Group Steering: Approaches Based on Power Moments

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Abstract

This paper considers the problem of steering a vast group of agents of which the dynamics are governed by a discrete-time first-order linear system. The group of agents are characterized as a probability density function and an occupation measure respectively in the paper and two corresponding treatments are given. We propose to use the power moments to characterize the density function/occupation measure of the agents. A moment system representation of the original system is put forward for control and an empirical control scheme corresponding to it is proposed. By the designed control law, the moment sequence of the control at each time step is positive, which ensures the existence of the control for the moment system. We then realize the control as an analytic form of function by a convex optimization scheme of which the existence and uniqueness of the solution have been proved in our previous paper. The terminal density is proved to converge to the desired terminal one, which distinguishes the proposed distribution steering scheme from other existing ones. An error analysis of the terminal density from the specified one is also provided. For the problem where the group of agents is characterized as an occupation measure, the control for each agent is determined by drawing independent and identically-distributed (i.i.d) samples from the realized analytic function. Finally we simulate both unconstrained and constrained controls of a vast group of agents, which validate our proposed algorithms.

Key words: Distribution steering; power moments; multiple agents.

1 Introduction

We are interested in the problem of steering the states of a vast group of agents, of which the dynamics are governed by a discrete-time stable first-order linear system, between an initial and a final probability densities or occupation measures without stochastic disturbance. We consider the linear dynamics of the \(i\)th agent

\[ x_i(k+1) = a(k)x_i(k) + u_i(k), \quad i = 1, \cdots, N \]

Since the system is stable and we assume \(a(k)\) to be positive, we have \(a(k) \in (0,1)\). The control input on the \(i\)th agent is denoted as \(u_i\), and \(x_i\) is its state. We assume that the agents are non-interactive and the volume of the agents is ignored. It means that the agents are allowed to occupy the same state and the collisions are ignored.

This poses a significant challenge both in theory and practice. From a theoretical standpoint, the challenge involves controlling numerous agents, distinguishing it from conventional control problems where only one object is typically controlled, and feedback control laws are developed based on the state of that single object. However, applying such principles to a vast group of agents becomes highly complex. In this scenario, the goal isn’t to manipulate the state of each individual agent to a predefined state, which is not only unnecessary but also computationally expensive. Instead, the objective is to control the entire group, ensuring it possesses specific global properties. As the number of agents approaches infinity, the problem becomes infinite-dimensional, making it intractable without dimension reduction or approximation. This intrinsic infinite-dimensionality adds depth and interest to the problem, presenting it as an open challenge. As Brockett asserted in [6], important limitations standing in the way of the wider use of optimal control can be circumvented by explicitly acknowledging that in most situations the apparatus implementing the control policy will be judged on its ability to cope with a distribution of initial states, rather than a single state. In practical terms, there are numerous scenarios demanding the control of a group of agents. These applications extend to steering swarms (such as UAVs or large collections of microsatellites), modeling the flow...
and collective motion of agents, and other ensemble control situations.

Numerous studies have addressed the challenge of steering groups of agents, leading to diverse findings. One line of research formulates the issue as stabilizing a discrete-time Markov process evolving within a compact subset of \( \mathbb{R}^d \) towards a target density [3, 4, 13]. These contributions are profound and thought-provoking. However, there still exist several disadvantages. The results primarily focus on stabilizing the system and lack the capability to steer the initial density to an arbitrary distribution in a specified limited number of steps. Additionally, due to the discretization of the domain, it is not quite feasible to propose a quantitative analysis of the difference between the terminal distribution obtained by the proposed algorithms and the desired one.

In prior research, a notable exploration revolves around the assumption that the distribution follows a Gaussian pattern [1, 2, 17–21, 31]. Specifically, the goal here is to manipulate the first two moments of Gaussian distributions to converge towards predetermined targets. However, these Gaussian methodologies primarily originate from the context of quantifying control uncertainty, where the distributions serve to characterize the uncertainty inherent in the system state. Nevertheless, in our treatment of the group steering problem, assuming the agents to be distributed according to a Gaussian pattern is infeasible. Such an assumption would fail to adequately represent the complexity of non-Gaussian scenarios, significantly constraining the efficacy of the proposed group steering algorithm in practical applications.

Considerable advancements in steering continuous-time systems have been made, notably by Chen, Georgiou, and Pavon, who introduced fundamental results using the Schrödinger Bridge strategy for Gaussian distributions [9, 10] and extended their findings to other types of distributions [11]. These results have further been expanded to nonlinear continuous-time systems with hard state constraints [8]. Additionally, a robust optimal density control strategy for robotic swarms is proposed in [24]. These contributions collectively enrich our understanding of steering problems across different system dynamics and constraints.

When the density of the agents is assumed to be Gaussian, we use the first and second order moments to characterize the density, which turns the problem into a finite-dimensional one. However, by generalizing the mean and covariance to all the power moments, we will have a more conceptual view of this problem. Controlling the system state as a distribution function, if only assumed to be Lebesgue integrable, is an uncountably infinite-dimensional problem. By probability theory, we note that a distribution function can be uniquely determined by its full power moment sequence [12, 32]. By controlling the full power moment sequence instead of the distribution of system state, the problem is reduced to a countably infinite-dimensional one. By properly truncating the first several terms of the power moment sequence for characterizing the density of the system state, the problem is now steering a truncated power moment sequence to another, which is finite-dimensional and tractable. Decomposing the distribution by the power moments provides a completely new perspective to the challenge proposed by Brockett in [6].

In this paper we provide what can be regarded as the first computable and implementable solution to the distribution steering problem of the discrete-time linear system within limited steps by power moments, where the specified initial and terminal distributions, including probability densities and occupation measures of the agents, are arbitrary (only assuming the existence of first several orders of power moments). The paper is organized as follows. In Section 2, we propose a moment system representation as a counterpart of the discrete-time linear system. Subsequently, we introduce a problem definition for group steering based on the moment system. Unlike conventional control problems, the positivity of Hankel matrices for both control inputs and system states’ moments is required, posing challenges for standard approaches such as optimal control. In the context of general distribution steering, as considered in [25], the system trajectory is predetermined. We propose an empirical scheme for determining the system trajectory. Additionally, we employ a density parametrization algorithm from our prior work [29] to realize control inputs as analytic functions using power moments. The results mentioned above were presented at the IFAC World Congress 2023 [28]. This paper delves into more comprehensive quantitative analyses of the proposed density steering problem, and extends its applicability to steering vast groups of agents in real-world scenarios. With the number of moment terms used approaching infinity, we propose that the terminal distribution obtained by the proposed steering scheme converges almost everywhere to the desired one. The density steering problem is intrinsically infinite-dimensional, where densities are not assumed to adhere to specific functions. Since the number of power moments we utilize is finite, the presence of errors in the terminal density is inevitable. Consequently, we propose a tight upper bound for the error of the terminal density relative to the specified one. Building upon the density-steering algorithm outlined in Section 2, we introduce an algorithm for steering a finite number of agents characterized as an occupation measure. Each agent’s control input is determined by drawing independent and identically distributed (i.i.d.) samples from the realized control functions, facilitated by an acceptance-rejection sampling strategy. Finally, we provide six examples to validate the effectiveness of the two proposed steering algorithms. The examples encompass densities such as Gaussian distributions, mixtures of Gaussians, mixtures of generalized logistic distributions, and mixtures of Laplacians.
Steer the group of agents as a probability density function

We first consider the problem of steering an initial density function to a terminal one. Then the steering of arbitrary occupation measures will be treated in the following section, of which there has not been a solution in the previous papers.

2.1 A moment system representation

With a vast group of agents, i.e., $N$ is large, a conventional approach is to approximate $x(k)$ and $u(k)$ as random variables for which the density functions are denoted as $q_k$ and $p_k$. It is a mean-field approximation of the state evolution of the agents. The problem of steering the group of agents is then turned to steering an initial density function to a terminal one. The density control problem is formulated as follows.

The dynamics of the system is

$$x(k + 1) = a(k)x(k) + u(k).$$

(2)

Given an initial probability density function $q_0(x)$ of the random variable $x(0)$, determine the control sequence, i.e., $(u(0), \ldots, u(K - 1))$ such that the terminal density function is $\tau(x)$ for $x(K)$.

However it is not always feasible for us to obtain a closed form of solution to this problem. If the distributions are not assumed to fall within specific classes, the problem is intrinsically infinite-dimensional. We note that the density function of $x(k + 1)$ can be written as

$$q_{k+1}(t) = \int_{\mathbb{R}} f_k (\xi, t - \xi) \, d\xi$$

$$= \int_{\mathbb{R}} q_k \left( \frac{\xi}{a(k)} \right) p_k \left( t - \xi \right) \, d\xi$$

(3)

$$\mathbb{E} \left[ x^l(k) \right] = \mathbb{E} \left[ x^l(k) \right] = \mathbb{E} \left[ x^l(k) \right] \mathbb{E} \left[ x^l(k) \right]$$

The new input vector is written as

$$\mathbb{E} \left[ u^l(k) \right] = \mathbb{E} \left[ u^l(k) \right] \mathbb{E} \left[ u^l(k) \right] \mathbb{E} \left[ u^l(k) \right]$$

(5)

where we have denoted

$$\mathbb{E} \left[ x^l(k) \right] = \int_{\mathbb{R}} x^l q_k(x) \, dx,$$

$$\mathbb{E} \left[ x^l(k) u^l(k) \right] = \int_{\mathbb{R}} x^l q_k(x) \, dx \int_{\mathbb{C}} u^l p_k(u) \, du,$$

and

$$\mathbb{E} \left[ u^l(k) \right] = \int_{\mathbb{C}} u^l p_k(u) \, du.$$

(6)

The matrix $A(\mathbb{U}(k))$ of the new system can then be written as (9).

The sets $\mathbb{S}$ and $\mathbb{C}$ are the support of $q_k(x)$ and $p_k(u)$ correspondingly. For control problems without state constraints, $\mathbb{S}$ can be chosen as the real line $\mathbb{R}$. Or it can be

There is a similar problem in non-Gaussian Bayesian filtering. In our previous results [29], we proposed a method of using the power moments to treat this intractable problem, mainly for characterizing the macroscopic property of the distributions. However, even it is theoretically feasible to characterize the distribution of the agents by the full power moment sequence, the problem is infinite dimensional. A common treatment is to truncate the first $2n$ moment terms $[7, 14]$, which turns the problem we treat to a truncated moment problem.

By the system equation (2), the power moments of the states up to order $2n$ are written as

$$\mathbb{E} \left[ x^l(k) \right] = \sum_{j=0}^{l} \left( \frac{l}{j} \right) \mathbb{E} \left[ x^j(k) u^{l-j}(k) \right]$$

for $l \in \mathbb{N}_0$ ($\mathbb{N}_0$ denotes the set of all nonnegative integers), $l \leq 2n$. It is difficult to treat the term $\mathbb{E} \left[ x^j(k) u^{l-j}(k) \right]$. However, we note that if $x(k)$ and $u(k)$ are independent from each other, i.e., $\mathbb{E} \left[ x^j(k) u^{l-j}(k) \right] = \mathbb{E} \left[ x^j(k) \right] \mathbb{E} \left[ u^{l-j}(k) \right]$, the dynamics of the moments can be written in a linear matrix equation

$$\mathcal{X}(k + 1) = \mathcal{A}(\mathbb{U}(k))\mathcal{X}(k) + \mathbb{U}(k)$$

(4)

where the new state vector is composed of the power moment terms up to order $2n$,
chosen as a compact subset of $\mathbb{R}$ according to its bounds. Similarly, $C$ can be chosen as $\mathbb{R}$ if the control input is unconstrained, or a compact subset of $\mathbb{R}$ if constrained.

By employing truncated power moments to describe the dynamics of the system (1), where $x(k)$ and $u(k)$ represent probability densities, we aim to reformulate the control problem as the manipulation of the power moments of these probability densities. The system (4) is denoted as the moment system associated with (1).

By the proposed moment system, the original density steering problem can be reduced to distribution steering by power moments, which is formulated as follows. The dynamics of the moment system is

$$X(k+1) = \mathcal{A}(U(k))X(k) + U(k)$$

where $X(k), U(k)$ are obtained by (7), (8). Given an arbitrary initial density $q_0(x)$, determine the control sequence $(u(0), \ldots, u(K-1))$ such that the first $2n$ order power moments of the terminal density are identical to those of an arbitrarily specified one, i.e.,

$$\int_{\mathbb{R}} x^l \tau(x) dx = \int_{\mathbb{R}} x^j q_K(x) dx$$

for $l = 1, \ldots, 2n$, where $\tau$ is the specified terminal density function.

Remark: The advantage of the proposed problem is twofold. Firstly, with this formulation, the problem becomes finite-dimensional, allowing for a closed-form solution. Secondly, it eliminates the need for the initial and terminal density functions of the agents to adhere to specific classes, enhancing the algorithm’s applicability across a diverse array of real-world scenarios.

However, when using the moment system for control, there is still a need to design control laws that satisfy

$$E \left[ x^j(k) u^{l-j}(k) \right] = E \left[ x^j(k) \right] E \left[ u^{l-j}(k) \right].$$

(11)

With a limited number of agents, it might be possible to propose feedback control laws for each agent, i.e., the control input is a function of the states of all the agents. However with a large number of agents, it is no longer tractable. In our density steering problem, $N$ is infinity (for the problem of steering occupation measures, $N$ is still a large number). A feasible control law needs to satisfy that the control vector is independent of the current state vector. In the next part of section, we propose an algorithm for steering power moments to desired ones.

2.2 An empirical control scheme for the moment system

In the previous section, controlling the group of agents has been reduced to controlling the moment system corresponding to it. Then the task is now to figure out an algorithm to determine a sequence of $(U(0), \ldots, U(K-1))$. Nonetheless, there exist two primary distinctions from conventional control problems. Firstly, in this problem, the system matrix of the moment system becomes a function of the control vector. Secondly, the sequence of elements in the control vector, denoted as $U(k)$, must adhere to the condition that the corresponding Hankel matrix

$$H_{U(k)} = 
\begin{bmatrix}
E[u^0(k)] & E[u^1(k)] & \cdots & E[u^n(k)] \\
E[u^1(k)] & E[u^2(k)] & \cdots & E[u^{n+1}(k)] \\
\vdots & \vdots & \ddots & \vdots \\
E[u^n(k)] & E[u^{n+1}(k)] & \cdots & E[u^{2n}(k)]
\end{bmatrix}$$

is positive definite. Here $H_{U}$ denotes the Hankel matrix of the vector $U$. We define such a subspace of $\mathbb{R}^{2n}$ as $\mathbb{V}_{2n+}^2 := \{ U \in \mathbb{R}^{2n} | H_U(k) > 0 \}$.

In prior studies, the distribution steering problems have consistently employed optimal control strategies. However, such an approach is not entirely feasible in this specific problem. The challenge lies in the necessity to ensure that both $X(k)$ and $U(k)$ belong to $\mathbb{V}_{2n+}^2$. To the best of our knowledge, there has yet to be a viable result capable of addressing the optimal control problem while imposing the constraint that states and control inputs must reside within $\mathbb{V}_{2n+}^2$ - in other words, ensuring that the corresponding Hankel matrices are positive definite.

Now, we formulate the problem we are to treat in this part of section. Let $U$ be the feasible set of control sequences $U := (U(0), \ldots, U(K-1))$, which satisfies

$$\sum_{k=0}^{K-1} E \left[ |U(k)| U(k) \right] < \infty$$
and effects the terminal system state \( x(K) \) to be distributed satisfying (10). Then the family \( U \) represents admissible control inputs which achieve the desired moment transfer. Denote the error of the moments from the specified ones as

\[
e(k) = X_T - X(k),
\]

where the elements of \( X_T \) are the power moments corresponding to the specified terminal density function \( \tau(x) \).

Simultaneously confining both \( \mathcal{U}(k) \) and \( X(k) \) to belong to the set \( \mathbb{V}_{n+}^{2n} \) is not always feasible. However, it’s worth noting that a sub-optimal solution to the control problem can be attained by first determining the trajectory of the state and subsequently obtaining the control inputs corresponding to this trajectory. A similar approach was also utilized in [25] to address the general distribution steering problem. We first determine the trajectory of the states.

**Lemma 2.1.** Given

\[
e(k_0) = X_T - X(k_0) \in \mathbb{V}_{n+}^{2n},
\]

we have

\[
X(k) = X(k_0) + \sum_{i=k_0}^{k-1} \omega_i e(k_0) \in \mathbb{V}_{n+}^{2n} \tag{13}
\]

for \( k = k_0 + 1, \cdots, K \) where \( \omega_i \in \mathbb{R}_+ \) for \( i = k_0, \cdots, K-1 \) and \( \sum_{i=k_0}^{K-1} \omega_i = 1 \).

**Proof.** The proof is straightforward. Since \( X(k_0), e(k_0) \in \mathbb{V}_{n+}^{2n} \), we have \( H_{\mathcal{U}(k_0)} > 0, H_{e(k_0)} > 0 \). We note that the sum of positive definite matrices is still positive definite. Since \( \omega_i > 0 \), we have \( \omega_i e(k_0) \in \mathbb{V}_{n+}^{2n} \). Then \( X(k) \in \mathbb{V}_{n+}^{2n} \).

Now it remains to prove that there exists a time step \( k_0 \) at which \( X_T - X(k_0) \in \mathbb{V}_{n+}^{2n} \).

**Proposition 2.2.** There exists a time step \( k_0 \) which satisfies \( X_T - X(k_0) \in \mathbb{V}_{n+}^{2n} \), assuming that \( X(k), 0 \leq k \leq k_0 \) are uncontrolled moment states, i.e., \( u(k) = 0, 0 \leq k \leq k_0 \).

**Proof.** We write the Hankel matrix form of \( X_T - X(k) \) as (14). Since \( u(k) = 0, 0 \leq k \leq k_0 \), we obtain (15).

Now it remains to prove that \( \exists \omega, H_{X_T - X(k_0)} > 0 \) with \( u(k) = 0, 0 \leq k \leq k_0 \). By definition of the positive definiteness, it is equivalent to prove that each leading principal minor, the determinant of leading principal submatrix, is positive. Denote the \( n_{th} \)-order leading principal submatrix of \( H_{X_T - X(k)} \) as \( H_i(k) \), and the corresponding minor as \( \det(H_i(k)) \). We note that each \( \det(H_i(k)) \) is a polynomial of \( \prod_{i=0}^{K} \omega(i) \), of which the degree is even. Therefore, if there exists no real zero for all the \( \det(H_i(k)) \), all \( k \in \mathbb{N}_0 \) satisfies \( H_{X_T - X(k_0)} > 0 \). Now we consider the case that \( \det(H_i(k)) \) has at least a real zero in \((0,1)\). We note that \( \det(H_i(k_0)) > 0 \) with \( k_0 \to +\infty \). Let \( \bar{k}_i \) be the smallest integer greater than the largest zero of the polynomial \( \det(H_i(k)) \). By the continuity of \( \det(H_i(k)) \), we have that \( \det(H_i(k)) > 0, k \in (\bar{k}_i, +\infty) \). Therefore, we have \( H_{X_T - X(k_0)} > 0 \), which ensures the positiveness of all \( H_i(k_0) \) and completes the proof.

By Proposition 2.2, it is possible to choose a time step \( k_0 \) which satisfies \( X_T - X(k_0) \in \mathbb{V}_{n+}^{2n} \). We assume that the system is uncontrolled before \( k_0 \), i.e. \( u(k) = 0, k \leq k_0 \). From step \( k_0 \), we impose controls on the system. Lemma 2.1 has proved the positiveness of \( X(k), k = k_0, \cdots, K \). Therefore it remains to determine the parameters \( \omega_{k_0}, k, k_0, \cdots, K-1 \) and the corresponding control inputs \( \mathcal{U}(k) \).

It is a non-trivial problem. We give an empirical scheme to treat it. To obtain a relatively smooth transition of states, it is desired that the \( \omega_i \)'s are close to each other. It is usually feasible for us to choose

\[
\omega_{k_0} = \cdots = \omega_{K-1} = \frac{1}{K-k_0}
\]

We note that to determine the trajectory of the system state a priori has also been adopted in the literature. In [25], prespecified system states, which was called to form a reference trajectory, were used to determine the control inputs. After that the parameters \( \omega_i \)'s are determined, the control inputs of the moment system \( \mathcal{U}(i) \) for \( i = k_0, \cdots, K-1 \) can then be calculated by solving the equation (4), provided with \( X(k), k = k_0 + 1, \cdots, K \) calculated by (13).

However sometimes the control inputs \( \mathcal{U}(k) \in \mathbb{V}_{n+}^{2n} \) by choosing the \( \omega_i \)'s to be all equal. It usually happens when the specified initial/terminal density has several modes (peaks). If so, we can choose a larger \( \omega_{k_0}/\omega_{K-1} \).

In conclusion, we have proposed to use the moment system to control the vast group of agents characterized as a probability density function. And an empirical control law has been proposed which ensures the existence of \( u(k) \). However, it is only feasible for us to obtain the power moments of the control inputs \( u(k) \). We need to obtain the \( u(k) \) given its power moments, which we call realization of the control inputs.
either where \( \mathbb{R} \)

is supported on a compact interval within \( \mathbb{R} \). According to the type of the control problem, \( \mathcal{C} \) is either \( \mathbb{R} \) (unconstrained control), or a compact interval on \( \mathbb{R} \) (constrained control). We define

\[
B(u) = \begin{bmatrix} 1 & u & \cdots & u^{n-1} & u^n \end{bmatrix}^\top
\]

and the linear integral operator

\[
\Theta : p(u) \mapsto \Sigma = \int_{\mathcal{D}} B(u)p(u)B^\top(u)du,
\]

where \( p(u) \) is defined on the space \( \mathcal{D}_{2n} \). Moreover, let

\[
\Sigma = \begin{bmatrix} 1 & E[u] & \cdots & E[u^n] \\
E[u] & E[u^2] & \cdots & E[u^{n+1}] \\
\vdots & \vdots & \ddots & \vdots \\
E[u^n] & E[u^{n+1}] & \cdots & E[u^{2n}] \end{bmatrix}
\]

where \( E[u^i], i = 1, \cdots, 2n, \) are the entries of the designed control \( \mathbb{U} \). Moreover, since \( \mathcal{D}_{2n} \) is convex, then so is \( \text{range}(\Theta) = \Theta \mathcal{D}_{2n} \).

Let

\[
\mathcal{L}_+ := \{ \Lambda \in \text{range}(\Theta) \mid B(u)^\top \Lambda B(u) > 0, u \in \mathbb{R} \}.
\]

For any \( r \in \mathcal{D} \) and any positive definite \( \Sigma \), there exists a unique \( \hat{\rho} \in \mathcal{D}_{2n} \) that minimizes (16) subject to \( \Theta(\hat{\rho}) = \Sigma \). Specifically,

\[
\hat{\rho} = \frac{r}{B \hat{\Lambda} AB}
\]

where \( \hat{\Lambda} \) is the unique solution to the minimization problem

\[
J_r(\Lambda) := \text{tr}(\Lambda \Sigma) - \int_{\mathcal{D}} r(u) \log [B(u)^\top \Lambda B(u)] du
\]

over all \( \Lambda \in \mathcal{L}_+ \) [29]. Here, \( \text{tr}(\cdot) \) denotes the trace of the matrix.
Then the density estimation is formulated as a convex optimization problem. Unlike other methods of moments, the power moments of our proposed density estimate are exactly identical to those specified, which makes it a satisfactory approach for realization of the control inputs. Since the prior density \( r(u) \) and the density estimate \( \hat{p}(u) \) are both supported on \( \mathbb{R} \), \( r(u) \) can be chosen as a Gaussian distribution (or a Cauchy distribution if \( \hat{p}(u) \) is assumed to be heavy-tailed).

We note that for the Hausdorff moment problem, where the control \( u \) is supported on a compact subset of \( \mathbb{R} \), the proofs and results for the Hamburger moment problem are also valid by substituting the domain \( \mathbb{R} \) by a compact interval on the real line. By doing this, we will obtain a density estimate \( \hat{p} \) in the form of (17). The prior \( r(u) \) can then be chosen as a truncated Gaussian or a truncated Laplacian.

We conclude the algorithm for density steering of the vast group of agents in this section as in Algorithm 1.

**Algorithm 1** Density Steering of a Vast Group of Agents.

**Input:** The maximal time step \( K \); the parameter of the system \( a(k) \) for \( k = 0, \ldots, K - 1 \); the initial system density \( q_0(x) \); the specified terminal density \( \tau(x) \).

**Output:** The controls \( u(k), k = 0, \ldots, K - 1 \).

1: \( k \leftarrow 0 \)
2: while \( k < K \) and \( e(k) \notin \mathbb{V}_r^n \) do
3: \( k \leftarrow k + 1 \)
4: Calculate \( \mathcal{X}(k) \) by (4) if \( k > 0 \) or by (5) if \( k = 0 \)
5: \( \text{if } e(k) \notin \mathbb{V}_r^n \text{ then} \)
6: \( \text{Calculate the states of the moment system } \mathcal{X}(i) \text{ for } i = k + 1, \ldots, K - 1 \text{ by (13) with } \omega_k = \cdots = \omega_{K-1} \)
7: \( \text{Calculate the controls of the moment system } u(i) \text{ for } i = k, \ldots, K - 1 \text{ by (4)} \)
8: \( \text{if } \exists i, u(i) \notin \mathbb{V}_r^n \text{ then} \)
9: \( \text{Back to Step 6, adjust } \omega_k, \ldots, \omega_{K-1} \)
10: end if
11: \( \text{Optimize the cost function (18) and obtain the analytic estimates of the densities } p_i(u) \text{ for } i = k, \ldots, K - 1 \)
12: \( \text{else} \quad u(k) = 0 \)
13: \( \text{end if} \)
14: Calculate the power moments of the system state \( x(k+1) \), i.e., \( \mathcal{X}(k+1) \)
15: \( k \leftarrow k + 1 \)
16: end while

2.4 Convergence and error analyses of the terminal density function

Through the proposed density steering scheme, the first \( 2n \)-order power moments of the terminal density function match those of the desired one. As \( n \) approaches infinity, the full moment sequence of the terminal density converges to that of the desired density. By Theorem 4.5.5 in [12], we can straightforwardly demonstrate that the terminal distribution obtained from our distribution steering algorithm converges almost everywhere to the true desired distribution as \( n \) tends to infinity. Assuming both the initial and desired terminal distributions are continuous, we can further conclude that the terminal density generated by the proposed algorithm converges to the desired distribution. This convergence to the desired density distinguishes our proposed algorithm from other existing methods.

Since we used the truncated power moments of the initial and terminal density functions for steering, there may exist an error between the terminal density and the desired one. We will propose an upper bound of error of the terminal density, as to characterize the maximal difference between the terminal density by our proposed algorithm and the specified one.

In [29], we proposed an error upper bound for the Hamburger moment problem in the sense of total variation distance, which is a measure widely used in the moment problem [26,27].

The total variation distance between the terminal density \( q_K(x) \) and the desired terminal density \( \tau(x) \) is defined as follows:

\[
V(q_K, \tau) = \sup_x \left| \int_{-\infty}^x (q_K - \tau) dx \right| = \sup_x |F_{q_K} - F_{\tau}|
\]

where \( F_{q_K} \) and \( F_{\tau} \) are the two distribution functions of \( q_K \) and \( \tau \).

Shannon-entropy is used to calculate the upper bound of the total variation distance in [27]. The Shannon-entropy [23] is defined as

\[
H[q] = -\int_c q(x) \log q(x) dx.
\]

We first introduce the Shannon-entropy maximizing distribution \( \tilde{q}_K \), of which the moments are the power moments of \( q_K \). It has the following density function [15],

\[
\tilde{q}_K(x) = \exp \left(-\sum_{i=0}^{2n} \lambda_i x^i \right)
\]

where \( \lambda_0, \ldots, \lambda_{2n} \) are determined by the following constraints,

\[
\int_c x^k \exp \left(-\sum_{i=0}^{2n} \lambda_i x^i \right) dx = \int_c x^k \tau(x) dx
\]
for } k = 0, 1, \cdots, 2n. \text{ By referring to } [27], \text{ the KL distance between the true density and the Shannon-entropy maximizing density can be written as }
\begin{align*}
KL(\tau \| \hat{q}_K) &= \int_C \tau(x) \log \left( \frac{\tau(x)}{\hat{q}_K(x)} \right) dx \\
&= -H[\tau] + \sum_{i=0}^{2n} \lambda_i \int_C x^k \tau(x) dx \\
&= H[\hat{q}_K] - H[\tau].
\end{align*}

Similarly, we can obtain \( KL(\tau \| q_K) = H[q_K] - H[\tau] \).

By [16, 27], we obtain
\begin{align*}
V(\hat{q}_K, q_K) &\leq 3 \left[ -1 + \left\{ 1 + \frac{4}{9} KL(q_k \| \hat{q}_K) \right\}^{1/2} \right]^{1/2} \\
&= 3 \left[ -1 + \left\{ 1 + \frac{4}{9} (H[q_K] - H[q_k]) \right\}^{1/2} \right]^{1/2}
\end{align*}

and
\begin{align*}
V(q_K, \tau) &\leq 3 \left[ -1 + \left\{ 1 + \frac{4}{9} (H[q_K] - H[\tau]) \right\}^{1/2} \right]^{1/2}
\end{align*}

Then the error upper bound of the terminal density can be written as
\begin{align*}
V(q_K, \tau) &= \sup_x |F_{q_K}(x) - F_{\tau}(x)| \\
&\leq \sup_x \{|F_{q_K}(x) - F_{q_k}(x)| + |F_{q_k}(x) - F_{\tau}(x)|\} \\
&\leq \sup_x \{|F_{q_k}(x) - F_{\tau}(x)| + \sup_x |F_{q_k}(x) - F_{\tau}(x)|\} \\
&\leq 3 \left[ -1 + \left\{ 1 + \frac{4}{9} (H[q_K] - H[q_k]) \right\}^{1/2} \right]^{1/2} \\
&+ 3 \left[ -1 + \left\{ 1 + \frac{4}{9} (H[q_K] - H[\tau]) \right\}^{1/2} \right]^{1/2}
\end{align*}

3 Steer the group as an occupation measure

In the preceding section, we introduced an algorithm designed to guide a vast group of agents, represented as a probability density function, to a specified terminal state. However, the characterization of agents as a density function is an approximation when the number of agents tends to infinity. To apply our proposed algorithm to real-world scenarios, where the number of agents is finite, we must address individual agents and provide specific control inputs for each. In this section, we employ the occupation measure to describe the agent group and present a control scheme aimed at steering an initial occupation measure towards a terminal one.

We first define the occupation measure [30] of the agents at time step } k \text{ by }
\begin{align*}
d\mu_k(x, u) &= \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i(k)) \delta(u - u_i(k)) dx du.
\end{align*}

Then the occupation measure of the group state can be written as
\begin{align*}
dq_k(x) &= \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i(k)) dx \quad (19)
\end{align*}
which is a marginal measure of } d\mu_k(x, u).

Another marginal measure, the occupation measure of
\begin{align*}
dp_k(u) &= \frac{1}{N} \sum_{i=1}^{N} \delta(u - u_i(k)) du \quad (20)
\end{align*}

The power moments of the occupation measures can then be written as
\begin{align*}
\mathbf{E}[x'(k)] &= \int_{\mathbb{R}} x dq_k(x) = \frac{1}{N} \sum_{i=1}^{N} x_i'(k), \quad (21)
\end{align*}
and
\begin{align*}
\mathbf{E}[u'(k)] &= \int_{\mathbb{R}} u dp_k(u) = \frac{1}{N} \sum_{i=1}^{N} u_i'(k). \quad (22)
\end{align*}

Now we define the occupation measure steering by power moments, which we will treat in this section.

The dynamics of the moment system is
\begin{align*}
X(k + 1) &= A(U(k))X(k) + U(k).
\end{align*}
where } X(k), U(k) \text{ are defined as } (21),(22). \text{ Given an arbitrary initial state } x(0), \text{ determine the control sequence } (u_i(0), \cdots, u_i(K - 1)) \text{ for each agent } i \text{ such that the power moments of the terminal occupation measure are identical to those of an arbitrarily specified state, i.e.,}
\begin{align*}
\mathbf{E}[x_T^2] &= \int_{\mathbb{R}} x^2 \tau(x) dx = \frac{1}{N} \sum_{i=1}^{N} x_i^2(K) \quad (23)
\end{align*}
for \( l = 1, \cdots, 2n \), where \( \tau(x) \) is the specified terminal density.

The main difference of the occupation measure steering problem from the density steering one lies in determining the control input for each agent. We naturally consider designing feedback control laws for the agents. It might be feasible with limited number of agents. However it is quite expensive and problematic with quite a large number of agents as in our problem. Moreover, as to implement the feedback control, we need to obtain the states of each agent and calculate the control inputs based on all of them, which requires us to install numerous sensors on the agents and we have to treat the issues of the communications between the agents. However, by our proposed algorithm using moments, it is not necessary to collect all states of the agents and to transmit them back to the center to calculate the control inputs at all time steps.

Since \( N \) is large, we consider first estimating the occupation measure of \( u(k) \) as a continuous function \( \hat{p}(u) \). Then we draw \( N \) i.i.d samples from it and assign them to \( u_i \), i.e., \( u_i \sim \hat{p}(u) \), \( i \in \mathbb{N}_0, i \leq N \). By the strong law of large numbers, we note that

\[
\frac{1}{N} \sum_{i=1}^{N} u_i(k) \frac{a_x}{\ln} \int_{\mathbb{R}} u^l \hat{p}_u(u) du, \text{ with } N \to +\infty, \quad (24)
\]

which means that the power moments of \( u(k) \) converge almost surely to the power moments \( \mathbb{E}(u) \) of the designed controls. Moreover, the sampling strategy ensures that the system state \( x(k) \) and the control input \( u(k) \) are independent from each other, hence (11) is satisfied. Then the problem comes to putting forward a sampling strategy. We consider using the acceptance-rejection sampling [5] strategy for this task.

The idea of acceptance-rejection sampling is that, even if it is not possible to directly sample from \( \hat{p} \), there exists another density \( \tilde{p} \) from which it is easy to sample. The task can be reduced to sampling from \( \tilde{p} \) directly and then rejecting the samples in a strategic way to make the remaining samples seemingly drawn from \( \hat{p} \). We call the density \( \tilde{p} \) the "candidate density" and \( \hat{p} \) the "target density".

We assume that

\[
e = \sup_{x \in \text{supp}(\hat{p})} \hat{p} < \infty \quad (25)
\]

and that we can calculate \( e \). In our paper, the support of both \( \hat{p} \) and \( \tilde{p} \) is \( \mathbb{C} \). As to satisfy (25), the candidate density needs to have heavier tails than the target density. Then we give the sampling algorithm in Algorithm 2.

| Algorithm 2 Sample \( u_i(k) \) from the Realized Control. |
|---|
| **Input:** The number of agents \( N \in \mathbb{N}_0 \); the realized control \( \hat{p}_u(u) \); the candidate density \( \tilde{p}_u(u) \) |
| **Output:** The controls \( u_i(k), i = 1, \cdots, N \) |
| 1: \( i \leftarrow 1 \) |
| 2: \( \text{while } i \leq N \text{ do} \) |
| 3: \( \text{Sample an } r_i \text{ from a uniform distribution } U[0, 1] \) |
| 4: \( \text{Sample a } u \in \text{supp}(\tilde{p}) \text{ from the candidate density } \tilde{p} \) |
| 5: \( \text{if } u \leq \frac{\tilde{p}}{e} \text{ then} \) |
| 6: \( u_i(k) \leftarrow u \) |
| 7: \( \text{else} \) |
| 8: \( \text{back to step 3} \) |
| 9: \( \text{end if} \) |
| 10: \( i \leftarrow i + 1 \) |
| 11: \( \text{end while} \) |

We note that in real applications, the controls \( u_i(k), i = 1, \cdots, N \) are sometimes bounded. In previous results, the problem has not been treated by the feedback control laws given the domain of the system state being the whole \( \mathbb{R} \). By the proposed Algorithm 2, we simply need to choose \( \tilde{p} \) and the corresponding \( \hat{p} \) to be truncated densities supported on specified bounded intervals.

We now adopt the proposed acceptance-rejection sampling strategy to update Algorithm 1 to treat the occupation measure steering problem, leading to Algorithm 3.

### 4 Numerical examples

In the previous sections, we proposed algorithms for steering a vast group of agents, either characterized as a probability density function or an occupation measure. In this section, we perform numerical simulations on different types of distributions, supported on \( \mathbb{R} \) or a compact subset of \( \mathbb{R} \), with multiple modes or a single mode, to validate our proposed algorithms.

We first simulate the steering of a vast group of agents as a probability density function. We begin with unconstrained control, i.e., the control inputs \( u(k) \) are not constrained.

#### 4.1 Unconstrained density steering of a vast group of agents

In Example 1, we simulate a typical problem which is to steer a Gaussian density to another in four steps. The initial Gaussian density is chosen as

\[
q_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad (26)
\]

\[
Algorithms 2 \text{ Sample } u_i(k) \text{ from the Realized Control. }
\text{Input: } \text{The number of agents } N \in \mathbb{N}_0; \text{ the realized control } \hat{p}_u(u); \text{ the candidate density } \tilde{p}_u(u) \text{. }
\text{Output: } \text{The controls } u_i(k), i = 1, \cdots, N \text{. }
1: \ i \leftarrow 1 \\
2: \text{while } i \leq N \text{ do} \\
3: \text{Sample an } r_i \text{ from a uniform distribution } U[0, 1] \\
4: \text{Sample a } u \in \text{supp}(\tilde{p}) \text{ from the candidate density } \tilde{p} \\
5: \text{if } u \leq \frac{\tilde{p}}{e} \text{ then} \\
6: \ u_i(k) \leftarrow u \\
7: \text{else} \\
8: \text{back to step 3} \\
9: \text{end if} \\
10: \ i \leftarrow i + 1 \\
11: \text{end while} \\
\]
the terminal density function is a mixture of generalized steps where the initial density function is a Gaussian and the specified terminal occupation measure \(d\tau(x)\) of the system state, i.e., \(X(k)\) for \(k = 0, \cdots, K - 1\), the initial occupation measure \(d\phi_0(x)\); the specified terminal occupation measure \(d\tau(x)\).

In Example 2, we simulate a steering problem in three steps where the initial density function is a Gaussian and the terminal density is complicated. In Example 3, we will simulate a steering problem in four steps which has not been treated in the previous papers.

### Algorithm 3 Steering Occupation Measure for a Large Group of Agents

**Input:** The number of agents \(N \in \mathbb{N}_0\); the maximal time step \(K\); the system parameter \(a(k)\) for \(k = 0, \cdots, K - 1\); the initial occupation measure \(d\phi_0(x)\); the specified terminal occupation measure \(d\tau(x)\).

**Output:** Control inputs for the \(i\)th target \(u_i(k), k = 0, \cdots, K - 1, i = 1, \cdots, N\).

1: \(k \leftarrow 0\)
2: while \(k < K\) and \(e(k) \notin \mathbb{V}_{2n}^i\) do
3: Calculate \(X(k)\) using (4) if \(k > 0\) or (5) if \(k = 0\)
4: Calculate \(e(k)\) using (12)
5: if \(e(k) \in \mathbb{V}_{2n}^i\) then
6: Calculate the states of the moment system \(X(i)\) for \(i = k + 1, \cdots, K - 1\) using (13) with \(\omega_k = \cdots = \omega_{K-1}\)
7: Calculate the controls of the moment system \(U(i)\) for \(i = k, \cdots, K - 1\) using (4)
8: if \(\exists i, U(i) \notin \mathbb{V}_{2n}^i\), then
9: Go back to Step 6, adjust \(\omega_k, \cdots, \omega_{K-1}\)
10: end if
11: Optimize the cost function (18) and obtain the analytic estimates of the densities \(\rho_i(u)\) for \(i = k, \cdots, K - 1\)
12: Execute Algorithm 2 and obtain the control inputs \(u_i(j)\) for all agents at time step \(j = k, \cdots, K - 1\)
13: else \(u_i(k) = 0, i = 1, \cdots, N\)
14: end if
15: Calculate the power moments of the system state \(x(k+1)\), i.e., \(X(k+1)\)
16: \(k \leftarrow k + 1\)
17: end while

and the terminal density is specified as

\[
\tau(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}. \tag{27}
\]

The system parameters \(a(k), k = 0, \cdots, 3\) are i.i.d. samples drawn from the uniform distribution \(U[0.5, 0.7]\). The states of the moment system, i.e., \(X(k)\) for \(k = 0, 1, 2, 3\) are given in Figure 1. The controls of the moment system, i.e., \(U(k)\) for \(k = 0, 1, 2, 3\) are given in Figure 2. We note that by our proposed algorithm, \(X(k), U(k) \in \mathbb{V}_{2n}^i\), which makes it feasible for us to realize the controls. The realized control inputs are given in Figure 3. The transition of the control inputs is smooth, which is satisfactory.

In Example 2, we simulate a steering problem in three steps where the initial density function is a Gaussian and the terminal density function is a mixture of generalized logistic distributions. The initial one is chosen as

\[
q_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
\]

The system parameters \(a(k), k = 0, \cdots, 3\) are i.i.d. samples drawn from the uniform distribution \(U[0.5, 0.7]\). The states of the moment system, i.e., \(X(k)\) for \(k = 0, 1, 2, 3\) are given in Figure 4. The controls of the moment system, i.e., \(U(k)\) for \(k = 0, 1, 2, 3\) are given in Figure 5. The realized controls in Figure 6 also show that the transition of the control inputs is smooth, even the specified terminal density is complicated.

In Example 3, we will simulate a steering problem in four steps which has not been treated in the previous papers.
Fig. 3. Realized control inputs $u(k)$ by $u(k)$ for $k = 0, 1, 2, 3$, which are obtained by our proposed control scheme.

Fig. 4. $X(k)$ at time steps $k = 0, 1, 2, 3$.

Fig. 5. $u(k)$ at time steps $k = 0, 1, 2, 3$.

Fig. 6. Realized control inputs $u(k)$.

The initial density is chosen as a Gaussian distribution

$$q_0(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}.$$

However the terminal density function is specified as a multi-modal density which is a mixture of two Laplacians

$$\tau(x) = \frac{0.7}{2} e^{||x-1||} + \frac{0.3}{2} e^{-||x+3||}.$$

The system parameters $a(k), k = 0, \ldots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$. The results are given in Figure 7, 8 and 9. In this example, the two modes are not close to each other as in Example 2. However the realized control inputs are still smooth, which validates the performance of our algorithm in steering a group to separate groups relatively far from each other.
4.2 Constrained density steering of a vast group of agents

In some scenarios, there are boundary conditions on the control inputs. A common constraint is that they are bounded by a compact interval \([a, b]\) on \(\mathbb{R}\). With our proposed algorithm, we are able to treat the density steering problem with bounded control inputs.

In Example 4, we will simulate a problem which is sort of problematic however important in real practice. The initial and the terminal densities are both chosen as a mixture of two Gaussian densities, which are

\[
q_0(x) = \frac{0.5}{\sqrt{2\pi} \cdot 2} e^{\frac{(x)^2}{2 \cdot 2^2}} + \frac{0.5}{\sqrt{2\pi} \cdot 2} e^{\frac{-(x+1)^2}{2 \cdot 2^2}},
\]

and

\[
\tau(x) = \frac{0.4}{\sqrt{2\pi}} e^{\frac{(x-1)^2}{2 \cdot 2^2}} + \frac{0.6}{\sqrt{2\pi}} e^{\frac{(x+1)^2}{2 \cdot 2^2}}.
\]

The density steering is performed in five steps. The control inputs are bounded on the interval \(\mathcal{C} = [-2, 2]\). The system parameters \(a(k), k = 0, \ldots, 4\) are i.i.d. samples drawn from the uniform distribution \(U[0.5, 0.7]\).

The results are given in Figure 10, 11 and 12. We note that at time step \(k = 0\), \(X_T - X(0) \notin \mathcal{V}_{+}^{2n}\). Then by the proposed algorithm, we don’t apply control inputs to the agents, which is shown in Figure 11. At time step \(k = 1\), we have \(X_T - X(1) \in \mathcal{V}_{+}^{2n}\). Hence we start applying controls starting from this step. It takes three steps for us to steer the density to the specified one.

The goal of the steering in this problem is to steer two distinct groups of agents to two specified terminal groups. The boundary of control inputs and the multiple modality of both initial and terminal densities make the problem a challenging one. By our proposed algorithm, we give a solution to this problem and the realized control inputs are still smooth, which is a satisfactory performance.
Fig. 12. Realized control inputs $u(k)$ at time steps $k = 1, 2, 3, 4$. The agents are uncontrolled at $k = 0$, i.e., $u(0) = 0$.

The moments is able to treat the occupation measure steering problem. The analytic function $u(k)$ at each time step is realized as control inputs $u_i(k)$ of each agent $i$. We can therefore obtain a terminal occupation measure to compare to the specified one. In the following part of this section, we simulate the occupation measure steering examples.

4.3 Occupation measure steering of a vast group of agents

We simulate examples on occupation measure steering in this part of section. In Example 5, we steer 2000 agents to a specified occupation measure. The initial states of each agent $x_i$ is drawn i.i.d. from the Gaussian distribution (26). The specified terminal occupation measure consists of 2000 i.i.d. samples drawn from the terminal distribution (27). The system parameter $a(k), k = 0, \cdots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$.

Figure 13 displays histograms of $u_i(k)$ at time step $k$ for each agent $i$. These histograms consist of 2000 i.i.d samples drawn from the realizations of $u(k)$ shown in Figure 3. Additionally, Figure 14 illustrates the histograms of the occupation measure for the states of the 2000 agents at time steps $k = 0, 1, 2, 3$. The power moments of order 1 to 4 for the terminal occupation measure, obtained using our proposed algorithm, are 1.21, 5.41, 14.44, 78.77 respectively. Notably, these values are close to the desired terminal distribution, with power moments of order 1 to 4 being 1.5, 13, 73 respectively. This outcome validates the effectiveness of our algorithm in steering the occupation measure.

In Example 6, we steer 2000 agents to a specified occupation measure. The initial states of each agent $x_i$ is drawn i.i.d. from the Gaussian distribution (26). The specified terminal occupation measure consists of 2000 i.i.d. samples drawn from the terminal distribution (28), of which the density is a mixture of two Gaussian densities. The system parameter $a(k), k = 0, \cdots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$.

The histograms of $u_i(k)$ at time step $k$ for each agent $i$ are given in Figure 15. They are 2000 i.i.d samples drawn from the realizations of $u(k)$ in Figure 6. Figure 16 gives the histograms of the occupation measure of the states of the 2000 agents at $k = 0, 1, 2, 3$. We note that the power moments of order 1 to 4 of (28) are 0.5, 3.88, 8.8, 52.8 respectively. The power moments of order 1 to 4 of the terminal occupation measure by our proposed algorithm are 0.54, 3.97, 9.0, 52.79 respectively, which are quite close to the specified ones. This
example validates the ability of the proposed scheme in treating distribution steering problems with complicated distributions.

Fig. 15. The histograms of \( u_i(k) \) at time step \( k \) for each agent \( i \). The upper left and right figures are \( u_i(0) \) and \( u_i(1), i = 1, \ldots, 2000 \) respectively. The lower left and right figures are \( u_i(2) \) and \( u_i(3) \) respectively.

Fig. 16. The histograms of the system states \( x(k) \) of 2000 Monte-Carlo simulations at time step \( k = 0, 1, 2, 3 \), together with the desired terminal distribution 28.

5 Concluding remarks

In this paper, we propose to use moments for the problem of steering a vast group of agents. The vast group of agents are characterized as probability density functions and occupation measures. We first treat the density steering problem. Without assuming the initial and terminal density to fall within specific function classes, the original problem is infinite-dimensional and intractable.

As to treat this problem, we propose a moment system representation of the original system, and reduce the original problem to the control of the moment system. Different from the conventional control problems, the elements of the control inputs are in the system matrix, and we have to ensure the Hankel matrix of the control inputs at each time step to be positive definite. Since it is not treatable by the existing control schemes including the commonly used optimal control, we propose an empirical control scheme to treat this problem. By doing this, it is feasible for us to realize the control inputs for the original system. The previous results were presented at IFAC World Congress 2023 [28]. In this paper, we consider more fundamental issues of the group steering problems and propose algorithms for real-world group steering problems. We propose that with the number of moment terms used approaching infinity, the terminal distribution obtained by the proposed scheme is almost everywhere equal to the desired one. We also propose an error upper bound of the terminal density by using our proposed algorithm, in the sense of the total variation distance. Our algorithm of steering an arbitrary initial distribution to another arbitrary one in limited steps for the discrete-time first-order ODE systems, without assuming their function classes, is the first one in the literature. Based on the proposed density-steering algorithm, we put forward an algorithm steering an arbitrary occupation measure representing the vast group of agents to another arbitrary one. We use acceptance-rejection sampling to draw i.i.d. samples from the realized control inputs \( u(k) \) and assign them to each agents. By doing this, the control inputs are independent of the current states of each agent.

This paper treats the group steering problem of the first-order time-variant linear system. However, the problem is much more complicated for multi-order and multivariate systems. For first-order systems, the control inputs \( u(k) \)'s are one-dimensional density functions. Hence the positive definiteness of the Hankel matrix of each \( U(k) \) is the necessary and sufficient condition of the existence of \( u(k) \) [22]. However for multi-dimensional densities, it is no longer valid. To derive control inputs \( U(k) \)'s for the moment system then to realize them will be a challenging problem. Results in subjects e.g. mathematical moment problem and real algebraic geometry shall be used to treat the group steering problem for those systems.

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