Convergence Analysis of the Grad’s Hermite Approximation to the Boltzmann Equation

Neeraj Sarna*, Jan Giesselmann, Manuel Torrilhon

Center for Computational Engineering & Department of Mathematics
RWTH Aachen University, Germany

Abstract

In [13], Grad proposed a Hermite series expansion for approximating the solution to the Boltzmann equation. But this approximation has been shown to suffer from instabilities which prohibits its convergence analysis and can also lead to non-converging solutions for curved domains. The instabilities mainly arise due to a set of poorly imposed boundary conditions which lead to an unbounded growth in the entropy function. Recently, through the framework proposed in [27], it has been possible to construct an arbitrary order entropy stable Hermite approximation, for the linearised Boltzmann equation. In the present work, we study the global convergence of these stable Hermite approximations, for initial boundary value problems, in a weighted $L^2$ space. Under certain assumptions on the regularity of the kinetic solution, we formulate an explicit convergence rate for the Hermite approximation. The presented convergence rate has been confirmed, with an acceptable accuracy, by numerical tests for several initial boundary value benchmark problems.

1 Introduction

The state of a gas, for all flow regimes, is accurately described by the well known Boltzmann equation which governs the evolution of a phase density functional. The phase density functional lives on a seven dimensional space, and needless to say, this high dimensionality, makes any numerical solution of the Boltzmann equation very computationally expensive. In the recent years, the Direct Simulation Monte Carlo (DSMC) method has proven to be a method of high fidelity for solving the Boltzmann equation; see [3] for details. But for low Mach (or low Knudsen number) regimes, the standard DSMC method proves to be expensive due to the inherent presence of the Monte Carlo noise and special variance reduction techniques are required to obtain acceptable results [17]. This motivates the search for other possible deterministic methods for solving the kinetic equation in this flow regime.

Solving the kinetic equation through a Galerkin approach involves approximating the kinetic solution in a suitably chosen finite dimensional space. The stability estimate for the kinetic equation usually motivates the choice of the approximation space whereas, the stability of the Galerkin approximation influences its convergences properties [28]. For the fully non-linear Boltzmann equation, the Galerkin-Hermite approximation proposed by Grad [13] does not enjoy any known stability estimates. But the present work is concerned with the linearised Boltzmann equation for which the Grad’s approximation, equipped with a set of boundary conditions proposed in [26], is entropy stable.

The convergence behaviour of moment approximations, particularly for boundary value problems, is not very well understood. Part of the lack of understanding originates from the misconception that

*sarna@mathcces.rwth-aachen.de
including more and more moments, in the moment approximation, leads to monotonically decreasing error in some chosen quantity of interest. But such a test case independent monotonic convergence behaviour is clearly not observed in practice [33]. For the first time in [28], the moment approximations were interpreted as a Galerkin method. Since the convergence behaviour of Galerkin methods entirely depends upon the regularity of the solution which in turn depends upon the computational experiment under consideration, such an interpretation is clearly more helpful in understanding the test case dependent convergence behaviour of the moment approximations. A reinterpretation of moment approximations, as a Galerkin method, not only helped the authors in [25, 28] to rigorously analyse their convergence, for the semi-conductor transport equation, but has also lead to the development of the adjoint based moment adaptivity [1].

One of the crucial step of the convergence analysis, will be the construction of a projection operator which provides a projection that satisfies the same boundary conditions as those satisfied by the moment approximation. For the boundary conditions proposed in [23, 26, 27], it will be clear that the projection will not be the same as the orthogonal projection onto the approximation space. The projection operator will allow us to split the approximation error into two parts; one part which will contain the error in the expansion coefficients (or the moments) and the other part which will be the projection error. We will try to bound the error in the expansion coefficients in terms of the error in the projection and the proof of convergence will then rely upon proving that the projection error tends to zero.

The convergence of the projection error requires certain regularity assumptions upon the kinetic solution. For defining the regularity, in the velocity space, we will make use of the Hermite Sobolev spaces, introduced in [32], which can be further related to the standard Sobolev spaces. We will also prove convergence rates for which it will be required to make certain assumptions upon the decay rates of the magnitude of the moments of the kinetic solution [9]. The proposed convergence rates will have explicit dependency upon these assumed decay rates which can be further related to the velocity space regularity of the kinetic solution.

The article has been structured in the following way. In the first section we discuss the properties of the linearised Boltzmann equation, the regularity assumptions upon the kinetic solution and the corresponding Hermite approximation. The second section focuses upon the convergence analysis and contains the main result of the paper. It particularly focuses upon defining a suitable projection operator, deriving the governing equation for the approximation error and developing bounds (and convergence rates) for the projection error. In the last section, we justify the proposed convergence rates with the help of simulations.

## 2 The Linearised Boltzmann Equation

Let \( f : (0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \), \((t, x, \xi) \mapsto \bar{f}(t, x, \xi)\), denote the phase density function, of a mono-atomic gas. We will mainly focus upon a three dimensional velocity space \( d = 3 \) and we will assume the physical space, \( \Omega \subset \mathbb{R}^3 \), to be the half space \( \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \). Strategies to extend the presented framework to domains with \( C^2 \) boundaries will be briefly discussed later in remark 7. The phase density function will be assumed to be normalised such that the density \( (\rho) \), the mean flow velocity \( (\mathbf{v}_i) \) and the temperature in energy units \( (\theta) \), of the gas, can be given as

\[
\rho = \int_{\mathbb{R}^d} \bar{f} d\xi, \quad \rho v_i = \int_{\mathbb{R}^d} \xi_i \bar{f} d\xi, \quad \rho v_i v_j + d\rho \theta = \int_{\mathbb{R}^d} \xi_i \xi_j \bar{f} d\xi.
\]

where the repeated indices \((\xi_i \xi_j \text{ and } v_i v_j)\) imply the use of Einstein’s summation convention. For convenience, we will assume all the quantities to be non-dimensionalised with appropriate powers of some reference density \( \rho_0 \), temperature \( \theta_0 \) and length scale \( L \). For low Mach flows, we can assume \( \bar{f} \) to be a small perturbation of an equilibrium ground state \( f_0 \) i.e. \( \bar{f} = f_0 + \varepsilon f \), where \( \varepsilon \) is some smallness
Parameter

\[ f_0(\xi) = \frac{1}{(\sqrt{2\pi})^d} \exp \left(-\frac{\xi \cdot \xi}{2}\right) \]  

(1)

One can conclude that the mean flow velocity \( (v_0) \) corresponding to \( f_0 \) is zero and therefore, the mean flow velocity resulting from the linearisation will be \( O(\varepsilon) \) which justifies the linearisation for low Mach flow regimes. The governing equation for \( f \) can be obtained by replacing the linearisation for \( \bar{f} \) into the Boltzmann equation and ignoring terms of order higher than \( O(\varepsilon) \) \cite{15}

\[ \mathcal{L}(f(t,x,\xi)) = 0 \quad \text{in} \quad (0,T] \times \Omega \times \mathbb{R}^d, \quad \mathcal{L} := \partial_t + \xi \cdot \partial_x - \frac{Q}{Kn} \]  

(2a)

\[ f(t = 0, x, \xi) = f_i(x, \xi), \quad \text{in} \quad \Omega \times \mathbb{R}^d \]  

(2b)

\[ f(t, x, \xi) = f_{in}(t, x, \xi) \quad \text{where} \quad t \in (0,T], x \in \partial \Omega, \xi_1 < 0. \]  

(2c)

Later, we will discuss the details related to the initial \( (f_i) \) and the boundary conditions \( (f_{in}) \). The operator \( Q \), is the linearisation of the Boltzmann collision operator about \( f_0 \)

\[ Q(f(\xi)) = \int_{\mathbb{R}^d} \int_{S^2} \sigma(|\xi - \xi'|, \kappa) f_0(\xi_0) f_0(\xi) \]  

\[ \left( \frac{f(\xi)}{f(\xi_0)} + \frac{f(\xi')}{f(\xi_0)} - \frac{f(\xi)}{f(\xi_0)} - \frac{f(\xi')}{f(\xi_0)} \right)_d\kappa d\xi, \]  

(3)

The post-collisional velocities \( \xi' \) and \( \xi' \) can be given in terms of the pre-collisional velocities \( \xi \) and \( \xi \)

\[ \xi' = \frac{\xi + \xi_0}{2} + \frac{|\xi - \xi'|}{2} \kappa, \quad \xi' = \frac{\xi + \xi_0}{2} - \frac{|\xi - \xi'|}{2} \kappa. \]  

(4)

The collision kernel, which depends upon the interaction potential between the molecules, is denoted by \( \sigma \geq 0 \) (positive by physical assumptions); the dependency on \( (t,x) \) has been hidden in (3) for brevity. The factor \( Kn \), appearing in (2a), is the well-known Knudsen number which is the result of non-dimensionalisation and is given as

\[ Kn = \frac{\sqrt{\theta_0}}{L\sigma_0\rho_0} \]

where \( \sigma_0 \) is some reference magnitude for the collision kernel \( \sigma \).

For further convenience, we define the Hilbert space

\[ K := L^2(\mathbb{R}^d, f_0^{-1}), \quad ||r||_K = \sqrt{\int_{\mathbb{R}^d} r^2 f_0^{-1} d\xi}, \quad \langle r, g \rangle_K = \int_{\mathbb{R}^d} r g f_0^{-1} d\xi. \]  

(5)

To include the space domain in \( K \), we also define

\[ X := L^2(\Omega; K), \quad ||r||_X = \sqrt{\int_{\Omega} ||r||_K^2 dx}, \quad \langle r, g \rangle_X = \int_{\Omega} \int_{\mathbb{R}^d} r g f_0^{-1} d\xi dx. \]  

(6)

We will assume that \( Q \) is bounded on \( K \). See remark 1 for physical assumptions which justify this boundedness of \( Q \). Using certain symmetry properties of the collision operator \cite{6}, it can be shown that for all types of collision kernels, \( Q \) is negative semi-definite

\[ \langle f, Q(f) \rangle_K \leq 0 \quad \forall \quad f \in \mathcal{D}(Q). \]  

(7)
The equality holds true if and only if \( f \) is the linearisation of the Maxwell-Boltzmann’s distribution about \( f_0 \)

\[
f(t,x,\xi) = f_{\text{lin}}(t,x,\xi) := \left( \dot{\rho}(t,x) + \ddot{v}_i(t,x)\xi + \frac{\ddot{\theta}(t,x)}{2} (\xi^2 - 3) \right) f_0(\xi),
\]

where \( \rho, v_i \) and \( \theta \) represent the deviation of \( \rho, v_i \) and \( \theta \) from their respective ground states up to \( O(\varepsilon) \).

To avoid any discontinuities in the given data, we will consider the initial and the boundary conditions to be continuously differentiable, and the initial conditions to be compatible with the boundary conditions

\[
f_i(.,\xi) \in C^1(\Omega) \quad \forall \; \xi \in \mathbb{R}^d, \quad f_{\text{in}}(.,\xi) \in C^1((0,T] \times \partial \Omega) \quad \forall \; (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^{-} \times \mathbb{R} \times \mathbb{R}
\]

\[
f_i(x,.)(t) \in C^1(\mathbb{R}^d) \quad \forall \; x \in \Omega, \quad f_{\text{in}}(t,x,.)(t) \in C^1(\mathbb{R}^{-} \times \mathbb{R} \times \mathbb{R}) \quad \forall \; x \in \Omega, t \in (0,T].
\]

Note that by assuming \( f_{\text{in}} \) to be independent of the kinetic solution (2c), we have excluded the gas-wall boundary conditions and are rather using the inflow type boundary conditions [6]. This assumption on \( f_{\text{in}} \) has been made merely for the sake of simplicity and can be dropped at the expense of technical complications to include the gas-wall interaction.

The existence and uniqueness of the solution to the kinetic IBVP ((2a), (2c) and (2b)), has been studied in many of the previous works; see [8, 11, 22]. Presently, we will rely upon the kinetic equation having a strong solution which leads to the assumption

**Assumption 1.** Considering the initial (2b) and the boundary conditions (2c) to the kinetic equation (2a) to satisfy (9) and (12), we will assume that there exists a strong solution to the initial boundary value problem ((2a), (2c) and (2b))

\[
f(.,\xi) \in C^1((0,T] \times \Omega), \quad \forall \; \xi \in \mathbb{R}^d.
\]

Multiplying (2a) by \( f^{-1}_0 \), integrating over \( \Omega \times \mathbb{R}^d \), using **Gauss-Theorem** and using (7), we can find an estimate for \( \|f(t,.,.)\|_{X} \)

\[
\frac{d}{dt} \|f(t,.,.)\|^2_{X} \leq -\int_{\partial \Omega} \int_{\mathbb{R}^{d-1}} \xi_1 f_{\text{in}}(t,x,\xi)|f(t,x,\xi)|^2 f^{-1}_0(\xi) d\xi ds, \quad t \in (0,T].
\]

Let us further assume that

\[
f_t \in X \quad \text{and} \quad -\int_{\partial \Omega} \int_{\mathbb{R}^{d-1}} \xi_1 f_{\text{in}}(t,x,\xi)_0 f^{-1}_0 d\xi ds \leq H(t),
\]

where \( H \in L^1((0,T]) \) is some function independent of the kinetic solution. Then, by integrating (11) over \( (0,T] \) and using our assumption (10), one can conclude that there is a unique \( f \in C((0,T];X) \). This will allow us to approximate \( f \) in a finite dimensional subspace of \( K \). For prescribing the boundary conditions to our moment approximation, we assume some additional velocity space regularity on the boundary conditions \( f_{\text{in}}(t,x,.) \in K \).

**Remark 1.** One of the possible ways to ensure the boundedness of \( Q \) on \( K \) is to decompose it as

\[
Q(f) = \bar{Q}(f) - v(\xi) f, \quad v(\xi) = \int_{\mathbb{R}_+^2} \sigma(|\xi - \xi_*|, \kappa) f_0(\xi_*) d\kappa d\xi_*
\]

where \( v(\xi) \) is the collision frequency and \( \bar{Q} \) is an integral operator, the explicit form of which can be found in [7]. Then, by having a bound for \( \bar{Q} \) on \( K \) and an upper bound for \( v(\xi) \), we can bound \( Q \) on \( K \). To have the aforementioned bounds on \( \bar{Q} \) and \( v(\xi) \), we need to make assumptions on the collision kernel \( \sigma \) and the interaction potential.

We will assume that the molecules interact through a purely repulsive inverse power law potential
given by \( \varphi(r) = \varphi_0 r^{1-\gamma} \) with \( \varphi_0 > 0, \gamma > 2 \) and \( r \) being the distance between the two interacting gas molecules. By assuming that the collision kernel corresponding to the inverse power law potentials has an angular cut-off, see \cite{7,14} for details, we can make the collision frequency \( \langle v(\xi) \rangle \) well defined which then allows us to decompose \( Q \) as in \( (13) \). To obtain boundedness of \( Q \) on \( K \) we recall the results presented in \cite{7,14}. Using the angular cut-off assumption, for inverse power law potentials, it can be shown that \( \check{Q} \) is bounded on \( K \) for all \( \gamma > 2 \) \cite{7,14}. For hard potentials \( (\gamma > 5) \), \( v(\xi) \) only has a lower bound and tends to infinity as \( \xi \to \infty \) \cite{14} (see also \cite{6}). Whereas for soft potentials \( (\gamma < 5) \) and Maxwell’s potential \( (\gamma = 5) \), \( v(\xi) \) is bounded from above \cite{14,15} and therefore we will assume \( \gamma \leq 5 \); this finally provides us with a \( Q \) which is bounded on \( K \). Note that for \( \gamma = 5 \), which is the Maxwell’s interaction potential, \( v(\xi) \) is a constant.

**Remark 2.** For the convergence analysis we only require two properties from the collision operator \( Q \) appearing in \( (2a) \) (i) boundedness on \( K \) and (ii) negative semi-definiteness \( (7) \). It can be easily shown that the linearised-BGK collision operator given by \cite{6}

\[
Q_{bgk} = (f_{\neq} - f),
\]

where \( f_{\neq} \) is as defined in \( (8) \), also satisfies both of these properties. Therefore, our analysis will also be valid for the linearised-BGK equation.

### 2.1 Velocity Space Regularity

The coming convergence analysis will rely on certain assumptions upon the velocity space regularity of the kinetic solution. To capture the regularity of an element of \( K \), we make use of the Hermite-Sobolev space, \( W^k_H(\mathbb{R}^d) \), which is the image of \( L^2(\Omega d) \) under the inverse of the Hermite Laplacian operator \( (\Delta_H)^k = (-2\Delta + \frac{1}{2} x_1, x_2)^k \) and is related to \( H^{2k}(\mathbb{R}^d) \) as \( (12) \).

\[
W^k_H(\mathbb{R}^d) \subset H^{2k}(\mathbb{R}^d), \quad k > 0.
\]

An expression for \( \|f(t, ...) f_0^{1/2}(\cdot)\|_{L_1(\Omega d) W^k_H(\mathbb{R}^d)} \) can be easily given by expanding the kinetic solution in terms of the eigenvectors of \( \Delta_H \). To give these eigenvectors, we first define the tensorial Hermite polynomials with the help of the multi-index \( \beta^{(i)} \)

\[
\psi_{\beta^{(i)}}(\xi) := \prod_{p=1}^d H_{\beta^{(i)}_p}(\xi_p), \quad \beta^{(i)} = \left( \beta_1^{(i)}, \ldots, \beta_d^{(i)} \right).
\]

where the Hermite polynomials \( (H_{\xi}) \) enjoy the property of orthogonality and recursion

\[
\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} H_{\xi_1}(\xi) H_{\xi_2}(\xi) \exp \left( -\frac{\xi^2}{2} \right) d\xi = \delta_{j_1} \quad \Rightarrow \quad \int_{\mathbb{R}^d} \psi_{\beta^{(i)}}(\xi) \psi_{\beta^{(i)}}(\xi) d\xi = \prod_{p=1}^d \delta_{\beta^{(i)}_p \beta^{(i)}_p}.
\]

\[
\sqrt{i+1} H_{\xi_{i+1}}(\xi) + \sqrt{i} H_{\xi_{i-1}}(\xi) = \xi H_{\xi_i}(\xi).
\]

It can be shown that \( \psi_{\beta^{(i)}} f_0^{1/2} \) are the eigenfunctions of \( \Delta_H \) with the eigenvalues \( 2\|\beta^{(i)}\|_H + d \) \cite{32}. The quantity \( \|\beta^{(i)}\|_H \) is the so-called degree of the basis function \( \psi_{\beta^{(i)}} \) and to express the kinetic solution as a series expansion, in terms of \( \psi_{\beta^{(i)}} f_0^{1/2} \), we define the \( \|\beta^{(i)}\|_H \)-th order moment of a function in \( K \) as

**Definition 2.1.** Let \( n(m) \) represent the total number of basis functions \( (\psi_{\beta^{(i)}}(\xi)) \) of degree \( m \). In the vector \( \psi_m(\xi) \in \mathbb{R}^{n(m)} \) we collect all those basis functions which have a particular degree \( m \). Using
\[ \psi_m(\xi), \text{ we define } \lambda_m : K \rightarrow \mathbb{R}^n(m) \text{ as} \]
\[ \lambda_m(r) = \int_{\mathbb{R}^d} \psi_m(\xi)r(\xi)d\xi, \quad \forall r \in K. \]

Thus, \( \lambda_m(r) \) represents a vector containing all the \( m \)-th order moments of \( r \). To collect all the moments of \( r \), which are of order less than or equal to \( M \) (\( m \leq M \)), we additionally define
\[ \Psi_M(\xi) = (\psi_0(\xi)', \psi_1(\xi)', \ldots, \psi_M(\xi)'), \quad \Lambda_M(r) = (\lambda_0(r)', \lambda_1(r)', \ldots, \lambda_M(r)')' \]
where \( \Psi_M(\xi) \in \mathbb{R}^{\mathbb{Z}^M}, \ Lambda_M : K \rightarrow \mathbb{R}^{\mathbb{Z}^M} \) with \( \mathbb{Z}^M = \sum_{m=0}^{M} n(m) \).

Using the above definition, and the orthogonality of the basis functions, the kinetic solution can be represented as
\[ f(t, x, \xi) = \sum_{m=0}^{\infty} \lambda_m(f(t, x, \xi)) \cdot \psi_m(\xi) f_0(\xi), \quad \tag{17} \]
\[ \Rightarrow \|f(t, \ldots, t_0, \ldots)\|_{L^2(\Omega; W^k_{\gamma}(\mathbb{R}^d))} := \sum_{m=0}^{\infty}(2m + d')^{2k}\|\lambda_m(f(t, \ldots, t))\|_{L^2(\Omega; \mathbb{R}^n(m))}, \quad t \in (0, T]. \quad \tag{18} \]

It is crucial to note that solely from the Sobolev index in \( W^k_{\gamma}(\mathbb{R}^d) \), we cannot develop a bound for \( \|\lambda_m(f(t, \ldots, t))\|_{L^2(\Omega; \mathbb{R}^n(m))} \) (or infer its decay rate) in terms of the moment order \( m \); see [9] for details and a related example
\[ f(t, \ldots, t_0, \ldots) \in L^2(\Omega; W^k_{\gamma}(\mathbb{R}^d)) \quad \Leftrightarrow \quad \|\lambda_m(f(t, \ldots, t))\|_{L^2(\Omega; \mathbb{R}^n(m))} \leq C m^{-(k+\frac{1}{2})}. \]

Since the decay rates, of the magnitude of the moments, will be needed for formulating explicit convergence rates for the Hermite approximation to the kinetic solution, we additionally define a sub-space of \( X \) which collects all those elements of \( X \) for which the magnitude of the moments decays at a certain rate

**Definition 2.2.** Let \( \tilde{\Omega} \subseteq \Omega \) then, for \( s > 0.5 \), we define a subspace of \( L^2(\tilde{\Omega}; K), X^s_{\tilde{\Omega}} \) as
\[ X^s_{\tilde{\Omega}} = \left\{ r \in L^2(\tilde{\Omega}; K) : \exists C > 0 : \|\lambda_m(r)\|_{L^2(\tilde{\Omega}; \mathbb{R}^n(m))} \leq C m^{-s}, m \in \mathbb{N} \right\}. \quad (19) \]

The relation in (18) permits us to infer the regularity of an element of \( X^s_{\tilde{\Omega}} \)
\[ r \in X^s_{\tilde{\Omega}} \Rightarrow r f_0^{-\frac{1}{2}} \in L^2(\Omega; W^k_{\gamma}(\mathbb{R}^d)) \subset L^2(\Omega; H^{2k}(\mathbb{R}^d)), \quad \forall k^{*} < s - \frac{1}{2}. \tag{20} \]

We will now make certain assumption upon the velocity space regularity of the kinetic solution (and its derivatives)

**Assumption 2.** We will assume that the kinetic solution to the initial boundary value problem ((2a), (2c) and (2b)) is such that there exist numbers \( s^e_{\xi}, s^{\text{io}}_{\xi} \) and \( s^{e/o}_{\xi} \) such that
\[ (\partial_\xi f)^e(t, \ldots, t) \in X^s_{\tilde{\Omega}} , \quad (\partial_\xi f)^o(t, \ldots, t) \in X^s_{\tilde{\Omega}} , \quad (\partial_\xi f)^e(t, \ldots, t) \in X^s_{\tilde{\Omega}} , \quad (\partial_\xi f)^o(t, \ldots, t) \in X^s_{\tilde{\Omega}} \]
\[ f^e(t, \ldots, t) \in X^s_{\tilde{\Omega}} , \quad f^o(t, \ldots, t) \in X^s_{\tilde{\Omega}}, \quad t \in (0, T], \quad i \in \{1, 2, 3\} \tag{21} \]
where \( .^e \) and \( .^o \) denote the even and the odd parts, of the various quantities, defined with respect to \( \xi \)
\[ f^o(\xi_1, \xi_2, \xi_3) = \frac{1}{2} (f(\xi_1, \xi_2, \xi_3) - f(-\xi_1, \xi_2, \xi_3)), \quad f^e(\xi_1, \xi_2, \xi_3) = \frac{1}{2} (f(\xi_1, \xi_2, \xi_3) + f(-\xi_1, \xi_2, \xi_3)). \]

\(^1\) throughout the article, \( C \) will denote a generic positive constant
Note that, merely for the sake of simplicity, we have assumed the same degree of regularity for the spatial derivatives of all the directions. The reason for allowing different regularity for the even and odd parts will become clear from the convergence analysis which follows. Whereas, the reason for choosing the projection operator. From the assumption in (23a) and with the help of (20), we can infer

\[ \langle \xi_t^e f^e(x, \xi), \eta^e \rangle = \langle \xi_t^o f^o(x, \xi), \eta^o \rangle = 0, \quad \forall \eta^e, \eta^o \in L^2(\Omega; W^2_H(\mathbb{R}^d)), \]

\[ \langle \xi_t^o f^o(x, \xi), \eta^e \rangle = \langle \xi_t^e f^e(x, \xi), \eta^o \rangle = 0, \quad \forall \eta^e, \eta^o \in L^2(\Omega; W^2_H(\mathbb{R}^d)). \]

(22)

\[ f^e(x, \xi) \in L^2(\Omega; W^2_H(\mathbb{R}^d)), \quad f^o(x, \xi) \in L^2(\Omega; W^2_H(\mathbb{R}^d)). \]

\[ \forall \xi_t^e < s_t^e, \quad \xi_t^o < s_t^o, \quad t \in (0, T]. \]

2.2 Hermite Approximation

From the assumption made upon the kinetic solution, (10), it is straightforward to conclude that, for all \( v \in K \), the kinetic solution satisfies

\[ \langle v(\cdot), L(f(t, x, \cdot)) \rangle_K = 0, \quad \forall (t, x) \in (0, T] \times \Omega \] (23a)

\[ \langle v(\cdot), f(t = 0, x, \cdot) \rangle_K = \langle v(\cdot), f_0(x, \cdot) \rangle_K, \quad \forall x \in \Omega \] (23b)

\[ \int_{\mathbb{R}^d} \int_{\xi_1 < 0} v(\xi) f(t, x, \xi) d\xi = \int_{\mathbb{R}^d} \int_{\xi_1 < 0} v(\xi) f_0(t, x, \xi) d\xi, \quad \forall (t, x) \in (0, T] \times \Omega \] (23c)

where the Hilbert space \( K \) is as defined in (5) and \( L \) is the integro-differential operator given in (2a).

Following a standard Galerkin technique, we can approximate the solution to the variational form in (23a), in a suitable finite dimensional subspace of \( K \), through

\[ \text{Find } f_M \in Y_M := \text{span}\{ \psi_{\beta} / f_0 \} / \| \psi_{\beta} / f_0 \|_1 \leq M \text{ such that} \]

\[ \langle v(\cdot), L(f_M(t, x, \cdot)) \rangle_K = 0, \quad \forall v \in Y_M \subset K, \quad \forall (t, x) \in (0, T] \times \Omega \] (24a)

\[ \langle v(\cdot), f_M(t = 0, x, \cdot) \rangle_K = \langle v(\cdot), f_0(x, \cdot) \rangle_K, \quad \forall v \in Y_M \subset K, \quad \forall x \in \Omega. \] (24b)

which can be looked upon as a mesh-less spectral Galerkin method to approximate the kinetic solution in the velocity space. The boundary conditions for \( f_M \) will be discussed later. As is clear from the definition of the basis functions itself, that the collision invariants (scaled with \( f_0 \)) are included in \( Y_M \) for any \( M \geq 2 \). Thus we will always consider \( M \geq 2 \) which ensures that mass, momentum and energy conservation is included in (24a).

Remark 3. The choice to approximate \( f \) through \( f_M \) is the same as that proposed by Grad [13]. By choosing \( v \) to be the different basis functions of \( Y_M \), in (24a), we can recover the linearised Grad’s moment equations; though, one can only recover the so-called full moment systems (Grad’s-10, 20 equations etc.) through (24a) [4, 33]. Choosing \( v = f_M \) in (24a), we can retrieve an entropy law for the moment approximation (\( f_M \)) which then proves the symmetric hyperbolicity of the resulting moment system [2, 16, 18].

Even and Odd basis: To prescribe a set of boundary conditions to \( f_M \), we will need to split it into an even and an odd part, in the velocity space. The even (and the odd) part has to be defined with respect to the velocity direction normal to the boundary of the domain. For the present choice of the domain, \( \Omega = \mathbb{R}^- \times \mathbb{R} \times \mathbb{R} \), the wall normal points in the \( x_1 \) direction. Therefore, we first define those
basis functions, of $K$, which are even and odd with respect to $\xi_1$ and then express the even and the odd parts of $f$ in terms of these basis functions.

**Definition 2.3.** Let $n_o(m)$ and $n_e(m)$ denote the total number of tensorial Hermite polynomials in $\psi_m(\xi)$ which are odd and even, with respect to $\xi_1$, respectively. Correspondingly, let $\psi_m^o(\xi) \in \mathbb{R}^{n_o(m)}$ and $\psi_m^e(\xi) \in \mathbb{R}^{n_e(m)}$ represent vectors containing those basis functions, out of $\psi_m(\xi)$, which are odd and even, with respect to $\xi_1$, respectively. Then, we define $\lambda_m^o : K \to \mathbb{R}^{n_o(m)}$ and $\lambda_m^e : K \to \mathbb{R}^{n_e(m)}$ as

$$\lambda_m^o(r) = \int_{\mathbb{R}^d} \psi_m^o(\xi)r(\xi)d\xi, \quad \lambda_m^e(r) = \int_{\mathbb{R}^d} \psi_m^e(\xi)r(\xi)d\xi, \quad \forall r \in K.$$  

To collect all the odd and the even moments of $r$ which have a degree less than or equal to $M$ ($m \leq M$), we define

$$\Psi_M^o(\xi) = (\psi_1^o(\xi)', \psi_2^o(\xi)', \ldots, \psi_M^o(\xi)')', \quad \Psi_M^e(\xi) = (\psi_1^e(\xi)', \psi_2^e(\xi)', \ldots, \psi_M^e(\xi)')',$$  

$\Lambda_M^o(r) = (\lambda_1^o(r)', \lambda_2^o(r)', \ldots, \lambda_M^o(r)')', \quad \Lambda_M^e(r) = (\lambda_1^e(r)', \lambda_2^e(r)', \ldots, \lambda_M^e(r)')'$

where $\Lambda_M^o : K \to \mathbb{R}^{M}, \Lambda_M^e : K \to \mathbb{R}^{M}, \Psi_M^o(\xi) \in \mathbb{R}^{M}$ and $\Psi_M^e(\xi) \in \mathbb{R}^{M}$. We represent the total number of odd and even moments of degree less than or equal to $M$ through $\Xi^o_M = \sum_{i=0}^{M} n_o(i)$ and $\Xi^e_M = \sum_{i=0}^{M} n_e(i)$ respectively.

From the definition of the approximation space $\mathcal{Y}_M \subset K$, in (24a), it is clear that we define it with the $l^1$ norm of the multi-index $\beta^{(i)}$. The approximation space, $\mathcal{Y}_M$, has the property that the total number of even variables are strictly greater than the total number of odd variables i.e. $\Xi^e_M > \Xi^o_M$; see appendix-B. Since the difference in the total number of odd and even basis functions will show up in the convergence analysis, we briefly discuss the motivation behind choosing the approximation space in this particular way. Our choice is motivated by a favourable rotational property which can be stated as [35]

$$\psi_m(O\xi) = O_m \psi_m(\xi)$$  

(25)

where $O \in \mathbb{R}^{d \times d}$ and $O_m \in \mathbb{R}^{n(m) \times n(m)}$ are orthogonal matrices with $O_m$ depending solely upon $m$ and $O$. The vector $\psi_m(\xi)$ is as given in definition 2.1. Thanks to (25), the numerical implementation overhead required to solve for $f_M$ significantly reduces since no special treatment is required for curved boundaries or unstructured meshes; see [34] for details. Furthermore, it can also be helpful in extending the analysis done for one physical space dimension to multi-dimensions and therefore this particular choice of the basis functions is very much favourable.

**Remark 4.** The approximation space can also be defined with the $l^\infty$ norm of the multi-index [25]

$$\mathcal{Y}_M^\infty := \text{span} \{ \psi_{\beta^{(i)} f_0} \} ||\beta^{(i)}||_{\infty} \leq M.$$  

Clearly, the above choice of the approximation space is not the same as $\mathcal{Y}_M$ given in (24a). The above approximation is similar to solving the linearised BE on a velocity grid with the grid points being the Gauss-Hermite quadrature points. But, if we define a similar vector as $\psi_m(\xi)$ for the basis functions $\psi_{\beta^{(i)}}$ in $\mathcal{Y}_M^\infty$. Then $\psi_m(\xi)$ does not satisfy the rotational property given in (25); see [25] for more details.

**Boundary Conditions and Stability:** The expressions for the boundary conditions, to be discussed next, become much simpler by defining the following matrices

**Definition 2.4.** We define the matrix $A_{\Psi}^{(p,q)} \in \mathbb{R}^{\Xi^e_M \times \Xi^e_M}$ which captures the weighted integral of the outer product of the two vectors $\Psi_{p}^e$ and $\Psi_{q}^e$ multiplied by $\xi_1$ on the full velocity space

$$A_{\Psi}^{(p,q)} = \int_{\mathbb{R}^d} \Psi_{p}^e(\xi_1) \Psi_{q}^e(\xi_1)' f_0 d\xi.$$
Similarly we define $A_{\psi}^{(p,q)} \in \mathbb{R}^{n,q} \times \mathbb{Z}^p$ which captures the weighted integral of the outer product of the two vectors $\Psi_p$ and $\Psi_q$ multiplied by $\xi_1$ on the full velocity space

$$A_{\psi}^{(p,q)} = \int_{\mathbb{R}^d} \Psi_p^{\xi_1} (\psi_q^{\xi_1})' f_0 d\xi_1.$$ 

Recall that both $\Psi_q^{e}(\xi)$ and $\psi_q^{e}(\xi)$ are vectors containing even basis functions. However, $\Psi_q^{e}(\xi)$ contains all the basis functions with degree less than equal to $q$, whereas, $\psi_q^{e}(\xi)$ only contains the basis function of degree equal to $q$.

**Definition 2.5.** We define the matrix $B_{\psi}^{(p,q)} \in \mathbb{R}^{n,q} \times \mathbb{Z}^p$ which captures the weighted integral of the outer product of the two vectors $\Psi_p^{o}(\xi)$ and $\Psi_q^{e}(\xi)$ on the half velocity space

$$B_{\psi}^{(p,q)} = 2 \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_p^{e}(\psi_q^{e})' f_0 d\xi_1.$$ 

Similarly we define $B_{\psi}^{(p,q)} \in \mathbb{R}^{n,q} \times \mathbb{Z}^p$ which captures the weighted integral of the outer product of the two vectors $\Psi_p^{o}(\xi)$ and $\psi_q^{e}(\xi)$ on the half velocity space

$$B_{\psi}^{(p,q)} = 2 \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_p^{e}(\psi_q^{e})' f_0 d\xi_1.$$ 

**Remark 5.** From the above definitions, we can deduce that $A_{\psi}^{(p,q)}$ (or $B_{\psi}^{(p,q)}$) is nothing but a set of columns of $A_{\psi}^{(p,q)}$ (or $B_{\psi}^{(p,q)}$).

$$B_{\psi}^{(p,q)} = \left( B_{\psi}^{(p,1)}, B_{\psi}^{(p,2)} \ldots B_{\psi}^{(p,q)} \right), \quad A_{\psi}^{(p,q)} = \left( A_{\psi}^{(p,1)}, A_{\psi}^{(p,2)} \ldots A_{\psi}^{(p,q)} \right)$$

A detailed discussion related to the derivation of the boundary conditions, which follow, can be found in [5, 13, 27]; here we only discuss the results briefly. We summarise the methodology used to derive the boundary conditions

(i) From the work done in [5, 13, 27] it is well established that we need to prescribe a set of boundary conditions to the odd moments ($A_{\psi}^{o}(f_M(t,x))$). In (23c), if we restrict ourselves to those $v$ which are odd in $\xi_1$ then we can derive a relation for $A_{\psi}^{o}(f_M(t,x))$

$$\int_{\mathbb{R}^{d-1}} \int_{\xi_1 < 0} v(\xi) f(t,x,\xi) d\xi = \int_{\mathbb{R}^{d-1}} \int_{\xi_1 < 0} v(\xi) f_{in}(t,x,\xi) d\xi, \quad \forall v \in \mathcal{Y}_{\psi}^{o}, (t,x) \in (0,T) \times \partial \Omega$$

(26)

where $\mathcal{Y}_{\psi}^{o} := \{ r \in \mathcal{Y}_{\psi} : r(\xi_1, \xi_2, \xi_2) = -r(-\xi_1, \xi_2, \xi_2) \}.$

(ii) By replacing $f$ by $f_M$ in (26), we arrive at a preliminary relation which will provide us with the final set of boundary conditions

$$\int_{\mathbb{R}^{d-1}} \int_{\xi_1 < 0} v(\xi) f_M(t,x,\xi) d\xi = \int_{\mathbb{R}^{d-1}} \int_{\xi_1 < 0} v(\xi) f_{in}(t,x,\xi) d\xi, \quad \forall v \in \mathcal{Y}_{\psi}^{o}, (t,x) \in (0,T) \times \partial \Omega.$$ 

(27)

For prescribing the boundary conditions, Grad[13] used the above relation but it leads to an unbounded growth of $\|f_M(t,\ldots)\|_X$. This is undesirable considering that the true solution belongs to $C((0,T];X)$; see [23, 26]. By suitably modifying the relation for $A_{\psi}^{o}(f_M(t,x))$, obtained through (27), with a term which will only depend upon the highest order even moment of $f_M$ ($A_{\psi}^{e}(f_M(t,x))$) we will arrive at a set of boundary conditions which will lead to the stability of $f_M$. 

2.2 Hermite Approximation
The variational form in (26) can also be expressed in terms of an operator defined over $l^2$, with the help of the following result

**Lemma 2.1.** For every $r \in K$, it holds

$$\int_{\mathbb{R}^d} \Psi_M^0 r^o d\xi = 2 \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_M^0 r^e d\xi + \mathcal{G}(r)$$

(28)

or equivalently

$$\Lambda_M^o(r) = \lim_{q \to \infty} B_q^{(M,q)} \Lambda_q^o(r) + \mathcal{G}(r)$$

(29)

where $r^o$ and $r^e$ are the odd and even parts of $r$, with respect to $\xi_1$, respectively, and $\mathcal{G} : K \to \mathbb{R}^{2^M}$ is given as

$$\mathcal{G}(r) = 2 \int_{\mathbb{R}^{d-1}} \int_{\xi_1 < 0} \Psi_M^o r(\xi) d\xi.$$

In (29), $\lim_{q \to \infty} B_q^{(M,q)}$ can be looked upon as an operator defined on $l^2$ with its range being in $\mathbb{R}^{2^M}$.

**Proof.** See appendix-A.

As a result of lemma 2.1, the variational expression in (26) is equivalent to

$$\Lambda_M^o(f(t,x,)) = \lim_{q \to \infty} B_q^{(M,q)} \Lambda_q^o(f(t,x,)) + \mathcal{G}(f_\in(t,x,)) \text{ in } (0,T] \times \partial \Omega$$

(30)

where $B_q^{(M,q)}$ is as given in definition 2.5 and $\mathcal{G}(\cdot)$ is the functional defined in lemma 2.1. Ignoring the contribution from all $\lambda_q^o(f)$, with $q > M$, we obtain a relation for $\Lambda_q^{(M)}(f_M)$

$$\Lambda_q^{(M)}(f_M(t,x,)) = B_q^{(M,q)} \Lambda_q^o(f_M(t,x,)) + \mathcal{G}(f_\in(t,x,)), \text{ in } (0,T] \times \partial \Omega$$

(31)

which is nothing but the relation in (27). To state the boundary conditions, which lead to stability, we make use of the matrix $A_q^{(M,M)}$ given in definition 2.4. We can split $A_q^{(M,M)}$ and $B_q^{(M,M)}$ as

$$A_q^{(M,M)} = \left( A_q^{(M,M-1)}, A_q^{(M,M)} \right), \quad B_q^{(M,M)} = \left( B_q^{(M,M-1)}, B_q^{(M,M)} \right)$$

where $A_q^{(M,M-1)}$ and $B_q^{(M,M-1)}$ will both be square matrices because $\Xi_e^{-1} = \Xi_o^M$; see appendix-B. The boundary conditions, which lead to stability, can now be given as

$$\Lambda_M^o(f_M(t,x,)) = R^{(M)} A_q^{(M,M)} \Lambda_M^o(f_M(t,x,)) + \mathcal{G}(f_\in(t,x,)), \quad R^{(M)} = B_q^{(M,M-1)} \left( A_q^{(M,M-1)} \right)^{-1}$$

(32)

where the square matrix $R^{(M)} \in \mathbb{R}^{2^M \times 2^M}$ can be proven to be s.p.d [26] and is the so-called *Onsager matrix* [23]. The invertibility of $A_q^{(M,M-1)}$ follows from the structure of $A_q^{(M,M)}$, see appendix-C for a detailed explanation.

Since we have made the assumption of continuous differentiability on the kinetic initial and the kinetic boundary conditions (9), the initial and the boundary conditions for $f_M$ will also be continuously differentiable; $f_i(\cdot,\xi) \in C^1(\Omega)$ and $\mathcal{G}(f_\in) \in C^1((0,T] \times \partial \Omega)$. Hence, due to the linearity of the moment system resulting from (24a), a strong solution to the initial boundary value problem ((24a), (24b) and (32)) can be ensured by using compatible initial and boundary conditions [24]. For the coming discussion, we will assume that the moment system has a strong solution.
While coming up with error bounds for the Hermite approximation, we will exploit the stability estimate of the highest order even moments of $f$ in Remark 6. We outline the convergence analysis, to be carried out in the present section, in a few standard steps

(i) **Projection Operator**: we will define a projection operator $\Pi_M : K \rightarrow Y_M$, where $Y_M$ is as defined in (24b), such that the projection satisfies the same boundary conditions as those satisfied by $f_M$ in (32). The stability of the Hermite approximation (33) can be exploited with such a projection operator.

(ii) **Decompose the error**: we decompose the approximation error into two parts

$$E_M = f - f_M = \underbrace{f - \Pi_M f}_{e_M} + \underbrace{\Pi_M f - f_M}_{P_M}$$

where $e_M \in Y_M$ is the error in the moments (or the expansion coefficients) and $P_M$ is the error in the projection.

(iii) **Bound for the projection error**: we develop a bound for $\|P_M(t,..)\|_X$, in terms of the moments of the kinetic solution. Also, the velocity space regularity assumption in (21) will provide us with a convergence rate for $\|P_M(t,..)\|_X \rightarrow 0$, as $M \rightarrow \infty$.

(iv) **Bound for the error in moments**: we derive a governing equation for $e_M$ in terms of $P_M$. This equation will be an inhomogeneous version of the variational form in (24a) with the inhomogeneity depending upon $L(P_M)$. A bound and the convergence rate for $\|e_M(t,..)\|_X$ will follow from a bound for $\|L(P_M(t,..))\|_X$ and the convergence rate for $\|P_M(t,..)\|_X$, respectively. The rate
at which \( \| E_M(t, \ldots) \|_{\mathbb{X}} \to 0 \), as \( M \to \infty \), will then be a consequence of the convergence rates for \( \| e_M(t, \ldots) \|_{\mathbb{X}}, \| P_M(t, \ldots) \|_{\mathbb{X}} \to 0 \).

Defining a suitable projection operator is the main hurdle in analysing the convergence of Grad’s Hermite approximation (24a). The eigenspectrum, of the flux matrix corresponding to the moment system resulting from (24a), allows us to only prescribe a set of boundary conditions to the odd moments [13]. Given the finite value of \( M \) in the Hermite approximation (24a), while formulating the boundary conditions (32), we ignore the contribution from moments which have an order higher than \( M \). As discussed below, this prohibits us from using the orthogonal projection operator. We note that a similar structure, for the flux matrix, also appears for the \( P_N \) equations of radiation transport and for the newly proposed Petrov-Galerkin scheme [10] which should require a similar set of boundary conditions. Therefore, we expect that the same projection operator can also be used to show convergence of those schemes.

### 3.1 The Projection Operator

We require the projection to be such that its odd moments (\( \Lambda_{M}^o(\cdot) \)) satisfy the same boundary conditions as those satisfied by \( f_M \) in (32). Else, as will be clear from the coming analysis, we will obtain error terms along the boundary in the equation for \( \| e_M(t, \ldots) \|_{\mathbb{X}} \), which cannot be bounded appropriately. We motivate the formalism of our projection operator based upon the observation that to derive a relation for \( \Lambda_{M}^o(f(t, x, \ldots)) \) in the interior of the domain (\( \Omega \)), we can simply replace \( \mathcal{G}(f_{in}(t, x, \ldots)) \) by \( \mathcal{G}(f(t, x, \ldots)) \) in (30)

\[
\Lambda_{M}^o(f(t, x, \ldots)) = \lim_{q \to \infty} B_{\Psi}^{(M,q)} \Lambda_{q}^o(f(t, x, \ldots)) + \mathcal{G}(f(t, x, \ldots)) \quad \forall (t, x) \in (0, T] \times \Omega.
\]

Therefore, if we define a projection, the odd moments of which satisfy the boundary relation (32) then, the projection can be extended to the interior of the domain by simply replacing \( \mathcal{G}(f_{in}(t, x, \ldots)) \) by \( \mathcal{G}(f(t, x, \ldots)) \). We finally define the projection operator as

**Definition 3.1.** Let \( r \in K \), then we define \( \hat{\Pi}_M : K \to \mathcal{Y}_M \) as the projection operator which is such that \( \hat{\Pi}_M f \) satisfies the same boundary conditions as those satisfied by \( f_M \)

\[
\hat{\Pi}_M r = \hat{\Lambda}_M(r) \cdot \Psi_M^o(\hat{\xi}) f_0(\hat{\xi}) + \Lambda_M(r) \cdot \Psi_M^o(\hat{\xi}) f_0(\hat{\xi}) \quad \text{with} \quad \hat{\Lambda}_M(r) = R^{(M)} \Lambda_q^{(M,M)} \Lambda_M^o(r) + \mathcal{G}(r)
\]

where \( R^{(M)} \), \( \mathcal{G} \) and \( \Lambda_q^{(M,M)} \) are as defined in (32), lemma 2.1 and definition 2.4 respectively. Similarly, we can define the orthogonal projection operator \( \Pi_M : K \to \mathcal{Y}_M \) as

\[
\Pi_M r = \Lambda_M^o(r) \cdot \Psi_M^o(\xi) f_0(\xi) + \Lambda_M^o(r) \cdot \Psi_M^o(\xi) f_0(\xi).
\]

By the above definitions, we can infer the reason behind not using the orthogonal projection operator for splitting the error in (35). The relation satisfied by \( \Lambda_{M}^o(f(t, x, \ldots)) \), at the boundary, is given by (30) and contains contribution from even moments with degree higher than \( M \) (\( \Lambda_q^o(f(t, x, \ldots)), q > M \)). As a result, the odd moments of \( \Pi_M f \), which are the same as \( \Lambda_M^o(f(t, x, \ldots)) \), do not satisfy the same boundary conditions as those satisfied by the odd moments of \( f_M \) in (32).

**Remark 7.** We have defined the projection operator (\( \hat{\Pi}_M \)) with respect to the velocity direction perpendicular to the boundary (\( \hat{\xi}_1 \)); this is implicit in the definition of \( \mathcal{G}(\cdot) \) and \( \Lambda_q^{(M,M)} \). Owing to the chosen physical space (\( \Omega = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \)), the direction normal to the wall remains the same at all boundary points which makes the definition of \( \hat{\Pi}_M \) straightforward. But, of course, for a general domain, the normal direction at every boundary point can vary. We briefly discuss a methodology to construct the projection operator for a domain with \( C^2 \) boundary. Consider \( \Omega \) to be some arbitrary domain with a \( C^2 \)
boundary. Then, for every point \( x_0 \in \partial \Omega \) we can define a line which passes through \( x_0 \) and points in the opposite direction of the normal \( (n(x_0)) \) perpendicular to the boundary

\[
L_{x_0} := \{x \in \Omega : x - x_0 = \alpha n(x_0), \alpha \in \mathbb{R}^- \}.
\]

Since the boundary is \( C^2 \), there exists some \( \delta > 0 \) such that

\[
\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial \Omega) \geq \delta \}
\]

has the property that no two lines \( L_{x_0} \) and \( L_{x_1} \), for any \( x_0, x_1 \in \partial \Omega \), will intersect within \( \Omega_\delta \).

Inside \( \Omega_\delta \), we use orthogonal projection \( \Pi_M \). Whereas, outside of \( \Omega_\delta \), we proceed as follows. For every \( x \in \Omega_\delta \), there exists a unique \( x_0 \) such that \( x \in L_{x_0} \). Let \( \Pi_{M_0} \) denote the projection operator accounting for the boundary conditions at \( x_0 \). Then at \( x \) we define the projection operator to be the linear combination of the projection operator which satisfies the boundary conditions, \( \Pi_{M_0} \), and the orthogonal projection operator \( \Pi_M \)

\[
\hat{\Pi}_M := \left(1 - \frac{|x-x_0|}{\delta}\right) \Pi_{M_0} + \frac{|x-x_0|}{\delta} \Pi_M.
\]

In this way, \( x \mapsto \hat{\Pi}_M(f_M(\cdot, x, \cdot)) \) satisfies the desired boundary conditions and is \( C^1 \).

**Remark 8.** Our choice to prescribe the initial conditions to \( f_M \) through (24b) is not unique. Rather, we can use any projection operator, to prescribe the initial conditions, which introduces an error that decays faster (or at least at the same rate) than the approximation error \( (E_M) \). From the upcoming convergence analysis it will be clear that both \( \hat{\Pi}_M \) and \( \Pi_M \) satisfy this criteria. For the sake of simplifying the coming analysis, instead of (24b), we will rather use the initial conditions

\[
f_M(t = 0, x, \xi) = \hat{\Pi}_M f(t, x, \xi).
\]

as a result of which the value of the error \( (e_M) \), defined in (35), will be zero at \( t = 0 \). Note that for practical applications, the implementation of the projection operator \( \hat{\Pi}_M \) can be cumbersome and therefore, one might want to prescribe the initial conditions through \( \Pi_M \) or some other, easier to implement, interpolation.

**Remark 9.** Along the boundary \( \mathcal{G}(f(t, x, \cdot)) = \mathcal{G}(f_M(t, x, \cdot)) \), which can be explicitly computed and therefore, by only knowing the value of the even moments \( (\Lambda_M^p(f(t, x, \cdot)) \) ) , \( \hat{\Pi}_M f \) can be fully determined. But, this is not true for the interior of the domain since \( \mathcal{G}(f(t, x, \cdot)) \) cannot be explicitly computed without knowing the complete kinetic solution.

**Remark 10.** Due to our definition of the projection operator \( \hat{\Pi}_M \), the error in projection \( (P_M) \), defined in (35), will not be orthogonal to \( \mathcal{G}_M \). This is in contrast to the analysis done in [9, 28].

### 3.2 Main Result

**Theorem 3.1.** The error in the moment approximation, \( E_M = f - f_M \), for all \( t \in (0, T] \), can be bounded as

\[
\|E_M(t, \cdot, \cdot)\|_X \leq \|f(t, \cdot, \cdot) - \hat{\Pi}_M f(t, \cdot, \cdot)\|_X + \int_0^t (A_1(\tau) + A_2(\tau) + A_3(\tau)) d\tau
\]  

(39)
We devote the next few pages to proving to be smaller and smaller, one needs to conduct an asymptotic analysis for the kinetic solution. For Remark 11. To understand how the constant $C_{Kn}$, in (41), behaves as the Knudsen number is chosen to be smaller and smaller, one needs to conduct an asymptotic analysis for the kinetic solution. For
3.3 Error Equation

To derive a bound for the approximation error ($\|E_M(t,\ldots)\|_X$) we first derive a bound for the error in the expansion coefficients ($\|e_M(t,\ldots)\|_X$) given in (35). For $\Pi_M f$, using the projection of the linearised BE (23a), we can easily find

$$\langle v(\cdot), \mathcal{L}(\Pi_M f(t,\cdot)) \rangle_K = \langle v(\cdot), \mathcal{L}(\hat{\Pi}_M f(t,\cdot) - f(t,\cdot)) \rangle_K \quad \forall v \in \mathcal{Y}_M, \quad \forall (t,x) \in (0,T] \times \Omega$$

where $\mathcal{Y}_M$ is as defined in (24b). Subtracting the above relation from (24a), and using the linearity of $\mathcal{L}$, we find

$$\langle v(\cdot), \mathcal{L}(e_M(t,\cdot)) \rangle_K = \langle v(\cdot), \mathcal{L}(f(t,\cdot) - \hat{\Pi}_M f(t,\cdot)) \rangle_K \quad \forall v \in \mathcal{Y}_M, \quad \forall (t,x) \in (0,T] \times \Omega$$

which can be looked upon as an inhomogeneous form of the variational form in (24a). From our choice of the initial conditions (38), it follows

$$e_M(t = 0,\ldots) = 0 \quad \Rightarrow \quad \Lambda_M(e_M(t = 0,\ldots)) = 0 \quad \forall x \in \Omega$$

where $\Lambda_M$ contains all the moments of order less than or equal to $M$ and is as given in definition 2.1. Due to the definition of the projection operator, the odd moments of $e_M$ satisfy the homogeneous version of the boundary conditions satisfied by the moment approximation (32)

$$\Lambda_M(e_M(t,\ldots)) = R^{(M)} A^{(M,M)}_\Psi \Lambda_M(e_M(t,\ldots)) \quad \forall (t,x) \in (0,T] \times \partial \Omega$$

which shows that when $f_{in}$, appearing in (2c), is independent of the kinetic solution, the boundary inhomogeneity ($\mathcal{Y}(f_{in}(t,\ldots))$) does not play a role in the error equation thus simplifying the analysis. Choosing $v = e_M$ in (42), integrating over $\Omega$, using Gauss-Theorem and the boundary conditions for $e_M$, we find

$$\|e_M(t,\ldots)\|_X \leq \int_\Omega \langle e_M(t,\ldots), \mathcal{L}(f(t,\ldots) - \hat{\Pi}_M f(t,\ldots)) \rangle_K d\Omega \quad \forall t \in (0,T] \quad (43)$$

which is a result of the stability of the variation form in (24a). Using the definition of $\mathcal{L}$ from (2a) and the fact that $\langle v, w \rangle_K = \langle v, \Pi_M w \rangle_K, \forall (v,w) \in \mathcal{Y}_M \times K$, we can spell out the expression on the right of the above inequality

$$\int_\Omega \langle e_M, \mathcal{L}(f - \hat{\Pi}_M f) \rangle_K d\Omega = \int_\Omega \langle e_M, (\Pi_M \partial^t f - \hat{\Pi}_M \partial^t f) \rangle_K d\Omega = \int_\Omega \langle e_M, (\Pi_M \partial^t f - \hat{\Pi}_M \partial^t f) + Q(f - \hat{\Pi}_M f) \rangle_K d\Omega.$$
\[
\sum_{i=1}^{d} \int_{\Omega} \langle e_M, \Pi_M (\xi_i (\partial_x f - \hat{\Pi}_M \partial_x f)) \rangle_K \, dx \leq \| e_M \|_X \sum_{i=1}^{d} \| \Pi_M (\xi_i (\partial_x f - \hat{\Pi}_M \partial_x f)) \|_X
\]

(44)

\[
\int_{\Omega} \langle e_M, Q(f - \hat{\Pi}_M f) \rangle_K \, dx \leq \| e_M \|_X \| Q \| \| f - \hat{\Pi}_M f \|_X
\]

We can further simplify \(\sum_{i=1}^{d} \| \Pi_M (\xi_i (\partial_x f - \hat{\Pi}_M \partial_x f)) \|_X\) by first noting the triviality

\[
\sum_{i=1}^{d} \| \Pi_M (\xi_i (\partial_x f - \hat{\Pi}_M \partial_x f)) \|_X \leq \sum_{i=1}^{d} \left( \| \Pi_M (\xi_i (\Pi_M \partial_x f - \hat{\Pi}_M \partial_x f)) \|_X + \| \Pi_M (\xi_i (\partial_x f - \Pi_M \partial_x f)) \|_X \right)
\]

(45)

After a straightforward but lengthy computation, it can be shown that (page-80, [28])

\[
\sum_{i=1}^{d} \| \Pi_M (\xi_i (\Pi_M \partial_x f - \hat{\Pi}_M \partial_x f)) \|_X \leq \| A_{M,M}^{(M,M)} \|_2 \sum_{i=1}^{d} \| (\Pi_M \partial_x f - \hat{\Pi}_M \partial_x f) \|_X
\]

(46)

Also, using the orthogonality and recursion of the Hermite polynomials we find

\[
\sum_{i=1}^{d} \| \Pi_M (\xi_i (\partial_x f - \Pi_M \partial_x f)) \|_X = \sum_{i=1}^{d} \| \Pi_M (\xi_i (\lambda_{M+1} (\partial_x f) \cdot \psi_{M+1}) f_0) \|_X
\]

(47)

Substituting (44), (45), (46) and (47) into (43), integrating over \((0,t]\), with \(t \leq T\), and using triangular inequality, for a non-zero \(e_M\), we find

\[
\| E_M(t, \ldots) \|_X \leq \| f(t, \ldots) - \hat{\Pi}_M f(t, \ldots) \|_X
\]

\[
+ \int_{0}^{t} \left( \| \Pi_M \partial_x f(t, \ldots) - \hat{\Pi}_M \partial_x f(t, \ldots) \|_X + \| A_{M,M}^{(M,M)} \|_2 \sum_{i=1}^{d} \| \Pi_M \partial_x f(t, \ldots) - \hat{\Pi}_M \partial_x f(t, \ldots) \|_X
\]

\[
+ \sqrt{M+1} \sum_{i=1}^{d} \| \lambda_{M+1} (\partial_x f(t, \ldots)) \|_{L^2(\Omega; \mathbb{R}^{n(M+1)})} + \frac{1}{K \nu} \| Q \| \| f(t, \ldots) - \hat{\Pi}_M f(t, \ldots) \|_X \right) \, d\tau
\]

(48)

The above expression is a bound for the approximation error in terms of the projection error of different quantities and the closure error. The rate of convergence of the closure error will trivially follow from the velocity space regularity assumption made upon the kinetic solution (21). Therefore, to obtain the final form of the error bound and the convergence rate, given in Theorem 3.1, we need to develop bounds for the projection error, in terms of the moments of the kinetic solution, and a bound for the norm of \(A_{M,M}^{(M,M)}\).

### 3.4 Projection Error

To derive bounds for the error in projection, we will need the norms of a few matrices and operators.
Lemma 3.1.

(i) Let \( \lim_{q \to \infty} B_{\Psi}^{(M,q)} \), with \( B_{\Psi}^{(M,q)} \) as defined in definition 2.5 and \( M \in \mathbb{N} \), be an operator defined on \( l^2 \). Then it holds

\[
\| \lim_{q \to \infty} B_{\Psi}^{(M,q)} \| \leq 1.
\]

where \( \lim_{q \to \infty} B_{\Psi}^{(M,q)} \) has to be understood in the same sense as described in lemma 2.1.

(ii) Let \( A_{\Psi}^{(M,M)} \) and \( A_{\Psi}^{(M,M)} \), \( M > 1 \), be as given in definition 2.4. Then

\[
\| (A_{\Psi}^{(M,M-1)})^{-1} A_{\Psi}^{(M,M)} \|_2 \leq C \sqrt{M}, \quad \| A_{\Psi}^{(M,M)} \|_2 \leq C \sqrt{M}.
\]

Proof. See appendix-D.

Lemma 3.2. Let \( r^e \in X_{\Omega}^e \) and \( r^o \in X_{\Omega}^o \). Then the error in the projection (\( \| \hat{\Pi}_{M}r - r \|_X \)) can be bounded as

\[
\| \hat{\Pi}_{M}r - r \|_X \leq \Theta^{(M)} \| \lambda_{M}^{e}(r) \|_{L^2(\Omega_{\mathbb{R}^n}(M))} + \frac{\sqrt{2}}{(2(M + 1) + d)e} \| r^e f_0 \|_{L^2(\Omega_{\mathbb{R}^n}(\mathbb{R}^d))} + \frac{1}{(2(M + 1) + d)e} \| r^o f_0 \|_{L^2(\Omega_{\mathbb{R}^n}(\mathbb{R}^d))}, k^e < s^e - \frac{1}{2}, \quad k^o < s^o - \frac{1}{2}.
\]

where \( \Theta^{(M)} = \| R^{(M)}A_{\Psi}^{(M,M)} - B_{\Psi}^{(M,M)} \|_2 \). Similarly, the difference between the orthogonal projection and the projection which satisfies the boundary conditions, \( \hat{\Pi}_{M}r - \Pi_{M}r \), can be bounded as

\[
\| \hat{\Pi}_{M}r - \Pi_{M}r \|_X \leq \Theta^{(M)} \| \lambda_{M}^{o}(r) \|_{L^2(\Omega_{\mathbb{R}^n}(M))} + \frac{1}{(2(M + 1) + d)e} \| r^o f_0 \|_{L^2(\Omega_{\mathbb{R}^n}(\mathbb{R}^d))}, k^e < s^o - \frac{1}{2}.
\]

As \( M \to \infty \), we have the convergence rate

\[
\| \hat{\Pi}_{M}r - r \|_X \leq CM^{-\bar{\omega}}, \quad \| \hat{\Pi}_{M}r - \Pi_{M}r \|_X \leq CM^{-(s^o - \frac{1}{2})}, \quad \bar{\omega} = \min \left\{ s^o - \frac{1}{2}, s^e - \frac{1}{2} \right\}.
\]

Proof. Express \( r \) as

\[
r = \sum_{m=1}^{\infty} \lambda_{m}^{o}(r) \cdot \psi_{m}^{e}(\xi), f_0(\xi) + \sum_{m=0}^{\infty} \lambda_{m}^{e}(r) \cdot \psi_{m}^{o}(\xi), f_0(\xi), \quad \Lambda_{M}^{e}(r) = \lim_{q \to \infty} B_{\Psi}^{(M,q)} \lambda_{M}^{e}(r) + \mathcal{G}(r).
\]

where the second relation of the above two is a result of lemma 2.1. The definition of \( \hat{\Pi}_{M}r \) implies

\[
\hat{\Pi}_{M}r = \hat{\Lambda}_{M}^{e}(r) \cdot \psi_{M}^{e}(\xi), f_0(\xi) + \lambda_{M}^{e}(r) \cdot \psi_{M}^{e}(\xi), f_0(\xi), \quad \hat{\Lambda}_{M}^{e}(r) = R^{(M)}A_{\Psi}^{(M,M)} \Lambda_{M}^{e}(r) + \mathcal{G}(r).
\]

Using \( \lim_{q \to \infty} B_{\Psi}^{(M,q)} \lambda_{M}^{e}(r) = \sum_{q=0}^{\infty} B_{\Psi}^{(M,q)} \lambda_{M}^{e}(r), \) (34) and subtracting \( r \) from \( \hat{\Pi}_{M}r \), we find

\[
\hat{\Pi}_{M}r - r = \left( (R^{(M)}A_{\Psi}^{(M,M)} - B_{\Psi}^{(M,M)}) \lambda_{M}^{e}(r) \right) \cdot \psi_{M}^{e}(\xi), f_0(\xi) - \sum_{q=M+1}^{\infty} \left( B_{\Psi}^{(M,q)} \lambda_{M}^{e}(r) \right) \cdot \psi_{M}^{e}(\xi), f_0(\xi) - \sum_{q=M+1}^{\infty} \left( \lambda_{q}^{e}(r) \cdot \psi_{q}^{e}(\xi) + \lambda_{q}^{o}(r) \cdot \psi_{q}^{o}(\xi) \right) f_0(\xi).
\]

(49)
where $B_{\psi}^{(M,M)}$ is as defined in definition 2.5. The matrices $B_{\psi}^{(M,q)}$ and the operator $\lim_{q \to \infty} B_{\psi}^{(M,q)}$, appearing in (49), can be looked upon as restrictions of $\lim_{q \to \infty} B_{\psi}^{(M,q)}$ given in lemma 3.1; thus all of their norms can be bounded by one. This implies

$$\|\hat{\Pi}_M r - r\|_X \leq \left(\Theta^{(M)}\right)^2 \|\lambda^e_M(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} + \sum_{q=M+1}^{\infty} \left(2\|\lambda^e_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} + \|\lambda^o_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))}\right)$$

$$\leq \left(\Theta^{(M)}\right)^2 \|\lambda^e_M(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} + \frac{2}{(2(M+1) + d)^{2M}} \sum_{q=M+1}^{\infty} (2q + d)^{2M} \|\lambda^e_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))}$$

$$+ \frac{1}{(2(M+1) + d)^{2M}} \sum_{q=M+1}^{\infty} (2q + d)^{2M} \|\lambda^o_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))}$$

$$\leq \left(\Theta^{(M)}\right)^2 \|\lambda^e_M(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} + \frac{2}{(2(M+1) + d)^{2M}} \|r^e f_0\|^2_{L^2(\Omega; W^{\nu}_{2,2}(R^d))}$$

$$+ \frac{1}{(2(M+1) + d)^{2M}} \|\lambda^o_M(r)\|^2_{L^2(\Omega; W^{\nu}_{2,2}(R^d))} \quad (50)$$

where for the last inequality we have used the definition

$$\|r^e f_0\|^2_{L^2(\Omega; W^{\nu}_{2,2}(R^d))} = \sum_{q=0}^{\infty} (2q + d)^{2M} \|\lambda^e_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))}. \quad (50)$$

This proves the bound for $\|\hat{\Pi}_M r - r\|_X$. To prove the convergence rate we use the first inequality in (50). Using the definition of $X^2_0$ and that of $R^{(M)}$ from (21) and (32) respectively and the results from lemma 3.1, we find

$$\left(\Theta^{(M)}\right)^2 \|\lambda^e_M(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} = \|R^{(M)} A_{\psi}^{(M,M)} - B_{\psi}^{(M,M)}\|^2_{L^2(\Omega; R^{n\nu}(\psi))}$$

$$\leq \left(\left(\Theta^{(M)}\right)^2 \|\lambda^e_M(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} + \|\lambda^o_M(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))}\right)^2$$

$$\leq C \frac{M^{2s^e - 1}}{M^{2s^o - 1}} \quad (52)$$

Additionally we have

$$\sum_{q=M+1}^{\infty} \left(2\|\lambda^e_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))} + \|\lambda^o_q(r)\|^2_{L^2(\Omega; R^{n\nu}(\psi))}\right)$$

$$\leq 2C \left(\frac{1}{(M+1)^{2s^e}} + \int_{M+1}^{\infty} \frac{1}{w^{2s^e}} dw\right) + C \left(\frac{1}{(M+1)^{2s^o}} + \int_{M+1}^{\infty} \frac{1}{w^{2s^o}} dw\right)$$

$$\leq 2C \left(\frac{1}{(M+1)^{2s^e}} + \frac{1}{(2s^e - 1)(M+1)^{2s^e - 1}}\right) + C \left(\frac{1}{(M+1)^{2s^o}} + \frac{1}{(2s^o - 1)(M+1)^{2s^o - 1}}\right)$$

$$\leq \frac{C}{M^{2\tilde{\omega}}}, \quad \tilde{\omega} = \min\left\{s^e - \frac{1}{2}, s^o - \frac{1}{2}\right\}. \quad (53)$$

The convergence rate for $\|r - \hat{\Pi}_M r\|_X$ follows from (52) and (53). Using the definition of $\Pi_M$ and $\hat{\Pi}_M$ from definition 3.1 we find

$$\hat{\Pi}_M r - \Pi_M r = \left(R^{(M)} A_{\psi}^{(M,M)} - B_{\psi}^{(M,M)}\right) \lambda^e_M(r) \cdot \psi_M f_0 - \sum_{q=M+1}^{\infty} \left(B_{\psi}^{(M,q)} \lambda^e_q(r) \cdot \psi_M(\xi) f_0(\xi)\right)$$

which implies
\[ \| \hat{\Pi}_M r - \Pi_M r \|_X^2 \leq \left( \Theta^{(M)} \right)^2 \| \lambda_M^s(r) \|_{L^2(\Omega; \mathbb{R}^{\nu(M)})}^2 + \sum_{q=M+1}^{\infty} \| \lambda_q^s(r) \|_{L^2(\Omega; \mathbb{R}^{\nu(q)})}^2. \]

The bound for \( \| \hat{\Pi}_M r - \Pi_M r \|_X^2 \) and its corresponding convergence rate then follows from (50), (52) and (53) by removing the contribution from the odd moments of order higher than \( M \).

Using the result from lemma 3.2 in the upper bound for \( E_M \) (48) proves the error bound in Theorem 3.1. To arrive at the convergence rate in Theorem 3.1, we firstly bound the closure error in Theorem 3.1 as

\[
\sqrt{M + 1} \| \lambda_{M+1}(f(t, \ldots)) \|_{L^2(\Omega; \mathbb{R}^{\nu(M+1)})} \leq \sqrt{M + 1} \left( \| \lambda_M^u(f(t, \ldots)) \|_{L^2(\Omega; \mathbb{R}^{\nu(M+1)})} + \| \lambda_M^u(f(t, \ldots)) \|_{L^2(\Omega; \mathbb{R}^{\nu(M+1)})} \right),
\]

(54)

Then, by using the convergence rate of the projection error and by noting that \( \| A^{(M)}_q \|_2 \leq C\sqrt{M} \) (see lemma 3.1) we arrive at the convergence rate given in Theorem 3.1.

3.5 Discussion

**Improved Boundary Conditions:** In [23], authors have tried to improve upon the model for the Onsager matrix (\( R^{(M)} \)), given in (32), for the regularised-13 moment system [30]. Through the error bounds of the projection error, given in lemma 3.2, we can understand how the convergence behaviour is influenced by the Onsager matrix. The error bound for the projection error (\( f - \hat{\Pi}_M f \)), given in the first inequality in (50), can be decomposed into two parts. One part which is independent of \( \Theta^{(M)} \) (and so of \( R^{(M)} \)), which we denote by \( \tilde{a} \), and the other part which is dependent on \( \Theta^{(M)} \) and is denoted by \( \tilde{a}_\Theta^{(M)} \)

\[ \tilde{a}_\Theta^{(M)} = \left( \Theta^{(M)} \right)^2 \| \lambda_M^s(f) \|_{L^2(\Omega; \mathbb{R}^{\nu(M)})}^2 + \sum_{q=M+1}^{\infty} \left( 2 \| \lambda_q^s(f) \|_{L^2(\Omega; \mathbb{R}^{\nu(q)})}^2 \right). \]

where \( \Theta^{(M)} \) is as defined in (40d). Clearly, changing the Onsager matrix cannot influence the convergence rate of \( \tilde{a} \). Hence, we conclude that for any given Onsager matrix, our moment approximation cannot converge faster than the rate at which \( \tilde{a} \) converges. Though, changing the Onsager matrix can provide us with a reduction in the value of \( \tilde{a}_\Theta^{(M)} \) which can lead to a smaller value for the error bound.

**Remark 12.** The presence of \( \tilde{a} \) in our error bound is a result of ignoring the contribution from all higher order even moments in (30) in order to derive (32). Presently, ignoring this contribution, appears to be necessary if we want to derive a set of boundary conditions through (30); though this particular way of deriving the boundary conditions might not be unique.

**Moment Decay Rates:** As the predicted convergence rate in Theorem 3.1 is influenced by the decay rates of the moments (\( s \)), it is crucial to understand which values of the different \( s \), appearing in (21), can we expect from the true kinetic solution. For the sake of the present discussion let us assume \( s = s_p = s^e \) (and same for the values of \( s_x, s_q^e \) etc). Due to the kinetic boundary conditions (2c), the kinetic solution can be discontinuous, in the velocity space, in certain parts of the space-time domain \((0, T] \times \Omega\). This motivates the decomposition \((0, T] \times \Omega = (0, T] \times \Omega^R \cup (0, T] \times \Omega^D) \) where in \((0, T] \times \Omega^R, f(t, x, \ldots) \) is atleast once weakly differentiable whereas in \((0, T] \times \Omega^D \) it is not
Theorem 3.1 is useful in predicting the convergence rate, of the Hermite approximation. For the ease of numerical implementation and the generation scheme is based upon a weak boundary implementation, which preserves the stability of the moment 500 elements. For temporal discretization, we use a fourth order explicit Runge-Kutta scheme. The DG 1 with the predicted convergence rate in (41), implies that the Hermite approximation might fail to converge.

The result in Theorem 3.1 is useful in predicting the convergence rate, of the Hermite approximation, for a given test case. But in practical applications, the true analytical solution is usually not known and thus for the convergence rate prediction one relies on a highly refined numerical scheme. Starting from smooth initial and boundary conditions, the true kinetic solution can be such that the discontinuity in the velocity space is concentrated in a very small region of the space time domain. Then, due to the limited availability of the computational resources, a numerically obtained reference solution might never capture such a discontinuity and thus show more regularity than the true solution. In such a case, the value of the observed decay rate will be very close to $s^R$. Up to our knowledge, a mathematical theory which could predict the value of $s$, given some initial conditions, boundary conditions and the Knudsen number, does not exist. Nevertheless, the numerical experiments conducted in the past do indicate that starting from smooth initial conditions, the discontinuity in the velocity space, for moderately high Knudsen numbers, can be localized in a very small part of the space-time domain; with the space part being located very close to the boundary. See [20, 31, 33] for details.

Remark 13. The norm of $A^{(M,M)}(\xi)$, as can be concluded from the convergence analysis, leads to a loss of half an order of convergence. The growth in the norm of $A^{(M,M)}$, with $M$, can be attributed to the recurrence relation of the Hermite polynomials. Use of the Hermite polynomials is further related to the velocity space ($\xi$) being unbounded.

On the contrary, in the radiation transport equation, the variable analogous to velocity lives on a unit sphere and is thus bounded. A moment approximation can therefore be developed with the help of spherical harmonics defined on the unit sphere. The recurrence relation of the spherical harmonics is such that the matrix analogous to $A^{(M,M)}$ is bounded by one. As a result, an additional half an order of convergence, at least for the Cauchy problem, is gained in the predicted convergence rate. See [9] for details.

4 Numerical Results

The goal of the following numerical experiments is to validate the convergence rate, with some acceptable accuracy, presented in Theorem 3.1. For the ease of numerical implementation and the generation of a reference solution, we will be using the BGK collision operator, given in (14), for all the numerical experiments. As mentioned earlier, the use of the BGK collision operator does not influence the predicted convergence rate in (41).

All of our numerical experiments will be one dimensional in the physical space. To discretize the 1D physical space we use a discontinuous galerkin (DG) discretization with first order polynomials and 500 elements. For temporal discretization, we use a fourth order explicit Runge-Kutta scheme. The DG scheme is based upon a weak boundary implementation, which preserves the stability of the moment approximation (33) on a spatially discrete level; see [34] for details. For test cases which involve steady states, we assume that the steady state is reached when $\|\partial_t f_M\|_2 \leq 10^{-8}$. Note that the error bound in
Theorem 3.1 is linear in $t$ and thus deteriorates for computations which take longer times. For all the coming numerical experiments which include steady states, we found that the time taken to reach the steady state was not significantly large ($T = 32$ being the maximum). Thus, we can still expect the error bound in Theorem 3.1 to be not extremely large and the predicted convergence rate to be reliable.

In (21), for simplicity, we have assumed a constant value for the decay rates of the moments for all times $t \in (0, T]$. But in realistic situations, such a time independent decay rate is usually not guaranteed. Therefore, in accordance with our error bound in Theorem 3.1, to find the values of the different $s$ given in (21), we use the quantities

$$N_m^{(x)} = \int_0^T \|\lambda_m(\partial_t f_{\text{ref}}(t, \ldots))\|_{L^2(\Omega; \mathbb{R}^{m(n)})} dt, \quad N_m^{(t)} = \int_0^T \|\lambda_m(\partial_x f_{\text{ref}}(t, \ldots))\|_{L^2(\Omega; \mathbb{R}^{m(n)})} dt$$

which are representatives of the average magnitude of the moments from $(0, T)$. In (55), $f_{\text{ref}}$ is a reference solution coming from a sufficiently large value of $M = M_{\text{ref}}$ in (24a). Its been observed in [9], that the decay behaviour of the quantities defined in (55) might show some artefacts for higher order moments. To remove these artefacts, we follow the methodology proposed in [9], i.e., we first compare the decay behaviour with that obtained through $M_{\text{ref}} - 1$. Then we keep only those data points corresponding to which the results between $M_{\text{ref}}$ and $M_{\text{ref}} - 1$ differ by less than 3 percent.

The time integrals in the definition for the various $N_m$ will be computed with the help of the trapezoidal rule. We will approximate $s^{T}$ (and similarly the other values of $s$) as the slope of the linear curve which has the minimum $L^2$ distance to the curve $(\log(M), \log(N_m))$ with $N_m$ being the same as $N_m$ but with a dependency only on the magnitude of the odd moments. Having computed the different values of $s$, the predicted convergence rate ($\omega_{\text{pre}}$) will follow from (41)

$$\omega_{\text{pre}} = \min \left\{ s^T - \frac{1}{2}, s^T - \frac{1}{2}, s^T - \frac{1}{2}, s^T - \frac{1}{2}, s^T - \frac{1}{2}, s^T - \frac{1}{2}, \frac{1}{2} \right\}$$

where $s^{T/2}$, computed through $N_m^{(T)}$, is the decay rate of the moments corresponding to $f_{\text{ref}}(t = T, \ldots)$ and accounts for the projection error appearing in the error bound (39). Similar to the computation of $s$, the obtained convergence rate, denoted by $\omega_{\text{obs}}$, will be given by the slope of the linear curve which minimises the $L^2$ distance to the curve $(\log(M), \log(\|E_M\|_{\infty}))$ where $E_M = f_{\text{ref}} - f_M$ with $f_{\text{ref}}$ being a reference solution. The error in the convergence rate prediction will then be given by

$$E_\omega = \omega_{\text{obs}} - \omega_{\text{pre}}$$

### 4.1 One Dimensional Velocity Space

In the present test case we will consider the velocity space and the physical space, both to be one dimensional i.e. $d = 1$ and $\Omega = (0, 1)$. For a one dimensional velocity space, the Hermite approximation simplifies to

$$f_M(t, x, \xi) = f_0(\xi) \sum_{m=0}^M \lambda_m(f_M(t, x, \cdot)) H_m(\xi).$$

Note that we have assumed $\Omega$ to be the half plane $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ for the analysis conducted in the previous section but we can extend the analysis to the domain being considered presently through the following argument. The projection operator ($\hat{\Pi}_M$), given in (3.1), has been defined with respect to the boundary conditions corresponding to $x = 1$. A similar projection operator can also be constructed for the boundary
conditions at $x = 0$ and by taking a linear combination of the projection operation defined with respect to $x = 0$ and $x = 1$, same results as those presented in Theorem 3.1 can also be obtained for the bounded domain being considered presently. Using the present one dimensional configuration, we will be looking into two different flow regimes, one with $Kn = 0.1$ and the other with $Kn \to \infty$. As discussed earlier, for $Kn = 0.1$, we expect sufficiently high regularity of the kinetic solution, in the velocity space, which would then lead to a nicely converging moment approximation as per Theorem 3.1. In the free streaming example, $Kn \to \infty$, we expect the kinetic solution to be discontinuous in the velocity space on a significant part of the space-time domain; this should result in a poorly converging moment approximation.

**Remark 14.** One can conclude that for all odd values of $M$, we will have the same number of odd and even moments. As a result, for all odd values of $M$, the matrix $A^{(M,M)}_{ww}$, defined in definition 2.4, will be a square matrix. Thus, the boundary conditions in (32) will be the same as the boundary conditions in (31) meaning the factor $\Theta^{(M)}$, appearing in the error bounds in Theorem 3.1, will be zero or equivalently, the boundary stabilisation error will be zero. On the contrary, for even values of $M$, the boundary stabilisation error will be non-trivial.

**Gaussian density in vacuum ($Kn = 0.1$):** As initial conditions we consider

$$f_t(x, \xi) = \frac{\rho_t}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right), \quad \rho_t = \exp\left[-(x-0.5)^2 \times 100\right]$$

which corresponds to a Gaussian density profile with all the higher order moments being zero. As boundary conditions, we consider vacuum at both the ends $x = 0$ and $x = 1$ which is equivalent to considering $f_n = 0$. As final time we consider $T = 0.3$ and the reference solution corresponds to $M_{ref} = 200$.

In Figure 1, we have shown the behaviour of the various quantities defined in (55). The decay rates of the different quantities defined in (55) are given in Table 1. Clearly, all the decay rates remain well above one, validating the observed regularity of the distribution function in the velocity space; see appendix-E. The variation of the approximation error has been shown in Figure 2 and for the convergence rates we found

$$\omega_{\text{pre}} = 0.97, \quad \omega_{\text{obs}} = 1.16, \quad E_\omega = \omega_{\text{obs}} - \omega_{\text{pre}} = 0.19$$

(56)

where $\omega_{\text{obs}}$ has been computed using different values of $M$ till 40. For the sake of validation, we also compute the convergence rates with the reference solution obtained through a discrete velocity method (DVM); see appendix-E for details of our DVM implementation. With DVM as the reference, we obtain $\omega_{\text{pre}} = 0.98$, $\omega_{\text{obs}} = 1.15$ and $E_\omega = \omega_{\text{obs}} - \omega_{\text{pre}} = 0.17$ which is very similar to the results in (56). Due to the similarity in the results obtained and the ease of computing higher order moments with a moment solution, for all the coming test cases, we will be using the moment solution as the reference.

| Quantity | Decay Rate |
|----------|------------|
| $N_m$    | 2.30 ($= s = s' = s''$) |
| $N_m^{(r)}$ | 1.95 ($= s_t = s'_t = s''_t$) |
| $N_m^{(x)}$ | 1.97 ($= s_s = s'_s = s''_s$) |
| $N_m^{(f)}$ | 1.73 ($= s_T = s'_T = s''_T$) |

**Table 1:** Decay rates for the time integrated magnitude of the moments defined in (55) corresponding to the gaussian density in vacuum test case; $Kn = 0.1$ and $T = 0.3$. The decay rates for the odd and even moments were found to be the same, for the present test case, and have thus been represented by one value.
4.1 One Dimensional Velocity Space

(a) decay of $N_m$ obtained through $M = 200$ for $Kn = 0.1$, $T = 0.3$ and the gaussian density in vacuum test case.

(b) decay of $N_m^{(t)}$ obtained through $M = 200$ for $Kn = 0.1$, $T = 0.3$ and the gaussian density in vacuum test case.

(c) decay of $N_m^{(x)}$ obtained through $M = 200$ for $Kn = 0.1$, $T = 0.3$ and the gaussian density in vacuum test case.

(d) decay of $N_m^{(T)}$ obtained through $M = 200$ for $Kn = 0.1$, $T = 0.3$ and the gaussian density in vacuum test case.

Figure 1: The plots depict the decay of the various quantities, defined in (55), obtained through a refined moment approximation ($M = 200$). All the plots are on a log-log scale and correspond to the gaussian density in vacuum test case.

Figure 2: Decay of the approximation error, on a log-log scale, for different values of $M$ corresponding to the gaussian density in vacuum test case; $Kn = 0.1$ and $T = 0.3$. 

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Density wave wall collision \((Kn = 0.1)\): In the present test case we are interested in a gaussian density wave, which collides with a wall and then disperses inside a region which is enclosed by two walls. As initial conditions, we consider a gaussian density profile moving towards the right with a gaussian velocity profile with all the other higher order moment being zero

\[
f_t(x, \xi) = (\rho_t(x) + v_t(x)\xi) f_0(\xi), \quad \rho_t(x) = v_t(x) = \exp \left[ - (x - 0.5)^2 \times 100 \right], \quad \forall (x, \xi) \in \Omega \times \mathbb{R}.
\]

At both the boundaries, \(x = 0\) and \(x = 1\), we consider fully accommodated walls the boundary conditions for which follow from the Maxwell’s accommodation model and can be given as

\[
f_{in}(t, x, \xi) = \hat{\rho}_{in}[f](t, x) f_0(\xi), \quad \forall (t, x, \xi) \in (0, T] \times \partial \Omega \times \mathbb{R}
\]

with \(\hat{\rho}_{in}[f]\) computed such that the gas velocity at both the walls remains zero, see [29] for details. As the final time we consider \(T = 0.3\) and as the reference we consider \(M_{ref} = 200\).

The decay rates of the various quantities defined in (55) are given in Table 2. The decay behaviour of \(N_m^{(x)}\) and \(N_m^{(T)}\) differs significantly for odd and even moments which is in contrast with the previous test case. Also, contrary to the previous test case, the approximation error shows an oscillatory convergence behaviour, see Figure 4. Such an oscillatory convergence behaviour is similar to that previously observed in [33]. Presently, we could not explain the fast convergence of the moment approximation corresponding to the even values of \(M\). Though, for odd values of \(M\), the error in prediction \((E_\omega)\), was found to be approximately the same as that for the previous test case, see Table 3.

| Quantity | Decay Rate (Even Moments) | Decay Rate (Odd Moments) |
|----------|--------------------------|--------------------------|
| \(N_m\)  | 2.156 (= \(s^r\))       | 2.053 (= \(s^s\))       |
| \(N_m^{(x)}\) | 1.476 (= \(s^x\))   | 0.995 (= \(s^s\))       |
| \(N_m^{(T)}\) | 2.165 (= \(s^T\))    | 1.974 (= \(s^s\))       |
| \(N_m^{(T)}\) | 1.704 (= \(s^T\))    | 1.488 (= \(s^s\))       |

Table 2: Decay rates for the odd and even moments of different quantities, defined in (55), corresponding to the density wave wall collision test case; \(Kn = 0.1\) and \(T = 0.3\).

| Values of \(M\) | \(\omega_{pre}\) | \(\omega_{obs}\) | \(E_\omega = \omega_{obs} - \omega_{pre}\) |
|-----------------|-----------------|-----------------|-----------------|
| Odd             | 0.476           | 0.685           | 0.209           |
| Even            | 0.476           | 1.003           | 0.527           |

Table 3: Observed and predicted convergence rates for the density wave wall collision test case; \(Kn = 0.1\) and \(T = 0.3\).

Steady state in free streaming regime \((Kn \rightarrow \infty)\): The initial conditions for the present test case are vacuum, i.e. \(f_t = 0\). The kinetic boundary conditions, (2c), will be considered to be given by

\[
f_{in}(t, x, \xi) = \left( \hat{\rho}_{in}(t, x) + \bar{v}_{in}(t, x)\xi + \frac{\hat{\sigma}_{m}(t, x)}{2} (\xi^2 - 1) \right) f_0(\xi), \quad (t, x, \xi) \in (0, T] \times \partial \Omega \times \mathbb{R}
\]

which is the deviation of the Maxwell-Boltzmann distribution function about \(f_0\) for a one dimensional velocity space.
4.1 One Dimensional Velocity Space

Figure 3: The plots depict the decay of the quantities defined in (55) obtained through a refined moment approximation \((M = 200)\). All the plots are on a log-log scale and correspond to the density wave wall collision test case.

Figure 4: The decay of error for different values of \(M\) corresponding to the density wave wall collision.
We will consider $\tilde{\rho}_n = \tilde{v}_n = 0$ and

$$
\tilde{\theta}_n(t,x) = \begin{cases} 
0, & t = 0 \\
n_x \exp \left[ -\frac{1}{1-(t-t_0)^2} + 1 \right], & t \in (0,1) \\
n_x, & t \geq 1
\end{cases}
$$

(58)

where $n_x$ is the normal to the boundary, pointing out of the domain. Ultimately we would like to have a $\tilde{\theta}_n(t)$ which is equal to one. But, starting from $t = 0$, we cannot immediately use $\tilde{\theta}_n(t) = 1$ or else we will generate an incompatible set of initial and boundary conditions. Such an incompatibility will lead to a discontinuity travelling from the wall into the domain which will ultimately lead to a violation of the assumptions made on the kinetic solution in (10).

For the present flow regime we suspect that using any numerically generated reference solution, either from a highly refined moment approximation or the DVM, would be inappropriate since such a reference solution might misrepresent the true kinetic solution by retaining too much velocity space regularity. Therefore, we would rather iterate into the steady state and compare our results with the exact steady state solution which can be easily constructed for the present flow regime and is independent of the physical space

$$
f(x,\xi) = \begin{cases} 
f_h(\xi) \left( \xi^2 - 1 \right), & \xi \leq 0 \\
\frac{f_0(\xi)}{\sqrt{\xi}} \left( \xi^2 - 1 \right), & \xi > 0
\end{cases}, \quad x \in \Omega.
$$

Thus, the exact steady state kinetic solution is discontinuous in the velocity space along the entire physical space which implies $f \notin X^s_{\Omega}, \forall s > 1$ or equivalently $f(x,.)f_0^{\frac{1}{2}}(.) \notin H^k(\mathbb{R}), \forall k \geq 1, x \in \Omega$. Since the initial and the boundary conditions are smooth, as we march into the steady state in time, we expect that the kinetic solution is discontinuous in the velocity space on a larger and larger part of the space-time domain. Therefore, as compared to the previous test case, from the convergence rates presented in Theorem 3.1, we expect the moment approximation to be poorly converging (or not converging at all) for the present test case.

In fig.5(a), we have shown the variation of the approximation error ($\|E_M\|$) for some particular values of $M$. We can identify two monotonically converging sub-sequences corresponding to odd and even values of $M$. Though, no improvement in accuracy is observed while moving from an even value of $M$ to a subsequent odd value. The value of the convergence rate was found to be 0.11 and 0.10 for the odd and even values of $M$ respectively. Thus, as compared to the previous test case ($Kn = 0.1$), the convergence rate has severely deteriorated, which is as expected. Moreover, the rate at which error decreases (with respect to $M$) was found to decrease as the value of $M$ is increased. We therefore speculate that as $M \to \infty$, the moment method might fail to converge (or converge very slowly), to the true solution.

### 4.2 Three Dimensional Velocity Space

For the test cases which follow, we will be considering a three dimensional velocity space, i.e., $d = 3$ in (2a) and $\Omega = (0,1) \times \mathbb{R} \times \mathbb{R}$. All the test cases considered will be such that $\tilde{\partial}_n f = \tilde{\partial}_n f = 0$ thus reducing problems to be one dimensional in the physical space. As was mentioned earlier, a result similar to Theorem 3.1, can be obtained for such a physical domain using a linear combination of projection operators defined with respect to $x_1 = 0$ and $x_1 = 1$. For all the test cases, the Knudsen number will be fixed at $Kn = 0.1$.

For both the following test cases, it can be shown that in the moment approximation, only those moments will have a non-trivial value which are coupled to the deviation in temperature (\(\tilde{\theta}\)) and density (\(\tilde{\rho}\)), see [33] for details. This further implies that only those moment variables which correspond to basis
Along \( x \) (which justifies the name of the test case):

\[
\psi_{\beta(i)}(x, \xi) = \begin{cases} 
0 & \text{if } \beta(i) \text{ has a zero or two in the last two components of the multi-index} \\
\text{a non-trivial value} & \text{if } \beta(i) \text{ has either a zero or a two in the last two components of the multi-index}
\end{cases}
\]

For odd and even values of \( M \), the moment approximation corresponding to odd values of \( M \) converges faster than the one corresponding to the even values of \( M \). Whereas, the moment approximation corresponding to odd values of \( M \) did much better, in terms of the decay rates for odd and even moments, for \( Kn \rightarrow \infty \) test case. The steady state solution is independent of the physical space.

**Harmonic Heat Inflow**: In the present test case we consider zero initial conditions and vacuum at \( x = 1 \)

\[
f_I(x, \xi) = 0, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^3, \quad f_m(t, x, \xi) = 0, \quad \forall (t, \xi) \in (0, T] \times \mathbb{R}^3, \quad x = 1.
\]

Along \( x = 0 \), we consider \( f_m \) to be such that the temperature corresponding to it is harmonic in time (which justifies the name of the test case):

\[
f_m(t, x, \xi) = \frac{\tilde{\Theta}_m(t)}{2} (\xi^2 - 3) f_0(\xi), \quad \tilde{\Theta}_m(t) = 1 - \cos(\pi t), \quad \forall (t, \xi) \in (0, T] \times \mathbb{R}^3, \quad x = 0 \quad (59)
\]

As the end time we consider \( T = 0.5 \). As a reference solution, we choose a highly refined moment solution corresponding to \( M_{ref} = 45 \). In Grad’s terminology, this corresponds to Grad’s-17296 moment equation (i.e. a total of 17296 variables) and contains in total 134 heat conduction variables. The behaviour of the time integrated magnitude of the moments, defined in (55), is shown in Figure 6 and the values of the decay rates obtained are given in Table 4. The decay rates for odd and even moments, for all the quantities, were found to be very similar and well above one.

The convergence behaviour was found to be oscillatory Figure 7, with the moment approximation corresponding to the odd values of \( M \) converging faster than the one corresponding to the even values of \( M \). The convergence rate for even values of \( M \) was found to be closer to the predicted convergence rate. Whereas, the moment approximation corresponding to odd values of \( M \) did much better, in terms of the convergence rate, than what was predicted.

**Heat Conduction**: In the present test case, we will study the steady state heat conduction between two infinitely large parallel plates placed at \( x_1 = 0 \) and \( x_1 = 1 \). As kinetic boundary conditions, we will
4 NUMERICAL RESULTS

Figure 6: The plots depict the variation of the magnitudes of the moments, defined in (55), obtained through a refined moment approximation (M = 45). All the plots are on a log-log scale and correspond to the harmonic heat inflow test case.

| Quantity     | Decay Rate (Even Moments) | Decay Rate (Odd Moments) |
|--------------|---------------------------|--------------------------|
| \( N_m \)   | 1.6615 = \( s^e \)       | 1.5578 = \( s^o \)       |
| \( N_{m}^{(x)} \) | 1.4454 = \( s^e_x \)   | 1.2644 = \( s^o_x \)   |
| \( N_{m}^{(t)} \) | 1.724 = \( s^e_t \)   | 1.611 = \( s^o_t \)   |
| \( N_{m}^{(T)} \) | 1.686 = \( s^e_T \)   | 1.6213 = \( s^o_T \)   |

Table 4: Decay rates for the odd and even moments of different quantities corresponding to the harmonic heat inflow test case.

consider the Maxwell’s accommodation model with an accommodation coefficient of one

\[
f_{in}(t, x, \xi) = \left( \tilde{\rho}_{in}[f](t, x) + \frac{\tilde{\theta}_{in}(t, x)}{2} (\xi_i \xi_i - 3) \right) f_0(\xi), \quad (t, x, \xi) \in (0, T] \times \partial \Omega \times \mathbb{R}
\] (60)
4.2 Three Dimensional Velocity Space

| Values of M | $\omega_{\text{pre}}$ | $\omega_{\text{obs}}$ | $E_\omega = \omega_{\text{obs}} - \omega_{\text{pre}}$ |
|-------------|-----------------------|-----------------------|-----------------------------------------------|
| Odd         | 0.445                 | 0.928                 | 0.476                                          |
| Even        | 0.445                 | 0.650                 | 0.215                                          |

Table 5: Observed and predicted convergence rates for the harmonic heat inflow test case.

![Figure 7](image)

Figure 7: The variation of error for different values of $M$ corresponding to the harmonic heat inflow test case.

With $\theta_{in}$ being the same as that given in (58) and $\rho_{in}[f]$ computed such that the normal velocity of the gas perpendicular to the wall remains zero, see [29] for details. The initial conditions will be chosen to be zero, $f_i = 0$, and as a reference we will consider $M_{\text{ref}} = 45$. The decay rates of the moments of different quantities can be found in Table 6. The decay rate for odd moments of $N_m^{(x)}$ was found to differ significantly from that of the even moments and was the lowest among all the other decay rates; thus indicating a comparatively low regularity in $(\partial x_1 f)^\alpha$.

```
| Quantity | Decay Rate (Even Moments) | Decay Rate (Odd Moments) |
|----------|---------------------------|--------------------------|
| $N_m$    | 1.597 ($= s^{(s)}$)      | 1.545 ($= s^{(s)}$)      |
| $N_m^{(x)}$ | 1.529 ($= s^{(s)}_x$) | 1.064 ($= s^{(s)}_x$) |
| $N_m^{(t)}$ | 1.762 ($= s^{(s)}_t$) | 1.690 ($= s^{(s)}_t$) |
| $N_m^{(y)}$ | 1.535 ($= s^{(s)}_y$) | 1.662 ($= s^{(s)}_y$) |
```

Table 6: Decay rates for the odd and even moments of different quantities corresponding to the heat conduction test case; $Kn = 0.1$.

From Figure 9, which shows the variation of the error with $M$, we can observe two sub-sequences of moment approximations which converge monotonically at different rates; this oscillatory convergence behaviour is the same as the one observed in [33] and is also similar to the behaviour observed in the previous test case. This fast convergence behaviour, of a particular sub-sequence, cannot presently be explained with the help of the error bounds given in Theorem 3.1. The observed and the predicted convergence rate, for both even and odd values of $M$ can be found in Table 7. The error in convergence rate prediction was found to be similar to the previous test case for all the even values of $M$ but for the odd values of $M$, the moment approximation did much better than what was predicted.
4 NUMERICAL RESULTS

(a) decay of $N_m$ obtained through $M = 45$ for the heat conduction problem with $Kn = 0.1$.

(b) decay of $N_m^{(i)}$ obtained through $M = 45$ for the heat conduction problem with $Kn = 0.1$.

(c) decay of $N_m^{(x)}$ obtained through $M = 45$ for the heat conduction problem with $Kn = 0.1$.

(d) decay of $N_m^{(T)}$ obtained through $M = 45$ for the heat conduction problem with $Kn = 0.1$.

Figure 8: The decay of the time integrated magnitude of the moments \((55)\) obtained through $M = 45$ for the heat conduction experiment with $Kn = 0.1$.

| Values of M | $\omega_{pre}$ | $\omega_{obs}$ | $E_{\omega} = \omega_{obs} - \omega_{pre}$ |
|-------------|----------------|----------------|--------------------------------------|
| Odd         | 0.529          | 0.985          | 0.456                                |
| Even        | 0.529          | 0.682          | 0.153                                |

Table 7: Observed and predicted convergence rates for the heat conduction problem.

4.3 Discussion

In all of the test cases, we found that the Hermite approximation converged a bit faster than predicted. A possible explanation for this is as follows. Recall the splitting of the space-time domain as given in subsubsection 3.5.
Figure 9: The variation of error with the different values of M corresponding to the heat conduction experiment.

\[ f(t, \ldots) \in X^{R}_{\Omega^{R}}, \quad f(t, \ldots) f_0^{-\frac{1}{2}} \in L^2(\Omega^{R}; H^{2k^R}(\mathbb{R}^d)), \quad s^R > 1, 2k^R < 2s^R - 1, t \in (0, T] \]
\[ f(t, \ldots) \in X^{D}_{\Omega^{D}}, \quad f(t, \ldots) f_0^{-\frac{1}{2}} \in L^2(\Omega^{D}; H^{2k^D}(\mathbb{R}^d)), \quad s^D \leq 1, 2k^D < 2s^D - 1, t \in (0, T]. \]

where \( \Omega \times (0, T] = (\Omega^{D} \times (0, T]) \cup (\Omega^{R} \times (0, T]) \). With the definition of the \( L^2 \) norms, it is straightforward to see that the error bound in (39) can be decomposed into two parts: one part with norms defined over \( \Omega^{R} \), label this as \( E_{R} \), and the other part with norms defined over \( \Omega^{D} \), label this as \( E_{D} \). Clearly, as the value of \( M \) is increased, the value of \( E_{R} \) decays at a faster rate as compared to \( E_{D} \). If the true solution is such that \( E_{R} \) is much bigger than \( E_{D} \) then the slow decay rate of \( E_{D} \) can only influence the convergence rate, of the Hermite approximation, for sufficiently high values of \( M \). Therefore, it is possible that in all our numerical experiments the value of \( M \) was not high enough to see the influence of \( E_{D} \) which resulted in a higher convergence rate than predicted. Note that, if in practice, the contribution from \( E_{D} \) is of the order of the machine precision, then its contribution into the error bound will never be seen numerically.

5 Conclusion

Using a Galerkin type approach, under certain regularity assumptions on the kinetic solution, the global convergence of the Hermite approximation to the linearised Boltzmann equation was proved. The speed of convergence was quantified by proving convergence rates which, as was expected, depend on the decay rates of the moments of the exact solution; these decay rates were further related to the regularity of the kinetic solution in the velocity space. The derived convergence rates were found to be accurate for flow regimes where the exact solution is smooth in velocity space on a large part of the space time domain. Since, intuitively, the discontinuity in the kinetic solution is damped out by collisions, our analysis is capable of predicting the convergence rates of the Hermite approximation for moderately high Knudsen numbers.

Numerical experiments of benchmark problems were conducted, which involve one dimensional and three dimensional velocity spaces. For a moderately high Knudsen number \( (Kn = 0.1) \), the observed convergence rate coincided, with decent accuracy, with the one predicted by the a priori analysis. For a test case involving free streaming (no-collisions), the Hermite approximation was found to converge very poorly which was as expected, since, in this case, the moments of the exact solution decay very slowly. All the test cases involving the three dimensional velocity space showed oscillatory convergence in the
Hermite approximation with two sub-sequences converging monotonically but with different convergence rates. Although the analysis failed to predict the oscillatory convergence behaviour, it accurately predicts the convergence rate of the slowly converging sub-sequence. Understanding the oscillatory convergence behaviour is an interesting task which is beyond the scope of this paper.

6 Acknowledgements

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Appendix A Proof of Lemma 2.1

By splitting the integral over \( \xi_1 \), we have

\[
\int_{\mathbb{R}^d} \Psi_M^\nu r^\nu d\xi = \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_M^\nu r^\nu d\xi + \frac{1}{2} \mathcal{G}(r). \tag{61}
\]

Using \( \int_{\mathbb{R}^d} \Psi_M^\nu r^\nu d\xi = 0 \) and the definition of \( r^\nu \) and \( r^e \)

\begin{align*}
r^\nu &= \frac{1}{2} (r(\xi_1, \xi_2, \xi_3) - r(-\xi_1, \xi_2, \xi_3)), \\
r^e &= \frac{1}{2} (r(\xi_1, \xi_2, \xi_3) + r(-\xi_1, \xi_2, \xi_3)), \\
r &= r^\nu + r^e.
\end{align*}

we find

\[
\int_{\mathbb{R}^d} \Psi_M^\nu r^\nu d\xi = \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_M^\nu r^\nu d\xi + \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_M^\nu r^e d\xi + \frac{1}{2} \mathcal{G}(r)
\]

\[
= \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_M^\nu r^\nu d\xi + \frac{1}{2} \int_{\mathbb{R}^{d}} \Psi_M^\nu r^e d\xi + \frac{1}{2} \mathcal{G}(r)
\]

which then provides us with (28). To derive (29), we express \( r^\nu \) and \( r^e \) as

\[
r^\nu = \sum_{m=1}^{\infty} \lambda_m^\nu(r) \cdot \Psi_m^\nu f_0(\xi), \\
r^e = \sum_{m=0}^{\infty} \lambda_m^e(r) \cdot \Psi_m^e f_0(\xi)
\]

and replace the above expansion into (28) to get the desired result. Note that \( \lambda_m^\nu(r) = \lambda_m^\nu(r^\nu) \) and \( \lambda_m^e(r) = \lambda_m^e(r^e) \) which is a consequence of the orthogonality of the Hermite polynomials (16a).

We can consider \( \lim_{q \to \infty} B^{(M,q)}_q \) to be an operator defined over \( l^2 \) in the sense of

\[
(\lim_{q \to \infty} B^{(M,q)}_q) x := (\lim_{q \to \infty} B^{(M,q)}_q x), \quad \forall x \in l^2.
\]

We now show that \( \lim_{q \to \infty} B^{(M,q)}_q \) is well defined on \( l^2 \) which is equivalent to showing that the limit, \( q \to \infty \), is well defined. Let \( x \in l^2 \) and let \( x^q \in \mathbb{R}^q \) be a vector containing the first \( q \) elements of \( x \). To extend \( x^q \) by zeros, we additionally define \( x^q \in l^2 \) which has the same first \( q \) elements as \( x \) and all the other elements zero. From the definition of \( B^{(M,q)}_q \), definition 2.5, we find

\[
B^{(M,q)}_q x^q = 2 \int_{\mathbb{R}^{d-1}} \int_{\xi_1 > 0} \Psi_M^\nu g^q d\xi, \text{ where } g^q = (\Psi^\nu \cdot x^q) f_0.
\]
Trivially, $\bar{x}^q$ converges to $x$ in $l^2$. This implies that $g^q$ converges in $K$; recall $K = L^2(\mathbb{R}^d, f_0^{-1})$. Then, by the continuity of the inner product of $L^2(\mathbb{R}^+ \times \mathbb{R}^2, f_0^{-1})$, we have the convergence of $B_{\mu}^{(M,q)}x^q$ in $\mathbb{R}^Z$. 


Appendix B Relation even and odd moments

With the help of some straightforward computations, we can derive an expression for \( n_o(p) \) and \( n_e(p) \)

\[
\begin{align*}
n_o(p) &= \begin{cases} 
\sum_{i=0}^{p-1} (2i+1), & p \text{ odd} \\
\sum_{i=0}^{p-1} (2i+2), & p \text{ even}
\end{cases}, \\
n_e(p) &= \begin{cases} 
\sum_{i=0}^{p} (2i+2), & p \text{ odd} \\
\sum_{i=0}^{p} (2i+1), & p \text{ even}
\end{cases}.
\end{align*}
\]

From (62), we can conclude that \( n_e(p) > n_o(p) \) for all \( p \in \mathbb{N} \). Precisely speaking

\[
\begin{align*}
n_o(p+2) - n_o(p) &= n_e(p+1) - n_o(p) = p+2 \\
n_o(p+1) &= n_e(p) \quad \Rightarrow \quad \Xi^M_o = \Xi^{M-1}_e \quad \text{or} \quad \Xi^M_e > \Xi^M_o.
\end{align*}
\]

Appendix C Structure of \( A_{\Psi}^{(M,M)} \)

From the definition of \( A_{\Psi}^{(M,M)} \) it is clear that it contains blocks of the integral

\[
D^{(k,l)} = \int_{\mathbb{R}^d} \psi'^e_k(\xi) \xi_l \psi'^e_k(\xi) \ d\xi,
\]

where, the second relation is a result of only considering basis functions upto degree \( M \) in our moment approximation (24a). Using the recursion of the Hermite polynomials (16b), we conclude

\[
\psi'^e_k(\xi) \xi_l = d^{(k,k-1)} \psi'^e_{k-1}(\xi) + d^{(k,k+1)} \psi'^e_{k+1}
\]

where \( \psi'^e_{k+1} \) is vector containing the first \( n_o(k) \) components of \( \psi'^e_{k+1} \). The matrices \( d^{(k,k-1)}, d^{(k,k+1)} \in \mathbb{R}^{n_o(k) \times n_o(k)} \) are diagonal matrices containing the square root entries appearing in the recursion relation. Using the orthogonality of the basis functions, the matrix \( D^{(k,l)} \) can be written as

\[
D^{(k,l)} = \begin{cases} 
\int_{\mathbb{R}^d} \psi'^e_{k-1}(\xi) \psi'^e_{k-1}(\xi) \ d\xi = d^{(k,k-1)}, & l = k-1 \\
\int_{\mathbb{R}^d} \psi'^e_{k+1}(\xi) \psi'^e_{k+1}(\xi) \ d\xi = d^{(k,k+1)} \quad \text{\( \emptyset \)}, & l = k+1 \\
0, & \text{else}
\end{cases}
\]

Note that \( D^{(k,k-1)} \in \mathbb{R}^{n_o(k) \times (n_o(k)-1)} \), where \( n_e(k-1) = n_o(k) \), whereas \( D^{(k,k+1)} \in \mathbb{R}^{n_o(k) \times n_o(k)} \). Since, \( n_e(k) = n_o(k+1) \), \( A_{\Psi}^{(M,M)} \) will consist of blocks of \( D^{(k,k-1)} \) on its main diagonal and blocks of \( D^{(k,k+1)} \) on its off diagonal with no entries below the main diagonal. The matrix plot of last few columns of \( A_{\Psi}^{(M,M)} \) has been shown in Figure 10. From (62), we can conclude

\[
d^{(k,k-1)}_{ii} = \sqrt{\left( \beta^{(1,o)}_k \right)_i}, \quad d^{(k,k+1)}_{ii} = \sqrt{\left( \beta^{(1,o)}_k \right)_i} + 1, \quad i \in \{1, \ldots, n_o(k)\}
\]

where \( \beta^{(1,o)}_k \) are as defined below

**Definition C.1.** Let \( \beta^{o}_k \in \mathbb{R}^{n_o(k) \times d} \) be such that each row of \( \beta^{o}_k \) contains the multi-index of the odd basis functions contained in \( \psi'^o_k(\xi) \). And, let \( \beta^{(1,o)}_k \in \mathbb{R}^{n_o(k)} \) represent the first column of \( \beta^{o}_k \).
Note that all the entries in $\mathbf{p}^{(1,o)}_k$ are odd. Therefore, all the entries along the diagonal of $d^{(k,k+1)}$ and $d^{(k,k-1)}$ are square roots of even and odd numbers respectively. It can be shown that the number of times one appears in $\mathbf{p}^{(1,o)}_k$ is equal to $k + 2$. Thus, $d^{(k,k-1)}$ has the structure

$$
\begin{bmatrix}
\tilde{d}^{(k,k-1)} & 0 \\
0 & I^{k+2}
\end{bmatrix}
$$

(66)

where $\tilde{d}^{(k,k-1)} \in \mathbb{R}((n, (k)-(k+2))) \times (n, (k)-(k+2))$ and $I^{k+2}$ is an identity matrix of size $(k+2) \times (k+2)$. From (64), (65) and (66) we can conclude that

$$
D^{(k,k-1)} = \begin{pmatrix}
\tilde{d}^{(k,k-1)} & 0 \\
0 & I^{k+2}
\end{pmatrix},
\quad D^{(k,k+1)} = \begin{pmatrix}
d^{(k,k+1)} & 0
\end{pmatrix}.
$$

(67)

The matrix $A^{(M,M-1)}_{\Psi}$, which can be constructed by ignoring the contribution from $D^{(M-1,M)}$ into $A^{(M,M)}_{\Psi}$, is upper triangular with blocks of $D^{(k,k-1)}$ along its diagonal. Since $D^{(k,k-1)}$ contains square roots of odd numbers along its diagonal, which are all non-zero, the invertibility of $A^{(M,M-1)}_{\Psi}$ follows.

**Appendix D  Norms of Matrices and Operators**

We will need the result

**Lemma D.1.** Let $A \in \mathbb{R}^{n \times n}$, $n \geq 1$, be given by

$$
A_{ij} = \sqrt{2i-1} \delta_{ij} + \sqrt{2i} \delta_{(i+1)j}, \quad \forall \quad i, j \in \{1, \ldots, n\}.
$$

Then, the solution $x \in \mathbb{R}^n$, to the linear system

$$
A_{ij} x_j = \delta_{in}
$$

is such that $\|x\|_2 = 1$.

**Proof.** For $n = 1$, the result is trivial and so we consider the $n > 1$ case. The existence of a solution to (68) is clear from the structure of $A$ itself. From the first $n-1$ equations of the linear system (68) it follows

$$
x_i \sqrt{2i-1} + x_{i+1} \sqrt{2i} = 0 \quad \forall \quad i \in \{1, 2, \ldots, n-1\}
$$

with which we can express any $x_p$ ($p \geq 2$) in terms of $x_1$ as

$$
x_p = (-1)^{p-1} \prod_{k=1}^{p-1} \sqrt{\frac{2k-1}{2k}} x_1 = (-1)^{p-1} \sqrt{\frac{(2p-3)!!}{(2p-2)!!}} x_1, \quad p \in \{2, \ldots, n\}.
$$

(69)

Thus

$$
\|x\|_2^2 = x_1^2 \left(1 + \sum_{p=2}^{n} \frac{(2p-3)!!}{(2p-2)!!}ight) = x_1^2 \sum_{p=0}^{n-1} \frac{1}{2^p p}.
$$

(70)

From the last equation in (68) and using (69) we have

$$
x_n = \frac{1}{\sqrt{2n-1}} \Rightarrow x_1 = (-1)^{n-1} \sqrt{\frac{(2n-2)!!}{(2n-1)!!}}.
$$

(71)
Using the value of $x_1$ from (71) in (70), we find

$$
\|x\|_2^2 = \frac{(2n-2)!!}{(2n-1)!!} \sum_{p=0}^{n-1} \frac{1}{2^p p!}.
$$

It can be shown through induction that

$$
\sum_{p=0}^{n-1} \frac{1}{2^p p!} = \frac{(2n-1)!!}{(2n-2)!!} \Rightarrow \|x\|_2^2 = 1.
$$
(i) Norm of \( \lim_{q \to \infty} A^{(M,q)}_q \): We define \( \mathcal{H}^+ := L^2(\mathbb{R}^+ \times \mathbb{R} \times f_0^{-1}) \) with the norm \( \| \cdot \|_{\mathcal{H}^+} = \| L^{2}(\mathbb{R}^+ \times \mathbb{R} \times f_0^{-1}) \) and the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}^+} = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R} \times f_0^{-1})} \). Let \( L^2 \colon \mathbb{R}^M \to \mathbb{R}^{2M} \) be given as \( L = \lim_{q \to \infty} A^{(M,q)}_q \); the fact that the range of \( L \) will be contained in \( \mathbb{R}^{2M} \) can be concluded by using the Cauchy-Schwartz inequality. Define \( y \in \mathbb{R}^{2M} \) as

\[
y = Lx = 2 \langle \psi_M f_0, r \rangle_{\mathcal{H}^+} \quad \text{where} \quad r = \sum_{m=0}^{m} x_m \cdot \psi_m f_0
\]

\[
x = (x_0, x_1, \ldots, x_k, \ldots), \quad x_k \in \mathbb{R}^{n(k)}.
\]

Note that the functions, \( \sqrt{2} \psi_0 f_0 \), are orthonormal under \( \langle \cdot, \cdot \rangle_{\mathcal{H}^+} \). This implies \( \| r \|_{\mathcal{H}^+}^2 = \frac{1}{2} \| x \|_2^2 \).

The orthogonal projection of \( r \) onto \( \{ \sqrt{2} \psi_m f_0 \}_{m \leq M} \) can be given as

\[
\mathcal{P} r = \sum_{m=1}^{M} y_m \cdot \psi_m f_0, \quad y = (y_1, y_2, \ldots, y_M)^t, \quad y_k \in \mathbb{R}^{n(k)}.
\]

Therefore, it holds \( \| \mathcal{P} r \|_{\mathcal{H}^+} \leq \| r \|_{\mathcal{H}^+} \). Since \( \| \mathcal{P} r \|_{\mathcal{H}^+}^2 = \frac{\| y \|_2^2}{2} \) and \( \| r \|_{\mathcal{H}^+}^2 = \frac{\| x \|_2^2}{2} \), we obtain \( \| y \|_2^2 \leq \| x \|_2^2 \), which immediately proves that \( \| L \| \leq 1 \).

(ii) Norm of \( A^{(M,M)}_q \): Let \( A = A^{(M,M)}_q \left( A^{(M,M)}_q \right)^t \). Since every row of \( A^{(M,M)}_q \) contains two entries, one on the main diagonal and one on the off diagonal (see appendix-C), every row of \( A \) will contain a maximum of three entries. Since the maximum magnitude of entries in \( A^{(M,M)}_q \) is \( O(\sqrt{M}) \), the maximum magnitude of the entries, in \( A \), will be \( O(M) \). The Gershgorins circle theorem then implies that the maximum eigenvalue of \( A \) will be \( O(M) \) which implies \( \| A^{(M,M)}_q \|_2 \leq C \sqrt{M} \).

(iii) Norm of \( \left( A^{(M,M-1)}_q \right)^{-1} A^{(M,M)}_q \|_2 \): In the coming discussion we will assume \( M \) to be even; for \( M \) being odd, the proof follows along similar lines and will not be discussed for brevity. From the structure of \( A^{(M,M)}_q \), shown in Figure 10, it is clear that it only has a contribution from \( D^{(M-1,M)} \in \mathbb{R}^{n_o(M-1) \times n_o(M)} \), with \( D^{(M-1,M)} \) as defined in (67). Let \( X \in \mathbb{R}^{2M \times n_o(M-1)} \) represent those columns of \( \left( A^{(M,M-1)}_q \right)^{-1} \) which get multiplied with \( D^{(M-1,M)} \) appearing in \( A^{(M,M)}_q \). As a result

\[
\left\| \left( A^{(M,M-1)}_q \right)^{-1} A^{(M,M)}_q \right\|_2 = \| XD^{(M-1,M)} \|_2 \leq \| X \|_2 \| D^{(M-1,M)} \|_2
\]

From (65) it follows that \( \| D^{(M-1,M)} \|_2 \leq C \sqrt{M} \). We will now show that \( X \) is a unitary matrix which will then prove our claim.

Let \( x^{(\omega)} \) denote the \( \omega \)-th column of \( X \) with \( \omega \in \{ 1, \ldots, n_o(M-1) \} \). We can decompose \( x^{(\omega)} \) as

\[
x^{(\omega)} = \left( x^{(\omega)}_{n_o(0)}, x^{(\omega)}_{n_o(1)}, \ldots, x^{(\omega)}_{n_o(M-1)} \right)^t, \quad x^{(\omega)}_{n_o(q)} \in \mathbb{R}^{n_o(q)}.
\]

The different values of \( x^{(\omega)} \), for different values of \( \omega \), can be found by solving the system of
From the entries of definition C.1 the half intervals with a hope to be more accurate along done with the help of a DG scheme. The quadrature points, in the velocity space, have been defined on (grid points based upon the Gauss-Legendre quadrature points defined over the two half space intervals For the one dimensional velocity space, we consider a truncated velocity space Appendices E DISCRETE VELOCITY METHOD

The set of remaining equations can be given as

$$D^{(k,k-1)} x^{(\omega)}_{n_r(k-1)} + D^{(k,k+1)} x^{(\omega)}_{n_r(k+1)} = 0, \quad \forall k \in \{1, \ldots, M-2\}, \quad D^{(M-1,M-2)} x^{(\omega)}_{n_r(M-1)} = 0$$  \hspace{1cm} (72)

where $I^{\omega}_{n_r(M-1)}$ is a diagonal matrix of size $n_r(M-1) \times n_r(M-1)$ such that $(I^{\omega}_{n_r(M-1)})_{ii} = \delta_{i\omega}$ and $D^{(k,k-1)}$ (and $D^{(k,k+1)}$) are as defined in (67). From (72) we conclude

$$x^{(\omega)}_{n_r(M-1)} = 0 \Rightarrow x^{(\omega)}_{n_r(M-2q-1)} = 0, \quad \forall q \in \{1, \ldots, \frac{M}{2}\}.$$  

The set of remaining equations can be given as

$$D^{(k,k-1)} x^{(\omega)}_{n_r(k-1)} + D^{(k,k+1)} x^{(\omega)}_{n_r(k+1)} = 0, \quad \forall k \in \{1, 3, \ldots, M-3\}$$  \hspace{1cm} (74)

The orthogonality of solutions to (74) is clear from the structure of the linear system itself. Therefore, to prove our claim, we need to show that

$$\|x^{(\omega)}\|_2 = 1 \quad \forall \omega \in \{1, \ldots, n_r(M-1)\}$$  \hspace{1cm} (75)

for which we will claim that solving (74), for a given $\omega$, is equivalent to solving a system of the type (68); the result will then follow from lemma D.1. From the entries of $d^{(k,k-1)}$ and $d^{(k,k+1)}$ defined in (65), it follows that the system in (74) is equivalent to

$$\begin{pmatrix}
1 & \sqrt{2} & 0 & 0 & \ldots & \ldots & \ldots \\
0 & \sqrt{3} & \sqrt{4} & 0 & \ldots & \ldots & \ldots \\
0 & 0 & \ddots & \ddots & 0 & \ldots & \ldots \\
0 & 0 & 0 & \sqrt{(\beta_{M-1}^{(1,\omega)})j} - 2 & \sqrt{(\beta_{M-1}^{(1,\omega)})j - 1} & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \sqrt{(\beta_{M-1}^{(1,\omega)})j} & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\begin{pmatrix}
x^{(\omega)}_{n_r(M-2q)} \\
x^{(\omega)}_{n_r(M-2q-1)} \\
\vdots \\
x^{(\omega)}_{n_r(M-2)} \\
x^{(\omega)}_{n_r(M-1)} \\
\delta_{j,\omega}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}$$  \hspace{1cm} (76)

where $\beta_{k}^{(1,\omega)}$ is as defined in definition C.1, $q = \left(\frac{(\beta_{M-1}^{(1,\omega)})j + 1}{2}\right)$ and for every $\omega$, $j \in \{1, \ldots, n_r(M-1)\}$. For $j = \omega$, the system in (76) is the same as (68) and hence (75) follows.

Appendix E DISCRETE VELOCITY METHOD

For the one dimensional velocity space, we consider a truncated velocity space $\xi \in (-5, 5)$ with the grid points based upon the Gauss-Legendre quadrature points defined over the two half space intervals $(-5, 0)$ and $(0, 5)$. We consider 100 quadrature points in both the intervals and 500 elements along the physical space. Similar to the moment approximation, the discretization in the physical space has been done with the help of a DG scheme. The quadrature points, in the velocity space, have been defined on the half intervals with a hope to be more accurate along $\xi = 0$ around which the kinetic solution can be discontinuous; see [21, 31] for details of the discrete velocity method.
E.1 Gaussian density in vacuum:

Figure 11 shows a comparison between the variation of the kinetic solution obtained through a highly refined moment approximation $M = 200$ and a highly refined DVM. Both the solutions show acceptable similarity.

(a) Comparison between the deviation in density ($\lambda_0(f(t = 0.3, x, .))$) obtained through $M = 200$ and DVM; $Kn = 0.1$ and $T = 0.3$.

(b) Variation of $f$ obtained through the moment approximation ($M = 200$); $Kn = 0.1$ and $T = 0.3$.

(c) Variation of $f$ obtained through DVM; $Kn = 0.1$ and $T = 0.3$.

(d) Log values of the point wise relative error in the kinetic solution obtained through $M = 200$ and the DVM; $Kn = 0.1$ and $T = 0.3$.

Figure 11: *In (a) we compare the variation of density obtained through a highly refined moment solution ($M = 200$) and the DVM for the 1D physical and velocity space with $Kn = 0.1$. Both the methods show excellent agreement, in density variation, when highly refined. From fig-(b) and (c) it is clear that the physical space is dominated by a kinetic solution which is continuous in the velocity space.*
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