Testing isomorphism between tuples of subspaces

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Abstract

Given two tuples of subspaces, can you tell whether the tuples are isomorphic? We develop theory and algorithms to address this fundamental question. We focus on isomorphisms in which the ambient vector space is acted on by either a unitary group or general linear group. If isomorphism also allows permutations of the subspaces, then the problem is at least as hard as graph isomorphism. Otherwise, we provide a variety of polynomial-time algorithms with Matlab implementations to test for isomorphism. Keywords: subspace isomorphism, Grassmannian, Bargmann invariants, $H^*$-algebras, quivers, graph isomorphism MSC2020: 14M15, 05C60, 81R05, 16G20

1 Introduction

The last decade has seen a surge of work to find arrangements of points in real and complex Grassmannian spaces that are well spaced in some sense; for example, one may seek optimal codes, which maximize the minimum distance between points, or designs, which offer an integration rule. While precursor work in this vein appeared between the '50s and '80s by Rankin [56, 55], Grey [31], Seidel [58], Welch [70], and Levenshtein [50], the seminal paper by Conway, Hardin and Sloane [13] arrived later in 1996. The recent resurgence of interest in this problem has been largely stimulated by emerging applications in multiple description coding [63], digital fingerprinting [53], compressed sensing [3], and quantum state tomography [57].

There have been several fruitful approaches to studying arrangements of points in the Grassmannian. First, it is natural to consider highly symmetric arrangements of points. Such arrangements were extensively studied in [65, 11, 8, 68, 64, 10] in the context of designs, and later, symmetry was used to facilitate the search for optimal codes [37, 38, 39, 48, 7, 45, 40]. In many cases, the symmetries that underly optimal codes can be abstracted to weaker combinatorial structures that produce additional codes. For example, one may use strongly regular graphs to obtain optimal codes in $\text{Gr}(1, R^d)$ [69], or use Steiner systems to obtain optimal codes in $\text{Gr}(1, C^d)$ [28]. In this spirit, several infinite families of optimal codes have been constructed from combinatorial designs [42, 47, 27, 24, 23, 20, 29, 19]. In some cases, it is even possible to construct optimal codes from smaller codes [3, 66, 6, 46]. Researchers

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have also leveraged different computational techniques to find new arrangements \cite{11, 36, 41} and to study various properties of known arrangements \cite{12, 22, 21, 52, 51}. See \cite{4, 26, 67} for surveys of many of these results.

To date, the vast majority of this work has focused on the special case of projective spaces, and it is easy to explain this trend: it is harder to interact with points in general Grassmannian spaces. To illustrate this, suppose you are given two tuples \( A \) and \( B \) of \( r \)-dimensional subspaces of \( \mathbb{R}^d \). You are told that the subspaces in \( A \) were drawn independently and uniformly at random from the Grassmannian \( \text{Gr}(r, \mathbb{R}^d) \), and that \( B \) was drawn according to one of two processes: either there exists an orthogonal transformation \( g \in \text{O}(d) \) such that \( B = g \cdot A \), or \( B \) was also drawn independently and uniformly at random. How can you tell which process was used to construct \( B \)? In the special case where \( r = 1 \), the lines are almost surely not orthogonal, and one may leverage this feature to select vector representatives of the lines and then compute a canonical form of the Gramian (i.e., the reduced signature matrix discussed in \cite{67}) that detects whether there exists \( g \in \text{O}(d) \) such that \( B = g \cdot A \). However, if \( r > 1 \), it is not obvious how to find such an invariant. On the other hand, we benefit from the fact that many optimal configurations – like so-called equiangular tight frames – share certain properties – like not being pairwise orthogonal – with generic configurations, allowing results about generic configurations to be applied to certain optimal configurations.

This obstruction has had substantial ramifications on progress toward optimal codes in more general Grassmannian spaces. In particular, Sloane maintains an online catalog \cite{60} of putatively optimal codes in \( \text{Gr}(r, \mathbb{R}^d) \) for \( r \in \{1, 2, 3\} \) and \( d \in \{3, \ldots, 16\} \). Suppose one were to find a code for \( r \in \{2, 3\} \) that is competitive with Sloane’s corresponding putatively optimal code. Are these codes actually the same up to rotation? If researchers cannot easily answer this question, then they are less inclined to contribute to the hunt for optimal codes in these more general Grassmannian spaces. While there are several works in the literature that treat related problems \cite{61, 71, 54, 43, 59, 34}, the particular problem we identify has yet to be treated. The primary purpose of this paper is to help close this gap with both theory and code.

Notationally, we let \( \mathbb{F} \) denote an arbitrary field. Every \( \Gamma \leq \text{GL}(d, \mathbb{F}) \) has a natural action on \( \text{Gr}(r, \mathbb{F}^d) \). Given an involutive automorphism \( \sigma \) of \( \mathbb{F} \), we consider the Hermitian form defined by \( \langle x, y \rangle = \sum_i \sigma(x_i)y_i \), and we let \( \text{U}(d, \mathbb{F}, \sigma) \) denote the subgroup of all \( g \in \text{GL}(d, \mathbb{F}) \) such that \( \langle gx, gy \rangle = \langle x, y \rangle \) for all \( x, y \in \mathbb{F}^d \). For example, \( \text{U}(d, \mathbb{F}, \sigma) \) contains all \( d \times d \) permutation matrices. Over any field, the identity is an involutive automorphism and over quadratic extensions one may choose the only nontrivial field automorphism as the involution (see, e.g., \cite{32, 33}). Specifically, we also adopt the standard notations for the orthogonal group \( \text{O}(d) = \text{U}(d, \mathbb{R}, \text{id}) \) and the unitary group \( \text{U}(d) = \text{U}(d, \mathbb{C}, \overline{\text{id}}) \). For \( \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\} \), we say that generic points in \( \text{Gr}(r, \mathbb{F}^d) \) satisfy property \( P \) if there exists an open and dense subset \( S \subseteq \mathbb{F}^{d \times r} \) such that for every \( A \in S \), it holds that \( V := \text{im} A \in \text{Gr}(r, \mathbb{F}^d) \) and \( V \) has property \( P \).

Our problem can be viewed as an instance of a more general, fundamental problem:

**Problem 1** (Common orbit). Given a \( G \)-set \( X \) and two points \( x, y \in X \), determine whether there exists \( g \in G \) such that \( g \cdot x = y \).

One attractive approach to solving the common orbit problem is to construct an invariant, that is, a function \( f \colon X \to S \) for some set \( S \) such that \( f(x) = f(y) \) only if there
exists $g \in G$ such that $g \cdot x = y$. In particular, $f(x)$ is determined by the orbit $G \cdot x$. If $f$ always returns different values for different orbits, then we say $f$ is an injective invariant. Observe that an injective invariant provides a complete solution to common orbit, since one may simply compare $f(x)$ with $f(y)$.

We will study four types of common orbit problems with $X = (\text{Gr}(r, F^d))^n$. In particular, for each $\Gamma \in \{U(d, F, \sigma), \text{GL}(d, F)\}$, we consider the following actions on $X$:

$$G = \Gamma \times S_n, \quad (g, \pi) \cdot (x_i)_{i \in [n]} = (g \cdot x_{\pi^{-1}(i)})_{i \in [n]}; \quad G = \Gamma, \quad g \cdot (x_i)_{i \in [n]} = (g \cdot x_i)_{i \in [n]}.$$

Here, $S_n$ denotes the symmetric group on $n$ letters. In the following section, we show that any solution to common orbit in the case of $G = \Gamma \times S_n$ can be used to solve graph isomorphism, thereby suggesting that this case is computationally hard. Next, Section 3 treats the case $G = \Gamma \in \{O(d), U(d)\}$. First, we show how to obtain a canonical choice of Gramian for generic real planes (i.e., points in $\text{Gr}(2, R^d)$), before finding injective invariants using ideas from the representation theory of $H^*$-algebras. Finally, Section 4 treats the case $G = \Gamma = \text{GL}(d, F)$, where the common orbit problem frequently reduces to solving a linear system. In the special case where $F = C$, we provide conditions under which generic subspaces allow for such a simple solution. Matlab implementations of Algorithm 2 and Lemma 11 may be downloaded from [44].

## 2 Isomorphism up to permutation

An important instance of common orbit is when $G = S_m \times S_n$ acts on $X = \{0, 1\}^{m \times n}$ by $(g, h) \cdot x = gxh^{-1}$. If we restrict $X$ to only include matrices for which each column has exactly two 1s and no two columns are equal, then $X$ corresponds to the set of incidence matrices of simple graphs on $m$ vertices and $n$ edges, and the common orbit problem corresponds to graph isomorphism:

**Problem 2** (Graph isomorphism). Given two simple graphs $G$ and $H$, determine whether $G \sim H$, that is, there exists a bijection $f: V(G) \to V(H)$ between the vertices that preserves the edges; i.e., for every $u, v \in V(G)$, it holds that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$.

In general, a decision problem is a pair $(P, M)$ where $P$ maps problem instances to answers $P: Q \to \{\text{yes, no}\}$ and $M: Q \to \mathbb{N}$ measures the size of the problem instance. For example, for the graph isomorphism problem, $P$ is the set of all $(G, H)$, where $G$ and $H$ are both simple graphs, $P(G, H)$ returns whether $G$ and $H$ are isomorphic, and if we represent $G$ and $H$ in terms of their incidence matrices, we are inclined to take $M(G, H) = |V(G)||E(G)| + |V(H)||E(H)|$. We say a decision problem $(P, M)$ is GI-hard if, given a blackbox that computes $P(x)$ in time that is polynomial in $M(x)$, one may solve graph isomorphism with runtime that is polynomial in the numbers of vertices and edges in the input graphs $G$ and $H$. For example, if we restrict the input set of graph isomorphism to only consider $(G, H)$ for which $G$ and $H$ are regular graphs, then the resulting subproblem is known to be GI-hard [12].

This section is concerned with two different isomorphism problems between tuples of subspaces. In both cases, we focus our attention on a discrete set of problem instances.
Given a field $F$, take $0, 1 \in F$ and let $Q(r, d, n, F)$ denote the set of $(A, B) \in (\{0, 1\}^{d \times r})^n \times (\{0, 1\}^{d \times r})^n$ such that $\text{rank } A_i = \text{rank } B_i = r$ for every $i \in [n]$. The standard representation of $(A, B) \in Q(r, d, n, F)$ uses $2rdn$ bits. With this, we may define our decision problems:

1. $\mathcal{P}_1(r, F, \sigma) = (P, M)$, where $F$ is an arbitrary field with involutive automorphism $\sigma$, $Q = \bigcup_{d,n \geq 1} Q(r, d, n, F)$, $P(A, B)$ returns whether there exists $(g, \pi) \in U(d, F, \sigma) \times S_n$ such that $(g, \pi) \cdot (\text{im } A_i)_{i \in [n]} = (\text{im } B_i)_{i \in [n]}$, where $d = d(A, B)$ and $n = n(A, B)$, and $M(A, B) = 2 \cdot r \cdot d(A, B) \cdot n(A, B)$.

2. $\mathcal{P}_2(F) = (P, M)$, where $F$ is an arbitrary field, $Q = \bigcup_{r,d,n \geq 1} Q(r, d, n, F)$, $P(A, B)$ returns whether there exists $(g, \pi) \in \text{GL}(d, F) \times S_n$ such that $(g, \pi) \cdot (\text{im } A_i)_{i \in [n]} = (\text{im } B_i)_{i \in [n]}$, where $d = d(A, B)$ and $n = n(A, B)$, and $M(A, B) = 2 \cdot r(A, B) \cdot d(A, B) \cdot n(A, B)$.

In words, $\mathcal{P}_1(r, F, \sigma)$ concerns isomorphism up to unitary and permutation for any fixed rank $r$, whereas $\mathcal{P}_2(F)$ concerns isomorphism up to linear automorphism and permutation, but with the rank no longer fixed. As we will see, both problems are hard. (The fact that $r$ is fixed for one problem and not for the other is an artifact of our proof of hardness.)

**Theorem 3.**

(a) For every $r \in \mathbb{N}$ and every field $F$ with involutive automorphism $\sigma$, it holds that $\mathcal{P}_1(r, F, \sigma)$ is GI-hard.

(b) For every field $F$, it holds that $\mathcal{P}_2(F)$ is GI-hard.

**Proof.** (a) Fix $r$, $F$ and $\sigma$. We will use a $\mathcal{P}_1(r, F, \sigma)$ oracle to efficiently solve graph isomorphism. Given two simple graphs $G$ and $H$, we return no if $V(G)$ and $V(H)$ are of different size, or if $E(G)$ and $E(H)$ are of different size. Otherwise, put $n := |V(G)|$, $e := |E(G)|$ and $d := re$, and for each graph, arbitrarily label the vertices and edges with members of $[n]$ and $[e]$, respectively. We use this labeling of $G$ to determine $A$. Specifically, for each $j \in [n]$, define $A_j \in F^{d \times r}$ to consist of $e$ blocks of size $r \times r$, where for each $i \in [e]$, the $i$th block of $A_j$ is $I_r$ if $j$ is a vertex in edge $i$, and otherwise the block is zero. Define $B$ similarly in terms of our labeling of $H$. Given $(A, B)$, the $\mathcal{P}_1(r, F, \sigma)$ oracle returns whether there exists $(g, \pi) \in U(d, F, \sigma) \times S_n$ such that $(g, \pi) \cdot (\text{im } A_i)_{i \in [n]} = (\text{im } B_i)_{i \in [n]}$, and we will output this answer as our solution to graph isomorphism. It remains to show that $G \sim H$ if and only if there exists $(g, \pi) \in U(d, F, \sigma) \times S_n$ such that $(g, \pi) \cdot (\text{im } A_i)_{i \in [n]} = (\text{im } B_i)_{i \in [n]}$. For ($\Rightarrow$), observe that a graph isomorphism determines a choice of $\pi \in S_n$ as well as a permutation of edges. This permutation of edges can be implemented as a block permutation matrix $g \in U(d, F, \sigma)$ so that $g A_{x^{-1}(i)} = B_i$, which then implies $(g, \pi) \cdot (\text{im } A_i)_{i \in [n]} = (\text{im } B_i)_{i \in [n]}$. For ($\Leftarrow$), we first define two additional graphs $G'$ and $H'$, both on vertex set $[n]$. For $G'$, say $i \leftrightarrow j$ if there exists $x \in \text{im } A_i$ and $y \in \text{im } A_j$ such that $(x, y) \neq 0$. Define $H'$ similarly in terms of $B$. By our construction of $A$ and $B$, it holds that $G' \sim G$ and $H' \sim H$. Furthermore, the existence of $(g, \pi) \in U(d, F, \sigma) \times S_n$ such that $(g, \pi) \cdot (\text{im } A_i)_{i \in [n]} = (\text{im } B_i)_{i \in [n]}$ implies that $G' \sim H'$, meaning $G \sim H$, as desired.

(b) Fix $F$. We will use a $\mathcal{P}_2(F)$ oracle to efficiently solve graph isomorphism for regular graphs, which suffices by [72]. Without loss of generality, we may put $n := |V(G)| = |V(H)|$,
$d := |E(G)| = |E(H)|$, and let $r$ denote the common degree of $G$ and $H$. For each graph, arbitrarily label the vertices and edges with members of $[n]$ and $[d]$, respectively, and let $(e_i)_{i \in [d]}$ denote the identity basis in $F^d$. For each $j \in [n]$, select $A_j \in F^{d \times r}$ so that its column vectors are the $r$ members of $(e_i)_{i \in [d]}$ that correspond to edges $i$ incident to vertex $j$. Define $B$ similarly in terms of our labeling of $H$. Given $(A, B)$, the $P_2(F)$ oracle returns whether there exists $(g, \pi) \in \text{GL}(d, F) \times S_n$ such that $(g, \pi) \cdot (\text{im} A_i)_{i \in [n]} = (\text{im} B_i)_{i \in [n]}$, and we will output this answer as our solution to graph isomorphism. It remains to show that $G \sim H$ if and only if there exists $(g, \pi) \in \text{GL}(d, F) \times S_n$ such that $(g, \pi) \cdot (\text{im} A_i)_{i \in [n]} = (\text{im} B_i)_{i \in [n]}$. For $(\Rightarrow)$, the isomorphism determines a permutation matrix $g \in \text{GL}(d, F)$ and a permutation $\pi \in S_n$ such that such that $(g, \pi) \cdot (\text{im} A_i)_{i \in [n]} = (\text{im} B_i)_{i \in [n]}$. For $(\Leftarrow)$, we first define two additional graphs $G'$ and $H'$, both on vertex set $[n]$. For $G'$, say $i \leftrightarrow j$ if $\text{im} A_i \cap \text{im} A_j \neq \{0\}$, and define $H'$ similarly in terms of $B$. By our construction of $A$ and $B$, it holds that $G' \sim G$ and $H' \sim H$. Furthermore, the existence of $(g, \pi) \in \text{GL}(d, F) \times S_n$ such that $(g, \pi) \cdot (\text{im} A_i)_{i \in [n]} = (\text{im} B_i)_{i \in [n]}$ implies that $G' \sim H'$, meaning $G \sim H$, as desired.  

Of course, Theorem 3 does not mean that solving $P_1(r, F, \sigma)$ or $P_2(F)$ is always hopeless. As an example, the Bargmann invariants computed in Section 3.1 are ordered lists of numbers; if the histograms of these numbers are not equal, then the lines cannot be isomorphic up to permutation.

## 3 Isomorphism up to linear isometry

While the previous section demonstrated that certain isomorphism problems are hard, this section will show that isomorphism up to linear isometry is relatively easy. This would have taken Halmos by surprise, as he considered this problem to be difficult even for triples of subspaces [35]. Throughout this section, we assume $F \in \{\mathbb{R}, \mathbb{C}\}$ without mention, meaning $U(d, F, \sigma) \in \{O(d), U(d)\}$.

### 3.1 Lines

Chien and Waldron [10] provide an injective invariant for tuples of lines in $F^d$ up to isometric isomorphism. Given a tuple $(v_i)_{i \in [n]}$ of unit vectors in $F^d$ that span each line in the tuple $\mathcal{L} = (\ell_i)_{i \in [n]}$, define the $m$-vertex Bargmann invariants or $m$-products by

$$
\Delta(v_{i_1}, \ldots, v_{i_m}) := \langle v_{i_1}, v_{i_2} \rangle \langle v_{i_2}, v_{i_3} \rangle \cdots \langle v_{i_m}, v_{i_1} \rangle, \quad i_1, \ldots, i_m \in [n].
$$

Denoting $P_i := v_i v_i^*$, we see that $\Delta(v_{i_1}, \ldots, v_{i_m}) = \text{tr}(P_{i_1} \cdots P_{i_m})$, and so the choice of $v_i \in \ell_i$ is irrelevant. Furthermore, as their name suggests, these quantities are invariant to isometric isomorphism, since for $Q \in U(d, F, \sigma)$, the orthogonal projection onto $Q \cdot \ell_i$ is $QP_i Q^*$, and $\text{tr}(QP_{i_1} Q^* \cdots QP_{i_m} Q^*) = \text{tr}(P_{i_1} \cdots P_{i_m})$.

Given the 2-products, one may define the frame graph $G(\mathcal{L})$ on $[n]$ in which we draw an edge $i \leftrightarrow j$ when $\ell_i$ and $\ell_j$ are not orthogonal; we note that the frame graph has also been referred to as the correlation network [32]. Letting $E$ denote the edge set of the frame graph, then the indicator functions of the edge sets of Eulerian subgraphs of $G(\mathcal{L})$ form a

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1Our definition differs slightly from [10] since our inner product is conjugate-linear in the first argument.
subspace $\mathcal{E} \subseteq \mathbb{F}_2^E$. Given a maximal spanning forest $F$ of $G(\mathcal{L})$, then each edge in $E \setminus E(F)$ completes a unique cycle with this forest, and the indicator functions of the edge sets of these cycles form a basis for $\mathcal{E}$. Let $C(F)$ denote the set of these cycles. With these notions, we may enunciate the main result of [10] (for unweighted lines).

Proposition 4 (Corollary 3.2 in [10], cf. Theorem 2 in [30]). Given a tuple $\mathcal{L}$ of lines in $\mathbb{F}^d$, select any maximal spanning forest $F$ of the frame graph $G(\mathcal{L})$. Then $\mathcal{L}$ is determined up to isometric isomorphism by its 2-products and each $m$-product corresponding to a cycle in $C(F)$.

Proof. Let $\mathcal{L}$ and $\mathcal{L}'$ be $n$-tuples of lines in $\mathbb{F}^d$. As noted above, if $\mathcal{L}$ and $\mathcal{L}'$ are isometrically isomorphic, then all of their $m$-products must be equal.

For the other direction, select an $n$-tuple of unit vectors $\{v_i\}_{i=1}^n$ in $\mathbb{F}^d$ that span the lines in $\mathcal{L}$. Let $\{u_i\}_{i=1}^n$ be another $n$-tuple of unit vectors in $\mathbb{F}^d$ such that the 2-products and each $m$-product corresponding to a cycle in $C(F)$ (corresponding to a spanning forest $F$ of $\mathcal{L}$) of each tuple of vectors are equal. We would like to show that $\{v_i\}_{i=1}^n$ and $\{u_i\}_{i=1}^n$ are the same modulo $U(d, \mathbb{F}, \sigma)$ and choice of basis vectors. Since the spectral theorem implies tuples of vectors are the same modulo $U(d, \mathbb{F}, \sigma)$ if and only if their Gramians are component-wise equal, it suffices to show that there exist unimodular $\eta_i$ for $i \in [n]$ such that for all $i, j \in [n]$

$$\langle u_i, u_j \rangle = \overline{\eta_j} \langle v_i, v_j \rangle.$$  

(1)

If $i$ and $j$ are in different components of $G(\mathcal{L})$ (where we are using $i$ as shorthand for $\ell_i$), then $\langle v_i, v_j \rangle = 0$ and [11] yields no restriction on the values of $\eta_i$ and $\eta_j$. Thus, we may assume without loss of generality that $G(\mathcal{L})$ is connected and $F$ is a spanning tree with root $r$. Since 2-products are equal,

$$|\langle u_i, u_j \rangle|^2 = \langle u_i, u_j \rangle \langle u_j, u_i \rangle = \langle v_i, v_j \rangle \langle v_j, v_i \rangle = |\langle v_i, v_j \rangle|^2$$

for all $i, j \in [n]$. For $i \in [n]$ such that $ri$ is an edge in $F$, let $\eta_i$ be the necessarily unimodular scalar such that $\langle u_r, u_i \rangle = \eta_i \langle v_r, v_i \rangle$. Now for $j \in [n]$ such that $ri$ and $ij$ are edges in $F$ but not $rj$ let $\eta_j$ be the necessarily unimodular scalar such that [11] holds. Continue this process inductively, setting the $\eta_k$ for vertices $k$ at distance 3, 4, . . . from $r$. Since $F$ is spanning, we have uniquely defined $\eta_i$ for each $i \in [n]$. However, we now need to verify that [11] holds for any $ij$ that is an edge in $G(\mathcal{L})$ but not $F$. Let $ij$ be such an edge; it lies in a unique cycle in $C(F)$, say with vertex sequence $i, j, k_3, k_4, \ldots, k_m, i$. Since each edge but $ij$ lies in $F$,

$$\langle v_i, v_j \rangle \langle v_j, v_{k_3} \rangle \langle v_{k_3}, v_{k_4} \rangle \cdots \langle v_{k_m}, v_i \rangle = \langle u_i, u_j \rangle \langle u_j, u_{k_3} \rangle \langle u_{k_3}, u_{k_4} \rangle \cdots \langle u_{k_m}, u_i \rangle$$

$$= \langle u_i, u_j \rangle \overline{\eta_j} \eta_{k_3} \langle v_j, v_{k_3} \rangle \eta_{k_3} \eta_{k_4} \langle v_{k_3}, v_{k_4} \rangle \cdots \eta_{k_m} \eta_i \langle v_{k_m}, v_i \rangle$$

$$= \eta_i \overline{\eta_j} \langle u_i, u_j \rangle \langle v_j, v_{k_3} \rangle \langle v_{k_3}, v_{k_4} \rangle \cdots \langle v_{k_m}, v_i \rangle,$$

implying that [11] holds for $ij$, as desired. \qed

Generically (or for equiangular tight frames and certain other optimal configurations), none of the inner products $\langle v_i, v_j \rangle$ equal zero. In this case, the frame graph is complete, and so we may take $F$ to be the star graph in which $1 \leftrightarrow j$ for every $j \neq 1$. Then $C(F)$
Lemma 6. Also leverage consequences of the fact that \( \text{SO}(2) \) is abelian: values of a cross Gramian between two planes are either all equal or all distinct. We will and to do so, we exploit several features of this special case. For example, the singular values of a cross Gramian between two planes are either all equal or all distinct. We refer to \( G \) as the normalized Gramian of \( A \). Since the Gramian of \( (v_i)_{i \in [n]} \) is invariant to isometries acting on \( (v_i)_{i \in [n]} \), normalizing the Gramian removes any ambiguity introduced by selecting \( v_i \in \ell_i \), and so the normalized Gramian is a generically injective invariant for \( (\text{Gr}(1,F^d))^n \) modulo \( U(d,F,\sigma) \). Notice that the entries of \( G \) are the triple products corresponding to \( C(F) \), and so this conclusion may also be viewed in terms of Proposition 4.

At this point, we can treat the case of lines from two related but different perspectives: Generically (and for certain optimal configurations), it suffices to compute the normalized Gramian, but in general, we must appeal to more intricate Bargmann invariants. In what follows, we will see that a similar story holds for general subspaces.

3.2 Real, nowhere orthogonal planes

We say two subspaces \( U, V \subseteq F^d \) are nowhere orthogonal if \( U \cap V^\perp = U^\perp \cap V = \{0\} \). By counting dimensions, one may conclude that subspaces are nowhere orthogonal only if they have the same dimension. Given bases \((u_i)_{i \in [r]} \) and \((v_i)_{i \in [r]} \) for \( U \) and \( V \), nowhere orthogonality is equivalent to the cross Gramian \( \langle (u_i, v_j) \rangle_{i,j \in [r]} \) being invertible. As one might expect, nowhere orthogonality is a generic property of subspaces of common dimension; we provide a short proof in the real case:

**Lemma 5.** Two generic \( r \)-dimensional subspaces of \( \mathbb{R}^d \) are nowhere orthogonal.

**Proof.** Given \( A_1, A_2 \in \mathbb{R}^{d \times r} \), then \( \text{im} A_1 \) and \( \text{im} A_2 \) are nowhere orthogonal subspaces of dimension \( r \) if and only if \( f(A_1, A_2) := \det(A_1^* A_2) \neq 0 \). Since the polynomial \( f \) is nonzero at \( A_1 = A_2 = [r, 0] \), it follows that \( f \neq 0 \), and so \( f^{-1}(\mathbb{R} \setminus \{0\}) \) is a generic set, as desired. \( \square \)

In this section, we consider the special case of nowhere orthogonal 2-dimensional subspaces of \( \mathbb{R}^d \). This case is particularly relevant to the study of real equi-isoclinic planes, which have received some attention recently \[15, 17, 16, 18, 46\]. In general, subspaces are said to be equi-isoclinic if there exists \( \theta > 0 \) such that every principal angle between any two of the subspaces equals \( \theta \). (Note that equi-isoclinic subspaces with \( \theta < \frac{\pi}{2} \) are nowhere orthogonal.) Such subspaces were introduced by Lemmens and Seidel \[49\], and at times, they emerge as arrangements of points in the Grassmannian that maximize the minimum chordal distance \[14\]. In fact, most of Sloane’s chordal-distance codes of real planes \[60\] are nowhere orthogonal, and well over half have the property that all cross Gramians have a minimum singular value greater than \( 10^{-4} \).

In what follows, we obtain a normalized Gramian for real, nowhere orthogonal planes, and to do so, we exploit several features of this special case. For example, the singular values of a cross Gramian between two planes are either all equal or all distinct. We will also leverage consequences of the fact that \( \text{SO}(2) \) is abelian:

**Lemma 6.** If \( A \in \text{SO}(2) \) and \( B \in \text{O}(2) \), then \( A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) and \( AB = BA^{\det B} \).
Algorithm 1: Canonical Gramian between real, nowhere orthogonal planes

**Data:** Gramian \( A \in (\mathbb{R}^{2 \times 2})^{n \times n} \) of orthobases of \( n \) nowhere orthogonal planes in \( \mathbb{R}^d \)

**Result:** Gramian \( G \in (\mathbb{R}^{2 \times 2})^{n \times n} \) of another choice of orthobases

Put \( R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) and \( S = \text{diag}(R, \ldots, R) \)

**if** there exists \((k,l)\) such that \( A_{kl} \) has distinct singular values **then**

- Let \((k,l)\) be the first such indices, lexicographically
- Compute the singular value decomposition \( A_{kl} = W_k \Sigma V^* \) and put \( \tilde{W}_k = W_k R \)
- For \( j \neq k \), compute polar decompositions \( W_k^* A_{kj} = P_j W_j^* \) and \( \tilde{W}_k^* A_{kj} = \tilde{P}_j \tilde{W}_j^* \)
- Put \( D = \text{diag}(W_1, \ldots, W_n) \) and \( \tilde{D} = \text{diag}(\tilde{W}_1, \ldots, \tilde{W}_n) \)
- Put \( G = \min(D^* AD, \tilde{D}^* A\tilde{D}) \), lexicographically

**else**

- For each \((i,j)\), find \( \alpha_{ij} > 0 \) such that \( H_{ij} := \alpha_{ij} A_{ij} \in O(2) \)
- Put \( H = (H_{ij})_{i,j \in [n]} \) and \( D = \text{diag}(H_{11}, \ldots, H_{1n}) \)

  **if** there exists \((k,l)\) such that \( \det(DHD^*)_{kl} = -1 \) **then**

  - Let \((k,l)\) be the first such indices, lexicographically
  - Put \( Q = (DHD^*)_{kl} R \)\(^{-1/2} \) // either square root may be selected
  - Put \( E = \text{diag}(QH_{11}, \ldots, QH_{1n}) \)
  - Put \( G = \min(EAE^*, SEAE^* S) \), lexicographically

  **else**

  - Put \( G = \min(DAD^*, SDAD^* S) \), lexicographically

**end**

**end**

\[ \begin{align*}
\text{Proof.} \quad & \text{The first claim follows from the fact that } [z s \ -s c]^{-1} = [c \ -s \ z \ s] \text{ when } c^2 + s^2 = 1. \text{ For the second claim, if } \det B = 1, \text{ then since } SO(2) \text{ is abelian, we have } AB = BA. \text{ If } \det B = -1, \text{ then put } R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } C = BR. \text{ Then } C \in SO(2), \text{ and so the first claim gives}
\end{align*} \]

\[ AB = ABRR = ACR = CAR = CRRAR = CRA^{-1} = BA^{-1}. \]

\[ \square \]

**Theorem 7.** The function implemented by Algorithm\( \square \) is an injective invariant for nowhere orthogonal tuples in \((\text{Gr}(2, \mathbb{R}^d))^n \) modulo \( O(d) \).

**Proof.** First, if two different inputs \( A_1 \) and \( A_2 \) produce the same output \( G \), then by the construction of \( G \) in both cases, there must exist block diagonal unitary matrices \( U_1 \) and \( U_2 \) such that \( G = U_1 A_1 U_1^* = U_2 A_2 U_2^* \). It then follows that \( A_2 = (U_2^* U_1) A_1 (U_2^* U_1)^* \), that is, \( A_1 \) and \( A_2 \) are equivalent. It remains to show that equivalent inputs produce identical outputs.

Take any tuple \((U_i)_{i \in [n]} \) in \( O(2) \) and put \( U = \text{diag}(U_1, \ldots, U_n) \). We will first show that \( UAU^* \) produces the same output as \( A \). Throughout, we use \( ^\sharp \) to denote the version of calculations that come from \( UAU^* \), e.g., \( A^\sharp = UAU^* \). First, \( A_{kl} \) has the same singular values as \( A_{kl}^\sharp = U_k A_{kl} U_k^* \), and so \( (k,l)^\sharp \) exists if and only if \( (k,l) \) exists for the first condition in Algorithm\( \square \). Suppose \( (k,l)^\sharp = (k,l) \) does exist. Next, there are four choices of left singular vectors of \( U_k A_{kl} U_k^* \), namely, \( W_k^\sharp \in \{ \pm U_k W_k, \pm U_k W_k R \} \). As such, there exists \( \epsilon \in \{ \pm 1 \} \) and \( t \in \{ 0, 1 \} \) such that

\[ W_k^\sharp = \epsilon U_k W_k R^t, \quad \tilde{W}_k^\sharp = \epsilon U_k W_k R^{t+1}. \]
Since each $A_{kj}$ is invertible, the polar decompositions are unique, and we have

$$ (W^*_{k}A_{kj})^t = eR^t W^*_{k}U_{kj}A_{kj}U^*_{j} = eR^t W^*_{k}A_{kj}U^*_{j} = \begin{cases} \epsilon P_{j} W^*_{j}U^*_{j} & \text{if } t = 0 \\ \epsilon P_{j} W^*_{j}U^*_{j} & \text{if } t = 1. \end{cases} $$

Either way, the polar decomposition gives $W^*_j = eU_j W_j R^t$. Similarly, $\bar{W}^*_j = eU_j W_j R^{t+1}$.

With this we see that

$$ (D^* AD)^{\sharp}_{ij} = (W^*_{ij}A_{ij}W_j)^{\sharp} = R^t W^*_{ij}U_{ij}A_{ij}U^*_{j}W_jR^t = (W^*_{ij}R^t)^{*}A_{ij}(W_jR^t), $$
and similarly $$(\bar{D}^* A\bar{D})^{\sharp}_{ij} = (\bar{W}^*_{ij}R^{t+1})^{*}A_{ij}(W_jR^t)$$.

It follows that \{(D^* AD)^{\sharp}, (\bar{D}^* A\bar{D})^{\sharp}\} = \{D^* AD, \bar{D}^* A\bar{D}\}$, and so $\bar{G} = G$.

Next, we suppose that no such $(k,l)$ exists. Since $A_{ij}^t = U_i A_{ij}U_j^*$ with $U_i, U_j \in O(2)$, then $\alpha_{ij}^t = \alpha_{ij}$ and so $H_{ij}^t = U_i H_{ij}U_j^*$. Next,

$$ (DHD^*)^{\sharp}_{kl} = (H_{1k}H_{kl}H_{1l}^*)^t = U_1 H_{1k}U_{kl}^*U_kH_{kl}U_{i1}^*U_1H_{1l}^*U_1 = U_1(DHD^*)_{kl}U_1^*, $$
and so $\text{det}(DHD^*)^t_{kl} = \text{det}(DHD^*)_{kl}$. For the remainder of the proof, put $T = DHD^*$ and define $t$ to be 1 if $\text{det} U_i = -1$, and otherwise 0.

Suppose there exists $(k,l)^{\sharp} = (k,l)$ such that $T_{kl} = -1$. Then $Q^* = \pm(U_1T_{kl}U_1^*)^{-1/2}$, and since $Q^* \in SO(2)$, Lemma 6 gives

$$ (EAE^{\sharp})_{ij}^* = \alpha_{ij}^{-1}\text{det} T_{ij}(Q^*)^{T_{ij}(Q^*)^t} = \alpha_{ij}^{-1}U_1T_{ij}U_1^*(Q^*)^t = \alpha_{ij}^{-1}U_1T_{ij}U_1^*(Q^*)^t. $$

If $\text{det} T_{ij} = 1$, then this reduces to

$$ (EAE^{\sharp})_{ij}^* = \alpha_{ij}^{-1}U_1T_{ij}U_1^*(Q^*)^t = \alpha_{ij}^{-1}U_1T_{ij}U_1^* = \alpha_{ij}^{-1}T_{ij}^t = \alpha_{ij}^{-1}R^t T_{ij}R^t. $$

Otherwise, $\text{det} T_{ij} = -1$, and so

$$ (EAE^{\sharp})_{ij}^* = \alpha_{ij}^{-1}U_1T_{ij}U_1^*(Q^*)^{-2} = \alpha_{ij}^{-1}U_1T_{ij}U_1^*T_{ij}U_1^*R = \alpha_{ij}^{-1}U_1T_{ij}T_{kl}U_1^*R. $$

Since $\text{det}(T_{ij}T_{kl}) = 1$, Lemma 6 then gives

$$ (EAE^{\sharp})_{ij}^* = \alpha_{ij}^{-1}U_1T_{ij}T_{kl}U_1^*R = \alpha_{ij}^{-1}(T_{ij}T_{kl})^tU_1^t = \alpha_{ij}^{-1}R^t T_{ij}T_{kl}R^{t+1}. $$

Similarly,

$$ (EAE^{\sharp})_{ij} = \begin{cases} \alpha_{ij}^{-1}T_{ij} & \text{if } \text{det} T_{ij} = 1 \\ \alpha_{ij}^{-1}T_{ij}T_{kl} & \text{if } \text{det} T_{ij} = -1, \end{cases} $$
meaning \{(EAE^*)^{\sharp}, S(EAE^*)^{\sharp}S\} = \{EAE^*, \text{SEA}E^*S\}$, and so $G^t = G$.

In the final case, we have $\text{det}(DHD^*)^{\sharp}_{ij} = \text{det}(DHD^*)_{ij} = 1$ for every $i, j \in [n]$. Here, Lemma 6 gives

$$ T_{ij}^{\sharp} = U_1T_{ij}U_1^* = T_{ij}^{\text{det} U_1} = R^t T_{ij}R^t, $$
meaning \{(T^t, ST^tS) = \{T, STS\}, and so $G^t = G$. \hfill \Box$

A Matlab implementation of Algorithm 1 may be downloaded from [44].

We note that another (uglier) algorithm produces a normalized Gramian for generic rank-$r$ subspaces, but the algorithm we found does not produce a Gramian if any two of the subspaces are isoclinic (for example). Due to this failure, we decided to not report the details of this algorithm.
3.3 $H^\ast$-algebras and generalized Bargmann invariants

In pursuit of an injective invariant for $(\text{Gr}(r, F^d))^n$ modulo $\text{U}(d, F, \sigma)$, we consider traces of products of matrices, generalizing Bargmann invariants and building on the approaches in $[61, 71, 54, 43, 30, 59]$. There are large upper bounds on the number of traces of products that must be computed to generate an injective invariant on a single (1-tuple) $d \times d$ matrix, like $4d^2$ $[54]$, and we prove in Lemma 8 that there is a lower bound on Bargmann invariants that must be computed in general to provide an injective invariant for tuples of lines. Thus, our goal of this section is to give (two different) algorithms to compute injective invariants that generalize Bargmann invariants and require a reasonable number of computations. Neither requires genericity of the subspaces.

First, we clarify how we must use these invariants with the help of a lemma:

**Lemma 8.** Consider any function $f: (\text{Gr}(1, F^d))^d \to F^m$ such that each coordinate function of $f$ is a fixed Bargmann invariant. Then $f$ is an injective invariant of $(\text{Gr}(1, F^d))^d$ modulo $\text{U}(d, F, \sigma)$ only if $m \geq (d - 1)!/2$.

**Proof.** Select $\epsilon \in \{\pm\}$ and consider the lines $L_\epsilon$ spanned by the vectors 

$e_1 + e_2, \ldots, e_{d-1} + e_d, e_d + \epsilon e_1$.

For both choices of $\epsilon$, the frame graph $G(L_\epsilon)$ is the cycle graph $C_d$ of length $d$, and so the maximal spanning forest $F$ is a path graph. The 2-products of $L_\epsilon$ are those of $L_\epsilon$, but the $d$-product corresponding to the lone cycle $C_d \in C(F)$ has the same sign as $\epsilon$. As such, $L_\epsilon$ is not isomorphic to $L_\epsilon$ modulo $\text{U}(d, F, \sigma)$ by Proposition 4. The Bargmann invariants that do not vanish on $L_\epsilon$ are the ones that correspond to closed walks along $C_d$. Of these, the Bargmann invariants that distinguish $L_\epsilon$ from $L_\epsilon$ are closed walks with odd winding number around $C_d$. Overall, distinguishing $L_\epsilon$ from $L_\epsilon$ requires a Bargmann invariant whose closed walk is supported on all of $C_d$.

Now select $\pi \in S_d$ and $\epsilon \in \{\pm\}$ and consider the lines $\pi \cdot L_\epsilon$ obtained by permuting the tuple $L_\epsilon$ according to $\pi$. Distinguishing $\pi \cdot L_\epsilon$ from $\pi \cdot L_\epsilon$ for every $\pi \in S_d$ requires Bargmann invariants whose closed walks are supported on each of the length-$d$ cycles in the complete graph $K_d$. The result follows from the fact that there are $(d - 1)!/2$ such cycles. \[\square\]

Considering $(d - 1)!/2$ is far too large for efficient computation, we instead accept a different type of injective invariant: Given a tuple $L$ of $n$ lines, return a collection $W$ of walks on $K_n$ as well as the Bargmann invariant of $w$ evaluated at $L$ for each $w \in W$. Note that this is the form provided by Proposition 4 at least if $F$ were selected canonically; this can be accomplished by iteratively growing $F$ from edges in lexicographic order.

The remainder of this section considers two different generalizations of the Bargmann invariants, and we use these invariants to distinguish between tuples of subspaces modulo isometric isomorphism. Our results for both generalizations apply ideas from the representation theory of $H^\ast$-algebras. For what follows, we remind the reader that $F \in \{\mathbb{R}, \mathbb{C}\}$.

**Definition 9.** We say $\mathcal{A}$ is an $H^\ast$-algebra over $(F, \sigma)$ if

(H1) $(\mathcal{A}, +, \times, F)$ is a finite-dimensional associative algebra with unity,
(H2) \( \ast : \mathcal{A} \to \mathcal{A} \) is a conjugate-linear involutory antiautomorphism, and

(H3) \((\cdot, \cdot) : \mathcal{A} \times \mathcal{A} \to F\) is a Hermitian form on \( \mathcal{A} \) such that

\[(xy, z) = (y, x^* z) = (x, zy^*) \quad \forall x, y, z \in \mathcal{A}.
\]

A representation of an \( H^*\)-algebra \( \mathcal{A} \) is a \( * \)-algebra homomorphism \( f : \mathcal{A} \to F^{k \times k} \). The corresponding character \( \chi_f : \mathcal{A} \to F \) is given by \( \chi_f(x) = tr(f(x)) \). Two representations \( f, g : \mathcal{A} \to F^{k \times k} \) are equivalent if there exists \( U \in U(k, F) \) such that \( g(x) = U f(x) U^* \).

**Proposition 10** (Theorem 3 in [30]). Two representations of an \( H^*\)-algebra are equivalent if and only if their characters are equal.

Given \( S \subseteq F^{k \times k} \), let \( \mathcal{A}(S) \) denote the smallest algebra with unity containing \( S \).

**Lemma 11.** Consider tuples \( (A_i)_{i \in [n]} \) and \( (B_i)_{i \in [n]} \) over \( F^{k \times k} \) for which there exists \( \pi \in S_n \) such that \( A_{i}^* = A_{\pi(i)} \) and \( B_{i}^* = B_{\pi(i)} \) for every \( i \in [n] \). Select words \( (w_j(x_1, \ldots, x_n))_{j \in [m]} \) in noncommuting variables \( x_i \) such that the evaluation \( (E_j := w_j(A_1, \ldots, A_n))_{j \in [m]} \) is a basis for \( \mathcal{A}((A_i)_{i \in [n]}) \). (Here, evaluating the word of length zero produces the identity matrix.) There exists \( U \in U(k, F, \sigma) \) such that \( U A_i U^* = B_i \) for every \( i \in [n] \) if and only if

(i) the evaluation \( (F_j := w_j(B_1, \ldots, B_n))_{j \in [m]} \) is a basis for \( \mathcal{A}((B_i)_{i \in [n]}) \),

(ii) \( tr(E_i^* E_j) = tr(F_i^* F_j) \) for every \( i, j \in [m] \),

(iii) \( tr(E_i^* E_j E_k) = tr(F_i^* F_j F_k) \) for every \( i, j, k \in [m] \), and

(iv) \( tr(E_i^* A_j) = tr(F_i^* B_j) \) for every \( i \in [m], j \in [n] \).

**Proof.** (\( \Rightarrow \)) Suppose there exists \( U \in U(k, F, \sigma) \) such that \( U A_i U^* = B_i \) for every \( i \in [n] \). Then \( UE_i U^* = F_i \) for every \( i \in [m] \), and (i)–(iv) follow immediately.

(\( \Leftarrow \)) First, the assumed existence of \( \pi \in S_n \) implies that \( \mathcal{A}((A_i)_{i \in [n]}) \) and \( \mathcal{A}((B_i)_{i \in [n]}) \) are \( H^*\)-algebras. Indeed, both algebras inherit (H3) from \( F^{k \times k} \) by taking \( (x, y) = \text{tr}(x^* y) \). By (i), there is a unique linear \( f : \mathcal{A}((A_i)_{i \in [n]}) \to \mathcal{A}((B_i)_{i \in [n]}) \) that maps \( E_i \mapsto F_i \) for every \( i \in [m] \). Next, (ii) implies that for every \( x \in \mathcal{A}((A_i)_{i \in [n]}) \), it holds that \( f(x) \) is the unique \( y \in \mathcal{A}((B_i)_{i \in [n]}) \) such that \( \text{tr}(E_i^* x) = \text{tr}(F_i^* y) \) for every \( i \in [m] \). This combined with (iii) and (iv) then imply that \( f \) maps \( E_j E_k \mapsto F_j F_k \) for every \( j, k \in [m] \) and \( A_j \mapsto B_j \) for every \( j \in [n] \). The former implies that \( f \) is an algebra isomorphism, since decomposing \( x = \sum_i a_i E_i \) and \( y = \sum_j b_j E_j \) gives \( xy = \sum_{ij} a_i b_j E_i E_j \), which \( f \) then maps to \( \sum_{ij} a_i b_j F_i F_j = f(x) f(y) \). Since \( f : A_i \mapsto B_i \) for every \( i \in [n] \), the assumed existence of \( \pi \in S_n \) implies that \( f \) is a \( *\)-algebra isomorphism. Indeed, letting \( Rw \) denote the reversal of the word \( w \), then since \( f \) is an algebra isomorphism, \( f \) maps

\[ (w(A_1, \ldots, A_n))^* = (Rw)(A_1^*, \ldots, A_n^*) = (Rw)(A_{\pi(1)}, \ldots, A_{\pi(n)}) \]

to \((Rw)(B_{\pi(1)}, \ldots, B_{\pi(n)}) = (w(B_1, \ldots, B_n))^* \), and so \( f(x^*) = f(x)^* \) by linearity. At this point, we consider two representations of \( \mathcal{A}((A_i)_{i \in [n]}) \), namely, the identity map and \( f \). Since the identity matrix resides in both \( \mathcal{A}((A_i)_{i \in [n]}) \) and \( \mathcal{A}((B_i)_{i \in [n]}) \) by definition, (ii) together with linearity gives that the characters of these representations are equal, and so Proposition 10 implies the existence of \( U \in U(k, F, \sigma) \) such that \( f(x) = U x U^* \). Since \( f : A_i \mapsto B_i \) for every \( i \in [n] \), we are done. \( \square \)
Lemma 12. and then report traces of the form (ii)–(iv). In the following, we show that a certain (obvious) evaluation specifies a collection of words \((w_j)_{j \in [m]}\) such that \((w_j(A_1, \ldots, A_n))_{j \in [m]}\) is a basis for \(\mathcal{A}((A_i)_{i \in [n]})\).

Proof. Assume it holds for \(l\). Let \(A_{j}w_j(A_1, \ldots, A_n)\) be a basis among these evaluations. Let \(w_{m_{\text{old}}} = 1\) (the word of length zero).

Initialize \(m_{\text{old}} = 0\) and \(m_{\text{new}} = 1\)

while \(m_{\text{new}} > m_{\text{old}}\) do

Update \(m_{\text{old}} = m_{\text{new}}\)

for \(i \in [n]\) and \(j \in [m_{\text{old}}]\) do

if \(A_iw_j(A_1, \ldots, A_n)\) is linearly independent of \((w_l(A_1, \ldots, A_n))_{l \in [m_{\text{new}}]}\) then

Put \(w_{m_{\text{new}}+1} = x_iw_j\) and update \(m_{\text{new}} = m_{\text{new}} + 1\)

end

end

end

Algorithm 2: Canonical basis for matrix algebra from finite generating set

Data: Matrices \((A_i)_{i \in [n]}\) in \(F^{k \times k}\)

Result: Words \((w_j)_{j \in [m]}\) such that \((w_j(A_1, \ldots, A_n))_{j \in [m]}\) is a basis for \(\mathcal{A}((A_i)_{i \in [n]})\)

Put \(w_1 = 1\) (the word of length zero)

Initialize \(m_{\text{old}} = 0\) and \(m_{\text{new}} = 1\)

while \(m_{\text{new}} > m_{\text{old}}\) do

Update \(m_{\text{old}} = m_{\text{new}}\)

for \(i \in [n]\) and \(j \in [m_{\text{old}}]\) do

if \(A_iw_j(A_1, \ldots, A_n)\) is linearly independent of \((w_l(A_1, \ldots, A_n))_{l \in [m_{\text{new}}]}\) then

Put \(w_{m_{\text{new}}+1} = x_iw_j\) and update \(m_{\text{new}} = m_{\text{new}} + 1\)

end

end

end

In [30], Proposition 10 is used to prove (Theorem 4 in [30]) that calculating the traces of the evaluations of every possible word on the generating matrices of length between one and \(4k^2\) (i.e., on the order of \(n^k k^2\)) is an injective invariant. Our goal in what follows is to prune the list of necessary words to evaluate.

Overall, to determine a tuple of matrices in \(F^{k \times k}\) up to unitary equivalence, it suffices to specify a collection of words \((w_i)_{i \in [m]}\) that can be used to span the corresponding \(H^\sigma\)-algebra, and then report traces of the form (ii)–(iv). In the following, we show that a certain (obvious) choice of words, i.e., the result of Algorithm 2 is invariant to conjugation by unitary matrices and computable in polynomial time.

Lemma 12. Given \((A_i)_{i \in [n]}\) in \(F^{k \times k}\), Algorithm 2 returns words \((w_j)_{j \in [m]}\) such that the evaluation \((w_j(A_1, \ldots, A_n))_{j \in [m]}\) is a basis for \(\mathcal{A}((A_i)_{i \in [n]})\). Given \((UA_iU^*)_{i \in [n]}\) for some \(U \in U(k, F, \sigma)\), Algorithm 2 returns the same words \((w_j)_{j \in [m]}\). Algorithm 2 terminates after at most \(m \leq k^2\) iterations of the while loop, and each iteration can be implemented in a way that costs \(O(mnk^4)\) operations.

Proof. First, consider the set of evaluations of all words at \((A_i)_{i \in [n]}\). This set spans \(\mathcal{A}((A_i)_{i \in [n]})\), which is a subspace of \(F^{k \times k}\), and therefore has finite dimension. It follows that there exists a basis among these evaluations. Let \(L\) denote the smallest possible length of the longest word in a basis.

For the moment, let us remove the constraint \(m_{\text{new}} > m_{\text{old}}\) of the while loop. We claim that after the \(l\)th iteration of the unconstrained while loop, \(\text{span}(w_j(A_1, \ldots, A_n))_{j \in [m_{\text{new}}]}\) contains all evaluations of words of length \(l\). By our initialization \(w_1 = 1\), this holds for \(l = 0\). Assume it holds for \(l \geq 0\). Then every word of length \(l + 1\) has the form \(x_iw\), where \(w\) is a word of length \(l\). Evaluating then produces \(A_iw(A_1, \ldots, A_n)\). By the induction hypothesis, \(w(A_1, \ldots, A_n)\) can be expressed as a linear combination of \((w_j(A_1, \ldots, A_n))_{j \in [m_{\text{old}}]}\). Since we test all of \((A_iw_j(A_1, \ldots, A_n))_{j \in [m_{\text{old}}]}\) for linear independence in order to select \((w_j)_{j \in [m_{\text{new}}]}\), it follows that \(\text{span}(w_j(A_1, \ldots, A_n))_{j \in [m_{\text{new}}]}\) contains \(A_iw(A_1, \ldots, A_n)\).

Now suppose that the \(l\)th iteration of the unconstrained while loop resulted in \(m_{\text{new}} = m_{\text{old}}\). Then no new words were added to \(\{w_j\}\) in the \(l\)th iteration. In fact, for every \(i \in [n]\)
and \(j \in [m_{\text{old}}]\), it holds that \(A_i w_j(A_1, \ldots, A_n)\) resides in \(\text{span}(w_i(A_1, \ldots, A_n))_{i \in [m_{\text{old}}]}\), and so no new words will also be added in any future iteration. Considering \((w_i(A_1, \ldots, A_n))_{i \in [m_{\text{new}}]}\) forms a basis for \(\mathcal{A}'(\{A_i\})_{i \in [n]}\) by the end of the \(L\)th iteration, it follows that the original while loop with constraint \(m_{\text{new}} > m_{\text{old}}\) terminates with a basis after \(L + 1\) iterations.

Now suppose we were instead given \((UA_i U^*)_{i \in [n]}\) for some \(U \in U(k, F, \sigma)\). Since the map \(x \mapsto UxU^*\) is a linear isometry over \(F^{k \times k}\), it follows that \(UA_i U^* w_j(UA_1 U^*, \ldots, UA_n U^*) = UA_i w_j(A_1, \ldots, A_n) U^*\) is linearly independent of \((w_i(A_1, \ldots, A_n))_{i \in [m_{\text{new}}]}\) if and only if \(A_i w_j(A_1, \ldots, A_n)\) is linearly independent of \((w_i(A_1, \ldots, A_n))_{i \in [m_{\text{new}}]}\). As a consequence, Algorithm 2 returns the same words \((w_j)_{j \in [m]}\).

For the final claim, recall that the while loop terminates after \(L + 1\) iterations. To estimate this number of iterations, let \(m_l\) denote the dimension of \(\text{span}(w_i(A_1, \ldots, A_n))_{i \in [m_{\text{new}}]}\) after the \(l\)th iteration of the while loop, i.e., \(m_l = m_{\text{new}}\). Then

\[
1 = m_0 < m_1 < \cdots < m_L = m_{L+1} = m.
\]

It follows that \(L < m\), and so the while loop terminates after at most \(m \leq k^2\) iterations, as claimed. One may implement each iteration of the while loop by first multiplying every matrix \(A_i\) by every matrix \(w_j(A_1, \ldots, A_n)\), costing \(nm_{\text{old}} \cdot O(k^3) = O(nk^3)\) operations, then vectorizing the matrices \((w_i(A_1, \ldots, A_n))_{i \in [m_{\text{new}}]}\) and the \(nm_{\text{old}}\) matrix products to form the columns of a \(k^2 \times (m_{\text{old}} + nm_{\text{old}})\) matrix, computing the row echelon form of this matrix in \(O(k^4(m_{\text{old}} + nm_{\text{old}})) = O(mnk^4)\) operations, and then finally using the pivot columns of the result to decide which words to add to \(\{w_j\}\).

Matlab implementations of Algorithm 2 and Lemma 11 may be downloaded from [44].

While the per-iteration cost of Algorithm 2 scales poorly with \(k\), we will find that this cost can sometimes be improved dramatically. At the moment, the main takeaway should be that Algorithm 2 always returns the desired basis in polynomial time.

### 3.3.1 Projection algebras

Taking inspiration from [30], and in light of Lemma 11, there is a natural choice of invariant to determine tuples of subspaces up to isometric isomorphism.

**Theorem 13.** There exists an injective invariant for \((\text{Gr}(r, F^d))^n\) modulo \(U(d, F, \sigma)\) that, given a tuple of orthogonal projection matrices, can be computed in \(O(nd^8 + r^2d^8)\) operations.

**Proof.** Let \(A_i\) denote the orthogonal projection onto the \(i\)th subspace, run Algorithm 2 to determine words \((w_j)_{j \in [m]}\) that produce a basis \((E_j)_{j \in [m]}\) for the algebra \(\mathcal{A}'(\{A_i\})_{i \in [n]}\), and then compute the traces prescribed in Lemma 11(ii)–(iv). By Lemmas 11 and 12, the words \((w_j)_{j \in [m]}\) together with the traces (ii)–(iv) form an injective invariant for \((\text{Gr}(r, F^d))^n\) modulo \(U(d, F, \sigma)\). Since Algorithm 2 ensures that \(E_1 = I\), then the traces in (ii) are already captured by the traces in (iii).

To compute these traces, it is helpful to perform some preprocessing. For each projection \(A_i\), we find a decomposition of the form \(A_i = T_i T_i^*\) with \(T_i \in F^{d \times r}\) in \(O(rd^2)\) operations. (It
suffices to draw Gaussian vectors \((g_j)_{j \in [r]}\) in \(O(rd)\) operations, then compute \((A_i g_j)_{j \in [r]}\) in \(O(rd^2)\) operations, then perform Gram–Schmidt in \(O(dr^2)\) operations.) Every trace that we need to compute can be expressed as the trace of a product of \(A_i\)'s. We will apply the cyclic property of the trace and compute matrix–vector products whenever possible. For example, the trace of \(A_1 A_2\) is given by

\[
\text{tr}(A_1 A_2) = \text{tr}(T_1 T_1^* T_2 T_2^*) = \text{tr}(T_1^* T_2 T_2^* T_1) = \sum_{j \in [r]} e_j^* T_1^* T_2 T_2^* T_1 e_j,
\]

where \((e_j)_{j \in [r]}\) denotes the identity basis in \(F^r\). We compute the right-hand side by first computing \(T_1 e_j\) in \(O(rd)\) operations, then \(T_2^* (T_1 e_j)\) in \(O(rd)\) operations, etc. In our case, each word has length at most \(m\), and so each term of the above sum can be computed in \(O(rd m)\) operations.

Overall, we compute the words in \(O(m^2 nd)\) operations (by Lemma \[12\]), then we compute \((T_i)_{i \in [n]}\) in \(O(nr d^2)\) operations, and then each of the \(m^3\) traces in (iii) and each of the \(mn\) traces in (iv) costs \(O(r^2 dm)\) operations. In total, this invariant costs \(O(m^2 nd^4 + nr d^2 + m^4 r^2 d + m^2 nr^2 d)\) operations. Since \(m \leq d^2\), this operation count is \(O(nd^8 + r^2 d^9)\).

While this invariant can be computed in polynomial time, the runtime is sensitive to the ambient dimension \(d\).

### 3.3.2 Quivers and cross Gramian algebras

Consider any sequence \((A_i)_{i \in [n]}\), where each \(A_i\) is an isometric embedding of some \(r\)-dimensional vector space \(V_i\) over \(F\) into \(F^d\). That is, \(A_i : V_i \rightarrow F^d\) and \((\text{im } A_i)_{i \in [n]} \in (\text{Gr}(r, F^d))^n\). For every \((i, j) \in [n]^2\), we then have a mapping \(A_i^* A_j : V_j \rightarrow V_i\). Together, \((V_i)_{i \in [n]}, (A_i^* A_j)_{i, j \in [n]}\) forms a representation of a so-called quiver \(Q = (Q_0, Q_1, s, t)\) defined by \(Q_0 = [n], Q_1 = [n]^2, s : (i, j) \mapsto j,\) and \(t : (i, j) \mapsto i\). The corresponding quiver algebra \(F Q\) enjoys a representation over \(V := \bigoplus_{i \in [n]} V_i\) with maps

\[
f_{ij} : V \xrightarrow{\pi_j} V_j \xrightarrow{A_i^* A_j} V_i \xleftarrow{\pi_i^*} V,
\]

where \(\pi_i\) denotes the coordinate projection from \(V\) to \(V_i\). As we will see, these endomorphisms over \(V\) generate an \(H^*\)-algebra that provides more efficient invariants.

**Theorem 14.** There exists an injective invariant for \((\text{Gr}(r, F^d))^n\) modulo \(U(d, F, \sigma)\) that, given a Gramian of orthobases of subspaces, can be computed in \(O(r^8 n^5 + r^6 n^3)\) operations.

**Proof.** Denote the subspaces by \((\text{im } A_i)_{i \in [n]}\), where each \(A_i \in F^{d \times r}\) has orthonormal columns, and put \(A = [A_1 \cdots A_n]\). By assumption, we are given the Gramian \(A^* A\). Letting \(\Pi_i\) denote the \(rn \times rn\) orthogonal projection matrix onto the \(i\)th block of \(r\) coordinates in \(F^{rn}\), then the matrix representation of \(f_{ij}\) is \(A_{ij} := \Pi_i A^* A \Pi_j\).

Given \((A_{ij})_{i, j \in [n]}\) and \((B_{ij})_{i, j \in [n]}\) of this form, suppose there exists \(U \in U(rn, F, \sigma)\) such that \(UA_{ij} U^* = B_{ij}\) for every \(i, j \in [n]\). Then since \(A_{ii} = B_{ii} = \Pi_i\), it holds that \(U\) is necessarily block diagonal. Furthermore,

\[
UA^* AU^* = U \left( \sum_{ij} \Pi_i A^* A \Pi_j \right) U^* = \sum_{ij} UA_{ij} U^* = \sum_{ij} B_{ij} = B^* B.
\]

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As such, unitary equivalence between \((A_{ij})_{i,j \in [n]}\) and \((B_{ij})_{i,j \in [n]}\) implies block unitary equivalence between the orthobasis Gramians \(A^*A\) and \(B^*B\). The implication also goes in the other direction: Given a block diagonal \(U \in U(rn, F, \sigma)\) such that \(UA^*AU^* = B^*B\), then

\[
UA_{ij}U^* = U\Pi_i A^* \Pi_j U^* = \Pi_i U A^* A U^* \Pi_j = \Pi_i B^* B \Pi_j = B_{ij}.
\]

It remains to test whether there exists \(U \in U(rn, F, \sigma)\) such that \(UA_{ij}U^* = B_{ij}\) for every \(i,j \in [n]\), which leads us to consider Lemma \([11]\)

Note that \(A_{ii}^* = A_{jj}\) and similarly for \(B\), and so \((A_{ij})_{i,j \in [n]}\) and \((B_{ij})_{i,j \in [n]}\) satisfy the hypothesis of Lemma \([11]\) with \(\pi(i,j) = (j,i)\). Since \(\sum_i A_{ii} = I\), we may run a version of Algorithm \([2]\) that instead initializes with all words \(x_{ij}\) of length 1 whose evaluations \(A_{ij}\) are nonzero; these evaluations are linearly independent since they have disjoint support. Since all words of positive length evaluate as a matrix in \(F^{rn \times rn}\) that is supported on some \(r \times r\) block, it follows that the resulting basis can be indexed as \((E_{ijk})_{i,j \in [n], k \in [m_{ij}]\}}\), where \(E_{ijk}\) is the \(k\)th basis element that is supported in the \((i,j)\)th \(r \times r\) block. For example, it holds that \(E_{ij1} = A_{ij}\) whenever \(A_{ij} \neq 0\).

To see how efficient this choice of invariants is, we first describe how to reduce the per-iteration cost of Algorithm \([2]\) to \(O(r^6n^3)\). First, we take all products between \(A_{ij}\)’s and evaluations of existing words. For each word, there are at most \(n\) different \(A_{ij}\)’s that will produce a nonzero product, and so the total number of products is at most \(r^2n^3\), each costing \(O(r^3)\) operations. Next, the evaluations of existing words and the resulting products can be partitioned according to their support before testing linear independence. For each \(i,j \in [n]\), the total number of these matrices that are supported on the \((i,j)\)th \(r \times r\) block is at most \(r^2 + nr^2\) (at most \(r^2\) from the existing words, and at most \(nr^2\) from the resulting products), and it costs \(O(r^6n)\) operations to compute the corresponding row echelon form. We perform this for each of the \(n^2\) blocks to identify new words to add. All together, the per-iteration cost is \(O(r^2n^3 \cdot r^3 + n^2 \cdot r^6n) = O(r^6n^3)\). Our bound on the total number of iterations is \(r^2n^2\), meaning we obtain the desired words after \(O(r^8n^5)\) operations.

Next, we point out the complexity of computing the traces (ii)–(iv). First, since \(E_{iii} E_{ijk} = E_{ijk}\), the traces in (ii) are examples of traces in (iii). Next, for every \((i,j) \in [n]\), we either have \(E_{iji} = A_{ij}\) or \(x_{ij}\) is not one of the words in \((w_{ij})_{j \in [m_i]}\). As such, the traces in (iv) are captured by both the words and the traces in (iii). Of the traces in (iii), the only ones that are possibly nonzero take the form

\[
\text{tr}(E_{jia}^* E_{jkb} E_{kic})
\]

for some \(i,j,k \in [n], a \in [m_{ji}], b \in [m_{jk}]\) and \(c \in [m_{ki}]\). Since \(m_{ij} \leq r^2\) for every \(i,j \in [n]\), we therefore have at total of at most \(n^3r^6\) traces to compute, each costing \(O(r^3)\) operations. These \(O(r^9n^3)\) operations contribute to the total of \(O(r^8n^5 + r^9n^3)\) operations it takes to compute this invariant.

Interestingly, the cross Gramian algebra introduced in the above proof can be used to obtain a new (short) proof of Proposition \([4]\).

**Proof of Proposition \([4]\)** Consider \(A = [a_1 \cdots a_n]\), where each \(a_i\) is a unit vector spanning the corresponding line in \(\mathcal{L}\). Then the cross Gramian algebra is generated by \(A_{ij} = \langle a_i, a_j\rangle e_i e_j^*\). Observe that every product of these matrices is either 0 or a multiple of \(e_i e_j^*\) for some
Given a maximal spanning forest $F$ of the frame graph $G(\mathcal{L})$, we select the following words in noncommuting variables $(x_{ij})_{i,j \in [n]}$: For each $(i, j) \in [n]^2$ such that $i$ and $j$ belong to a common component of $G(\mathcal{L})$, select the unique directed path in $F$ from $j$ to $i$ with vertices denoted by $j = i_0 \to i_1 \to \cdots \to i_l = i$, and then put

$$w_{ij} := x_{i_0,i_{i_1-1}}x_{i_1,i_{i_2-1}} \cdots x_{i_{i_l-1},i_l}.$$ 

In particular, $w_{ii} = x_{ii}$ for every $i \in [n]$. Then the evaluation of $w_{ij}$ is a nonzero multiple of $e_i e_j^*$, and all of these evaluations together form a basis for the algebra.

Now consider the traces in Lemma 11(ii)–(iv). Every trace in (ii) and (iii) is either 0 or nonzero only if the corresponding directed paths form a closed walk along the edges of $F$, in which case each edge of $F$ is traversed as many times in one direction as it is in the other direction. Meanwhile, a trace in (iv) is nonzero only if the corresponding directed paths form a closed walk comprised of a directed path in $F$ and a directed edge in $G(\mathcal{L})$. If the path has length 1, then the result is a 2-product, and otherwise the result is an $m$-product corresponding to a cycle in $C(F)$.

4 Isomorphism up to linear automorphism

In this section, we test isomorphism between tuples of subspaces modulo the action of $GL(d, F)$. In the previous section, we devised injective invariants to test for isomorphism; that is, given a representation $A$ of some tuple of subspaces, we compute the invariant $f(A)$, and since the invariant is injective modulo some group action, we know that $f(A) = f(B)$ if and only if $A$ and $B$ represent the same tuple of subspaces modulo that group action. In this section, we instead find a function $g$ such that $g(A, B)$ determines whether $A$ and $B$ represent the same tuple of subspaces modulo linear automorphism. To accomplish this, let $A, B \in F^{d \times r \times n}$ be such that $A = [A_1 \cdots A_n]$ represents the tuple of $r$-dimensional subspaces $(\text{im} A_i)_{i \in [n]}$, and similarly for $B$. Then $X \in GL(d, F)$ satisfies $X \cdot \text{im} A_i = \text{im} B_i$ for every $i \in [n]$ if and only if for every $i \in [n]$, there exists $Y_i \in GL(r, F)$ such that $X A_i = B_i Y_i$. For a fixed $A$ and $B$, this suggests a homogeneous linear system with variables $(X, Y_1, \ldots, Y_n)$. It turns out that testing for isomorphism reduces to solving this linear system provided $A$ is trivially stabilized:

**Definition 15.** We say that $(A_i)_{i \in [n]}$ in $F^{d \times r}$ is **trivially stabilized** if each $A_i$ has rank $r$ and if, under the action of $GL(d, F)$, the intersection of the stabilizers of the points $(\text{im} A_i)_{i \in [n]}$ equals $\{a I_d : a \in F^x\}$.

**Lemma 16.** Given trivially stabilized $(A_i)_{i \in [n]}$ in $F^{d \times r}$ and any $(B_i)_{i \in [n]}$ in $F^{d \times r}$, there exists $X_0 \in GL(d, F)$ such that $X_0 \cdot \text{im} A_i = \text{im} B_i$ for every $i \in [n]$ if and only if the solution set to the homogeneous linear system

$$(X, Y_1, \ldots, Y_n) \in F^{d \times d} \times (F^{r \times r})^n, \quad X A_i - B_i Y_i = 0, \quad i \in [n]$$

is 1-dimensional and the $X$-coordinate of some nonzero solution resides in $GL(d, F)$. 

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Proof. First, \((\Leftarrow)\) follows immediately from taking \(X_0\) to be the \(X\)-coordinate of the assumed nonzero solution. Indeed, this choice ensures that \(B_iY_i = X_0A_i\) has rank \(r\), meaning \(Y_i \in \text{GL}(r,F)\). For \((\Rightarrow)\), note that by assumption, there exist \(\tilde{Y}_1, \ldots, \tilde{Y}_n \in \text{GL}(r,F)\) such that \(B_i = X_0A_i\tilde{Y}_i^{-1}\) for every \(i \in [n]\). Then \(XA_i = BY_i\) if and only if \(XA_i = X_0A_i\tilde{Y}_i^{-1}Y_i\), if and only if \(X_0^{-1}XA_i = A_i\tilde{Y}_i^{-1}Y_i\). That is, \((X, Y_1, \ldots, Y_n)\) is a solution to \(XA_i = B_iY_i\) precisely when \((X_0^{-1}X, \tilde{Y}_1^{-1}Y_1, \ldots, \tilde{Y}_n^{-1}Y_n)\) is a solution to \(UA_i = A_iV_i\). Since this transformation is invertible, the dimensions of these solution sets are identical. By assumption, \((A_i)_{i \in [n]}\) is trivially stabilized, and so the solutions \((U, V_1, \ldots, V_n)\) in \(\text{GL}(d,F) \times (\text{GL}(r,F))^n\) take the form \(aI_d, aI_r, \ldots, aI_r\) for some \(a \in F^\times\). As such, the solution set has dimension at least 1. If the dimension were larger, then there would exist another nonzero solution \((U, V_1, \ldots, V_n)\), but since \((I_d + \epsilon U, I_r + \epsilon V_1, \ldots, I_r + \epsilon V_n)\) resides in \(\text{GL}(d,F) \times (\text{GL}(r,F))^n\) for sufficiently small \(\epsilon > 0\), this violates the hypothesis that \((A_i)_{i \in [n]}\) is trivially stabilized. It follows that the solution set has dimension exactly 1. Furthermore, the \(U\)-coordinate of any nonzero solution takes the form \(aI_d\) for some \(a \in F^\times\), and so the corresponding the \(X\)-coordinate is \(aX_0 \in \text{GL}(d,F)\).

Overall, we can solve isomorphism up to linear automorphism for subspaces that are trivially stabilized. In what follows, we study trivially stabilized subspaces in the special case where \(F = C\) and \(r\) divides \(d\). Let \(n^*(r,d)\) denote the smallest \(n\) for which \(n\) generic points in \(\text{Gr}(r, C^d)\) are trivially stabilized. Then we have the following:

**Theorem 17.** Suppose \(r < d\) and \(r\) divides \(d\). Then

\[
n^*(r,d) = \begin{cases} 
  d/r + 1 & \text{if } r = 1 \\
  d/r + 2 & \text{if } r > 1 \text{ and } d/r > 2 \\
  d/r + 3 & \text{if } r > 1 \text{ and } d/r = 2.
\end{cases}
\]

**Proof.** We will iteratively make use of the observation that, for any \(X, Y \in \text{GL}(d, C)\), it holds that

\[Y \cdot (X^{-1} \cdot \text{im } A_i) = X^{-1} \cdot \text{im } A_i\]

if and only if \((XYX^{-1}) \cdot \text{im } A_i = \text{im } A_i\), meaning \((\text{im } A_i)_{i \in [n]}\) is trivially stabilized if and only if \((X^{-1} \cdot \text{im } A_i)_{i \in [n]}\) is trivially stabilized.

Suppose \(n \geq d/r\). Then generically, \(A_1, \ldots, A_n \in C^{d \times r}\) has the property that \(A := [A_1 \cdots A_{d/r}] \in \text{GL}(d, C)\). As such, we may assume that \(A^{-1}\) exists. For every \(i \in [d/r]\), it holds that \(A_i^{-1}A_i = I_r\) zero except for its \(i\)th \(r \times r\) block, which equals \(I_r\). As such, for each \(i \in [d/r]\), the stabilizer of \(A_i^{-1}A_i\) is \(\text{GL}(r, C) \times \text{GL}(d-r, C)\), where \(\text{GL}(r, C)\) acts on these \(r\) coordinates and \(\text{GL}(d-r, C)\) acts on the other \(d-r\) coordinates. The intersection of these stabilizers is \((\text{GL}(r, C))^{d/r}\), with each \(\text{GL}(r, C)\) acting on a different batch of \(r\) coordinates. Since this intersection is nontrivial, we have \(n^*(r, d) > d/r\).

Now suppose \(n \geq d/r + 1\). Then generically, \(A_{d/r+1}\) has the property that each \(r \times r\) block \(B_i\) of \(A^{-1}A_{d/r+1}\) resides in \(\text{GL}(r, C)\). As such, taking \(B = \text{diag}(B_1, \ldots, B_{d/r})\), then \((B^{-1} \cdot (A^{-1} \cdot \text{im } A_i))_{i \in [d/r]} = (A^{-1} \cdot \text{im } A_i)_{i \in [d/r]}\). Furthermore, \(B^{-1}A^{-1}A_{d/r+1}\) consists of \(d/r\) blocks that all equal \(I_r\), and so the only members of \((\text{GL}(r, C))^{d/r}\) that reside in the stabilizer of \(\text{im } B^{-1}A^{-1}A_{d/r+1}\) take the form \(\text{diag}(X, \ldots, X)\) for \(X \in \text{GL}(r, C)\). Notably, this stabilizer is trivial when \(r = 1\), meaning \(n^*(r, d) = d/r + 1\) in this case. It remains to treat the case where \(r > 1\).
Now suppose \( r > 1 \) and \( n \geq d/r + 2 \). Then generically, \( B^{-1}A^{-1}A_{d/r+2} = [C_1; C_2; \ldots; C_{d/r}] \) has the property that \( C_1 \in \text{GL}(r, \mathbb{C}) \), and so \( \text{im } B^{-1}A^{-1}A_{d/r+2} = \text{im}[I_r; C_2C_1^{-1}; \ldots; C_{d/r}C_1^{-1}] \).

Next, \( C_2C_1^{-1} \) generically has \( r \) distinct eigenvalues, meaning we can write \( C_2C_1^{-1} = SDS^{-1} \) for some \( S \in \text{GL}(r, \mathbb{C}) \) and diagonal \( D \). (This is where we use the fact that \( \mathbf{F} = \mathbb{C} \).) Then

\[
\text{im } B^{-1}A^{-1}A_{d/r+2} = \text{im}[S; SD; C_3C_1^{-1}S; \ldots; C_{d/r}C_1^{-1}S].
\]

Taking \( T = \text{diag}(S, \ldots, S) \), then \( (T^{-1}(B^{-1}A^{-1}\cdot \text{im } A_i))_{i \in [d/r+1]} = (B^{-1}A^{-1}\cdot \text{im } A_i)_{i \in [d/r+1]} \), whereas \( \text{im } T^{-1}B^{-1}A^{-1}A_{d/r+2} = \text{im}[I_r; D; S^{-1}C_3C_1^{-1}S; \ldots; S^{-1}C_{d/r}C_1^{-1}S] \). Notice from the first two blocks that \( \text{diag}(X, \ldots, X) \) is in the stabilizer of \( \text{im } T^{-1}B^{-1}A^{-1}A_{d/r+2} \) only if \( X \) is diagonal. At this point, we treat the case in which \( d/r > 2 \). Since generically, it holds that an eigenvector of \( C_2C_1^{-1} \) is not simultaneously an eigenvector of \( C_3C_1^{-1} \), we know that \( S^{-1}C_3C_1^{-1}S \) is not diagonal. As such, if \( X \) is diagonal and \( XS^{-1}C_3C_1^{-1}SX^{-1} = S^{-1}C_3C_1^{-1}S \), then it must hold that \( X = aI_r \) for some \( a \in \mathbb{C}^\times \). Overall, \( n^*(r, d) = d/r + 2 \) if \( r > 1 \) and \( d/r > 2 \), whereas \( n^*(r, d) > d/r + 2 \) if \( r > 1 \) and \( d/r = 2 \).

Finally, suppose \( r > 1 \), \( d/r = 2 \), and \( n = d/r + 3 \). Then generically, we may write

\[
\text{im } T^{-1}B^{-1}A^{-1}A_{d/r+3} = \text{im}[I; E]
\]

such that \( E \) is not diagonal, and so the only diagonal \( X \) such that \( XEX^{-1} = E \) has the form \( aI_r \) for some \( a \in \mathbb{C}^\times \), implying the result.

\[ \square \]

## 5 Discussion

This paper studied the problem of testing isomorphism between tuples of subspaces with respect to various notions of isomorphism. Several open problems remain:

- Is there a canonical choice of Gramian for equi-isoclinic subspaces of dimension \( r > 2 \)? What about the complex case?

- How many (generalized) Bargmann invariants are required to solve isomorphism up to linear isometry?

- What is \( n^*(r, d) \) when \( r \) does not divide \( d \)? What about when \( \mathbf{F} = \mathbb{R} \)?

- Can one solve isomorphism up to linear automorphism for all tuples of subspaces?

- How should one compute the symmetry group of a given tuple of subspaces?

Some of the ideas in the paper may have interesting applications elsewhere. For example, there has been a lot of work to develop symmetric arrangements of points in the Grassmannian \([65, 11, 8, 68, 64, 10, 37, 38, 39, 48, 7, 45, 40, 29, 19]\). What are the projection and cross Gramian algebras of these arrangements? It would also be interesting to see if some of the techniques presented in this paper could be used to treat other emerging problems involving invariants to group actions, e.g. \([2, 9]\).
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