{X}$, and
2. the stabilizer in $\Gamma$ of every element of $\mathcal{X}$ is trivial.

It follows from (1) and (2) that if $X$ is an arbitrary polyhedron in $\mathcal{X}$, the restriction of $q$ to $\text{int } X$ is a homeomorphism of $\text{int } X$ onto a subset of $M$, which will be denoted $C_X$; and that for any two elements $X, X'$ of $\mathcal{X}$ we have either $C_X = C_X'$ or $C_X \cap C_{X'} = \emptyset$. Thus $A := \{C_X : X \in \mathcal{X}\}$ is a partition of $M$; it is locally finite since $q$ is a covering map, and since $\mathcal{X}$ is locally finite by the definition of a polyhedral complex. For each $H \in A$, let us choose an element $D_H$ of $\mathcal{X}$ such that $C_{D_H} = H$; and let us set $\Phi_H = q'|D_H$. Then $(M, A, (D_H)_{H \in A}, (\Phi_H)_{H \in A})$ is a generalized polyhedral complex with underlying space $M$. To verify Condition (6) of Definition 2.8, given an element $H$ of $A$ and a face $F$ of $D_H$, we set $C = C_F$, so that $D_C = \gamma(F)$ for some $\gamma \in \Gamma$, and define the isometry $\iota$ to be $\gamma|F$. The generalized polyhedral complex defined in this way will be denoted $K^\mathcal{X}$. It is well defined up to modifying the convex polyhedra $D_H$ within their isometry classes, and modifying the maps $\Phi_H$ by precomposing them with isometries between convex polyhedra.

Remarks and Notation 2.11. If $n \geq 2$ is an integer, and $S$ is a non-empty finite subset of a hyperbolic $n$-manifold $M$, we will associate with $S$ a generalized polyhedral complex $K_S$ such that $|K_S| = M$. For this purpose we write $M = H^n/\Gamma$, where $\Gamma \leq \text{Isom}(H^n)$ is discrete, and torsion-free, and let $q : H^n \to M$ denote the quotient map. Then $\tilde{S} := q^{-1}(S)$ is a non-empty, locally finite subset of $H^n$. Hence by Proposition 2.6, $\mathcal{X}_S$ is a well-defined polyhedral complex. It is clear that the polyhedral complex $\mathcal{X}_S$ is invariant under $\Gamma$, which is Condition (1) of 2.10 with $\mathcal{X} = \mathcal{X}_S$. We claim that Condition (2) of 2.10 also holds with $\mathcal{X} = \mathcal{X}_S$; to prove this, suppose that an element $\gamma$ of $\Gamma$ stabilizes a polyhedron $E \in \mathcal{X}_S$. The set of Voronoi regions having $E$ as a face is non-empty by the definition of $\mathcal{X}_S$, and is finite.
since $X_S$ is locally finite by the definition of a polyhedral complex. Since this set is invariant under $\gamma$, it follows that there is an integer $m > 0$ such that any Voronoi region $X$ having $E$ as a face is invariant under $\gamma^m$. But int $X$ contains a unique point $P$ of $\tilde{S}$ by 2.7. Since $\tilde{S}$ is $\Gamma$-invariant, it follows that $\gamma^m$ fixes $P$ and is therefore the identity; as $\Gamma$ is torsion-free, this implies that $\gamma = 1$, and Condition (2) is established.

As Conditions (1) and (2) hold, 2.10 gives a well-defined generalized polyhedral complex $K^{X_S}$ with $|K^{X_S}| = M$. We now define $K_S$ to be $K^{X_S}$.

We shall also write $A_S$ for $A^{K_S}$, and $A_S^d$ for $A^{d,K_S}$ for any $d \geq 0$. For each $H \in A_S$ we shall write $D_{H,S}$ for $D^{K_S}_H$, and $\Phi_{H,S}$ for $\Phi^{K_S}_H$.

According to 2.7, each point of $\tilde{S}$ lies in a unique open $n$-cell of $X_S$, and each open $n$-cell contains a unique point of $\tilde{S}$. Since, by 2.10, every polyhedron in $X_S$ has trivial stabilizer in $\Gamma$, it follows that each point of $S$ lies in a unique $n$-cell of $K_S$, and that each $n$-cell of $K_S$ contains a unique point of $S$. In particular we have $\#(A^0_S) = \#(S)$, and $S \subset A^3_S$.

**Definition 2.12.** Let $S$ be a non-empty finite subset of a hyperbolic 3-manifold. We define a dot system for $S$ to be a subset $T$ of $|A^2_S| \subset K_S$ which contains at most one point of each 2-cell of $K_S$.

**Notation and Remarks 2.13.** Let $S$ be a non-empty finite subset of a hyperbolic 3-manifold, and let $T$ be a dot system for $S$. We will denote by $\mathcal{P}_{S,T}$ the set of all ordered pairs $(H,u)$ such that $H \in A^2_S$ and $u \in \Phi_{H,S}^{-1}(T) \subset \Phi_{H,S}^{-1}(|A^2_S|) \subset \partial D_{H,S}$. For each $H \in A^3_S$ we will denote by $p_{H,S}$ the unique point of $S$ lying in $H$ (see 2.11), and by $\tilde{p}_{H,S}$ the unique point of $\operatorname{int} D_{H,S}$ which is mapped to $p_{H,S}$ by $\Phi_{H,S}$.

For each $(H, u) \in \mathcal{P}_{S,T}$, let $\ell_{H,u}$ denote the line segment in the convex polyhedron $D_{H,S}$ joining the points $\tilde{p}_{H,S}$ and $u$. We set

$$G^{S,T} = S \cup \bigcup_{(H,u) \in \mathcal{P}_{S,T}} \Phi_{H,S}(\ell_{H,u}) \subset M.$$ 

If $\tau$ is a point in $T$, and if $C$ denotes the 2-cell of $K_S$ containing $\tau$, then according to 2.9, the star of $C$ in $K_S$ has exactly two elements of dimension 3. Hence there are exactly two elements of $\mathcal{P}_{S,T}$, which we denote $(H_i, u_i)$ for $i = 1, 2$, such that $\Phi_{H_i,S}(u_i) = \tau$. Now $A_\tau := \{\tau\} \cup \Phi_{H_1,S}^{\operatorname{int}}(\ell_{H_1,u_1}) \cup \Phi_{H_2,S}^{\operatorname{int}}(\ell_{H_2,u_2})$ is an open arc in $M$. Topologically, $G^{S,T}$ has the structure of a graph (possibly with loops and multiple edges) in which the vertices are the points of $S$ and the open edges are the arcs of the form $A_\tau$ for $\tau \in T$.

It follows from this construction that for any $H \in A^3_S$, the valence of the vertex $p_{H,S}$ in the graph $G^{S,T}$ is the number of two-dimensional faces of $D_{H,S}$ whose interiors are mapped by $\Phi_{H,S}$ onto 2-cells that meet $T$.

**Proposition 2.14.** Let $\Theta$ be a compact three-dimensional submanifold-with-boundary of a hyperbolic 3-manifold $M$. Let $S$ be a finite subset of $\Theta$, and let $T \subset M$ be a dot system for $S \subset M$. Suppose that for every 2-cell $C$ of $K_S$ with $C \cap \Theta \neq \emptyset$, we have $T \cap C \cap \Theta \neq \emptyset$. Then the following conclusions hold.
(1) Any two points of $S$ that lie in the same component of $\Theta$ also lie in the same component of the graph $G^{S,T}$.

(2) If $p$ is a point of $S$, and if $G_p$ and $\Theta_p$ respectively denote the components of $G^{S,T}$ and $\Theta$ that contain $p$, then

$$\text{Im}(\pi_1(\Theta_p, p) \to \pi_1(M, p)) \subseteq \text{Im}(\pi_1(G_p, p) \to \pi_1(M, p)),$$

where the unlabeled arrows denote inclusion homomorphisms.

**Proof.** Suppose that $M, \Theta, S$ and $T$ satisfy the hypotheses of the proposition. We will prove the following assertion:

**2.14.1.** Let $\gamma : [0, 1] \to \Theta$ be a path in $\Theta$ whose endpoints lie in $S$. Then $\gamma$ is fixed-endpoint homotopic in $M$ to a path in $G^{S,T}$.

Suppose for the moment that 2.14.1 is true. If a point $p \in S$ is given, applying 2.14.1 to an arbitrary closed path in $M$ based at $p$ establishes Conclusion (2) of the proposition. On the other hand, if $p$ and $q$ are points of $S$ that lie in the same component of $\Theta$, applying 2.14.1 to a path in $\Theta$ joining $p$ to $q$ gives a path in $G^{S,T}$ joining $p$ to $q$, and Conclusion (1) follows. Thus the proof of the proposition will be complete once 2.14.1 is proved.

Suppose that $\gamma$ satisfies the hypothesis of 2.14.1. Set $p_j = \gamma(j)$ for $j = 0, 1$. Since $p_0, p_1 \in S \subseteq |A^S_2|$ by 2.11, we may assume after a small fixed-endpoint homotopy that $\gamma$ maps $(0, 1)$ into $\text{int} \Theta$, that $\gamma([0, 1])$ is disjoint from $|A^S_2|$ and $|A^S_3|$, and that $\gamma$ is transverse to the $2$-manifold $|A^S_3|$.

If $\gamma^{-1}(|A^S_3|) = \emptyset$, then $\gamma([0, 1])$ must be contained in some $3$-cell $H$ of $K_S$. Since $H \cap S = \{p_{H,S}\}$ by 2.13, $\gamma$ is a closed path based at $p_{H,S}$, and in view of the contractibility of $H$ it is fixed-endpoint homotopic to the constant path at $p_{H,S}$. This gives the conclusion of 2.14.1 in this case.

For the rest of the proof we will assume that $\gamma^{-1}(|A^S_3|) \neq \emptyset$. We may then write $\gamma^{-1}(|A^S_3|) = \{s_i : 0 < i < n\}$, where $n \geq 2$ and $0 = s_0 < \cdots < s_n = 1$. For $0 < i < n$, let $C_i$ denote the $2$-cell of $K_S$ containing $\gamma(s_i)$. Since $\gamma$ is a path in $\Theta$, we have $C_i \cap \Theta \neq \emptyset$. The hypothesis of the proposition then implies that $C_i$ contains a unique point $\tau_i$ of $T$.

For each $i \in \{1, \ldots, n-1\}$ let us fix a path $\alpha_i$ in $C_i$ from $\gamma(s_i)$ to $\tau_i$. We define $\alpha_0$ and $\alpha_n$ to be the constant paths at $p_0$ and $p_1$ respectively.

For $i = 1, \ldots, n$ we have $\gamma((s_{i-1}, s_i)) \subset H_i$ for some $H_i \in A^3_S$. Note that $p_0 = p_{H_1,S}$ and that $p_1 = p_{H_n,S}$. We set $D_i = D_{H_i,S}$ and $\Phi_i = \Phi_{H_i,S}$ for $i = 1, \ldots, n$. There is a unique map $\tilde{\gamma}_i$ from $[s_{i-1}, s_i]$ to the convex polyhedron $D_i$ such that $\Phi_i \circ \tilde{\gamma}_i = \gamma|_{[s_{i-1}, s_i]}$.

For $i = 1, \ldots, n-1$ we have $\gamma(s_i) \in C_i$. Hence for $i = 2, \ldots, n$ the point $\tilde{\gamma}_i(s_{i-1})$ lies in the interior of a face $U_i$ of $D_i$, while for $i = 1, \ldots, n-1$ the point $\tilde{\gamma}_i(s_i)$ lies in the interior of a face $U'_i$ of $D_i$; and $\Phi_i$ maps int $U_i$ homeomorphically onto $C_{i-1}$ for $i = 2, \ldots, n$, while $\Phi_i$ maps int $U'_i$ homeomorphically onto $C_i$ for $i = 1, \ldots, n-1$. There therefore exist paths $\beta_i$ in int $U_i$ for $i = 2, \ldots, n$, and paths $\beta'_i$ in int $U'_i$ for $i = 1, \ldots, n-1$, such that $\Phi_i \circ \beta_i = \alpha_{i-1}$ for $i = 2, \ldots, n$ and $\Phi_i \circ \beta'_i = \alpha_i$ for $i = 1, \ldots, n-1$. We define $\beta_1$ and $\beta'_n$ to be the constant
paths at \( \hat{p}_0 := \hat{p}_{H_0,S} \) and \( \hat{p}_1 := \hat{p}_{H_0,S} \) respectively; we then have \( \Phi_i \circ \beta_i = \alpha_{i-1} \) and \( \Phi_i \circ \beta_i' = \alpha_i \) for \( i = 1, \ldots, n \).

For \( i = 1, \ldots, n \) we may now choose a homotopy \( (\delta_{i,t} : [s_{i-1}, s_i] \to \mathcal{D}_i)_{0 \leq t \leq 1} \) such that \( \delta_{i,0} = \tau_i \), and such that \( \delta_{i,t}(s_{i-1}) = \beta_i(t) \) and \( \delta_{i,t}(s_i) = \beta_i'(t) \) for \( 0 \leq t \leq 1 \). Now \( \varepsilon_i := \delta_{i,1} \) maps \( s_{i-1} \) to \( u_i := \beta_i(1) \) and maps \( s_i \) to \( u_i' := \beta_i'(1) \).

For \( i = 2, \ldots, n \) we have \( \Phi_i(u_i) = \alpha_{i-1}(1) = \tau_{i-1} \in \mathcal{T} \); thus we have \( (H_i, u_i) \in \mathcal{P}_{S,T} \); so that the line segment \( \ell_{H_i,u_i} \subset \mathcal{D}_i \) is defined. Likewise, for \( i = 1, \ldots, n - 1 \) we have \( \Phi_i(u_i') = \alpha_i(1) = \tau_i \in \mathcal{T} \); thus we have \( (H_i, u_i') \in \mathcal{P}_{S,T} \); so that the line segment \( \ell_{H_i,u_i'} \subset \mathcal{D}_i \) is defined. We set \( L_1 = \ell_{H_1,u_1}, L_n = \ell_{H_n,u_n}, \) and \( L_i = \ell_{H_i,u_i} \cup \ell_{H_i,u_i'} \) for any \( i \) with \( 1 < i < n \). Then \( L_i \subset \mathcal{D}_i \) is connected for \( i = 1, \ldots, n \); this is obvious for \( i = 1, n \), and for \( 1 < i < n \) it follows from the observation that the segments \( \ell_{H_i,u_i} \) and \( \ell_{H_i,u_i'} \) share (at least) the endpoint \( \hat{p}_{H_i,S} \). (It may happen that \( u_i = u_i' \), but this does not affect the argument.)

For \( i = 1, \ldots, n \), since \( u_i \) and \( u_i' \) lie in the connected set \( L_i \), we may choose a map \( \zeta_i : [s_{i-1}, s_i] \to L_i \) such that \( \zeta_i(s_{i-1}) = u_i \) and \( \zeta_i(s_i) = u_i' \). Since \( \mathcal{D}_i \) is contractible, there is a homotopy \( (\eta_{i,t} : [s_{i-1}, s_i] \to \mathcal{D}_i)_{0 \leq t \leq 1} \), constant on \( \{s_{i-1}, s_i\} \), such that \( \eta_{i,0} = \varepsilon_i \) and \( \eta_{i,1} = \zeta_i \).

For each \( i \in \{1, \ldots, n\} \), let us now define \( (\theta_{i,t} : [s_{i-1}, s_i] \to \mathcal{D}_i)_{0 \leq t \leq 1} \) to be the composition of the homotopies \( \delta_{i,t} \) and \( \eta_{i,t} \) (so that \( \theta_{i,0} = \delta_{i,2t} \) for \( 0 \leq t \leq 1/2 \) and \( \theta_{i,t} = \eta_{i,2t-1} \) for \( 1/2 \leq t \leq 1 \)). Then we have a well-defined homotopy \( (\Lambda_t : [0,1] \to \mathcal{M})_{0 \leq t \leq 1} \), constant on \( \{0,1\} \), given by setting \( \Lambda_t(s) = \Phi_i \circ \theta_{i,t}(s) \) whenever \( 1 \leq i \leq n \) and \( s_{i-1} \leq s \leq s_i \). We have \( \Lambda_0 = \gamma_i \) and the definition of \( \mathcal{G}^{S,T} \) gives that \( \Lambda_1([0,1]) \subset \mathcal{G}^{S,T} \).}

3. Volumes of certain convex sets in \( \mathbf{H}^3 \)

In the introduction to this paper we sketched the proof of the central result Proposition 4.10. One essential step in the sketch involved computing the volume of the set that was denoted in the introduction by \( Z \cap \text{nbhd}_\rho(T) \). Here \( \rho \) is a positive real number and \( T \) is a point in \( \mathbf{H}^3 \) (so that \( \text{nbhd}_\rho(T) \) is a ball centered at \( T \)); and \( Z \) is an “ice cream cone” defined as the convex hull of the union of \( \{T\} \) with another ball (a “scoop of ice cream”). The goal of this section is to prove Lemma 3.10, which provides a formula for the volume of such an intersection in what will turn out to be the crucial case: this is the case in which the scoop has radius less than \( \rho \), and the ball \( \text{nbhd}_\rho(T) \) contains the center of the scoop but does not contain the entire scoop. (These conditions are expressed by a chain of inequalities in the hypothesis of Lemma 3.10.)

Along the way to proving Lemma 3.10, we will give formulae for the volumes of various other kinds of sets in \( \mathbf{H}^3 \).

**Notation 3.1.** We define a strictly increasing function \( B(r) = \pi(\sinh(2r) - 2r) \) for \( r > 0 \). Geometrically, \( B(r) \) gives the volume of a ball in \( \mathbf{H}^3 \) of radius \( r \).

We define a solid cap to be a subset of \( \mathbf{H}^3 \) which is the intersection of a closed ball with a closed half-space which meets the interior of the ball. (In particular, a closed ball is itself a solid cap in this sense.)
Now let real numbers $r$ and $w$ be given, with $r > 0$. Let $N$ be an open ball of radius $r$ centered at a point $O \in \mathbb{H}^3$, and let $\Pi$ be a plane whose distance from the center of $N$ is $|w|$. Let $\mathcal{H}$ be a closed half-space which is bounded by $\Pi$, contains $O$ if $w \leq 0$, and does not contain $O$ if $w > 0$. Then the set $\overline{N} \cap \mathcal{H}$ is a solid cap if $|w| < r$, and is otherwise either a one-point set or the empty set. The volume of $\overline{N} \cap \mathcal{H}$, which depends only on $r$ and $w$, will be denoted by $\kappa(r, w)$.

For the case $w > 0$, an analytic expression for $\kappa(r, w)$ is given in [9, Section 14]. (Note that we have $\kappa(r, w) = 0$ whenever $w \geq r$. ) We can calculate $\kappa(r, w)$ in the case $w \leq 0$ by observing that $\kappa(r, 0) = B(r)/2$, and that $\kappa(r, w) = B(r) - \kappa(r, |w|)$ when $w < 0$. This method of calculating values of $\kappa$ is used in the course of computations referred to in Section 5 of this paper.

Proposition 3.3 below gives a formula for an altitude of a hyperbolic triangle in terms of its sides; the function $\eta$ defined in 3.2 is needed for this formula.

**Notation and Remarks 3.2.** We define a function $\eta$ on $(0, \infty)^3$ by

$$
\eta(r_1, r_2, D) = \frac{2(\cosh r_1)(\cosh r_2)(\cosh D) - (\cosh^2 r_1 + \cosh^2 r_2 + \cosh^2 D) + 1}{\sinh^2 D}.
$$

(The labeling of the coordinates in $(0, \infty)^3$ as $r_1$, $r_2$ and $D$ is chosen for consistency with the geometric application to be given in Proposition 3.3 below.) Note that for any $r_1, r_2, D > 0$ we have

$$(\sinh^2 D)((\sinh^2 r_1) - \eta(r_1, r_2, D)) = ((\cosh r_1)(\cosh r_2) - \cosh D)^2 \geq 0,$$

so that

$$(3.2.1) \quad \eta(r_1, r_2, D) \leq \sinh^2 r_1$$

for all $r_1, r_2, D > 0$.

We set

$$\mathcal{V} = \{(r_1, r_2, D) \in (0, \infty)^3 : \eta(r_1, r_2, D) \geq 0\}.$$ 

Note that $\mathcal{V}$ is closed in the subspace topology of $(0, \infty)^3 \subset \mathbb{R}^3$.

It follows from (3.2.1) that for every $(r_1, r_2, D) \in \mathcal{V}$ we have $(\cosh r_1)/\sqrt{1 + \eta(r_1, r_2, D)} \geq 1$. We may therefore define a non-negative-valued function $\sigma$ on $\mathcal{V}$ by

$$\sigma(r_1, r_2, D) = \text{arccosh} \left( \frac{\cosh r_1}{\sqrt{1 + \eta(r_1, r_2, D)}} \right).$$

**Proposition 3.3.** Let $P_1P_2E$ be a hyperbolic triangle. Set $r_i = |P_iE|$ for $i = 1, 2$, and $D = |P_1P_2|$. Let $U$ denote the perpendicular projection of $E$ to the line containing $P_1$ and $P_2$ (so that $U$ may or may not lie on the segment $P_1P_2$). Then $(r_1, r_2, D) \in \mathcal{V}$, and $\sinh |EU| = \sqrt{\eta(r_1, r_2, D)}$. 
Proof. Let \( \alpha \) denote the angle of the given triangle at the vertex \( P_1 \). The hyperbolic law of cosines gives
\[
\cos \alpha = \frac{\cosh r_1 \cosh D - \cosh r_2}{\sinh r_1 \sinh D}.
\]
Writing \( \sin^2 \alpha = 1 - \cos^2 \alpha \), substituting the right side of (3.3.1) for \( \cos \alpha \) and simplifying, we obtain \( \sin^2 \alpha = \eta(r_1, r_2, D)/\sinh^2 r_1 \). Hence \( \eta(r_1, r_2, D) \geq 0 \), i.e. \( (r_1, r_2, D) \in \mathcal{V} \), and \( (\sinh r_1)(\sin \alpha) = \sqrt{\eta(r_1, r_2, D)} \). But the hyperbolic law of cosines, applied to the right triangle \( P_1UE \), gives \( \sinh |EU| = \sinh r_1 \sin \alpha \), and the conclusion follows.

Our next goal is to give a formula for the intersection of two balls in \( \mathbb{H}^3 \).

**Notation 3.4.** We define a function \( V_{\text{ens}} \) on \( \mathcal{V} \) by
\[
V_{\text{ens}}(x, y, z) = \kappa(x, \sigma(x, y, z)) + \kappa(y, z - \sigma(x, y, z)).
\]

**Proposition 3.5.** Suppose that \( P_1 \) and \( P_2 \) are distinct points of \( \mathbb{H}^3 \), and set \( D = \text{dist}(P_1, P_2) \). Let positive numbers \( r_1 \) and \( r_2 \) be given, and suppose that \( r_2 < \min(D, r_1) \), \( D < r_1 + r_2 \), and \( r_1 < r_2 + D \). Then \((r_1, r_2, D) \in \mathcal{V} \), and
\[
\text{vol}(\text{nbhd}_{r_1}(P_1) \cap \text{nbhd}_{r_2}(P_2)) = V_{\text{ens}}(r_1, r_2, D).
\]

**Proof.** For \( i = 1, 2 \) set \( N_i = \text{nbhd}_{r_i}(P_i) \). The hypothesis implies that the open balls \( N_1 \) and \( N_2 \) have non-empty intersection and that neither is contained in the other. Hence \( \Delta := \partial N_1 \cap \partial N_2 \) is a closed disk. If \( \Pi \) denotes the plane containing \( \Delta \), the closed half-spaces bounded by \( \Pi \) may be labeled \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) in such a way that \( \overline{N}_1 \cap \overline{N}_2 \) is the union of the solid caps \( K_1 := \mathcal{H}_1 \cap \overline{N}_1 \) and \( K_2 := \mathcal{H}_2 \cap \overline{N}_2 \). We have \( K_1 \cap K_2 = \Delta \). Hence
\[
\text{vol}(\overline{N}_1 \cap \overline{N}_2) = \text{vol} K_1 + \text{vol} K_2.
\]

Let \( U \) denote the center of \( \Delta \). Then \( U, P_1 \) and \( P_2 \) lie on a line \( L \), which meets \( \Pi \) perpendicularly at \( U \). We choose a point \( E \) on the circle \( \partial \Delta \); then \( \ell := |EU| \) is the radius of \( \Delta \). For \( i = 1, 2 \), we denote by \( u_i \) the distance from \( P_i \) to the plane \( \Pi \).

We consider the right triangle \( EUP_i \) for \( i = 1, 2 \). Its hypotenuse has length \( |EP_i| = r_i \), and its other side lengths are \( |EU| = \ell \) and \( |UP_i| = u_i \). The hyperbolic law of cosines therefore gives \( \cosh r_i = (\cosh u_i)(\cosh \ell) \), i.e. \( r_i > \ell \) and
\[
u_i = \text{arccosh} \left( \frac{\cosh r_i}{\cosh \ell} \right).
\]

The hypothesis implies that \( r_2 < r_1 \), which with (3.5.2) gives
\[
u_2 < \nu_1.
\]

The triangle \( P_1P_2E \) has side lengths \( r_1 = |P_1E|, r_2 = |P_2E| \) and \( D = |P_1P_2| \); furthermore, the line segment \( EU \), whose length is \( \ell \), is contained in the plane \( \Pi \) and is therefore perpendicular to \( L \) at \( U \). It therefore follows from Proposition 3.3 that \( (r_1, r_2, D) \in \mathcal{V} \) (which is the first assertion of the present proposition), and that \( \sinh \ell = \sqrt{\eta(r_1, r_2, D)} \). Hence
\[
\cosh \ell = \sqrt{1 + \eta(r_1, r_2, D)}.
\]
It follows from (3.5.2), (3.5.4), and the definition of the function \( \sigma \) on its domain \( \mathcal{V} \) (see 3.2) that

\[(3.5.5) \quad u_1 = \sigma(r_1, r_2, D).\]

We claim that

\[(3.5.6) \quad P_1 \in \text{int} \mathcal{H}_2 = \mathbb{H}^3 - \mathcal{H}_1.\]

Assume that (3.5.6) is false, so that \( P_1 \in \mathcal{H}_1 \). Since \( P_1 \) is the center of \( N_1 \), we then have \( P_1 \in K_1 \subset \overline{N_1} \cap \overline{N_2} \), so that \( P_1 \in \overline{N_2} \). This means that \( \text{dist}(P_1, P_2) \leq r_2 \), which is a contradiction, since by hypothesis \( \text{dist}(P_1, P_2) = D > r_2 \). Thus (3.5.6) is established.

Now since, by (3.5.6), the center of the radius-\( r_1 \) ball \( N_1 \), which lies at a distance \( u_1 \) from \( \Pi \), does not lie in the half-space \( \mathcal{H}_1 \), it follows from the definition of \( \kappa \) given in 3.1 that \( \text{vol} \ K_1 = \kappa(r_1, u_1) \). Combining this with (3.5.5), we obtain

\[(3.5.7) \quad \text{vol} K_1 = \kappa(r_1, \sigma(r_1, r_2, D)).\]

Next note that since \( U, P_1, P_2 \in L \) and \( P_1 \neq P_2 \), one of the following cases must occur: (a) \( P_1 \) lies strictly between \( P_2 \) and \( U \), or \( U = P_1 \); (b) \( U \) lies strictly between \( P_1 \) and \( P_2 \); (c) \( P_2 \) lies between \( P_1 \) and \( U \), or \( U = P_2 \). If (a) holds, then \( u_2 = \text{dist}(U, P_2) - \text{dist}(U, P_1) = u_1 \), which contradicts (3.5.3). Hence (b) or (c) must hold.

If (b) holds, then \( P_1 \) and \( P_2 \) lie in the interiors of distinct half-spaces bounded by \( \Pi \). Since \( P_1 \in \text{int} \mathcal{H}_2 \) by (3.5.6), we have \( P_2 \in \text{int} \mathcal{H}_1 \) in this case. Thus the center of the radius-\( r_2 \) ball \( N_2 \), which lies at a distance \( u_2 \) from \( \Pi \), does not lie in the half-space \( \mathcal{H}_2 \). It therefore follows from the definition of \( \kappa \) given in 3.1 that

\[(3.5.8) \quad \text{vol} K_2 = \kappa(r_2, u_2) \text{ if (b) holds.}\]

If (c) holds, then \( P_1 \) and \( P_2 \) lie in the same closed half-space bounded by \( \Pi \). Since \( P_1 \in \text{int} \mathcal{H}_2 \) by (3.5.6), we have \( P_2 \in \mathcal{H}_2 \) in this case. Thus the center of the radius-\( r_2 \) ball \( N_2 \), which lies at a distance \( u_2 \) from \( \Pi \), lies in the half-space \( \mathcal{H}_2 \). It therefore follows from the definition of \( \kappa \) given in 3.1 that

\[(3.5.9) \quad \text{vol} K_2 = \kappa(r_2, -u_2) \text{ if (c) holds.}\]

On the other hand, since \( \text{dist}(P_1, P_2) = D \), we have \( u_2 = D - u_1 \) if (b) holds, and \( u_2 = u_1 - D \) if (c) holds. In view of (3.5.8) and (3.5.9), it follows that in all cases we have \( \text{vol} K_2 = \kappa(r_2, D - u_1) \). Combining this with (3.5.5), we obtain

\[(3.5.10) \quad \text{vol} K_2 = \kappa(r_2, D - \sigma(r_1, r_2, D))\]

in all cases. The conclusion of the proposition follows from (3.5.1), (3.5.7), (3.5.10), and the definition of \( V_{\text{ens}} \).

\[\square\]

3.6. By a right circular cone \( \mathcal{K} \subset \mathbb{H}^3 \) we mean a set which is the convex hull of \( \{T\} \cup \Delta \), where \( T \) (the apex of \( \mathcal{K} \)) is a point, \( \Delta \) is a closed disk contained in a plane \( \Pi \), and the line joining \( T \) to the center of \( \Delta \) is perpendicular to \( \Pi \). A line segment joining \( T \) to a point of \( \partial \Delta \) will be called a generator of \( \mathcal{K} \); the line segment joining \( T \) to the center of \( \Delta \) will be
called the *axis* of $\Delta$; and the angle between a generator and the axis will be called the *angle* of $\mathscr{K}$.

The functions defined in the next subsection will be used in Proposition 3.8 to compute the volume of a right circular cone in $H^3$.

**Notation 3.7.** If $a$ is a positive number and $\beta$ is any real number, we set

\[
\psi(a, \beta) = \arccosh \left( \frac{\cosh a}{\sqrt{1 + (\sinh^2 a)(\sin^2 \beta)}} \right)
\]

and

\[
V_{\text{cone}}(a, \beta) = \frac{B(a)}{2} (1 - \cos \beta) - \kappa(a, \psi(a, \beta)).
\]

**Proposition 3.8.** If $\mathscr{K}$ is a right circular cone whose generator has length $a$ and whose angle is $\beta$, then the axis of $\mathscr{K}$ has length $\psi(a, \beta)$, and we have

\[
\text{vol } \mathscr{K} = V_{\text{cone}}(a, \beta).
\]

**Proof.** Let $T$ denote the apex of $\mathscr{K}$, let $O$ denote the center of the base $\Delta$ of $\mathscr{K}$, and let $P$ be a point on $\partial \Delta$. Then $l := |OP|$ is the radius of $\Delta$, and $h := |TO|$ is the length of the axis. Applying the hyperbolic law of sines to the right triangle $TOP$, we obtain

\[
\sinh l = (\sin \beta)(\sinh a);
\]

applying the hyperbolic Pythagorean theorem to this same triangle, we obtain

\[
(\cosh h)(\cosh l) = \cosh a.
\]

In view of the definition of $\psi(a, \beta)$ it follows that $h = \psi(a, \beta)$, which is the first assertion of the proposition.

Now let $N$ denote the ball of radius $a$ centered at $T$, so that $\text{vol } N = B(a)$. Let $\Sigma$ denote the sector of $N$ consisting of all radii of $N$ that meet $\Delta$. Then $\Sigma$ meets the sphere $S := \partial N$ in a spherical cap centered at the point $Q$ where $S$ meets the ray originating at $T$ and passing through $O$. Let us equip $S$ with the standard spherical metric in which its total area is $4\pi$. In terms of this metric we have $S \cap \Sigma = \text{bdh}_\beta(Q)$, so that the area of $S \cap \Sigma$ in the standard spherical metric is $2\pi(1 - \cos \beta)$. Hence

\[
\text{vol } \Sigma = \left( \frac{\text{area}(\Sigma \cap S)}{\text{area } S} \right) \cdot \text{area } S = \frac{B(a)}{2} (1 - \cos \beta).
\]

Let $\mathcal{H}$ and $\mathcal{H}'$ denote the half-spaces in $H^3$ bounded by the plane $\Pi$ containing $\Delta$, labeled in such a way that $O \in \mathcal{H}'$. Since $h$ is the distance from $T$ to $\Pi$, 3.1 gives $\text{vol}(\Sigma \cap \mathcal{H}) = \text{vol}(N \cap \mathcal{H}) = \kappa(a, h)$. But the sets $\Sigma \cap \mathcal{H}$ and $\mathscr{K} = \Sigma \cap \mathcal{H}'$ meet precisely in $\Delta$ and have union $\Sigma$; hence

\[
\text{vol } \mathscr{K} = \text{vol}(\Sigma) - \text{vol}(\Sigma \cap \mathcal{H}) = \frac{B(a)}{2} (1 - \cos \beta) - \kappa(a, h);
\]

since $h = \psi(a, \beta)$, the right-hand side of this last equality is by definition equal to $V_{\text{cone}}(a, \beta)$. \qed

The next subsection introduces the functions that appear in Lemma 3.10, which is the main result of this section.
Notation 3.9. If $r$ and $D$ are real numbers with $0 < r < D$, we set
\[ \omega(r, D) = \arccosh \left( \frac{\cosh D}{\cosh r} \right) \]
and
\[ \theta(r, D) = \arcsin \left( \frac{\sinh r}{\sinh D} \right). \]
We then have $\omega(r, D) > 0$ and $0 < \theta(r, D) < \pi/2$.

We set $V_0 = \{(\rho, r, D) \in V : r < D\}$. Since $V_{\text{lens}}$ has domain $\mathcal{V}$, and since $\psi, V_{\text{cone}}$ and $\kappa$ all have domain $(0, \infty) \times (-\infty, \infty)$, while $\omega(r, D)$ and $\theta(r, D)$ are defined when $0 < r < D$ and are positive-valued, we may define a function $\phi$ with domain $V_0$ by
\[ \phi(\rho, r, D) = V_{\text{lens}}(\rho, r, D) + V_{\text{cone}}(\omega(r, D), \theta(r, D)) - \kappa(r, D - \psi(\omega(r, D), \theta(r, D))). \]

It follows from the definitions of the functions $V_{\text{lens}}, V_{\text{cone}}, \kappa$ and $\psi$ that these functions are continuous on their respective domains. Hence $\phi$ is continuous on $V_0$.

Lemma 3.10. Let $T$ and $Q$ be distinct points in $\mathbb{H}^3$. Set $D = \text{dist}(T, Q)$. Let $\rho$ and $r$ be positive numbers such that $r < D < \rho < D + r$. Then $(\rho, r, D) \in \mathcal{V}_0$. Furthermore, if $Z \subset \mathbb{H}^3$ denotes the convex hull of $\{T\} \cup \text{nbhd}_r(Q)$, then
\[ \text{vol}(Z \cap \text{nbhd}_\rho(T)) = \phi(\rho, r, D). \]

Proof. Set $J = \text{nbhd}_r(Q)$. Since $D > r$, there is a right circular cone $\mathcal{K}$ which has apex $T$, and whose generators are tangent to $J$. We have $Z = J \cup \mathcal{K}$. Let $a$ denote the common length of all generators of $\mathcal{K}$. If $U$ is a point of tangency between a generator of $\mathcal{K}$ and the ball $J$, then the triangle $TUQ$ has a right angle at $U$; its hypotenuse $TQ$ has length $D$, and its sides $TU$ and $UQ$ have respective lengths $a$ and $r$. The hyperbolic Pythagorean theorem therefore gives $\cosh D = (\cosh a)(\cosh r)$, i.e.
\[ (3.10.1) \quad a = \omega(r, D). \]

In particular $(3.10.1)$ implies that $a < D$, which in view of the hypothesis implies that $a < \rho$. Since $\mathcal{K} \subset \text{nbhd}_a(T)$, it follows that $\mathcal{K} \subset \text{nbhd}_\rho(T)$. Hence
\[ Z \cap \text{nbhd}_\rho(T) = (J \cup \mathcal{K}) \cap \text{nbhd}_\rho(T) = \mathcal{K} \cup (J \cap \text{nbhd}_\rho(T)). \]

It follows that
\[ \text{vol}(Z \cap \text{nbhd}_\rho(T)) = \text{vol}(\mathcal{K}) + \text{vol}(J \cap \text{nbhd}_\rho(T)) - \text{vol}(\mathcal{K} \cap (J \cap \text{nbhd}_\rho(T))) \]
\[ = \text{vol}(\mathcal{K}) + \text{vol}(J \cap \text{nbhd}_\rho(T)) - \text{vol}(\mathcal{K} \cap J), \]
where the last equality again follows from the inclusion $\mathcal{K} \subset \text{nbhd}_\rho(T)$.

The angle $\beta$ of the right circular cone $\mathcal{K}$ is the angle of the right triangle $TUQ$ at the vertex $T$. The hyperbolic law of sines gives $\sin \beta = (\sinh r)/(\sinh D)$, i.e.
\[ (3.10.3) \quad \beta = \theta(r, D). \]
According to Proposition 3.8, we have \( \text{vol} \mathcal{K} = V_{\text{cone}}(\omega(r, D), \theta(r, D)) \), which with (3.10.1) and (3.10.3) gives
\[
(3.10.4) \quad \text{vol} \mathcal{K} = V_{\text{cone}}(\omega(r, D), \theta(r, D)).
\]
Since the hypotheses of the present lemma imply in particular that \( r < \min(D, \rho) \), \( D < r + \rho \) and \( \rho < r + D \), we may apply Proposition 3.5, with \( T, Q, \rho \) and \( r \) playing the respective roles of \( P_1, P_2, r_1 \) and \( r_2 \), to deduce that \( (\rho, r, D) \in \mathcal{V} \), and that
\[
(3.10.5) \quad \text{vol}(J \cap \text{nbhd}_\rho(T)) = V_{\text{lens}}(\rho, r, D).
\]
If \( \Pi \) denotes the plane containing the base \( \Delta \) of the cone \( \mathcal{K} \), it follows from the definition of \( \mathcal{K} \) that \( \mathcal{K} \cap J \) is the intersection of \( J \) with a half-space \( \mathcal{H} \) bounded by \( \Pi \), and that the center \( Q \) of the ball \( J \) does not lie in \( \mathcal{H} \). Hence if \( d \) denotes the distance from \( Q \) to \( \Pi \), we have \( d = |QO| = D - |TO| \). Since \( TO \) is the axis of \( \mathcal{K} \), it follows from the first assertion of Proposition 3.8 that \( |TO| = \psi(a, \beta) \). Hence \( \text{vol}(\mathcal{K} \cap J) = \kappa(r, D - \psi(a, \beta)) \), which with (3.10.1) and (3.10.3) gives
\[
(3.10.6) \quad \text{vol}(\mathcal{K} \cap J) = \kappa(r, D - \psi(\omega(r, D), \theta(r, D))).
\]
Since \( r < D \) by hypothesis, and since we have observed that \( (\rho, r, D) \in \mathcal{V} \), we have \( (\rho, r, D) \in \mathcal{V}_0 \). The equality \( \text{vol}(Z \cap \text{nbhd}_\rho(T)) = \phi(\rho, r, D) \) follows from (3.10.2), (3.10.4), (3.10.5), (3.10.6), and the definition of \( \phi(\rho, r, D) \).

\[\square\]

4. The central results

The central results of this paper are Proposition 4.10 and its Corollary 4.11. The statements and proofs of these results involve some functions that were used in [6], of which we review the definitions in Subsection 4.1. They also involve some basic notions about hyperbolic manifolds and metric spaces which are discussed in Subsections 4.2—4.6.

Notation and Remarks 4.1. As in [6], for any \( n \geq 2 \) and any \( r > 0 \) we shall denote by \( h_n(r) \) the distance from the barycenter to a vertex of a regular hyperbolic \( n \)-simplex \( \mathcal{D}_{n,r} \) with sides of length \( 2r \). Note that since \( \mathcal{D}_{n,r} \) has diameter \( 2r \), we have \( h_n(r) \leq 2r \) for any \( n \geq 2 \) and any \( r > 0 \).

Formulae for \( h_2(r) \) and \( h_3(r) \) are given in [9, Subsection 9.1].

For \( r > 0 \) we define a function \( \text{density}(r) \) (denoted \( d_3(r) \) in [6]), by
\[
\text{density}(r) = (3\beta(r) - \pi)(\sinh(2r) - 2r)/\tau(r),
\]
where the functions
\[
\beta(r) = \text{arcsec}(\text{sech}(2r) + 2) \quad \text{and} \quad \tau(r) = 3 \int_{\beta(r)}^{\text{arcsec} 3} \text{arcsech}((\sec t) - 2) \, dt
\]
respectively give the dihedral angle and the volume of \( \mathcal{D}_{3,r} \).

For any \( r > 0 \) we set \( b(r) = B(r)/\text{density}(r) \).
It is clear from the geometric definitions that $\beta(r)$ and $\tau(r)$, and therefore $\text{density}(r)$ and $b(r)$, are continuous for $r > 0$.

**Definition and Remark 4.2.** Let $X$ be a metric space, and let $\delta$ be a positive real number. A $\delta$-discrete set for $X$ is defined to be a subset $S$ of $X$ such that for any two distinct elements $x, y$ of $S$ we have $\text{dist}(x, y) \geq \delta$. A complete metric space $X$ is compact if and only if for every $\delta > 0$ there is a positive integer $N$ such that every $\delta$-discrete set in $X$ has cardinality at most $N$.

**Definition and Notation 4.3.** A point $p$ of a hyperbolic 3-manifold $M$ is said to be $\epsilon$-thin for a given $\epsilon > 0$ if there is a homotopically non-trivial loop based at $p$ having length less than $\epsilon$. The open set of $M$ consisting of all $\epsilon$-thin points is denoted $M_{\text{thin}}(\epsilon)$, and we set $M_{\text{thick}}(\epsilon) = M - M_{\text{thin}}(\epsilon)$.

**Definitions 4.4.** A Margulis number for an orientable hyperbolic 3-manifold $M$ is defined to be a positive number $\epsilon$ such that, for every point $p \in M$ and for any two loops $\alpha$ and $\beta$ based at $p$ such that the elements $[\alpha]$ and $[\beta]$ of $\pi_1(M, p)$ do not commute, we have $\max(\text{length } \alpha, \text{length } \beta) \geq \epsilon$. If the strict inequality $\max(\text{length } \alpha, \text{length } \beta) > \epsilon$ holds for all $\alpha$ and $\beta$ such that $[\alpha]$ and $[\beta]$ do not commute, we will say that $\epsilon$ is a strict Margulis number for $M$.

The following proposition is a minor variant on a standard result: see for example [13, Proposition 4.9]. (The hypothesis that $\epsilon$ is a strict Margulis number, rather than just a Margulis number, gives the additional information that $M_{\text{thin}}(\epsilon)$ is the interior of a smooth submanifold of $M$.) Most of the conclusions of this proposition will be needed in the present paper; the final sentence will be needed in one of the sequels to this paper.

**Proposition 4.5.** Let $M$ be a finite-volume orientable hyperbolic 3-manifold, and let $\epsilon$ be a positive number. Then $M_{\text{thick}}(\epsilon)$ is compact. If in addition we assume that $\epsilon$ is a strict Margulis number for $M$, then $M_{\text{thin}}(\epsilon)$ is the interior of a smooth submanifold of $M$, which is closed as a subset of $M$ and has only finitely many components; and each of these components is diffeomorphic to either $D^2 \times S^1$ or $T^2 \times [0, \infty)$. Hence $M_{\text{thick}}(\epsilon)$ is a connected 3-manifold-with-boundary, and the inclusion homomorphism $\pi_1(M_{\text{thick}}(\epsilon)) \to \pi_1(M)$ is surjective. Furthermore, if $\epsilon$ is a strict Margulis number for $M$, and if $M$ is written as a quotient $H^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(H^3)$ is discrete and torsion-free with finite covolume, and if $q : H^3 \to M$ denotes the quotient map, then each component of $q^{-1}(M_{\text{thin}}(\epsilon))$ is convex.

**Proof.** Let $\delta > 0$ be given. Set $\delta' = \min(\epsilon, \delta)$. If $S$ is any $\delta$-discrete set in $M_{\text{thick}}(\epsilon) \subset M$, and if $p_1, \ldots, p_m$ are distinct points of $S$, then the sets $\text{nbhd}_{\delta'/2}(p_1), \ldots, \text{nbhd}_{\delta'/2}(p_m)$ are pairwise disjoint, and each of these sets is isometric to a ball of radius $\delta'/2$ in $H^3$. Hence $m \leq \text{vol}(M)/B(\delta'/2)$. This shows that $\lceil \text{vol}(M)/B(\delta'/2) \rceil$ is an upper bound for the cardinality of any $\delta$-discrete set in $M_{\text{thick}}(\epsilon)$, and hence $M_{\text{thick}}(\epsilon)$ is compact.

To prove the second assertion, write $M = H^3/\Gamma$, where $\Gamma$ is a torsion-free discrete subgroup of $\text{Isom}_+(H^3)$, and let $q : H^3 \to M$ denote the quotient map. Suppose that $\epsilon$ is a strict Margulis number for $M$, and for each non-trivial element $x$ of $\Gamma$ set $Y(x) = \{ P \in H^3 :$
dist\( (P, x \cdot P) \leq \varepsilon \)\}. Let \( \mathcal{C} \) denote the set of all maximal abelian subgroups of \( \Gamma \), and for each \( C \in \mathcal{C} \) set \( W(C) = \bigcup_{1 \neq x \in C} Y(x) \). Now if \( C \in \mathcal{C} \) is given, the non-trivial elements of \( C \) all have either the same axis \( A_C \) or the same parabolic fixed point \( T_C \); and for each \( x \in C - \{1\} \), the set \( Y(x) \) either is empty, or is a closed hyperbolic cylinder centered at \( A_C \) (i.e. a set of the form \( \{ P \in \mathbb{H}^3 : \text{dist}(P, A_C) \leq R \} \) for some \( R > 0 \)), or is a closed horoball based at \( T_C \). Furthermore, \( C \) has only finitely many elements \( x \) such that \( Y(x) \neq \emptyset \). It follows that for each \( C \in \mathcal{C} \) we have \( W(C) = Y(x_C) \) for some \( x_C \in C - \{1\} \). If \( C \) and \( C' \) are distinct elements of \( \mathcal{C} \), then \( x_C \) and \( x_{C'} \) do not commute, and if \( P \) is a point of \( W(C) \cap W(C') \), we have \( \text{max}(\text{dist}(P, x_C \cdot P), \text{dist}(P, x_{C'} \cdot P)) \leq \varepsilon \); this is impossible since \( \varepsilon \) is a strict Margulis number for \( M \). Hence \( (W(C))_{C \in \mathcal{C}} \) is a disjoint family of subsets of \( \mathbb{H}^3 \). It is also a locally finite family, because if \( C_1, C_2, \ldots \) is a sequence of distinct elements of \( \mathcal{C} \), and \( P_1, P_2, \ldots \) is a sequence of points such that \( P_i \in W(C_i) \) and \( P_i \to P_\infty \in \mathbb{H}^3 \), then for each \( i \geq 1 \) there is a non-trivial element \( x_i \) of \( C_i \) with \( \text{dist}(P_i, x_i \cdot P_i) \leq \varepsilon \); then \( x_1, x_2, \ldots \) is a sequence of distinct elements of \( \Gamma \) for which the sequence \( (\text{dist}(P_\infty, x_i \cdot P_\infty))_{i \geq 1} \) is bounded, contradicting the discreteness of \( \Gamma \). Hence \( \bigcup_{C \in \mathcal{C}} W(C) \subset \mathbb{H}^3 \) is a smooth 3-manifold-with-boundary, closed as a subset of \( \mathbb{H}^3 \), and its components are cylinders and horoballs; it is clearly \( \Gamma \)-invariant, and therefore the ratio \( \text{vol}(W(C))/\text{vol}(M) < \infty \), the manifold \( W(C)/\Gamma \) is diffeomorphic to \( D^2 \times S^1 \) if \( W(C) \) is a cylinder, and to \( T^2 \times [0, \infty) \) if \( W(C) \) is a horoball. The compactness of \( M_{\text{thick}}(\varepsilon) \) implies that \( V \) has only finitely many components.

The final assertion now follows from the observation that a cylinder or horoball in \( \mathbb{H}^3 \) is convex.

**Definitions 4.6.** Let \( M \) be a hyperbolic 3-manifold and let \( \varepsilon \) be a positive real number. We define an \( \varepsilon \)-**thick** \( \varepsilon \)-**discrete set** for \( M \) to be an \( \varepsilon \)-discrete set for \( M \) which is contained in \( M_{\text{thick}}(\varepsilon) \). By a **maximal** \( \varepsilon \)-**thick** \( \varepsilon \)-**discrete set** for \( M \) we mean simply an \( \varepsilon \)-thick \( \varepsilon \)-discrete set for \( M \) which is not properly contained in any other \( \varepsilon \)-thick \( \varepsilon \)-discrete set for \( M \).

In the statements and proofs of Proposition 4.7, Lemma 4.8 and Proposition 4.10, we will freely use the machinery and notation developed in Section 2: if \( \mathcal{I} \) is a locally finite subset of \( \mathbb{H}^3 \), then for each point \( P \in \mathcal{I} \), the set \( X_{P, \mathcal{I}} \) will be defined as in 2.3, and will often be denoted simply by \( X_P \) when the set \( \mathcal{I} \) is understood. If \( S \) is a finite subset of a hyperbolic 3-manifold \( M \), the generalized polyhedral complex \( K_S \) will be defined as in 2.11, as will the convex polyhedron \( \mathcal{D}_{H,S} \) and the map \( \Phi_{H,S} \) for every cell \( H \) of \( K_S \), and the set \( \mathcal{A}^d_S \) of cells of \( K_S \) for every integer \( d \geq 0 \).

**Proposition 4.7.** Let \( \varepsilon > 0 \) be given, and suppose that \( \mathcal{I} \) is an \( \varepsilon \)-**discrete set** in \( \mathbb{H}^3 \) (so that in particular \( \mathcal{I} \) is a locally finite subset of \( \mathbb{H}^3 \)). Then for every \( P \in \mathcal{I} \), we have

\[
B(\varepsilon/2)/\text{vol}(X_{P,\mathcal{I}} \cap \text{nbhd}_{h_3(\varepsilon/2)}(P)) \leq \text{density}(\varepsilon/2).
\]
Proof. We shall extract this from the proof of [6, Theorem 1]. (The statement and proof of [6, Theorem 1] are given in arbitrary spaces of constant curvature, but in the following discussion we specialize to the hyperbolic case.)

If \( \mathcal{S} \) is an \( \varepsilon \)-discrete set in \( H^n \), then for each \( P \in \mathcal{S} \), the ball \( \text{nbhd}_{\varepsilon/2}(P) \) is contained in the Voronoi region \( X_{P,\mathcal{S}} \). In the terminology of [6], the balls \( \text{nbhd}_{\varepsilon/2}(P) \) for \( P \in \mathcal{S} \) are said to form a “packing of spheres of radius \( \varepsilon/2 \).” The set \( X_{P,\mathcal{S}} \) is referred to as the “Voronoi-Dirichlet cell,” or “V-D cell,” of the ball \( \text{nbhd}_{\varepsilon/2}(P) \), and the “density” of \( \text{nbhd}_{\varepsilon/2}(P) \) in \( X_{P,\mathcal{S}} \) is defined to be \( \text{vol}(\text{nbhd}_{\varepsilon/2}(P))/\text{vol}(X_{P,\mathcal{S}}) \). Theorem 1 of [6] asserts that this “density” is bounded above by the quantity \( d_n(\varepsilon/2) := \text{vol}(L \cap \mathcal{D}_{n,\varepsilon/2})/\text{vol}(\mathcal{D}_{n,\varepsilon/2}) \), where \( L \) denotes the union of the balls of radius \( \varepsilon/2 \) centered at the vertices of \( \mathcal{D}_{n,\varepsilon/2} \). The quantity \( d_3(\varepsilon/2) \) is equal to \( \text{density}(\varepsilon/2) \) as defined in 4.1.

In the proof of [6, Theorem 1], \( r \) denotes the quantity which is denoted \( \varepsilon/2 \) in the statement of the present lemma and \( S \) denotes the ball \( \text{nbhd}_{\varepsilon/2}(P) \), while \( \tilde{S} \) denotes the ball \( \text{nbhd}_{h_3(\varepsilon/2)}(P) \) and \( Z \) denotes the set \( X_{P,\mathcal{S}} \). The sentence in the proof beginning on line 3 of p. 259 reads: “We shall show that the density of \( S \) in \( Z \cap \tilde{S} \) and a fortiori in \( Z \), is less than or equal to...d_n(\varepsilon/2)...” In our notation, the “density of \( S \) in \( Z \cap \tilde{S} \)” is \( \text{vol}(\text{nbhd}_{\varepsilon/2}(P))/\text{vol}(X_{P,\mathcal{S}} \cap \text{nbhd}_{h_3(\varepsilon/2)}(P)) = B(\varepsilon/2)/\text{vol}(X_{P,\mathcal{S}} \cap \text{nbhd}_{h_3(\varepsilon/2)}(P)) \). Thus in the three-dimensional case the latter quantity is bounded above by \( d_3(\varepsilon/2) = \text{density}(\varepsilon/2) \).

Lemma 4.8. Let \( M \) be a finite-volume orientable hyperbolic 3-manifold, and let \( \varepsilon \) be a positive number. Then:

1. there exists a maximal \( \varepsilon \)-thick \( \varepsilon \)-discrete set for \( M \), and any such set is finite; and
2. if \( S \) is any maximal \( \varepsilon \)-thick \( \varepsilon \)-discrete set for \( M \), then for every 3-cell \( H \) of \( K_S \) we have \( \text{vol} H \geq b(\varepsilon/2) \).

Furthermore, if we write \( M = H^3/\Gamma \), where \( \Gamma \leq \text{Isom}_+(H^3) \) is discrete and torsion-free and has finite covolume, and let \( q : H^3 \rightarrow M \) denote the quotient map, then for any maximal \( \varepsilon \)-thick \( \varepsilon \)-discrete set \( S \) for \( M \) we have

3. \( \tilde{S} := q^{-1}(S) \) is an \( \varepsilon \)-discrete set for \( H^3 \),
4. \( \text{dist}(E, \tilde{S}) < \varepsilon \) for every \( E \in q^{-1}(M_{\text{thick}}(\varepsilon)) \);
5. for every \( P \in \tilde{S} \) we have \( X_{P,\tilde{S}} \supset \text{nbhd}_{\varepsilon/2}(P) \) and \( X_{P,\tilde{S}} \cap q^{-1}(M_{\text{thick}}(\varepsilon)) \subset \text{nbhd}_\varepsilon(P) \);
6. for every \( P \in \tilde{S} \) we have \( \text{vol}(X_{P,\tilde{S}} \cap \text{nbhd}_{h_3(\varepsilon/2)}(P)) \geq b(\varepsilon/2) \).

Proof. Conclusion (1) follows from the compactness of \( M_{\text{thick}}(\varepsilon) \), which is the first assertion of Proposition 4.5.

To verify Conclusions (2)–(6), we take \( M \) to be written in the form \( H^3/\Gamma \), where \( \Gamma \) is discrete and torsion-free, and we let \( q : H^3 \rightarrow M \) denote the quotient map. Set \( \tilde{S} = q^{-1}(S) \). To prove (3) we must show that if \( P_1 \) and \( P_2 \) are distinct points of \( \tilde{S} \), then \( \text{dist}(P_1, P_2) \geq \varepsilon \). Set \( p_i = q(P_i) \) for \( i = 1, 2 \). If \( p_1 \neq p_2 \), then since \( p_1 \) and \( p_2 \) are distinct points of the \( \varepsilon \)-discrete set \( S \), we have \( \text{dist}(p_1, p_2) \geq \varepsilon \) and hence \( \text{dist}(P_1, P_2) \geq \varepsilon \). If \( p_1 = p_2 \), then a geodesic path from \( P_1 \) to \( P_2 \) is projected by \( q \) to a closed path based at \( p_1 \), which is homotopically non-trivial.
since $P_1 \neq P_2$. Since $p_1 \in S \subset M_{\text{thick}}(\varepsilon)$, this closed path must have length at least $\varepsilon$, and hence $\text{dist}(P_1, P_2) \geq \varepsilon$ in this case as well.

To prove (4), let a point $E \in q^{-1}(M_{\text{thick}}(\varepsilon))$ be given. Set $e = q(E)$. Since $e \in M_{\text{thick}}(\varepsilon)$, we have $S \cup \{e\} \subset M_{\text{thick}}(\varepsilon)$; the maximality of $S$ then implies that $S \cup \{e\}$ is not an $\varepsilon$-discrete set, and hence that $\text{dist}(e, w) < \varepsilon$ for some $w \in S$. A path in $M$ from $e$ to $w$ which has length less than $\varepsilon$ lifts to a path of length less than $\varepsilon$ from $E$ to $W$ for some $W \in q^{-1}(\{w\}) \subset q^{-1}(S)$, so that $\text{dist}(E, S) \leq \text{dist}(E, W) < \varepsilon$, which gives (4).

To prove the first part of Assertion (5), let $P \in S$ be given. Consider an arbitrary point $E \in \text{nbhd}_{\varepsilon/2}(P)$. If $P' \in S$ distinct from $P$, then by Conclusion (3) we have $\text{dist}(P, P') \geq \varepsilon$. Hence $\text{dist}(E, P') \geq \text{dist}(P, P') - \text{dist}(E, P) > \varepsilon - \varepsilon/2 = \varepsilon/2$, and therefore $\text{dist}(E, P) < \text{dist}(E, P')$ for every point $P' \neq P$ in $S$. By the definition of the Voronoi region $X_P$ it follows that $E \in X_P$; this establishes the inclusion $X_P \supset \text{nbhd}_{\varepsilon/2}(P)$.

To prove the second part of Assertion (5), consider an arbitrary point $E \in X_P \cap q^{-1}(M_{\text{thick}}(\varepsilon))$. Since $E \in q^{-1}(M_{\text{thick}}(\varepsilon))$, we have $\text{dist}(E, S) < \varepsilon$ by Conclusion (4); that is, for some point $P' \in S$ we have $\text{dist}(E, P') < \varepsilon$. Now since $E \in X_P$, we have $\text{dist}(E, P) \leq \text{dist}(E, P') < \varepsilon$, so that $E \in \text{nbhd}_{\varepsilon}(P)$. This establishes the inclusion $X_P \cap q^{-1}(M_{\text{thick}}(\varepsilon)) \subset \text{nbhd}_{\varepsilon}(P)$, and completes the proof of (5).

To prove (6), we apply Proposition 4.7, letting $S$, which is an $\varepsilon$-discrete set by Assertion (3), play the role of $\mathcal{S}$ in that proposition. This gives $B(\varepsilon/2)/\text{vol}(X_P \cap \text{nbhd}_{\varepsilon/2}(P)) \leq \text{density}(\varepsilon/2)$, so that $\text{vol}(X_P \cap \text{nbhd}_{\varepsilon/2}(P)) \geq B(\varepsilon/2)/\text{density}(\varepsilon/2) = b(\varepsilon/2)$, and (6) is established.

Finally, to prove Conclusion (2), we note that according to the definition of the generalized polyhedral complex $K_S$ (see 2.10 and 2.11), each cell $H$ of $K_S$ is the homeomorphic image under $q$ of the Voronoi region $X_P$ for some $P \in S$. We then have $\text{vol}H = \text{vol}X_P \geq \text{vol}(X_P \cap \text{nbhd}_{\varepsilon/2}(P)) \geq b(\varepsilon/2)$, by Conclusion (6). $\square$

The following technical result, Lemma 4.9, will be needed as background for the statement of Proposition 4.10. The statements of Lemma 4.9, Proposition 4.10, and Corollary 4.11 involve the set $\mathcal{Y}_0$ and the function $\phi$ that were defined in Subsection 3.9.

**Lemma 4.9.** Let $\varepsilon$ and $R$ be positive numbers such that $2\varepsilon < R < 5\varepsilon/2$. Then for every number $D$ in the interval $[R/2 - \varepsilon/4, \varepsilon]$, we have $(R - D, \varepsilon/2, D) \in \mathcal{Y}_0$.

**Proof.** First consider the case of a point $D \in (R/2 - \varepsilon/4, \varepsilon]$. Set $\rho = R - D$ and $r = \varepsilon/2$. The inequalities $R > 2\varepsilon$ and $R/2 - \varepsilon/4 \leq D \leq \varepsilon$ then imply that $r < D < \rho < D + r$. It therefore follows from the first assertion of Lemma 3.10 (applied to any two points of $H^3$ separated by a distance $D$) that $(\rho, r, D) \in \mathcal{Y}_0$.

It remains to establish the assertion in the case $D = D_0 := R/2 - \varepsilon/4$. We have shown that the continuous map $F : D \mapsto (R - D, \varepsilon/2, D)$ carries the interval $(R/2 - \varepsilon/4, \varepsilon]$ into $\mathcal{Y}_0 \subset \mathcal{Y} \subset (0, \infty)^3$. The hypothesis of the present lemma directly implies that $F(D_0) = (R/2 + \varepsilon/4, \varepsilon/2, R/2 - \varepsilon/4) \in (0, \infty)^3$, and since $\mathcal{Y}$ is closed in the subspace topology of
Among the conclusions of the following proposition, only Conclusion (5) is quoted later in the paper (in the proof of Corollary 4.11). The other conclusions are either needed for the proof of (5), or needed for applications in the sequel to this paper, or both.

**Proposition 4.10.** Let $M$ be a finite-volume orientable hyperbolic 3-manifold. Let $\varepsilon$ and $R$ be positive numbers with $2\varepsilon < R < 5\varepsilon/2$ (so that by Lemma 4.9 we have $(R-D,\varepsilon/2, D) \in \mathcal{Y}_0$ for every $D \in [R/2-\varepsilon/4, \varepsilon]$). Let $c$ be a positive number such that (a) $\phi(R-D,\varepsilon/2, D) \geq c$ for every $D \in [R/2-\varepsilon/4, \varepsilon]$, and (b) $B(\varepsilon/2) \geq c$.

Set $\Theta = M_{\text{thick}}(\varepsilon)$, and let $S \subset \Theta$ be a maximal $\varepsilon$-thick $\varepsilon$-discrete set for $M$ (which exists by Conclusion (1) of Lemma 4.8). Let $C \subset A^3_S$ denote the set of all 2-cells of $K_S$ which meet $\Theta$. For each $C \in C$, select a point $\tau_C \in C \cap \Theta$, and define a dot system $\mathcal{T}$ for $S$ (see 2.12) by setting $T = \{\tau_C : C \in C\}$. Then:

1. The cardinality of $S$ is at most $\lfloor \text{vol}(M)/b(\varepsilon/2) \rfloor$.
2. For every $H \in A^3_S$, the number of two-dimensional faces of $\mathcal{D}_{H,S}$ (see 2.11) whose interiors are mapped by $\Phi_{H,S}$ onto cells in $C$ is at most

$$\left\lfloor \frac{B(R) - b(\varepsilon/2)}{c} \right\rfloor.$$  

3. Each component of the graph $\mathcal{G}^{S,T}$ (see 2.13) has first betti number bounded above by

$$(4.10.1) \quad 1 + \left\lfloor \frac{\text{vol} M}{b(\varepsilon/2)} \right\rfloor \cdot \left( \frac{1}{2} \left\lfloor \frac{B(R) - b(\varepsilon/2)}{c} \right\rfloor - 1 \right).$$

4. If $\varepsilon$ is a strict Margulis number for $M$, then $\mathcal{G}^{S,T}$ is connected, and the inclusion homomorphism $\pi_1(\mathcal{G}^{S,T}) \to \pi_1(M)$ is surjective.

5. If $\varepsilon$ is a strict Margulis number for $M$, we have

$$\text{rank} \pi_1(M) \leq 1 + \left\lfloor \frac{\text{vol} M}{b(\varepsilon/2)} \right\rfloor \cdot \left( \frac{1}{2} \left\lfloor \frac{B(R) - b(\varepsilon/2)}{c} \right\rfloor - 1 \right).$$

**Proof.** According to 2.11, we have $\#(S) = \#(A^3_S)$. Since the 3-cells of $K_S$ are pairwise disjoint, we have $\text{vol} M \geq \sum_{H \in A^3_S} \text{vol} H$. But by Conclusion (2) of Lemma 4.8 we have $\text{vol} H \geq b(\varepsilon/2)$ for each $H \in A^3_S$, and hence $\text{vol} M \geq \#(A^3_S) \cdot b(\varepsilon/2) = \#(S) \cdot b(\varepsilon/2)$. This implies (1).

To prove (2), we write $M = H^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(H^3)$ is discrete and torsion-free, and let $q : H^3 \to M$ denote the quotient map. We set $\tilde{S} = q^{-1}(S)$.

If $H \in A^3_S$ is given, let us fix an open cell $\tilde{H}$ that is mapped homeomorphically onto $H$ by $q$. We have $\tilde{H} = \text{int} X_{P,\tilde{S}}$ for some $P \in \tilde{S}$. The number of two-dimensional faces of $\mathcal{D}_{H,S}$ whose interiors are mapped by $\Phi_{H,S}$ onto cells in $C$ is equal to $\#(\mathcal{F})$, where $\mathcal{F}$ denotes the set of all two-dimensional faces of $X_P = X_{P,\tilde{S}}$ whose interiors are mapped by $q$ onto cells in $C$. For each $F \in \mathcal{F}$, we will denote by $T_F$ the unique point of $\text{int} F$ which is mapped to $\tau_C$ by $q$.  

$(0, \infty)^3 \subset \mathbb{R}^3$ (see 3.2), it follows that $F(D_0) \in \mathcal{Y}$.
Let $F \in \mathcal{F}$ be given. Then $F$ is a codimension-1 face of the Voronoi region $X_P$. It therefore follows from Assertion (3) of Proposition 2.4 that $F$ is the intersection of $X_P$ with another Voronoi region, which we write as $X_{Q_F}$ for some $Q_F \in \tilde{S}$. (In the following argument, $X_P$ and $X_{Q_F}$ will play the role of the objects that were denoted by $X$ and $Y$ when this argument was sketched in the introduction.) In particular, $\text{int} \ X_{Q_F}$ is disjoint from $X_P$ for each $F \in \mathcal{F}$. On the other hand, if $F$ and $F'$ are distinct elements of $\mathcal{F}$, the elements $X_{Q_F}$ and $X_{Q_{F'}}$ of $\mathcal{X}_{\tilde{S}}$ are distinct and therefore have disjoint interiors. Thus $(\text{int} \ X_{Q_F})_{F \in \mathcal{F}}$ is a disjoint family of subsets of $\mathbb{H}^3 - X_P$.

Let $N \subset \mathbb{H}^3$ denote the ball of radius $R$ centered at $P$. Set $L = N \cap X_P$. By the hypothesis and an observation made in $4.1$, we have $R > 2\varepsilon > \varepsilon \geq h_3(\varepsilon/2)$, so that $\text{nbhd}_{h_3(\varepsilon/2)}(P) \cap X_P \subset L$. Since Conclusion (6) of Lemma 4.8 gives $\text{vol}(X_P \cap \text{nbhd}_{h_3(\varepsilon/2)}(P)) \geq b(\varepsilon/2)$, we have in particular that $\text{vol} L \geq b(\varepsilon/2)$. Hence

\begin{equation}
\text{vol}(N - L) \leq B(R) - b(\varepsilon/2).
\end{equation}

Since $(\text{int} \ X_{Q_F})_{F \in \mathcal{F}}$ is a disjoint family of subsets of $\mathbb{H}^3 - X_P \subset \mathbb{H}^3 - L$, we have a disjoint family $(N \cap \text{int} \ X_{Q_F})_{F \in \mathcal{F}}$ of subsets of $N - L$. In view of (4.10.2) it follows that

\begin{equation}
\sum_{F \in \mathcal{F}} \text{vol}(N \cap X_{Q_F}) \leq B(R) - b(\varepsilon/2).
\end{equation}

Now for each $F \in \mathcal{F}$, the point $T_F$ is a common point of the Voronoi regions $X_{P,\tilde{S}}$ and $X_{Q_F,\tilde{S}}$; it follows from the definition of a Voronoi region (see 2.3) that $\text{dist}(T_F, P) = \text{dist}(T_F, Q_F)$. We denote the common value of these distances by $D_F$.

By hypothesis we have $\tau_C \in C \cap \Theta$ for every $C \in \mathcal{C}$. Hence $T_F \in F \cap q^{-1}(\Theta) \subset X_P \cap q^{-1}(\Theta)$. But according to Conclusion (5) of Lemma 4.8 we have $X_P \cap q^{-1}(\Theta) \subset \text{nbhd}_{\varepsilon}(P)$. Thus $D_F = \text{dist}(T_F, P) < \varepsilon$. On the other hand, Conclusion (5) of Lemma 4.8 also gives that $X_P \supset \text{nbhd}_{\varepsilon/2}(P)$, and since $T_F \in F \subset \partial X_P$, it follows that $D_F = \text{dist}(T_F, P) \geq \varepsilon/2$. Thus we have

\begin{equation}
\varepsilon/2 \leq D_F \leq \varepsilon.
\end{equation}

The triangle inequality implies that

\begin{equation}
\text{nbhd}_{R-D_F}(T_F) \subset \text{nbhd}_R(P) = N.
\end{equation}

On the other hand, for each $F \in \mathcal{F}$, we have $T_F \in F \subset X_{Q_F}$, and Conclusion (5) of Lemma 4.8 implies that $\text{nbhd}_{\varepsilon/2}(Q_F) \subset X_{Q_F}$. Hence if we define $Z_F \subset \mathbb{H}^3$ to be the convex hull of \{T_F\} $\cup$ $\text{nbhd}_{\varepsilon/2}(Q_F)$, the convexity of $X_{Q_F}$ (see Proposition 2.4, Assertion (1)) implies that $Z_F \subset X_{Q_F}$. With (4.10.5), this gives

\begin{equation}
N \cap X_{Q_F} \supset Z_F \cap \text{nbhd}_{R-D_F}(T_F).
\end{equation}

We claim that

\begin{equation}
\text{vol}(Z_F \cap \text{nbhd}_{R-D_F}(T_F)) \geq c.
\end{equation}
To prove (4.10.7), we distinguish two cases. It follows from (4.10.4) that either \( \varepsilon / 2 \leq D_F \leq R/2 - \varepsilon / 4 \), or \( R/2 - \varepsilon / 4 < D_F \leq \varepsilon \). To prove (4.10.7) in the case where \( \varepsilon / 2 \leq D_F \leq R/2 - \varepsilon / 4 \), we note that in this case we have \( R - D_F \geq D_F + \varepsilon / 2 \); since \( \text{dist}(T_F, Q_F) = D_F \), it follows that \( \text{nbhd}_{R-D_F}(T_F) \supset \text{nbhd}_{\varepsilon / 2}(Q_F) \). By definition the set \( Z_F \) also contains \( \text{nbhd}_{\varepsilon / 2}(Q_F) \), and hence \( \text{vol}(Z_F \cap \text{nbhd}_{R-D_F}(T_F)) \geq \text{vol} \text{nbhd}_{\varepsilon / 2}(Q_F) = B(\varepsilon / 2) \). But according to the hypothesis we have \( B(\varepsilon / 2) \geq c \), and (4.10.7) is established in this case.

Now consider the case in which \( R/2 - \varepsilon / 4 < D_F \leq \varepsilon \). If we set \( D = D_F \), \( \rho = R - D_F \), and \( r = \varepsilon / 2 \), the inequalities \( R > 2 \varepsilon \) and \( R/2 - \varepsilon / 4 < D_F \leq \varepsilon \) then imply that \( r < D < \rho < D + r \). In addition, since in addition we have \( \text{dist}(T_F, Q_F) = D \), we may apply Lemma 3.10, with \( Q_F \) and \( T_F \) playing the respective roles of \( Q \) and \( T \) (so that \( Z_F \) plays the role of \( Z \)) to deduce that \( (R - D_F, \varepsilon / 2, D_F) = (\rho, r, D) \in \mathcal{Y}_0 \), and that \( \text{vol}(Z_F \cap \text{nbhd}_{R-D_F}(T_F)) = \phi(\rho, r, D) = \phi(R - D_F, \varepsilon / 2, D_F) \). Since the boundary of the ball \( \text{nbhd}_{R-D_F}(T_F) \) has volume 0, we have \( \text{vol}(Z_F \cap \text{nbhd}_{R-D_F}(T_F)) = \phi(R - D_F, \varepsilon / 2, D_F) \). Since \( D_F \in (R/2 - \varepsilon / 4, \varepsilon) \subseteq [R/2 - \varepsilon / 4, \varepsilon] \), condition (a) of the hypothesis gives \( \phi(R - D_F, \varepsilon / 2, D_F) \geq c \), and (4.10.7) is established in this case as well.

Now, combining (4.10.4) and (4.10.7), we find that \( \text{vol}(N \cap X_{Q_F}) \geq c \) for every \( F \in \mathcal{F} \). Hence the left hand side of (4.10.3) is bounded below by \( c \cdot \#(\mathcal{F}) \). With (4.10.3) we then obtain
\[
\#(\mathcal{F}) \leq \frac{1}{c} \cdot (B(R) - b(\varepsilon / 2)),
\]
which gives (2).

To prove (3), let \( \alpha \) denote the upper bound for \( \#(S) \) given by (1). According to 2.13, the vertex set of \( \mathcal{G}^{S,T} \) is equal to \( S \), and hence has cardinality at most \( \alpha \). Now let \( \beta \) denote the quantity given by (2) which, for any given \( H \in \mathcal{A}_S^2 \), bounds the number of two-dimensional faces of \( \mathcal{D}_{H,S} \) whose interiors are mapped by \( \Phi_{H,S} \) onto 2-cells belonging to \( C \). Then according to 2.13, each vertex of \( \mathcal{G}^{S,T} \) has valence at most \( \beta \).

In particular, if \( \mathcal{G} \) is any component of \( \mathcal{G}^{S,T} \) then the set \( \mathcal{V} \) of vertices of \( \mathcal{G} \) has cardinality at most \( \alpha \), and each vertex of \( \mathcal{G} \) has valence at most \( \beta \). If \( E \) denotes the number of edges of \( \mathcal{G} \), the first betti number of \( \mathcal{G} \) is
\[
1 - \#(\mathcal{V}) + E = 1 - \#(\mathcal{V}) + \frac{1}{2} \sum_{v \in \mathcal{V}} \text{valence}(v) = 1 + \sum_{v \in \mathcal{V}} \left( \frac{\text{valence}(v)}{2} - 1 \right) \leq 1 + \alpha \left( \frac{\beta}{2} - 1 \right).
\]
This proves (3).

Now assume that \( \varepsilon \) is a strict Margulis number for \( M \). Then by Proposition 4.5, \( \Theta \) is a connected 3-manifold-with-boundary, and the inclusion homomorphism \( \pi_1(\Theta, p) \rightarrow \pi_1(M, p) \) is surjective. It follows from the construction of the dot system \( \mathcal{T} \) in the statement of the present proposition that for every 2-cell \( C \) with \( C \cap \Theta \neq \emptyset \) we have \( \mathcal{T} \cap C \cap \Theta \neq \emptyset \). Thus \( M \), \( \Theta \), \( S \) and \( \mathcal{T} \) satisfy the hypotheses of Proposition 2.14. Since \( \Theta \) is connected, it follows from Assertion (1) of Proposition 2.14 that all the points of \( S \) lie in the same component of \( \mathcal{G}^{S,T} \). But according to 2.13, the vertex set of the graph \( \mathcal{G}^{S,T} \) is \( S \), and therefore every component of \( \mathcal{G}^{S,T} \) contains at least one point of \( S \). Hence \( \mathcal{G}^{S,T} \) is connected.
Now let us choose a point \( p \in S \). Since \( \Theta \) and \( G^{S,T} \) are connected, Assertion (2) of Proposition 2.14 becomes \( \text{Im}(\pi_1(\Theta, p) \to \pi_1(M, p)) \leq \text{Im}(\pi_1(G^{S,T}, p) \to \pi_1(M, p)) \), where the unlabeled arrows denote inclusion homomorphisms. Since the inclusion homomorphism \( \pi_1(\Theta, p) \to \pi_1(M, p) \) is surjective, it now follows that the inclusion homomorphism \( \pi_1(G^{S,T}, p) \to \pi_1(M, p) \) is also surjective. This establishes (4).

To prove Assertion (5) we need only note that if \( \varepsilon \) is a strict Margulis number for \( M \), then it follows from Assertion (4) that the rank of \( \pi_1(M) \) is bounded above by the rank of the fundamental group of the connected graph \( G^{S,T} \), which is in turn equal to the first betti number of \( G^{S,T} \) and is therefore bounded above by (4.10.1) according to Assertion (3).

**Corollary 4.11.** Let \( M \) be a finite-volume orientable hyperbolic 3-manifold. Let \( \varepsilon \) be a (not necessarily strict) Margulis number for \( M \), and let \( R \) be a number such that \( 2\varepsilon < R < 5\varepsilon/2 \) (so that by Lemma 4.9 we have \( (R - D, \varepsilon/2, D) \in \mathcal{V}_0 \) for every \( D \in [R/2 - \varepsilon/4, \varepsilon] \)). Let \( c \) be a positive number such that (a) \( \phi(R - D, \varepsilon/2, D) > c \) for every \( D \in [R/2 - \varepsilon/4, \varepsilon] \), (b) \( B(\varepsilon/2) > c \), and (c) \( (B(R) - b(\varepsilon/2))/c \) is not an integer. Then we have

\[
\text{rank } \pi_1(M) \leq 1 + \left( \frac{\text{vol } M}{b(\varepsilon/2)} \right) \cdot \left( \frac{1}{2} \left\lfloor \frac{B(R) - b(\varepsilon/2)}{c} \right\rfloor - 1 \right).
\]

**Proof.** Choose a strictly monotone increasing sequence \( (\varepsilon_i)_{i \geq 1} \) converging to \( \varepsilon \). Since \( \varepsilon \) is a Margulis number for \( M \), and \( \varepsilon_i < \varepsilon \) for each \( i \geq 1 \), each \( \varepsilon_i \) is a strict Margulis number. Condition (b) of the hypothesis implies that \( B(\varepsilon_i/2) > c \) for all sufficiently large \( i \).

Since \( R > 2\varepsilon_i \) for every \( i \), it follows from Lemma 4.9 that we have \( (R - D_i, \varepsilon_i/2, D) \in \mathcal{V}_0 \) for every \( i \geq 1 \) and every \( D \in [R/2 - \varepsilon/4, \varepsilon] \).

We claim that for \( i \) sufficiently large, we also have \( \phi(R - D_i, \varepsilon_i/2, D) > c \) for every \( D \in [R/2 - \varepsilon_i/4, \varepsilon_i] \). If this is false, we may assume after passing to a subsequence that for each \( i \geq 1 \) there is a number \( D_i \in [R/2 - \varepsilon_i/4, \varepsilon_i] \subset [R/2 - \varepsilon/4, \varepsilon] \) such that \( \phi(R - D_i, \varepsilon_i/2, D_i) \leq c \). After again passing to a subsequence we may assume that the sequence \( (D_i)_{i \geq 1} \) converges to a limit \( D_\infty \in [R/2 - \varepsilon/4, \varepsilon] \). Lemma 4.9 gives \( (R - D_\infty, \varepsilon/2, D_\infty) \in \mathcal{V}_0 \), and since \( \phi \) is continuous on \( \mathcal{V}_0 \) (see 3.9), we have \( \phi(R - D_\infty, \varepsilon/2, D_\infty) \leq c \); this contradicts Condition (a) of the hypothesis, and our claim is established.

Thus we have shown that for sufficiently large \( i \), Conditions (a) and (b) of the hypotheses of Proposition 4.10 hold when \( \varepsilon \) is replaced by \( \varepsilon_i \). Since each \( \varepsilon_i \) is a strict Margulis number, it now follows from Assertion (5) of Proposition 4.10 that

\[
(4.11.1) \quad \text{rank } \pi_1(M) \leq 1 + \left( \frac{\text{vol } M}{b(\varepsilon_i/2)} \right) \cdot \left( \frac{1}{2} \left\lfloor \frac{B(R) - b(\varepsilon_i/2)}{c} \right\rfloor - 1 \right)
\]

for sufficiently large \( i \). But since the function \( b \) is continuous by 4.1, we have \( (B(R) - b(\varepsilon_i/2))/c \to (B(R) - b(\varepsilon/2))/c \) as \( i \to \infty \); and Condition (c) of the hypothesis then guarantees that \( \lfloor (B(R) - b(\varepsilon_i/2))/c \rfloor = \lfloor (B(R) - b(\varepsilon/2))/c \rfloor \) for sufficiently large \( i \). The conclusion of the corollary therefore follows upon taking limits in (4.11.1). \( \square \)
5. Numerical calculations

In this section we apply Corollary 4.11 to the proofs of Proposition 5.2 and Theorem 5.4, which provide the concrete estimates stated in the introduction.

The statement and proof of the following lemma involve the set $\mathcal{Y}_0$, and the functions $\eta, \sigma, V_{\text{lens}}, \omega, \theta, \psi, V_{\text{cone}}$ and $\phi$, which were defined in Section 3.

**Lemma 5.1.** Set $\varepsilon = \log 3$ and $R = 2\log 3 + 0.15$. Let $I$ denote the interval $[R/2 - \varepsilon/4, \varepsilon/4] = [3(\log 3)/4 + 0.075, \log 3]$, so that by Lemma 4.9 we have $(R - D, \varepsilon/2, D) \in \mathcal{Y}_0$ for every $D \in I$. Then we have $\phi(R - D, \varepsilon/2, D) > 0.496$ for every $D \in I$.

**Proof.** Since $(R - D, \varepsilon/2, D) \in \mathcal{Y}_0$ for every $D \in I$, we may define functions $H, \Sigma, W_{\text{lens}}, \Psi, W_{\text{cone}},$ and $\Phi$ on $I$ by $H(D) = \eta(R - D, \varepsilon/2, D), \Sigma(D) = \sigma(R - D, \varepsilon/2, D), W_{\text{lens}}(D) = V_{\text{lens}}(R - D, \varepsilon/2, D),\Psi(D) = \psi(\omega(\varepsilon/2, D), \theta(\varepsilon/2, D)), W_{\text{cone}}(D) = V_{\text{cone}}(\omega(\varepsilon/2, D), \theta(\varepsilon/2, D)),$ and $\Phi(D) = \phi(R - D, \varepsilon/2, D).$ We shall begin by finding upper and lower bounds for $H, \Sigma,$ and $\Psi,$ and lower bounds for $W_{\text{cone}}, W_{\text{lens}},$ and $\Phi,$ on certain subintervals of $I$.

From the definitions of $H, \Sigma$, $W_{\text{lens}}, \Psi, W_{\text{cone}},$ and $\Phi$ given above, and the definitions of $\eta, \sigma, V_{\text{lens}}, \psi, V_{\text{cone}}$ and $\phi$ given in Section 3, we find that, for each $D \in I,$ we have

\begin{equation}
H(D) = \frac{2 \cosh(R - D) \cosh(\varepsilon/2) \cosh D - (\cosh^2(R - D) + \cosh^2(\varepsilon/2) + \cosh^2 D) + 1}{\sinh^2 D},
\end{equation}

\begin{equation}
\Sigma(D) = \arccosh \left( \frac{\cosh(R - D)}{\sqrt{1 + H(D)}} \right),
\end{equation}

\begin{equation}
W_{\text{lens}}(D) = \kappa(R - D, \Sigma(D)) + \kappa(\varepsilon/2, D - \Sigma(D)),
\end{equation}

\begin{equation}
\Psi(D) = \arccosh \left( \frac{\cosh \omega(\varepsilon/2, D)}{\sqrt{1 + (\sinh^2 \omega(\varepsilon/2, D))(\sin^2 \theta(\varepsilon/2, D))}} \right),
\end{equation}

\begin{equation}
W_{\text{cone}}(D) = \frac{B(\omega(\varepsilon/2, D))}{2}(1 - \cos \theta(\varepsilon/2, D)) - \kappa(\omega(\varepsilon/2, D), \Psi(D))),
\end{equation}

and

\begin{equation}
\Phi(D) = W_{\text{lens}}(D) + W_{\text{cone}}(D) - \kappa(\varepsilon/2, D - \Psi(D)).
\end{equation}

For any nondegenerate closed subinterval $[D_-, D_+]$ of $I$, we set

\begin{equation}
H^-(D_-, D_+) = \frac{2 \cosh(R - D_+) \cosh(\varepsilon/2) \cosh D_- - (\cosh^2(R - D_-) + \cosh^2(\varepsilon/2) + \cosh^2 D_+) + 1}{\sinh^2 D_+}
\end{equation}
and
\[
H^+(D_-, D_+) = \frac{2 \cosh(R - D_-) \cosh(\varepsilon/2) \cosh D_+ - \left( \cosh^2(R - D_+) + \cosh^2(\varepsilon/2) + \cosh^2 D_- \right) + 1}{\sinh^2 D_-}
\]

These definitions, together with (5.1.1), imply that
\[
(5.1.7) \quad H^-(D_-, D_+) \leq H(D) \leq H^+(D_-, D_+) \text{ for every } D \in [D_-, D_+].
\]

We define a good interval to be a nondegenerate closed subinterval \([D_-, D_+]\) of \(I\) such that (1) \(H^-(D_-, D_+) > -1\), (2) \(H^+(D_-, D_+) < \sinh^2(R - D_-)\) and (3) \(\sinh \omega(\varepsilon/2, D_-) > \sinh \omega(\varepsilon/2, D_+) \cdot \sin \theta(\varepsilon/2, D_+)^2\).

Let \([D_-, D_+]\) be any good interval. It follows from Conditions (1) and (2) of the definition of a good interval, together with (5.1.7), that \(-1 < H^-(D_-, D_+) \leq H^+(D_-, D_+) < \sinh^2(R - D_+) < \sinh^2(R - D_-)\). Hence we may define quantities \(\Sigma^-(D_-, D_+)\) and \(\Sigma^+(D_-, D_+)\) by setting
\[
\Sigma^-(D_-, D_+) = \arccosh\left( \frac{\cosh(R - D_+)}{\sqrt{1 + H^+(D_-, D_+)^2}} \right)
\]
and
\[
\Sigma^+(D_-, D_+) = \arccosh\left( \frac{\cosh(R - D_-)}{\sqrt{1 + H^-(D_-, D_+)^2}} \right).
\]

These definitions, together with (5.1.2) and (5.1.7), imply that
\[
(5.1.8) \quad \Sigma^-(D_-, D_+) \leq \Sigma(D) \leq \Sigma^+(D_-, D_+)
\]
for every good interval \([D_-, D_+]\) and every \(D \in [D_-, D_+]\).

Now, for every good interval \([D_-, D_+]\), set
\[
W^\text{\,lens}_\ast(D_-, D_+) = \kappa(R - D_+, \Sigma^+(D_-, D_+)) + \kappa(\varepsilon/2, D_+ - \Sigma^-(D_-, D_+)).
\]

Since the function \(\kappa\) is (weakly) monotone increasing in its first (positive-valued) argument and monotone decreasing in its second (real-valued) argument, it follows from (5.1.3), (5.1.8), and the definition of \(W^\text{\,lens}_\ast\) that for every good interval \([D_-, D_+]\) and every \(D \in [D_-, D_+]\) we have
\[
(5.1.9) \quad W^\text{\,lens}_\ast(D) \geq W^\text{\,lens}_\ast(D_-, D_+).
\]

Recall from 3.9 that the function \(\theta\) takes its values in \((0, \pi/2)\). Condition (3) of the definition of a good interval, together with the positivity of the sine function on \((0, \pi/2)\), implies that for any good interval \([D_-, D_+]\) we have
\[
1 + (\sinh^2 \omega(\varepsilon/2, D_-))(\sin^2 \theta(\varepsilon/2, D_-)) < 1 + (\sinh^2 \omega(\varepsilon/2, D_+))(\sin^2 \theta(\varepsilon/2, D_+)) < \cosh^2 \omega(\varepsilon/2, D_-) < \cosh^2 \omega(\varepsilon/2, D_+).
\]

Hence for each good interval \([D_-, D_+]\) we may define quantities \(\Psi^-(D_-, D_+)\) and \(\Psi^+(D_-, D_+)\) by
\[
\Psi^-(D_-, D_+) = \arccosh\left( \frac{\cosh \omega(\varepsilon/2, D_-)}{\sqrt{1 + (\sinh^2 \omega(\varepsilon/2, D_+))(\sin^2 \theta(\varepsilon/2, D_+))}} \right)
\]
and
\[
\Psi^+(D_-, D_+) = \arccosh\left( \frac{\cosh \omega(\varepsilon/2, D_+)}{\sqrt{1 + (\sinh^2 \omega(\varepsilon/2, D_-))(\sin^2 \theta(\varepsilon/2, D_-))}} \right).
\]
5.1.4

for every good interval \([D_-, D_+]\) and for every \(D \in [D_-, D_+].\)

Next, for every good interval \([D_-, D_+],\) set

\[
W_{\text{cone}}(D_-, D_+) = \frac{B( \omega(\varepsilon/2, D_+))}{2} (1 - \cos \theta(\varepsilon/2, D_-)) - \kappa(\omega(\varepsilon/2, D_+), \Psi(D_-, D_+)).
\]

It follows from this definition, together with (5.1.5), (5.1.10), the monotonicity properties of \(\kappa, \omega \text{ and } \theta\) mentioned above, and the monotonicity of the cosine function on \((0, \pi/2),\) that

(5.1.11)

\[
W_{\text{cone}}(D) \geq W_{\text{cone}}^-(D_-, D_+)
\]

for every good interval \([D_-, D_+]\) and for every \(D \in [D_-, D_+].\)

Now, for every good interval \([D_-, D_+],\) set

\[
\Phi^-(D_-, D_+) = \Psi^-(D_-, D_+) - \kappa(\varepsilon/2, D_- - \Psi^+(D_-, D_+)).
\]

It follows from this definition, together with (5.1.9), (5.1.11), (5.1.10), and the monotonicity properties of \(\kappa\) mentioned above, that

(5.1.12)

\[
\Phi(D) \geq \Phi^-(D_-, D_+)
\]

for every good interval \([D_-, D_+]\) and for every \(D \in [D_-, D_+].\)

In view of (5.1.12), in order to prove the lemma it suffices to show that every point of \(I\) lies in a good interval \([D_-, D_+]\) such that \(\Phi^-(D_-, D_+) > 0.496.\)

For this purpose, we denote by \((\delta_1, \ldots, \delta_{46})\) the 46-tuple of positive constants

\[
(0.17, 0.14, 0.12, 0.10, 0.09, 0.08, 0.07, 0.06, 0.05, 0.045, 0.040, 0.035, 0.030, 0.025, 0.022, 0.020, 0.018, 0.016, 0.014, 0.012, 0.010, 0.0084, 0.007, 0.006, 0.005, 0.0042, 0.0035, 0.0030, 0.0025, 0.0022, 0.0019, 0.0016, 0.0013, 0.0011, 0.0009, 0.00075, 0.0006, 0.0005, 0.0004, 0.0003, 0.00025, 0.00020, 0.00015, 0.00010, 0.00005, 0.00002).
\]

We set \(D_0 = R/2 - \varepsilon/4, D_i = \varepsilon - \delta_i \) for \(i = 1, \ldots, 46\) and \(D_{47} = \varepsilon.\) We have \(R/2 - \varepsilon/4 = D_0 < D_1 < \cdots < D_{47} = \varepsilon,\) so that \(I\) is the union of the intervals \([D_0, D_1], \ldots, [D_{46}, D_{47}].\) By direct calculation we find that for \(i = 1, \ldots, 47\) the interval \([D_{i-1}, D_i]\) is good and \(\Phi^-(D_{i-1}, D_i) > 0.496.\) (To say that \([D_{i-1}, D_i]\) is good means that the quantities \(H^-(D_{i-1}, D_i) + 1, \sinh^2(R - D_i) - H^+(D_{i-1}, D_i), \) and \(\sinh(\varepsilon/2, D_{i-1}) - \sinh(\varepsilon/2, D_i) \cdot \sin \theta(\varepsilon/2, D_i)\) are positive. The smallest values of these respective quantities, as \(i\) ranges from 1 to 47, are 0.75 . . . , 2.22511 . . . , and 0.31 . . . , and are respectively achieved when \(i = 1, i = 47,\) and \(i = 1.\) The smallest value of \(\Phi^-(D_{i-1}, D_i)\) is equal to 0.49603 . . . , and is achieved when \(i = 45.)\)
As was mentioned in the introduction, a group $G$ is said to be $k$-semifree for a given positive integer $k$ if each subgroup of $G$ having rank at most $k$ is a free product of free abelian groups.

**Proposition 5.2.** Let $M$ be a finite-volume orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 2-semifree. Then

$$\text{rank } \pi_1(M) < 1 + \lambda_0 \cdot \text{vol } M,$$

where $\lambda_0 = \frac{156}{b((\log 3)/2))} = 167.781 \ldots$.

**Proof.** According to [2, Corollary 4.2], $\log 3$ is a Margulis number for $M$. We will apply Corollary 4.11, taking $\varepsilon = \log 3$, $R = 2 \log 3 + 0.15$, and $c = 0.496$. Note that we have $2\varepsilon < R < 5\varepsilon/2$, as required for Corollary 4.11.

Lemma 5.1 asserts that the constant $c$ satisfies Condition (a) of Corollary 4.11. The constant $c$ also satisfies Condition (b) of that corollary, because $B((\log 3)/2) = 0.73 \ldots > c$. We have $(B(R) - b(\varepsilon/2))/c = 314.62 \ldots$, so that Condition (c) of Corollary 4.11 holds and $[(B(R) - b(\varepsilon/2))/c] = 314$. The assertion now follows from Corollary 4.11. $\square$

The transition from Proposition 5.2 to the homology bounds given by Theorem 5.4 will involve the following lemma.

**Lemma 5.3.** Let $M$ be a non-compact, finite-volume, orientable hyperbolic 3-manifold. Suppose that for some prime $p$ we have $\dim H_1(M, \mathbb{F}_p) \geq 3$. Then $\text{vol } M > 2.848$.

**Proof.** If $M$ has two cusps, this follows from [1, Theorem 3.6], which asserts that the smallest volume of an orientable two-cusped hyperbolic 3-manifold is $V_{\text{oct}} = 3.66 \ldots$, the volume of a regular ideal hyperbolic octahedron. If $M$ has more than two cusps, it is a standard consequence of Thurston’s hyperbolic Dehn filling theorem [5, Chapter E] that there is a hyperbolic manifold $M'$ having exactly two cusps which can be obtained from $M$ by Dehn filling, and that $\text{vol } M' < \text{vol } M$. Since $\text{vol } M' \geq V_{\text{oct}}$, we have $\text{vol } M > V_{\text{oct}}$ in this case.

If $M$ has one cusp and $\text{vol } M \leq 2.848$, then [14, Theorem 1.2] asserts that $M$ is one of the manifolds m003, m004, m006, m007, m009, m010, m011, m015, m016 or m017 in the SnapPea census. One calculates, using [7], that if $M$ is any of these manifolds then $H_1(M; \mathbb{F}_p)$ is a direct sum of $\mathbb{Z}$ with a (possibly trivial) cyclic group. Hence $\dim H_1(M, \mathbb{F}_p) \leq 2$ for every prime $p$. $\square$

**Theorem 5.4.** Let $M$ be any finite-volume orientable hyperbolic 3-manifold. Then:

1. for any prime $p$ we have

$$\dim H_1(M; \mathbb{F}_p) < \lambda_1 \cdot \text{vol } M,$$

where

$$\lambda_1 = \frac{1}{1.22} + \frac{156}{b((\log 3)/2)} = 168.601 \ldots;$$
(2) if $M$ is non-compact, and $p$ is any prime, we have
\[ \text{dim } H_1(M; \mathbb{F}_p) < \lambda'_1 \cdot \text{vol } M, \]
where
\[ \lambda'_1 = \frac{1}{2.848} + \frac{156}{b((\log 3)/2)} = 168.132 \ldots; \]
and
(3) if $M$ is compact, we have
\[ \text{dim } H_1(M; \mathbb{F}_2) < \lambda''_1 \cdot \text{vol } M, \]
where
\[ \lambda''_1 = \frac{1}{3.77} + \frac{156}{b((\log 3)/2)} = 168.046 \ldots. \]

Proof. We set $V = \text{vol } M$. Let $p$ be a prime, and set $h_{M,p} = \text{dim } H_1(M; \mathbb{F}_p)$. We must show:

(5.4.1) \[ h_{M,p} < \lambda_1 V; \]
(5.4.2) \[ h_{M,p} < \lambda'_1 V \text{ if } M \text{ is non-compact}; \]
(5.4.3) \[ h_{M,p} < \lambda''_1 V \text{ if } M \text{ is compact and } p = 2. \]

According to [14, Theorem 1.3], we have $\text{vol } M > 0.94$ for any finite-volume orientable hyperbolic 3-manifold $M$. Hence in the case where $h_{M,p} \leq 10$, we have $h_{M,p} < 11 \cdot V$, which is stronger than each of the inequalities (5.4.1)—(5.4.3).

The rest of the proof will be devoted to the case in which $h_{M,p} \geq 11$.

Since in particular $h_{M,p} \geq 4$, it follows from [2, Lemma 5.2] that $\pi_1(M)$ is 2-semifree. According to Proposition 5.2, we therefore have rank $\pi_1(M) < 1 + \lambda_0 \cdot V$, where $\lambda_0 = 156/(b((\log 3)/2))$. In particular we have $h_{M,p} < 1 + \lambda_0 \cdot V$, which we rewrite as

(5.4.4) \[ h_{M,p} < \left( \frac{1}{V} + \lambda_0 \right) V. \]

Consider the subcase in which $M$ is compact and $p = 2$. Since $h_{M,2} \geq 11$, it then follows from [13, Proposition 13.4] that $V > 3.77$, which with (5.4.4), gives $h_{M,2} < \lambda''_1 V$. This proves (5.4.3).

Now consider the subcase in which $M$ is non-compact. Since $h_{M,p} \geq 11 > 3$, Lemma 5.3 gives $V > 2.848$. With (5.4.4), this gives $h_{M,p} < \lambda'_1 V$, and (5.4.2) is proved.

To prove (5.4.1), we first notice that in the subcase where $M$ is non-compact, the asserted inequality follows from (5.4.2). If $M$ is compact, we use [2, Theorem 1.1], which implies that if $M$ is a closed, orientable hyperbolic 3-manifold with $\text{dim } H_1(M; \mathbb{F}_p) \geq 4$ for some prime $p$, then $\text{vol } M > 1.22$. In the notation of the present proof, since $h_{M,p} \geq 11 > 4$, we have $V > 1.22$; with (5.4.4), this gives $h_{M,p} < \lambda_1 V$, and (5.4.1) is proved. \qed
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