AN OVERVIEW AND SUPPLEMENTS TO THE THEORY OF FUNCTIONAL RELATIONS FOR ZETA-FUNCTIONS OF ROOT SYSTEMS

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Abstract. We give an overview of the theory of functional relations for zeta-functions of root systems, and show some new results on functional relations involving zeta-functions of root systems of types $B_r, D_r, A_3$ and $C_2$. To show those new results, we use two different methods. The first method, for $B_r, D_r, A_3$, is via generating functions, which is based on the symmetry with respect to Weyl groups, or more generally, on our theory of lattice sums of certain hyperplane arrangements. The second method for $C_2$ is more elementary, using partial fraction decompositions.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers, $\mathbb{Z}$ the set of rational integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers.

Let $V$ be an $r$-dimensional real vector space with the inner product $\langle \ , \ \rangle$, and $\Delta \subset V$ be a reduced root system of rank $r$. Let $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ be its fundamental system. Denote by $\Delta_+, \Delta_-$ the set of all positive roots and of all negative roots, respectively, and $n = |\Delta_+|$. For any $\alpha \in \Delta$, we denote by $\alpha^\vee$ the associated coroot.

Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be the set of fundamental weights defined by $\langle \alpha^\vee_i, \lambda_j \rangle = \delta_{ij}$ (Kronecker’s delta). Let $Q^\vee$ be the coroot lattice, $P$ the weight lattice, $P_+$ the set of integral dominant weights, and $P_{++}$ the set of integral strongly dominant weights, respectively, defined by

$$Q^\vee = \bigoplus_{i=1}^r \mathbb{Z} \alpha_i^\vee, \quad P = \bigoplus_{i=1}^r \mathbb{Z} \lambda_i, \quad P_+ = \bigoplus_{i=1}^r \mathbb{N}_0 \lambda_i, \quad P_{++} = \bigoplus_{i=1}^r \mathbb{N} \lambda_i.$$

The zeta-function of the root system $\Delta$ is defined by an $r$-ple series in $n$ variables. It is defined by

$$\zeta_r(s; \Delta) = \sum_{m_1=1}^\infty \cdots \sum_{m_r=1}^\infty \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, m_1 \lambda_1 + \cdots + m_r \lambda_r \rangle^{-s_\alpha}, \quad (1.1)$$

where $s = (s_\alpha)_{\alpha \in \Delta_+}$ is a complex vector. This multiple series is convergent absolutely when $\Re s_\alpha > 1$ for all $\alpha$, and can be continued meromorphically to the whole space $\mathbb{C}^n$ (see [26, Theorem 3]). When the root system is of type $X_r$ ($X = A, B, C, D, E, F$ or $G$), we write the associated zeta-function as $\zeta_r(s; X_r)$.

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When $X = A, B, C$ and $D$, the explicit form of (1.1) was given in [12], where the recursive structure of those zeta-functions was also discussed. The case of type $G_2$ was studied in [16, 23].

The notion of zeta-functions of root systems is a generalization of the following three kinds of multiple zeta-functions.

1°. It is a multi-variable generalization of Witten zeta-functions. Let $g$ be a semisimple Lie algebra, and define

$$(1.2) \quad \zeta_W(s, g) = \sum \dim \varphi^{-s},$$

where the sum runs over all equivalent classes of finite dimensional irreducible representations of $g$. Zagier [12] introduced (1.2) under the name of the Witten zeta-function, after the work of Witten [41]. Since there is a one-to-one correspondence between irreducible representations and dominant weights, using Weyl’s dimension formula we find that

$$(1.3) \quad \zeta_W(s, g) = K(g)^s \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha \in \Delta_+} (\alpha^\vee, m_1 \lambda_1 + \cdots + m_r \lambda_r)^{-s},$$

where $K(g) = \prod_{\alpha \in \Delta_+} (\alpha^\vee, \lambda_1 + \cdots + \lambda_r)$ and $\Delta(g)$ is the root system corresponding to $g$.

Remark 1. Witten originally considered the zeta values associated not with Lie algebras, but with Lie groups. Multi-variable version of such zeta-functions associated with Lie groups has been studied in [20, 24].

2°. Tornheim [38] considered special values of the double series

$$(1.4) \quad \zeta_{MT,2}(s_1, s_2, s_3) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \prod_{1 \leq i < j \leq 3} (m_i + m_j)^{-s_i - s_j},$$

at positive integer points. From our viewpoint, this function is nothing but the zeta-function of the root system of type $A_2$. Later, the second author [26] introduced the $C_2$-analogue, that is

$$(1.5) \quad \zeta(s_1, s_2, s_3; C_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3},$$

and then the second and the third authors [30] considered more general zeta-functions of root systems of type $A_r$, which are of the form

$$(1.6) \quad \zeta_r(s; A_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1 \leq i < j \leq r+1} (m_i + \cdots + m_j)^{-s_{ij}},$$

where $s = (s_{ij})_{1 \leq i < j \leq r+1}$. On the other hand, as another generalization of (1.4), the second author [26] also introduced the Mordell–Tornheim multiple zeta-function

$$(1.7) \quad \zeta_{MT,r}(s_1, \ldots, s_r, s_{r+1}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1} \cdots m_r^{-s_r} (m_1 + \cdots + m_r)^{-s_{r+1}}.$$
of (1.6), because if we put $s_{ij} = 0$ for all $(i, j) \neq (1, 2), (2, 3), \ldots, (r, r + 1), (1, r + 1)$ in (1.5), then it reduces to (1.6).

3°. The Euler–Zagier multiple zeta-function is defined by

$$\zeta_{EZ,r}(s_1, \ldots, s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1}(m_1 + m_2)^{-s_2} \cdots (m_1 + \cdots + m_r)^{-s_r}$$

(Hoffman [5], Zagier [42]). Special values of (1.7) at positive integer points (in the domain of absolute convergence) are called multiple zeta values (MZV), and have been studied extensively in these decades. We find that, if we put $s_{ij} = 0$ for all $(i, j) \neq (1, 2), (1, 3), \ldots, (1, r + 1)$ in (1.5), then it reduces to (1.7). It is also possible to understand (1.7) as special cases of zeta-functions of root systems of type $C_r$ (see [21]).

From the above observation we can expect that we will be able to construct a unified theory of multiple zeta-functions from the viewpoint of root systems. This expectation is embodied when we consider the problem of functional relations, which we now explain.

In the study of MZV, a central problem is to find various relations among those values. In fact, a lot of such relations are known (duality, sum formula, Ohno relation, Le–Murakami relation, . . . ). Then it is a natural question to ask: whether those relations are valid only at integer points, or valid also at other values continuously as functional relations? This question was raised by the second author (cf. [28, p.161]). In the frame of Euler–Zagier multiple zeta-functions, no such functional relation is known, except for the classical harmonic product formula (and perhaps the functional equations for double zeta-functions, see [27], [15, 18]). However, if we observe the landscape from the wider standpoint of zeta-functions of root systems, we are able to find various functional relations whose specialization gives relations among MZVs. The aim of the present article is to survey the known results on functional relations among zeta-functions of root systems, and report some new results in this direction.

In the next section, we summarize various previous results on this topic. In particular we mention the main general result (Theorem 2) in [25]. Then in Sections 3 to 5, we give some explicit examples of this Theorem 2 in the cases of types $B_r$, $D_r$, and $A_3$. In the last section we present another method of obtaining functional relations, for zeta-functions of type $C_2$.

2. A SURVEY ON PREVIOUS METHODS

The first affirmative answer to the question mentioned at the end of the introduction is the following result of the third author [40]. Let $\zeta(s)$ be the Riemann zeta-function and let

$$\zeta_2(s_1, s_2, s_3; A_2) = \zeta_{MT,2}(s_1, s_2, s_3)$$

defined by (1.4). Then, for $k, l \in \mathbb{N}_0$,

$$\zeta_2(k, l, s; A_2) + (-1)^k \zeta_2(k, s, l; A_2) + (-1)^l \zeta_2(l, s, k; A_2)$$

$$= 2 \sum_{j=0}^{[k/2]} \binom{k + l - 2j - 1}{l - 1} \zeta(2j) \zeta(s + k + l - 2j)$$

$$+ 2 \sum_{j=0}^{[l/2]} \binom{k + l - 2j - 1}{k - 1} \zeta(2j) \zeta(s + k + l - 2j)$$

holds for any $s \in \mathbb{C}$. 

The above formula is actually a little different from the form given in (30). Inspired by (31), Nakamura [34] published an alternative method, which gives the above form. From the expression in (40), it is possible to deduce the above form, just by using a certain elementary lemma (see [29], Lemma 2.1).

The basic idea in (40) is to consider the series with additional factor \((-u)^{-n}\), where \(u \geq 1\) and \(n \in \mathbb{N}\). Because of this additional factor, the series is convergent nicely. And at the end of the proof take the limit \(u \to 1\) to obtain the relation of ordinary multiple zeta-functions. This method is sometimes called the \(u\)-method. Using the same method, the second and the third authors [30] proved some functional relations involving the zeta-function of \(A_3\).

Nakamura’s method in [34] is different. His argument starts with the expression

\[ 
\sum_{m_1 \neq 0, m_2 \neq 0} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3} = \int_0^1 \sum_{m_1 \neq 0} e^{2\pi i m_1 x} m_1^{-s_1} \sum_{m_2 \neq 0} \frac{e^{2\pi i m_2 x}}{m_2^{s_2}} \sum_{n \neq 0} \frac{e^{-2\pi i nx}}{n^{s_3}} dx, 
\]

which was used by Zagier in his lecture at Kyushu University in 1999, and then uses properties of Bernoulli polynomials. By the same technique, Nakamura [35] obtained functional relations among zeta-functions of \(B_2, A_3, A_2\), and of \(A_2\) with characters.

Yet another (rather elementary) method was proposed by Zhou, Bradley and Cai [43], who stated certain functional relations among zeta-functions of \(A_3\). Inspired by [43], Ikeda and Matsuoka [8] gave functional relations among zeta-functions of \(A_3, A_2, A_2\) and \(A_1\) with correcting some inaccurate point in [43].

All of the methods mentioned above have the same feature, that is, in general, some of the variables in the results are forced to be 0. To avoid this unsatisfactory restriction, the authors [10] introduced the idea of considering infinite series of polylogarithm type (that is, with an additional exponential factor on the numerator), and combining this additional flexibility with the \(u\)-method to obtain more general form of functional relations. This technique was inspired by the work of the second and the third authors [32]. Applying this idea, the authors proved various functional relations among zeta-functions of \(A_3, B_3, C_2, C_3\) and \(G_2\) (see [14], [10], [23]).

It is noted that the second and the third authors [31] introduced another new method of finding functional relations among certain multiple zeta-functions including the zeta-function of type \(A_2\). This method can be regarded as a kind of multiple analogue of Hardy’s one (see [6], also [37], Section 2.2) of proving the functional equation for the Riemann zeta-function. However, at present, it is unclear whether we can apply this method to zeta-functions of root systems of general types.

Another approach to functional relations for the zeta-function of type \(A_2\), due to Onodera [36] is also to be mentioned.

On the other hand, when one observes the form of the left-hand side of (2.1), one may feel that there may be some underlying connection with the action of the associated Weyl group. This is in fact true, and the authors developed an alternative (more structural) approach of finding functional relations.

Let \(I \subset \{1, 2, \ldots, r\}\), and \(\Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi\). Let \(V_I\) be the subspace of \(V\) spanned by \(\Psi_I\). Then \(\Delta_I = \Delta \cap V_I\) is the root system in \(V_I\) whose fundamental system is \(\Psi_I\). For \(\Delta_I\), we denote the corresponding coroot lattice, weight lattice etc. by \(\Lambda^*_I = \bigoplus_{\alpha \in I} \mathbb{Z}\alpha^*_I\), \(P_I = \bigoplus_{\alpha \in I} \mathbb{Z}\alpha\) etc. Let \(t : Q^*_I \rightarrow Q^*_I\) be the natural embedding, and \(t^* : P \rightarrow P_I\) the projection induced from \(t\); that is, for \(\lambda \in P\), \(t^*(\lambda)\) is defined as a unique element of \(P_I\) satisfying \(\langle t(q), \lambda \rangle = \langle q, t^*(\lambda) \rangle\) for all \(q \in Q^*_I\).
Let $\sigma_\alpha$ be the reflection with respect to $\alpha$, and denote by $W = W(\Delta)$ the Weyl group of $\Delta$, namely the group generated by $\{\sigma_i \mid 1 \leq i \leq r\}$, where $\sigma_i = \sigma_{e_i}$. For $w \in W$, we put $\Delta_w = \Delta_+ \cap w^{-1}\Delta_-$. Let $W_I$ be the subgroup of $W$ generated by all the reflections associated with the elements in $\Psi_I$, and $W_I^+ = \{w \in W \mid \Delta_{I+}^+ \subset w\Delta_{I+}^+\}$.

The fundamental Weyl chamber is defined by

$$C = \{v \in V \mid \langle \alpha_i^+, v \rangle \geq 0 \text{ for } 1 \leq i \leq r\}.$$  

Then $W$ acts on the set of Weyl chambers $\{wC \mid w \in W\}$ simply transitively. For any subset $A \subset \Delta$, let $H_{A^\vee}$ be the set of all $v \in V$ which satisfies $\langle \alpha^\vee, v \rangle = 0$ for some $\alpha \in A$. In particular, $H_{\Delta^\vee}$ is the set of all walls of Weyl chambers.

For $s = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^n$, define the action of $W$ to $s$ by $(ws)_\alpha = s_{w^{-1}\alpha}$ for $w \in W$ with the convention that, if $\alpha \in \Delta_-$, then we understand that $s_\alpha = -s_\alpha$. Define

$$S(s, y; I; \Delta) = \sum_{y \in V} \frac{1}{\prod_{\alpha \in \Delta_+} (\langle \alpha^\vee, \lambda \rangle^{s_\alpha})} e^{2\pi i \langle \gamma, \lambda \rangle} \prod_{\alpha \in \Delta_+} (\langle \alpha^\vee, \lambda \rangle^{s_\alpha})^{-1},$$

where $y \in V$. This sum was first introduced in [19, (110)]. Moreover in [19, Theorems 5 and 6], we showed

$$S(s, y; I; \Delta) = \sum_{w \in W_I^+} \left( \prod_{\alpha \in \Delta_{I-1}} (-1)^{s_\alpha} \right) \zeta_\sigma(w^{-1}s, w^{-1}y; \Delta).$$

Also in the same article we proved a certain multiple integral expression of $S(s, y; I; \Delta)$, and noticed that when $I = \emptyset$ and $s_\alpha$ are positive integers, the integrand of that expression can be written in terms of classical Bernoulli polynomials. This observation led us to the definition of Bernoulli functions of root systems and their generating functions. However it requires extremely huge task if we want to calculate the integral expression given in [19] more explicitly. This situation was improved in [13], in which more accessible expressions were given when $I = \emptyset$ (see [13, Theorem 4.1]).

To define Bernoulli functions of root systems, we need some more notations.

Let $\Delta^* = \Delta_+ \setminus \Delta_{I+}$ and $d = |I^c|$. We may find $V_I = \{\gamma_1, \ldots, \gamma_d\} \subset \Delta^*$ such that $V = V_I \cup \Psi_I$ becomes a basis of $V$. Let $V_I = V(\Delta^*)$ be the set of all such bases. In particular, $V = V_\emptyset$ be the set of all linearly independent subsets $V = \{\beta_1, \ldots, \beta_r\} \subset \Delta_+$.

For $V \in V_I$, the lattice $L(V^\vee) = \bigoplus_{\beta \in V} \mathbb{Z}\beta^\vee$ is a sublattice of $Q^\vee$. Let $\{\mu_i^\vee\}_{i \in \Psi_I}$ be the dual basis of $V^\vee = V_I^\vee \cup \Psi_I^\vee$, namely $\langle \gamma_i^\vee, \mu_i^\vee \rangle = \delta_{ki}$, $\langle \alpha_i^\vee, \mu_i^\vee \rangle = \delta_{ij}$, and $\langle \gamma_k^\vee, \mu_i^\vee \rangle = \langle \alpha_i^\vee, \mu_k^\vee \rangle = 0$. Let $p_{V_I^\vee}$ be the projection defined by

$$p_{V_I^\vee}(v) = v - \sum_{\gamma \in V_I} \mu_\gamma^\vee \langle \gamma^\vee, v \rangle = \sum_{\alpha \in \Psi_I} \mu_\alpha \langle \alpha^\vee, v \rangle,$$

for $v \in V$.

Next we introduce a generalization of the notion of “fractional part” of real numbers. Let $R$ be the set of all linearly independent subsets $R = \{\beta_1, \ldots, \beta_{r-1}\} \subset \Delta$, and let $\delta_{R^\vee} = \bigoplus_{i=1}^r \mathbb{R}\beta_i^\vee$ be the hyperplane passing through $R^\vee \cup \{0\}$. We fix a non-zero vector $\phi \in V \setminus \bigcup_{R \in R} \delta_{R^\vee}$.

Then $\langle \phi, \mu_\beta^\vee \rangle \neq 0$ for all $V \in V$ and $\beta \in V$. For $y \in V$, $V \in V$ and $\beta \in V$, we define

$$\{y\}_{V, \beta} = \begin{cases} \{y, \mu_\beta^\vee\}, & \langle \phi, \mu_\beta^\vee \rangle > 0, \\ 1 - \{\gamma, \mu_\beta^\vee\}, & \langle \phi, \mu_\beta^\vee \rangle < 0, \\ \end{cases}$$

where $\{\cdot\}$ on the right-hand side denotes the usual fractional part of real numbers.
Using these notions, we now define Bernoulli functions of the root system $\Delta$ associated with $I$ and their generating functions.

**Definition 2.1 (25, Definition 2.2).** For $t_I = (t_\alpha)_{\alpha \in \Delta^*} \in \mathbb{C}^n$ and $\lambda \in P_I$, let

$$F(t_I, y, \lambda; I; \Delta)$$

(2.7)

$$= \sum_{V \in V_I} \left( \prod_{\gamma \in \Delta^* \setminus V_I} t_\gamma - \sum_{\beta \in V_I} t_\beta (\gamma^\vee, p_{V_I}(\gamma)) \right) \frac{t_\gamma}{|Q^\vee/L(V^\vee)|} \sum_{q \in Q^\vee/L(V^\vee)} \exp(2\pi \sqrt{-1}(y + q, p_{V_I}(\lambda)))$$

$$\times \prod_{\beta \in V_I} \frac{t_\beta \exp(\{y + q\}_{V, \beta})}{e^{t_\beta} - 1},$$

and define Bernoulli functions $P(k, y, \lambda; I; \Delta)$ of the root system $\Delta$ associated with $I$ by the expansion

$$F(t_I, y, \lambda; I; \Delta) = \sum_{k \in \mathbb{N}^{\Delta^* \setminus V_I}} P(k, y, \lambda; I; \Delta) \prod_{\alpha \in \Delta^* \setminus V_I} \frac{t_\alpha^k}{k_\alpha!}. \tag{2.8}$$

**Theorem 2 (25, Theorem 2.3).** Let $s_\alpha = k_\alpha \in \mathbb{N}$ for $\alpha \in \Delta^*$ and $s_\alpha \in \mathbb{C}$ for $\alpha \in \Delta^*_{I+}$.

We assume

(1) If $\alpha$ belongs to an irreducible component of type $A_1$, then the corresponding $k_\alpha \geq 2$.

Then we have

$$S(s, y; I; \Delta) = (-1)^{|\Delta^*|} \left( \prod_{\alpha \in \Delta^* \setminus V_I} \frac{(2\pi \sqrt{-1})^{k_\alpha}}{k_\alpha!} \right)$$

$$\times \sum_{\lambda \in P_{I+}} \left( \prod_{\alpha \in \Delta^*_{I+}} \frac{1}{(\alpha^\vee, \lambda)^{s_\alpha}} \right) P(k, y, \lambda; I; \Delta). \tag{2.9}$$

When $I = \emptyset$, this result was obtained in (3.10)]. In this case $P_I = P_\emptyset = \{0\}$, and so there is only one $\lambda$, that is $\lambda = 0$. For general $I$, however, we have to consider the above sum with respect to $\lambda \in P_+ +$, so the situation becomes much more complicated. In fact, to prove Theorem 2 for general $I$, we had to develop the theory of certain lattice sums of hyperplane arrangements [22], and using the result in [22] we proved the general case of Theorem 2 in [25].

Combining this theorem with [24], we obtain another tool of showing functional relations among zeta-functions of root systems. For this purpose, it is necessary to know the explicit form of $P(k, y, \lambda; I; \Delta)$. In view of (2.8), this can be done if we know the explicit form of $F(t_I, y, \lambda; I; \Delta)$. This point will be discussed in the next two sections.

### 3. Generating Functions ($B_{r-1} \subset B_r$ and $D_{r-1} \subset D_r$ Cases)

In [25, Sections 3 and 4], we gave the explicit forms of the generating function $F(t_I, y, \lambda; I; \Delta)$ in the cases of the root systems of $A_r$ and $C_r$ with $I = \{2, \ldots, r\}$, and in the cases of arbitrary root systems with $|I| = 1$.

In this section we give the generating functions in the rest cases of classical root systems, that is, $B_r$ and $D_r$ cases with $I = \{2, \ldots, r\}$, following the method in [25, Sections 3 and 4].
The other cases, in general, seem much more complicated. As a first step to consider general cases, in the next section, we present the generating function of type \( I = \{1, 3\} \).

Let \( I \subset \{1, \ldots, r\} \) with \(|I^c| = 1\) and put \( I^c = \{k\} \). Then \( \Psi_I = \{\alpha_i\}_{i \in I} \), and we see that

\[
\Delta^* = \{\alpha^* = \sum_{i=1}^r a_i \alpha_i^* \in \Delta^*_1 \mid a_k = \langle \alpha^*, \lambda_k \rangle \neq 0\}.
\]

Since \(|\mathcal{V}_I| = 1\) in the present case, we have

\[
(3.1) \quad \mathcal{V}_I = \{\mathcal{V} = \{\beta\} \cup \Psi_I\}_{\beta \in \Delta^*}.
\]

For \( \mathcal{V} = \{\beta\} \cup \Psi_I \in \mathcal{V}_I \) and \( \gamma \in \Delta^* \setminus \{\beta\} \), we have the transposes \( p_{\mathcal{V}_I}^* \) of the projections \( p_{\mathcal{V}_I} \) (defined by (2.5)) as

\[
(3.2) \quad p_{\mathcal{V}_I}^*(\gamma^*) = \langle \gamma^*, \mu_{\beta} \rangle \beta^*.
\]

We put \( b_i = b_i(\beta) = \langle \beta^*, \lambda_i \rangle \) \((1 \leq i \leq r)\) so that

\[
(3.3) \quad \beta^* = \sum_{i=1}^r b_i \alpha_i^* \quad \mu_{\beta} = \frac{\lambda_k}{b_k}.
\]

Write \( y = y_1 \alpha_1^* + \cdots + y_r \alpha_r^* \) and \( \lambda = \sum_{i \neq k} m_i \lambda_i \in P_I \). Then we obtain the following form of the generating function (under the identification \( y = (y_i)_{1 \leq i \leq r} \) and \( \lambda = (m_i)_{1 \leq i \in \{\lambda \}} \))

\[
(3.4) \quad F(\{(t_{\beta})_{\beta \in \Delta^*}, (y_i)_{1 \leq i \leq r}, (m_i)_{1 \leq i \in \{\lambda \}} \}; I; \Delta)
\]

\[
= \sum_{\beta \in \Delta^*} \prod_{\gamma \in \Delta^* \setminus \{\beta\}} \frac{1}{t_{\gamma} - \langle \gamma^*, \lambda \rangle} t_{\beta} - 2\pi \sqrt{-1} \left( p_{\mathcal{V}_I}^*(\gamma^*), \lambda \right)
\]

\[
\times \frac{1}{b_k} \sum_{0 \leq a_k < b_k} \exp\left(2\pi \sqrt{-1} \sum_{i \neq k} m_i \left(y_i - \frac{b_i}{b_k}(y_k + a_k)\right)\right)
\]

\[
\times \frac{t_{\beta} \exp\left(\frac{y_k + a_k}{b_k}\right)}{e^t - 1}
\]

(see [25, (25)]). Using the above expressions, we will give explicit forms of generating functions of types \( B_r \) and \( D_r \).

### 3.1. \( B_r \) Case

We realize \( \Delta^*_Y = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq r\} \cup \{2\epsilon_j \mid 1 \leq j \leq r\} \) and \( \Psi^Y = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{r-1} - \epsilon_r, 2\epsilon_r\} \). Then

\[
(3.5) \quad \lambda_i = \begin{cases} 
\epsilon_1 + \cdots + \epsilon_i & (1 \leq i \leq r-1), \\
(\epsilon_1 + \cdots + \epsilon_r)/2 & (i = r).
\end{cases}
\]

Choose \( I = \{2, \ldots, r\} \) and \( I^c = \{1\} \) as in the following diagram.

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_r
\end{array}
\]

Then

\[
(3.6) \quad \alpha_1 = \alpha_2 = \cdots = \alpha_3.
\]

Then

\[
(3.7) \quad \Psi_I^Y = \{\alpha_2^Y = \epsilon_2 - \epsilon_3, \ldots, \alpha_{r-1}^Y = \epsilon_{r-1} - \epsilon_r, \alpha_r^Y = 2\epsilon_r\}
\]

and \( \Delta^* = \{\epsilon_i \pm \epsilon_j \mid 2 \leq j \leq r\} \cup \{2\epsilon_1\} \). Hence

\[
(3.8) \quad \mathcal{V}_I^Y = \{\{\epsilon_1 - \epsilon_j\} \cup \Psi_I^Y\}_{2 \leq j \leq r} \cup \{\{\epsilon_1 + \epsilon_j\} \cup \Psi_I^Y\}_{2 \leq j \leq r} \cup \{\{2\epsilon_1\} \cup \Psi_I^Y\}.
\]
For $\beta^V = e_i - e_j$ $(2 \leq j \leq r)$, we see that

\begin{equation}
(3.9) \quad b_l = \langle \beta^V, \lambda_i \rangle = \begin{cases} 
1 & \text{if } (1 \leq l < j), \\
0 & \text{if } (j \leq l \leq r), 
\end{cases}
\end{equation}

and in particular $\mu_{\beta}^V = \lambda_1 = e_1$. Therefore for $\gamma^V = e_1 - e_i \in \Delta^* (i \neq j)$ and $\gamma^V = e_1 + e_i \in \Delta^*$,

\begin{equation}
(3.10) \quad p_{V_i}^* (\gamma^V) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1)(e_1 - e_j)
\end{equation}

and

\begin{equation}
(3.11) \quad p_{V_i}^* (2e_1) = 2e_1 - (2e_1, \lambda_1)(e_1 - e_j)
\end{equation}

Next for $\beta^V = e_1 + e_j$ $(2 \leq j \leq r)$, we see that

\begin{equation}
(3.12) \quad b_l = \langle \beta^V, \lambda_i \rangle = \begin{cases} 
1 & \text{if } (1 \leq l < j), \\
2 & \text{if } (j \leq l \leq r), \\
1 & \text{if } (l = r), 
\end{cases}
\end{equation}

and in particular $\mu_{\beta}^V = \lambda_1 = e_1$. Therefore for $\gamma^V = e_1 + e_i \in \Delta^* (i \neq j)$ and $\gamma^V = e_1 - e_i \in \Delta^*$,

\begin{equation}
(3.13) \quad p_{V_i}^* (\gamma^V) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1)(e_1 + e_j)
\end{equation}

and

\begin{equation}
(3.14) \quad p_{V_i}^* (2e_1) = 2e_1 - (2e_1, \lambda_1)(e_1 + e_j)
\end{equation}

Finally for $\beta^V = 2e_1$, we see that

\begin{equation}
(3.15) \quad b_l = \langle \beta^V, \lambda_i \rangle = \begin{cases} 
2 & \text{if } (1 \leq l < r), \\
1 & \text{if } (l = r), 
\end{cases}
\end{equation}

and in particular $\mu_{\beta}^V = \lambda_1/2 = e_1/2$. Therefore for $\gamma^V = e_1 \pm e_i \in \Delta^* (i \neq j)$, $\gamma^V = e_1 \pm e_i \in \Delta^*$,

\begin{equation}
(3.16) \quad p_{V_i}^* (\gamma^V) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1/2)(2e_1)
\end{equation}

Using the above data of $p_{V_i}^* (\gamma^V)$ and (3.5), we can calculate

\[ (p_{V_i}^* (\gamma^V), \lambda) = (p_{V_i}^* (\gamma^V), m_2 \lambda_2 + \cdots + m_r \lambda_r). \]

Let

\[ m_{j,k} = \begin{cases} 
m_j + \cdots + m_k & \text{if } (j \leq k), \\
0 & \text{otherwise}. 
\end{cases} \]
(Notice that this notation is different from $m_{j,k}$ used in [25, (32)].) Then we obtain that, when $\beta^\vee = e_1 - e_j$ ($2 \leq j \leq r$), then

$$\langle p_{V_i}^*(\gamma^\vee), \lambda \rangle =\begin{cases} -m_{i,j-1} & (\gamma^\vee = e_1 - e_i, i < j) \\ m_{j,i-1} & (\gamma^\vee = e_1 - e_i, j < i) \\ m_{i,j-1} + 2m_{j,r-1} + m_r & (\gamma^\vee = e_1 + e_i, i \leq j) \\ m_{j,i-1} + 2m_{i,r-1} + m_r & (\gamma^\vee = e_1 + e_i, j < i) \\ 2m_{j,r-1} + m_r & (\gamma^\vee = 2e_1), \end{cases}$$

when $\beta^\vee = e_1 + e_j$ ($2 \leq j \leq r$), then

$$\langle p_{V_i}^*(\gamma^\vee), \lambda \rangle =\begin{cases} -(m_{i,j-1} + 2m_{j,r-1} + m_r) & (\gamma^\vee = e_1 - e_i, i \leq j) \\ -(m_{j,i-1} + 2m_{i,r-1} + m_r) & (\gamma^\vee = e_1 - e_i, j < i) \\ m_{i,j-1} & (\gamma^\vee = e_1 + e_i, i \leq j) \\ -m_{j,i-1} & (\gamma^\vee = e_1 + e_i, j < i) \\ -2m_{j,r-1} + m_r & (\gamma^\vee = 2e_1), \end{cases}$$

and when $\beta^\vee = 2e_1$, then

$$\langle p_{V_i}^*(\gamma^\vee), \lambda \rangle = \pm \left( m_{i,r-1} + \frac{1}{2} m_r \right)$$

for $\gamma^\vee = e_1 \pm e_i$.

Therefore by (3.1) we now obtain the explicit form of the generating function in the $B_r$ case. We write $t_{e_1} = t_{e_i} = t_{e_j}$ for $2 \leq i \leq r$ and $t_{e_1} = t_1$. Then the explicit form is as follows.

$$F(t_1, (t_{\pm i})_{2 \leq i \leq r}, (y_i)_{1 \leq j \leq r}, (m_i)_{2 \leq i \leq r}; \{2, \ldots, r\}; B_r)$$

$$= \sum_{j=2}^{r} \prod_{2 \leq i < j} \frac{t_{-i}}{t_{-i} - t_{-j} + 2\pi \sqrt{-1}m_{i,j-1}} \prod_{j \leq i \leq r} \frac{t_{-i}}{t_{-i} - t_{-j} - 2\pi \sqrt{-1}m_{j,i-1}}$$

$$\times \prod_{2 \leq i \leq j} \frac{t_{+i}}{t_{+i} - t_{-j} - 2\pi \sqrt{-1}(m_{i,j-1} + 2m_{j,r-1} + m_r)}$$

$$\times \prod_{j \leq i \leq r} \frac{t_{+i}}{t_{+i} - t_{-j} - 2\pi \sqrt{-1}(m_{j,i-1} + 2m_{i,r-1} + m_r)}$$

$$\times \frac{t_1}{t_1 - 2t_{-j} - 2\pi \sqrt{-1}(2m_{j,r-1} + m_r)}$$

$$\times \exp\left(2\pi \sqrt{-1}\left(\sum_{i=2}^{r-1} m_i (y_i - y_1) + \sum_{i=j}^{r} m_i y_i\right)\right) \frac{t_{-j} \exp(t_{-j} \{y_i\})}{e^{t_{-j} - 1}}$$

$$+ \sum_{j=2}^{r} \prod_{2 \leq i < j} \frac{t_{-i}}{t_{-i} - t_{+j} + 2\pi \sqrt{-1}(m_{i,j-1} + 2m_{j,r-1} + m_r)}$$

$$\times \prod_{j \leq i \leq r} \frac{t_{-i}}{t_{-i} - t_{+j} + 2\pi \sqrt{-1}(m_{j,i-1} + 2m_{i,r-1} + m_r)}$$

$$\times \prod_{2 \leq i < j} \frac{t_{+i}}{t_{+i} - t_{+j} - 2\pi \sqrt{-1}(m_{i,j-1} + 2m_{j,r-1} + m_r)}$$

$$\times \prod_{j \leq i \leq r} \frac{t_{+i}}{t_{+i} - t_{+j} + 2\pi \sqrt{-1}(m_{j,i-1} + 2m_{i,r-1} + m_r)}$$

$$\times \frac{t_1}{t_1 - 2t_{+j} + 2\pi \sqrt{-1}(2m_{j,r-1} + m_r)}$$

$$\times \exp\left(2\pi \sqrt{-1}\left(\sum_{i=2}^{j-1} m_i (y_i - y_1) + \sum_{i=j}^{r-1} m_i (y_i - 2y_1) + m_r (y_r - y_1)\right)\right)\right)$$
\[ \times \frac{t_{ij} \exp(t_{ij} \{y_i\})}{e^{t_{ij}} - 1} + \prod_{2 \leq i \leq r} \frac{t_{i-1}}{t_{i-1} - \frac{1}{2} t_1 + \pi \sqrt{-1} (2m_{i, r-1} + m_r)} \times \prod_{2 \leq i \leq r} \frac{t_{i+1}}{t_{i+1} - \frac{1}{2} t_1 - \pi \sqrt{-1} (2m_{i, r-1} + m_r)} \times \frac{1}{2} \left( \exp \left( 2\pi \sqrt{-1} \left( \sum_{i=2}^{r-1} m_i (y_i - y_1) + m_r (y_r - \frac{1}{2} y_1) \right) \right) \times \frac{t_1 \exp(t_1 \{\frac{1}{2} y_1\})}{e^{t_1} - 1} + \exp \left( 2\pi \sqrt{-1} \left( \sum_{i=2}^{r-1} m_i (y_i - y_1) + m_r (y_r - \frac{1}{2} (y_1 + 1)) \right) \right) \times \frac{t_1 \exp(t_1 \{\frac{1}{2} (y_1 + 1)\})}{e^{t_1} - 1} \right). \]

3.2. \( D_r \) Case. We realize \( \Delta^\vee = \{ e_i \pm e_j \mid 1 \leq i < j \leq r \} \) and \( \Psi^\vee = \{ e_1 - e_2, \ldots, e_{r-1} - e_r, e_{r-1} + e_r \} \). Then

\[ \lambda_i = \begin{cases} e_1 + \cdots + e_i & (1 \leq i \leq r - 2), \\ \frac{1}{2} (e_1 + \cdots + e_{r-1} - e_r) & (i = r - 1), \\ \frac{1}{2} (e_1 + \cdots + e_{r-1} + e_r) & (i = r). \end{cases} \] (3.18)

Choose \( I = \{2, \ldots, r\} \) and \( I^c = \{1\} \) as in the following diagram.

\[
\begin{array}{c}
\alpha_1 \\
\hline
\alpha_2
\end{array}
\]

Then

\[ \Psi^\vee_I = \{ \alpha_2^\vee = e_2 - e_3, \ldots, \alpha_{r-1}^\vee = e_{r-1} - e_r, \alpha_r^\vee = e_{r-1} + e_r \} \] (3.19)

and \( \Delta^*\vee = \{ e_1 \pm e_j \mid 2 \leq j \leq r \} \). Hence

\[ \Psi^\vee_I = \{ \{ e_1 - e_j \} \cup \Psi^\vee_I \}_{2 \leq j \leq r} \cup \{ \{ e_1 + e_j \} \cup \Psi^\vee_I \}_{2 \leq j \leq r} \] (3.21)

For \( \beta^\vee = e_1 - e_j \) (2 \leq j \leq r), we see that

\[ b_l = \langle \beta^\vee, \lambda_i \rangle = \begin{cases} 1 & (l < j) \\ 0 & (\text{otherwise}) \end{cases} \] (3.22)

and in particular \( \mu^\vee_\beta = \lambda_1 = e_1 \). Therefore for \( \gamma^\vee = e_1 - e_i \in \Delta^*\vee (i \neq j) \) and \( \gamma^\vee = e_1 + e_i \in \Delta^*\vee \),

\[ p^\vee_I(\gamma^\vee) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1)(e_1 - e_j) = e_1 \pm e_i - (e_1 - e_j) = \begin{cases} e_j \pm e_i \in \Delta^\vee & (i \neq j) \\ 2e_i & (i = j). \end{cases} \] (3.23)

Next for \( \beta^\vee = e_1 + e_j \) (2 \leq j \leq r), we see that

\[ b_l = \langle \beta^\vee, \lambda_i \rangle = \begin{cases} 0 & (l = r - 1, j = r) \\ 2 & (j \leq l \leq r - 2) \\ 1 & (\text{otherwise}) \end{cases} \] (3.24)
and in particular \( \mu^Y = \lambda_1 = e_1 \). Therefore for \( \gamma^Y = e_1 + e_i \in \Delta^{*Y} \) (\( i \neq j \)) and \( \gamma^Y = e_1 - e_i \in \Delta^{*Y}, \)

\[
p^*_Y \gamma (\gamma^Y) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1)(e_1 + e_j)
= e_1 \pm e_i - (e_1 + e_j)
= \begin{cases} 
eg e_j \pm e_i \in \Delta^Y \
eg 2e_j \end{cases} (i \neq j)
\]

(3.25)

Using the same notation as in the \( B_r \) case, we can now calculate \( p^*_Y \gamma (\gamma^Y), \lambda \) and write down the explicit form of the generating function as follows.

\[
F((t_{\pm i})_{2 \leq i \leq r}, (y_j)_{1 \leq j \leq r}, (m_i)_{2 \leq i \leq r}; \{2, \ldots, r\}; D_r) = \sum_{j=2}^{r-1} \prod_{2 \leq i < j} t_{i-j} - 2\pi \sqrt{-1} m_{i,j-1} \prod_{j < i \leq r} t_{i-j} - 2\pi \sqrt{-1} m_{j,i-1}
\]

\[
\times \prod_{2 \leq i \leq r} t_{i-j} - 2\pi \sqrt{-1} (m_{i,r-2} + m_r)
\]

\[
\times t_{i-r} - 2\pi \sqrt{-1} (m_r - m_{r-1}) \exp(2\pi \sqrt{-1} \sum_{i=2}^{r-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_i y_i) \frac{t_r \exp(t_r \{y_1\})}{e^{t_r} - 1}
\]

\[
+ \sum_{j=2}^{r-1} \prod_{2 \leq i \leq j} t_{i-j} + 2\pi \sqrt{-1} (m_{i,j-1} + 2m_{j,r-2} + m_{r-1,r}) \prod_{j < i \leq r} t_{i-j} + 2\pi \sqrt{-1} (m_{j,i-1} + 2m_{i,r-2} + m_{r-1,r})
\]

\[
\times t_{i-r} + 2\pi \sqrt{-1} (m_{r-2} + m_r)
\]

\[
\times \prod_{2 \leq i < j} t_{i-j} - 2\pi \sqrt{-1} m_{i,j-1} \prod_{j < i \leq r} t_{i-j} - 2\pi \sqrt{-1} m_{j,i-1}
\]

\[
\times \exp(2\pi \sqrt{-1} \sum_{i=2}^{r-1} m_i(y_i - y_1) + \sum_{i=j}^{r-2} m_i(y_i - 2y_1))
\]

\[
\]
and in particular $\mu_{\beta}^V = \lambda_2$. Therefore for $\gamma^V \in \Delta^* \setminus \{\beta^V\}$,

$$
p_{I}^{V_{I}}(\gamma^V) = \gamma^V - \langle \gamma^V, \lambda_2 \rangle \beta^V
$$

and

$$
= \gamma^V - \beta^V.
$$
By putting \( t_{e_i - e_j} = t_{ij} \) for \( 1 \leq i \leq 2 < j \leq 4 \), we obtain the generating function as follows.

\[
F((t_{13}, t_{23}, t_{23}, t_{14}), (y_1, y_2, y_3), (m_1, m_3); \{1, 3\}; A_3) =
\frac{t_{13}}{t_{23} - t_{13} + 2\pi\sqrt{-1}m_1} \frac{t_{14}}{t_{24} - t_{13} - 2\pi\sqrt{-1}m_3}
\times \frac{t_{24} - t_{13} - 2\pi\sqrt{-1}(m_3 - m_1)}{t_{24}}
\times \exp \left( 2\pi\sqrt{-1} \left(m_1(y_1 - y_2) + m_3y_3 \right) \right) \frac{t_{13} \exp(t_{13}\{y_2\})}{e^{t_{13}} - 1}
\]

\[
= \frac{t_{14}}{t_{24} - t_{23} - 2\pi\sqrt{-1}m_1} \frac{t_{13}}{t_{23} - t_{24} - 2\pi\sqrt{-1}m_3}
\times \frac{t_{24} - t_{23} - 2\pi\sqrt{-1}(m_3 - m_1)}{t_{23}}
\times \exp \left( 2\pi\sqrt{-1} \left(m_1y_1 + m_3(y_3 - y_2) \right) \right) \frac{t_{13} \exp(t_{23}\{y_2\})}{e^{t_{23}} - 1}
\]

\[
+ \frac{t_{13}}{t_{23} - t_{14} + 2\pi\sqrt{-1}m_3} \frac{t_{23}}{t_{24} - t_{14} + 2\pi\sqrt{-1}(m_1 + m_3)}
\times \frac{t_{24} - t_{14} + 2\pi\sqrt{-1}m_1}{t_{24}}
\times \exp \left( 2\pi\sqrt{-1} \left(m_1(y_1 - y_2) + m_3(y_3 - y_2) \right) \right) \frac{t_{13} \exp(t_{24}\{y_2\})}{e^{t_{24}} - 1}
\]

\[
\times \exp \left( 2\pi\sqrt{-1} \left(m_1(y_1 - y_2) + m_3(y_3 - y_2) \right) \right) \frac{t_{14} \exp(t_{14}\{y_2\})}{e^{t_{14}} - 1}.
\]

5. Several examples of functional relations

From the explicit forms of generating functions given in the preceding two sections, we can deduce various functional relations. From the results proved in Section 3, we can show the general forms of functional relations in the cases of types \( B_r \) and \( D_r \), similar to [23 Theorem 3.2], though we do not give the statement in the present paper. We here give explicit examples of generating functions of type \( B_3 \) and also of type \( A_3 \) (\( \simeq D_3 \)).

5.1. \( B_3 \) Case. We write zeta-functions of root systems of type \( B_2 \) and type \( B_3 \) as

\[
\zeta_2(s_1, s_2, s_3, s_4; B_2) = \sum_{m_1, m_2 \geq 1} \frac{1}{m_1^s m_2^s (m_1 + m_2)^s (2m_1 + m_2)^s},
\]

\[
\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9; B_3) = \sum_{m_1, m_2, m_3 \geq 1} \frac{1}{m_1^s m_2^s m_3^s (m_1 + m_2 + m_3)^s (2m_2 + m_3)^s (2m_1 + 2m_2 + m_3)^s}.
\]

These series converge absolutely for \( \Re s_1, \ldots, \Re s_9 \geq 1 \) because \( m_1 + m_2 \geq 2\sqrt{m_1m_2} \) and \( m_1 + m_2 + m_3 \geq 3\sqrt{m_1m_2m_3} \).
Here we give explicit forms of functional relations among them as follows. Let \( r = 3, \Delta = \Delta(B_3), I = \{2, 3\} \) and \((y_1, y_2, y_3) = (0, 0, 0)\) in (3.17). Then we have

\[
\begin{align*}
(5.1) \quad F(t_1, t_{\pm 2}, t_{\pm 3}), 0, (m_2, m_3); \{2, 3\}; B_3) &= \\
&= t_{-3} \frac{t_{-3}}{t_{-3} - t_{-2} - 2\pi \sqrt{-1}m_2 t_{+2} - t_{-2} - 2\pi \sqrt{-1}(2m_2 + m_3)} \\
&\times \frac{t_{+3}}{t_{+3} - t_{-2} - 2\pi \sqrt{-1}(m_2 + m_3)} \\
&\times \frac{t_{-2}}{t_1 - 2t_{-2} - 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{-2}} - 1} \\
&+ \frac{t_{-2} - t_{-3} + 2\pi \sqrt{-1}m_2 t_{+2} - t_{-3} - 2\pi \sqrt{-1}(m_2 + m_3)}{t_{-3} t_{+3}} \\
&\times \frac{t_{-3}}{t_{+3} - t_{-2} - 2\pi \sqrt{-1}m_2 t_1 - t_{-2} - 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{-2}} - 1} \\
&+ \frac{t_{-2} - t_{+3} + 2\pi \sqrt{-1}(m_2 + m_3) t_{-2} + t_{-2} + 2\pi \sqrt{-1}(m_2 + m_3)}{t_{+3} t_{+2}} \\
&\times \frac{t_{+2}}{t_{+2} - 2t_{+3} + 2\pi \sqrt{-1}m_2 t_{+3} - 2t_{+2} + 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{+3}} - 1} \\
&+ \frac{t_{-2} - \frac{1}{2}t_1 + \pi \sqrt{-1}(2m_2 + m_3) t_{-3} - \frac{1}{2}t_1 + \pi \sqrt{-1}m_3}{t_{+3} t_{+2}} \\
&\times \frac{t_{+2}}{t_{+2} - \frac{1}{2}t_1 - \pi \sqrt{-1}(2m_2 + m_3) t_{+3} - \frac{1}{2}t_1 - \pi \sqrt{-1}m_3} \\
&\times \frac{1}{2} \left( \frac{t_1}{e^{t_1} - 1} + (-1)^{m_3} t_1 \exp \left( \frac{1}{2} t_1 \right) \right).
\end{align*}
\]

Hence we can compute \( P(k, y, \lambda; I; B_3) \). For example, we obtain

\[
(5.2) \quad P((2, 1, 1, 1), 0, (m_2, m_3); \{2, 3\}; B_3) = \\
= \frac{1}{16\pi^6 m_2^3 m_3^2 (m_2 + m_3)} + \frac{1}{16\pi^6 m_2^3 (m_2 + m_3)(2m_2 + m_3)^3} \\
- \frac{(-1)^{m_3}}{16\pi^6 m_2^3 (m_2 + m_3)} \\
- \frac{1}{4\pi^6 m_2^3 (2m_2 + m_3)^2} \frac{(-1)^{m_3}}{16\pi^6 m_2 m_3^3 (m_2 + m_3)^2} \\
- \frac{1}{24\pi^4 m_2^3 (2m_2 + m_3)^2} \frac{(-1)^{m_3}}{4\pi^6 m_2^3 (2m_2 + m_3)^4} \\
+ \frac{1}{12\pi^4 m_2^3 (2m_2 + m_3)^2} + \frac{1}{16\pi^6 m_2 m_3^2 (2m_2 + m_3)^3} \\
+ \frac{1}{16\pi^6 m_2 (m_2 + m_3)(2m_2 + m_3)^4}.
\]
Therefore, from Theorem 2 and by using the relation

\[
\sum_{m,n \geq 1} \frac{(-1)^n}{m^s n^s (m+n)^s (2m+n)^s} = - \sum_{m,n \geq 1} \frac{1}{m^s n^s (m+n)^s (2m+n)^s} + 2 \sum_{m,n \geq 1} \frac{1}{m^s (2n)^s (m+2n)^s (2m+2n)^s}
\]

we obtain the functional relation

\[
(5.3) \quad \zeta(1, s_2, s_3, 1, s_5, s_6, 1, 1, 2; B_3) - \zeta(1, s_2, s_3, 1, s_5, s_6, 1, 1, 2; B_3) + \zeta(s_2, 1, 2, 1, 1, s_5, s_6; B_3) + \zeta(2, 1, 2, 1, 1, s_5, s_6; B_3) - \zeta(1, s_2, 1, 2, s_5, s_6; B_3) + \zeta(1, s_5, s_6; B_3)
\]

\[
= (-1)^6 \left(2 \sqrt{2} \pi \right)^6 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P((2,1,1,1,1,1,0,1); m_2, m_3; \{2,3\}; B_3)}{m_2^s m_3^s (m_2+m_3)^s (2m_2+m_3)^s}
\]

\[
= 2 \zeta(2, s_2 + 2, s_3 + 3, s_5 + 1, s_6; B_2) + 2 \zeta(2, s_2 + 1, s_3 + 3, s_5 + 1, s_6; B_2) - \frac{2^{s_3-s_6}}{4} \zeta(2, s_3 + 4, s_2 + 6, s_5; B_2)
\]

\[
+ 2 \zeta(2, s_2 + 1, s_3 + 3, s_5 + 2, s_6; B_2) + 4 \pi^2 \zeta(2, s_3 + 2, s_5, s_6 + 2; B_2) - \frac{2^{s_3-s_6}}{4} \zeta(2, s_3 + 2, s_2 + 6, s_5; B_2)
\]

\[
+ 2 \zeta(2, s_2 + 1, s_3 + 3, s_5 + 2, s_6; B_2) + 10 \zeta(2, s_2 + 1, s_3 + 1, s_5 + 1, s_6 + 2; B_2).
\]

Setting \((s_2, s_5, s_6) = (1, 1, 2)\), we obtain

\[
(5.4) \quad 2 \zeta(1, 1, 2, 1, 1, s_5, s_6, 1, 1, 2; B_3)
\]

\[
= \frac{2}{2} \zeta(3, s_3 + 3, 2, 2; B_2) + 2 \zeta(3, s_3 + 3, 2, 5; B_2)
\]

\[
- 10 \zeta(2, s_3 + 4, 2, 2; B_2) - \frac{2^{s_3}}{16} \zeta(2, s_3 + 4, 1, 4, 1; B_2)
\]

\[
- 2 \zeta(2, s_3 + 3, 3, 2; B_2) + 4 \pi^2 \zeta(2, s_3 + 2, 1, 4; B_2)
\]

\[
- \frac{2^{s_3-s_6}}{24} \zeta(2, s_3 + 2, 1, 4, 1; B_2) - \frac{2^{s_3}}{16} \zeta(2, s_3 + 2, 1, 6, 1; B_2)
\]

\[
+ 2 \zeta(2, s_3 + 3, 5; B_2) + 10 \zeta(2, s_3 + 2, 6; B_2)
\]

which corresponds to the result for the \(C_3\) case (see [25 (97)]).

Here we recall the known fact that

\[
(5.5) \quad \zeta(2, a, b, c, d; B_2) \in \mathbb{Q} \left[\pi^2, \{\zeta(2j+1)\}_{j \in \mathbb{N}}\right]
\]

for \(a, b, c, d \in \mathbb{N}\) with \(2 \nmid (a + b + c + d)\), which was given by the third-named author (see [39]). Similar to [25 (100)], setting \(s_3 = 2k - 1\) \((k \in \mathbb{N})\) in \(5.4\), we obtain from \(5.5\) that

\[
(5.6) \quad \zeta(3, 1, 1, 2, 1, 1, 2k - 1, 1, 1, 2; B_3) \in \mathbb{Q} \left[\pi^2, \{\zeta(2j+1)\}_{j \in \mathbb{N}}\right]
\]

for \(k \in \mathbb{N}\). For example, we obtain

\[
\zeta(3, 1, 1, 2, 1, 1, 1, 1, 2; B_3) = \frac{9 \pi^4}{320} \zeta(7) - \frac{1429 \pi^2}{384} \zeta(9) + \frac{4355}{128} \zeta(11).
\]
\[
\zeta_3(1, 1, 2, 1, 1, 3, 1, 1, 2; B_3) = -\frac{7\pi^4}{320}\zeta(9) + \frac{5143\pi^2}{1536}\zeta(11) - \frac{15833}{512}\zeta(13),
\]
\[
\zeta_3(1, 1, 2, 1, 1, 5, 1, 1, 2; B_3) = \frac{23\pi^8}{2419200}\zeta(7) + \frac{11\pi^6}{20160}\zeta(9) - \frac{941\pi^4}{15360}\zeta(11) + \frac{16121\pi^2}{2048}\zeta(13) - \frac{74079}{1024}\zeta(15).
\]

5.2. \textbf{A}_3 \textbf{Case} and \textbf{D}_3 \textbf{Case}. Here we deduce an explicit functional relation involving the zeta-function of \textbf{A}_3, from the result proved in Section 4. From (2.8) and (4.9), we can compute \(P(k, y, \lambda; 1, 3; A_3)\). For example, we obtain
\[
P((1, 1, 1, 1), 0, (m_1, m_3); \{1, 3\}; A_3)
\]
\[
= \frac{1}{8\pi^4m_1^2m_3(m_1 + m_3)} + \frac{1}{8\pi^4m_1^2m_3(m_3 - m_1)}
\]
\[
- \frac{1}{8\pi^4m_1m_3^2(m_3 - m_1)} + \frac{1}{8\pi^4m_1m_3^2(m_1 + m_3)}
\]
\[
- \frac{1}{8\pi^4m_1m_3(m_3 - m_1)^2} + \frac{1}{8\pi^4m_1m_3(m_1 + m_3)^2}.
\]

When \(m_1 = m_3 = m\), we have
\[
P((1, 1, 1, 1), 0, (m, m); \{1, 3\}; A_3) = \frac{7}{32\pi^4m^4} - \frac{1}{48\pi^2m^2}.
\]

Renaming the variables in the case \(r = 3\) of (1.2), we write
\[
\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6; A_3)
\]
\[
= \sum_{m_1, m_2, m_3 = 1}^{\infty} \frac{1}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1 + m_2)^{s_4}(m_2 + m_3)^{s_5}(m_1 + m_2 + m_3)^{s_6}}.
\]

This series converges absolutely for \(\Re s_1, \ldots, \Re s_6 \geq 1\) because \(m_1 + m_2 + m_3 \geq 3\sqrt{m_1m_2m_3}\).

Then, from Theorem 2, we obtain the functional relation
\[
\zeta_3(s_1, 1, s_3, 1, 1, 1; A_3) - \zeta_3(1, 1, 1, s_1, s_3, 1; A_3)
\]
\[
+ \zeta_3(s_1, 1, 1, 1, s_1, 1; A_3) + \zeta_3(s_1, 1, 1, 1, 1; A_3)
\]
\[
- \zeta_3(1, 1, 1, s_1, 1, 1; A_3) + \zeta_3(1, s_1, 1, 1, 1, s_3; A_3)
\]
\[
= (2\pi\sqrt{-1})^4
\]
\[
\times \left( \sum_{m_1, m_2, m_3 = 1}^{\infty} \frac{1}{m_1^{s_1}m_2^{s_2}m_3^{s_3}}P((1, 1, 1, 1), 0, (m_1, m_3); \{1, 3\}; A_3)
\]
\[
+ \sum_{m = 1}^{\infty} \frac{1}{m^{s_1+s_3}}P((1, 1, 1, 1), 0, (m, m); \{1, 3\}; A_3) \right)
\]
\[
= 2\left\{ \zeta_2(s_1 + 2, s_3 + 1, 1; A_2) + \zeta_2(s_1 + 2, 1, s_3 + 1; A_2)
\right.
\]
\[
- \zeta_2(s_3 + 1, 1, s_1 + 2; A_2) - \zeta_2(s_1 + 1, s_3 + 2; A_2)
\]
\[
+ \zeta_2(s_3 + 2, 1, s_1 + 1; A_2) + \zeta_2(s_1 + 1, s_3 + 2, 1; A_2)
\]
\[
- \zeta_2(s_1 + 1, 2, s_3 + 1; A_2) - \zeta_2(s_3 + 1, 2, s_1 + 1; A_2)
\right\}
\]

\[ + \zeta_2(s_1 + 1, s_3 + 1, 2; A_2) \right) + \zeta(s_1 + s_3 + 4) \]
\[ - \frac{\pi^2}{3} \zeta(s_1 + s_3 + 2). \]

If we set \((s_1, s_3) = (1, 1)\), we see that
\[ \{ \zeta_2(k, l, m; A_2) \mid k, l, m \in \mathbb{N} \text{ with } k + l + m = 6 \} \]
appear on the right-hand side. Using the partial fraction decomposition formula
\[ \frac{1}{X^{p+q}} = \sum_{i=0}^{p-1} \binom{q-1+i}{i} X^{p-i} Y^q + \sum_{i=0}^{q-1} \binom{p-1+i}{i} Y^{q-i} X^p \quad (p, q \in \mathbb{N}), \]
we obtain
\[ \zeta_2(k, l, m; A_2) = \sum_{i=0}^{k-1} \binom{l-1+i}{i} \zeta_{\mathbb{E}_2, 2}(k-i, l+m+i) \]
\[ + \sum_{i=0}^{l-1} \binom{k-1+i}{i} \zeta_{\mathbb{E}_2, 2}(l-i, k+m+i) \]
(see Huard et al. [1, (1.6)]). It is well-known that \( \zeta_{\mathbb{E}_2, 2}(p, q) \) \((p, q \in \mathbb{N}, q \geq 2, p + q \leq 6)\) can be expressed in terms of \( \{ \zeta(m) \mid 2 \leq m \leq 6 \} \), by using \( \mathbb{E}_2 \)-Face, an interface for evaluation of Euler sums (see [3]). Using these results, we can consequently obtain
\[ (5.9) \]
\[ 2 \zeta_2(1, 1, 1, 1, 1; A_3) = 4 \zeta(3)^2 - \frac{31}{5670} \pi^6. \]

We emphasize that this formula is a new result. It cannot be deduced from the known functional relation for type \( A_3 \) [20, Theorem 9].

Here we give a comment on the \( D_3 \) case. Although generally we consider \( r \geq 4 \) for \( D_r \) cases, the generating function (3.26) is valid even when \( r = 3 \). In this case, we see that the Dynkin diagram (3.19) reduces to
\[ (5.10) \]
and this coincides with (4.1). In fact, if we read \( t_{13}, t_{14}, t_{23}, t_{24}, y_1, y_2, y_3, m_1, m_3 \) as \( t_{-3}, t_{+2}, t_{-2}, t_{+3}, y_2, y_1, y_3, m_2, m_3 \) respectively in (4.9), then we obtain the generating function for \( D_3 \) which comes from (3.26).

6. ANOTHER TYPE OF FUNCTIONAL RELATION FOR \( \zeta_2(s; C_2) \)

In this section, we give another type of functional relation for the zeta-function of type \( C_2 \) defined by
\[ \zeta_2(s_1, s_2, s_3, s_4; C_2) = \sum_{m, n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}} \]
which absolutely converges in the region \( \Re s_j \geq 1 \) \((j = 1, 3, 4)\) and \( \Re s_2 \geq 0 \).

We first summarize the progress of research on this function.

As stated in Section 1, the second author defined (6.1) inspired by Zagier’s observation on Witten’s zeta-function in [42], and showed its analytic continuation in [26]. Based on
this result, the third author evaluated (6.1) at any positive integer point \((s_1, s_2, s_3, s_4) = (k_1, k_2, k_3, k_4)\) where \(k_1 + k_2 + k_3 + k_4\) is odd (see \([39]\)). Furthermore, we gave general forms of functional relations for zeta-functions of root systems, and explicit examples for them including (6.1) (see \([10, \text{Section 3}], [11, \text{Section 3}]\)). More explicit expressions for functional relations for \(\zeta_2(s; C_2)\) were given by Nakamura \([35]\) and the authors \([14]\). As for their character analogues, see \([13]\).

It is emphasized that these functional relations include Witten’s volume formulas which imply \(\zeta_2(2k, 2k, 2k, 2k; C_2) \in \mathbb{Q} \cdot \pi^{8k} \ (k \in \mathbb{N})\). On the other hand, these give no information on \(\zeta_2(2k-1, 2k-1, 2k-1, 2k-1; C_2) \ (k \in \mathbb{N})\), because in these cases, the functional relations vanish. We here give a new type of functional relations between \(\zeta_2(s; C_2)\) and double zeta-functions of Euler–Zagier type, and evaluate \(\zeta_2(s; C_2)\) at positive integer points from the known result on double zeta values. This result especially gives an explicit expression formula for \(\zeta_2(1, 1, 1, 1; C_2)\) in terms of \(\zeta(s)\) and polylogarithms (see (6.10) below).

The result in this section is given by considering partial fraction decompositions and partial summations of harmonic sums, and the method here is totally different from that stated in the preceding sections. The main technique is similar to that used in \([39]\) (and methodologically some common feature with \([8, 43]\), though they studied the case of type \(A_r\)).

Let

\[
\phi(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s} = (2^{1-s} - 1)\zeta(s),
\]

\[
\zeta_{EZ,2}(s_1, s_2; \sigma_1, \sigma_2) = \sum_{1 \leq m < n} \frac{\sigma_1^m \sigma_2^n}{m^{s_1} n^{s_2}} \ (\sigma_1, \sigma_2 \in \{\pm 1\}).
\]

Note that \(\zeta_{EZ,2}(s_1, s_2; 1, 1) = \zeta_{EZ,2}(s_1, s_2)\). We see that the function \(\zeta_{EZ,2}(s_1, s_2; \sigma_1, \sigma_2)\) can be continued meromorphically to the whole space \(\mathbb{C}^2\). In fact, this function is a special case of twisted double zeta-functions, whose analytic properties have been studied by many authors. When \((\sigma_1, \sigma_2) = (-1, -1)\), the proof of the continuation is included in the results of Akiyama and Ishikawa \([11]\) or of de Crisenoy \([4]\), and when \((\sigma_1, \sigma_2) = (\pm 1, \mp 1)\), it is included in \([9]\) or \([13]\).

By considering partial fraction decompositions and the partial summation, we prove the following result.

**Proposition 4.** For \(a, b, c \in \mathbb{N}\) and \(s \in \mathbb{C}\),

\[
\zeta_2(a, b, c; C_2) = \sum_{\ell=0}^{b-1} \binom{c - 1 + \ell}{\ell} (-1)^\ell \\
\times \left\{ (-1)^a \sum_{j=0}^{b-\ell-2} \binom{a - 1 + j}{j} \zeta_{EZ,2}(s + a + c + \ell + j, b - \ell - j) \\
+ \sum_{j=0}^{a-2} \binom{b - \ell - 1 + j}{j} (-1)^j \zeta(s + b + c + j) \zeta(a - j) \\
- (-1)^a \binom{a + b - \ell - 2}{b - \ell - 1} \\
\times \{ \zeta_{EZ,2}(1, s + a + b + c - 1) + \zeta(s + a + b + c) \} \right\}
\]
\begin{align*}
&\sum_{\ell=0}^{c-1} \binom{b-1+\ell}{\ell} 2^{s+b+\ell-1} \\
&\times \left\{ (-1)^a \sum_{j=0}^{c-\ell-2} \binom{a-1+j}{j} \zeta_{EZ,2}(s+a+b+\ell+j, c-\ell-j) \\
&\quad + \sum_{j=0}^{a-2} \binom{c-\ell-1+j}{j} (-1)^j \zeta(s+b+c+j) \zeta(a-j) \\
&\quad - (-1)^a \binom{a+c-\ell-2}{c-\ell-1} \right\} \\
&\times \{ \zeta_{EZ,2}(1, s+a+b+c-1) + \zeta(s+a+b+c) \} \\
&\times \left\{ (-1)^b \sum_{\ell=0}^{c-1} \binom{b-1+\ell}{\ell} 2^{s+b+\ell-1} \\
&\times \left\{ (-1)^a \sum_{j=0}^{c-\ell-2} \binom{a-1+j}{j} \right. \\
&\quad \times \zeta_{EZ,2}(s+a+b+\ell+j, c-\ell-j; -1, 1) \\
&\quad + \sum_{j=0}^{a-2} \binom{c-\ell-1+j}{j} (-1)^j \phi(s+b+c+j) \zeta(a-j) \\
&\quad - (-1)^a \binom{a+c-\ell-2}{c-\ell-1} \right. \\
&\quad \times \{ \zeta_{EZ,2}(1, s+a+b+c-1, 1, -1) + \phi(s+a+b+c) \} \right\}. \\
\end{align*}

**Proof.** First we recall the partial fraction decomposition formula

\begin{equation}
\frac{1}{X^p(X+Y)^q} = \sum_{i=0}^{p-1} \binom{q-1+i}{i} \frac{(-1)^i}{Y^{q+i}X^{p-i}} \\
+ (-1)^p \sum_{i=0}^{q-1} \binom{p-1+i}{i} \frac{1}{Y^{p+i}(X+Y)^{q-i}}
\end{equation}

for \( p, q \in \mathbb{N} \) (see, for example, [2, Lemma 1]). Furthermore, we consider the following “incomplete” version of (6.2), that is,

\begin{equation}
\frac{1}{X^p(X+Y)^q} = \sum_{i=0}^{p-2} \binom{q-1+i}{i} \frac{(-1)^i}{Y^{q+i}X^{p-i}} \\
+ (-1)^p \binom{p+q-2}{p-1} \frac{1}{XY^{p+q-2}(X+Y)} \\
+ (-1)^q \sum_{i=0}^{q-2} \binom{p-1+i}{i} \frac{1}{Y^{p+i}(X+Y)^{q-i}}
\end{equation}

where the empty sum is interpreted as 0.
For \( a, b, c \in \mathbb{N} \) and \( s \in \mathbb{C} \), using (6.2) with \((X, Y, p, q) = (m + n, n, b, c)\), we have

\[
(6.4) \quad \zeta_2(a, s, b, c; C_2) = \sum_{i=0}^{b-1} \binom{c - 1 + i}{i} (-1)^i \sum_{m,n=1}^{\infty} \frac{1}{m^a n^{s+c+i}(m+n)^{b-i}}
\]

\[
(6.5) \quad + (-1)^b \sum_{i=0}^{c-1} \binom{b - 1 + i}{i} \sum_{m,n=1}^{\infty} \frac{1}{m^a n^{s+b+i}(m+2n)^{c-i}},
\]

where we denote the right-hand side by \( I_1 + I_2 \). Applying (6.3) with \((X, Y, p, q) = (m, n, a, b - i)\) to \( I_1 \), we have

\[
I_1 = \sum_{i=0}^{b-1} \binom{c - 1 + i}{i} (-1)^i \times \left\{ (-1)^a \sum_{j=0}^{b-i-2} \binom{a - 1 + j}{j} \zeta_{E2,2}(s + a + c + i + j, b - i - j) \right. \\
+ (-1)^{a-1} \sum_{j=0}^{a-2} \binom{a + b - i - 2}{b - i - 1} \sum_{m,n=1}^{\infty} \frac{1}{mn^{s+a+b+c-2}(m+n)} \\
+ \sum_{j=0}^{a-2} \binom{b - i - 1 + j}{j} (-1)^j \zeta(s + b + c + j) \zeta(a - j) \right\},
\]

and the second sum in the curly parentheses can be evaluated as

\[
(6.6) \quad \sum_{m,n=1}^{\infty} \frac{1}{mn^{s+a+b+c-2}(m+n)} = \sum_{n=1}^{\infty} \frac{1}{n^{s+a+b+c-1}} \sum_{m=1}^{\infty} \frac{1}{m} - \frac{1}{m + n} = \sum_{1 \leq m \leq n} \frac{1}{mn^{s+a+b+c-1}} = \zeta_{E2,2}(1, s + a + b + c - 1) + \zeta(s + a + b + c).
\]

As for \( I_2 \), corresponding to the decomposition

\[
\sum_{m,n=1}^{\infty} \frac{1}{m^a n^{s+b+i}(m+2n)^{c-i}} = 2^{s+b+i} \sum_{m,n=1}^{\infty} \frac{1}{m^{2a}(2n)^{s+b+i}(m+2n)^{c-i}}
\]

\[
= 2^{s+b+i-1} \sum_{m,n=1}^{\infty} \frac{1}{m^{a-n^a+b+i}(m+n)^{c-i}}
\]

\[
+ 2^{s+b+i-1} \sum_{m,n=1}^{\infty} \frac{(-1)^n}{m^a n^{s+b+i}(m+n)^{c-i}},
\]

we write \( I_2 = I_{21} + I_{22} \). Applying (6.3) with \((X, Y, p, q) = (m, n, a, c - i)\) to \( I_{21} \), we have

\[
I_{21} = (-1)^b \sum_{i=0}^{c-1} \binom{b - 1 + i}{i} 2^{s+b+i-1} \sum_{m,n=1}^{\infty} \frac{1}{m^a n^{s+b+i}(m+n)^{c-i}}
\]
\[= (-1)^b \sum_{i=0}^{c-1} \binom{b-1+i}{i} 2^{s+b+i-1} \]
\[\times \left\{ (-1)^a \sum_{j=0}^{c-i-2} \binom{a-1+j}{j} \right\} \zeta_{E_2}(s + a + b + i + j, c - i - j) \]
\[+ (-1)^{a-1} \left( a + c - i - 2 \right) \sum_{m,n=1}^{\infty} \frac{1}{m n^{s+a+b+c-2}(m+n)} \]
\[+ \sum_{j=0}^{a-2} \binom{c - i - 1 + j}{j} (-1)^j \zeta(s + b + c + j) \zeta(a - j) \right\} \]
\[= (-1)^b \sum_{i=0}^{c-1} \binom{b-1+i}{i} 2^{s+b+i-1} \]
\[\times \left\{ (-1)^a \sum_{j=0}^{c-i-2} \binom{a-1+j}{j} \right\} \zeta_{E_2}(s + a + b + i + j, c - i - j) \]
\[+ (-1)^{a-1} \left( a + c - i - 2 \right) \sum_{j=0}^{a-2} \binom{c - i - 1 + j}{j} (-1)^j \zeta(s + b + c + j) \zeta(a - j) \right\}, \]

where the last equality follows by using (6.6). Similarly we obtain

\[I_{22} = (-1)^b \sum_{i=0}^{c-1} \binom{b-1+i}{i} 2^{s+b+i-1} \]
\[\times \left\{ (-1)^a \sum_{j=0}^{c-i-2} \binom{a-1+j}{j} \right\} \zeta_{E_2}(s + a + b + i + j, c - i - j; -1, 1) \]
\[+ (-1)^{a-1} \left( a + c - i - 2 \right) \zeta_{E_2} \]
\[\sum_{j=0}^{a-2} \binom{c - i - 1 + j}{j} (-1)^j \zeta(s + b + c + j) \zeta(a - j) \right\}. \]

Combining these results, we consequently obtain the assertion. \( \square \)

**Remark 5.** Partial fraction decompositions are very useful tool of finding functional relations. For example, (6.4) is a very simple consequence of partial fraction decomposition, but this formula is already a functional relation among the eta-functions of types \( C_2 \) and \( A_2 \). (Note that (6.4) first appeared in [39, (10)].) Generally speaking, any double shuffle relations and associated functional equations can be shown by using partial fraction decompositions (see [17]).
Example 6. Setting \((a, b, c) = (1, 1, 1)\) \((1, 2, 1)\) and \((3, 3, 3)\) in Proposition 4 we obtain
\[
\zeta_2(1, s, 1; C_2) = (1 - 2^s)\{\zeta_{EZ,2}(1, s + 2) + \zeta(s + 3)\}
- 2^s\{\zeta_{EZ,2}(1, s + 2; 1, -1) + \phi(s + 3)\},
\]
\[
\zeta_2(1, s, 1; C_2) = 2^{s+1}\{\zeta_{EZ,2}(1, s + 3) + \zeta(s + 4)\}
+ 2^{s+1}\{\zeta_{EZ,2}(1, s + 3; 1, -1) + \phi(s + 4)\}
- \zeta_{EZ,2}(s + 2, 2),
\]
\[
\zeta_2(3, s, 3; C_2) = 2^{s+2}\{\zeta_{EZ,2}(s + 7, 2) + (2^{s+2} - 1)\zeta_{EZ,2}(s + 6, 3)
+ 3(1 - 2^{s+6})\{\zeta_{EZ,2}(1, s + 8) + \zeta(s + 1)\}
+ (2^{s+2} \cdot 39 + 3)\zeta(2)\zeta(s + 7)
+ (4 - 2^{s+2} \cdot 31)\zeta(3)\zeta(s + 6)
- 192\{\zeta_{EZ,2}(1, s + 8; 1; -1) + \phi(s + 9)\} + 156\zeta(2)\phi(s + 7)
- 124(3)\phi(s + 6)
+ 2^{s+2}\{19\zeta_{EZ,2}(s + 7, 2; -1, 1) + \zeta_{EZ,2}(s + 6, 3; -1, 1)\}.
\]

In particular, setting \(s = 1\) in (6.7) and \(s = 0\) in (6.8), we have
\[
\zeta_2(1, 1, 1; C_2) = -2\zeta_{EZ,2}(1, 3; 1, -1) - \frac{1}{2}\zeta(2)^2 + \frac{7}{4}\zeta(4),
\]
\[
\zeta_2(1, 0, 2, 1; C_2) = 2\zeta_{EZ,2}(1, 3; 1, -1).
\]

Here we recall
\[
\zeta_{EZ,2}(1, 3; 1, -1) = 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{12}(\log 2)^4 - \frac{15}{8}\zeta(4)
+ \frac{7}{4}\zeta(3)\log 2 - \frac{1}{2}\zeta(2)(\log 2)^2,
\]
where \(\text{Li}_4(\cdot)\) denotes the polylogarithm of order 4 (see [2] Section 4)). Therefore we obtain
\[
\zeta_2(1, 1, 1, 1; C_2) = \frac{17}{10}\zeta(2)^2 - 4\text{Li}_4\left(\frac{1}{2}\right) - \frac{7}{2}\zeta(3)\log 2
+ \zeta(2)(\log 2)^2 - \frac{1}{6}(\log 2)^4,
\]
\[
\zeta_2(1, 0, 2, 1; C_2) = -\frac{3}{2}\zeta(2)^2 + 4\text{Li}_4\left(\frac{1}{2}\right) + \frac{7}{2}\zeta(3)\log 2
- \zeta(2)(\log 2)^2 + \frac{1}{6}(\log 2)^4.
\]

Similarly we can express, for example,
\[
\zeta_2(1, 2, 2, 1; C_2) \quad \text{and} \quad \zeta_2(3, 3, 3; C_2)
\]
in terms of values of \(\zeta(s)\) and \(\zeta_{EZ,2}(s_1, s_2; \sigma_1, \sigma_2)\). However, it is unclear whether we can express these values in terms of single series like (6.10) and (6.11).

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