A RECURSION FORMULA FOR $k$-SCHUR FUNCTIONS

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Abstract. The Bernstein operators allow to build recursively the Schur functions. We present a recursion formula for $k$-Schur functions at $t = 1$ based on combinatorial operators that generalize the Bernstein operators. The recursion leads immediately to a combinatorial interpretation for the expansion coefficients of $k$-Schur functions at $t = 1$ in terms of homogeneous symmetric functions.

1. Introduction

The study of Macdonald polynomials led to the discovery of symmetric functions, $s^{(k)}_{\lambda}(t)$, indexed by partitions whose first part is no larger than a fixed integer $k \geq 1$, and depending on a parameter $t$. Experimentation suggested that when $t = 1$, the functions $s^{(k)}_{\lambda} := s^{(k)}_{\lambda}(1)$ play the fundamental combinatorial role of the Schur basis in the symmetric function subspace $\Lambda^k = \mathbb{Z}[h_1, \ldots, h_k]$; that is, they satisfy properties generalizing classical properties of Schur functions such as Pieri and Littlewood-Richardson rules. The study of the $s^{(k)}_{\lambda}$ led to several characterizations [6, 7, 9] (conjecturally equivalent) and to the proof of many of these combinatorial conjectures. We thus generically call the functions $s^{(k)}_{\lambda}$ $k$-Schur functions (at $t = 1$), but in this article consider only the definition presented in [9].

The Bernstein operators $B_n = \sum_{i \geq 0} (-1)^i h_{n+i} e_i^\perp$ (see Section 3 for the relevant definitions) allow to build the Schur functions recursively. That is, if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, we have

$$B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_\ell} = s_{\lambda},$$

(1.1)

where $s_{\lambda}$ is the Schur function indexed by the partition $\lambda$.

We present in this article a recursion for $k$-Schur functions that generalizes this recursion. It is based on a combinatorial generalization of the Bernstein operators,

$$B^{(k)}_{\lambda_1} B^{(k)}_{\lambda_2} \cdots B^{(k)}_{\lambda_\ell} = s^{(k)}_{\lambda},$$

(1.2)

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition such that $\lambda_1 \leq k$. In this case, the operators $B^{(k)}_i$ are only defined on certain subspaces of $\Lambda^k$, preventing for instance the study of the commutation relations that they could have satisfied. The term combinatorial is used to emphasize the fact that the operators $B^{(k)}_i$ are defined through their action on certain $k$-Schur functions, an action which is combinatorially much in the spirit of the action of the usual $B_i$’s on Schur functions.

Formula (1.2) leads immediately to a combinatorial interpretation for the expansion coefficients of $k$-Schur functions in terms of homogeneous symmetric functions. This interpretation is particularly relevant given that no Jacobi-Trudi type determinantal formula has yet been obtained for $k$-Schur functions.

Key words and phrases. Symmetric functions, Schur functions, Bernstein operators, $k$-Schur functions.

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2. Definitions

2.1. Basic definitions. Most of the definitions in this subsection are taken from [12]. A partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \) is a non-increasing sequence of positive integers. The degree of \( \lambda \) is \( |\lambda| = \lambda_1 + \cdots + \lambda_m \) and the length \( \ell(\lambda) \) is the number of parts \( m \). Each partition \( \lambda \) has an associated Ferrers diagram with \( \lambda_i \) lattice squares in the \( i \text{th} \) row, from the bottom to top. For example,

\[
\lambda = (4, 2, 1) = \begin{array}{ccc}
\cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & & \\
\end{array}
\]  

(2.1)

Given a partition \( \lambda \), its conjugate \( \lambda' \) is the diagram obtained by reflecting \( \lambda \) about the main diagonal. A partition \( \lambda \) is "\( k \)-bounded" if \( \lambda_1 \leq k \). Any lattice square in the Ferrers diagram is called a cell, where the cell \((i, j)\) is in the \( i \text{th} \) row and \( j \text{th} \) column of the diagram. We say that \( \lambda \sqsubseteq \mu \) when \( \lambda_i \leq \mu_i \) for all \( i \). The dominance order \( \triangleright \) on partitions is defined by \( \lambda \triangleright \mu \) when \( \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \) for all \( i \), and \(|\lambda| = |\mu|\).

A skew diagram \( \mu/\lambda \), for any partition \( \mu \) containing the partition \( \lambda \), is the diagram obtained by deleting the cells of \( \lambda \) from \( \mu \). The thick frames below represent \((5,3,2,1)/(4,2)\).

![Skew Diagram](image)

The degree of a skew diagram \( \mu/\lambda \) is \(|\mu| - |\lambda|\). We say that the skew diagram \( \mu/\lambda \) of degree \( \ell \) is a horizontal (resp. vertical) \( \ell \)-strip if it never has two cells in the same column (resp. row).

A cell \((i,j)\) of a partition \( \gamma \) with \((i + 1, j + 1) \not\in \gamma \) is called "extremal". A "removable" corner of partition \( \gamma \) is a cell \((i,j)\) \( \in \gamma \) with \((i,j + 1),(i + 1,j) \not\in \gamma \) and an "addable" corner of \( \gamma \) is a square \((i,j) \not\in \gamma \) with \((i,j - 1),(i - 1,j) \in \gamma \) (note that \((1,\gamma_1 + 1),(\ell(\gamma) + 1,1)\) are also considered to be addable corners). All removable corners are extremal. In the figure below we have labeled all addable corners with \( a \), labeled extremal cells \( e \), and framed the removable corners.

![Skew Diagram](image)

The hook-length of a cell \( c = (i,j) \) in a partition \( \lambda \) is \( \lambda_i - j + \lambda'_j - i + 1 \). That is, the number of cells in \( \lambda \) to the right of \( c \) plus the number of cells in \( \lambda \) above \( c \) plus one. If, as above, \( \lambda = (5,3,3,2) \), the hook-length of the cell \((1,2)\) is 7. We will say that a cell is \( k \)-bounded if its hook-length is not larger than \( k \).
Recall that a \(k+1\)-core is a partition that does not contain any \(k+1\)-hooks (see [3] for a discussion of cores and residues). An example of a 6-core (with the hook-length of each cell indicated) is:

\[
\begin{array}{cccccccc}
1 & 4 & 2 & 1 \\
5 & 3 & 2 \\
7 & 5 & 4 & 1 \\
\end{array}
\]

The \(k + 1\)-residue" of square \((i, j)\) is \(j - i \mod k + 1\). That is, the integer in this square when squares are periodically labeled with \(0, 1, \ldots, k\), where zeros lie on the main diagonal. Here are the 5-residues associated to \((6, 4, 3, 1, 1, 1)\):

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
4 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

We will need the following basic result on cores [1, 8].

**Proposition 1.** Let \(\gamma\) be a \(k+1\)-core.

1. Let \(c\) and \(c'\) be extremal cells of \(\gamma\) with the same \(k+1\)-residue (\(c'\) weakly north-west of \(c\)).
   a. If \(c\) is at the end of its row, then so is \(c'\).
   b. If \(c\) has a cell above it, then so does \(c'\).

2. Let \(c\) and \(c'\) be extremal cells of \(\gamma\) with the same \(k+1\)-residue (\(c'\) weakly south-east of \(c\)).
   a. If \(c\) is at the top of its column, then so is \(c'\).
   b. If \(c\) has a cell to its right, then so does \(c'\).

3. A \(k+1\)-core \(\gamma\) never has both a removable corner and an addable corner of the same \(k+1\)-residue.

2.2. **Bijection: \(k+1\)-cores and \(k\)-bounded partitions.** Let \(\mathcal{C}^{k+1}\) and \(\mathcal{P}^k\) respectively denote the collections of \(k+1\) cores and \(k\)-bounded partitions. There is a bijective correspondence between \(k+1\)-cores and \(k\)-bounded partitions that was defined in [8] by the map:

\[ p : \mathcal{C}^{k+1} \rightarrow \mathcal{P}^k \quad \text{where} \quad p(\gamma) = (\lambda_1, \ldots, \lambda_\ell), \]

with \(\lambda_i\) denoting the number of cells with a \(k\)-bounded hook in row \(i\) of \(\gamma\). Note that the number of \(k\)-bounded hooks in \(\gamma\) is \(|\lambda|\). The inverse map \(c = p^{-1}\) relies on constructing a certain skew diagram \(\gamma/\rho\) from \(\lambda\), and setting \(c(\lambda) = \gamma\). These special skew diagrams are defined:

**Definition 2.** For \(\lambda \in \mathcal{P}^k\), the “\(k\)-skew diagram of \(\lambda\)” is the diagram \(\lambda/k = \gamma/\rho\) where

(i) the number of cells in row \(i\) of \(\lambda/k\) is \(\lambda_i\) for \(i = 1, \ldots, \ell(\lambda)\)

(ii) no cell of \(\lambda/k\) has hook-length exceeding \(k\)

(iii) all cells of \(\rho\) have hook-lengths exceeding \(k+1\) (when considered in \(\lambda\)).

A convenient algorithm for constructing the diagram of \(\lambda/k\) is given by successively attaching a row of length \(\lambda_i\) to the bottom of \((\lambda_1, \ldots, \lambda_{i-1})/k\) in the leftmost position so that no hook-lengths exceeding \(k\) are created.

**Example 3.** Given \(\lambda = (4, 3, 2, 2, 1, 1)\) and \(k = 4\),

\[
\begin{array}{cccccccc}
\lambda = & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array} \quad \Leftrightarrow \quad \lambda/k = & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array} \quad \Leftrightarrow \quad c(\lambda) = & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]
2.3. **Affine symmetric group.** The affine symmetric group \( \tilde{S}_{k+1} \) is generated by the \( k+1 \) elements \( \sigma_0, \ldots, \sigma_k \) satisfying the affine Coxeter relations:

\[
\sigma_i^2 = id, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad (i - j \neq \pm 1 \mod k + 1), \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.
\]

Here, and in what follows, \( \sigma_i \) is understood as \( \sigma_i \mod k+1 \) if \( i \geq k + 1 \). Elements of \( \tilde{S}_{k+1} \) are called affine permutations, or simply permutations. The length of an element \( \sigma \in \tilde{S}_{k+1} \) is the smallest number \( \ell \) such that \( \sigma \) can be written as \( \sigma = \sigma_{i_1} \cdots \sigma_{i_\ell} \) for some \( i_1, \ldots, i_\ell \in \{0, 1, \ldots, k\} \).

Let \( \tilde{S}_{k+1} \) be the subgroup of \( \tilde{S}_{k+1} \) generated by \( \sigma_1, \ldots, \sigma_k \) (and thus isomorphic to the usual symmetric group on \( k \) elements). It is known that the minimal length (left) coset representatives in the quotient \( \tilde{S}_{k+1}/S_{k+1} \) are in bijective correspondence with \( k+1 \)-cores (and thus with \( k \)-bounded partitions). There is a natural action of the affine symmetric group on cores that accounts for this relation \([13, 11]\):

**Definition 4.** The “operator \( \sigma_i \)” acts on a \( k + 1 \)-core by:

- (a) removing all removable corners with \( k + 1 \)-residue \( i \) if there is at least one removable corner of \( k + 1 \)-residue \( i \)
- (b) adding all addable corners with \( k + 1 \)-residue \( i \) if there is at least one addable corner with \( k + 1 \)-residue \( i \)

In that correspondence, if \( \sigma = \sigma_{i_1} \cdots \sigma_{i_\ell} \) is a minimal length coset representative in \( \tilde{S}_{k+1}/S_{k+1} \), then \( \sigma_{i_1} \cdots \sigma_{i_\ell}(\emptyset) \) is the corresponding core (where \( \emptyset \) is the empty core). Given this correspondence, in what follows we will not distinguish between cores and minimal length coset representatives in \( \tilde{S}_{k+1}/S_{k+1} \). We will also write \( \sigma \in \tilde{S}_{k+1}/S_{k+1} \) to mean that \( \sigma \in \tilde{S}_{k+1} \) is a minimal length coset representative. Note that \( \sigma_i \) is not defined when there are no addable or removable corners of residue \( i \). In this case, the corresponding permutation is not a minimal length coset representative in \( \tilde{S}_{k+1}/S_{k+1} \), and will thus be of no concern to us.

**Example 5.** Given \( k = 3 \), the product \( \sigma_2 \sigma_3 \sigma_1 \sigma_0 \in \tilde{S}_4 \) corresponds to the 4-core \( \gamma = (3, 1, 1) \), since:

\[
\sigma_2 \sigma_3 \sigma_1 \sigma_0(\emptyset) = \sigma_2 \sigma_3 \sigma_1(\color{red}{\begin{array}{c}3 \\ 1 \\ 1 \end{array}}) = \sigma_2 \sigma_3(\color{red}{\begin{array}{c}2 \\ 1 \\ 1 \end{array}}) = \sigma_2(\color{red}{\begin{array}{c}2 \\ 1 \\ 1 \end{array}}) = \color{green}{\begin{array}{c}2 \\ 1 \\ 1 \end{array}}.
\]

Given \( r < s \in \mathbb{Z} \), we can define the transposition \( t_{r,s} \in \tilde{S}_{k+1} \) by

\[
t_{r,s} = \sigma_r \sigma_{r+1} \cdots \sigma_{s-2} \sigma_{s-1} \sigma_{s-2} \cdots \sigma_{r+1} \sigma_r
\]

It is easy to see that \( t_{r,s} \) is an involution. Furthermore, if \( s - r < k + 1 \), then it is easy to see using the Coxeter relations that

\[
t_{r,s} = \sigma_{s-1} \sigma_{s-2} \cdots \sigma_{r+1} \sigma_r \sigma_{r+1} \cdots \sigma_{s-2} \sigma_{s-1}.
\]

Now, given two \( k + 1 \)-cores \( \delta \) and \( \gamma \), we say that \( \delta \preceq \gamma \) if there exists a transposition \( t_{r,s} \), such that \( t_{r,s} \delta = \gamma \) and \( |p(\gamma)| = |p(\delta)| + 1 \). The transitive closure of this relation corresponds in \( \tilde{S}_{k+1}/S_{k+1} \) to the (strong) Bruhat order.

**Example 6.** Let \( k = 3 \) and \( \lambda = (2, 1, 1) \). Apply \( t_{1,3} \) to \( \iota(\lambda) = (3, 1, 1) = \gamma \).

\[
t_{1,3}(\gamma) = \sigma_1 \sigma_2 (\begin{array}{c}2 \\ 1 \\ 1 \end{array}) = \sigma_1 (\begin{array}{c}2 \\ 1 \\ 1 \end{array}) = \delta
\]

Also note that \( t_{1,3}(\delta) = \gamma \). Since \( p(\delta) = (1, 1, 1) \), then we have that \( (1, 1, 1) \preceq (3, 1, 1) \).

Equivalently, it can be shown that \( \delta \preceq \gamma \) iff \( \delta \subset \gamma \) and \( |p(\gamma)| = |p(\delta)| + 1 \). It is also known \([13, 11]\) that the Bruhat order, the transitive closure of this relation, is given by \( \delta \preceq \gamma \) iff \( \delta \subset \gamma \).

The following result is proved in \([5]\). Note that a ribbon is a connected skew-diagram that does not contain any \((2, 2)\) subdiagram.
Proposition 7. Let $\delta \preceq \gamma$ be $k + 1$-cores with $t_r, s \delta = \gamma$ and $0 < r < s$. Then

1. $s - r < k + 1$.
2. Each connected component of $\gamma / \delta$ is a ribbon with $s - r$ cells in diagonals of $k + 1$-residues $r, r + 1, \ldots, s - 1$.
3. The components are translates of each other and their heads lie on “consecutive” diagonals of $k + 1$-residue $s - 1$.

2.4. $k$-Schur functions. We now present the characterization of $k$-Schur functions given in [9].

Definition 8. Let $\gamma$ be a $k + 1$-core with $m$ $k$-bounded hooks and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of $m$. A “$k$-tableau” of shape $\gamma$ and “$k$-weight” $\alpha$ is a filling of $\gamma$ with integers $1, 2, \ldots, r$ such that

(i) rows are weakly increasing and columns are strictly increasing
(ii) the collection of cells filled with letter $i$ are labeled by exactly $\alpha_i$ distinct $k + 1$-residues.

Example 9. The 3-tableaux of 3-weight $(1, 3, 1, 2, 1, 1)$ and shape $(8, 5, 2, 1)$ are:

Remark 10. When $k$ is large, a $k$-tableau $T$ of shape $\gamma$ and $k$-weight $\mu$ is a semi-standard tableau of weight $\mu$ since no two diagonals of $T$ will have the same residue.

We denote the set of all $k$-tableaux of shape $\epsilon (\mu)$ and $k$-weight $\alpha$ by $T^k_\alpha (\mu)$, and define the “$k$-Kostka numbers” as:

$$K^{(k)}_{\mu \alpha} = |T^k_\alpha (\mu)|.$$ \hspace{1cm} (2.5)

As is the case for the Kostka number, they are such that $K^{(k)}_{\mu \alpha} = K^{(k)}_{\mu \gamma}$ if $\gamma$ is a permutation of $\alpha$, and satisfy a triangularity property.

Property 11. For any $k$-bounded partitions $\lambda$ and $\mu$,

$$K^{(k)}_{\mu \lambda} = 0 \quad \text{when} \quad \mu \nleq \lambda \quad \text{and} \quad K^{(k)}_{\mu \mu} = 1.$$ \hspace{1cm} (2.6)

Thus the matrix $|K^{(k)}|_{\mu, \lambda}$ (with $\mu$ and $\lambda$ running over all $k$-bounded partitions of a given degree) is invertible, naturally giving rise to a family of symmetric functions.

Definition 12. The “$k$-Schur functions”, indexed by $k$-bounded partitions, are defined as forming the unique basis of $\Lambda^k = \mathbb{Z}[h_1, \ldots, h_k]$ such that:

$$h_\lambda = s^{(k)}_\lambda + \sum_{\mu : \mu \succ \lambda} K^{(k)}_{\mu \lambda} s^{(k)}_\mu \quad \text{for all} \quad \lambda \quad \text{such that} \quad \lambda_1 \leq k.$$ \hspace{1cm} (2.7)

From this $k$-tableau characterization, many properties of $k$-Schur functions are derived in [9]. Of these properties, the only one relevant to this work is the $k$-Pieri rule, which we now present in the form given in [9].

Any proper subset $A \subseteq \mathbb{Z}_{k+1} = \{0, \ldots, k\}$ decomposes into unions $I_1 \cup I_2 \cup \cdots \cup I_m$ of maximal cyclic intervals. For each cyclic component $[a, b]$ of $A$, we will let $\sigma_A$ be equal to the product of factors $\sigma_{\sigma_{b+1}} \cdots \sigma_a$ (observe the descending order of the indices). For instance, if $k = 8$ and $A = \{0, 1, 3, 4, 6, 8\}$, we have that the cyclic components are $[8, 1] = \{8, 0, 1\}$, $[3, 4] = \{3, 4\}$ and $[6, 6] = \{6\}$. The corresponding element $\sigma_A$ is thus equal to $\sigma_1 \sigma_0 \sigma_8 \sigma_4 \sigma_3 \sigma_6 = \sigma_4 \sigma_3 \sigma_6 \sigma_1 \sigma_0 \sigma_8 = \cdots$ (the components commute among themselves).
Proposition 13. Let $h_\ell$ be the $\ell$th complete symmetric function. Furthermore, let $\lambda$ be a $k$-bounded partition, and $\gamma = c(\lambda)$ be its corresponding $k+1$-core. Then, if $\ell \leq k$, the $k$-Schur functions satisfy the $k$-Pieri rule:

$$h_\ell s_\lambda^{(k)} = \sum_A s_{p(\sigma_A(\gamma))}^{(k)}$$

where the sum is over all subsets $A$ of $\mathbb{Z}_{k+1}$ of cardinality $\ell$ such that $p(\sigma_A(\gamma))$ is a partition of size $\ell + |\lambda|$.

Remark 14. This formulation of the $k$-Pieri rule is equivalent to the one presented in [9]. In that case, the $k$-Pieri rule can be interpreted as

$$h_\ell s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)},$$

where the sum is over all $k$-bounded partitions $\mu$ such that $c(\mu)/c(\lambda)$ is a horizontal strip with exactly $\ell = |\mu| - |\lambda|$ distinct residues. In [5], it is shown that this condition is equivalent to $c(\mu)$ being equal to $\sigma_A(c(\lambda))$ for $A$ a subset of $\mathbb{Z}_{k+1}$ of cardinality $\ell$.

Example 15. We illustrate the $k$-Pieri rule for $k = 6$ by doing the product of $h_4$ and $s_{(4,3,2,2,2,1)}^{(6)}$. First, we give the Ferrers diagram of the $k + 1$-core $c(\lambda)$ associated to $\lambda = (4,3,2,2,2,1)$, with residues on the addable positions and frames on the $k$-bounded cells.

We then show the possible subsets $A$ of $\mathbb{Z}_{k+1}$ of cardinality $\ell = 4$ such that $p(\sigma_A(c(\lambda)))$ is a partition of 18:

$$h_4 s_{(4,3,2,2,2,1)}^{(6)} = s_{(6,3,3,2,2,1,1)}^{(6)} + s_{(5,3,3,2,2,2,1)}^{(6)} + s_{(5,4,3,2,2,2)}^{(6)} + s_{(5,4,2,2,2,2,1)}^{(6)} + s_{(4,4,3,2,2,2,1)}^{(6)}$$

3. Bernstein operators

The notation used in this section is taken from [12]. For $m$ a nonnegative integer, the operator $e_m^\perp$ is defined such that given any symmetric functions $f$ and $g$,

$$\langle e_m^\perp f, g \rangle = \langle f, e_m g \rangle,$$

with $e_m$ the $m^{th}$ elementary symmetric function and $\langle \cdot, \cdot \rangle$ the unique scalar product with respect to which the Schur functions are orthonormal. It can be shown that the operator $e_m^\perp$ has the following simple action on a Schur function

$$e_m^\perp s_\lambda = \sum_\mu s_\mu$$
where the sum is over all partition \( \mu \) such that \( \lambda / \mu \) is a \( m \)-vertical strip.

For \( n \) a nonnegative integer, the Bernstein operator is [14]

\[
B_n = \sum_{i \geq 0} (-1)^i h_{n+i} e_i^+, \tag{3.3}
\]

where \( h_m \) is the \( m \)-th complete symmetric function. The Bernstein operators allow to build the Schur functions recursively. That is, for \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \),

\[
B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_\ell} \cdot 1 = s_\lambda. \tag{3.4}
\]

Or equivalently, if \( \hat{\lambda} = (\lambda_2, \ldots, \lambda_\ell) \), then

\[
B_{\lambda_1} s_{\hat{\lambda}} = s_\lambda. \tag{3.5}
\]

4. The main formula

Note that for the remainder of the article, as it was the case in the previous section, \( \hat{\lambda} \) will stand for the partition \( \lambda \) without its first part.

Before being able to describe analogs of these operators for the \( k \)-Schur functions, we need some definitions. Let \( \gamma \) be a \( k+1 \)-core, and let \( x \) be the cell corresponding to the leftmost \( k \)-bounded cell in the first row of \( \gamma \). If \( x \) lies in column \( j \), then let the main subpartition of \( \hat{\gamma} \) (relatively to \( \gamma \)), be the subpartition of \( \hat{\gamma} \) made out of the columns of \( \hat{\gamma} \) from column \( j \) up to column \( \gamma_2 \) (that is, from column \( j \) rightward). For instance, let \( k = 6 \), and consider the 7-core \( \gamma = (5, 5, 3, 3, 2, 2, 1, 1, 1) \). As illustrated in the following diagram, where the \( k \)-bounded cells of \( \gamma \) are in bold face, the leftmost \( k \)-bounded cell, \( x \), in the first row of \( \gamma \) is in column 3. Therefore, the main subpartition of \( \hat{\gamma} \) is the partition filled with \( \circ \)'s in the diagram.

Remark 16. It is important to realize that the concept of main subpartition is only defined for a \( \delta \) such that \( \delta \preceq \omega \) for a given \( k+1 \)-core \( \omega \). When using the term main subpartition of \( \hat{\gamma} \), it is understood that the larger partition is in this case \( \gamma \).

Remark 17. The cells in the main subpartition of \( \hat{\gamma} \) are all \( k \)-bounded. When \( k \) is large enough, the main subpartition of \( \hat{\gamma} \) coincides with \( \hat{\gamma} \).

Remark 18. If in \( \hat{\gamma} \) there are columns to the left of its main subpartition, then they are all strictly larger than the largest column of the main subpartition. This is because the cell to the left of \( x \) in \( \gamma \) (see the example above) would not have otherwise a hook-length larger than \( k+1 \).

Recall that \( \delta \preceq \omega \) iff \( \delta \subseteq \omega \) and the number of \( k \)-bounded hooks in \( \delta \) is one less than that in \( \omega \). Also recall from Proposition 7 that if \( \delta \preceq \omega \), then \( \omega / \delta \) is a union of identical ribbons (of size smaller or equal to \( k \)) whose heads (southeast-most cell of the ribbon) occur on consecutive diagonals of a certain \( k+1 \)-residue. A ribbon will be horizontal if, as its name suggests, it coincides with a horizontal partition \( (n) \) for some \( n \).

Definition 19. Let \( \gamma \) be a core such that the main subpartition of \( \hat{\gamma} \) is of length \( m \). We will say that the core \( \delta \) can be obtained by removing a vertical \( (k, \ell) \)-strip from \( \hat{\gamma} \) if there exists a sequence of cores \( \hat{\gamma} = \omega^{(1)} \supseteq \omega^{(2)} \supseteq \cdots \supseteq \omega^{(\ell+1)} = \delta \) such that

(1) \( \omega^{(i+1)} \subsetneq \omega^{(i)} \) for all \( i = 1, \ldots, \ell \).
$(2)$ $\omega^{(i)}/\omega^{(i+1)}$ is a union of horizontal ribbons, the lowest of which appears in a row $r_i$ with $1 \leq r_i \leq m$.

$(3)$ $r_1, \ldots, r_\ell$ are all distinct.

**Example 20.** Let $k = 5$ and consider $\gamma = (6,6,3,3,1,1,1)$. It can be checked that the length of the main subpartition of $\hat{\gamma}$ is $m = 2$. The $k + 1$-core $\delta = (5,4,3,2,1,1,1,1,1)$ can be obtained from $\hat{\gamma}$, by removing the following vertical $(5,2)$-strip:

$$\gamma = \omega^{(1)} = (6,6,3,3,1,1,1,1,1) \supset \omega^{(2)} = (6,4,3,3,1,1,1,1,1) \supset \omega^{(3)} = (5,4,3,2,1,1,1,1,1) = \delta.$$ 

In the following sequence of Ferrers diagrams, we see that all the conditions for a vertical $(5,2)$-strip are satisfied. The framed cells correspond to successive ribbons having their lowest occurrence in different rows and within the first $m = 2$ rows.

![Ferrers diagrams](image)

**Remark 21.** By Remark 18 in condition $(2)$ of Definition 19, the lowest ribbons are always contained entirely in the main subpartition of $\hat{\gamma}$.

**Lemma 22.** Suppose we have a vertical $(k, \ell)$-strip $\hat{\gamma} = \omega^{(1)} \supset \omega^{(2)} \supset \cdots \supset \omega^{(\ell+1)} = \delta$, whose lowest ribbons occur in rows $r_1, \ldots, r_\ell$. Then, there exists a sequence $\hat{\gamma} = \omega^{(1)} \supset \omega^{(2)} \supset \cdots \supset \omega^{(\ell+1)} = \delta$, whose lowest ribbons occur in rows $r_1 > \cdots > r_\ell$. That is, removing a vertical $(k, \ell)$-strip can always be done in a certain order (by removing the ribbon whose lowest ribbon is the highest, then the one whose lowest ribbon is the second highest, and so on).

**Proof.** Suppose we have $\omega^{(i+1)} \prec \omega^{(i)} \prec \omega^{(i-1)}$, with both $\omega^{(i-1)}/\omega^{(i)}$ and $\omega^{(i)}/\omega^{(i+1)}$ given by a union of horizontal ribbons, the lowest of which are respectively $R_1$ and $R_2$. By Proposition 4 we have $\omega^{(i)} = t_{r,s}(\omega^{(i-1)})$, where $r$ (resp. $s - 1$) is the residue of the leftmost (resp. rightmost) cell in $R_1$. Similarly, we have $\omega^{(i+1)} = t_{r',s'}(\omega^{(i)})$, where $r'$ (resp. $s'-1$) is the residue of the leftmost (resp. rightmost) cell in $R_2$. If $R_1$ does not sit on top of $R_2$ (in which case $R_1$ would necessarily have to be removed first), we have that the cyclic intervals $[r,s-1]$ and $[r',s'-1]$ are disjoint and not contiguous. This is because $R_1$ and $R_2$ belong to the main subpartition of $\hat{\gamma}$ (which does not have repeated diagonals of the same residue) and because $r \neq s' \mod k + 1$ (otherwise there would be a hook of length $k + 1$ in the core $\omega^{(i-1)}$). Observe that in this case,

$$t_{r,s}t_{r,s}(\omega^{(i-1)}) = t_{r,s}(t_{r,s}(\omega^{(i-1)})) \prec t_{r,s}(\omega^{(i-1)}) \quad \Longrightarrow \quad t_{r',s'}(\omega^{(i-1)}) \prec \omega^{(i-1)}$$

and

$$t_{r,s}(\omega^{(i-1)}) \prec \omega^{(i-1)} \quad \Longrightarrow \quad t_{r,s}t_{r,s}(\omega^{(i-1)}) = t_{r,s}(t_{r,s}(\omega^{(i-1)})) \prec t_{r,s}(\omega^{(i-1)}),$$

since there is no interference in the adding and deleting process involved in acting with the transpositions. Therefore,

$$\omega^{(i+1)} = t_{r,s}t_{r,s}(\omega^{(i-1)}) \prec t_{r,s}(\omega^{(i-1)}) \prec \omega^{(i-1)}$$

leads to

$$\omega^{(i+1)} = t_{r,s}t_{r,s}(\omega^{(i-1)}) \prec t_{r,s}(\omega^{(i-1)}) \prec \omega^{(i-1)},$$

meaning that the ribbons $R_1$ and $R_2$ (with their translates) can be removed in any order. The general result then follows by applying this idea again and again.

The following results will show that removing a $(k, \ell)$-strip from a $k + 1$-core $\hat{\gamma}$ removes a $\ell$-vertical strip from the $k$-bounded partition associated to $\hat{\gamma}$. 

---

[4]: Footnote

[18]: Footnote
Lemma 23. Let $\gamma$ and $\delta$ be two $k+1$-cores such that $|p(\gamma)| = |p(\delta)| + 1$ and such that $\delta \subset \gamma$. If $\gamma/\delta$ is a union of horizontal ribbons, the highest of which appears in row $i$, then $p(\gamma) = p(\delta) + e_i$, where $e_i$ is the vector with a 1 in position $i$ and 0 everywhere else.

Proof. First note that if a row of $\gamma/\delta$ does not contain a horizontal ribbon then the number of $k$-bounded cells in that row is the same in $\gamma$ and in $\delta$. This is because the hook-length of a cell in that row is changed by at most one cell, preventing a change of the hook length from more than $k + 1$ to less than $k + 1$ (recall that a $k + 1$-core does not contain cells with hook-lengths of $k + 1$). As for the remaining rows, recall that the head of the ribbons occur in consecutive diagonals of the same residues. The proof is then illustrated in the following example at $k = 3$, where the cells in bold face are the horizontal ribbons in $\gamma/\delta$, and the cells with an $x$ are the cells that went from not being $k$-bounded in $\gamma$ to being $k$-bounded in $\delta$.

One simply needs to observe that in each row that contains a ribbon, the number of cells with an $x$ is equal to the number of cells in bold face (except in the highest such row). Since $|p(\gamma)| = |p(\delta)| + 1$, this implies that it must differ by one in the highest row that contains a ribbon. □

Proposition 24. Let $\delta$ and $\hat{\gamma}$ be $k+1$-cores such that $\delta$ can be obtained by removing a vertical $(k, \ell)$-strip from $\hat{\gamma}$. Then $p(\hat{\gamma})/p(\delta)$ is a vertical $\ell$-strip (in the usual sense).

Proof. From the previous lemma, we simply need to show that when going from $\hat{\gamma}$ to $\delta$ by removing horizontal ribbons, two ribbons will never occur in the same row. From Lemma 22 it is possible to choose $\hat{\gamma} = \omega^{(1)} \supset \omega^{(2)} \supset \cdots \supset \omega^{(\ell+1)} = \delta$ such that the horizontal ribbons are removed from top to bottom in the main subpartition. Let $\omega^{(i)}/\omega^{(i+1)}$ contain a given ribbon $\hat{R}$ and all its translates. The rightmost cell of $\hat{R}$, the translate of $\hat{R}$ in the main subpartition, has a residue $r$ that is not contained in any ribbon above it in the main subpartition (from Remark 21 and 17). Therefore, when going from $\hat{\gamma}$ to $\omega^{(i)}$, no cells of residue $r$ are removed or added, and thus the rightmost cell of $\hat{R}$ is also extremal in $\hat{\gamma}$. By Proposition 1 this means that the rightmost cell of $\hat{R}$ has to be at the end of its row in $\hat{\gamma}$ since the rightmost cell of $\hat{R}$ is also at the end it its row in $\hat{\gamma}$ (no two ribbons can occur in the same row of the main subpartition by definition of $(k, \ell)$-strip). Therefore, no horizontal ribbons can ever occur to the right of $\hat{R}$. □

We can now define the recursion for $k$-Schur functions that extends formula (3.5).

Definition 25. Let $V_{k,\ell}$ be the $\mathbb{Z}$-linear span of $k$-Schur functions whose first part is not larger than $r$. Given a partition $\nu$ such that $s_{\nu}^{(k)} \in V_{k,\ell}$, let $\gamma = (r, \nu_1, \nu_2, \ldots)$ and $\gamma = \mathfrak{c}(\lambda)$. Then the linear operator $e_{\ell,r}$ is defined on $V_{k,\ell}$ to be such that

$$e_{\ell,r}^{+} s_{\nu}^{(k)} = \sum_{\mu} s_{\mu}^{(k)},$$

where the sum is over all $k$-bounded partition $\mu$ such that $\mathfrak{c}(\mu)$ can be obtained by removing a vertical $(k, \ell)$-strip from $\hat{\gamma}$. If there is no such $\mu$, the result is simply defined to be zero.

Note that we only use the symbol $e_{\ell,r}^{+}$ in analogy with $e_{\ell,r}$. That is, to the best of our knowledge, $e_{\ell,r}^{+}$ is not the adjoint of multiplying by some symmetric function $e_{\ell,r}$ with respect to any scalar product. We should also point out that the operator $e_{\ell,r}^{+}$ does in fact depends on $r$, since $r$ appears in the definition of the $k + 1$-core $\gamma$ (and since extracting a $(k, \ell)$-strip from $\hat{\gamma}$ actually depends on $\gamma$).
Remark 26. By Proposition \[\ref{prop:mu},\] the \(\mu\)'s such that \(s_\mu^{(k)}\) occur in the action of \(e_{\ell,r}^\perp\) on \(s_\nu^{(k)}\) are such that \(\nu/\mu\) is a vertical \(\ell\)-strip. These \(\mu\)'s are thus a subset of the \(\mu\)'s such that \(s_\mu\) occur in the action of \(e_{\ell}^\perp\) on \(s_\nu\).

Example 27. Let \(\nu = (4, 3, 2, 2, 1)\) and \(k = 6\). Then \(\lambda = (4, 4, 3, 2, 2, 1)\) and \(\gamma = c(\lambda) = (6, 6, 3, 2, 2, 1)\). Hence \(\hat{\gamma} = (6, 3, 2, 2, 1)\) =

where the framed cells correspond to the main subpartition. If we apply \(e_{1,4}^\perp\), the vertical \((k, \ell)\)-strips need to be of length \(\ell = 1\). The following diagrams show the vertical \((k, 1)\)-strips that can be obtained, with the \(k\)-bounded cells marked with \(\circ\) (thus corresponding to the \(k\)-bounded partitions).

From here, we obtain that
\[e_{1,4}^\perp s_\nu^{(6)} = s_{(4,3,2,1,1)}^{(6)} + s_{(4,2,2,2,1)}^{(6)} + s_{(3,3,2,2,1)}^{(6)} \tag{4.2}\]

Now, for \(r = 1, \ldots, k\), let
\[B_r^{(k)} = \sum_{\ell \geq 0} (-1)^\ell h_{r+\ell} e_{\ell,r}^\perp \tag{4.3}\]

Note that this operator is only defined on \(V^{(k,r)}\). The main result of this article is then the following.

Theorem 28. Let \(\lambda = (\lambda_1, \lambda_2, \ldots)\) be a \(k\)-bounded partition. Then
\[B_{\lambda_1}^{(k)} s_\lambda^{(k)} = s_\lambda^{(k)} \tag{4.4}\]

Example 29. Using \(k\) and \(\lambda\) as in Example \[\ref{ex:example27},\] we will show that:
\[B_{4}^{(6)} s_\lambda^{(6)} = s_\lambda^{(6)} \tag{4.5}\]

By definition, our equation amounts to:
\[B_{1}^{(6)} s_\lambda^{(6)} = \sum_{\ell \geq 0} (-1)^\ell h_{4+\ell} e_{\ell,4}^\perp s_\lambda^{(6)} \tag{4.6}\]

According to the diagram of equation \[\ref{eq:27},\] we only need to consider vertical \((k, \ell)\)-strips up to \(\ell = 2\).

For the first term in the sum, acting with \(e_{0,4}^\perp\) on \(s_\lambda^{(6)}\) gives \(s_\lambda^{(6)}\), since we have to extract a vertical \((k, 0)\)-strip (which amounts to doing nothing). The action of \(e_{1,4}^\perp\) was explained in example \[\ref{ex:example27}.\] To compute the action of \(e_{2,4}^\perp\) on \(s_\lambda^{(6)}\), we present here the diagrams of the cores that can be obtained by removing a vertical \((k, 2)\)-strips from \(\lambda:\)
Remark 33. In example 31, the changeable cells of the two

Example 34. In example 34, the changeable cells of the two OX diagrams are located in the same positions: in the fifth and sixth cells of the first row and in the third cell of the second row.
Proof of Theorem 28. Equation (4.4) can be rewritten as
\[ \sum_{\ell \geq 0} (-1)^\ell h_{\ell,1+\ell} \epsilon_{\ell,1} \delta_s^{(k)} = \delta_s^{(k)}. \] (4.8)

Using the action of \( h_{\ell,1+\ell} \) and \( \epsilon_{\ell,1} \) on \( k \)-Schur functions, this is equivalent to
\[ \sum_{(\delta,A) \in D^{(k)}} (-1)^{|A| - |\lambda|} s^{(k)}_{p(\sigma_A(\delta))} = s^{(k)}_{\lambda}. \] (4.9)

Let \( \gamma = \epsilon(\lambda) \). We will show in Lemma 3.7 that the pair \((\hat{\gamma}, B)\), where \( B \) is the subset of \( Z_{k+1} \) of size \( \lambda_1 \) such that \( \sigma_B(\hat{\gamma}) = \gamma \), is the unique pair of \( D^{(k)}_\lambda \) that does not have a changeable cell. Given that the pair \((\hat{\gamma}, B)\) corresponds to a term \(+ s^{(k)}_{\lambda}\) in the l.h.s. of (4.9), to prove Theorem 28 suffices to show that
\[ \sum_{(\delta,A) \in C^{(k)}_\lambda} (-1)^{|A|} s^{(k)}_{p(\sigma_A(\delta))} = 0, \] (4.10)
where we recall that \( C^{(k)}_\lambda \) is the set of \((\delta, A) \in D^{(k)}_\lambda \) that have a changeable cell.

This result will readily follow if there exists an involution \( \varphi : C^{(k)}_\lambda \to C^{(k)}_\lambda \) that maps the pair \((\delta, A)\) to a pair \((\delta', A')\) such that \( \sigma_A(\delta) = \sigma_A(\delta') \) and \( (-1)^{|A|} = -(-1)^{|A'|} \). Such a sign-reversing involution will be constructed in the next section (see Definition 4.4 and Proposition 4.10).

By applying Theorem 28 again and again, we obtain a combinatorial interpretation for the expansion coefficients of \( k \)-Schur functions in terms of homogeneous symmetric functions.

Corollary 35. Let \( \lambda \) be a \( k \)-bounded partition such that \( \epsilon(\lambda) = \gamma \). Suppose that the sequence \( S = (\gamma^{(0)}, \ldots, \gamma^{(r)}) \), with \( \emptyset = \gamma^{(0)} \subseteq \gamma^{(1)} \subseteq \cdots \subseteq \gamma^{(r)} = \gamma \), is such that for all \( i = 1, \ldots, r \), we have that \( \gamma^{(i-1)} \) can be obtained by removing a \((k, \ell_i)\)-vertical strip from \( \gamma^{(i)} \) for some \( \ell_i \in \{0, \ldots, k\} \). Define part\((S)\) to be the partition corresponding to the rearrangement of the sequence \((\ell_1 + p_1, \ldots, \ell_r + p_r)\), where \( p_i \) is the length of the first row of \( p(\gamma^{(i)}) \) (equivalently, \( p_i \) is the number of \( k \)-bounded cells in the first row of \( \gamma^{(i)} \)). Finally, define \( \text{sgn}(S) \) to be \((-1)^{\ell_1 + \cdots + \ell_r}\). Then
\[ s^{(k)}_{\lambda} = \sum_S \text{sgn}(S) h_{\text{part}(S)}, \] (4.11)
where the sum is over all possible sequences \( S \) of the form given above.

Example 36. The sequences \( S \) in the previous corollary can be interpreted as certain fillings of \( \gamma = \epsilon(\lambda) \), as we will illustrate with an example. If \( k = 4 \) and \( \lambda = (2, 2, 2, 1) \), the possible sequences \( S \) are seen to be in correspondence with the following fillings of \( \gamma = (3, 2, 2, 1) \):

In these diagrams, the partition \( \gamma^{(i)} \) for a sequence \( S = (\gamma^{(0)}, \ldots, \gamma^{(r)}) \) can be obtained by reading the subdiagram containing the letters up to \( i \) in the diagram. The framed cells containing letter \( i \) indicate the location of the \((k, \ell_i)\)-vertical strip extracted from \( \gamma^{(i)} \) to obtain \( \gamma^{(i-1)} \). For instance, the fifth diagram of the second row corresponds to the sequence \( S' = (\emptyset, (1, 1), (1, 1, 1), (3, 2, 2, 1)) \). We illustrate this with the following figure:

\[ \emptyset \xleftarrow{1} \begin{array}{cccc} & & & \\ & & & \\ & & & \\ \end{array} \xleftarrow{0} \begin{array}{cccc} & & & \\ & & & \\ & & & \\ \end{array} \xleftarrow{2} \begin{array}{cccc} & & & \\ & & & \\ & & & \\ \end{array} \]
Each diagram corresponds to a $k+1$-core in the sequence, and the number above each arrow indicates the size of the $(k,\ell)$-vertical strip that is extracted to obtain the $k+1$-core that follows in the sequence. The bold face numbers in each diagram correspond to the $k$-bounded cells in each first row. We then see that $\text{part}(S') = (4,2,1)$, coming from the composition $(1+1,1+0,2+2)$, and $\text{sgn}(S') = (-1)^{1+0+2} = -1$.

We thus have

$$
\begin{align*}
s^{(4)}_{(2,2,1)} &= h_2 h_2 h_2 h_1 - h_3 h_2 h_2 - h_3 h_2 h_2 - h_3 h_2 h_1 h_1 + h_3 h_2 h_2 + h_4 h_2 h_1 - h_3 h_2 h_1 h_1 \\
&\quad + h_3 h_2 h_2 + h_3 h_2 h_1 - h_3 h_2 h_1 h_1 - h_4 h_2 h_1 - h_4 h_2 h_1 + h_4 h_3 \\
&= h_2 h_2 h_2 h_1 - 2 h_3 h_2 h_1 h_1 + h_3 h_3 h_1 + h_4 h_1 h_1 h_1 - h_4 h_2 h_2
\end{align*}
$$

5. The involution

**Lemma 37.** The only pair in $D^{(k)}_\lambda$ whose corresponding diagram does not have a changeable cell is $(\hat{\gamma}, B)$, where $B$ is the unique subset of $\mathbb{Z}_{k+1}$ of size $\lambda_1$ such that $\sigma_B(\hat{\gamma}) = \gamma$.

**Proof.** It was proven in [9] that the $k$-Schur functions obey

$$
h_\ell s^{(k)}_\nu = s^{(k)}_{(\ell,\nu)} + \sum_{\mu} s^{(k)}_\mu
$$

(5.1)

where the sum is over some $\mu$’s that are larger than $(\ell,\nu)$ in dominance order. That there exists a unique subset $B$ of $\mathbb{Z}_{k+1}$ of size $\lambda_1$ such that $\sigma_B(\hat{\gamma}) = \gamma$ follows from that equation when translated in the language of cores (see the $k$-Pieri rule (2.8)). By the construction of the core associated to a $k$-bounded partition (see Example [3]), $\sigma_B(\hat{\gamma}) = \gamma$ corresponds to $\hat{\gamma}$ with cells added in all the columns of $\hat{\gamma}$ plus possibly some extra columns. Therefore, the $OX$ diagram of $(\hat{\gamma}, B)$ has an $X$ at the top of every column of $\sigma_B(\hat{\gamma}) = \gamma$. Since there are no $O$ cells (no cells were removed), there are no $OX$ cells and thus there are no changeable cells.

The main observation in the previous paragraph is that acting with $\sigma_B$ adds a cell on top of every column of $\hat{\gamma}$. If $|A| = |B|$ and $A$ adds a cell on top of every column of $\hat{\gamma}$, then it is easy to see that we must have $A = B$. This is because in this case, starting from the second row, the cores $\sigma_A(\hat{\gamma})$ and $\sigma_B(\hat{\gamma})$ are equal since $\sigma_A(\hat{\gamma})/\gamma$ and $\sigma_B(\hat{\gamma})/\gamma$ are horizontal strips. To have the same number of $k$-bounded cells, $\sigma_A(\hat{\gamma})$ and $\sigma_B(\hat{\gamma})$ must thus be equal, which gives that $A = B$. Therefore, if $A \neq B$ and $|A| = |B|$, when acting with $\sigma_A$ on $\hat{\gamma}$, some columns of $\hat{\gamma}$ will not contain an $X$, and thus some changeable cells will be present.

If $\delta \neq \hat{\gamma}$, then the $OX$ diagram of $(\delta, A)$ will contain some $O$ cells. The only possible case without changeable cells is the case where, weakly to the right of a certain column $c$, the columns are entirely filled with $O$ cells (since a cell below an $O$ cell is changeable, as is an $OX$ cell), and where $X$ cells appear on top of every column of $\hat{\gamma}$ to the left of column $c$.

But this is impossible: from the previous paragraph there are at most $|B|$ distinct residues in the columns above $\hat{\gamma}$, and we have to add $|A| > |B|$ residues in a subset of those columns. \qed
Remark 38. The elements of $B$ are the residues of the $X$'s that sit on top of the main subpartition of $\hat{\gamma}$ plus the residues of the $X$'s to the right of the main subpartition. This is because by definition of the main subpartition of $\hat{\gamma}$, the $\lambda_1$ $k$-bounded cells in the first row of $\gamma$ start exactly in the leftmost column of the main subpartition. There are thus exactly $\lambda_1 = |B|$ distinct residues at the top of the columns of $\gamma$ starting from the leftmost column of the main subpartition.

Example 39. Let $k = 6, \lambda = (4, 4, 3, 3, 2, 1, 1)$ and $\gamma = c(\lambda)$. The following Ferrers diagrams illustrate Remark 38. The first is the Ferrers diagram of $\gamma$, where the $k$-bounded cells are given with their residues. The second is the $OX$ diagram of the unique pair $(\hat{\gamma}, B)$ that has no changeable cells, with the main subpartition in framed cells:

$$\gamma = \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8
\end{array}$$

$$\hat{\gamma}, B = \begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times
\end{array}$$

In this case, $B = \{1, 3, 4, 6\}$, recovered from the bold faced $X$ in the second diagram.

Lemma 40. Let $(\delta, A) \in C^{(k)}_{\lambda}$. Then, in the $OX$ diagram associated to $(\delta, A)$, there are no $O$ cells to the right of the rightmost changeable cell in $(\delta, A)$. Furthermore, the rightmost changeable cell in $(\delta, A)$ is in the main subpartition of $\hat{\gamma}$.

Proof. Let $c$ be the rightmost changeable cell in $(\delta, A)$. By definition of a changeable cell, to the right of $c$ in the $OX$ diagram of $(\delta, A)$, every column either does not contain any $O$ and finishes with an $X$ or is entirely made out of $O$'s. Also observe that, to the right of $c$, no $X$ cells can appear to the right of an $O$ cell.

Suppose there are some $O$ cells to the right of $c$. We have in this case $\delta \subset \hat{\gamma}$. From the previous comment, the last column of $\delta$ is entirely made out of $O$'s. Now, since $\sigma_A(\delta)/\delta$ is a horizontal strip and, as we have seen in the proof of Lemma 37, $\gamma/\hat{\gamma}$ has a cell over all columns of $\hat{\gamma}$, we have that $\sigma_A(\delta) \subset \gamma$ (no $X$ cells can appear to the right of an $O$ cell, and thus no $X$ cell will appear in the columns to the right of $\delta \subset \hat{\gamma}$). But this implies that $\sigma_A(\delta) \subset \gamma$ in the Bruhat order, and thus $|p(\sigma_A(\delta))| < |p(\gamma)|$ which is a contradiction. This gives the first claim.

In the case where there are some $O$ cells by the definition of the pair $(\delta, A)$ there are some $O$ cells in the main subpartition of $\hat{\gamma}$, and thus the first claim immediately implies that the rightmost changeable cell in $(\delta, A)$ is in the main subpartition of $\hat{\gamma}$.

Finally, in the case where there are no $O$ cells, we have $\delta = \hat{\gamma}$, $|A| = |B|$, and $A \neq B$, with $B$ as in Lemma 37. Suppose that $c$ is not in the main subpartition of $\hat{\gamma}$. Then, in the $OX$ diagram of $(\hat{\gamma}, A)$ there are $X$'s sitting on top of all columns of the main subpartition. Since, by the argument about the Bruhat order given before in the proof, we must have $\sigma_A(\hat{\gamma}) \not\subseteq \sigma_B(\hat{\gamma}) = \gamma$, and since $\gamma/\hat{\gamma}$ has a cell over every column of $\hat{\gamma}$, this implies that the first row of $\sigma_A(\hat{\gamma})$ is larger than the first row of $\gamma$. By Remark 38 all the residues in $B$ are thus contained in $A$. Since $|A| = |B|$, this gives the contradiction $A = B$.

We now describe the involution $\varphi$ (which will only be shown to be an involution in Proposition 46).

Definition 41. Let $(\delta, A) \in C^{(k)}_{\lambda}$. The involution $\varphi$ is defined according to the two following cases:

I Suppose the rightmost changeable cell $c$ in the $OX$ diagram of $(\delta, A)$ is an $OX$ cell, and let $i$ be the row in which it is found. In this case, the cells with an $OX$ in row $i$ are the only ones with an $O$. They correspond to a horizontal ribbon $R$. The involution $\varphi$ then changes the cells where $R$ and its translates are located into empty cells. This can be translated in the $(\delta, A)$ language in the following way. Let $r$ be the residue of $c$ and let $r'$ be the residue of the leftmost $OX$ cell in row $i$. The involution is then $\varphi : (\delta, A) \mapsto (t_{r', r+1}(\delta), A \setminus \{r\})$. 


II Suppose the rightmost changeable cell $c$ in the OX diagram of $(\delta, A)$ is an empty cell, and let $i$ be the row in which it is found. In this case, it can be shown that $c$ has a residue $r$ that does not belong to $A$. Let $b$ be the leftmost changeable cell in row $i$ whose residue $r'$ is such that $\{r', r' + 1, \ldots, r - 1, r\} \subseteq A \cup \{r\}$. The involution changes the cells in row $i$ from $b$ to $c$ into OX cells (plus the corresponding translates). In the $(\delta, A)$ language, this means that $\varphi: (\delta, A) \mapsto (t_{r', r'+1}(\delta), A \cup \{r\})$.

Example 42. Let $k = 5$, $\lambda = (3, 3, 3, 3, 2, 2, 1)$, $\delta = \tilde{\gamma} = c(\lambda)$ and $A = \{2, 3, 5\}$. We are in the case II situation since there are no OX cells. The leftmost changeable cell of $(\delta, A)$ is framed in the corresponding Ferrers diagram, where the $k$-bounded cells of $\delta$ are given with their residues. Then the action of the involution is:

\[
(\delta, A) = \begin{array}{c}
X \\
X \\
X \\
X \\
X
\end{array} \quad \xrightarrow{\varphi} \quad (t_{5,1}(\delta), A \cup \{0\}) = \begin{array}{c}
X \\
X \\
X \\
X \\
X
\end{array}
\]

In the resulting OX diagram, the rightmost changeable cell is an OX cell and we are thus in the case I situation. The definition of the involution then takes us back to $(\delta, A)$.

Observe that in $(t_{5,1}(\delta), A \cup \{0\})$, there is a horizontal ribbon of residues 5 and 0. Considering also the ribbon of residue 1 below this ribbon, we can form a vertical $(5, 2)$-strip. Take out this vertical $(5, 2)$-strip from $\tilde{\gamma}$, to form $\delta = (7, 6, 5, 4, 3, 2, 2, 1)$. Adding the residues $A = \{1, 2, 3, 4, 5\}$ leads to a case I situation, where again the rightmost changeable cell has been framed:

\[
(\delta, A) = \begin{array}{c}
X \\
X \\
X \\
X \\
X
\end{array} \quad \xrightarrow{\varphi} \quad (t_{1,2}(\delta), A \setminus \{1\}) = \begin{array}{c}
X \\
X \\
X \\
X \\
X
\end{array}
\]

Again, we see that the action of $\varphi$ on the resulting diagram takes us back to the initial pair $(\delta, A)$.

Lemma 43. In case I, residue $r' - 1$ does not belong to $A$.

Proof. In row $i$ of $\delta$, there is an extremal cell of residue $r' - 1$ at the end of its row. Suppose $r' - 1 \in A$. Let $\sigma_A = \sigma_{A'} \sigma_{A''}$ where $r' - 1$ belongs to $A'$ and $r' - 2$ (if it exists) belongs to $A''$. In $\sigma_{A''}(\delta)$, in row $i$, the extremal cell of residue $r' - 1$ is still at the end of its row. Now, when time comes to act with $\sigma_{r'-1}$, for $\sigma_A$ to increase the number of $k$-bounded hooks by $|A|$, there needs to be an addable corner of residue $r' - 1$ in $\sigma_{A''}(\delta)$. We will show now that this addable corner is above row $i$, which will lead to the contradiction that in $\sigma_{A''}(\delta)$ there is an extremal cell of residue $r' - 1$ above row $i$ that is not at the end of its row (see Proposition II).
Since there is no changeable cell to the right of $c$, in the $OX$ diagram associated to $(\delta, A)$, every column to the right of $c$ ends with an $X$ (and none of them contains an $O$ by Lemma 40). Suppose there is an $X$ of residue $r' - 1$ below $c$ in the $OX$ diagram of $(\delta, A)$, and thus sitting on top of a cell of residue $r'$. We have seen that $c$ belongs to the main subpartition and thus by Remark 17 this $X$ can only occur in the first row of the diagram. But this means that $X$’s are also found in the first row up until at least residue $r$, or else we find a cell having hook-length equal to $k + 1$. Let the cell with an $X$ of residue $r$ in the first row of the diagram be called $a$. By a previous comment, every column between $c$ and $a$ ends with an $X$, and thus this amounts to exactly $k+1-(i-1)=k-i+2$ (recall that $i$ is the row of $c$) distinct residues of the $X$’s. Recall that $|A|$ is the number of residues to add to $\delta$, thus $|A| \geq k - i + 2$. Now, if the first column of the main subpartition is of height $h_1$, we have $\lambda_1 \leq k - h_1$. The number of horizontal ribbons removed when going from $\gamma$ to $\delta$ is at most $h_1 - i + 1$ (one in each row above $c$ in the main subpartition plus the one in row $i$). Since $|A| = \lambda_1 + \text{the number of horizontal ribbons removed}$, then $|A| \leq k - h_1 + (h_1 - i + 1) = k - i + 1$. This is a contradiction. 

\textbf{Lemma 44.} In case II, residue $r' - 1$ does not belong to $A$.

\textbf{Proof.} If the cell immediately to the left of $b$ exists and is free from above (therefore changeable), then its residue cannot be in $A$ because otherwise this would violate the definition of $\varphi$ in case II.

Now suppose that $r' - 1 \in A$, and let $\sigma_A = \sigma_A' \sigma_A''$ where $r' - 1$ belongs to $A'$ and $r' - 2$ (if it exists) belongs to $A''$. Whether there is or not a cell to left of $b$, in $\sigma_A''(\delta)$ the cell above $b$ is an addable corner of residue $r' - 1$ (since the cell immediately to the left of $b$, as we just saw, cannot be free from above). Thus $\sigma_A$ adds an $X$ above $b$, which leads to the contradiction that $b$ is not changeable. 

\textbf{Lemma 45.} In case II, residue $r$ does not belong to $A$.

\textbf{Proof.} Suppose that $r \in A$, and let $\sigma_A = \sigma_A' \sigma_A''$ where $r$ belongs to $A'$ and $r - 1$ (if it exists) belongs to $A''$. Note that acting with $\sigma_A''$ cannot add an $X$ above $c$, since $c$ is changeable, and cannot add an $X$ to the right of $c$ since if $r + 1 \in A$, then $r + 1 \in A'$. Therefore, when $\sigma_r$ acts, the cell $c$ is a removable corner. This cannot be if $\sigma_A$ is to increase the number of $k$-bounded hooks by $|A|$. 

\textbf{Proposition 46.} The map $\varphi : \mathcal{C}_\lambda^{(k)} \rightarrow \mathcal{C}_\lambda^{(k)}, (\delta, A) \mapsto (\delta', A')$ is a well-defined sign-reversing involution (that is, the cardinalities of $A$ and $A'$ are different modulo 2) such that $\sigma_A(\delta) = \sigma_A'(\delta')$.

\textbf{Proof.} Consider case I. In this case, $(\delta', A') = (t_{r',r+1}(\delta), A\setminus\{r\})$. Suppose we have the $(k, \ell)$-strip $\gamma = \omega^{(1)} \supset \omega^{(2)} \supset \cdots \supset \omega^{(\ell+1)} = \delta$, and that $\omega^{(i)} \cup \omega^{(i+1)}$ contains the horizontal ribbon $R$. Thus, from Proposition 7 we have $\omega^{(i)} = t_{r',r+1}(\omega^{(i+1)})$. We will now see that

$$\gamma = \omega^{(1)} \supset \cdots \supset \omega^{(i)} = t_{r',r+1}(\omega^{(i+1)}) \supset t_{r',r+1}(\omega^{(i+2)}) \supset \cdots \supset t_{r',r+1}(\omega^{(\ell+1)}) = t_{r',r+1}(\delta) \quad (5.2)$$

is a vertical $(k, \ell - 1)$-strip. By definition, in the $OX$ diagram associated to $(\delta, A)$ the cells corresponding to $R$ are $OX$ cells. This implies that there are no $O$ cells below $R$, and thus that $R$ does not sit on top of any of the ribbons that occur later in the vertical $(k, \ell)$-strip. Furthermore, by Lemma 10 there are no $O$ cells to the right of $c$, and thus neither there are horizontal ribbons to the right of $R$. 

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Therefore, by Lemma 22, the ribbon $R$ (corresponding to $t_{r',r+1}$) could have been extracted last, which gives that (5.2) is a vertical $(k, \ell - 1)$-strip. Now, since $c$ is filled with an $OX$, we have $r \in A$. Therefore, $|A \setminus \{r\}| = |A| - 1$ and $\varphi$ is a sign-reversing map. Given that $r' - 1 \notin A$ by Lemma 13, we can let $\sigma_{A \setminus \{r\}} = \sigma_{D'}\sigma_{D''}$, where $\sigma_{D''} = \sigma_{r-1}\sigma_{r-2}\cdots\sigma_{r-1}\sigma_{r'}$. Therefore, using
\begin{equation}
 t_{r',r+1} = \sigma_{r'}\sigma_{r+1}\cdots\sigma_{r-2}\sigma_{r-1}\sigma_{r-2}\cdots\sigma_{r-1}\sigma_{r'}
\end{equation}
we find
\begin{equation}
\sigma_{A'}(\delta') = \sigma_{A \setminus \{r\}}t_{r',r+1}(\delta) = \sigma_{D'}\sigma_{r-1}\cdots\sigma_{r'}(\delta) = \sigma_{A}(\delta).
\end{equation}
Therefore case I is a well defined sign-reversing map such that $\sigma_{A}(\delta) = \sigma_{A'}(\delta')$.

Now, we consider case II. In this case $(\delta', A') = (t_{r',r+1}(\delta), A \cup \{r\})$. From Lemma 15, we have $|A \cup \{r\}| = |A| + 1$, and thus $\varphi$ is again a sign-reversing map. By Lemma 15, $r' - 1 \notin A$, so we can let $\sigma_{A \cup \{r\}} = \sigma_{D'}\sigma_{D''}$, where $\sigma_{D''} = \sigma_{r-1}\cdots\sigma_{r'}$. Therefore, using the same idea as before, we find
\begin{equation}
\sigma_{A}'(\delta') = \sigma_{A \cup \{r\}}t_{r',r+1}(\delta) = \sigma_{D'}\sigma_{r-1}\cdots\sigma_{r'}(\delta) = \sigma_{A}(\delta).
\end{equation}
By definition, we have that $|p(\sigma_{A}(\delta))| = |p(\delta)| + |A|$. We also have that
\begin{equation}
|p(\sigma_{A'}(\delta'))| = |p(\sigma_{A \cup \{r\}}t_{r',r+1}(\delta))| \leq |p(t_{r',r+1}(\delta))| + |A| + 1
\end{equation}
since $\sigma_{A \cup \{r\}}$ can increase the degree of $t_{r',r+1}(\delta)$ by at most the cardinality of $A \cup \{r\}$. Using (5.5), we then find that $|p(t_{r',r+1}(\delta))| \geq |p(\delta)| - 1$. On the other hand, from the definition of case II, we see that applying $\sigma_{r-1}\cdots\sigma_{r'}$ on $\delta$ removes $r - r' + 1$ $k$-bounded cells from $\delta$. This gives, using
\begin{equation}
 t_{r',r+1} = \sigma_{r-1}\cdots\sigma_{r'}\cdots\sigma_{r-1}\sigma_{r},
\end{equation}
that $|p(t_{r',r+1}(\delta))| \leq |p(\delta)| - 1$. Therefore, $|p(t_{r',r+1}(\delta))| = |p(\delta)| - 1$, and thus, $t_{r',r+1}(\delta) = \delta$.

By Proposition 7, this corresponds to removing a horizontal ribbon $R$ in row $i$, since applying $t_{r',r+1}$ to $\delta$ removes among other things the extremal cells of residues $r', \ldots, r$ in row $i$. Note that by Lemma 11, the lowest occurrence of $R$ is in the main partition of $\gamma$, and no more $OX$’s are found in row $i$ (thus $R$ is the only horizontal ribbon in its row). As a consequence, if the vertical $(k, \ell)$-strip $\gamma = \omega^{1} \supset \omega^{2} \supset \cdots \supset \omega^{(\ell+1)} = \delta$ is associated to $\delta$, then the vertical $(k, \ell + 1)$-strip $\gamma = \omega^{1} \supset \omega^{2} \supset \cdots \supset \omega^{(\ell+1)} = \delta \supset t_{r',r+1}(\delta)$ is associated to $t_{r',r+1}(\delta)$. Therefore case II is also a well defined sign-reversing map such that $\sigma_{A}(\delta) = \sigma_{A'}(\delta')$.

Finally, observe that the map $\varphi$ is such that the rightmost changeable cell of $(\delta, A)$ corresponds to the rightmost changeable cell of $(\delta', A')$. By construction, and by Lemma 15 which insures that case II is the inverse of case I, $\varphi$ is thus an involution. \hfill \Box

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