Uniform factorial decay estimate for the remainder of rough Taylor expansion

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We establish a uniform factorial decay estimate for the Taylor approximation of solutions to controlled differential equations in the $p$-variation metric. As part of the proof, we also obtain a factorial decay estimate for controlled paths which is interesting in its own right.

**Keywords:** Controlled differential equation; Rough paths; Taylor expansion; Factorial Decay.

**AMS MSC 2010:** 60H10; 34H05.

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## 1 Introduction

For a controlled differential equation of the form
\begin{align}
\frac{dY_t}{dt} &= f(Y_t) \frac{dX_t}{t} \\
Y_0 &= y_0
\end{align}
(1.1)
where $X : [0,T] \to \mathbb{R}^d$ is a path with finite 1-variation and $f : \mathbb{R}^c \to L(\mathbb{R}^d, \mathbb{R}^e)$ is a smooth vector field, we are interested in estimating the Taylor remainder
\begin{align}
Y_t - Y_s &= \sum_{k=1}^{N} f^{\otimes k}(Y_s) \int_{s<s_1<...<s_k<t} dX_{s_1} \otimes ... \otimes dX_{s_k} \\
\equiv \int_{s<s_1<...<s_N<t} f^{\circ N}(Y_s) - f^{\circ N}(Y_s) dX_{s_1} \otimes ... \otimes dX_{s_N},
\end{align}
(1.2)
(1.3)
where $f^{\circ m} : \mathbb{R}^e \to L \left((\mathbb{R}^d)^{\otimes m}, \mathbb{R}^e \right)$ is defined inductively by
\begin{align}
f^{\circ 1} &= f \\
f^{\circ k+1} &= D(f^{\circ k}) f.
\end{align}

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The functions \( f^{\circ k} \) can also be expressed in terms of iterative applications of the vector field \( f \) as differential operators \([3]\). The iterated integrals in (1.2) will appear numerous times and we shall use the shorthand

\[
X_{s,t}^k := \int_{s < s_1 < \ldots < s_k < t} dX_{s_1} \otimes \ldots \otimes dX_{s_k}.
\]  

(1.4)

Since the \( 1 \)-variation norm of \( X \) equals to the \( L^1 \) norm of the derivative of \( X \), we have (see for example \([4]\))

\[
\|X\|_{1-\text{var};[s,t]} = \sup_{s < t_1 < \ldots < t_n < t} \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|.
\]  

and \( \|f^{\circ N}\|_{\infty} \) denotes \( \sup_{x \in \mathbb{R}^d} |f^{\circ N}(x)| \) with \( |\cdot| \) being the operator norm

\[
\|f^{\circ N}(x)\| = \sup_{v \in \mathbb{R}^{d \otimes N}} \left\| f^{\circ N}(x)(v) \right\|.
\]  

Estimates of the form (1.5) have application both as a theoretical tool for analysing the equation (1.1) and as a practical numerical scheme for constructing the solution. The estimate (1.5), when the \( 1 \)-variation metric is replaced by the \( p \)-variation metric, has been shown in \([2]\) \((p < 3)\), \([5]\) \((p < 3)\) and \([4]\) \((p \geq 1)\) without the factorial decay factor. We shall prove such estimate with the factorial decay factor. The estimates of Davie \([2]\), Gubinelli \([5]\), Friz and Victoir \([4]\) as well as our estimates below gives a numerical scheme for approximating a solution to (1.1) in \( O(1) \) time steps.

**Theorem 1.1.** Let \( p \geq 1. \) Let \( X = (1, X^1, \ldots, X^{|\gamma|}) \) be a \( p \)-weak geometric rough path. Let \( f \) be a \( \text{Lip}(\gamma - 1) \) vector field where \( \gamma > p \). Let \( Y \) be a solution to the differential equation

\[
dY_t = f(Y_t) dX_t
\]  

(1.6)

defined in the sense of \([3]\). Then there exists a constant \( C_p \) depending only on \( p \) such that

\[
\left| Y_t - Y_s - \sum_{k=1}^{|\gamma|} f^{\circ k}(Y_s) X^k_{s,t} \right| \leq \frac{1}{(\frac{|\gamma|}{p})!} \beta^{\frac{|\gamma|}{p}} M_{p,\gamma} \|f\|_{\text{var}} \|X\|_{p-\text{var};[s,t]},
\]  

(1.7)

where

\[
M_{p,\gamma} = 2C_p \left( \|f\|_{\text{Lip}(\gamma - 1) \cup [p])} \right)^{|p|+1} \left( |X|_{p-\text{var}} \vee 1 \right)^{|p|+1};
\]  

\[
\|f\|_{\text{var}} = \max_{|\gamma| - |p| + 1 \leq m \leq |\gamma|} \left\| f^{\circ m_{\text{min}}(\gamma - m, 1)} \right\|_{\text{Lip}(\min(\gamma - m, 1))};
\]  

(1.8)

\[
\beta = \left( 1 + \sum_{r=2}^{\infty} \frac{2}{r-1} \wedge 1 \right)^{\frac{|p|+1}{p}}.
\]  

(1.9)

We refer the readers to Definition 9.16 and Definition 10.2 in \([3]\) for the definition of \( \text{Lip}(\gamma) \) vector fields and weak geometric rough paths respectively. We shall however recall the definition of \( p \)-variation and some basic notations in Section 2.

**Remark 1.2.** If the equation (1.6) has more than one solution, then any solution must satisfy (1.7).
Remark 1.3. Taking the biggest $\gamma$ may not yield the best estimate for the left hand side of (1.7). In general the term $\|f\|_{\gamma}$ could grow factorially fast in $\gamma$. Since a Lip($\gamma$) function is also Lip($\gamma'$) for all $\gamma' < \gamma$, we may choose $\gamma'$ which optimises the estimate (1.7).

The proof for (1.5) relies heavily on the relation between the $1$-variation of the path and the $L^1$ norm of its derivative. Proving an estimate of the form (1.5) for the $p$-variation metric, even without the factorial decay factor, requires the clever idea of Young[9]. The integration with respect to a path can be expressed in terms of the limit of a Riemann sum as the size of partition converges to zero. Young’s idea was to estimate the Riemann sum with respect to a partition by removing points from the partition successively. This idea had been used in [6] to show that, for $p < 2$, the $n$-th order iterated integral of a path $X$ is uniformly bounded by

$$\left(1 + 4^{\frac{1}{p}} \zeta\left(\frac{2}{p}\right)\right)^n \left(\frac{1}{n!}\right)^{\frac{1}{p}} \|X\|_{p-\text{var},[0,T]}^{n \int_0^T (1 + 4^{\frac{k}{p}} \zeta\left(\frac{2}{p}\right)) \frac{k}{p}\, dt}$$

where $\zeta$ is the classical zeta function. T. Lyons’ proof for the $p \geq 2$ case in [7] is slightly different and used the neoclassical inequality ([7],[1])

$$\sum_{k=0}^{N} \frac{1}{\Gamma(k/p + 1) \Gamma((n-k)/p + 1)} \left(a + b\right)^{(n-k)/p} \leq p \frac{1}{\Gamma(n/p + 1)} \left(a + b\right)^{n/p}$$

(1.11)

to obtain a uniform bound of the form

$$\beta^{n-1} \frac{1}{\Gamma(n/p + 1)} \|X\|_{p-\text{var},[0,T]}^{n \int_0^T (1 + 4^{\frac{k}{p}} \zeta\left(\frac{2}{p}\right)) \frac{k}{p}\, dt}$$

where $\Gamma$ is the Gamma function and $\beta$ is as defined in (1.9).

2 The Proof

2.1 Notations and basic definitions

For each $k \in \mathbb{N}$, we equip a norm on $(\mathbb{R}^d)^{\otimes k}$ by identifying it with $\mathbb{R}^{d^k}$. Let

$$T_1^N (\mathbb{R}^d) = 1 \oplus \mathbb{R}^d \oplus \ldots \oplus (\mathbb{R}^d)^N.$$ 

If $\pi_k$ denotes the projection operator $T_1^N (\mathbb{R}^d) \to (\mathbb{R}^d)^{\otimes k}$, then we define a norm on $T_1^N (\mathbb{R}^d)$ by

$$\|x\| = \max_{1 \leq k \leq N} \|\pi_k(x)\|^{\frac{1}{k}}.$$ 

Definition 2.1. Let $T > 0$ and $p \geq 1$. A path $X : [0,T] \to T_1^{[p]}(\mathbb{R}^d)$ has finite $p$-variation if for all $0 < s < t < T$,

$$\|X\|_{p-\text{var},[s,t]} := \sup_{s < t_1 < \ldots < t_n < t} \max_{1 \leq k \leq [p]} \left(\sum_{i=0}^{n-1} \|\pi_k(X_{t_i}^{-1}X_{t_{i+1}})\|^{\frac{1}{p}}\right)^{\frac{1}{p}} < \infty$$

(2.1)

where $X^{-1}$ denote the unique multiplicative inverse of $X \in T_1^{[p]}(\mathbb{R}^d)$. We will denote $\|X\|_{p-\text{var},[0,T]}$ by $\|X\|_{p-\text{var}}$.

We first recall Lyons’ extension theorem, which will be used repeatedly in the following form:

Fact 2.2. (Theorem 2.2.1 in [7]) Let $p \geq 1$ and $X = (1, X^1, \ldots, X^{[p]})$ be a $p$-weak geometric rough path. Then for all $N \geq [p] + 1$, there exists a unique continuous
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path $X = (1, X^1, \ldots, X^N) \in T_1^N ([0, 1])$ which extends $X$, $X_0 = (1, 0, \ldots, 0)$ and for all $[p] \leq l \leq N$,

$$\|\pi_l (X_{t_l}^{-1} X_{t_{l+1}})\| \leq \frac{\beta^{|l-1|}}{(\frac{l}{p})!} \|X\|_{p-var, [s, t]}, \quad (2.2)$$

**Remark 2.3.** We will denote $X_s^{-1} X_r$ by $X_{s,r}$ and $\pi_l (X_{s,r})$ by $X_{s,r}^l$. In particular, $X_{s,u} \otimes X_{u,t} = X_{s,t}$ and so, for any $s < u < t$,

$$X_{s,t}^m = \sum_{i=0}^m X_{s,u}^{m-i} \otimes X_{u,t}^i. \quad (2.3)$$

Note that for paths with finite 1-variation, the $(X^k)_{k \geq 1}$ defined in this theorem are exactly the iterated integrals of $X$. Hence no confusion will arise by using the same notation as in (1.4).

**Remark 2.4.** If $r \geq [p]$, then for any $m \geq 0$,

$$X_{s,t}^m = \lim_{|P| \to 0} \sum_{i=0}^{n-1} \sum_{k=1}^r X_{s,i}^{m-k} \otimes X_{i,i+1}^k \quad (2.4)$$

where the limit is taken as the mesh size of the partition $P = (s < t_1 < \ldots < t_{n-1} < t)$ goes to zero. By convention, for any $s < t$, $X_{s,t}^0 = 1$ and $X_{s,t}^m = 0$ if $m < 0$. In the case $r = m$, (2.4) follows directly from (2.3). For $r < m$, note that the sum over $k$ from $r + 1$ to $m$ in (2.4) vanishes after the taking of limit, due to (2.2). See [5] for details.

2.2 The proof

The following lemma is a factorial decay estimate for the Taylor remainder of a controlled path in the sense of Gubinelli [5]. This lemma is interesting in its own right. We interpret it as the dual counterpart of Fact 2.2.

**Lemma 2.5.** Let $p \geq 1$ and $\gamma > p$. Let $\{1, X^1, \ldots, X^{[p]}\}$ be a $p$-weak geometric rough path. Let $Y^{(i)}$ be a function $[0, T] \to L (\langle R^d \rangle ^{\otimes i}, R^c)$ and $(Y^{(0)}, Y^{(1)}, \ldots, Y^{(\gamma)})$ satisfies, for $[\gamma - p] \leq m \leq [\gamma]$,

$$|Y_t^{(m)} - \sum_{l=0}^{[\gamma]-m} Y_s^{(l+m)} X_{s,t}^l| \leq \frac{1}{([\gamma]-m)!} M \beta^{[\gamma]-m} \|X\|_{p-var, [s, t]}, \quad (2.5)$$

for all $s \leq t$ and for $0 \leq m \leq [\gamma - p] - 1$, the limit

$$\lim_{|P| \to 0} \sum_{i=0}^{n-1} \sum_{k=1}^r Y_{s,i}^{(m+l)} X_{i,i+1}^k, \quad (2.6)$$

where $|P| \to 0$ denotes the limit as the mesh size of a partition $P$ on $[s, t]$ goes to zero, exists and equals

$$Y_t^{(m)} - Y_s^{(m)}. \quad (2.7)$$

For $l \geq [p] + 1$, let $X^l$ denote the projection to $\langle R^d \rangle ^{\otimes l}$ of the unique extension of $(1, X^1, \ldots, X^{[p]})$ given in Fact 2.2. Then (2.5) holds for all $0 \leq m \leq [\gamma]$.

**Proof.** We will carry out backward induction on $k$ starting from $[\gamma - p]$ and moving down to 0.

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The base induction step of \( k = [\gamma - p] \) holds because of the assumption. We will assume from now onwards that \( k \leq [\gamma - p] - 1 \). It is useful to bear in mind that

\[
[\gamma] - [p] \leq [\gamma - p] \leq [\gamma] - [p] + 1.
\]

For the induction step, note that by (2.4) and the equality of (2.6) and (2.7),

\[
Y_t^{(k)} = \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+l)} X_l l^\ell \tag{2.8}
\]

\[
= \lim_{|P| \to 0} \sum_{i=0}^{n-1} \sum_{l_2=1}^{\lfloor \gamma \rfloor - l_2} \left( \sum_{l_1=0}^{\lfloor \gamma \rfloor - l_2} Y_s^{(k+l_1+l_2)} X_{l_1} l^\ell \right) X_{l_2 l, t} l^\ell \tag{2.9}
\]

where the limit is taken as the mesh size of the partition \( P = (s < t_1 < \ldots < t_{n-1} < t) \) goes to zero.

We first show that the term

\[
\sum_{i=0}^{n-1} \sum_{l_2=1}^{\lfloor \gamma \rfloor - l_2} \sum_{l_1=0}^{\lfloor \gamma \rfloor - l_2} Y_s^{(k+l_1+l_2)} X_{l_1} l^\ell X_{l_2 l, t} l^\ell \tag{2.10}
\]

is in fact independent of the partition \( P \).

\[
\sum_{i=0}^{n-1} \sum_{l_2=1}^{\lfloor \gamma \rfloor - l_2} \sum_{l_1=0}^{\lfloor \gamma \rfloor - l_2} Y_s^{(k+l_1+l_2)} X_{l_1} l^\ell X_{l_2 l, t} l^\ell = \sum_{i=0}^{n-1} \sum_{r=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+r)} X_{s, t_1} l^\ell - \sum_{r=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+r)} X_{s, t} l^\ell = \sum_{r=1}^{\lfloor \gamma \rfloor - k} Y_s^{(k+r)} X_{s, t} l^\ell
\]

where we have used (2.3) in the third line. Let

\[
\left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_l l^\ell \right)^P = \left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_l l^\ell \right)^{P \setminus \{t_j\}}
\]

Since (2.10) is independent of the partition,

\[
\left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_l l^\ell \right)^P - \left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_l l^\ell \right)^{P \setminus \{t_j\}} \tag{2.11}
\]

\[
= \sum_{i=1}^{\lfloor \gamma \rfloor - k} Y_{t_1 t_2} X_{t_1, t_2} l^\ell + \sum_{r=1}^{\lfloor \gamma \rfloor - k} Y_{t_1 t} X_{t_1, t} l^\ell \tag{2.12}
\]
By induction hypothesis, (2.5) which holds for $m > k$ and Theorem 2.2.1 in [7],

$$
\left| \sum_{l_2=1}^{[\gamma] - k} \left( Y_{t_j}^{(k+l_2)} - \sum_{l_1=0}^{[\gamma] - k - l_2} Y_{t_{j-1}}^{(k+l_1+l_2)} X_{t_{j-1}, t_j} \right) \right| X_{t_j, t_{j+1}}^2 \\
\leq \left| \sum_{l_2=1}^{[\gamma] - k} \left( \frac{1}{(\frac{[\gamma] - k - l_2}{p})!} M^{[\gamma] - k - l_2} \|X\|_{p, \text{var}, [t_{j-1}, t_j]} \right) \right| \\
\times \beta^{l_2 - 1} \|X\|_{p, \text{var}, [t_j, t_{j+1}]} \\
\leq \frac{1}{(\frac{[\gamma] - k}{p})!} \beta^{[\gamma] - k} \|X\|_{p, \text{var}, [t_{j-1}, t_j]} \|X\|_{p, \text{var}, [t_{j-1}, t_{j+1}]}.
$$

(2.13)

(2.14)

where the final line is obtained by the neoclassical inequality (1.11), proved in [1].

Let $\omega(s, t) = \|X\|_{p, \text{var}, [s, t]}^p$. We now choose $j$ such that, for $|P| \geq 2$,

$$
\omega(t_{j-1}, t_{j+1}) \leq \left( \frac{2}{|P| - 1} \wedge 1 \right) \omega(s, t)
$$

which exists since

$$
\sum_{i=1}^{n-1} \omega(t_{i-1}, t_{i+1}) \leq 2 \omega(s, t)
$$

and also that

$$
\omega(t_{j-1}, t_{j+1}) \leq \omega(s, t)
$$

for all $j$. Then as $\gamma - k \geq [p] + 1$, (2.14) is less than or equal to

$$
\frac{1}{(\frac{[\gamma] - k}{p})!} \beta^{[\gamma] - k} \left( \frac{2}{n-1} \wedge 1 \right)^{\frac{[\gamma] - k}{p}} \|X\|_{p, \text{var}, [s, t]}.
$$

By removing points successively from $P$ and using that $\left( Y_s^{(k)} - \sum_{l=0}^{[\gamma] - k} Y_s^{(k+l)} X_{s,t}^l \right)_{s,t} = 0$, we have

$$
\left| \left( Y_s^{(k)} - \sum_{l=0}^{[\gamma] - k} Y_s^{(k+l)} X_{s,t}^l \right) \right| \leq \frac{1}{(\frac{[\gamma] - k}{p})!} \beta^{[\gamma] - k} \sum_{n=2}^{\infty} \left( \frac{2}{n-1} \wedge 1 \right)^{\frac{[\gamma] - k}{p}} \|X\|_{p, \text{var}, [s, t]}^{[\gamma] - k}
$$

(2.15)

where the final line follows from (1.9).

By taking limit as $|P| \to 0$, (2.5) follows for $m = k$. \hfill \Box

For the differential equation

$$
dY_t = f(Y_t) \, dX_t
$$

we wish to apply Lemma 2.5 to $(Y, f^{(1)}(Y), \ldots, f^{(\gamma)}(Y))$. Using the standard estimates for rough differential equations, it turns out that it suffices to verify the assumption of Lemma 2.5 for paths with finite 1-variation. To do so, we need the following lemma.
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**Lemma 2.6.** Let $X : [0, T] \to \mathbb{R}^d$ be a path with finite 1-variation. Let $f$ be a Lip$(\gamma - 1)$ vector field. Let $Y_t$ be a solution to the differential equation (2.15). Then

\[
\begin{align*}
&f^m (Y_t) - f^m (Y_s) - \sum_{k=1}^{[\gamma] - m} f^m (Y_s) X^k_{s,t} \\
&= \begin{cases}
\int_s \leq s_1 \leq \leq s_{[\gamma] - m} \leq t & f^m (Y_s) - f^m (Y_{s}) \, dX_{s_1} \otimes \ldots \otimes dX_{s_{[\gamma] - m}}, \quad 0 \leq m < [\gamma] \\
0 & m = [\gamma].
\end{cases}
\end{align*}
\]

Proof. We will prove it by backward induction, starting from $[\gamma]$.

For the induction step, note first that by the fundamental theorem of calculus,

\[
\begin{align*}
\int_s^t f^m (Y_u) \, dX_u &= \int_s^t D (f^m) (Y_u) \, dX_u \\
&= \int_s^t D (f^m) (Y_u) \, dY_u \\
&= f^m (Y_t) - f^m (Y_s).
\end{align*}
\]

Then by (2.16) and the induction hypothesis,

\[
\begin{align*}
f^m (Y_t) - f^m (Y_s) &= \sum_{k=1}^{[\gamma] - m} f^m (Y_s) X^k_{s,t} \\
&= \int_s^t f^{m+1} (Y_{s_{[\gamma] - m}}) \, dX_{s_{[\gamma] - m}} - \sum_{k=1}^{[\gamma] - m} f^{m+1} (Y_{s_{[\gamma] - m}}) X^{k-1}_{s_{[\gamma] - m}} \, dX_{s_{[\gamma] - m}} \\
&= \int_s \leq s_1 \leq \leq s_{[\gamma] - m} \leq t f^m (Y_{s_1}) - f^m (Y_{s}) \, dX_{s_1} \otimes \ldots \otimes dX_{s_{[\gamma] - m}}.
\end{align*}
\]

\[\square\]

Proof of Theorem 1. The only thing to prove is that $(Y, f^1 (Y), \ldots, f^{[\gamma]} (Y))$ satisfies the assumptions of Lemma 2.5.

For each $s \leq t$, let $x^{s,t} : [s, t] \to \mathbb{R}^d$ be a continuous path with finite 1-variation such that for $1 \leq l \leq [p]$,

\[
(x^{s,t})^l_{s,t} = X^l_{s,t},
\]

where we use the notation from (1.4) and

\[
\int_s^t |dx_u| \leq c_p \|X\|_{\text{var}, [s, t]} \quad \text{(2.18)}
\]

for a function $c_p$ of $p$ which is specified in [3] along with the existence of $x^{s,t}$.

Consider the differential equation

\[
\begin{align*}
dY_{u}^{s,t} &= f (Y_{u}^{s,t}) \, dx_u^{s,t} \\
Y_{s}^{s,t} &= Y_{s}.
\end{align*}
\]

By Theorem 10.16 in [3], there exists a solution $Y^{s,t}$ of (2.19) such that the following estimate holds

\[
\|Y_t - Y_{t}^{s,t}\| \leq C_p \|f\|_{\text{Lip}(\gamma - 1) \cap [p]} \|X\|_{p - \text{var}, [s, t]} \quad \text{(2.20)}
\]
To estimate (2.23) for \( m \),

\[
\left| f^m(Y_t) - \sum_{k=0}^{[\gamma]-m} f^m(Y_s) X^k_{s,t} \right| \leq \left| f^m(Y_t) - f^m(Y_{t^{\gamma}}) \right| + \left| f^m(Y_{t^{\gamma}}) - \sum_{k=0}^{[\gamma]-m} f^m(Y_s) (x^{s,t})^k_{s,t} \right| \tag{2.21}
\]

By (2.20), for \( 0 \leq m \leq [\gamma] - 1 \),

\[
\left| f^m(Y_t) - f^m(Y_{t^{\gamma}}) \right| \leq C_p \left| f^m |_{\text{Lip}(1)} \right| |Y_t - Y_{t^{\gamma}}|^{\gamma - [\gamma]} \tag{2.22}
\]

If \( [\gamma - p] \leq m \leq [\gamma] - 1 \), then \( \gamma - m \leq [p] \) and so

\[
\left| f^m(Y_t) - f^m(Y_{t^{\gamma}}) \right| \leq C_p \left| f^m |_{\text{Lip}(1)} \right| |f^{\gamma([p]+1)\gamma([\gamma]-[p])}||X||_{p-\text{var},[s,t]}^{[\gamma]-[p]} \tag{2.24}
\]

To estimate (2.23) for \( m = [\gamma] \), we note that

\[
\left| f^{[\gamma]}(Y_t) - f^{[\gamma]}(Y_{t^{\gamma}}) \right| \leq f^{[\gamma]} |_{\text{Lip}([\gamma]-[\gamma])} |Y_t - Y_{t^{\gamma}}|^{\gamma - [\gamma]} \leq C_p f^{[\gamma]} |_{\text{Lip}([\gamma]-[\gamma])} |f^{\gamma([p]+1)\gamma([\gamma]-[p])}||X||_{p-\text{var},[s,t]}^{[\gamma]-[p]} \tag{2.26}
\]

In particular, we have

\[
\left| f^{[\gamma]}(Y_t) - f^{[\gamma]}(Y_{t^{\gamma}}) \right| \leq C_p f^{[\gamma]} |_{\text{Lip}([\gamma]-[\gamma])} |f^{\gamma([p]+1)\gamma([\gamma]-[p])}||X||_{p-\text{var},[s,t]}^{[\gamma]-[p]} \tag{2.27}
\]

To estimate the second term in (2.21), we use Lemma 2.6 to see that for \( [\gamma - p] \leq m \leq [\gamma] \),

\[
\left| f^m(Y_{t^{\gamma}}) - \sum_{k=0}^{[\gamma]-m} f^m(Y_s) (x^{s,t})^k_{s,t} \right| \leq \left| \int_{s \leq t_1 \leq \cdots \leq t_\gamma \leq t} f^{[\gamma]}(Y_{s_1^{[\gamma]}}) \cdots f^{[\gamma]}(Y_{s_{[\gamma]-1]^{[\gamma]}}) \, dx^{s_1^{[\gamma]}t_1} \cdots dx^{s_{[\gamma]-1}^{[\gamma]}t_{[\gamma]-1}} \right| \tag{2.25}
\]

For some function \( C_p \) depending on \( p \) only.

Note that by (2.17) and \( m \geq [\gamma - p] \geq [\gamma] - [p] \),

\[
\left| \int_{s \leq t_1 \leq \cdots \leq t_\gamma \leq t} f^{[\gamma]}(Y_{s_1^{[\gamma]}}) \cdots f^{[\gamma]}(Y_{s_{[\gamma]-1]^{[\gamma]}}) \, dx^{s_1^{[\gamma]}t_1} \cdots dx^{s_{[\gamma]-1}^{[\gamma]}t_{[\gamma]-1}} \right| \leq C_p \left| f^{[\gamma]} |_{\text{Lip}([\gamma]-[\gamma])} \right| |X||_{p-\text{var},[s,t]}^{[\gamma]-[p]} \tag{2.26}
\]

(2.25)
where in the third line we have used the $\gamma - \lfloor \gamma \rfloor$ Hölder continuity of $f^{\circ(\lfloor \gamma \rfloor)}$ with (2.18) and in the final line we have used Theorem 10.16 in [3].

Combining (2.21), (2.23) and (2.26), we have for $[\gamma - p] \leq m \leq \lfloor \gamma \rfloor$,

$$\left| f^{\circ(m)} (Y_t) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_s) X^k_{s,t} \right| \leq 2C_p \max_{[\gamma - p] + 1 \leq m \leq \lfloor \gamma \rfloor} \left| f^{\circ(\min(\gamma - m, 1))} (f)_{\text{Lip}}(\min(\gamma - m, 1)) \right| [p] + 1 \left[ \|X\|_{\text{p-var}} \lor 1 \right] f^{\circ([p] + 1)} (x_{u,v}) \right| \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) (x_{u,v})^k_{u,v} \bigg| \quad \text{for } k \in [p] + 1 \right| \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) X^k_{u,v} \right|.
$$

Here since $[\gamma - p] \leq m \leq \lfloor \gamma \rfloor$ so $[\gamma] - m \leq [p]$ and

$$([\gamma] - m)! \leq [p]!.$$

Therefore, by changing the constant $C_p$, we rewrite (2.28) in the form of the right hand side of (2.5). It now suffices to show (2.7). Note first that for $0 \leq m \leq [\gamma - p] - 1$ and $s \leq u \leq v \leq t$,

$$\left| f^{\circ(m)} (Y_v) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) X^k_{u,v} \right| \leq \left| f^{\circ(m)} (Y_v) - f^{\circ(m)} (Y_{u,v}) \right| + \left| f (Y_{u,v}) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) (x_{u,v})^k_{u,v} \right| + \left| \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) X^k_{u,v} \right| \quad \text{for } k \in [p] + 1 \right| \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) X^k_{u,v} \right|.
$$

The estimate (2.22) still holds with $(s, t)$ replaced by $(u, v)$ and (2.26) would hold with the constant $C_p$ now depending on $\gamma$ as well. For the final term in (2.31),

$$\left| \sum_{k=[p] + 1}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)} (Y_u) (x_{u,v})^k_{u,v} \right| \leq 2\lfloor \gamma \rfloor c_p \max_{0 \leq m \in [\gamma]} \sup_{s \leq u \leq t} \left| f^{\circ(m)} (Y_u) \right| \left( \|X\|_{\text{p-var}} \lor 1 \right) \left[ \|X\|_{\text{p-var}} \lor 1 \right]$$

where we used Fact 2.2 and

$$\left| (x_{u,v})^k_{u,v} \right| \leq c_p^k \left( \int_u^v |dx^u_t| \right)^k \leq c_p^k \left| X \right|_{\text{p-var}}^k.$$

Therefore, combining with (2.22) and (2.26), we have for some constants $C_{f,p,X,s,t,\gamma}$ and $C'_{f,p,X,s,t,\gamma}$.
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independent of \( u, v \) such that when \( |u - v| \) is sufficiently small,

\[
\left| f^{cm}(Y_u) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{s(m+k)}(Y_u) X^k_{u,v} \right| \\
\leq C_{f,p,X,s,t,\gamma} \left( \|X\|_{p-var,[u,u]}^\gamma + \|X\|_{p-var,[u,v]}^\gamma + \|X\|_{p-var,[u,v]}^{\gamma-n} \right) \\
\leq C'_{f,p,X,s,t,\gamma} \left( \|X\|_{p-var,[u,u]}^\gamma \right)
\]

Denote the expression in (2.29) as \( E (u, v) \). Let \( \lim_{|P| \to 0} \) denote the limit as the mesh size of a partition \( P \) on \([s, t]\) goes to zero. Then for \( m \leq |\gamma - p| - 1 \),

\[
\lim_{|P| \to 0} \sum_{i=0}^{n-1} \sum_{i=1}^{\lfloor \gamma \rfloor - m} E(t_i, t_{i+1}) \\
\leq C'_{f,p,X,s,t,\gamma} \lim_{|P| \to 0} \sum_{i=0}^{n-1} \|X\|_{p-var,[t_i, t_{i+1}]}^{\gamma([p] + 1)} \\
\leq C_{f,p,X,\gamma} \lim_{|P| \to 0} \max_i \|X\|_{p-var,[t_i, t_{i+1}]}^{\gamma([p] + 1) - p} \sum_{i=0}^{n-1} \|X\|_{p-var,[t_i, t_{i+1}]}^p
\]

Since for \( s < u < t \),

\[
\|X\|_{p-var,[s,u]}^p + \|X\|_{p-var,[u,t]}^p \leq \|X\|_{p-var,[s,t]}^p
\]

(2.33) is bounded by

\[
C_{f,p,X,\gamma} \lim_{|P| \to 0} \max_i \|X\|_{p-var,[t_i, t_{i+1}]}^{\gamma([p] + 1) - p} \|X\|_{p-var,[s,t]}^p
\]

which equals 0 by the uniform continuity of the map \((u, v) \to \|X\|_{p-var,[u,v]}^p\) (See [8]). Finally,

\[
\lim_{|P| \to 0} \sum_{i=0}^{n-1} \sum_{i=1}^{\lfloor \gamma \rfloor - m} f^{s(m+l)}(Y_{t_i}) X^l_{t_i, t_{i+1}} \\
= \lim_{|P| \to 0} \sum_{i=0}^{n-1} f^{cm}(Y_{t_{i+1}}) \left( \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{s(m+k)}(Y_{t_i}) X^k_{t_i, t_{i+1}} \right) \\
= f^{cm}(Y_t) + f^{cm}(Y_s).
\]

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