Well-posedness of SVI solutions to singular-degenerate stochastic porous media equations arising in self-organised criticality

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Abstract

We consider a class of generalised stochastic porous media equations with multiplicative Lip-
schitz continuous noise. These equations can be related to physical models exhibiting self-organised
criticality. We show that these SPDEs have unique SVI solutions which depend continuously on the
initial value. In order to formulate this notion of solution and to prove uniqueness in the case of a
slowly growing nonlinearity, the arising energy functional is analysed in detail.

Keywords: singular-degenerate SPDE, stochastic variational inequalities, generalised porous media,
self-organised criticality

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1 Introduction

We consider a class of singular-degenerate generalised stochastic porous media equations

\[ dX_t \in \Delta(\phi(X_t)) \, dt + B(t, X_t) \, dW_t, \]
\[ X_0 = x_0, \]  

(1.1)
on a bounded, smooth domain \( \Omega \subseteq \mathbb{R}^d \) with zero Dirichlet boundary conditions and \( x_0 \in H^{-1} \), where \( H^{-1} \) is the dual of \( H_0^1(\Omega) \). In the following, \( W \) is a cylindrical Wiener process on some separable Hilbert space \( \mathcal{H} \), and the diffusion coefficients \( B : [0, T] \times H^{-1} \times \Omega \to L_2(\mathcal{H}) \) take values in the space of

Hilbert-Schmidt operators \( L_2(\mathcal{H}) \). The nonlinearity \( \phi : \mathbb{R} \to \mathbb{R}^2 \) is the subdifferential of a convex lower-semicontinuous symmetric function \( \psi : \mathbb{R} \to \mathbb{R} \) (sometimes called “potential”), which grows at least linearly and at most quadratically for \( |x| \to \infty \). As paradigmatic examples, we mention the maximal monotone extensions of

\[ \phi_1(x) = \text{sgn}(x) \left( 1 - \mathbb{1}_{(-1,1)}(x) \right) \]  
\[ \phi_2(x) = x \left( 1 - \mathbb{1}_{(-1,1)}(x) \right), \]

(1.2)

which are encountered in the context of self-organised criticality. Indeed, equation (1.1) with the first
nonlinearity in (1.2) is related to a particle model which was first introduced by Bak, Tang and Wiesenfeld
in their celebrated works [3] and [4]. We refer to Section 1.2 below for details and references.

The main merits of this article are as follows. First, we give a meaning to (1.1) by defining a suitable
notion of solution and proving the existence and uniqueness of such solutions. Second, we extend the
applicability of the framework of SVI solutions, which features several properties which are desirable
independently of the specific equation presented above. For instance, it applies to stochastic partial
differential equations (SPDE) with a very general nonlinear drift term, and solutions for general initial
data can be identified by means of the equation and not only in a limiting sense.

We briefly outline the strategy that we are going to apply. First, we rewrite (1.1) into the form

\[ dX_t \in -\partial \varphi(X_t) \, dt + B(t, X_t) \, dW_t, \]

(1.3)

which incorporates the multivalued function \( \phi \) into an energy functional \( \varphi : H^{-1} \to [0, \infty] \). For example,
in case of the nonlinearity \( \phi_1 \) in (2.5), we define

\[ \varphi(u) = \begin{cases} 
\|\psi(u)\|_{TV}, & \text{if } u \text{ is a finite Radon measure on } \Omega, \\
+\infty, & \text{else},
\end{cases} \]

(1.4)
where $\psi$ is the anti-derivative of $\phi$, i.e., $\partial \psi = \phi$. For the precise definition of a convex function of a measure, we refer to Section 3 below. We then derive a stochastic variational inequality (SVI) from (1.5) and define a corresponding notion of solution, see Definition 2.5 below. In order to construct such a solution we first show that $\varphi$ as defined above is lower-semicontinuous, which then allows to show the convergence of an approximating sequence gained by a Yosida approximation of the nonlinearity and the addition of a viscosity term. Furthermore, in the proof of uniqueness, it is crucial to show that $\varphi$ can be well approximated by its values on $L^2$, which we ensure by showing that it coincides with the lower-semicontinuous hull of $\varphi|_{L^2}$ in $H^{-1}$. To this end, we will construct approximating sequences by an interplay of mollification and shifts, inspired by the construction of [11, Lemma A6.7]. This constitutes one technical focus of this work.

The structure of this article is as follows: In the subsequent sections of the introduction, we will give a brief overview on the mathematical literature concerning the solution theory of generalised stochastic porous media equations, and we will point out how equation (1.1) is motivated by the physics literature. In Section 2 we state the precise assumptions and formulate the first main result of this article, in which the well-posedness of Equation (1.1) is established (see Theorem 2.7 below). We prove the lower-semicontinuity of the abovementioned energy functional $\varphi$ and the property of $\varphi$ being the lower-semicontinuous hull of $\varphi|_{L^2}$ in $H^{-1}$ in Section 3, the latter of which is the second main result (see Theorem 3.8 below). In Section 4 the well-posedness result will be proved, following the arguments of [37, Section 2].

1.1 Mathematical Literature

In the recent decades, stochastic porous media equations have been very present in the mathematical literature. For the original case

$$dX_t = \Delta \phi(X_t)dt + B(t, X_t)dW_t,$$

(1.5)

where $\phi(r) = r^{[m]} := |r|^{m-1}r$ for $r \in \mathbb{R}$ and $m \geq 1$ ($m = 1$ representing the stochastic heat equation), a concisely summarised well-posedness analysis can be found in [53], which goes back to the work of Krylov and Rozovskii [45] and Pardoux [51]. In [54], the theory is extended to the fast diffusion case $m \in (0, 1)$, and other nonlinear functions $\phi$ are considered. A setting with a more general monotone and differentiable nonlinearity is considered in [9].

A severe additional difficulty arises when one considers the limit case $m = 0$, in which $\phi$ becomes multivalued. The first articles treating this type of porous medium equations, [10] and [8], either require $\phi$ to be surjective or more restrictions on the initial state or the noise. In [41], the $m = 0$ limit of (1.5) can be treated, but one has to restrict to more regular initial data or to the concept of limiting solutions. For general initial conditions, this notion of solution contains no characterisation in terms of the equation, which is often necessary for further work such as stability results (see e.g. [39]).

In [7] and later in [13, 36], the concept of stochastic variational inequalities (SVIs) and a corresponding notion of solution have been used to overcome these issues. We note that in [36], an identification of a functional as lower-semicontinuous hull was needed in the context of $p$-Laplace type equations with a $C^2$ potential, going back to results from [2, 27]. In [37], the existence and uniqueness of SVI solutions was proven for the $m = 0$ limit of (1.5), for which a refinement of previous methods became necessary, because the naive choice for the energy functional does not lead to an energy space with adequate compactness properties. The arising difficulties when setting up the energy functional are similar to the ones mentioned above for $\varphi$ from [14]. They have been overcome in [37] by using the specific shape of the nonlinearity, which allows to set the energy functional to

$$\varphi(u) = \begin{cases} \|u\|_{TV}, & \text{if } u \text{ is a finite Radon measure on } \mathcal{O}, \\ +\infty, & \text{else} \end{cases}$$

for $u \in H^{-1}$, which allows to use structural properties of the TV norm. With more regularity or structural assumptions on the noise and/or the initial state, more regularity for SVI solutions or the existence of strong solutions can be proved, as e.g. in [37, 38, 13, 32]. For the regularisation by noise of quasi-linear SPDE with possibly singular drift terms, we also mention the works [31, 43].

We next mention several different approaches to stochastic porous media equations. The article [14] considers the equation on an unbounded domain, the works [6, 18] use an approach via Kolmogorov
equations. In [12], an operatioral approach to SPDE is introduced which can be applied to generalised stochastic porous media equations with continuous nonlinearities. In [35, 21] and [19], stochastic porous media equations are solved in the sense of kinetic or entropy solutions, respectively. Previous works in those directions are, e.g., [16, 22] and [17, 26, 44]. [38] makes use of a rough path approach leading to pathwise rough kinetic/entropy solutions and including regularity results, with [30, 16] as some of the related preceding works.

Regarding the construction and analysis of the energy functional arising in the context of SVIs, we rely on techniques from [23, 50] on convex functionals of Radon measures. For the deterministic theory on porous medium equations, we refer to [50] and [57]. Regarding results on the long-time behaviour of singular-degenerate SPDE, see e.g. [28, 33] for the existence of random attractors, [40, 20, 48] for ergodicity and [34, 11] for finite-time extinction in the case of purely multiplicative noise.

1.2 Self-organised criticality (SOC)

The model (1.1) can to some extent be associated with processes exhibiting self-organised criticality (SOC). This concept postulates that many randomly driven processes featuring a critical threshold, at which relaxation events are triggered, possess a non-equilibrium statistical invariant state, in which intermittent events can be observed, the size of which is distributed by a power law. SOC has been initially discussed in view of certain cellular automaton models, which are introduced and explained in much detail in [3] and [4], as well as later by [52]. In these models, particles can be interpreted as units of granular material piling up, which coined the notion of “sandpile models”. Other applications, where self-organised critical behaviour has been observed, are the size of landslides [49], earthquakes (the famous Gutenberg-Richter law, see [42]) and stock prices [47].

In [25] and [24], the abovementioned sandpile models are related to a model similar to (1.1), i.e. a stochastic process in a continuous function space where mass of a continuously distributed size is both added and subtracted. In contrast to the assumptions mentioned above, the potential in [25] is only one-sided. As this leads to a process just forced towards $-\infty$, where no avalanches would occur, we consider symmetric potentials instead.

The underlying mechanisms of SOC have been a matter of lively discussion in the literature, see e.g. [58] for a review. The present work is supposed to contribute to this question by noting that SPDEs with singular-degenerate drift and additive noise incorporate several characteristic properties of the original sandpile models, such as deterministic dynamics which are locally switched on at a certain threshold. However, they also differ from them in other perspectives, such as the non-discrete structure. By setting up a theory for those processes, we ultimately hope to gain insight into their long-time statistics, see e.g. [48]. Thereby, we aim to investigate whether SOC extends to the continuous setting and potentially set the stage for new ways of explaining this statistical effect.

1.3 General notation

Unless specified differently, function or measure spaces will be understood to be defined on a smooth, bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$. We write $L^p = L^p(\Omega)$ for the usual Lebesgue spaces with norm $\|\cdot\|_{L^p}$ and scalar product $\langle \cdot, \cdot \rangle_{L^2}$ if $p = 2$. The Lebesgue measure is denoted by $dx$, and a measure with density $h \in L^1$ with respect to $dx$ is denoted by $h \, dx$. Furthermore, $H^1_0 = H^1_0(\Omega)$ denotes the Sobolev space of $L^2$ functions whose first-order weak derivatives exist and are in $L^2$, and which have zero trace, with norm $\|u\|_{H^1_0} = \|\nabla u\|_{L^2}$. The full space analogues $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$ are defined correspondingly. Furthermore, let $H^{-1}$ denote the topological dual of $H^1_0$. We use $-\Delta$ to denote the corresponding Riesz isomorphism, which gives rise to the inner product

$$\langle u, v \rangle_{H^{-1}} = \langle u, (-\Delta)^{-1} v \rangle_{H^1_0} \quad \text{for all } u, v \in H^{-1},$$

where the notation $\nu \cdot (u, v)_{V'} = \nu(u, v)_{V'}$ denotes evaluating a functional $u$ belonging to the dual space $V'$ of a Banach space $V$ at a vector $v \in V$.

Moreover, we let $C^0 = C^0(\Omega)$ denote the set of all continuous functions on $\Omega$ vanishing at the boundary, while we write $C^0_c = C^0_c(\Omega)$ for continuous functions with compact support. The same notation applies to spaces $C^k$ of $k$ times continuously differentiable functions.
For $m \in [0, 1]$ we define the set
\[ L^{m+1} \cap H^{-1} := \left\{ v \in L^{m+1} : \exists C \geq 0 \text{ such that } \int v \eta \, dx \leq C \| \eta \|_{H^1_0} \text{ for all } \eta \in C^1_c \right\}. \]

Note that $L^2 = L^2 \cap H^{-1}$ by the Cauchy-Schwarz and Poincaré inequalities. To each $v \in L^{m+1} \cap H^{-1}$, one can injectively assign a map
\[ C^1 \ni \eta \mapsto \int v \eta \, dx. \quad (1.6) \]

By continuity, (1.6) can be injectively extended to a bounded linear functional on $H^1_0$, which we call $\iota_m(v)$. The resulting map $\iota_m : L^{m+1} \cap H^{-1} \to H^{-1}$ is thus injective.

Let $\mathcal{M} = \mathcal{M}(\mathcal{O})$ be the space of all signed Radon measures on $\mathcal{O}$ with finite total variation, which is isomorphic to the dual space $(C^0_0)'$ via
\[ \mathcal{M} \ni \mu \mapsto \bar{\mu} \in (C^0_0)', \]
\[ \bar{\mu}(f) = \int f \, d\mu. \]

This allows us to use $(C^0_0)'$ and $\mathcal{M}$, as well as $\bar{\mu}$ and $\mu$ interchangeably. The variation measure of $\mu \in \mathcal{M}$ is denoted by $|\mu| := \mu_+ + \mu_-$ and the total variation of $\mu$ is given by
\[ \|\mu\|_{TV} = |\mu|(\mathcal{O}). \]

Note that the total variation is also the operator norm if the measure is interpreted as an element of $(C^0_0)'$ by the Riesz-Markov representation theorem (see e.g. [29, Theorem 1.200]). We define the space of measures of bounded energy by
\[ \mathcal{M} \cap H^{-1} := \left\{ \mu \in \mathcal{M} : \exists C \geq 0 \text{ such that } \int \eta(x) \, d\mu(x) \leq C \| \eta \|_{H^1_0} \text{ for all } \eta \in C^1_c(\mathcal{O}) \right\}. \]

By a density argument, restricting a measure $\mu \in \mathcal{M} \cap H^{-1}$ to a function on $C^1_0$ is an injective operation. Moreover, by continuity $\mu|_{C^1_0}$ can be injectively extended to a bounded linear functional on $H^1_0$, which we call $\iota(\mu)$. The resulting map $\iota : \mathcal{M} \cap H^{-1} \to H^{-1}$ is thus injective.

**Remark 1.1.** Let $0 \leq m' < m \leq 1$ and $v \in L^{m+1} \cap H^{-1}$. Then, it is easy to see that $v \in L^{m'+1} \cap H^{-1}$, $v \, dx \in \mathcal{M} \cap H^{-1}$ and
\[ \iota_m(v) = \iota_{m'}(v) = \iota(v \, dx). \]

In general, constants may vary from line to line, but are always positive and finite.

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# 2 Assumptions and main result

**Assumptions 2.1.** We require the following assumptions throughout this article.

(A1) $W$ is a cylindrical Itô-Wiener process in some separable Hilbert space $U$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t \geq 0}$, which means the following: There is a Hilbert-Schmidt embedding $J$ from $U$ to another Hilbert space $U_1$, which can be chosen to be bijective (see e.g. [53, Remark 2.5.1]). Defining $Q_1 := JJ^*$, $Q_1$ is linear, bounded, non-negative definite, symmetric and has finite trace, so that we obtain a classical $Q_1$-Wiener process $W$ on $U_1$. Moreover, for an operator $\tilde{B} : U \to H^{-1}$ we have
\[ \tilde{B} \in L_2(U, H^{-1}) \Leftrightarrow \tilde{B} \circ J^{-1} \in L_2 \left( Q_1^{1\frac{1}{2}}(U_1), H^{-1} \right), \quad (2.1) \]
such that if (2.1) is satisfied, we can define
\[
\int_0^T \tilde{B} \, d\tilde{W}_t := \int_0^T \tilde{B} \circ J^{-1} d\tilde{W}_t.
\]

(A2) The diffusion coefficients \( B : [0, T] \times H^{-1} \times \Omega \to L_2(U, H^{-1}) \) take values in the space of Hilbert-Schmidt operators, are progressively measurable and satisfy
\[
\|B(t, v) - B(t, w)\|_{L_2(U, H^{-1})}^2 \leq C \|v - w\|_{H^{-1}}^2, \quad \text{for all } v, w \in H^{-1}, \quad (2.2)
\]
\[
\|B(t, v)\|_{L_2(U, L^2)}^2 \leq C(1 + \|v\|_{L^2}^2), \quad \text{for all } v \in L^2, \quad (2.3)
\]
\[
\|B(t, 0)\|_{L_2(U, H^{-1})}^2 \leq C, \quad (2.4)
\]
for some constant \( C > 0 \) and all \((t, \omega) \in [0, T] \times \Omega\).

(A3) The so-called potential \( \psi : \mathbb{R} \to [0, \infty) \) is convex and lower-semicontinuous, and we assume \( \psi(0) = 0 \), which then implies \( 0 \in \partial \psi(0) \). For simplicity, we furthermore impose the symmetry assumption
\[
\psi(x) = \psi(-x) \quad \text{for all } x \in \mathbb{R}.
\]

(A4) Define \( \phi = \partial \psi : \mathbb{R} \to 2^\mathbb{R} \), the subdifferential of \( \psi \), and assume for all \( r \in \mathbb{R} \)
\[
\inf \{ |\eta|^2 : \eta \in \phi(r) \} \leq C(1 + |r|^2). \quad (2.5)
\]
In case that
\[
\lim_{|x| \to \infty} \frac{\psi(x)}{|x|} \to \infty, \quad (2.6)
\]
i.e. \( \psi \) is superlinear, we require

(A5) There exists \( m \in (0, 1] \), such that
\[
\psi(v) \in L^1(O) \quad \text{if and only if } \quad v \in L^{m+1}(O).
\]
In case that the potential is sublinear, i.e. that there exists a constant \( C > 0 \) such that
\[
\psi(x) \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}, \quad (2.7)
\]
we require

(A5') There exists \( y > 0 \) such that \( \psi(y) > 0 \).

Note that by convexity, Assumption [A5'] implies that
\[
\psi(x) \geq \frac{\psi(y)}{y} |x| - \psi(y) \quad \text{for all } x \in \mathbb{R}.
\]

Next, we define the energy functional for the notion of solution we are going to consider.

**Definition 2.2.** Let Assumptions (A1) be satisfied and recall the definition of \( \iota \) and \( \iota_m \) in Section 1.3

(i) In the case of a superlinear potential, i.e. if (2.6) is satisfied, we define for \( u \in H^{-1} \) the functional
\[
\varphi(u) = \begin{cases}
\int \psi(\tilde{u}) \, dx, & \text{if } \exists \tilde{u} \in L^{m+1} \cap H^{-1} \text{ such that } u = \iota_m(\tilde{u}), \\
+\infty, & \text{else},
\end{cases} \quad (2.8)
\]
where \( m \) is the exponent from (A5).

(ii) In the case of a sublinear potential, i.e. if (2.7) is satisfied, we define for \( u \in H^{-1} \) the functional
\[
\varphi(u) = \begin{cases}
\|\psi(\mu)\|_{TV}, & \text{if } \exists \mu \in \mathcal{M} \cap H^{-1} \text{ such that } u = \iota(\mu), \\
+\infty, & \text{else},
\end{cases} \quad (2.9)
\]
where the construction of a nonlinear functional of a measure, which is needed in (2.9), is given in Definition 3.3 below.
Remark 2.3. (i) Note that these definitions are unambiguous due to the injectivity of \( \iota_m \) and \( \iota \).

(ii) In principle we have defined different functionals \( (\varphi_m^{(i)})_{m \in (0,1]} \) and \( \varphi^{(ii)} \). The common denominator is justified, since for \( 0 < m' < m \leq 1 \) and \( v \in L^{m+1} \subset L^{m'+1} \), we have
\[
\varphi_m^{(i)}(\iota_m(v)) = \varphi_m^{(i)}(\iota_{m'}(v)) = \varphi^{(ii)}(v \, dx),
\]
where the first equality is due to Remark 1.1, the injectivity of \( \iota_m \), and the construction of \( \varphi \) and the second equality will become clear by Remark 2.4 below.

Remark 2.4. The choice of the energy functional in Definition 2.2 allows us to reformulate Equation (1.1) as a gradient flow, i.e. to rewrite it in the form
\[
\frac{dX_t}{dt} = -\partial \varphi(X_t) dt + B(t, X_t) dW_t,
\]
where the subdifferential is well-defined due to Proposition 3.7 below. More precisely, let a "classical" solution to (1.1) with \( x_0 \in H^{-1} \) be defined as in Remark 2.3 below. The choice of the energy functional in Definition 2.2 allows us to reformulate (1.1) as a gradient flow, i.e. to rewrite it in the form
\[
\frac{dX_t}{dt} = -\partial \varphi(X_t) dt + B(t, X_t) dW_t,
\]
in the sense of Appendix C.

**Proof.** We only need to show that \( \Delta v_t \in -\partial \varphi(\iota_1(X_t)) \) \( \mathbb{P} \)-almost surely for all \( t \in [0, T] \), which is done by verifying the subdifferential inequality
\[
\varphi(u) \geq \varphi(\iota_1(X_t)) + H^{-1}(u - \iota_1(X_t), -\Delta v_t)_{H^{-1}}
\]
for arbitrary \( u \in H^{-1} \) and for \( (t, \omega) \in [0, T] \times \Omega \), for which the abovementioned properties of classical solutions are satisfied. For \( \varphi(u) = \infty \), there is nothing to show. For the superlinear case with Assumption 2.4(A5) satisfied for \( m \in (0, 1] \), we consider \( u = \iota_m(w) \) with \( w \in L^{m+1} \cap H^{-1} \), which is equivalent to \( \varphi(u) < \infty \).

We compute
\[
\varphi(u) = \iota_m(w) - \iota_{m+1}(w) = \iota_{m+1}(w) - \iota_{m+1}(X_t) + \iota_{m+1}(X_t) - \iota_m(w) = \langle u - \iota_1(X_t), (\Delta) v_t \rangle_{H^{-1}}.
\]

In the sublinear case, i.e. 2.7, is satisfied, let \( u = \iota(\mu) \), \( \mu \in \mathcal{M} \cap H^{-1} \). Let \( (\mu_n)_{n \in \mathbb{N}} \) be the sequence of approximating measures for \( \mu \) given by Theorem 3.3 below. For \( n \in \mathbb{N} \), let \( u_n \in L^2 \cap H^{-1} \) be the density of \( \mu_n \). Then, using Theorem 3.3, Remark 1.1 and Remark 2.3, we compute
\[
\varphi(u) = \varphi(\iota_1(X_t)) = \lim_{n \to \infty} \varphi(\iota_1(u_n)) = \varphi(\iota_1(X_t))
\]
and
\[
\varphi(u) = \varphi(\iota_1(X_t)) = \lim_{n \to \infty} \varphi(\iota_1(u_n)) = \varphi(\iota_1(X_t))
\]
for arbitrary \( u \in H^{-1} \), for which the abovementioned properties of classical solutions are satisfied. For \( \varphi(u) = \infty \), there is nothing to show. For the superlinear case with Assumption 2.4(A5) satisfied for \( m \in (0, 1] \), we consider \( u = \iota_m(w) \) with \( w \in L^{m+1} \cap H^{-1} \), which is equivalent to \( \varphi(u) < \infty \).

This concludes the proof. \( \square \)
Now we are in the position to formulate the notion of solution we want to consider.

**Definition 2.5 (SVI solution).** Given Assumptions [2.1] let \( x_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1}) \), \( T > 0 \) and \( \varphi \) be defined as in Definition [2.2]. We say that an \( \mathcal{F}_t \)-adapted process \( X \in L^2(\Omega; C([0, T]; H^{-1})) \) is an SVI solution to (1.1) if the following conditions are satisfied:

(i) (Regularity) \( \varphi(X) \in L^1([0, T] \times \Omega) \).

(ii) (Variational inequality) For each \( \mathcal{F}_t \)-progressively measurable process \( G \in L^2([0, T] \times \Omega; H^{-1}) \) and each \( \mathcal{F}_t \)-adapted process \( Z \in L^2(\Omega; C([0, T]; H^{-1})) \cap L^2([0, T] \times \Omega; L^2) \), solving the equation

\[
Z_t - Z_0 = \int_0^t G_s \, ds + \int_0^t B(s, Z_s) \, dW_s \quad \text{for all } t \in [0, T],
\]

we have

\[
\mathbb{E} \| X_t - Z_t \|^2_{H^{-1}} \leq \mathbb{E} \| x_0 - Z_0 \|^2_{H^{-1}} + 2 \mathbb{E} \int_0^t \varphi(X_r) \, dr
- 2 \mathbb{E} \int_0^t \langle G_r, X_r - Z_r \rangle_{H^{-1}} \, dr
+ C \mathbb{E} \int_0^t \| X_r - Z_r \|^2_{H^{-1}} \, dr
\]

for some \( C > 0 \).

**Remark 2.6.** If \((X, \eta)\) is a strong solution to (2.10) in \( H^{-1} \), as defined in Appendix [C], then \( X \) is an SVI solution to (1.1).

**Proof.** For (i) from Definition 2.5, we first note that \( \varphi(0) = 0 \) and for \( s \in [0, T] \)

\[
0 \leq \varphi(X_s) \leq \varphi(0) + \langle \eta_s, 0 - X_s \rangle_{H^{-1}} = - \langle \eta_s, X_s \rangle_{H^{-1}}
\]

by the subdifferential inequality. Hence, using the assumptions on \((X, \eta)\), we can compute

\[
\mathbb{E} \int_0^T |\varphi(X_s)| \, ds = \mathbb{E} \int_0^T \varphi(X_s) \, ds \leq \int_0^T \| \eta_s \|_{H^{-1}} \| X_s \|_{H^{-1}} \, ds
\]

\[
\leq \frac{1}{2} \mathbb{E} \int_0^T \| \eta_s \|^2_{H^{-1}} \, ds + \frac{1}{2} \mathbb{E} \int_0^T \| X_s \|^2_{H^{-1}} \, ds
\]

\[
\leq \frac{1}{2} \mathbb{E} \int_0^T \| \eta_s \|^2_{H^{-1}} \, ds + \frac{T}{2} \mathbb{E} \left( \sup_{s \in [0, T]} \| X_s \|_{H^{-1}} \right)^2 < \infty,
\]

as required. For (ii) let \( G \) and \( Z \) be given as in 2.3. Then Itô’s formula (e.g. [53, Theorem 4.2.5]) implies for all \( t \in [0, T] \)

\[
\mathbb{E} \| X_t - Z_t \|^2_{H^{-1}} = \mathbb{E} \| x_0 - Z_0 \|^2_{H^{-1}} + 2 \mathbb{E} \int_0^t \langle \eta_r - G_r, X_r - Z_r \rangle_{H^{-1}} \, dr
+ \mathbb{E} \int_0^t \| B(r, X_r) - B(r, Z_r) \|^2_{L_2(U, H^{-1})} \, dr.
\]
Since \( \eta_r \in -\partial \varphi(X_r) \) \((\mathbb{P} \otimes dt)\)-almost everywhere, we have
\[
\langle \eta_r, X_r - Z_r \rangle_{H^{-1}} \leq \varphi(Z_r) - \varphi(X_r) \quad dt \otimes d\mathbb{P}\text{-a.e.}
\]
Using moreover the Lipschitz condition \(2.2\) on \(B\), we obtain for all \(t \in [0, T]\)
\[
\mathbb{E}\|X_t - Z_t\|^2_{H^{-1}} \leq \mathbb{E}\|x_0 - Z_0\|^2_{H^{-1}} + 2E \int_0^t (\varphi(Z_r) - \varphi(X_r))dr
\]
\[
- 2E \int_0^t \langle G_r, X_r - Z_r \rangle_{H^{-1}} dr
\]
\[
+ E \int_0^t C \|X_r - Z_r\|^2_{H^{-1}} dr,
\]
which is equivalent to \(2.13\).

With the concept of SVI solutions at hand, we can state the main result of this article:

**Theorem 2.7.** Given Assumptions \(2.1\) let \(x_0 \in L^2(\Omega, F_0; H^{-1})\) and \(T > 0\). Then there is a unique SVI solution \(X\) to \(1.1\). For two SVI solutions \(X, Y\) with initial conditions \(x_0, y_0 \in L^2(\Omega, F_0; H^{-1})\), we have
\[
\sup_{t \in [0, T]} \mathbb{E}\|X_t - Y_t\|^2_{H^{-1}} \leq C \mathbb{E}\|x_0 - y_0\|^2_{H^{-1}}.
\]

The proof of this theorem will be given in Section 3 below.

### 3 Properties of the energy functional

The aim of this section is to make Definition \(2.2\) rigorous by recalling the concept of convex functionals on measures, and to prove certain properties of the energy functional defined in Definition \(2.2\), which are needed for the proof of the main theorem. We start with some basic concepts concerning convex functionals.

**Definition 3.1.** Let \(f : \mathbb{R} \to [0, \infty]\) be a convex and lower-semicontinuous function with \(f(0) = 0\). We then define its convex conjugate \(f^* : \mathbb{R} \to [0, \infty]\) by
\[
f^*(x) = \sup_{y \in \mathbb{R}} (xy - f(y)),
\]
and its recession function \(f_\infty : \mathbb{R} \to [0, \infty]\) by
\[
f_\infty(x) = \lim_{t \to \infty} \frac{f(tx)}{t}.
\]

**Remark 3.2.** Note that \(f_\infty\) and \(f^*\) are convex. If \(f\) is symmetric, so are \(f_\infty\) and \(f^*\), the latter of which can be seen by computing
\[
f^*(x) = \sup_{y \in \mathbb{R}} (xy - f(y)) = \sup_{y \in \mathbb{R}} (-xy - f(-y)) = f^*(-x).
\]
Moreover, \(f_\infty\) is positively homogeneous.

For the notion of solution that we are aiming at, we need the concept of a convex function of a measure, which has been developed in \(23\).

**Definition 3.3.** Let \(\psi\) satisfy \(2.7\) as well as Assumptions \(2.1[A3], [A5]\). Define the set
\[
\mathcal{D}_\psi = \{v \in C^0_c(\mathcal{O}) : \psi^*(v) \in L^1(\mathcal{O})\}
\]
and let \(\mu \in \mathcal{M}(\mathcal{O})\). We then define the positive measure \(\psi(\mu) \in \mathcal{M}(\mathcal{O})\) by
\[
\int_\mathcal{O} \eta \psi(\mu) := \mathcal{M}(\mathcal{O}) (\psi(\mu), \eta)_{C^0_c(\mathcal{O})} := \sup \left\{ \int_\mathcal{O} v\eta \, d\mu - \int_\mathcal{O} \psi^*(v) \eta \, dx : v \in \mathcal{D}_\psi \right\}
\]
for \(\eta \in C^0_c(\mathcal{O}), \eta \geq 0\), and for general \(\eta \in C^0_c(\mathcal{O})\) we set
\[
\mathcal{M}(\mathcal{O}) (\psi(\mu), \eta)_{C^0_c(\mathcal{O})} = \mathcal{M}(\mathcal{O}) (\psi(\mu), \eta \vee 0)_{C^0_c(\mathcal{O})} - \mathcal{M}(\mathcal{O}) (\psi(\mu), (-\eta) \vee 0)_{C^0_c(\mathcal{O})},
\]
according to \(23\) Theorem 1.1.
Remark 3.4. As argued in [23, Lemma 1.1], one can write for \( \mu \in \mathcal{M}(\mathcal{O}) \)
\[
\int_{\mathcal{O}} \psi(\mu) = \|\psi(\mu)\|_{TV} = \sup \left\{ \int_{\mathcal{O}} v \, d\mu - \int_{\mathcal{O}} \psi^*(v) \, dx : v \in \mathcal{D}_\psi \right\}.
\]

Remark 3.5. Let \( \mu \in \mathcal{M}(\mathcal{O}) \) with Lebesgue decomposition \( \mu^a + \mu^s \), where \( \mu^a \) has the density \( h \in L^1(\mathcal{O}) \) with respect to the Lebesgue measure. Then, by [23, Theorem 1.1], we have
\[
\int_{\mathcal{O}} \eta \, \psi(\mu) = \int_{\mathcal{O}} \eta(x) \psi(h(x)) \, dx + \int_{\mathcal{O}} \eta \psi_\infty(\mu^s),
\]
where the recession function \( \psi_\infty \) is defined as in [32]. In particular, this formulation shows the useful fact that
\[
\psi(\mu) = \psi(\mu^a) + \psi(\mu^s).
\]

Our next aim is to prove the lower-semicontinuity of the energy functional defined in Definition 2.2 and Definition 3.3. First, we show that the Radon measure \( \psi(\mu) \) constructed in Definition 3.3 controls the norm of its original measure \( \mu \) in the following way.

**Lemma 3.6.** Let \( \psi \) satisfy (2.7) as well as Assumptions [2.1 (A5')] [A5']. Let \( \mu \in \mathcal{M}(\mathcal{O}) \) and let \( y > 0 \) such that \( \psi(y) > 0 \) as demanded in Assumption 2.1 (A5'). Then
\[
\|\psi(\mu)\|_{TV} \geq \frac{\psi(y)}{y} \|\mu\|_{TV} - \psi(y) |\mathcal{O}|.
\]

**Proof.** For \( \mu \in \mathcal{M}(\mathcal{O}) \), denote by \( \mu = \mu^a + \mu^s \) the Lebesgue decomposition of \( \mu \) with respect to Lebesgue measure, and let \( h = \frac{d\mu^a}{dx} \) be the Radon-Nikodym derivative of \( \mu^a \). As \( \psi_\infty(\mu^s) \) is singular by [23] Theorem 4.2, we can use the decomposition (3.4) to obtain
\[
\|\psi(\mu)\|_{TV} = \int_{\mathcal{O}} \psi(h) \, dx + \|\psi_\infty(\mu^s)\|_{TV}.
\]
We now estimate the summands separately. For the absolutely continuous part we obtain using Assumption 2.1 (A5')
\[
\int_{\mathcal{O}} \psi(h) \, dx \geq \frac{\psi(y)}{y} \int_{\mathcal{O}} |h| \, dx - \psi(y) |\mathcal{O}|
= \frac{\psi(y)}{y} \|\mu^a\|_{TV} - \psi(y) |\mathcal{O}|.
\]
For the singular part, we note by Lemma A.5 that for \( v \in C^0_\infty(\mathcal{O}) \) being in \( \mathcal{D}\psi_\infty \) is equivalent to \( -\psi_\infty(1) \leq v \leq \psi_\infty(1) \), and for such \( v \), \( \psi_\infty^*(v) \equiv 0 \). Thus, we get with Corollary A.4 with \( k := \frac{\psi(y)}{y} \)
\[
\int_{\mathcal{O}} \psi_\infty(\mu^s) = \sup_{v \in \mathcal{D}_\psi_\infty} \left( \int_{\mathcal{O}} v \, d\mu^s - \int \psi_\infty^*(v) \, dx \right)
\geq \sup_{v \in C^0_\infty(\mathcal{O})} \int_{\mathcal{O}} v \, d\mu^s
= k \sup_{-k \leq v \leq k} \int_{\mathcal{O}} v \, d\mu^s = k \|\mu^s\|_{TV}.
\]
Thus, we can continue (3.6) by
\[
\|\psi(\mu)\|_{TV} \geq \frac{\psi(y)}{y} \|\mu^a\|_{TV} + k \|\mu^s\|_{TV} - \psi(y) |\mathcal{O}|
= \frac{\psi(y)}{y} \|\mu\|_{TV} - \psi(y) |\mathcal{O}|,
\]
as required. \( \square \)
Proposition 3.7. In both settings of Definition 2.2 \( \varphi : H^{-1} \to [0, \infty] \) is convex and lower-semicontinuous.

**Proof.** In the superlinear case, i.e. Definition 2.2 (i) applies, convexity and lower-semicontinuity of \( \varphi \) are proved in [5, p. 68]. In the sublinear case, i.e. Definition 2.2 (ii) applies, convexity becomes clear by Remark 3.3. It remains to prove lower-semicontinuity in the sublinear case.

**Step 1:** As a preparatory step, we establish weak* lower-semicontinuity of the functional \( \tilde{\varphi} : \mathcal{M}(O) \to [0, \infty) \),

\[ \tilde{\varphi}(\mu) = \|\psi(\mu)\|_{TV}, \]

for which we have

\[ \tilde{\varphi}|_{\mathcal{M}(O) \cap H^{-1}} = \varphi \circ \iota. \]

Consider \( \mu_n \to \mu \) weakly* for \( n \to \infty \). We can assume that \( \psi(\mu_n) \) contains a subsequence which is bounded in TV norm (otherwise there is nothing to show). Then we select a subsequence \( (\mu_{n_k})_{k \in \mathbb{N}} \) such that \( \|\psi(\mu_{n_k})\|_{TV} \to \liminf_{n \to \infty} \|\psi(\mu_n)\|_{TV} \) for \( k \to \infty \), from which we can choose a nonrelabeled subsequence \( (\psi(\mu_{n_k}))_{k \in \mathbb{N}} \) which converges weakly* to some \( \nu \in \mathcal{M}(O) \) (e.g. by [1, Satz 6.5]). By [23 Lemma 2.1], we get that

\[ \mathcal{M}(O)(\psi(\mu), \eta)_{C^0_0(O)} = \lim_{k \to \infty} \mathcal{M}(O)(\psi(\mu_{n_k}), \eta)_{C^0_0(O)} \leq \lim_{k \to \infty} \|\psi(\mu_{n_k})\|_{TV} \|\eta\|_{C^0_0(O)} \]

for \( \eta \in C^0_0(O), \eta \geq 0 \). Now, using that \( \psi(\rho) \) is a positive measure for any \( \rho \in \mathcal{M}(O) \) by (3.3), we obtain

\[ \|\psi(\mu)\|_{TV} = \sup_{\eta \in C^0_0(O)} \mathcal{M}(O)(\psi(\mu), \eta)_{C^0_0(O)} \leq \sup_{\eta \in [0,1]} \lim_{n \to \infty} \mathcal{M}(O)(\psi(\mu_n), \eta)_{C^0_0(O)} \]

\[ \leq \sup_{\eta \in C^0_0(O)} \lim_{k \to \infty} \|\psi(\mu_{n_k})\|_{TV} = \liminf_{n \to \infty} \|\psi(\mu_n)\|_{TV}, \]

as required.

**Step 2:** Assume now that \( (u_n)_{n \in \mathbb{N}} \subset H^{-1}, u \in H^{-1} \), and \( u_n \to u \) for \( n \to \infty \). Being the only non-trivial case, we can assume that \( (u_n)_{n \in \mathbb{N}} \) contains a subsequence (which we call again \( (u_n) \)) for which \( (\tilde{\varphi}(u_n))_{n \in \mathbb{N}} \) is bounded. Thus, there are measures \( \mu_n \in \mathcal{M}(O) \cap H^{-1} \) such that

\[ u_n(\eta) = \int_O \eta \, d\mu_n \quad \text{for all} \quad \eta \in C^1_0(O). \]

By definition of \( \varphi \), \( \varphi(u_n) = \|\psi(\mu_n)\|_{TV} \), such that Lemma 3.4 implies that \( \|\mu_n\|_{TV} \) is bounded. Thus, there is \( \tilde{\mu} \in \mathcal{M}(O) \) an again nonrelabeled subsequence \( (\mu_{n_k})_{n \in \mathbb{N}} \) such that \( \mu_{n_k} \rightharpoonup \tilde{\mu} \). For \( \eta \in C^1_0(O) \subseteq C^0_0(O) \) we have

\[ \int_O \eta \, d\tilde{\mu} = \lim_{n \to \infty} \int_O \eta \, d\mu_n = \lim_{n \to \infty} u_n(\eta) \leq \|u\|_{H^{-1}} \|\eta\|_{H^1_0(O)}, \]

so \( \tilde{\mu} \in \mathcal{M}(O) \cap H^{-1} \) and \( u = \iota(\tilde{\mu}) \). Using the weak* lower-semicontinuity of \( \tilde{\varphi} \) from Step 1, we get

\[ \varphi(u) = \tilde{\varphi}(\tilde{\mu}) \leq \liminf_{n \to \infty} \tilde{\varphi}(\mu_n) = \liminf_{n \to \infty} \varphi(u_n). \]

As this argument works for any bounded subsequence of \( (u_n)_{n \in \mathbb{N}}, (3.7) \) is also true for the original sequence \( (u_n)_{n \in \mathbb{N}} \). \( \square \)

As one can see from the definition of the energy functional \( \varphi \) in the second part of Definition 2.2 it has an explicit representation on \( H^{-1} \setminus \mathcal{M}(O) \), where it is \( \infty \), and on \( L^1(O) \cap H^{-1} \), where it is an integral. However, whenever we evaluate \( \varphi \) for general measures in \( \mathcal{M}(O) \cap H^{-1} \), e.g. in the uniqueness part of the proof of Theorem 2.7, we need an approximation reducing it to evaluations on \( L^1(O) \) functions. This will be made precise in the following theorem, the proof of which will take the rest of this section.

**Theorem 3.8.** Assume that \( \psi \) satisfies (2.7) as well as Assumptions 2.1 (A3) (A5). Let \( \mu \in \mathcal{M}(O) \cap H^{-1} \). Then there exists a sequence \( (\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(O) \cap H^{-1} \) such that for all \( n \in \mathbb{N} \), \( \mu_n \) has a density \( u_n \in L^2(O) \) with respect to the Lebesgue measure, and such that

\[ \iota(\mu_n) \to \iota(\mu) \quad \text{in} \quad H^{-1} \quad \text{for} \quad n \to \infty, \]
Theorem 3.8 implies that for all \( u \) we have also proven "\( \geq \beta \) since for each \( x \) and for \( n \rightarrow \infty \) we set
\[
\varphi(\iota(\mu_n)) \rightarrow \varphi(\iota(\mu)),
\]
where we used the notation \( \iota \) for the embedding \( \mathcal{M}(\mathcal{O}) \cap H^{-1} \rightarrow H^{-1} \) (cf. Section 1.3).

**Corollary 3.9.** Theorem 3.8 implies that \( \varphi \) is the lower-semicontinuous hull of \( \varphi_{L^2(\mathcal{O})} \) in \( H^{-1} \), which means that
\[
\varphi = \sup \{ \beta : H^{-1} \rightarrow [0, \infty] \mid \beta \text{ convex and lower-semicontinuous}, \beta|_{L^2(\mathcal{O})} \leq \varphi|_{L^2(\mathcal{O})} \}, \tag{3.10}
\]
where \( \sup \) denotes the pointwise supremum.

**Proof.** First note that \( \varphi \) itself satisfies the constraints for \( \beta \) on the right-hand side of (3.10), which yields "\( \leq \)" in (3.10). For the other direction, we first extend \( \varphi_{L^2(\mathcal{O})} \) to a function \( \tilde{\varphi} \) on \( H^{-1} \) by
\[
\tilde{\varphi}(u) = \begin{cases} \varphi(u) & \text{if } u \in L^2(\mathcal{O}) \cap H^{-1}, \\ +\infty & \text{else}. \end{cases}
\]

Obviously the last constraint in (3.10) then simplifies to
\[
\beta \leq \tilde{\varphi}.
\]

Moreover, we note from [15, Theorem 9.1] that convex functions \( f : H \rightarrow [0, \infty] \), where \( H \) is a real Hilbert space, are lower-semicontinuous if and only if they are weakly sequentially lower-semicontinuous. Thus, we can replace the set on the right-hand side of (3.10) by
\[
A := \{ \beta : H^{-1} \rightarrow [0, \infty] \mid \beta \text{ convex and weakly sequentially lower-semicontinuous}, \beta \leq \tilde{\varphi} \}. \tag{3.11}
\]

Theorem 3.8 implies that for all \( u \in H^{-1} \)
\[
\varphi(u) \geq \liminf_{u_n \rightarrow u} \tilde{\varphi}(u_n).
\]

Since for each \( \beta \in A \)
\[
\beta(u) \leq \liminf_{u_n \rightarrow u} \beta(u_n) \leq \liminf_{u_n \rightarrow u} \tilde{\varphi}(u_n),
\]
we have also proven "\( \geq \)" in (3.10).

We will approach Theorem 3.8 by giving an explicit construction for the sequence \((\mu_n)_{n \in \mathbb{N}}\), inspired by the construction in [11, Lemma A6.7]. It will rely on applying the original functional to modified functions, which is why we first introduce several modifications to functions on \( \mathcal{O} \).

We next introduce further notation and recall some concepts relying on the regularity of the boundary.

**Notations 3.10.** Since the domain \( \mathcal{O} \) is bounded and smooth, its boundary is locally the graph of a smooth function. More precisely, we recall from [11, Section A6.2] that for each \( y \in \partial \mathcal{O} \) there is a neighbourhood \( \tilde{U} \subset \mathbb{R}^d \), an orthonormal system \( e_1, \ldots, e_d \) of \( \mathbb{R}^d \), \( r, h \in \mathbb{R} \) with \( r > h > 0 \), and a smooth bounded function \( g : \mathbb{R}^{d-1} \rightarrow \mathbb{R} \), such that with the notation
\[
x_{d} := (x_1, \ldots, x_{d-1}), \quad \text{for } x = \sum_{i=1}^{d} x_i e_i,
\]
we have
\[
\tilde{U} = \{ x \in \mathbb{R}^d : |x_d - y| < r \text{ and } |x_d - g(x_d)| < h \},
\]
and for \( x \in \tilde{U} \)
\[
x_d = g(x_d) \text{ if and only if } x \in \partial \mathcal{O},
x_d \in (g(x_d), g(x_d) + h) \text{ if and only if } x \in \mathcal{O}, \text{ and}
x_d \in (g(x_d) - h, g(x_d)) \text{ if and only if } x \notin \mathcal{O}.
\]

For technical reasons we set
\[
U = \left\{ x \in \tilde{U} : |x_d - y| < \frac{r}{2} \text{ and } |x_d - g(x_d)| < \frac{h}{2} \right\}. \tag{3.12}
\]
The boundary $\partial \mathcal{O}$ is covered by those open sets $U$ belonging to all possible reference points $y$. As $\partial \mathcal{O}$ is compact, we can choose a finite subcovering $(U^j)_{j=1}^l$, and for each $U^j$, we denote the elements belonging to it by a superindex $j$, e.g., $y^j, e^j, g^j, h^j, U^j$. At last, we fix an open set $U^0$ with $\overline{U^0} \subset \mathcal{O}$, such that $\overline{\mathcal{O}} \subset \bigcup_{j=0}^l U^j$ and we set $e_0^0 := 0$.

Subordinate to the covering $\bigcup_{j=0}^l U^j$, let now $\zeta^0, \ldots, \zeta^l$ be a partition of unity on $\overline{\mathcal{O}}$, i.e. $0 \leq \zeta^j \leq 1$, $\zeta^j \in C_0^\infty(\mathbb{R}^d)$, supp$(\zeta^j) \subseteq U^j$ for all $j = 0, \ldots, l$, and

$$\sum_{j=0}^l \zeta^j = 1 \text{ on } \overline{\mathcal{O}}.$$ 

For $\eta : \mathcal{O} \to \mathbb{R}$, we define $\eta_{\text{ext}} : \mathbb{R}^d \to \mathbb{R}$ by

$$\eta_{\text{ext}}(x) = \begin{cases} \eta(x) & \text{if } x \in \mathcal{O} \\ 0 & \text{else.} \end{cases}$$

We briefly recall the concept of mollifications and regularisations, respectively.

**Definition 3.11.** Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a symmetric Dirac kernel, i.e.

$$\text{supp}(\rho) \subseteq B_1^d(0), \int_{\mathcal{O}} \rho \, dx = 1, \rho(x) = \rho(-x),$$

and let $(\rho_s)_{s > 0}$ be the corresponding family of convolution kernels, i.e.

$$\rho_s(x) = \frac{1}{\sqrt{s}} \rho \left( \frac{x}{s} \right).$$

Let $\eta \in L^2(\mathcal{O})$ and $\nu \in L^2(\mathbb{R}^d)$. Then, we define the mollified functions $\rho_s * \eta, \rho_s * \nu : \mathbb{R}^d \to \mathbb{R}$ by

$$\rho_s * \nu(x) = \int_{\mathbb{R}^d} \rho_s(x-y) \nu(y) \, dy \quad \text{and} \quad \rho_s * \eta(x) = \int_{\mathbb{R}^d} \rho_s(x-y) \eta_{\text{ext}}(y) \, dy.$$ 

**Remark 3.12.**

(i) With the notation of Definition 3.11 we have (see e.g. [1, Section 2.13])

$$\rho_s * \nu, \rho_s * \eta \in C^\infty(\mathbb{R}^d)$$

(ii) Note that the second notation is consistent with the extension, i.e. for $\eta \in L^2(\mathcal{O})$, we have

$$\rho_s * \eta = \rho_s * \eta_{\text{ext}}.$$ 

**Definition 3.13.** Let $\mu \in \mathcal{M}(\mathcal{O}), \nu \in \mathcal{M}(\mathbb{R}^d)$ and $\rho_s$ as in Definition 3.11. We then define the measure $\mu_{\text{ext}} \in \mathcal{M}(\mathbb{R}^d)$ by

$$\mu_{\text{ext}}(A) = \mu(A \cap \mathcal{O}) \quad \text{for } A \in \mathcal{B}(\mathbb{R}^d),$$

and the regularisation $\rho_s * \nu \, dx \in \mathcal{M}(\mathbb{R}^d)$ of $\nu$, whose density with respect to the Lebesgue measure is given by

$$\rho_s * \nu(x) = \int_{\mathbb{R}^d} \rho_s(x-y) \, d\nu(y).$$

**Remark 3.14.** Let $\eta \in C_0^\infty(\mathbb{R}^d)$ and $\nu \in \mathcal{M}(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \eta(x) \rho_s * \nu(x) \, dx = \int_{\mathbb{R}^d} \rho_s * \eta(x) \, \nu(dx),$$

which can be seen by applying Fubini’s theorem.

As a first modification, we introduce the following weighted shifts.

**Definition 3.15.** Let $\varepsilon > 0$ and $\eta : \mathcal{O} \to \mathbb{R}$. Then we define $\eta_{\varepsilon} : \mathcal{O} \to \mathbb{R}$ by

$$\eta_{\varepsilon}(x) = \sum_{j=0}^l \zeta^j(x) \eta_{\text{ext}}(x - \varepsilon e^j_d),$$

where we recall that $e_d^j$ is set to 0.
Remark 3.16. By this construction, we achieve that \( \eta_\varepsilon = 0 \) on a \( w(\varepsilon) \)-neighbourhood of \( \partial \mathcal{O} \) with

\[
w(\varepsilon) := \min \left\{ \text{dist}(U^0, \mathcal{O}^c), \min_{j=1, \ldots, d} \left( \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon}{2L^j}, \frac{h^j}{4L^j} \right\} \right) \right\} > 0,
\]

where \( L^j \) denotes the Lipschitz constant of \( g^j \) defined in Notations 3.10.

Proof. The number \( w(\varepsilon) \) is obviously strictly positive by the construction of the covering \( (U^j)_j \). To show the support property, let \( j \in \{0, 1, \ldots, l\} \) and \( U^j : U^j \cap (\mathcal{O}^c) \). By definition, \( \eta_{\text{ext}}(x - \varepsilon e^j) = 0 \) if \( x \in U^j \setminus U^j \). By the definition of \( \zeta^j \), we furthermore conclude that \( \zeta^j(x)\eta_{\text{ext}}(x - \varepsilon e^j) = 0 \) for \( x \notin U^j \). Consequently,

\[
\eta_\varepsilon : x \mapsto \sum_{j=0}^l \zeta^j(x)\eta_{\text{ext}}(x - \varepsilon e^j)
\]
is supported on

\[
U_\varepsilon := \bigcup_{j=0}^l U^j,
\]
such that it remains to show that \( \text{dist}(U_\varepsilon, \mathcal{O}^c) \geq w(\varepsilon) \), or equivalently, that \( \text{dist}(U^j, \mathcal{O}^c) \geq w(\varepsilon) \) for all \( j \in \{0, \ldots, l\} \).

For \( j = 0 \), this is trivial by construction of \( U^0 = U^0 \) and \( w(\varepsilon) \). For \( j = 1, \ldots, l \), using the coordinate system \( (x^j, x^l) \) we can rewrite

\[
U^j = \{ x \in U^j : x^j > g^j(x^j) + \varepsilon \}.
\]

Hence, we can compute for any \( x \in U^j \), i.e. \( x = (x^j, g^j(x^j) + \varepsilon') \) for some \( \varepsilon' \in (\varepsilon, \frac{h^j}{2L^j}) \), and \( y \in \partial \mathcal{O} \cap \tilde{U}^j \)

\[
\|x - y\|^2 = \|x^j - y^j\|^2 + |g(x^j) + \varepsilon' - g(y^j)|^2 \geq \|x^j - y^j\|^2 + (\varepsilon' - |g(x^j) - g(y^j)|)^2,
\]
where \( \|\cdot\| \) denotes the Euclidean norm both in \( \mathbb{R}^d \) and in \( \mathbb{R}^{d-1} \). Letting \( L^j \) be the Lipschitz constant of \( g^j \), we can then argue that either \( \|x^j - y^j\| > \frac{\varepsilon}{2L^j} \) or

\[
|g(x^j) - g(y^j)| \leq L^j \frac{\varepsilon}{2L^j} = \frac{\varepsilon}{2},
\]
such that \( \text{dist}(U^j, \partial \mathcal{O} \cap \tilde{U}^j) \) is at least \( \min \left\{ \frac{\varepsilon}{2}, \frac{h^j}{4L^j} \right\} \). By similar arguments, we can obtain from the construction of \( U^j \) in (3.12) (note that \( r^j > h^j \) by construction) that

\[
\text{dist}(U^j, \mathcal{O}) \geq \min \left\{ \frac{h^j}{4L^j}, \frac{h^j}{4L^j} \right\}
\]
such that we can conclude

\[
\text{dist}(U^j, \partial \mathcal{O}) = \min\{\text{dist}(U^j, \partial \mathcal{O} \cap \tilde{U}^j), \text{dist}(U^j, \partial \mathcal{O} \cap (\tilde{U}^j)^c)\} \geq \min\{\text{dist}(U^j, \partial \mathcal{O} \cap \tilde{U}^j), \text{dist}(U^j, (\tilde{U}^j)^c)\} \geq \min \left\{ \frac{\varepsilon}{2}, \frac{h^j}{4L^j}, \frac{h^j}{4L^j} \right\} \geq w(\varepsilon).
\]

\[ \square \]

Corollary 3.17. Let \( \eta \in H^1_0(\mathcal{O}) \). Then \( \eta_\varepsilon \in H^1_0(\mathcal{O}) \), since its extension is in \( H^1(\mathbb{R}^d) \) by construction and it is supported in \( \mathcal{O} \) by the previous remark.

For the later approximation of functionals and measures we need that the previously described procedure is linear and keeps the \( H^1_0(\mathcal{O}) \) norm controlled, as is proved in the following lemmata.

Lemma 3.18. Let \( \varepsilon > 0 \), \( \lambda \in \mathbb{R} \) and \( \eta, v : \mathcal{O} \to \mathbb{R} \). Then

\[
(\eta + \lambda v)_\varepsilon = \eta_\varepsilon + \lambda v_\varepsilon.
\]
Proof. We have for $x \in \mathcal{O}$

$$(\eta + \lambda v)_\varepsilon (x) = \sum_{j=0}^{l} \zeta^j (x) (\eta + \lambda v)_{\text{ext}} (x - \varepsilon e^j_d)$$

where $C$ may depend on $\varepsilon e^j_d$, the number of covering sets $l$, the Poincaré constant of the domain $\mathcal{O}$ and the spatial dimension $d$.

Lemma 3.19. Let $\varepsilon > 0$ and $\eta \in H^{1}_{0}(\mathcal{O})$. Then

$$\| \eta \|_{H^{1}_{0}(\mathcal{O})} \leq C \| \eta \|_{H^{1}_{0}(\mathcal{O})},$$

where $C$ only depends on the localising functions $(\zeta^j)^{l}_{j=0}$, the number of covering sets $l$, the Poincaré constant of the domain $\mathcal{O}$ and the spatial dimension $d$.

Proof. In the following, let $V^j = U^j \cap \mathcal{O}$ and $U^{j}_\varepsilon := \mathcal{O} \cap ((U^j \cap \mathcal{O}) + \varepsilon e^j_d)$ as before. We first note

$$\| \eta \|_{H^{1}_{0}(\mathcal{O})} = \left\| \sum_{j=0}^{l} \zeta^j \eta^j \right\|_{H^{1}_{0}(\mathcal{O})} = \sum_{j=0}^{l} \| \zeta^j \eta^j \|_{H^{1}_{0}(\mathcal{O})}, \quad (3.17)$$

where we write

$$\eta^j \in H^{1}(\mathbb{R}^d), \quad \eta^j(x) = \eta_{\text{ext}}(x - \varepsilon e^j_d).$$

We now analyse the summands separately, where we make use of the fact that for all $j \in \{1, \ldots, l\}$, $\zeta^j \in C_c^{\infty}(U^j)$ and $\zeta^j \eta^j$ is supported on $V^j$. In the following, $(\partial_i)^d_{i=1}$ represent the weak partial derivatives of first order. We then compute for $i \in \{1, \ldots, d\}$

$$\| \partial_i (\zeta^j \eta^j) \|_{L^2(\mathcal{O})} = \| \partial_i (\zeta^j \eta^j) \|_{L^2(V^j)}$$

$$\leq \left\| \partial_i (\zeta^j) \eta^j \right\|_{L^2(V^j)} + \left\| \zeta^j \partial_i \eta^j \right\|_{L^2(V^j)}$$

$$\leq C \left\| \eta^j \right\|_{L^2(\mathcal{O})} + \left( \int_{V^j} \left| \partial_i (\zeta^j \eta_{\text{ext}}(x - \varepsilon e^j_d)) \right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq C \left\| \eta \right\|_{L^2(\mathcal{O})} + \left( \int_{U^{j}_\varepsilon} \left| \partial_i (\eta)(x - \varepsilon e^j_d) \right|^2 dx \right)^{\frac{1}{2}}$$

This yields

$$\| \zeta^j \eta^j \|_{H^{1}_{0}(\mathcal{O})}^2 = \sum_{i=1}^{d} \left\| \partial_i (\zeta^j \eta^j) \right\|_{L^2(\mathcal{O})}^2$$

$$\leq \sum_{i=1}^{d} \left( C \left\| \eta \right\|_{L^2(\mathcal{O})} + \left\| \partial_i \eta \right\|_{L^2(\mathcal{O})} \right)^2$$

$$\leq C \left\| \eta \right\|_{H^{1}_{0}(\mathcal{O})}^2 + 2 \sum_{i=1}^{d} \left| \partial_i \eta \right|_{L^2(\mathcal{O})}^2 \leq C \left\| \eta \right\|_{H^{1}_{0}(\mathcal{O})}^2,$$

where $C$ may depend on $d, \mathcal{O}$ (through the Poincaré constant) and $\zeta^j$. Thus, we can continue (3.17) by

$$\| \eta \|_{H^{1}_{0}(\mathcal{O})} \leq \sum_{j=0}^{l} \| \zeta^j \eta^j \|_{H^{1}_{0}(\mathcal{O})}$$

$$\leq (l + 1) C \left\| \eta \right\|_{H^{1}_{0}(\mathcal{O})},$$

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as required.

As a second modification, we mollify $\eta_c$. We note that by Remark 3.16, $\rho \eta_c(x) = 0$ if $\text{dist}(x, \partial \Omega) \leq \frac{w(\varepsilon)}{2}$ and $0 < \delta \leq \frac{w(\varepsilon)}{2}$, so that in this case we can restrict $\rho \eta_c$ to $\Omega$ to get a $C^1_c(\Omega)$ function. By a slight abuse of notation, we then write

$$(\rho \eta_c)|\Omega = \rho \eta_c \in C^1_c(\Omega) \subseteq H^1_0(\Omega) \cap C_0(\Omega).$$

(3.18)

Also for this step, we have to ensure linearity, which is clear, and an estimate on the $H^1_0(\Omega)$ norm, which is done in the following lemma.

**Lemma 3.20.** Let $\varepsilon > 0$ and $0 < \delta \leq \frac{w(\varepsilon)}{2}$. Then

$$\| \rho \eta_c \|_{H^1_0(\Omega)} \leq C \| \eta \|_{H^1_0(\Omega)}$$

for all $\eta \in H^1_0(\Omega)$, where $C$ is the constant from Lemma 3.18.

**Proof.** For any $g \in L^2(\Omega)$ such that $\rho \eta_c = 0$ on $\Omega^c$ we can compute

$$\| \rho \eta_c g \|_{L^2(\Omega)}^2 = \| \rho \eta_c \|_{L^2(\Omega)}^2$$

$$= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \rho(x-y) g_\text{ext}(y) \, dy \right)^2 \, dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x-y) (g_\text{ext}(y))^2 \, dy \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x-y) \, dx (g_\text{ext}(y))^2 \, dy$$

$$= \| g_\text{ext} \|_{L^2(\mathbb{R}^d)}^2 = \| g \|_{L^2(\Omega)}^2,$$

where in the third step we could apply Jensen’s inequality since $\rho \eta_c(x-y) \, dy$ is a probability measure for each $x \in \mathbb{R}^d$. By Remark 3.16 for all $i \in \{1, \ldots, d\}$, $\rho \eta_c (\partial_i \eta_c)$ vanishes outside of $\Omega$ if $0 < \delta \leq \frac{w(\varepsilon)}{2}$. Hence $g$ in (3.19) can be replaced by each partial derivative $\partial_i \eta_c$ which yields

$$\| \rho \eta_c \|_{H^1_0(\Omega)}^2 = \sum_{i=1}^d \| (\rho \eta_c) \|_{L^2(\Omega)}^2$$

$$= \sum_{i=1}^d \| \rho \eta_c \|_{L^2(\Omega)}^2$$

$$\leq \sum_{i=1}^d \| \partial_i \eta_c \|_{L^2(\Omega)}^2 = \| \eta_c \|_{H^1_0(\Omega)}^2 \leq C \| \eta \|_{H^1_0(\Omega)}^2,$$

where the second equality can be found e.g. in [1] Section 2.23 and the last inequality is the statement of Lemma 3.19.

To allow the same approximation in the realm of measures, we continue by checking the compatibility of the presented modifications on functions $\eta \in C^0_c(\Omega)$ in the corresponding norm.

**Lemma 3.21.** Let $\varepsilon > 0$, $0 < \delta \leq \frac{w(\varepsilon)}{2}$, and $\eta \in C^0_c(\Omega)$. Then, the functions $\eta_c$ and $(\rho \eta_c)|\Omega$ are in $C^0_c(\Omega)$. Furthermore, we have

$$\| \rho \eta_c \|_{\infty} \leq \| \eta_c \|_{\infty} \leq \| \eta \|_{\infty},$$

(3.20)

where $\| \cdot \|_{\infty}$ denotes the supremum norm.

**Proof.** Note that for $0 < \delta \leq \frac{w(\varepsilon)}{2}$, $\rho \eta_c \in C^0_c(\Omega)$ by construction, Remark 3.12 and Remark 3.16. To obtain (3.20), we estimate for arbitrary $x \in \Omega$

$$| \eta_c(x) | \leq \sum_{j=0}^t \zeta_j(x) | \eta_\text{ext}(x - \varepsilon \eta_c) | \leq \sum_{j=0}^t \zeta_j(x) \| \eta \|_{\infty} \leq \| \eta \|_{\infty},$$

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which yields the second relation. The first one can be seen by
\[ |\rho_t \ast \nu_t(x)| \leq \int_{\mathbb{R}^d} \rho_t(x - y) \|\nu_t\|_\infty \, dy = \|\nu_t\|_\infty, \]
which concludes the proof. \(\square\)

This allows to define the following approximating objects for \(u \in \mathcal{M}(\mathcal{O}) \cap H^{-1}\).

**Definition 3.22.** Let \(\varepsilon > 0\), \(0 < \delta \leq \frac{\omega(\varepsilon)}{2}\) and \(\mu \in \mathcal{M}(\mathcal{O}) \cap H^{-1}\), and let \(u := \iota(\mu) \in H^{-1}\), which means that
\[
H^{-1}\langle u, \eta \rangle_{H^1_0(\mathcal{O})} = \mathcal{M}(\mathcal{O}) \langle \mu, \eta \rangle_{C^0_0(\mathcal{O})} \tag{3.21}
\]
for \(\eta \in C^0_0(\mathcal{O})\). We define \(u_\varepsilon, u_{\varepsilon, \delta} \in H^{-1}\) by
\[
H^{-1}\langle u_\varepsilon, \eta \rangle_{H^1_0(\mathcal{O})} = H^{-1}\langle u, \eta \rangle_{H^1_0(\mathcal{O})}
\]
\[
H^{-1}\langle u_{\varepsilon, \delta}, \eta \rangle_{H^1_0(\mathcal{O})} = H^{-1}\langle u, \rho \ast \eta \rangle_{H^1_0(\mathcal{O})} \tag{3.22}
\]
for \(\eta \in H^1_0(\mathcal{O})\) and \(\mu_\varepsilon, \mu_{\varepsilon, \delta} \in \mathcal{M}(\mathcal{O})\) by
\[
\mathcal{M}(\mathcal{O}) \langle \mu_\varepsilon, \eta \rangle_{C^0_0(\mathcal{O})} = \mathcal{M}(\mathcal{O}) \langle \mu_\varepsilon, \nu \rangle_{C^0_0(\mathcal{O})}
\]
\[
\mathcal{M}(\mathcal{O}) \langle \mu_{\varepsilon, \delta}, \eta \rangle_{C^0_0(\mathcal{O})} = \mathcal{M}(\mathcal{O}) \langle \mu_\varepsilon \ast \rho_\delta \ast \eta \rangle_{C^0_0(\mathcal{O})} \tag{3.23}
\]
for \(\eta \in C^0_0(\mathcal{O})\). Note that the newly defined objects are bounded linear functionals by Lemmas 3.18, 3.19, 3.20 and 3.21. Furthermore, by (3.21), we get for \(\eta \in C^0_0(\mathcal{O})\)
\[
H^{-1}\langle u_\varepsilon, \eta \rangle_{H^1_0(\mathcal{O})} = \mathcal{M}(\mathcal{O}) \langle \mu_\varepsilon, \eta \rangle_{C^0_0(\mathcal{O})} \quad \text{and} \quad H^{-1}\langle u_{\varepsilon, \delta}, \eta \rangle_{H^1_0(\mathcal{O})} = \mathcal{M}(\mathcal{O}) \langle \mu_{\varepsilon, \delta}, \eta \rangle_{C^0_0(\mathcal{O})},
\]
such that we have the desirable equalities
\[
u_\varepsilon = \iota(\mu_\varepsilon) \quad \text{and} \quad \nu_{\varepsilon, \delta} = \iota(\mu_{\varepsilon, \delta}),
\]
which allows to use \(u_\varepsilon\) and \(\mu_\varepsilon\) as well as \(u_{\varepsilon, \delta}\) and \(\mu_{\varepsilon, \delta}\) interchangeably if there is no risk of confusion.

In order to analyse how \(\phi\) acts on the approximating measures, we introduce a different construction for a measure \(\hat{\mu}_{\varepsilon, \delta}\), which will be shown to coincide with \(\mu_{\varepsilon, \delta}\) from (3.23). For this, we will need that if \(\mu\) is absolutely continuous with respect to the Lebesgue measure, so is \(\mu_\varepsilon\), which we will prove by computing its density:

**Lemma 3.23.** Let \(\varepsilon > 0\), \(h \in L^1(\mathcal{O})\) and \(\mu := h \, dx \in \mathcal{M}(\mathcal{O})\). Then \(\mu_\varepsilon\) has the density
\[
\mathcal{O} \ni x \mapsto \sum_{j=0}^{l} \zeta_j(x + \varepsilon e_j) h_{\text{ext}}(x + \varepsilon e_j)
\]
with respect to the Lebesgue measure.

**Proof.** For \(\eta \in C^0_0(\mathcal{O})\), we compute
\[
\int_{\mathcal{O}} \eta \, d\mu_\varepsilon = \int_{\mathcal{O}} \left( \sum_{j=0}^{l} \zeta_j(x) h_{\text{ext}}(x - \varepsilon e_j) \right) \, \mu(dx)
\]
\[
= \int_{\mathbb{R}^d} \left( \sum_{j=0}^{l} \zeta_j(x) h_{\text{ext}}(x - \varepsilon e_j) \right) \, h_{\text{ext}}(x) \, dx
\]
\[
= \sum_{j=0}^{l} \int_{\mathbb{R}^d} \zeta_j(x + \varepsilon e_j) h_{\text{ext}}(x + \varepsilon e_j) \, dx
\]
\[
= \int_{\mathcal{O}} \eta(x) \sum_{j=0}^{l} \zeta_j(x + \varepsilon e_j) h_{\text{ext}}(x + \varepsilon e_j) \, dx,
\]
as required. The switching of integration domains is possible as the integrands are supported on \(\mathcal{O}\) by Remark 3.10 or by assumption, respectively. \(\square\)
The alternative construction of $\mu_{\varepsilon, \delta}$ is then given by the following definition and lemma.

**Definition 3.24.** Let $\varepsilon, \delta > 0$. We then define the measure

$$
\tilde{\mu}_{\varepsilon, \delta} := \left((\rho_\delta \ast \mu_{\text{ext}})|_O \, dx\right)_\varepsilon,
$$

(3.24)

which is in $\mathcal{M}(O)$ since it is absolutely continuous with respect to the Lebesgue measure and since $\mu$ is finite.

**Lemma 3.25.** For $\varepsilon > 0$, $0 < \delta \leq \frac{w(\varepsilon)}{2}$, the measure $\tilde{\mu}_{\varepsilon, \delta}$ coincides with $\mu_{\varepsilon, \delta}$.

**Proof.** We apply $\tilde{\mu}_{\varepsilon, \delta}$ to $\eta \in C_0^0(O)$ and obtain

$$
\int_O \eta \, d\tilde{\mu}_{\varepsilon, \delta} = \int_O \eta \, d((\rho_\delta \ast \mu_{\text{ext}})|_O \, dx)_\varepsilon
$$

$$
= \int_O \eta \, d(\rho_\delta \ast \mu_{\text{ext}})|_O \, dx
$$

$$
= \int_{\mathbb{R}^d} (\eta \rho_\delta) \ast \mu_{\text{ext}} \, dx
$$

$$
= \int_{\mathbb{R}^d} \rho_\delta \ast (\eta \rho_\delta) \, dx
$$

$$
= \int_O \rho_\delta \ast \eta \, d\mu,
$$

where for the last step, we made use of Remark 3.12 (ii) and of Remark 3.16. We conclude by noticing that the last term is precisely the definition of $M(O) \langle \mu_{\varepsilon, \delta}, \eta \rangle_{C_0^0(O)}$. □

In the rest of this section, we will argue that the sequence

$$
\left(\mu_{\frac{1}{n}, \frac{1}{n} w(\frac{1}{n})}\right)_{n \in \mathbb{N}},
$$

is an approximation of $\mu \in M(O) \cap H^{-1}$ in the sense of Theorem 3.8. First we address the regularity of $\mu_{\varepsilon, \delta}$, where $\varepsilon > 0$ and $0 < \delta \leq \frac{w(\varepsilon)}{2}$.

**Lemma 3.26.** For all $\varepsilon > 0$, $0 < \delta \leq \frac{w(\varepsilon)}{2}$, the approximating measures $\mu_{\varepsilon, \delta}$ have a bounded density with respect to Lebesgue measure.

**Proof.** The fact that $\mu_{\varepsilon, \delta}$ has a density with respect to Lebesgue measure follows from its characterisation in Definition 3.24. This density is bounded in space since

$$
\left| \sum_{j=0}^l \zeta_j^\varepsilon(x + \varepsilon e_j^\varepsilon)(\rho_{\frac{1}{n}} \ast \mu_{\text{ext}})(x) \right| \leq l \sup_{x \in O} \left| \rho_{\frac{1}{n}} \ast \mu(x) \right| \leq l \sup_{x \in \mathbb{R}^d} \left| \rho_{\frac{1}{n}} \ast \mu(x) \right| \|\mu\|_{TV}.
$$

□

The first part of the following proposition allows to deduce property (3.8), while the second part is needed for the further proof of (3.9).

**Proposition 3.27.** Let $\rho$ be as in (3.14) and $0 < \delta \leq \frac{w(\varepsilon)}{2}$.

1. For $\eta \in H_0^1(O)$, we have

$$
\rho_{\delta_\varepsilon} \ast \eta \rightarrow \eta \quad \text{for } \varepsilon \rightarrow 0 \text{ in } H_0^1(O).
$$

(3.25)

2. For $\eta \in C_0^0(O)$, we have

$$
\rho_{\delta_\varepsilon} \ast \eta \rightarrow \eta \quad \text{for } \varepsilon \rightarrow 0 \text{ in } C_0^0(O).
$$

(3.26)
Proof. Throughout this proof, we will write $\delta$ instead of $\delta_x$, always assuming that $0 < \delta \leq \frac{\omega(x)}{2}$.

Proof of part 7 It is enough to show that for all $i \in \{1, \ldots, d\}$

$$\|\partial_i (\rho_k * \eta) - \partial_i \eta\|_{L^2(O)} \to 0 \text{ for } \varepsilon \to 0.$$  \hfill (3.27)

By the density of $C_c^\infty(O)$ in $H^1_0(O)$, for any $\beta > 0$ we can choose $\varphi \in C_c^\infty(O)$ such that

$$\max \left\{ \|\varphi - \eta\|_{L^2(O)}, \|\partial_i \varphi - \partial_i \eta\|_{L^2(O)} \right\} \leq \frac{\beta}{6(l + 1)C},$$  \hfill (3.28)

where

$$\tilde{C} := \max \left\{ \max_{j=1, \ldots, l} \sup_{R^d} \|\partial_i \zeta^j\|, 1 \right\}.$$

As $\varphi_{ext}, \zeta^j \in C_c^1(O)$ for each $j \in \{1, \ldots, l\}$, we can choose $\varepsilon_0 > 0$ small enough, such that for all $x \in R^d$ and $y, z \in B_{\varepsilon_0}(x)$

$$|\partial_i \zeta^j(y)\varphi_{ext}(z) - \partial_i \zeta^j(x)\varphi_{ext}(x)| \leq \frac{\beta}{6(l + 1)|O|^2}$$  \hfill (3.29)

and

$$|\zeta^j(y)\partial_i \varphi_{ext}(z) - \zeta^j(x)\partial_i \varphi_{ext}(x)| \leq \frac{\beta}{6(l + 1)|O|^2}.$$  \hfill (3.30)

We approach (3.27) by splitting the term under consideration into the more convenient pieces

$$\|\partial_i (\rho_k * \eta) - \partial_i \eta\|_{L^2(O)} = \|\rho_k * \partial_i \eta - \partial_i \eta_{ext}\|_{L^2(R^d)}$$

$$= \|\rho_k * \partial_i (\eta - \varphi_{ext}) + \rho_k * \partial_i \varphi_{ext} - \partial_i \eta_{ext}\|_{L^2(R^d)}$$

$$\leq \|\rho_k * \partial_i (\eta - \varphi_{ext})\|_{L^2(R^d)} + \|\rho_k * \partial_i \varphi_{ext} - \partial_i \eta_{ext}\|_{L^2(R^d)} + \|\partial_i \varphi_{ext} - \partial_i \eta_{ext}\|_{L^2(R^d)}$$

$$= (I) + (II) + (III).$$

We estimate the summands separately. For the first one we get with the convolution estimate (e.g. [1 Section 2.13])

$$(I) \leq \|\partial_i (\eta - \varphi_{ext})\|_{L^2(R^d)}$$

$$= \|\partial_i \left[ \sum_{j=0}^{l} \zeta^j \eta_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) - \sum_{j=0}^{l} \zeta^j \varphi_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) \right] \|_{L^2(R^d)}$$

$$= \|\partial_i \left[ \sum_{j=0}^{l} \zeta^j (\eta_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) - \varphi_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d)) \right] \|_{L^2(R^d)}$$

$$\leq \sum_{j=0}^{l} \|\partial_i \zeta^j \left( \eta_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) - \varphi_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) \right) \|_{L^2(R^d)}$$

$$+ \sum_{j=0}^{l} \|\zeta^j \left( \partial_i \eta_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) - \partial_i \varphi_{ext} \cdot (\cdot \cdot - \varepsilon e^j_d) \right) \|_{L^2(R^d)}$$

$$\leq \sum_{j=0}^{l} \left( \sup_{R^d} \|\partial_i \zeta^j\| \|\eta_{ext} - \varphi_{ext}\|_{L^2(R^d)} + \|\partial_i \eta_{ext} - \partial_i \varphi_{ext}\|_{L^2(R^d)} \right)$$

$$\leq \frac{\beta}{3},$$

where we used (3.28) in the last step. For the second term, we recall that $(\zeta^j)^j_{j=0}$ is a partition of unity.
on the support of \( \varphi \). Thus, we can compute

\[
(II) = \left\| \rho_\delta \ast \partial_i \left( \sum_{j=0}^{l} \zeta_j \varphi_{\text{ext}}(x - \varepsilon e_d^j) \right) - \partial_i \left( \sum_{j=0}^{l} \zeta_j \varphi_{\text{ext}} \right) \right\|_{L^2(\mathbb{R}^d)}
\]

\[
\leq \sum_{j=0}^{l} \left\| \rho_\delta \ast \partial_i \left( \zeta_j \varphi_{\text{ext}}(x - \varepsilon e_d^j) \right) - \partial_i \left( \zeta_j \varphi_{\text{ext}} \right) \right\|_{L^2(\mathbb{R}^d)}
\]

\[
= \sum_{j=0}^{l} \left\| \rho_\delta \ast \left( \partial_i \zeta_j \varphi_{\text{ext}}(x - \varepsilon e_d^j) \right) - \partial_i \zeta_j \varphi_{\text{ext}} - \zeta_j \partial_i \varphi_{\text{ext}} \right\|_{L^2(\mathbb{R}^d)}
\]

\[
\leq \sum_{j=0}^{l} \left\| \rho_\delta \ast \left( \partial_i \zeta_j \varphi_{\text{ext}}(x - \varepsilon e_d^j) \right) - \partial_i \zeta_j \varphi_{\text{ext}} \right\|_{L^2(\mathbb{R}^d)} + \sum_{j=0}^{l} \left\| \rho_\delta \ast \left( \zeta_j \partial_i \varphi_{\text{ext}}(x - \varepsilon e_d^j) \right) - \zeta_j \partial_i \varphi_{\text{ext}} \right\|_{L^2(\mathbb{R}^d)}
\]

\[
= \sum_{j=0}^{l} (IV)_j + \sum_{j=0}^{l} (V)_j.
\]

\( (IV)_j \) and \( (V)_j \) are treated analogously, so we only show the estimate for \( (V)_j \), where we choose \( \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 \) as for (3.29). Noting that \( \rho_\delta \) integrates to 1 for any \( \delta > 0 \) and using Jensen’s inequality in the second step, we obtain

\[
(V)_j^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_\delta(x-y) \left| \zeta_j(y) \partial_i \varphi_{\text{ext}}(y - \varepsilon e_d^j) - \zeta_j(x) \partial_i \varphi_{\text{ext}}(x) \right|^2 \, dy \, dx
\]

\[
\leq \int_{\mathbb{R}^d} \int_{\mathcal{B}_\delta(x)} \rho_\delta(x-y) \left| \zeta_j(y) \partial_i \varphi_{\text{ext}}(y - \varepsilon e_d^j) - \zeta_j(x) \partial_i \varphi_{\text{ext}}(x) \right|^2 \, dy \, dx.
\]

As \( \partial_i \varphi_{\text{ext}} \) is supported on \( \mathcal{O} \) and, for the analogous step for \( (IV)_j \), so is \( \varphi_{\text{ext}} \), we can argue as in the proof of Remark 3.10 to see that the integrand of the outer integral is supported on \( \mathcal{O} \). Thus, we can restrict the integration domain to obtain

\[
(V)_j = \int_{\mathcal{B}_\delta(x)} \int_{\mathbb{R}^d} \rho_\delta(x-y) \left| \zeta_j(y) \partial_i \varphi_{\text{ext}}(y - \varepsilon e_d^j) - \zeta_j(x) \partial_i \varphi_{\text{ext}}(x) \right|^2 \, dy \, dx
\]

\[
\leq \int_{\mathcal{O}} \frac{\beta^2}{36 (l+1)^2 |\mathcal{O}|} \int_{\mathbb{R}^d} \rho_\delta(x-y) \, dy \, dx
\]

\[
= \frac{\beta^2}{36 (l+1)^2} \cdot
\]

While we have used (3.30) in the second step, the estimate for \( (IV)_j \) uses (3.29) instead and gets the same result. We conclude

\[
(II) = \sum_{j=0}^{l} \left( (IV)_j + (V)_j \right) \leq \frac{\beta}{3}.
\]

Finally the estimate

\[
(III) \leq \frac{\beta}{3}
\]

is obvious by property (3.28). Collecting (3.31), (3.32), and (3.33), we obtain

\[ \| \partial_i (\rho_\delta \ast \eta_e - \eta) \|_{L^2(\mathcal{O})} \leq \beta \]

only by choosing \( \varepsilon \) small enough and adapting \( 0 < \delta \leq \frac{\varepsilon_0}{2} \), which proves (3.25).

**Proof of part 2** Since \( \eta \) is now assumed to be continuous and to have compact support, it is uniformly continuous. For arbitrary \( \beta > 0 \), we can thus fix \( \varepsilon_0 > 0 \) such that for all \( x, y \in \mathbb{R}^d \)

\[ |x - y| \leq \varepsilon_0 \quad \text{implies} \quad |\eta_{\text{ext}}(x) - \eta_{\text{ext}}(y)| \leq \frac{\beta}{l+1} \]
For \( \varepsilon \leq \frac{1}{2} \varepsilon_0 \), we use \( \delta \leq \frac{\varepsilon_0}{2} \leq \varepsilon \) by (3.16) to calculate for \( x \in \mathcal{O} \)

\[
|\rho_\delta * \eta_\varepsilon(x) - \eta(x)| = \left| \int_{B_\delta(x)} \rho_\delta(x-y) \left( \sum_{j=0}^l \zeta_j(y)(\eta_{\text{ext}}(y-\varepsilon \varepsilon_d) - \eta(x)) \right) dy \right|
\]

\[
\leq \int_{B_\delta(x)} \rho_\delta(x-y) \sum_{j=0}^l \left| \eta_{\text{ext}}(y-\varepsilon \varepsilon_d^j) - \eta_{\text{ext}}(x) \right| dy
\]

\[
\leq \int_{B_\delta(x)} \rho_\delta(x-y) \sum_{j=0}^l \frac{\beta}{l+1} dy = \beta,
\]

where for the second step we observe that for \( y \in B_\delta(x) \), we have

\[
| (y - \varepsilon \varepsilon_d^j) - x | \leq \delta + \varepsilon \leq 2 \varepsilon \leq \varepsilon_0.
\]

This proves (3.20). \( \square \)

We now turn to prove Property (3.9). Recall the definition of a convex function of a measure from Definition 3.3. We need some more lemmas on measures obtained by this technique, the first of which can be found in [23, Equation (2.11)].

**Lemma 3.28.** Let \( \psi \) satisfy (2.7) as well as conditions Assumptions (A3), (A5'). Let \( \mu \in \mathcal{M}(\mathbb{R}^d) \) and let \( (\rho_\delta)_{\delta > 0} \) be a family of mollifying kernels as specified in (5.15) and (5.14). Then

\[
\int_{\mathbb{R}^d} \psi(\rho_\delta * \mu) \, dx \leq \int_{\mathbb{R}^d} \psi(\mu) \quad \text{for all } \delta > 0.
\] (3.35)

**Remark 3.29.** Given the assumptions on \( \psi \), the theory of Definition 3.3 indeed also applies to finite measures on \( \mathbb{R}^d \) (cf. [23, p. 202]).

**Lemma 3.30.** Let \( \psi \) satisfy (2.7) as well as conditions Assumptions (A3), (A5'). For \( \mu \in \mathcal{M}(\mathcal{O}) \) we have

\[
\int_{\mathbb{R}^d} \psi(\mu_{\text{ext}}) = \int_{\mathcal{O}} \psi(\mu).
\] (3.36)

**Proof.** We define

\[
\mathcal{D}_1 := \left\{ \int_{\mathcal{O}} v \, d\mu - \int_{\mathcal{O}} \psi^*(v) \, dx : v \in L^1(\mu), \psi^*(v) \in L^1(\mathcal{O}) \right\}
\]

and

\[
\mathcal{D}_2 := \left\{ \int_{\mathbb{R}^d} v \, d\mu_{\text{ext}} - \int_{\mathbb{R}^d} \psi^*(v) \, dx : v \in L^1(M_{\text{ext}}), \psi^*(v) \in L^1(\mathbb{R}^d) \right\},
\]

which allows us to write

\[
\int_{\mathcal{O}} \psi(\mu) = \sup \mathcal{D}_1 \quad \text{and} \quad \int_{\mathbb{R}^d} \psi(\mu_{\text{ext}}) = \sup \mathcal{D}_2.
\]

We note that for \( v \) satisfying the conditions of \( \mathcal{D}_1 \), \( v_{\text{ext}} \) satisfies the conditions of \( \mathcal{D}_2 \), while the involved integrals agree due to the definition of \( \mu_{\text{ext}} \) and \( \psi^*(0) = 0 \). This yields “\( \geq \)”. Conversely, for \( v \) satisfying the conditions of \( \mathcal{D}_2 \) we can define \( \tilde{v} = v|_{\mathcal{O}} \). \( \tilde{v} \) satisfies the conditions of \( \mathcal{D}_1 \). Furthermore, we have

\[
\int_{\mathcal{O}} \tilde{v} \, d\mu = \int_{\mathbb{R}^d} v \, d\mu_{\text{ext}} \quad \text{and} \quad \int_{\mathcal{O}} \psi^*(\tilde{v}) \, dx \leq \int_{\mathbb{R}^d} \psi^*(v) \, dx \quad \text{due to } \psi^* \geq 0.
\]

Thus, we have found an element in \( \mathcal{D}_1 \) being larger than or equal to

\[
\int_{\mathbb{R}^d} v \, d\mu_{\text{ext}} - \int_{\mathbb{R}^d} \psi^*(v) \, dx,
\]

which yields “\( \leq \)”, completing the proof. \( \square \)
The key tool to prove the approximation property (3.38) is the following proposition. Although the statement can be proved for arbitrary $\mu \in \mathcal{M}(\mathcal{O})$, we can read off the construction in (3.24) that it is sufficient to consider Radon measures which are absolutely continuous with respect to the Lebesgue measure.

**Proposition 3.31.** Let $\varepsilon > 0$ and $0 < \delta \leq \frac{w(x)}{2}$. Let $h \in L^1(\mathcal{O})$ and set $\mu = h \, dx$. Then,

$$
\|\psi(\mu_{\varepsilon, \delta})\|_{TV} \leq \|\psi(\mu)\|_{TV}.
$$

(3.37)

**Proof.** Step 1: Recall Notations (3.10) and let $V^j = U^j \cap \mathcal{O}$. We want to approach (3.37) by estimating

$$
\int_{\mathcal{O}} \xi_{\alpha} \psi(\mu_{\varepsilon}),
$$

where $(\xi_{\alpha})_{\alpha > 0} \subset C^0_c(\mathbb{R}^d)$ is a sequence of non-negative cut-off functions compactly supported in $\mathcal{O}$, which converge to 1 pointwise in $\mathcal{O}$ for $\alpha \to 0$, and each of which is monotonically increasing on each $V^j$ in $e_d$ direction.

For instance, these functions can be constructed for $\alpha > 0$ as

$$
\xi_{\alpha}(x) = \begin{cases}
0 & \text{if dist}(x, \mathcal{O}^c) < \frac{\alpha}{2} \\
\frac{\alpha}{\alpha} (\text{dist}(x, \mathcal{O}^c) - \frac{\alpha}{2}) & \text{if } \frac{\alpha}{2} < \text{dist}(x, \mathcal{O}^c) < \alpha \\
1 & \text{if dist}(x, \mathcal{O}^c) > \alpha.
\end{cases}
$$

We briefly argue that $\xi_{\alpha}$ satisfies the abovementioned criteria. Note that since $e_d^0$ is set to zero, there is nothing to show for $j = 0$. Hence, let $j \in 1, \ldots, l$ and fix first $x, y \in V^j$ where $y = x + \varepsilon e_d^j$ for some $\varepsilon > 0$. Our goal is to show that

$$
\xi_{\alpha}(x) \leq \xi_{\alpha}(y).
$$

It is enough to show that

$$
\text{dist}(x, \partial \mathcal{O} \cap \tilde{U}^j) \leq \text{dist}(y, \partial \mathcal{O} \cap \tilde{U}^j),
$$

(3.38)

since $\xi_{\alpha}(x)$ is obviously monotonically increasing with the distance of $x$ to $\partial \mathcal{O}$ and for any other boundary point $b \in \partial \mathcal{O} \setminus \tilde{U}^j$, we have

$$
\|x - b\|_{\mathbb{R}^d} \geq \frac{h_y}{2} \geq \text{dist}(x, \partial \mathcal{O} \cap \tilde{U}^j),
$$

which yields

$$
\text{dist}(x, \mathcal{O}^c) = \text{dist}(x, \partial \mathcal{O} \cap \tilde{U}^j).
$$

**Argument for (3.38):** Consider a point $z = (z_d, z_d) = (z_d, g(z_d))$ in $(e_d^j, e_d^j)$ coordinates on $\partial \mathcal{O} \cap \tilde{U}^j$. If $z_d \leq x_d(< y_d)$, we have that

$$
\|x - z\|^2 = \|x_d - z_d\|^2 + |x_d - z_d|^2 \\
\leq \|x_d - z_d\|^2 + |y_d - z_d|^2 = \|y - z\|^2,
$$

where $\|\cdot\|$ denotes the Euclidean norm in both $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$. If $z_d > x_d$, we obtain by the intermediate value theorem $\lambda \in (0, 1)$ such that $z^* = \lambda z_d + (1 - \lambda) d$ and $g(z^*) = x_d$. Then we can compute with the same convention for $\|\cdot\|$,

$$
\|x - (z^*, x_d)\|^2 \leq \|x_d - z_d\|^2 \leq \|x_d - z_d\|^2 + |y_d - z_d|^2 = \|y - z\|^2.
$$

Thus, for any $z \in \partial \mathcal{O} \cap \tilde{U}^j$ on the graph, we can always find $(z^*, g(z^*))$ such that

$$
\|x - (z^*, g(z^*))\| \leq \|y - z\|
$$

(clearly in the first case one would set $z^* = x_d$). This yields (3.38).

In the following argument, we will need $\xi_{\alpha}(x) \geq (\xi_{\alpha})_{ext}(x - \varepsilon e_d^j)$ for $x \in V^j$, where $x - \varepsilon e_d^j$ is not a priori in $\mathcal{O}$. However, since $\xi_{\alpha} = 0$ outside of $\mathcal{O}$, it is clear that the statement is valid even if $x - \varepsilon e_d^j \notin \mathcal{O}$. 21
By the convexity of $\psi$, the construction of $(\zeta^j)_{j=0}^l$ and Lemma 3.23, we can estimate

$$\int_{\mathcal{O}} \xi_\alpha \psi(\mu_\varepsilon) = \int_{\mathcal{O}} \xi_\alpha(x) \psi \left( \sum_{j=0}^l \zeta^j(x + \varepsilon e_d^j) h_{\text{ext}}(x + \varepsilon e_d^j) \right) dx$$

$$\leq \int_{\mathcal{O}} \xi_\alpha(x) \sum_{j=0}^l \zeta^j(x + \varepsilon e_d^j) \psi(h_{\text{ext}}(x + \varepsilon e_d^j)) dx$$

$$= \int_{\mathbb{R}^d} \xi_\alpha(x) \sum_{j=0}^l \zeta^j(x + \varepsilon e_d^j) \psi(h_{\text{ext}}(x + \varepsilon e_d^j)) dx$$

$$= \int_{\mathbb{R}^d} \psi(h_{\text{ext}}(x)) \sum_{j=0}^l \xi_\alpha(x - \varepsilon e_d^j) \zeta^j(x) dx. \quad (3.39)$$

We note that $\sum_{j=0}^l \xi_\alpha(x - \varepsilon e_d^j) \zeta^j(x)$ is supported on $\mathcal{O}$ by Remark 3.16. Furthermore, by the construction of $\xi_\alpha$, we have

$$\xi_\alpha(x - \varepsilon e_d^j) \leq \xi_\alpha(x)$$

for all $x \in V^j$, so this holds especially for $x \in \mathcal{O}$ for which $\zeta^j(x) > 0$. Thus, we can continue

$$\xi_\alpha(x) \sum_{j=0}^l \xi_\alpha(x - \varepsilon e_d^j) \zeta^j(x) \psi(h(x)) dx$$

$$\leq \int_{\mathcal{O}} \sum_{j=0}^l \zeta^j(x) \xi_\alpha(x) \psi(h(x)) dx$$

$$= \int_{\mathcal{O}} \xi_\alpha(x) \psi(h(x)) dx = \int_{\mathcal{O}} \xi_\alpha \psi(\mu). \quad (3.40)$$

For a positive Radon measure $\mu$, we have $\mu(\mathcal{O}) = \sup \{ \mu(K) : K \subseteq \mathcal{O} \text{ compact} \}$. Since any such $K$ is included in

$$K_\alpha := \{ x \in \mathcal{O} : \text{dist}(x, \mathcal{O}^c) \geq \alpha \}$$

for $\alpha$ small enough, we can as well write $\mu(\mathcal{O}) = \lim_{\alpha \to 0} \mu(K_\alpha)$. Then, noting that $\xi_\alpha \geq 1_{K_\alpha}$, we can argue by definition of the Radon measure of compact sets that

$$\mu(\mathcal{O}) \geq \int_{\mathcal{O}} \xi_\alpha d\mu \geq \mu(K_\alpha) \xrightarrow{\alpha \to 0} \mu(\mathcal{O}),$$

thus $\mu(\mathcal{O}) = \lim_{\alpha \to 0} \int_{\mathcal{O}} \xi_\alpha d\mu$.

Hence, we conclude by

$$\int_{\mathcal{O}} \psi(\mu_\varepsilon) = \lim_{\alpha \to 0} \int_{\mathcal{O}} \xi_\alpha \psi(\mu_\varepsilon)$$

$$\leq \lim_{\alpha \to 0} \int_{\mathcal{O}} \xi_\alpha \psi(\mu) = \int_{\mathcal{O}} \psi(\mu). \quad (3.41)$$

**Step 2:** With the help of Lemma 3.28 and Lemma 3.30, we obtain for $0 < \delta \leq \frac{m(x)}{2}$:

$$\int_{\mathcal{O}} \psi(\mu_\varepsilon, \delta) = \int_{\mathcal{O}} \psi((\mu_\delta \ast \mu_{\text{ext}})|_{\mathcal{O}}) dx$$

$$\leq \int_{\mathcal{O}} \psi((\mu_\delta \ast \mu_{\text{ext}})|_{\mathcal{O}}) dx$$

$$= \int_{\mathbb{R}^d} \psi(\mu_\delta \ast \mu_{\text{ext}}) 1_{\mathcal{O}} dx$$

$$\psi \geq 0$$

$$\int_{\mathbb{R}^d} \psi(\mu_\delta \ast \mu_{\text{ext}}) dx$$

$$\leq \int_{\mathbb{R}^d} \psi(\mu_{\text{ext}}) dx$$

$$\leq \int_{\mathbb{R}^d} \psi(\mu),$$

which finishes the proof.
Corollary 3.32. Together with Remark 3.4, Proposition 3.31 immediately implies
\[ \limsup_{\varepsilon \to 0} \int_{\Omega} \psi(\mu_{\varepsilon, \delta_\varepsilon}) \leq \int_{\Omega} \psi(\mu) \]
as long as \( 0 < \delta_\varepsilon \leq \frac{w(\varepsilon)}{2} \).

Proof of Theorem 3.8. For \( \mu \) as in Theorem 3.8, we show that the sequence \((\mu_n)_{n \in \mathbb{N}} := (\mu_{\frac{1}{n}, \frac{1}{2}w(\frac{1}{n}))_{n \in \mathbb{N}} \)
where \( w \) was defined in Remark 3.16, meets all requirements.

By construction, \( \mu_n \in M(\Omega) \cap H^{-1} \) for all \( n \in \mathbb{N} \), and by Lemma 3.29, the density of \( \mu_n \) is bounded and thus in \( L^2(\Omega) \). Property 3.8 is proved in the first part of Proposition 3.27. For Property 3.9, note that Corollary 3.32 especially shows that \((\mu_n)_{n \in \mathbb{N}} \) is uniformly bounded in the TV norm, which means that it contains a subsequence that converges weakly* to \( \psi(\mu) \) by Proposition 3.27 and [23, Lemma 2.1]. Since this argument can be carried out for any subsequence, we get weak* convergence for the whole sequence and, also by [23, Lemma 2.1],
\[ \| \psi(\mu_{\frac{1}{n}, \frac{1}{2}w(\frac{1}{n})) \|_{TV} \rightarrow \| \psi(\mu) \|_{TV} \text{ as } n \rightarrow \infty. \]
This yields 3.9 and thereby concludes the proof.

4 Proof of the main result

Throughout this section, we work under Assumptions 2.1.

We first solve a modified SPDE by the variational approach, which will yield \( \varepsilon \)-approximate solutions. Moreover, we show improved regularity for those approximations, which is used later to prove their convergence to a limit in \( L^2(\Omega; C([0, T]; H^{-1})) \) for \( \varepsilon \to 0 \).

We consider the SPDE
\[ dX_t^\varepsilon = \varepsilon \Delta X_t^\varepsilon dt + \Delta \phi(\delta X_t^\varepsilon)dt + B(t, X_t^\varepsilon)dW_t, \]
\[ X_0^\varepsilon = x_0, \]
where we use the notation for the Yosida approximation of Appendix D and assume \( x_0 \in L^2(\Omega, \mathcal{F}_0; L^2) \). Now and in the following we omit the domain \( \Omega \) when using Lebesgue and Sobolev spaces as well as spaces of continuous or continuously differentiable functions, as introduced in Section 1.3.

Lemma 4.1. For all \( T > 0 \), Problem (4.1) gives rise to a solution in sense of Definition 3.1 with respect to the Gelfand triple \( V := L^2 \hookrightarrow H^{-1} \hookrightarrow (L^2)' = V' \).

Proof. We prove that (4.1) fits into the framework of Appendix B with the operator
\[ A(u) = \Delta(\varepsilon u + \phi'(u)) \text{ for } u \in L^2. \]
In [53, Example 4.1.11], it is shown that an operator \( A \) of the form \( u \mapsto \Delta(\Psi(u)) \) satisfies the four properties of Appendix B with respect to the Gelfand triple \( L^p \hookrightarrow H^{-1} \hookrightarrow (L^p)' \), if the following conditions are satisfied.

(\( \Psi1 \)) \( \Psi \) is continuous.
(\( \Psi2 \)) For all \( s, t \in \mathbb{R} \) we have
\[ (t-s)(\Psi(t) - \Psi(s)) \geq 0. \]
(\( \Psi3 \)) There exist \( p \in [2, \infty), a \in (0, \infty), c \in [0, \infty) \) such that for all \( s \in \mathbb{R} \) we have
\[ s\Psi(s) \geq a |s|^p - c. \]
There exist \( c_3, c_4 \in (0, \infty) \) such that for all \( s \in \mathbb{R} \)

\[
|\Psi(s)| \leq c_4 + c_3 |s|^{p-1},
\]

where \( p \) is as in (Ψ3).

We briefly check (Ψ1) – (Ψ4) for \( \Psi := \varepsilon \text{Id}_\mathbb{R} + \phi^s \). The first condition is satisfied by Lemma 4.2, the second one by the maximal monotonicity of \( \phi^s \), together with Corollary 2.1. Using \( \phi^s(0) = 0 \) and again the monotonicity of \( \phi^s \), we obtain \( s\phi^s(s) \geq 0 \) and thereby

\[
s\Psi(s) \geq s\varepsilon \text{Id}_\mathbb{R}(s) = \varepsilon |s|^2.
\]

Thus, (Ψ3) is satisfied for \( p = 2, a = \varepsilon \) and \( c = 0 \). (Ψ4) is then clear by Lemma 4.2. Thus, Theorem 1.2 is applicable as required.

The following lemma provides an important estimate on the regularity of these approximate solutions and corresponds to Lemma B.1:

**Lemma 4.2.** Let \( \varepsilon > 0 \), \( x_0 \in L^2(\Omega,F_0; L^2) \) and \( T > 0 \). Then for the solution \( (X_t^\varepsilon)_{t \in [0,T]} \) to (4.1) we have

\[
\mathbb{E} \sup_{t \in [0,T]} \|X^\varepsilon_t\|^2_2 + \varepsilon \mathbb{E} \int_0^T \|X^\varepsilon_r\|^2_{H^1_0} \, dr \leq C(\varepsilon \|x_0\|^2_2 + 1)
\]

with a constant \( C > 0 \) independent of \( \varepsilon \).

**Remark 4.3 (Notation).** In order to avoid confusion, we will distinguish two different embeddings of \( H^1_0 \) into \( H^{-1} \) in the following proof and sometimes later if there is some risk of misunderstanding. First, via the Riesz isomorphism \( -\Delta \) on \( H^1_0 \) with respect to the scalar product \( \langle u, v \rangle_{H^1_0} = \langle \nabla u, \nabla v \rangle_{L^2} \), since \( \Delta : H^1_0 \to H^{-1} \) is defined by

\[
H^{-1} \langle \Delta u, v \rangle_{H^1_0} = -\langle \nabla u, \nabla v \rangle_{L^2} \quad \text{for } u \in H^1_0.
\]

Second, via the duality mapping \( I' : L^2 \to H^{-1} \), where

\[
H^{-1} \langle I'u, v \rangle_{H^1_0} = \langle I'u, v \rangle_{L^2},
\]

using \( I \) for the canonical embedding \( H^1_0 \hookrightarrow L^2 \).

Since the identification of \( H^1_0 \) functions as \( L^2 \) functions is unambiguous, we will usually not mention the embedding \( I \). For taking the classical Laplacian of a \( C^2 \) function without embedding it into \( H^{-1} \), we write \( \Delta_{\text{class}} \).

**Example 4.4.** For \( u \in C^2_0 \), we have \( -\Delta u = I'(-\Delta_{\text{class}} u) \). As a proof, one can test with \( h \in H^1_0 \) and get

\[
H^{-1} \langle I'(-\Delta_{\text{class}} u), h \rangle_{H^1_0} = \langle -\Delta_{\text{class}} u, h \rangle_{L^2} = \langle \nabla u, \nabla h \rangle_{L^2} = H^{-1} \langle -\Delta u, h \rangle_{H^1_0}.
\]

**Proof of Lemma 4.2.** Let \( (e_i)_{i \in \mathbb{N}} \subset C^2_0 \) be a sequence of smooth eigenvectors to \( -\Delta_{\text{class}} \), i.e., \( -\Delta_{\text{class}} e_i = \lambda_i e_i \) for some \( (\lambda_i)_{i \in \mathbb{N}} \subset (0, \infty) \), such that \( (I' e_i)_{i \in \mathbb{N}} \) is an orthonormal basis in \( H^{-1} \). Such a sequence can be obtained by first choosing an \( L^2 \)-orthonormal basis of \( (-\Delta_{\text{class}}) \)-eigenvectors \( (\tilde{e}_i)_{i \in \mathbb{N}} \subset C^2_0 \subset L^2 \), where

\[
-\Delta_{\text{class}} \tilde{e}_i = \lambda_i \tilde{e}_i \quad \text{for some } \lambda_i > 0.
\]

Then, setting

\[
e_i = \sqrt{\lambda_i} \tilde{e}_i \quad \text{for } i \in \mathbb{N}
\]

keeps (4.2) true for \( \tilde{e}_i \) replaced by \( e_i \) and makes \( (I' e_i)_{i \in \mathbb{N}} \) an orthonormal basis in \( H^{-1} \) as required. The latter can be seen by computing for \( i, j \in \mathbb{N} \)

\[
\langle I' e_i, I' e_j \rangle_{H^{-1}} = \left( I' \left( \frac{\sqrt{\lambda_i}}{\lambda_i} (-\Delta_{\text{class}} \tilde{e}_i) \right), I' \left( \frac{\sqrt{\lambda_j}}{\lambda_j} \tilde{e}_j \right) \right)_{H^{-1}}.
\]

Using (4.4)

\[
\langle -\Delta \tilde{e}_i, I' \tilde{e}_j \rangle_{H^{-1}} = H^{-1}_0 \langle \tilde{e}_i, I' \tilde{e}_j \rangle_{H^{-1}} = \langle \tilde{e}_i, \tilde{e}_j \rangle_{L^2} = \delta_{ij}.
\]
We further let $P^n : H^{-1} \to H_n := \text{span}\{e_1, \ldots, e_n\}$ be the orthogonal projection onto the span of the first $n$ eigenvectors, i.e.

$$P^n(y) = \sum_{i=1}^n \langle y, I' e_i \rangle_{H^{-1}} e_i.$$ 

Recall that the unique variational solution $X^\varepsilon$ to (1.1) is constructed in [53] as a (weak) limit in $L^2([0, T] \times \Omega; L^2)$ of the solutions to the Galerkin approximation

$$\begin{align*}
dX^n_t &= \varepsilon P^n \Delta X_t^n dt + P^n \Delta \phi^\varepsilon(X_t^n) dt + P^n B(t, X_t^n) dW_t^n \\
X_0^n &= P^n x_0,
\end{align*}$$

in $H_n$, where for simplicity we omit the $\varepsilon$-dependence of $X^n$, and for an orthonormal basis $(g_i)_{i \in \mathbb{N}}$ of $U$ (as defined in Assumption 2.1[A1]) we let

$$W^n_t = \sum_{i=1}^n \langle J^{-1}(W_t), g_i \rangle_{L^2} g_i.$$ 

We now first note that for $x \in H_n$ we have $\Delta_{\text{class}} x \in H_n$ and thus

$$-\Delta x = I'(-\Delta_{\text{class}} x) \in \text{span}\{I' e_1, \ldots, I' e_n\}.$$ 

We conclude

$$P^n(-\Delta x) = \begin{cases} 
P^n(-\Delta x) \\
(I')^{-1} \left( \sum_{i=1}^n \langle -\Delta x, I' e_i \rangle_{H^{-1}} I' e_i \right) \\
(I')^{-1} (-\Delta x) \\
(I')^{-1} (I'(-\Delta_{\text{class}} x)) \\
-\Delta_{\text{class}} x. 
\end{cases}$$

Using these considerations, we can now first analyse some of the terms appearing when applying the finite-dimensional Ito formula to $e^{-Kt} \|X_t^n\|^2_{L^2}$. Using $X^n \in H_n$, we first compute

$$\langle X^n, P^n(-\Delta X^n) \rangle_{L^2} = \langle X^n, -\Delta_{\text{class}} X^n \rangle_{L^2} = \|X^n\|_{H^1_0}^2.$$ 

We note by Lemma [D.3] and (2.5) that

$$|\phi^\varepsilon(X^n)|^2 \leq C(1 + (X^n)^2),$$

so $\phi^\varepsilon(X^n) \in L^2$ since $X^n \in H_n \subseteq L^2$. Thus, $\phi^\varepsilon(X^n) \in H^1_0$ by [53] Theorem 2.1.11], and we can compute

$$\begin{align*}
\langle X^n, P^n(-\Delta \phi^\varepsilon(X^n)) \rangle_{L^2} &= \langle X^n, \sum_{i=1}^n \langle -\Delta \phi^\varepsilon(X^n), I' e_i \rangle_{H^{-1}} e_i \rangle_{L^2} \\
&= \sum_{i=1}^n \langle -\Delta \phi^\varepsilon(X^n), I' e_i \rangle_{H^{-1}} \langle X^n, e_i \rangle_{L^2} \\
&= \langle -\Delta \phi^\varepsilon(X^n), \sum_{i=1}^n \langle X^n, e_i \rangle_{L^2} I' e_i \rangle_{H^{-1}} \\
&= \langle -\Delta \phi^\varepsilon(X^n), \sum_{i=1}^n H^1_0 \langle X^n, I' e_i \rangle_{H^{-1}} I' e_i \rangle_{H^{-1}} \\
&= \langle -\Delta \phi^\varepsilon(X^n), -\Delta X^n \rangle_{H^{-1}} \\
&= H^{-1} (-\Delta \phi^\varepsilon(X^n), X^n)_{H^1_0}.
\end{align*}$$
Again by [59] Theorem 2.1.11, we obtain for all $r \in [0,t]$

$$
\mu^{-1} \langle \Delta \phi^x (X^n_r), X^n_r \rangle \big|_{H^2_0}^2 = - \langle \nabla X^n_r, \nabla \phi^x (X^n_r) \rangle_{L^2} = - (\phi^x)' (X^n_r) \|X^n_r\|_{H^2_0}^2 \leq 0,
$$

where we used that $(\phi^x)' (X^n_r) \geq 0$ almost everywhere by the monotonicity of $\phi^x$. Along with the finite-dimensional Ito formula, this can be used to estimate

$$
e^{-Kt} \|X^n_t\|_{L^2}^2 = \|P^n x_0\|_{L^2}^2 + 2 \int_0^t e^{-Kr} \langle X^n_r, eP^n (\Delta X^n) + P^n (\Delta \phi^x (X^n)) \rangle_{L^2} dr
$$

$$
+ 2 \int_0^t e^{-Kr} \langle X^n_r, P^n B(r, X^n) \rangle_{L^2} dr
$$

$$
+ \int_0^t e^{-Kr} \|P^n B(r, X^n)\|_{L^2(U,L^2)}^2 dr - K \int_0^t e^{-Kr} \|X^n_t\|_{L^2}^2 dr
$$

$$
\leq \|P^n x_0\|_{L^2}^2 - 2\varepsilon \int_0^t e^{-Kr} \|X^n_t\|_{H^2_0}^2 dr
$$

$$
+ 2 \int_0^t e^{-Kr} \langle X^n_r, P^n B(r, X^n) \rangle_{L^2} dr
$$

$$
+ \int_0^t e^{-Kr} \|P^n B(r, X^n)\|_{L^2(U,L^2)}^2 dr - K \int_0^t e^{-Kr} \|X^n_t\|_{L^2}^2 dr. \tag{4.3}
$$

Using lemma [57, 1] the Burkholder-Davis-Gundy inequality (see e.g. [53, Appendix D]) and (2.3), we get for the stochastic integral term in (4.3)

$$
E \sup_{t \in [0,T]} \left| \int_0^t \left\langle e^{-\frac{Kt}{2}} X^n_r, e^{-\frac{Kt}{2}} P^n B(r, X^n) \right\rangle_{L^2} dr \right|
$$

$$
\leq 3 E \left( \int_0^T \left\| e^{-\frac{Kt}{2}} X^n_r \right\|_{L^2}^2 \left\| e^{-\frac{Kt}{2}} P^n B(r, X^n) \right\|_{L^2(U,L^2)}^2 dr \right)^{\frac{1}{2}}
$$

$$
\leq 3 E \left( \int_0^T \left\| e^{-\frac{Kt}{2}} B(r, X^n) \right\|_{L^2(U,L^2)}^2 dr \right)^{\frac{1}{2}}
$$

$$
\leq 3 E \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \int_0^T \left\| e^{-\frac{Kt}{2}} B(r, X^n) \right\|_{L^2(U,L^2)}^2 dr \right)^{\frac{1}{2}}
$$

$$
\leq 3 E \left[ \frac{1}{12} \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right) + \frac{6}{2} \int_0^T e^{-Kt} \|B(r, X^n)\|_{L^2(U,L^2)}^2 dr \right]
$$

$$
\leq \frac{1}{4} E \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right) + 9E \int_0^T e^{-Kr} C(1 + \|X^n_r\|_{L^2}^2) dr
$$

$$
= \frac{1}{4} E \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right) + 9CE \int_0^T e^{-Kr} \|X^n_r\|_{L^2}^2 dr + \tilde{C}
$$

We can now estimate from (4.3) and the previous calculation that

$$
E \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right) + K E \int_0^T e^{-Kr} \|X^n_r\|_{L^2}^2 dr + 2\varepsilon \int_0^T e^{-Kr} \|X^n_r\|_{H^2_0}^2 dr
$$

$$
\leq 3 E \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right) + K E \int_0^t e^{-Kr} \|X^n_r\|_{L^2}^2 dr + 2\varepsilon \int_0^t e^{-Kr} \|X^n_r\|_{H^2_0}^2 dr
$$

$$
\leq 3 \left( E \|x_0\|_{L^2}^2 + \frac{1}{4} E \sup_{r \in [0,T]} \left( e^{-Kt} \|X^n_r\|_{L^2}^2 \right) + C E \int_0^T e^{-Kr} \|X^n_r\|_{L^2}^2 dr + \tilde{C} \right),
$$

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where we absorbed the second-to-last term in \([43](4.3)\) into the terms with the constants \(C\) and \(\tilde{C}\). Thus, we get

\[
\mathbb{E} \sup_{\tau \in [0,T]} \left( e^{-K\tau} \| X^n_\tau \|^2_{L^2} \right) + \varepsilon \int_0^T e^{-K\tau} \| X^n_\tau \|_{H^1_0}^2 \, d\tau
\]

\[
\leq 4\mathbb{E} \| x_0 \|^2_{L^2} + 4(C - K) \mathbb{E} \int_0^T e^{-K\tau} \| X^n_\tau \|_{L^2}^2 \, d\tau + \tilde{C},
\]

and by choosing \(K\) large enough and multiplying by \(e^{KT}\), we absorb in the constant, we obtain

\[
\mathbb{E} \sup_{\tau \in [0,T]} \| X^n_\tau \|_{L^2}^2 + \varepsilon \mathbb{E} \int_0^T \| X^n_\tau \|_{H^1_0}^2 \, d\tau \leq C(\mathbb{E} \| x_0 \|_{L^2}^2 + 1).
\]

Thus, \((X^n)_{n \in \mathbb{N}}\) is bounded in \(L^2(\Omega; L^\infty([0,T]; L^2))\) and in \(L^2(\Omega \times [0,T]; H^1_0)\). The latter is a Hilbert space, thus we can extract a weakly converging subsequence whose limit can be identified with the unique weak \(L^2(\Omega \times [0,T]; L^2)\) limit \(X^\varepsilon\). Furthermore, we can interpret the former as the dual space of \(L^2(\Omega; L^1([0,T]; L^2))\) which is separable. Thus, we can extract a weak* converging subsequence whose limit can again be identified with \(X^\varepsilon\). By weak (respectively weak*) lower-semicontinuity of the norms, we can thus pass to the limit \(n \to \infty\) to obtain the required inequality.

\[\left(\text{Remark 4.5.}\right]\] The Laplacian on \(L^2\) to the formal dual space \((L^2)'\) is defined in such a way that, using the notation of Remark 4.3 for \(x \in H^1_0 \subset L^2\) and \(y \in L^2\) we have

\[
(L^2)'(-\Delta x, y)_{L^2} = (-\Delta x, y')_{H^{-1}}.
\]

For the sake of simplicity, we will from now on omit writing \(I'\) for the duality embedding \(L^2 \hookrightarrow H^{-1}\), if there is no risk of confusion.

\[\left(\text{Proof of Theorem 2.7.}\right]\] The proof will be carried out in three steps. We first construct a solution candidate as a limit of solutions to \([4.1]\). Then we show that this limit indeed is an SVI solution and we conclude by showing uniqueness, which relies on the same construction which was already used to show the existence of a solution.

Step 1: We first show that the solutions \((X^\varepsilon)_{\varepsilon \geq 0}\) to \([4.1]\) form a Cauchy sequence in \(L^2(\Omega; C([0,T]; H^{-1}))\) for \(\varepsilon \to 0\). To this end, we first consider two of those solutions \(X^\varepsilon_1, X^\varepsilon_2\) with respective initial condition \(x_0^1, x_0^2 \in L^2(\Omega; H^1_0; L^2)\). By subsequently applying the Ito formula for the squared norm in Hilbert spaces (see e.g. [53] Theorem 4.2.5) and the finite-dimensional Ito formula (see e.g. [53] IV §3), and using 4.3 we then have for \(K > 0\)

\[
e^{-Kt} \| X^\varepsilon_1 - X^\varepsilon_2 \|^2_{H^{-1}} = \| I'(x_0^1 - x_0^2) \|^2_{H^{-1}}
\]

\[
+ 2 \int_0^t e^{-Kr} \langle \varepsilon_1 \Delta X^\varepsilon_1 - \varepsilon_2 \Delta X^\varepsilon_2, I'(X^\varepsilon_1 - X^\varepsilon_2) \rangle_{H^{-1}} \, d\tau
\]

\[
+ 2 \int_0^t e^{-Kr} \langle \Delta \phi^\varepsilon_1(X^\varepsilon_1) - \Delta \phi^\varepsilon_2(X^\varepsilon_2), I'(X^\varepsilon_1 - X^\varepsilon_2) \rangle_{H^{-1}} \, d\tau
\]

\[
+ 2 \int_0^t e^{-Kr} \langle X^\varepsilon_1 - X^\varepsilon_2, B(r, X^\varepsilon_1) - B(r, X^\varepsilon_2) \rangle_{L^2(U; H^{-1})} \, d\tau
\]

\[
- K \int_0^t e^{-Kr} \| X^\varepsilon_1 - X^\varepsilon_2 \|^2_{H^{-1}} \, d\tau.
\]

Using the definition of \(I'\) and \(-\Delta\) for the first step and Corollary 4.9 for the second step, we note that

\[
\langle \Delta \phi^\varepsilon_1(X^\varepsilon_1) - \Delta \phi^\varepsilon_2(X^\varepsilon_2), I'(X^\varepsilon_1 - X^\varepsilon_2) \rangle_{H^{-1}} =
\]

\[
= - \int_C (\phi^\varepsilon_1(X^\varepsilon_1) - \phi^\varepsilon_2(X^\varepsilon_2))(X^\varepsilon_1 - X^\varepsilon_2) \, dx
\]

\[
\leq C(\varepsilon_1 + \varepsilon_2) \left( 1 + \|X^\varepsilon_1\|^2_{L^2} + \|X^\varepsilon_2\|^2_{L^2} \right)
\]

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and
\[
\langle \varepsilon_1 \Delta X_i^\varepsilon, \varepsilon_2 \Delta X_j^\varepsilon, I'(X_i^\varepsilon - X_j^\varepsilon) \rangle_{H^{-1}} = -\int_\mathcal{O} \langle \varepsilon_1 X_i^\varepsilon - \varepsilon_2 X_j^\varepsilon, (X_i^\varepsilon - X_j^\varepsilon) \rangle dx \\
\leq C(\varepsilon_1 + \varepsilon_2) \left( \|X_i^\varepsilon\|^2_{L^2} + \|X_j^\varepsilon\|^2_{L^2} \right),
\]
df \otimes d\mathbb{P}\text{-almost everywhere. Using this and the Lipschitz property } (2.2) \text{ of } B \text{ we obtain}
\[
e^{-Kt} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} = \|I'(x_0^\varepsilon - x_0^2)\|^2_{H^{-1}} \\
+ C(\varepsilon_1 + \varepsilon_2) \left( 1 + \|X_i^\varepsilon\|^2_{L^2} + \|X_j^\varepsilon\|^2_{L^2} \right) \\
+ 2 \int_0^t e^{-Kr} (X_i^\varepsilon - X_j^\varepsilon, B(r, X_i^\varepsilon) - B(r, X_j^\varepsilon)) dW_r dy \\
+ C \int_0^t e^{-Kr} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} dr \\
- K \int_0^t e^{-Kr} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} dr.
\]
We can use (2.2), Lemma 4.2 and, as in (4.4), the Burkholder-Davis-Gundy inequality to obtain
\[
\mathbb{E} \sup_{t \in [0, T]} \left( e^{-Kt} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} \right) \leq 2\mathbb{E} \left\| I'(x_0^\varepsilon - x_0^2) \right\|^2_{H^{-1}} \\
+ C(\varepsilon_1 + \varepsilon_2) \left( \mathbb{E} \|x_0^\varepsilon\|^2_{L^2} + \mathbb{E} \|x_0^2\|^2_{L^2} + 1 \right) \tag{4.7}
\]
for K large enough, where we use the assumption that \( x_0^\varepsilon, x_0^2 \in L^2 \). If we assume that \( x_0^\varepsilon = x_0^2 = x_0 \), (4.7) implies
\[
\mathbb{E} \sup_{t \in [0, T]} \left( e^{-Kt} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} \right) \leq C(\varepsilon_1 + \varepsilon_2) (\mathbb{E} \|x_0\|^2_{L^2} + 1),
\]
and thus, by completeness there exists a process \( X \in L^2(\Omega; C([0, T]; H^{-1})) \) satisfying
\[
\begin{align*}
\mathbb{E} \sup_{t \in [0, T]} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} &\to 0 & \text{for } \varepsilon \to 0 \\
X_0 &= x_0.
\end{align*}
\]
In particular, we have for each \( t \in [0, T] \) that \( X_i^\varepsilon \to X_t \) for \( \varepsilon \to 0 \) in \( L^2(\Omega; H^{-1}) \). Since \( X_i^\varepsilon \) is \( \mathcal{F}_t \)-measurable by construction (see Theorem 3.2), so is \( X_t \), which makes \( X \) an adapted process. If the initial condition is indeed in \( L^2 \), this will be the candidate for an SVI solution.

It remains to construct a solution candidate if the initial state is not in \( L^2 \) but a general \( H^{-1} \) functional. To this end, we first notice that for two different initial conditions \( x_0^1, x_0^2 \in L^2(\Omega, \mathcal{F}_0; L^2) \), we can construct the limit of the approximate solutions in \( L^2(\Omega; C([0, T]; H^{-1})) \) as before, call them \( X^1 \) and \( X^2 \), respectively, and take the limit \( \varepsilon_1, \varepsilon_2 \to 0 \) in (4.7) to obtain
\[
\mathbb{E} \sup_{t \in [0, T]} \left( e^{-Kt} \|X_i^\varepsilon - X_j^\varepsilon\|^2_{H^{-1}} \right) \leq 2\mathbb{E} \left\| I'(x_0^\varepsilon - x_0^2) \right\|^2_{H^{-1}} \tag{4.8}
\]
Let now \( x_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1}) \) and select a sequence \( (x_0^n)_{n \in \mathbb{N}} \subset L^2(\Omega, \mathcal{F}_0; L^2) \) such that \( x_0^n \to x_0 \) in \( L^2(\Omega; H^{-1}) \) for \( n \to \infty \). Let \( (X^{\varepsilon, n})_{\varepsilon > 0, n \in \mathbb{N}} \) be the unique variational solutions to the respective initial conditions \( (x_0^n)_{n \in \mathbb{N}} \), for which Lemma 4.2 applies. We first construct the sequence \( (X^{\varepsilon, n})_{n \in \mathbb{N}} \) as the unique limits in \( L^2(\Omega; C([0, T]; H^{-1})) \) obtained as in the argument above, and notice that it is a Cauchy sequence by (4.8). Thus, we obtain another limit \( X \in L^2(\Omega; C([0, T]; H^{-1})) \) which we identify as a solution to (1.1) in the sense of Definition 2.5 in the following step.

Step 2: We show that the limit process satisfies the properties of Definition 2.5. Let \( \varepsilon > 0 \), \( x_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1}) \) and \( (x_0^n)_{n \in \mathbb{N}} \subset L^2(\Omega, \mathcal{F}_0; L^2) \) such that \( x_0^n \to x \in L^2(\Omega; H^{-1}) \) for \( n \to \infty \). Let \( (X^{\varepsilon, n})_{\varepsilon > 0, n \in \mathbb{N}} \) be the solutions to (1.1) with initial values \( x_0^n \). For part (i) of Definition 2.5 we apply
Ito’s formula as in (4.6) to obtain
\[
e^{-Kt} \|X^\varepsilon,n_t\|_{H^{-1}}^2 = \|I'(x_0)\|_{H^{-1}}^2 + 2 \int_0^t e^{-Kr} \langle \varepsilon \Delta X^\varepsilon,n, I'(X^\varepsilon,n) \rangle_{H^{-1}} \, dr
+ 2 \int_0^t e^{-Kr} \langle \Delta \phi^\varepsilon(X^\varepsilon,n), I'(X^\varepsilon,n) \rangle_{H^{-1}} \, dr
+ 2 \int_0^t e^{-Kr} \langle X^\varepsilon,n, B(r, X^\varepsilon,n) \rangle_{H^{-1}} \, dr
+ \int_0^t e^{-Kr} \|B(r, X^\varepsilon,n)\|_{L_2(U,H^{-1})}^2 \, dr
- K \int_0^t e^{-Kr} \|X^\varepsilon,n_t\|_{H^{-1}}^2 \, dr.
\]
(4.9)

Note that we have
\[
\langle \varepsilon \Delta X^\varepsilon,n, I'(X^\varepsilon,n) \rangle_{H^{-1}} = -\varepsilon \|X^\varepsilon,n\|_{L_2} \leq 0.
\]

With the notation of Appendix D and setting
\[
\varphi^\varepsilon(v) = \begin{cases} 
\int_\mathcal{O} \psi^\varepsilon(v) dx, & v \in L^2, \\
+\infty, & \text{otherwise},
\end{cases}
\]
(4.10)

for \(v \in H^{-1}, m \in [0,1]\), we can use \(\phi^\varepsilon = \partial \psi^\varepsilon\) and the fact that \(\phi^\varepsilon(X^\varepsilon,n) \in H_0^1 dt \otimes \mathbb{P}\)-almost everywhere by Lemma 1.2 Lemma 1.2 and the chain rule for Sobolev functions (see e.g. [59 Theorem 2.1.11]), to obtain
\[
\langle \Delta \phi^\varepsilon(X^\varepsilon,n), I'(X^\varepsilon,n) \rangle_{H^{-1}} = \langle -\Delta \phi^\varepsilon(X^\varepsilon,n), 0 - I'(X^\varepsilon,n) \rangle_{H^{-1}}
\leq \varphi^\varepsilon(0) - \varphi^\varepsilon(X^\varepsilon,n) = -\varphi^\varepsilon(X^\varepsilon,n).
\]
(4.11)

Furthermore, we can use (2.2) and (2.4) to obtain
\[
\|B(t, X^\varepsilon,n)\|_{L_2(U,H^{-1})}^2 \leq 2 \left( \|B(t, X^\varepsilon,n) - B(t,0)\|_{L_2(U,H^{-1})}^2 + \|B(t,0)\|_{L_2(U,H^{-1})}^2 \right)
\leq C(1 + \|X^\varepsilon,n\|_{H^{-1}}^2).
\]

Thus, (4.9) implies
\[
\mathbb{E}\left(e^{-Kt} \|X^\varepsilon,n_t\|_{H^{-1}}^2\right) \leq \mathbb{E}\left(x_0^\varepsilon\right)_{H^{-1}}^2 - 2\mathbb{E} \int_0^t e^{-Kr} \varphi^\varepsilon(X^\varepsilon,n) \, dr
+ (C - K)\mathbb{E} \int_0^t e^{-Kr} \|X^\varepsilon,n_t\|_{H^{-1}}^2 \, dr + \int_0^t C e^{-Kr} \, ds.
\]

Choosing \(K\) large enough, we get
\[
\mathbb{E}\left(e^{-Kt} \|X^\varepsilon,n_t\|_{H^{-1}}^2\right) \leq \mathbb{E}\left(x_0^\varepsilon\right)_{H^{-1}}^2 + 2e^{-Kt}\mathbb{E} \int_0^t \varphi^\varepsilon(X^\varepsilon,n) \, dr
\]
(4.12)

and thus
\[
\mathbb{E} \int_0^t \varphi^\varepsilon(X^\varepsilon,n) \, dr \leq C + \mathbb{E}\left(x_0^\varepsilon\right)_{H^{-1}}^2 \leq \tilde{C} < \infty,
\]
(4.13)

for some \(C, \tilde{C} > 0\). Note that \(\tilde{C}\) can be chosen independent of \(\varepsilon\) and \(n\) by the convergence of \((x_0^n)_{n \in \mathbb{N}}\) to \(x_0\). By Assumption 2.1 (A4) we can use Corollary D.7 to obtain for \(v \in L^2\)
\[
|\varphi^\varepsilon(v) - \varphi(v)| \leq \int_\mathcal{O} |\psi^\varepsilon(v) - \psi(v)| \, dx
\leq \int_\mathcal{O} C\varepsilon(1 + v^2) \, dx
= C\varepsilon(1 + \|v\|_{L^2}^2).
\]
(4.14)
Since $X^{ε,n} ∈ L^2 dt ⊗ P$-almost everywhere by Lemma 4.2, this leads to

\[
E \int_0^t \varphi'(X^{ε,n}_s) \, ds \geq E \int_0^t \varphi(\phi_{ε}^{k,n}(s)) \, ds - CεE \int_0^t 1 + \|X^{ε,n}_s\|^2_{L^2} \, ds. \tag{4.15}
\]

With these statements about fixed values of $ε$, we can now consider the limit $ε \to 0$. Taking into account that $X^{ε,n} \to X^n$ in $L^2(Ω; C([0,t]; H^{-1}))$ and thus in $L^2(Ω × [0,t]; H^{-1})$, we have a sequence $(ε_k)_{k\in\mathbb{N}} \subset [0,∞)$ with $ε_k \to 0$ for $k \to ∞$, such that $X^{ε_k,n} \to X^n$ pointwise $dt ⊗ P$-almost everywhere. We can thus use the lower-semicontinuity of $φ$, the Fatou lemma and (4.15) to obtain

\[
E \int_0^t \varphi(X^{n}_s) \, ds \leq \liminf_{k \to ∞} \int_0^t \varphi(X^{ε_k,n}_s) \, ds
\]

\[
\leq \liminf_{k \to ∞} \int_0^t \varphi(\phi_{ε}^{k,n}(s)) \, ds + Cε\int_0^t 1 + \|X^{ε,n}_s\|^2_{L^2} \, ds.
\]

As, by Lemma 4.2, the last term converges to 0 for $k \to ∞$ and $n \in \mathbb{N}$ fixed, we can deduce that

\[
E \int_0^t \varphi(X^{n}_s) \, ds \leq \liminf_{k \to ∞} \int_0^t \varphi(\phi_{ε}^{k,n}(s)) \, ds. \tag{4.16}
\]

Thus, taking $\liminf_{k \to ∞}$ in (4.13) for the subsequence $(ε_k)_k$ from (4.16) and then $\liminf_{n \to ∞}$, using lower-semicontinuity of $φ$ as in the first two steps from (4.16), we obtain

\[
E \int_0^t \varphi(X_s) \, ds < ∞,
\]

as required.

For the variational inequality part, let $G, Z, t$ be as in Definition 2.5(iii) Ito’s formula (e.g. [53] Theorem 4.2.5)) then implies

\[
E \|X^{ε,n}_t - Z_t\|^2_{H^{-1}} = E \|x_0^n - Z_0\|^2_{H^{-1}} + 2E \int_0^t \langle εΔX^{ε,n}_s + Δφ(X^{ε,n}_s) - G_r, X^{ε,n}_s - Z_r \rangle_{H^{-1}} \, ds + E \int_0^t \|B(r, X^{ε,n}_s) - B(r, Z_r)\|^2_{L^2(U, H^{-1})} \, ds.
\]

Analogous to (4.11), we have

\[
\langle Δφ(X^{ε,n}_s), X^{ε,n}_s - Z_r \rangle_{H^{-1}} + \varphi(\phi_{ε}^{n}(s)) \leq \varphi(Z_r) \tag{4.18}
\]

dt ⊗ $P$-almost everywhere, where we recall that both $X^{ε,n}$ and $Z$ are in $L^2 dt ⊗ P$-almost everywhere. Moreover, using the weighted Young inequality,

\[
\langle εΔX^{ε,n}_r, X^{ε,n}_r - Z_r \rangle_{H^{-1}} \leq ε\|ΔX^{ε,n}_r\|_{H^{-1}} \|X^{ε,n}_r - Z_r\|_{H^{-1}} \leq \frac{1}{2}ε^{\frac{2}{3}} \|X^{ε,n}_r\|^2_{H^{-1}} + \frac{1}{2}ε^{\frac{2}{3}} \|X^{ε,n}_r - Z_r\|^2_{H^{-1}}. \tag{4.19}
\]

dt ⊗ $P$-almost everywhere. Hence, by (2.25), (4.18) and (4.19),

\[
E \|X^{ε,n}_t - Z_t\|^2_{H^{-1}} + 2E \int_0^t \varphi(\phi_{ε}^{n}(s)) \, ds
\]

\[
\leq E \|x_0^n - Z_0\|^2_{H^{-1}} + 2E \int_0^t \varphi(Z_r) \, ds
\]

\[
- 2E \int_0^t \langle G_r, X^{ε,n}_r - Z_r \rangle_{H^{-1}} \, ds + Cε \int_0^t \|X^{ε,n}_r - Z_r\|^2_{H^{-1}} \, ds
\]

\[
+ 2E \int_0^t \frac{1}{2}ε^{\frac{2}{3}} \|ΔX^{ε,n}_r\|^2_{H^{-1}} + \frac{1}{2}ε^{\frac{2}{3}} \|X^{ε,n}_r - Z_r\|^2_{H^{-1}} \, ds. \tag{4.20}
\]
As for (4.16), there is a subsequence $\varepsilon_k$ such that

$$
\mathbb{E} \int_0^t \varphi(X^n_{\varepsilon_k}) \, dt \leq \liminf_{k \to \infty} \mathbb{E} \int_0^t \varphi^n(X^n_{\varepsilon_k}) \, dt. \tag{4.21}
$$

We notice that by $Z \in L^2 dt \otimes \mathbb{P}$-almost everywhere, we have $\varphi'(Z_r) \leq \varphi(Z_r)$ due to Corollary [D.5]. Moreover, any other term in (4.20) converges because $X^{\varepsilon,n} \to X^n$ in $L^2(\Omega; C([0,T]; H^{-1}))$, the requirement of $G$ belonging to $L^2(\Omega \times [0,T]; H^{-1})$ and Lemma [4.2]. Thus, we can take $\liminf_{k \to \infty}$ in (4.20) for the subsequence $(\varepsilon_k)_k$ to obtain

$$
\mathbb{E} \int_0^t \varphi(X^n_{\varepsilon_k}) \, dt \leq \frac{1}{2} \mathbb{E} \|X^n_{\varepsilon_k} - Z_t\|_{H^{-1}}^2 + \frac{1}{2} \mathbb{E} \|x_0^n - Z_0\|_{H^{-1}}^2 + \mathbb{E} \int_0^t \varphi(Z_r) \, dr

- \mathbb{E} \int_0^t \langle G_r, X^n_{\varepsilon_k} - Z_t \rangle_{H^{-1}} \, dr + \frac{1}{2} CE \int_0^t \|X^n_{\varepsilon_k} - Z_t\|_{H^{-1}}^2 \, dr.
$$

Now taking $\liminf_{n \to \infty}$, using the lower-semicontinuity of $\varphi$ and convergence of all the other terms, yields (2.13), as required.

**Step 3:** It remains to show that the solution constructed in the previous step is unique. To this end, let $x_0, y_0 \in L^2(\Omega, F_0; H^{-1})$, $(y^n_0)_{n \in \mathbb{N}} \subset L^2(\Omega, F_0; L^2)$ satisfying $y^n_0 \to y_0$ in $L^2(\Omega; H^{-1})$ for $n \to \infty$. Let $X$ be an arbitrary SVI solution to (1.1) with initial condition $x_0$ and let $(Y^{\varepsilon,n})_{\varepsilon > 0, n \in \mathbb{N}}$ be the solutions to (4.1) with respective initial conditions $(y^n_0)_{n \in \mathbb{N}}$. We first check that

$$
Z = Y^{\varepsilon,n} \quad \text{and} \quad G = \varepsilon \Delta Y^{\varepsilon,n} + \Delta \phi'(Y^{\varepsilon,n}) \tag{4.22}
$$

are admissible choices for (2.13). First,

$$
Y^{\varepsilon,n} \in L^2(\Omega; C([0,T]; H^{-1}))
$$

by construction and

$$
Y^{\varepsilon,n} \in L^2(\Omega \times [0,T]; L^2)
$$

by Lemma [4.2] with norm bounded uniformly in $\varepsilon$. Also by Lemma [4.2] we have

$$
\mathbb{E} \int_0^T \|\varepsilon \Delta Y^{\varepsilon,n} \|_{H^{-1}}^2 \, dt = \varepsilon^2 \mathbb{E} \int_0^T \|Y^{\varepsilon,n} \|_{H^1}^2 \, dt < \infty.
$$

Finally, for the nonlinear term, we have by the chain rule for the composition of Lipschitz functions with $H^1_0$ functions (e.g. [29] Theorem 2.1.11) that almost everywhere in $\mathcal{O}$

$$
\nabla \phi'(Y^{\varepsilon,n}) = (\phi'(Y^{\varepsilon,n})) \nabla Y^{\varepsilon,n},
$$

such that we can compute using Lemma [D.2]

$$
\int_{\mathcal{O}} |\nabla \phi'(Y^{\varepsilon,n})|^2 \, dx = \int_{\mathcal{O}} |(\phi'(Y^{\varepsilon,n})) \nabla Y^{\varepsilon,n}|^2 \, dx \leq \frac{1}{\varepsilon^2} \|\nabla Y^{\varepsilon,n}\|_{L^2}^2
dt \otimes \mathbb{P}$\text{-}almost \text{everywhere}.\] Consequently,

$$
\|\phi'(Y^{\varepsilon,n})\|_{H^1_0}^2 \leq C(\varepsilon) \|Y^{\varepsilon,n}\|_{H^1_0}^2,
$$

such that we can conclude by Lemma [4.2]

$$
\mathbb{E} \int_0^T \|\Delta \phi'(Y^{\varepsilon,n})\|_{H^{-1}}^2 \, dt = \mathbb{E} \int_0^T \|\phi'(Y^{\varepsilon,n})\|_{H^1_0}^2 \, dt

\leq C(\varepsilon) \mathbb{E} \int_0^T \|Y^{\varepsilon,n}\|_{H^1_0}^2 \, dt

\leq \tilde{C}(\varepsilon) < \infty,
$$

which yields that the choices in (4.22) were admissible.
As a consequence, (2.13) yields for \( t \in [0, T] \)
\[
\mathbb{E} \| X_t - Y_t^{\varepsilon, n} \|_{H^{-1}}^2 + 2 \mathbb{E} \int_0^t \varphi(X_s) \, ds \\
\leq \mathbb{E} \| x_0 - y_0 \|_{H^{-1}}^2 + 2 \mathbb{E} \int_0^t \varphi(Y_s^{\varepsilon, n}) \, ds \\
- 2 \mathbb{E} \int_0^t \langle \varepsilon \Delta Y_s^{\varepsilon, n} + \phi(X_s) \rangle \, ds \\
+ C \mathbb{E} \int_0^t \| X_s - Y_s^{\varepsilon, n} \|_{H^{-1}}^2 \, ds.
\] (4.23)

For \( u \in L^2 \) and \( \varphi \) as in (4.10), we have, as in (4.11) and (4.18),
\[
\langle -\Delta \phi(Y_s^{\varepsilon, n}), u - Y_s^{\varepsilon, n} \rangle_{H^{-1}} + \varphi(Y_s^{\varepsilon, n}) \leq \varphi(\varepsilon(u)) \, dt \otimes \mathbb{P}-a. e.
\] (4.24)

Since \( Y_s^{\varepsilon, n} \in H_0^1 \subset L^2 \) \( dt \otimes \mathbb{P}\text{-a. e.} \) we can use Corollary [2.7] as in (4.14) to obtain \( dt \otimes \mathbb{P}\text{-almost everywhere} \)
\[
|\varphi(Y_s^{\varepsilon, n}) - \varphi(Y_t^{\varepsilon, n})| \leq C \varepsilon \left( 1 + \| Y_s^{\varepsilon, n} \|_{L^2}^2 \right).
\]

Thus, we can modify (4.24) and get
\[
\langle -\Delta \phi(Y_s^{\varepsilon, n}), u - Y_s^{\varepsilon, n} \rangle_{H^{-1}} + \varphi(Y_s^{\varepsilon, n}) \leq \varphi(u) + C \varepsilon \left( 1 + \| Y_s^{\varepsilon, n} \|_{L^2}^2 \right) \, dt \otimes \mathbb{P}\text{-a.e.}
\] (4.25)

Note that (4.25) is trivial if \( \varphi(u) = \infty \). Furthermore, (4.25) can be deduced analogously for \( u \in L^m(L^{m+1} \cap H^{-1}) \) in the superlinear setting, i.e. when \( \varphi \) is given by (2.3), with \( m \) as in Assumption 2.1 (A5) In the sublinear setting, i.e. \( \varphi \) is given by (2.9), and \( u \in \mathcal{M}(O) \cap H^{-1} \), we consider an approximating sequence \((\mu_j)_{j \in \mathbb{N}} \subset \mathcal{M} \cap H^{-1}\) with densities \((u_j)_{j \in \mathbb{N}} \subset L^2 \) given by Theorem 3.8 such that (4.25) is satisfied for all \( u_j, j \in \mathbb{N} \). We then pass to the limit \( j \to \infty \) and notice that \((\mu_j)_{j \in \mathbb{N}} \) has been constructed in such a way that both \( \varphi(u_j) \to \varphi(u) \) and
\[
\langle -\Delta \phi(Y_s^{\varepsilon, n}), u_j - Y_s^{\varepsilon, n} \rangle_{H^{-1}} \\
= H_0^1(\phi(\varepsilon(Y_s^{\varepsilon, n})), u_j - Y_s^{\varepsilon, n})_{H^{-1}} \\
\to H_0^1(\phi(\varepsilon(Y_s^{\varepsilon, n})), u - Y_s^{\varepsilon, n})_{H^{-1}} \\
= \langle -\Delta \phi(Y_s^{\varepsilon, n}), u - Y_s^{\varepsilon, n} \rangle_{H^{-1}}.
\]

Consequently, replacing \( u \) by \( X \) in (4.25), we have in any case
\[
\langle -\Delta \phi(Y_s^{\varepsilon, n}), X - Y_s^{\varepsilon, n} \rangle_{H^{-1}} + \varphi(Y_s^{\varepsilon, n}) \leq \varphi(X) + C \varepsilon \left( 1 + \| Y_s^{\varepsilon, n} \|_{L^2}^2 \right) \, dt \otimes \mathbb{P}-a.e.
\] (4.26)

Using (4.26) and the same estimate as in (4.19), we can modify (4.28) to obtain for \( t \in [0, T] \)
\[
\mathbb{E} \| X_t - Y_t^{\varepsilon, n} \|_{H^{-1}}^2 \leq \mathbb{E} \| x_0 - y_0 \|_{H^{-1}}^2 \\
+ 2 \mathbb{E} \int_0^t \varepsilon \frac{1}{2} \| \Delta Y_s^{\varepsilon, n} \|_{H^{-1}}^2 \, ds + \varepsilon \frac{1}{2} \| X_t - Y_t^{\varepsilon, n} \|_{H^{-1}}^2 \\
+ C \mathbb{E} \int_0^t \| X_t - Y_t^{\varepsilon, n} \|_{H^{-1}}^2 \, ds + C \varepsilon \mathbb{E} \left( 1 + \| Y_t^{\varepsilon, n} \|_{L^2}^2 \right) \, dt.
\]

Taking \( \varepsilon \to 0 \) and then \( n \to \infty \) yields
\[
\mathbb{E} \| X_t - Y_t \|_{H^{-1}}^2 \leq \mathbb{E} \| x_0 - y_0 \|_{H^{-1}}^2 \\
+ C \mathbb{E} \int_0^t \| X_t - Y_t \|_{H^{-1}}^2 \, ds \quad \text{for} \quad t \in [0, T],
\] (4.27)

where \( Y \) is the SVI solution which has been constructed from \((Y^{\varepsilon, n})\) in the limiting procedure of the first two steps of this proof. Gronwall’s inequality then yields \( X = Y \) if \( x_0 = y_0 \), and thus uniqueness of SVI solutions. Then, estimate (2.1) follows by applying Gronwall’s inequality to (4.27) with different initial values, which concludes the proof.
A Generalities on convex functions

We collect and prove some well-known or easy facts on convex functions defined on \( \mathbb{R} \).

**Lemma A.1.** Let \( f : \mathbb{R} \to [0, \infty) \) be convex with \( f(0) = 0 \) and \( x, y \in \mathbb{R} \setminus \{0\} \) with \( x < y \). Then

\[
\frac{f(x)}{x} \leq \frac{f(y)}{y}. \tag{A.1}
\]

In particular, for \( x > 0 \) this implies \( f(x) \leq f(y) \).

**Proof.** Note that by convexity, we have for \( \lambda \in (0, 1), x \in \mathbb{R} \)

\[
f(\lambda x) = f(\lambda x + (1 - \lambda)0) \leq \lambda f(x) + (1 - \lambda) f(0) = \lambda f(x). \tag{A.2}
\]

If \( x < 0 < y \), the statement is obvious by the nonnegativity of \( f \). If \( 0 < x < y \), we use (A.2) with \( \lambda := \frac{x}{y} \) to get

\[
\frac{f(x)}{x} \leq \frac{f(y)}{y},
\]

while for \( x < y < 0 \) we use (A.2) with \( \lambda := \frac{y}{x} \) to get

\[
\frac{f(y)}{y} \leq \frac{f(x)}{x},
\]

as required.

**Lemma A.2.** Let \( \psi \) satisfy Assumptions 2.1 and \( y > 0 \). Then, if \( \psi(y) > 0 \), we have

\[
\psi^*(-x) = \psi^*(x) \leq \psi(y) \quad \text{for} \ x \in \left[0, \frac{\psi(y)}{y}\right].
\]

**Proof.** By Remark 3.2, the last part of Lemma A.1 and the nonnegativity of \( \psi^* \), it is enough to show

\[
\psi^*(\psi(y) y) \leq \psi(y). \tag{A.3}
\]

To verify (A.3), we use the definition of \( \psi^* \) in (3.1) and distinguish three cases for \( y \in \mathbb{R} \). For \( y' \geq y \) we have by Lemma A.1

\[
\frac{\psi(y)}{y} y' - \psi(y') = y' \left( \frac{\psi(y)}{y} - \frac{\psi(y')}{y'} \right) \leq 0,
\]

for \( y' \leq 0 \) we have by the nonnegativity of \( \psi \)

\[
\frac{\psi(y)}{y} y' - \psi(y') \leq 0,
\]

and for \( y' \in (0, y) \) we have

\[
\frac{\psi(y)}{y} y' - \psi(y') \leq \frac{\psi(y)}{y} y = \psi(y),
\]

which yields the claim.

**Lemma A.3.** Let \( \psi \) satisfy Assumptions 2.1. For \( K = \text{dom}(\psi^*) := \{ x \in \mathbb{R} : \psi^*(x) < \infty \} \) we have

\[
\sup K = \lim_{t \to \infty} \frac{\psi(t)}{t} \quad \text{and} \quad \sup(-K) = \lim_{t \to \infty} \frac{\psi(-t)}{t}.
\]

**Proof.** We only prove the first statement, the second then becomes clear by symmetry. To this end, note first that the limit is actually a supremum, as \( \frac{\psi(t)}{t} \) is increasing (by (A.1)). Let now \( x \in K \), which means that \( xt - \psi(t) \leq c_x < \infty \) and thus \( \frac{\psi(t)}{t} \geq x - \frac{c_x}{t} \) for all \( t \in [0, \infty) \), which yields “\( \leq \)” by letting \( t \to \infty \).

Conversely, we have \( \frac{\psi(t)}{t} \in K \) for \( t > 0, \psi(t) > 0 \) by Lemma A.2. As \( \psi^*(0) = 0 \), this is true also if \( \psi(t) = 0 \), thereby proving “\( \geq \)”. \( \square \)
Corollary A.4. Let \( \psi \) satisfy Assumptions [27]. By Lemmas A.4 and A.5, we have that
\[
\psi_\infty(1) = \psi_\infty(-1) \geq \frac{\psi(y)}{y}
\]
for \( y > 0 \) with \( \psi(y) > 0 \).

Lemma A.5. Let \( \psi \) satisfy Assumptions [27]. For the convex conjugate of the recession function, we have
\[
\psi^\ast(x) := (\psi_\infty)^\ast(x) = \chi([\psi_\infty(1), \psi_\infty(1)])(x)
\]
for \( x \in \mathbb{R} \), where for an Interval \( I \) we have written
\[
\chi_I(x) = \begin{cases} 0, & \text{if } x \in I \\ +\infty, & \text{else.} \end{cases}
\]

Proof. In the superlinear case, i.e., (2.6) is satisfied, we have \( \psi_\infty = \chi(0) \) and thus \( \psi^\ast \equiv 0 \), as required. In the sublinear case, we first note that \( \psi_\infty \) is, by definition, positively homogeneous, which by symmetry amounts to absolute homogeneity. Thus
\[
\psi_\infty(x) = \psi_\infty(1) |x|,
\]
where \( \psi_\infty(1) > 0 \) by Corollary A.4, which allows to conclude by the definition of the convex conjugate. \( \square \)

B Variational solutions to nonlinear SPDE

Let \((\Omega, \mathcal{F}, \mathbb{P})\) a complete probability space, \(V \subset H \subset V'\) a Gelfand triple, \((W_t)_{t \in [0,T]}\) a cylindrical Itô-Wiener process taking values in another separable Hilbert space \((U, \langle \cdot, \cdot \rangle_U)\) with normal filtration \((\mathcal{F}_t)_{t \in [0,T]}\). Let\n\[
A : [0,T] \times V \times \Omega \to V', \ B : [0,T] \times V \times \Omega \to L_2(U,H),
\]
be progressively measurable and satisfy the following conditions:

(H1) (Hemicontinuity) For all \( u, v, x, \omega \in \Omega \) and \( t \in [0, T] \), the map
\[
\mathbb{R} \ni \lambda \mapsto \langle \lambda_A(t, u + \lambda v, \omega), x \rangle_V
\]
is continuous.

(H2) (Weak monotonicity) There exists \( c \in \mathbb{R} \), such that for all \( u, v \in V \)
\[
2 \langle A(u, u) - A(u, v), u - v \rangle_V + \|B(u) - B(v)\|_{L_2(U,H)}^2 \leq c \|u - v\|_H^2
\]
on \([0, T] \times \Omega\).

(H3) (Coercivity) There exist \( \alpha \in (1, \infty), c_1 \in \mathbb{R}, c_2 \in (0, \infty) \) and an \((\mathcal{F}_t)\)-adapted process \( f \in L^1([0,T] \times \Omega, dt \otimes \mathbb{P}) \), such that for all \( v \in V, t \in [0, T] \)
\[
2 \langle A(t,v), v \rangle_V + \|B(t)\|_{L_2(U,H)}^2 \leq c_1 \|v\|_H^\alpha - c_2 \|v\|_V^\alpha + f(t) \quad \text{on } \Omega. \quad \text{(B.1)}
\]

(H4) (Boundedness) There exist \( c_3 \in [0, \infty) \) and an \((\mathcal{F}_t)\)-adapted process
\[
g \in L^{\infty}([0,T] \times \Omega, dt \otimes \mathbb{P}),
\]
such that for all \( v \in V, t \in [0, T] \)
\[
\|A(t,v)\|_V \leq g(t) + c_3 \|v\|_V^{\alpha-1}
\]
on \( \Omega \), where \( \alpha \) is as in (H3).
We then consider the stochastic partial differential equation
\[
dX_t = A(t, X_t) dt + B(t, X_t) dW_t,
\]
for which we establish the following notion of solution:

**Definition B.1.** A continuous \( H \)-valued \( (\mathcal{F}_t) \)-adapted process \((X_t)_{t \in [0, T]} \) is called a (variational) solution of (B.2), if for its \( dt \otimes \mathbb{P} \)-equivalence class \( \hat{X} \) we have
\[
\hat{X} \in L^\alpha([0, T] \times \Omega, dt \otimes \mathbb{P}; V) \cap L^2([0, T] \times \Omega, dt \otimes \mathbb{P}; H),
\]
with \( \alpha \) as in (B.1), and \( \mathbb{P} \)-a. s.
\[
X_t = X_0 + \int_0^t A(s, X_s) ds + \int_0^t B(s, X_s) dW_s, \quad t \in [0, T],
\]
where \( \hat{X} \) is any \( V \)-valued progressively measurable \( dt \otimes \mathbb{P} \)-version of \( X \).

We then have the following well-posedness result (see [53, Theorem 4.4], relying on [45]).

**Theorem B.2.** Let \( X_0 \in L^2(\Omega, \mathcal{F}_0; \mathbb{P}; H) \). Then there exists a unique solution \( X \) to (B.2) in the sense of Definition B.1.

**C Strong solutions to gradient-type SPDE**

Let \( \varphi : H \to \mathbb{R} \) be a proper, lower-semicontinuous, convex function on a separable real Hilbert space \( H \). We consider an SPDE of the type
\[
dX_t = -\partial \varphi(X_t) dt + B(t, X_t) dW_t,
\]
where \( W \) is a cylindrical Wiener process in a separable Hilbert space \( U \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with normal filtration \((\mathcal{F}_t)_{t \geq 0}\) and \( B : [0, T] \times H \times \Omega \to L_2(U, H) \) is Lipschitz continuous, i.e. for all \( v, w \in H \)
\[
\|B(t, v) - B(t, w)\|_{L_2(U, H)} \leq C \|v - w\|_H^2,
\]
and for all \((t, \omega) \in [0, T] \times \Omega\). Furthermore, we assume that
\[
\|B(\cdot, 0)\|_{L_2(U, H)} \in L^2([0, T] \times \Omega).
\]

**Definition C.1.** Let \( x_0 \in L^2(\Omega, \mathcal{F}_0; H) \). An \( H \)-continuous, \( \mathcal{F}_t \)-adapted process \( X \in L^2(\Omega; C([0, T]; H)) \) for which there exists a selection \( \eta \in -\partial \varphi(X) \), \( dt \otimes \mathbb{P} \)-a. e., is said to be a strong solution to (C.1) if
\[
\eta \in L^2([0, T] \times \Omega; H)
\]
and \( \mathbb{P} \)-a. s.
\[
X_t = x_0 + \int_0^t \eta_r \; dr + \int_0^t B(r, X_r) \; dW_r \quad \text{for all } t \in [0, T].
\]

**D Yosida-Approximation of multivalued operators**

The theory of Yosida approximations can be applied to general maximal monotone operators from Banach spaces to their dual, see e.g. [5 Section 2]. However, we constrain ourselves to the case of the Hilbert space \( \mathbb{R} \).

Fix \( \varepsilon > 0 \). For a convex, lower-semicontinuous proper function \( \psi : \mathbb{R} \to [0, \infty) \) we define its Moreau-Yosida approximation \( \psi^\varepsilon : \mathbb{R} \to [0, \infty) \) by
\[
\psi^\varepsilon(r) = \inf_{s \in \mathbb{R}} \left( \frac{|r - s|^2}{2\varepsilon} + \psi(s) \right),
\]
(D.1)
Let \( \phi = \partial \psi : \mathbb{R} \to 2^\mathbb{R} \) be the subdifferential of \( \psi \). For each \( r \in \mathbb{R} \), we define the resolvent \( J^\varepsilon(r) \) as the unique solution \( s \) to
\[
s + \varepsilon \phi(s) \ni r.
\]
Hereby the resolvent is well-defined, since \( \phi \) is maximal monotone as a subdifferential (see e. g. [5, Theorem 2.8]), which implies that \( \text{Id}_\mathbb{R} + \varepsilon \phi \) is bijective. We then define the Yosida approximation \( \phi^\varepsilon : \mathbb{R} \to \mathbb{R} \) of \( \phi \) by
\[
\phi^\varepsilon(r) = \frac{1}{\varepsilon} (r - J^\varepsilon r).
\]  
(D.2)

We state and prove some properties of this approximation, most of which are true for general subpotential operators. The usage of additional assumptions will be highlighted.

**Proposition D.1.** We have \( \phi^\varepsilon(r) \in \phi(J^\varepsilon r) \).  
Furthermore, \( \psi^\varepsilon \) is continuous, convex and Gateaux differentiable, and \( \phi^\varepsilon = (\psi^\varepsilon)' \). In particular, \( \phi^\varepsilon \) is also maximal monotone.

*Proof.* The first claim is clear by construction. The remaining statements are proved in [5, Theorem 2.9]. \( \square \)

**Lemma D.2.** The Yosida approximation \( \phi^\varepsilon \) is Lipschitz continuous with Lipschitz constant \( \frac{1}{\varepsilon} \).

*Proof.* Fix \( x, y \in \mathbb{R} \). By definition of \( J^\varepsilon \), we have
\[
J^\varepsilon x - J^\varepsilon y + \varepsilon (\phi^\varepsilon(x) - \phi^\varepsilon(y)) = x - y.
\]
By multiplying with \( \phi^\varepsilon(x) - \phi^\varepsilon(y) \) and keeping (D.3) in mind, we obtain
\[
\varepsilon(\phi^\varepsilon(x) - \phi^\varepsilon(y))^2 \leq |\phi^\varepsilon(x) - \phi^\varepsilon(y)| \cdot |x - y|,
\]
which immediately yields the claim. \( \square \)

**Lemma D.3.** Defining \( |\phi(r)| := \inf \{ |\eta| : \eta \in \phi(r) \} \), we have \( |\phi^\varepsilon(r)| \leq |\phi(r)| \) for all \( r \in \mathbb{R} \).

*Proof.* By monotonicity of \( \phi \), we get for \( \eta \in \phi(r) \)
\[
0 \leq (r - J^\varepsilon(r))(\eta - \phi^\varepsilon(r)).
\]
Noting that \( r - J^\varepsilon(r) = \varepsilon \phi^\varepsilon(r) \), we can simplify
\[
0 \leq \varepsilon |\phi^\varepsilon(r)| \cdot |\eta| - \varepsilon(\phi^\varepsilon(r))^2
\]
to obtain the estimate. \( \square \)

The next lemma is proved in [5, Theorem 2.9]:

**Lemma D.4.** For each \( r \in \mathbb{R} \), we have
\[
\psi^\varepsilon(r) = \frac{1}{2\varepsilon} |r - J^\varepsilon(r)|^2 + \psi(J^\varepsilon r),
\]
in other words, the infimum in (D.1) is assumed at \( J^\varepsilon r \).

As an immediate consequence, we get

**Corollary D.5.** For each \( r \in \mathbb{R} \), we have
\[
\psi(J^\varepsilon r) \leq \psi^\varepsilon(r) \leq \psi(r).
\]

*Proof.* The first inequality is clear by Lemma D.4, the second one by setting \( r = s \) in (D.1). \( \square \)
Lemma D.6. For each \( r \in \mathbb{R} \), we have
\[
|\psi(r) - \psi^\varepsilon(r)| \leq \varepsilon |\phi(r)|^2 \quad \text{for all } r \in \mathbb{R}.
\] (D.4)

Proof. Fix an arbitrary \( r \in \mathbb{R} \). For any \( \eta \in \phi(r) \) we have, using Corollary D.3 in the first step and the subdifferential inequality in the second step,
\[
0 \leq \psi(r) - \psi(J^e r) \leq -\eta(J^e r - r) \leq |\eta| \varepsilon |\phi(r)|.
\]
Since \( \eta \in \phi(r) \) was arbitrary, we can pass to its infimum. Using Lemma D.3 we obtain (D.4). \hfill \Box

Corollary D.7. With Lemma D.6 under the additional assumption \(|\phi(r)|^2 \leq C(1 + |r|^2)\), we obtain
\[
|\psi(r) - \psi^\varepsilon(r)| \leq C \varepsilon(1 + r^2) \quad \text{for all } r \in \mathbb{R}.
\]

Lemma D.8. We have for all \( a, b, \varepsilon, \varepsilon_1, \varepsilon_2 > 0 \)
\[
(\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b)) (a - b) \geq -C(\varepsilon_1 + \varepsilon_2) \left( |\phi^{\varepsilon_1}(a)|^2 + |\phi^{\varepsilon_2}(b)|^2 \right).
\]

Proof. We compute
\[
(\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b)) (a - b) = (\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))(J^{\varepsilon_1} a - J^{\varepsilon_2} b) \\
+ (\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))(a - J^{\varepsilon_2} b) \\
\geq (\varepsilon_1 |\phi^{\varepsilon_1}(a)| - \varepsilon_2 |\phi^{\varepsilon_2}(b)|) \\
\geq -\frac{1}{2}(\varepsilon_1 + \varepsilon_2) \left( |\phi^{\varepsilon_1}(a)|^2 + |\phi^{\varepsilon_2}(b)|^2 \right),
\]
where the second step uses (D.3) for the first summand to be positive and (D.2) for the second summand. In the last step, we neglect the squared terms and use Young’s inequality for the mixed terms. \hfill \Box

Corollary D.9. Under the additional assumption \(|\phi(r)|^2 \leq C(1 + |r|^2)\), Lemma D.3 immediately yields
\[
(\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b)) (a - b) \geq -C(\varepsilon_1 + \varepsilon_2) \left( 1 + |a|^2 + |b|^2 \right).
\]

E Estimate on special quadratic variations

Lemma E.1. Let \( U, H \) be Hilbert spaces, \( Q : U \to U \) linear, bounded, non-negative definite and symmetric, \( W \) a (possibly cylindrical) \( Q \)-Wiener process on \( U \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and normal filtration \((\mathcal{F}_t)_{t \geq 0}\). Further let \( B : \Omega \times [0, T] \to L_2(Q^{\frac{1}{2}}(U), H) \) such that \( B \) is predictable and
\[
\mathbb{P} \left( \int_0^T \|B(s)\|_{L_2(Q^{\frac{1}{2}}(U), H)} \, ds < \infty \right) = 1,
\]
and \( f \) an \((\mathcal{F}_t)\)-adapted continuous \( H \)-valued process. Then, the quadratic variation of a stochastic integral on \( H \) of the form
\[
M_t = \int_0^t \langle f_r, B_r \, dW_r \rangle_H
\]
can be estimated from above by
\[
\langle M \rangle_t \leq \int_0^t \langle f_r \rangle_H^2 \|B_r\|_{L_2(Q^{\frac{1}{2}}(U), H)}^2 \, dr.
\]

Proof. If \( Q \) is of finite trace and thus \( W \) is a classical Wiener process, the statement follows from [53, Lemma 2.4.2] and [53, Lemma 2.4.3]. In case of a cylindrical Wiener process, we can compute, using the
notation of Assumption 2.1 \([A1]\),

\[
\langle M \rangle_t = \left\langle \int_0^t \left\langle f_r, B_r \circ J^{-1} d\tilde{W}_r \right\rangle_H \right\rangle_t \\
\leq \int_0^t \|f_r\|_H^2 \|B_r \circ J^{-1}\|_{L_2(Q_1^2(U), H)}^2 \, dr \\
= \int_0^t \|f_r\|_H^2 \|B_r\|_{L_2(Q_1^2(U), H)}^2 \, dr,
\]

where in the second step, we use the Lemma for the classical \(Q_1\)-Wiener process \(\tilde{W}\) on \(U_1\). The last step can be seen by the fact that for an orthonormal basis \((e_k)_{k \in \mathbb{N}}\) of \(Q_1^2(U)\), we have that \((J e_k)_{k \in \mathbb{N}}\) is an orthonormal basis of \(L_2(Q_1^2(U), H)\); see \([53, \text{section 2.5.2}]\) for details.

\[\square\]

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