Universal scattering laws for bouncing cosmology

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Many variants of the standard model of cosmology have been proposed where the Big Bang is replaced by a Big Bounce thanks to corrections to Einstein gravity at small scales. We introduce here the notion of a singularity scattering map, as we call it, which relates the large scale geometries before and after the bounce, and we then establish, for the class of so-called quiescent singularities, a full classification of all of the maps that enjoy suitable locality conditions. This classification uncovers universal laws for bouncing cosmologies (scaling of Kasner exponents, canonical transformation of matter) while leaving room for model-dependent junctions. Our framework encompasses scenarios based on modified gravity, ekpyrotic matter, or quantum gravity corrections (string theory, loop quantum cosmology). For a selection of these models we study explicitly the corresponding singularity scattering map; thanks to our classification the map is fully determined from Bianchi I solutions, yet controls how spatial inhomogeneities and anisotropies are transmitted through bounces without any symmetry assumption. Our systematic classification of singular scattering maps opens up the possibility for a fine study of observational implications of arbitrary bouncing scenarios, while remaining agnostic about the specific mechanism responsible for the bounce.

INTRODUCTION

Toward a unification of bouncing scenarios. An important class of proposals to resolve the initial singularity problem in Cosmology are bouncing scenarios where the Big Bang is replaced by a Big Bounce, and in which the Universe undergoes a contracting phase followed by the expanding phase that includes the present time. Such scenarios have been constructed through various modified gravity theories, matter violating energy conditions, or quantum gravity effects in string theory and loop quantum cosmology. In this Letter, we revisit this old problem by abstracting away all microscopic details of the model.

Formulation and classification. We propose the new notion of singularity scattering map, which precisely extracts the essence of the bounce and focuses on the Einstein equations themselves which we impose before and after the bounce. The relevant scattering map in a given scenario is determined, in a second stage only, from an extended model of gravity. Our main contribution is a characterization of all possible junction conditions that might arise from such a physical model. We refer to our companion paper [17] for proofs in $d = 3$ space dimensions. Interestingly, with our standpoint we are naturally led to distinguish between universal and model-dependent aspects of junction relations.

Universality. Our classification uncovers universal scattering laws for the geometry and matter across the bounce. We prove that Kasner exponents before and after the bounce obey the universal scaling relation

$$\left(g^{1/2}\hat{K}\right)_{\text{after}} = \gamma\left(g^{1/2}\hat{K}\right)_{\text{before}}$$

(1)

(for a constant $\gamma \in \mathbb{R}$), with $g$ the spatial metric in synchronous gauge, $g^{1/2}$ its volume factor, and $\hat{K}$ the traceless part of the extrinsic curvature. We also prove that matter, modeled away from the bounce as a minimally coupled and massless scalar field $\phi$, undergoes a canonical transformation. Essentially, as explicit in (11),

$$\Phi: (\pi_\phi, \phi)_\text{before} \mapsto (\pi_\phi, \phi)_\text{after} \text{ preserves } d\pi_\phi \wedge d\phi$$

(2)

where $\pi_\phi$ is the momentum conjugate to $\phi$. In addition, the metric after the bounce is entirely determined from $\Phi$.

Model-dependence. We then study the singularity scattering maps associated with the pre Big Bang scenario, loop quantum cosmology, and some modified matter models. The map for a given model encapsulates microscopic physics constrained by the universal scattering laws (1) and (2). It is determined by calculations in homogeneous (but anisotropic) Bianchi I universes, yet describes bounces with arbitrary spatial inhomogeneities. Throughout, our focus is on quiescent cosmological singularities, a class first identified by Barrow [4] (see also [1, 7]) so that the BKL oscillating behavior [5] is suppressed.

Perspectives. We do not attempt here to review the vast subject of bouncing cosmologies and we focus on the issues mentioned above only. For further material we refer to the review papers by Ashtekar [2], Brandenberger and Peter [6], and Gasperini and Veneziano [10].

SINGULARITY SCATTERING MAPS

Bounce hypersurface. We introduce here our notion of singularity scattering maps for quiescent singularities in a $(d + 1)$-dimensional spacetime ($d \geq 2$) with a scalar field. We focus on bouncing scenarios in which corrections to Einstein gravity are negligible away from the bounce locus, which we model as a spacelike singularity hypersurface labeled $t = 0$ (we also treat timelike singularities in [17]),
but are essential at (small) time scales \( t_b \) around this hypersurface. Provided spatial inhomogeneities are mild (see below) the spacetime, on each side, is well described at larger time scales by a solution of Einstein equations which is singular at the bounce hypersurface, while these two solutions are connected using a suitable junction condition.

**ADM formalism.** We work with a Gaussian (or synchronous gauge) foliation in which the metric reads
\[
g^{(d+1)} = -dt^2 + g(t, x),
\]
the bounce hypersurface being normalized to be at proper time \( t = 0 \). Each constant-time hypersurface is endowed with a Riemannian metric \( g = g_{ab} \) and an extrinsic curvature \( K = K^a_b \) such that \( K_{ac} = K_a^b g_{bc} \) is a symmetric two-tensor field. Here, \( a, b, \ldots \) are local coordinate indices on each time slice.

We consider the evolution of a self-gravitating massless scalar field \( \phi \) which obeys the matter evolution equation
\[
- \nabla^2 \phi + \text{Tr}(K) \partial_t \phi = - \Delta_g \phi, \tag{3a}
\]
where \( \Delta_g \phi = \nabla_a \nabla^a \phi \) is the Laplace operator on the spacelike slices. On the other hand, the ADM formulation of the Einstein equations reads
\[
\partial_t g_{ab} + 2 K_{ab} = 0,
\]
\[
\partial_t K^b_a - (\text{Tr}(K)) K^b_a = R^b_a - \partial_a \partial_b \phi, \tag{3b}
\]
\[
(\text{Tr}(K))^2 - (\partial_t \phi)^2 = -R + \partial_a \partial_b \phi, \quad \nabla_a K^b_a - \partial_b (\text{Tr}(K)) + \partial_a \partial_b \phi = 0,
\]
in which the first two equations are evolution equations in \( t \) and the latter two equations are constraints. (We normalize the speed of light to \( c = 1 \) and Newton constant to \( 8\pi G = 1 \).)

**Data on a bounce hypersurface.** Near the singularity, asymptotic profiles describing the main behavior of a solution are found by neglecting spatial derivatives compared to \( t \) derivatives, namely neglecting right-hand sides of (3), and then solving the resulting equations. This family of asymptotic profiles (denoted by a \( * \) subscript) reads
\[
\begin{align*}
g^\pm_s(t) &= e^{2(\log |t/t_s|)k_s^\pm} g^\pm, \\
\phi^\pm_s(t) &= \phi_0^\pm + \log |t/t_s| + \phi_1^\pm,
\end{align*}
\]
parametrized by singularity data \((g^\pm, k^\pm_0, \phi_0^\pm, \phi_1^\pm)\) prescribed on each side \( \pm = \text{sgn}(t) \) of the bounce. The asymptotic metric is expressed in terms of the matrix exponential of \((k^\pm_0)\), and \( t_s > 0 \) is a time scale used to express the profile in a dimensionally-consistent manner.

The singularity data must satisfy the constant trace relation \( \text{Tr} k^\pm = 1 \) together with an asymptotic form of the Hamiltonian and momentum constraints, that is,
\[
1 - k^\pm_a k^\pm_b = (\phi_0^\pm)^2, \quad \nabla^a k^\pm_b = \phi_0^\pm \partial_b \phi_1^\pm, \tag{5}
\]
where \( \nabla^\pm \) is the connection associated with \( g^\pm \). In our context, a data set \((g^\pm, k^\pm_0, \phi_0^\pm, \phi_1^\pm)\) is called quiescent if Kasner exponents \( k^\pm_0 \) (eigenvalues of \( k^\pm \)) are positive.

Alternatively, (4) can be expressed in an orthonormal coframe (vielbein) \( e_\pm^i \) for \( g^\pm \) that diagonalizes \( k^\pm \). The asymptotic metric is then \( g^\pm_{(d+1)} = -dt^2 + g^\pm_s \) with
\[
g^+_s = \sum_{i=1}^d |t/t_s|^{2k^+_i(x)} e^+_i(x) e^+_i(x), \quad \pm t > 0. \tag{6}
\]

The trace and Hamiltonian constraints read \( \sum_i k^+_i = 1 \) and \( 1 - \sum_i (k^-_i)^2 = (\phi_0^-)^2 \), respectively. As an example we could take \( e_\pm^i = dx^i \) in some coordinate system, in which case the profile is an exact solution of Einstein equations when the exponents \( k^+_i \) are constants.

In general, asymptotic profiles are not exact solutions.

**Validity of asymptotic profiles.** Asymptotic profiles are defined for all times \( \pm t \in (0, \infty) \), but are only good approximations in some range \( t_b \ll |t| < t_s \); indeed, curvature generically blows up as \( t \to 0^\pm \) so that corrections (e.g. higher-curvature corrections) become important at small time \( |t| \approx t_b \), while the spatial derivatives neglected in (3) stop being negligible at some large time scale \( t_s \) since they decay slower than \( 1/t^2 \) at \( |t| \to \infty \).

Our assumption of mild spatial inhomogeneities is that \( t_b \ll t_s \). Equivalently, we require that at time \( t_b \) the spatial derivative terms in (3) such as \( \Delta_g \phi/\phi, \partial_t \phi \) or \( R \) are parametrically smaller than the typical scale \( 1/t_b^2 \) of the left-hand sides, so that they remain smaller on some time interval \((t_b, t_s)\). Under this assumption we retrieve the data for the asymptotic profile as the (approximately constant for \( t_b \ll |t| \ll t_s \)) values
\[
(g^\pm, k^\pm_0, \phi_0^\pm, \phi_1^\pm) \Rightarrow (|t/t_s|^{2k^+_0} g, -tK, t \partial_t \phi, \phi - t \log |t/t_s| \partial_t \phi) \quad \text{as} \quad |t| \to t_b, t_s.
\]

In idealized setups where \( t_b = 0 \) (singular bounce) or \( t_s = \infty \) (spatially homogeneous case) the singularity data can be defined as the \( t \to 0^\pm \) or \( t \to \pm \infty \) limits, respectively.

**The new notion.** By construction, all the slices of the foliation are diffeomorphic to the \( t = 0 \) slice \( \mathcal{H} \). Let us denote by \( \mathbf{I}(\mathcal{H}) \) the set of all singularity data \((g^\pm, k^\pm_0, \phi_0^\pm, \phi_1^\pm)\), as described by the conditions (5). By definition, a singularity scattering map is then a local diffeomorphism-covariant map \( \mathbf{S} \colon \mathbf{I}(\mathcal{H}) \to \mathbf{I}(\mathcal{H}) \). General covariance and locality ensure that \( \mathbf{S} \) is characterized by its effect on any small ball, so that the notion of singularity scattering map does not depend upon the bounce hypersurface \( \mathcal{H} \). We then propose to introduce junction conditions associated with a given singularity scattering map \( \mathbf{S} \) and relating asymptotic data (7), by definition, as
\[
(g^\pm, k^\pm_0, \phi_0^\pm, \phi_1^\pm) = \mathbf{S}(g^\pm, k^\pm_0, \phi_0^\pm, \phi_1^\pm). \tag{8}
\]

**Mathematical advances.** The existence of solutions to the Einstein equations asymptotic to quiescent profiles (4) and satisfying the junction conditions (8) is proven in the companion paper [17] based on the earlier work [1, 7]. We also refer to [13–16] for recent progress on the theory of weak solutions with singularities. Our definition is
a generalization to singularity hypersurfaces of Israel’s junction conditions [11] for hypersurfaces across which the metric remains regular. Our junction conditions are reminiscent of the so-called kinetic relations for phase boundaries in fluid dynamics and material science [19, 15].

**SELECTED EXAMPLES OF BOUNCES**

*Reduction to Bianchi I.* We now study singularity scattering maps for several models (pre Big Bang, modified matter, etc.) of spatially homogeneous bounces, and exhibit the two features (1) and (2). As we argue in the next section, these are universal laws that can be derived model-independent based only on an ultralocality assumption (a notion we define in the next section). For now we work with a Bianchi I metric (with \( d \geq 2 \))

\[
g^{(d+1)} = -dt^2 + \omega(t)^{2/d} \sum_i e^{2\alpha_i(t)} dx^i dx^i, \tag{9}
\]

where the parameters \( \alpha_i \) describe the anisotropic stress and sum to zero, the volume factor is \( \omega := |g|^{1/2} \), and the average Hubble parameter is \( H = \dot{\alpha}_0(\log \omega)^{1/d} \).

*Asymptotic profiles.* As explained before (7), spatial homogeneity means that \( t_\infty = \text{namely the bounce is well-described for all } |t| \gg t_b \) by the asymptotic profiles (6) (from here on we normalize \( t_0 = 1 \)), which are exact Bianchi I solutions to Einstein equations with a free scalar field. Explicitly, in the notation (9) we consider bounces that are asymptotic (at \( t \rightarrow \pm \infty \)) to

\[
\omega = \pm \omega_0^{1/2} (t - t_0^\pm), \quad \phi = \phi_0^{1/2} \log |t - t_0^\pm| + \phi_1^\pm, \quad \alpha_i = (k_i^\pm - \frac{1}{2}) \log |t - t_0^\pm| + \nu_i^\pm, \quad (\phi_0^{1/2})^2 + (k_i^\pm)^2 = 1
\]

for some constants \( (t_0^\pm, \omega_0^\pm, k_i^\pm, \nu_i^\pm, \phi_0^\pm, \phi_1^\pm) \) such that \( \sum_i \nu_i^\pm = 0 \), the Kasner exponents \( k_i^\pm \) (eigenvalues of \( k^\pm \)) sum to 1, and \( |k^\pm|^2 = \text{Tr}(k^\pm)^2 \). We define Kasner radii \( r^\pm = r(\phi_0^\pm) = \sqrt{1 - \frac{\alpha^2}{d-1} (\phi_0^\pm)^2} = \sqrt{\frac{\alpha^2}{d-1} \text{Tr}(k^\pm)^2} \in [0, 1] \), where \( k^\pm = k \pm - \frac{1}{2} \) is the traceless extrinsic curvature. We illustrate an example of such a bounce in Figure 1.

We are interested in the map that relates the parameters describing the two limits. Invariance under time translations and coordinate redefinitions of each \( x^i \) ensures that \( t_0^\pm \), \( \omega_0^\pm \), \( \nu_i^\pm \) appear precisely as shifts of \( t_0^\pm \), \( \omega_0^\pm \), \( \nu_i^\pm \), respectively, so \( (t_0^\pm, \omega_0^\pm, \nu_i^\pm) \) only depend on \( (k_i^\pm, \phi_0^\pm, \phi_1^\pm) \). For simplicity, we focus on \( (k_i^\pm, \phi_0^\pm, \phi_1^\pm) \) and do not discuss the proper time offset and metric scale factors.

*Universal scattering laws.* The first scattering law (1) that we will exhibit in concrete bounces translates, in the Bianchi I notation, to \( \omega_0^\pm (k_i^\pm - \frac{1}{2}) \gamma \omega_0^\pm (k_i^\pm - \frac{1}{2}) \) for some \( \gamma \in \mathbb{R} \). It implies \( \omega_0^\pm r(\phi_0^\pm) = |\gamma| \omega_0^\pm r(\phi_0^\pm) \).

Next, we concentrate on the map describing scalar fields

\[
\Phi: (k^-, \phi_0^-, \phi_1^-) \rightarrow (\phi_0^+, \phi_1^+). \tag{11a}
\]

The second scattering law (2) states that, at fixed \( k^- = r^- \), the map \( \Phi \) is a canonical transformation in the sense that it preserves the volume form \( d(\phi_0^+/r(\phi_0^+)) \wedge d\phi_1 = d\phi_0^+ d\phi_1/r(\phi_0^+)^3 \) up to the sign \( \epsilon = \text{sgn} \gamma \):

\[
\det \begin{pmatrix} \frac{\partial \phi_0^+}{\partial \phi_0^+} & \frac{\partial \phi_0^+}{\partial \phi_1^+} \\ \frac{\partial \phi_1^+}{\partial \phi_0^+} & \frac{\partial \phi_1^+}{\partial \phi_1^+} \end{pmatrix} = \epsilon \partial \phi_0^+ \left( \frac{\phi_0^+}{r(\phi_0^+)^3} \right) = \epsilon \left( \frac{\epsilon}{r(\phi_0^+)^3} \right). \tag{11b}
\]

*Pre Big Bang scenario.* Let us turn to concrete models, starting with a singular bounce in which junction conditions are inspired from string theory [22, 9, 10]. It is most conveniently described with “string frame” fields \( g_{SF}, g_{SF}^{(d+1)} \) that obey suitably truncated metric-dilaton equations. The idea is to glue along \( t_{SF} = 0 \) two Bianchi I solutions of these equations related by scale-factor duality, assuming that higher derivative and/or higher loop corrections resolve the singularity. For \( \pm t_{SF} > 0 \),

\[
g_{SF}^{(d+1)} = -dt_{SF}^2 + \sum_{i=1}^d e^{2u_{i}^+} |t_{SF}|^{2\beta_i^+} dx^i dx^i, \quad \phi_{SF} = \log |g_{SF}^{(d+1)}|^{1/2} - \log |t_{SF}|,
\]

where the constants \( u_{i^+}, \beta_{i^+} \) obey \( \sum_i \beta_{i^+}^2 = 1 \) and each \( \beta_{i^+} \) is coordinate-invariant; they are sensitive to how the singularity is resolved and to values of \( \beta_- \).

The Einstein frame metric \( g^{(d+1)} = e^{-2\phi_{SF}/(d-1)} g_{SF}^{(d+1)} \), its proper time \( t \), and the canonically normalized scalar \( \phi = \phi_{SF}/\sqrt{d-1} = \text{then be computed is } (10) \) with

\[
t_0^\pm = 0, \quad \omega_0^\pm = \frac{d-1}{d-1} \gamma \omega_0^\pm (k^\pm - \frac{1}{2}) \Sigma_\pm \quad \text{for some } \gamma \in \mathbb{R}.
\]

It implies \( \omega_0^\pm r(\phi_0^\pm) = |\gamma| \omega_0^\pm r(\phi_0^\pm) \).

Next, we concentrate on the map describing scalar fields

\[
\Phi: (k^-, \phi_0^-, \phi_1^-) \rightarrow (\phi_0^+, \phi_1^+). \tag{11a}
\]
interesting junction, with \( \omega_c^+ (k^+_i - \frac{1}{a}) = -\omega_0^- (k^-_i - \frac{1}{a}) \)
\[
\phi_0^+ = -(2\sqrt{d-1} + (d+1)\phi_0^-)/(d+1 + 2\sqrt{d-1} \phi_0^-),
\]
and \( \phi_1^+ = (r^+/r^-)\phi_1^- + f(\beta^-) \) with a function \( f \) that depends on how the singularity is resolved. The canonical transformation \( \Phi \) is depicted in Figure 2. Regardless of \( f \), one checks that both laws (1) and (2) are obeyed.

For all other sign choices (except the trivial \( \beta_+ = \beta_- \)) these laws are violated. Given our general results in the next section, we learn that the junction conditions for other sign choices do not extend to non-homogeneous spacetimes (in fact, applying the transformation pointwise would violate the momentum constraint).

**Modified gravity and loop quantum cosmology.** Both in loop quantum cosmology [3, 23] and in quite general modified gravities [8] (Brans–Dicke theory, kinetic gravity braiding, mimetic gravity, etc.), the densitized shear \( \tilde{K} = (\sqrt{\text{det} g}) \) is continuous (up to a sign) across Bianchi I bounces. To derive this, the authors of [8] assumed that modifications of gravity are encapsulated in an effective stress-tensor, preserve spatial rotation invariance, and are strong enough to lead to a bounce but are negligible away from it.

This is precisely our first universal scattering law (1) (with \( \gamma = -1 \)), which we prove (cf. next section) **without any symmetry assumption.** It would be very interesting to determine the precise scattering maps for these models and check our second scattering law (2) directly.

**Bounces with modified matter.** We now consider Einstein gravity coupled to a scalar field with Lagrangian \( \mathcal{L}(\phi, X) \) where \( X = -|\nabla \phi|^2 = \dot{\phi}^2 \). It is beyond the scope of this Letter to investigate which Lagrangians lead to bouncing solutions: such bounces were found with ekpyrotic matter [12], ghost condensates, Brans–Dicke theory in Einstein frame, among others. For our purposes, Bianchi I solutions should asymptote to free scalar ones (10) at \( t \to \pm \infty \), for which \( \mathcal{X} \to 0 \) and \( |\phi| \to \infty \). To ensure this we demand \( \mathcal{L} \simeq X/2 \) (free scalar Lagrangian) in these limits. We illustrate this in Figure 1, postponing a detailed analysis of specific models to later work.

In the Bianchi I setting (9), the action (per comoving volume) reduces to
\[
S = \int \left( \mathcal{L}(\phi, \dot{\phi}^2) - \frac{d-1}{2d} \omega^2 + \frac{1}{2} \sum_{i=1}^d \dot{\alpha}_i^2 \right) \omega \; dt.
\]
As observed in [8], the equation of motion \( \partial_t (\omega \dot{\phi}_i) = 0 \) for \( \alpha_i \) states that \( \lambda_i = \omega \alpha_i \) are constants so their \( t \to \pm \infty \) limits \( \pm \omega_0^\pm (k^+_i - \frac{1}{a}) \) are equal. This proves our first scattering law (1) in this case, with \( \gamma = -1 \) for any modified matter Lagrangian \( \mathcal{L} \) that exhibits bounces.

Next, we switch to the Hamiltonian formalism with momenta \( \pi_\phi = 2\omega \dot{\phi} \mathcal{L}, \pi_\omega = -\frac{d-1}{2d} \omega + \pi_\omega = \omega \alpha_i \) conjugate to \( \phi, \omega, \alpha_i \). By Liouville’s theorem, the symplectic form \( \varpi = dx_\phi \wedge d\phi + dx_\omega \wedge d\omega + dx_\alpha \wedge d\alpha_i \) is time-invariant so its \( t \to \pm \infty \) limits coincide. The asymptotics \( \mathcal{L} \simeq X/2 \) and (10), including the Hamiltonian constraint, give
\[
\varpi_{\pm} = \pm d d\omega (\frac{1}{d-1} \lambda_i^2)^{1/2} \phi_0^\pm /r (\phi_0^\pm) ) \wedge d\phi_1^\pm + d\lambda_i \wedge d\nu_i^\pm,
\]
and these limits must coincide. At fixed \( \lambda \), this means \( \pm d (\phi_0^\pm /r^\pm) \wedge d\phi_1^\pm \) are equal, so the map \( \Phi \) is a canonical transformation as stated in (11) with \( \epsilon = \text{sgn} \gamma = -1 \). This establishes the second scattering law (2) for modified-matter bounces. One can check that \( \phi_0^+ \), \( \phi_0^- \) only depend on \( \phi_0^+, \phi_1^- \), and the scattering map takes the explicit form (14) given below in our model-independent analysis.

**UNIVERSALITY AND MODEL-DEPENDENCE**

**Classification of singularity scattering maps.** As observed in the analysis of quiescent cosmological singularities in [5, 1, 7], spatial derivatives can be neglected near a singularity, so that each spatial point undergoes an (almost) independent evolution in time. When describing a bounce as the junction of two singular solutions to Einstein equations along the bounce hypersurface, it is natural to assume that the same “ultralocality” property holds through the bounce. Namely, we focus on **ultralocal scattering maps**, as we call them, for which the value of \( (g^+, k^+, \phi_0^+, \phi_1^+) \) at a point \( x \) along the bounce hypersurface depends on \( (g^-, k^-, \phi_0^-, \phi_1^-) \) at the same point but is independent of (spatial) derivatives thereof.

This simple postulate has far reaching consequences, leading to a model-independent classification of singularity scattering maps: any ultralocal scattering map is either an anisotropic map \( S_{\beta_0, \gamma}^\text{ui} \) (14) **or an isotropic map** \( S_{\Delta, \varphi, \epsilon}^{\text{iso}} \) (15). We present here the classification in \( d \geq 2 \) spatial dimensions, generalizing our \( d = 3 \) proof [17].

**Proof sketch.** By general covariance the scalars read \( (\phi_0^\pm, \phi_1^\pm) = \Phi (\phi_0^-, \phi_1^-), \chi_m \) in terms of scalar invariants \( \chi_m := \text{Tr}(k^- r^-)^m, 3 \leq m < d \). Likewise the tensors \( k^\pm \) and \( \log(g^+(g^-)^{-1}) \) are linear combinations of \( (k^-)^n \), \( 0 \leq n < d \). A calculation then gives
\[
\nabla_a^+ k_b^+ = \Omega^{-1} \nabla_a^-(\Omega k_b^-) - X_b/2, \quad (13)
\]
where $X_k$ is a sum of (scalar) $\partial_0$ (scalar) terms and $\Omega = \sqrt{|g^+|/|g^-|}$. We want the momentum constraint $\nabla^a k^\pm b = \phi^\pm_0 \partial_0 \phi^\pm_1$ on the “-” side to imply the “+” one. This requires (1) to hold, namely $\Omega k^+ = \gamma k^-$, so that the right-hand side of (13) reduces to (scalar) $\partial_0$ (scalar) terms; in contrast, for $n > 1$, $\nabla_\alpha (\partial^- \eta^\nu)^{\mu}_\nu$ involves all derivatives of $k^\pm$. Tracking (scalar) $\partial_0$ (scalar) terms in detail yields solvable differential equations for the scalar coefficients $\sigma_n$ in $\log (g^+(g^-)^{-1}) = \sum n \sigma_n (k^- / r)^n$, which imply our second scattering law (2) in the form (11).

Anisotropic ultralocal scattering. Scattering maps for which $\gamma \neq 0$ are characterized by the canonical transformation $\Phi$ obeying (11) with $\epsilon = \text{sgn} \gamma$. Explicitly, $S^\text{an}_k : (g^-, k^-, \phi^\pm_0, \phi^\pm_1) \mapsto (g^+, k^+, \phi^+_0, \phi^+_1)$ reads

$$
(\phi^+_0, \phi^+_1) = \Phi(\chi_m, \phi^-_0, \phi^-_1),
\hat{k}^+ = \epsilon (r^+ / r^-)^{\hat{k}^-}, \quad g^+ = \gamma r^+ / r^- \exp \left( \sum_{n=2}^{d-1} \sigma_n \left( \frac{\chi_n^+ - \hat{k}^+}{\chi_n^+} \right) \right) g^-,
$$

where $\sigma_n = (\partial_{\chi_n^+}, \xi + 2 \epsilon (\phi^+_0 / r^+)^{\partial_{\chi_n^+} + \phi^+_1})(n+1)$, $\xi$ vanishes at $\phi^+_0 = \pm \sqrt{(d-1)/d}$, and $\partial_{\phi^+_0} \xi = -2 \epsilon (\phi^+_0 / r^+)^{\partial_{\phi^+_0} \phi^+_1}$.

Remarkably, (i) $S^\text{an}_k$ depends on a single canonical transformation $\Phi ; (\phi^+_0, \phi^+_1) \mapsto (\phi^+_0, \phi^+_1)$ parametrized by the scalar invariants $\chi_m$; (ii) the densitized trace-free part of the extrinsic curvature $(k^+ - \frac{1}{d} \delta) \sqrt{g^\pm}$ is unchanged up to a constant factor $\gamma$, and in particular its eigenvectors (Kasner frame) are preserved; (iii) the metric is scaled anisotropically in each eigenvector direction of $k^\pm$.

Isotropic ultralocal scattering. The second class of maps is obtained by taking $\gamma = 0$ in (1), namely $k^+ = 0$. The constraints (5) then fix $\phi^+_0$ and make $\phi^+_1$ constant, while the metric is arbitrary. The isotropic map reads

$$
S^\text{iso}_{\Delta, \varphi} : (g^-, k^-, \phi^-_0, \phi^-_1) \mapsto (g^+, k^+, \phi^+_0, \phi^+_1),
$$

for any constant $\varphi \in \mathbb{R}$, sign $\epsilon = \pm 1$, and function $\Delta = \sum_{n=2}^{d-1} \Delta_n (\phi^-_0, \phi^-_1, \chi_n) (k^-)^n$ with positive eigenvalues.

While the constant map $\Phi = (\phi^+_0, \phi^+_1)$ is not strictly speaking a canonical transformation since it sits at a singular point of the symplectic form (11), it can be realized as a limit of canonical transformations. However, the isotropic map (15) is not a limiting case of the anisotropic map (14) since (15) allows a much more general metric $g^+$.

The isotropic scattering map $S^\text{iso}_{\Delta, \varphi, \epsilon}$ physically describes an irreversible bouncing scenario in which almost all the information is lost: (i) The extrinsic curvature is a constant multiple of the identity, so the bounces produce an isotropic and homogeneous evolution. (ii) The two components of the matter field after the bounce are overall constants. (iii) However, the metric is scaled differently along the different eigenvectors of the extrinsic curvature.

Outlook on universality and model-dependence. Our notion of singularity scattering maps extracts the relevant macroscopic effects induced by a given microscopic model. Remarkably, by a simple ultralocality postulate we have established a full classification of these maps which proves universal laws while leaving room for model-dependent aspects to affect the Universe after the bounce. The classification explains precise universal laws obeyed by example models ranging from string theory to loop quantum cosmology and modified gravity: (1) continuity of the densitized shears $K^\pm \sqrt{g^\pm}$, and (2) canonical transformation of the scalar field.

We have seen explicitly that in the pre Big Bang scenario our approach selects the natural choice of signs $\beta_+ = -\beta_-$, which leads to an anisotropic scattering map characterized by a very explicit $\Phi$; cf. (12). In contrast, the scattering map is not explicit for modified matter models, and depends on the choice of Lagrangian, yet we have proven in homogeneous cases that it obeys the universal scattering laws and fits in our classification. We extend the analysis to non-homogeneous bounces in [18].

The key for our classification were the constraint equations and the fact that space derivatives are negligible near the singularity. Our method should thus generalize to other matter fields, a cosmological constant, bounces that do not asymptote to general relativity, Penrose’s cyclic cosmological model [20, 21], etc. In particular, for compressible fluids with general equations of state we find in [15] an interesting interplay between geometric singularities, fluid shock waves, and phase transition boundaries.

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