Mixing of Pseudoscalar Mesons

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Abstract

η−η′ mixing is discussed in the quark-flavor basis with the hypothesis that the decay constants follow the pattern of particle state mixing. On exploiting the divergences of the axial vector currents – which embody the axial vector anomaly – all mixing parameters are fixed to first order of flavor symmetry breaking. An alternative set of parameters is obtained from a phenomenological analysis. We also discuss mixing in the octet-singlet basis and show how the relevant mixing parameters are related to those in the quark-flavor basis. The dependence of the mixing parameters on the strength of the anomaly and the amount of flavor symmetry breaking is investigated. Finally, we present the divergences of the axial vector currents which embody the axial vector decay constants follow the pattern of particle state mixing. On exploiting the anomaly actually arises in QCD.

1. Introduction

η−η′ mixing is a subject of considerable interest that has been examined in many phenomenological investigations, see, e.g., [1–6]. New aspects of mixing, which mainly concern the proper definition of meson decay constants and the consistent extraction of mixing parameters from experimental data, have recently been discussed by Kaiser and Leutwyler [7] and by us [8,9,10]. The purpose of the present article is to review these new developments and to clarify the interplay between the UA(1) anomaly and flavor symmetry breaking. We will not comment on the question of how the UA(1) anomaly actually arises in QCD [11–15].

We start from the quantum mechanical picture of mixing as a superposition of basis states. This mixing is either described in the octet-singlet basis, e.g. [5–7],

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = U(\theta) \begin{pmatrix}
\eta_8 \\
\eta_0
\end{pmatrix},
\]

or in the quark-flavor basis, e.g. [1–4,8],

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = U(\phi) \begin{pmatrix}
\eta_q \\
\eta_s
\end{pmatrix},
\]

where \( U \) is a unitary matrix defined as

\[
U(\alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}.
\]

The basic states, \( \eta_8, \eta_0, \eta_q, \eta_s \), are assumed to be orthogonal states, i.e. mixing with heavier pseudoscalar mesons (e.g. possible glueballs) is ignored. They are further-assumed to be identifiable by their valence quark content which are either the SU(3)q octet and singlet combinations of quark-antiquark pairs or, for the \( \eta_q \) and \( \eta_s \), the combination \( q\bar{q} = (u\bar{u} + d\bar{d})/\sqrt{2} \) and \( s\bar{s} \), respectively. We stress that as long as state mixing is regarded, one may freely transform from one orthogonal basis to the other. The respective mixing angles are related to each other by \( \theta = \phi - \theta_{\text{ideal}} \) where \( \theta_{\text{ideal}} = \arctan \sqrt{2} \) is the ideal mixing angle.

The phenomenological analyses of decay or scattering processes often involve weak decay constants of \( \eta \) and \( \eta' \) mesons which are defined by

\[
(0|f_{\mu j}^\pi(p)|p) = i f_{\mu j}^\pi f_p p_i, \quad (i = 8, 0, q, s; \ P = \eta, \eta').
\]

Occasionally, it is assumed that the octet and singlet decay constants, follow the pattern of state mixing

\[
\begin{align*}
\eta_q &= f_\eta \cos \theta, \quad \eta_8 = -f_0 \sin \theta, \\
\eta_s &= f_\eta \sin \theta, \quad \eta_0 = f_0 \cos \theta,
\end{align*}
\]

However, a recent study within chiral perturbation theory [7] as well as a combined phenomenological analysis of the meson-photon transition form factors, the two-photon decay widths and radiative \( J/\psi \) decays [16] revealed that (5) is inadequate and theoretically inconsistent. The general parameterization [7]

\[
\begin{align*}
\eta_q &= f_\eta \cos \theta_b, \quad \eta_8 = -f_b \sin \theta_b, \\
\eta_s &= f_\eta \sin \theta_b, \quad \eta_0 = f_b \cos \theta_b,
\end{align*}
\]

is required. According to [7,16] the angles \( \theta_b \) and \( \theta_0 \) differ considerably as a consequence of flavor symmetry breaking. Analogously to (5), one may assume that the strange and non-strange decay constants follow the pattern of state mixing in the quark-flavour basis [8]

\[
\begin{align*}
\eta_q &= f_q \cos \phi, \quad \eta_8 = -f_s \sin \phi, \\
\eta_s &= f_q \sin \phi, \quad \eta_0 = f_s \cos \phi.
\end{align*}
\]

Alternatively, one may introduce the analogue of (6) with two mixing angles, \( \phi_q \) and \( \phi_s \), here as well. The phenomenological analysis carried through in [16] provided \( \phi_q \simeq 39.4^\circ \), and \( \phi_s \simeq 38.5^\circ \). The closeness of the two angles is sufficiently suggestive to ignore the little difference and to assume \( \phi_q = \phi_s = \phi \) [8]. A theoretical explanation for this fact is given by the OZI-rule which implies that the difference between \( \phi_q \) and \( \phi_s \) vanishes to leading order in the \( 1/N_c \).
expansion [7,10]. For consistency, we neglect all OZI-rule violating effects in the following. The decay constants \( f_\pi \) and \( f_\pi^0 \) then respect the simple mixing behaviour (7) which is equivalent to the hypothesis that both the basic states, \( \eta_q \) and \( \eta_q \), have vanishing vacuum transition matrix elements with opposite currents

\[
\langle 0| J^{\mu}_{\pi 5}| \eta_q \rangle = \langle 0| J^{\mu}_{\pi 5}| \eta_q \rangle = 0.
\]

The analogous relation in the octet-singlet basis does not hold. With regard to this central assumption the quark-flavor basis is distinct. It implies among other thinks that the decay constants of the mesons can be written as simple mass-independent superpositions of \( f_q \) and \( f_s \). We will explain in the following Sections that the proper use of this basis provides new insights in \( \eta - \eta' \) mixing and successful predictions.

In Section 2 we will present some technical details of the quark-flavor basis and discuss the determination of the basic mixing parameters \( f_q, f_s, \) and \( \phi \), their relations to the mixing parameters, \( f_q, f_s, \theta_q, \theta_s \), and \( \theta_0 \), in the octet-singlet basis, and the important role of the matrix elements of the anomaly operator. Next, in Section 3, we study the dependence of masses and mixing angles on the input parameters that classify the strength of the \( U_A(1) \) anomaly and the flavor symmetry breaking. Some phenomenological applications of the quark-flavor mixing scheme will be discussed in Section 4. Our article ends with a summary (Section 5).

2. The quark-flavor mixing scheme

As is well-known [13], the \( U_A(1) \) axial vector anomaly, that plays a crucial role for understanding the mixing behaviour of the pseudoscalar mesons, is embodied in the divergences of axial vector currents,

\[
\bar{q} \gamma_\mu j^{\mu}_{5} \gamma_5 q = 2m_q (\bar{u} \gamma_\mu \gamma_5 u) + \frac{2a}{4\pi} G \tilde{G}.
\]

Analogous relations hold for the other quark flavors. Here, \( G \) denotes the gluon field strength tensor and \( \tilde{G} \) its dual; \( m_q \) is the current mass of quark species \( i \). The vacuum-meson transition matrix elements of the axial vector current divergences are given by the product of the square of the meson mass, \( M_{\pi}^2 \), and the appropriate decay constant

\[
\langle 0| J^{\mu}_{\pi 5}| P \rangle = M_{\pi}^2 f_{\pi}. \tag{10}
\]

The mass factors, which necessarily appear quadratically here, can be viewed as the elements of the physical particle mass matrix

\[
M^2 = \begin{pmatrix}
M_{\eta}^2 & 0 \\
0 & M_{\eta'}^2
\end{pmatrix}. \tag{11}
\]

Transforming to the quark-flavor basis and exploiting the relations (10), one finds

\[
M_{\eta}^2 = U^\dagger(\phi) M^2 U(\phi) = \begin{pmatrix}
m_{\eta}^2 + 2a^2 & \sqrt{2}ya^2 \\
\sqrt{2}ya^2 & m_{\eta'}^2 + 2a^2
\end{pmatrix} \tag{12}
\]

The quark mass contributions to \( M_{\eta}^2 \) are defined as

\[
f_q = f_s, \quad f_s = \sqrt{2f_\pi^0 - f_\pi^0}. \tag{18}
\]

\[
\begin{align}
m_{\eta q}^2 &= \frac{\sqrt{2}}{f_q} \langle 0| m_q \bar{u} \gamma_\mu \gamma_5 u + m_d \bar{d} \gamma_\mu \gamma_5 d| \eta_q \rangle, \\
m_{\eta s}^2 &= \frac{2}{f_s} \langle 0| m_s \bar{s} \gamma_\mu \gamma_5 s| \eta_s \rangle, \tag{13}
\end{align}
\]

and \( a^2 \) parameterizes the anomaly contribution to the mass matrix,

\[
a^2 = \frac{1}{\sqrt{2f_q}} \langle 0| \frac{2a}{4\pi} G \tilde{G} | \eta_q \rangle. \tag{14}
\]

The anomaly mediates \( \eta_q \leftrightarrow \eta_s \) transitions (see Fig. 1) and therefore leads to \( \eta - \eta' \) mixing. The vacuum-\( \eta_q \) matrix elements of the anomaly operator \( G \tilde{G} \), are non-zero and in fact large because of the non-trivial properties of the QCD vacuum – there are strong gluonic fluctuations with pseudoscalar quantum numbers to which the \( \eta_q \) states can couple.

The parameter \( y = f_q/f_s \) measures the strength of flavor symmetry violation encoded in the decay constants. The symmetry of the mass matrix forces an important connection between \( y \) and anomaly matrix elements

\[
y = \frac{f_q}{f_s} = \frac{\sqrt{2} \langle 0| \frac{2a}{4\pi} G \tilde{G} | \eta_q \rangle}{\langle 0| \frac{2a}{4\pi} G \tilde{G} | \eta_s \rangle}. \tag{15}
\]

2.1. Determination of the basic mixing parameters

Equation (12) provides three relations which allow the determination of \( a^2, y \) and \( \phi \) for given masses of the physical mesons and quark mass terms:

\[
\sin \phi = \pm \sqrt{\frac{(M_{\eta}^2 - m_{\eta q}^2)(M_{\eta'}^2 - m_{\eta q}^2)}{(M_{\eta}^2 - \eta_q)(m_{\eta q}^2 - m_{\eta q}^2)}}, \tag{16}
\]

\[
a^2 = \frac{1}{2} \frac{(M_{\eta}^2 - m_{\eta q}^2)(M_{\eta'}^2 - m_{\eta q}^2)}{(m_{\eta q}^2 - m_{\eta q}^2)},
\]

\[
y = \sqrt{2} \frac{(M_{\eta}^2 - m_{\eta q}^2)(m_{\eta q}^2 - m_{\eta q}^2)}{(M_{\eta}^2 - m_{\eta q}^2)(m_{\eta q}^2 - m_{\eta q}^2)}. \tag{17}
\]

In order to determine the mixing parameters from (16) we take recourse to first order of flavor symmetry breaking and relate the quark mass terms to the pion and kaon masses which themselves are not affected by the anomaly

\[
m_{\eta q}^2 = M_{\pi}^2, \quad m_{ss}^2 = 2M_K^2 - M_{\pi}^2. \tag{17}
\]

Inserting these values into (16) one gets – to the given order – parameter-free results for the mixing parameters which are quoted in Table I. To the same order of flavor symmetry breaking one also has the theoretical estimate
Table I. Theoretical (to first order of flavor symmetry breaking) and phenomenological values of mixing parameters. The parameter $y$ is calculated using (16). The error estimate refers to the experimental uncertainties only. Table taken from [8].

| Source            | $f_3/f_s$ | $f_2/fs$ | $\phi$  | $y$  | $\alpha^2[\text{GeV}^2]$ |
|-------------------|------------|-----------|---------|------|-------------------------|
| Theory            | 1.00       | 1.41      | 42.4°   | 0.78 | 0.281                   |
| Phenomenology     | 1.07       | 1.34      | 39.3°   | 0.81 | 0.265                   |
|                   | ±0.02      | ±0.06     | ±1.0°   | ±0.03| ±0.010                  |

One may also determine the mixing parameters from phenomenology and look for consistency with or deviations from the first order theoretical results. The mixing angle $\phi$ can cleanly be extracted from a number of processes which have been analyzed in Refs [8,17,18]. The idea is to consider ratios of $\eta$ and $\eta'$ observables in which the dependence on form factors, decay constants etc. cancels, and only functions of the mixing angle (times kinematical pre-factors) appear. In Table II we compile the results of a detailed phenomenological analysis presented in [8]. It yields a weighted average for the mixing angle $\phi$

$$\phi_{av} = 39.3° \pm 1.0°.$$  

(19)

where the error refers to the experimental uncertainties only. Quite remarkably, the values for $\phi$ obtained from very different physical processes are all compatible with each other within the errors. This would not be the case in the octet-singlet scheme (if $\theta_8 = \theta_0 = \theta$ were assumed). This fact used to be a known problem in previous analyses (for instance [5,6,17,19], see also Table III) where values for $\phi$ varying from $\approx 45°$ to $\approx 32°$ (corresponding to $-10° \lesssim \phi \lesssim -23°$) have been found. The phenomenological value of $\phi$ in (19) does not differ substantially from the theoretical value quoted in Table I i.e. systematic effects from higher order flavor symmetry breaking corrections etc., are apparently not large.

Using the mass matrix (12) and the phenomenological value of $\phi$, one can evaluate phenomenological values for $\alpha^2$ and $y$. Finally, the two-photon decays of the $\eta$ and the $\eta'$ provide information on the decay constants. The PCAC results for the two-photon decay widths of the $\eta$ and $\eta'$ in terms of $\phi$, $f_3$, and $f_s$ can be written as [16],

$$I[P \to \gamma\gamma] = \frac{\alpha^2}{32\pi} \left( \frac{M_P^2}{f_3^2} \right).$$  

(20)

Here the effective decay constants which are only operative for the two-photon decays, are defined by

$$\frac{1}{f_3^{\text{eff}}} = \frac{1}{C_\pi} \left[ C_q \cos \phi - C_s \sin \phi \right];$$  

$$\frac{1}{f_s^{\text{eff}}} = \frac{1}{C_\pi} \left[ C_s \sin \phi + C_q \cos \phi \right];$$  

(21)

where $C_q = 1/(3\sqrt{2})$, $C_s = 5/(9\sqrt{2})$ and $C_s = 1/9$ are electrical charge factors. The basic decay constants $f_3$ and $f_s$ can be evaluated from (20) using the experimental values for the decay width [20]. The results are combined with the value of $y$ that we obtain from Eq. (16) in order to improve the accuracy of the value for $f_s$. The so-obtained results are also listed in Table I. The phenomenological values for $\phi$, $f_3$, and $f_s$ provide $f_s^{\text{eff}} = 1.02 f_s$ and $f_3^{\text{eff}} = 0.79 f_s$. As can be noticed from Table I there is no substantial deviation between the theoretical and the phenomenological set of parameters, i.e. again higher-order flavor-symmetry breaking corrections etc., absorbed in the phenomenological values, seem to be reasonably small. A recent lattice calculation [21] yielded $\gamma_{\text{lat}} \approx 0.71$ which is in reasonable agreement with our values, too.

Table II. Determination of the mixing angle $\phi$ from different experimental processes, according to Ref. [8,10] and references therein.

| $J/\psi \to \eta \eta'$ | $D_0 \to \eta \eta'$ | $\eta \to \rho \rho \to \eta \eta'$ | $\omega \to \eta \eta' \pi$ | $\pi^0 \to \eta \eta' n$ | $\eta \to \eta^0 \eta$ | $J/\psi \to \eta \eta' \gamma$ | Average |
|-------------------------|---------------------|---------------------------------|--------------------------|----------------------|--------------------------|--------------------------|---------|
| $39.9° \pm 2.9°$        | $41.3° \pm 5.3°$    | $35.3° \pm 5.5°$               | $43.1° \pm 3.0°$         | $36.5° \pm 1.4°$     | $37.4° \pm 1.8°$         | $39.0° \pm 1.6°$         | $39.3° \pm 1.0°$         |

Table III. Octet-singlet mixing parameters, evaluated in the quark-flavor scheme from the theoretical and phenomenological parameters given in Table I, and comparison with other results. The values given in parentheses are not quoted in the original literature but have been evaluated by us from information given therein. Crosses indicate approaches where the difference between $\theta$, $\theta_0$ and $\theta_8$ has been ignored.

| $\theta$ | $\theta_0$ | $\theta_8$ | $f_3/f_s$ | $f_s/f_s$ | Method |
|----------|------------|------------|-----------|-----------|--------|
| $-12.3°$ | $-21.0°$   | $-2.7°$    | 1.28      | 1.15      | quark scheme (theo.) [8] |
| $-15.4°$ | $-21.2°$   | $-2.9°$    | 1.26      | 1.17      | quark scheme (phen.) [8] |
| $-21.4°$ | $-21.0°$   | $-7.0°$    | 1.37      | 1.21      | energy-dependent scheme [22] |
| $-20.4°$ | $-20.5°$   | $-0.1°$    | 1.36      | 1.32      | VDM & phenomenology [23] |
| $-20.5°$ | $-20.5°$   | $-4°$      | 1.28      | 1.25      | $\chi$PT & phenomenology [7] |
| $-20°$   | $-22.2°$   | $-9.1°$    | 1.30      | 1.04      | GMO & $\gamma$ decays [24] |
| $-15.5°$ | $-19.5°$   | $-5.5°$    | [1.27]    | [1.17]    | transition form factors [16] |
| $-12.6°$ | $-19.5°$   | $-5.5°$    | [1.2]     | [1.1]     | phenomenology [18] |
| $-23° - 17°$ | $-20°$    | $-5°$      | [1.2]     | [1.1]     | quark model [25] |
| $-9°$    | $-20°$     | $-5°$      | [1.2]     | [1.1]     | anomaly & meson masses [4] |
2.2. The mixing parameters in the octet-singlet basis

Transforming the non-strange and strange axial-vector currents to octet and singlet ones, one can connect the octet-singlet decay constants defined in (4) to \( f_q \) and \( f_s \) with the result [8]

\[
f_k = \sqrt{1/3 f_q^2 + 2/3 f_s^2}, \quad \theta_k = \phi - \arctan(\sqrt{2} f_q/f_q^o),
\]

\[
f_0 = \sqrt{2/3 f_q^2 + 1/3 f_s^2}, \quad \theta_0 = \phi - \arctan(\sqrt{2} f_q^o/f_s),
\]

and thus

\[
\tan(\theta_0 - \theta_k) = \sqrt{2/3 (f_0 - f_k)/f_k^o}.
\]

It is easy to convince oneself that for \( \theta \neq \theta_0 \) the basic states \( \Psi_0 \) and \( \Omega_0 \) defined through (1) are not pure states in the sense of Eq. (8); the matrix elements \( \langle \Psi_0 | f^0_q | \Omega_0 \rangle \) are non-zero as a consequence of SU(3)_c violation (\( f_0 \neq f_k \)). In fact, defining decay constants analogously to (4), one finds

\[
f_q^0 = f_0^o = \frac{\sqrt{2}}{3} (f_0 - f_k).
\]

In Table III we show the results for the five mixing parameters required in the octet-singlet basis, evaluated from \( \phi, f_q, f_s \) and \( f_k \) and compare them to other results to be found in the literature. Fair agreement can be observed between all approaches that do not assume the equality of \( \theta_k \) and \( \theta_0 \) [4,7,8,16,22,23,25]. Note that in some cases [22,23] different parameterizations for the \( \eta^0 - \eta' \) mixing are given which lead, however, to the same physical results. The analysis presented in [18] basically determines the mixing angle \( \phi \) along the same lines as in [8]. It therefore leads to a value for the mixing angle \( \theta \) that is similar to our phenomenological one. In [18] flavor symmetry breaking is encoded in the constituent quark masses instead of decay constants but the size of the effects is similar in both cases. In the analyses [5,6,17,19,24], in which the differences between \( \theta_k, \theta_0 \) and \( \theta \) are ignored, values for \( \theta \) around \(-20^\circ\) have been obtained. This is close to our phenomenological value of \( \theta \) but quite different from the values for \( \theta_k \) and \( \theta_0 \). The reason for this will become clear in Section 2.3.

The general parameterization (5) combined with (22) and the theoretical estimate (18) implies the following “sum rules”

\[
f_q^{\psi^0} + f_q^{\psi^8} = f_k^2 \simeq \frac{1}{3} (4 f_k^2 - f_q^2),
\]

\[
f_q^{\eta^0} + f_q^{\eta^8} = f_0^2 + f_q^2 \sin(\theta_k - \theta_0) \simeq \frac{2\sqrt{2}}{3} (f_k^2 - f_q^2).
\]

Leutwyler and Kaiser [7] derived these relations within the framework of chiral perturbation theory from assumptions on flavor symmetry breaking similar to (18) which are required in order to fix the parameters of the chiral effective Lagrangian. Eqs. (25), (26) are not a consequence of the dynamical content of chiral perturbation theory [26]. It is to be noted that Eq. (26) appears to be rather sensitive to higher order flavor symmetry breaking effects e.g., as the comparison with our phenomenological values shows. This sensitivity is related to that of the mixing angle \( \theta_k \), see Table III. The singlet decay constants \( f_0^q \) and, consequently, \( f_q^0 \) and \( f_s^0 \), are renormalization-scale dependent [7] while the ratio \( y = f_q^0/f_s \) as well as all mixing angles are scale-independent. The anomalous dimension controlling the scale-dependence of \( f_0^q \) is of order \( x_0^2 \) and therefore leads to tiny effects in the basis decay constants which we discard. The sum rule for \( f_0^q \) analogous to (25) is to be modified in order to take into account the scale dependence [7]. This additional OZI-rule violating effect is also neglected.

2.3. Anomaly matrix elements

We have already pointed out the crucial role of the anomaly in the mass matrix (12). Vacuum-particle matrix elements of the anomaly operator \( GG \), which is occasionally termed the topological charge density, are also of importance in other processes. Therefore, we list various anomaly matrix elements evaluated in the quark-flavor mixing scheme:

\[
\langle \psi^0 | \frac{2 x}{4 \pi} GG | \eta \rangle = \sqrt{2/3} f_q (f_0 - f_k) y^2, \]

\[
\langle \psi^0 | \frac{2 x}{4 \pi} GG | \eta_0 \rangle = \sqrt{2/3} f_q (2 f_0 + f_k) y^2,\]

\[
\langle \psi^0 | \frac{2 x}{4 \pi} GG | \eta \rangle = -\sin \theta_k f_q \sqrt{2 f_0^2 + f_0^2 y^2},
\]

\[
\langle \psi^0 | \frac{2 x}{4 \pi} GG | \eta_0 \rangle = \cos \theta_k f_q \sqrt{2 f_0^2 + f_0^2 y^2}.
\]

The corresponding matrix elements for \( \eta_q \) and \( \eta_k \) can be obtained from (14) and (15).

A particular noteworthy result is the non-vanishing of the vacuum-\( \eta_q \) matrix element; only in the limit of exact flavor symmetry, \( f_q = f_k \), it becomes zero. This again demonstrates the impurity of the \( \eta_8 \) state as defined in (1) with \( \theta = \phi - \theta_{\text{ideal}} \). Interesting is also the ratio of the \( \eta \) and \( \eta' \) matrix elements

\[
\frac{\langle \psi^0 | \frac{x}{4 \pi} GG | \eta \rangle}{\langle \psi^0 | \frac{x}{4 \pi} GG | \eta' \rangle} = -\tan \theta_k.
\]

The ratio of the anomaly matrix elements can also be expressed in terms of the mixing angle \( \phi \) and the masses of the physical mesons. Using (2), (14) and (15), one finds

\[
\frac{\langle \psi^0 | \frac{x}{4 \pi} GG | \eta \rangle}{\langle \psi^0 | \frac{x}{4 \pi} GG | \eta' \rangle} = \frac{M_\eta^2}{M_{\eta'}^2} \cot \phi.
\]

This relation has already been obtained by Ball et al. [17] independent of the quark-flavor mixing scheme. However, its connection with the mixing behaviour of the decay constants has not been recognized in [17].

In the simple octet-singlet mixing scheme, defined by (1), (5), one implicitly assumes that the anomaly only mediates vacuum-\( \eta_0 \) transitions and that \( \langle 0 | GG | \eta_0 \rangle = 0 \). These assumptions imply the replacement of \( \theta_k \) in (28) by \( \theta \). Hence, analyses that are based on the assumption \( \langle 0 | GG | \eta_0 \rangle = 0 \), typically lead to large values of the mixing angle \( \theta \) which resemble that for \( \theta_k \) (e.g., [24]) and are in conflict with those values obtained from processes that only involve state mixing. Examples of such analyses are that of the Gell-Mann-Okubo formula or that of the decays \( J/\psi \to \gamma \eta, \gamma \eta' \). The latter
process will be discussed in Section 4.1 while a detailed investigation of the Gell-Mann-Okubo formula in the light of the new ideas on $\eta - \eta'$ mixing, can be found in [9,10].

3. Masses and mixing angles versus $a^2$ and $y$

The investigations presented in Section 2 clearly reveal the important role of SU(3)$_f$ symmetry breaking and the $U_A(1)$ anomaly in understanding and parameterizing $\eta - \eta'$ mixing.

To elucidate further the role of the anomaly we consider the strength of $a^2$ as a free parameter and evaluate the mixing angles and the $\eta$, $\eta'$ masses for given values of $a^2$ by diagonalizing the mass matrix (12). The values of the quark mass terms (13), (17) and of the flavor symmetry breaking parameter $y (= 0.78)$ are kept fixed. The results are plotted in Fig. 2.

In the limit $a^2 \to 0$ the non-diagonal elements of the mass matrix (12) become negligible, and the pure flavor states $\eta_0$ and $\eta_8$ are the mass eigenstates with the masses $M_0 = m_{qq} \approx M_Z$ and $M_8 = m_{ss} \approx \sqrt{2}M_K$. The mixing angle $\phi$ tends to zero and, correspondingly, $\theta \to -\theta_0$. The substantial difference between $\theta_0$ and $\theta_0$ remains constant, according to (23). For small values of $a^2$ the situation is similar to the case of vector mesons where mixing is only due to the weak, gluon-mediated, OZI-rule violating $q\bar{q} \to q'\bar{q}'$ transitions which are not enhanced by the $U(1)_A$ anomaly. One thus has almost “ideal mixing” between $\phi$ and $\eta$ mesons with a mixing angle $\phi_Y$ of only about $3.4^\circ \pm 0.2^\circ$ as determined, for instance, from the ratio of the $\phi \to \pi^0\gamma$ and $\phi \to \pi^+\gamma$ branching ratios [27].

Now, if $a^2$ is increased, $\eta - \eta'$ mixing becomes stronger. At $a^2 \simeq (0.26 - 0.28)$ GeV$^2$ the meson masses acquire their physical values and $\phi$ is about $40^\circ$; this is the physical region. If $a^2$ is amplified further and becomes much larger than the quark mass terms in (17), the mass matrix simplifies to

$$\begin{align*}
M_{\eta_\eta}^{2} & \to a^2 \begin{pmatrix} 2 & \sqrt{2}y \\ \sqrt{2}y & y^2 \end{pmatrix}.
\end{align*}$$

Diagonalization of this mass matrix yields $M^{\eta} \to \sqrt{2 + y^2}a$ while $M_{\eta}$ stays close to its physical value. For the mixing angles one finds $\phi = \arctan(\sqrt{2}/y)$ and $\theta_0 = 0$ while $\theta_0$ becomes positive. Although gluons are flavor-blind, mixing between different states is flavor-dependent because also the decay constants $f_i$ are involved, c.f. (15). On the other hand, if we now consider the additional limit of exact SU(3)$_f$ symmetry, i.e. $y \to 1$, one obtains a “democratic” mass matrix where all elements in (30) have the same strength (apart from trivial factors of $\sqrt{2}$ arising from the definition of the $\eta_0$ and $\eta_8$ basis states). Diagonalization of the mass matrix (30) then provides a mass $\eta$ Goldstone meson, and a heavy $\eta'$ meson with $M^{\eta'} \to \sqrt{3}a$. For the mixing angles one finds $\phi \to \theta_0^{\text{ideal}}$ and $\theta_0 \simeq \theta_8 \simeq \theta_0 \to 0$, i.e. in this case the octet-singlet basis becomes the physical one.

The results of a similar analysis of the mass matrix (12) where now the flavor symmetry breaking parameter $y$ is varied but the anomaly parameter $a^2$ is kept fixed, are shown in Fig. 3. In the limit $y \to 1$ one finds

---

**Fig. 2.** The mixing angles $\phi$, $\theta_8$, and $\theta_0$ and the masses $M_{\eta}$ and $M_{\eta'}$ vs. the strength of the anomaly parameter $a^2$ (for $y = 0.78$ and quark mass terms according to (17)). The hatched vertical band refers to the range $a^2 = 0.281 \pm 0.01$ GeV$^2$. The dotted lines indicate the physical $\eta$ and $\eta'$ masses. $a^2$ in GeV$^2$ and masses in GeV.

**Fig. 3.** The mixing angles $\phi$, $\theta_8$, and $\theta_0$ and the masses $M_{\eta}$ and $M_{\eta'}$ vs. $y$ (for $a^2 = 0.281$ GeV$^2$ and quark mass terms according to (17)). The hatched vertical band refers to the range $y = 0.78 \pm 0.03$. The dotted lines indicate the physical $\eta$ and $\eta'$ masses. All masses in GeV.
\[ \theta = \theta_8 = \theta_0 = \phi - \theta_{\text{ideal}}. \]
Both the (simple) octet-singlet mixing scheme, (1) and (5), and the quark-flavor one, (2) and (7), hold and are fully equivalent. If \( \gamma \) is getting smaller than 1 the splitting between the three mixing angles \( \theta, \theta_0 \) and \( \theta_0 \) sets in and becomes maximal in the (academic) limit \( \gamma \to 0 \).

4. Applications

4.1. Radiative decays of vector mesons

According to [28] the radiative decays of heavy S-wave quarkonia into \( \eta \) or \( \eta' \) proceed through the emission of the photon from the heavy quark which subsequently annihilates into lighter quark pairs through the effect of the anomaly, see Fig. 4. Making use of (27), one finds for the ratio of decay widths in the quark flavor scheme [8]

\[ R(\eta') = \frac{R(S_u \to \eta')}{R(S_u \to \gamma \eta')} = \text{cot}^2 \theta_y \left( \frac{k_{\gamma \eta'}}{k_{\eta'}} \right)^3 \]

where \( k_{\gamma \eta'} = \sqrt{(M^2 - m_\pi^2 - m_\eta^2)^2 - 4m_\pi^2 m_\eta^2/(2M)} \) denotes the final state’s three-momentum in the rest frame of the decaying particle. From the experimental value of \( R(\eta'/\phi) \) [20] one evaluates \( |\theta_y| = 22.0^\circ \pm 1.0^\circ \) or \( \phi = 39.0^\circ \pm 1.6^\circ \); the latter value has been used in the phenomenological determination of the basic mixing parameters [8], see Table II. From the phenomenological value of \( \theta_0 \) quoted in Table III, we predict \( R(\eta') = 5.8 \) and \( R(\gamma) = 6.5 \). The result for \( R(\eta') \) agrees with the experimental value \( 2.9^{+1.4}_{-1.3} \), within the uncomfortably large errors [29]. We repeat – in the simple octet-singlet mixing scheme, defined by (1) and (5), one would have interpreted \( \theta_0 \) in (31) as \( \theta \).

One may also consider radiative transitions \( \eta \) or \( \eta' \) and light vector mesons. In this case the \( U_A(1) \) anomaly does not contribute but, as an additional complication, vector meson mixing is to be taken into account. The recent measurement of the radiative \( \phi \) meson decays performed by KLOE [30] allows for another severe test of the quark-flavor mixing scheme now. The ratio of the corresponding decay widths, which is indeed measured by KLOE, is given by

\[ R(\phi) = \text{cot}^2 \phi \left( \frac{k_{\gamma \eta'}}{k_{\eta'}} \right)^3 \left[ 1 - 2 f_s \tan \phi \right] \]

from which we predict \( R(\phi) = (5.66 \pm 0.20) \cdot 10^{-3} \) (for \( \phi_y = 3.4^\circ \pm 0.2^\circ \)) in agreement with the KLOE result of \( (5.30 \pm 0.5 \pm 0.4) \cdot 10^{-3} \). This analysis can be extended to the case of \( \omega \) and \( \rho \) mesons [9] but the quality of the experimental data for these decays [20] needs improvement before the mixing scheme can be examined seriously.

4.2. The pseudoscalar meson photon transition form factor

The transition form factors between pseudoscalar mesons and photons at large momentum transfer are subject of intense theoretical interest, see e.g. [16,31–33]. Since the form factors are sensitive to the decay constants they also provide a crucial test of the quark-flavor mixing scheme. A leading twist analysis to next-to-leading order (NLO) QCD accuracy, based on QCD factorization, lead to (for \( P = n^\circ, \eta, \eta' \))

\[ F_{P\gamma}(\hat{Q}, \omega) = \frac{\hat{f}_{\gamma P}^P}{\sqrt{Q^2}} \int_{1 - \omega^2}^{1} d\xi \Phi_P(\xi, \mu_F)^2 \]

\[ \times \left[ 1 + \frac{2z_s(\mu_F)}{\pi} \frac{K(\omega, \xi, \hat{Q}/\mu_F)}{\mu} \right] \]

+ gluonic contribution for \( \eta, \eta' \),

\[ F_{P\gamma}(\hat{Q}, \omega) = \frac{\hat{f}_{\gamma P}^P}{\sqrt{Q^2}} \int_{1 - \omega^2}^{1} d\xi \Phi_P(\xi, \mu_F)^2 \]

\[ \times \left[ 1 + \frac{2z_s(\mu_F)}{\pi} \frac{K(\omega, \xi, \hat{Q}/\mu_F)}{\mu} \right] \]

where \( \hat{Q}^2 = (Q^2 + Q'^2)/2 \) is the average of the two photon virtualities and \( \omega = (Q^2 - Q'^2)/(Q^2 + Q'^2) \) the normalized difference. \( \Phi_P(\xi, \mu_F) \) are the light-cone distribution amplitudes where \( \xi \) is related to the parton momentum fractions and \( \mu_F \) is the factorization scale, contain the relevant non-perturbative input. For the case of the \( \eta \) and \( \eta' \) we employ the quark-flavor mixing scheme. Eq. (33) is then to be understood as a suitable superposition of the corresponding expressions for the \( \eta \) and \( \eta' \) transition form factors. To simplify matters it is, in agreement with experiment [16], assumed that the \( \eta \) and \( \eta_s \) distribution amplitudes are approximately equal. The \( \eta - \eta' \) mixing then reflects itself in the effective decay constants, \( \hat{f}_{\gamma P}^P \), defined by:

\[ \hat{f}_{\gamma P}^P = \frac{1}{C_{\xi}} [C_q f_{q} \cos \phi - C_s f_s \sin \phi] \]

\[ \hat{f}_{\gamma P}^P = \frac{1}{C_{\xi}} [C_q f_{q} \sin \phi + C_s f_s \cos \phi] \]

Here \( C_{\xi}, C_q, \) and \( C_s \) are defined after (21). In case of the pion one simply has \( \hat{f}_{\gamma P}^P = f_{\gamma P} \). The effective decay constants, \( \hat{f}_{\gamma P}^P \), differ from the \( f_{\gamma P} \) defined in (21) for the decays into two real photons. Their numerical values \( (\hat{f}_{\gamma P} = 0.98 f_{\gamma P}, \hat{f}_{\gamma S} = 1.62 f_{\gamma P}) \) are to be compared with the values for \( f_{\gamma P} \) quoted after (21). In particular, \( f_{\gamma P} \) is markedly different from \( f_{\gamma P} \) while by accident one has \( \hat{f}_{\gamma P} \approx f_{\gamma P} \). This lead to some confusion in early attempts to interpret the experimental data [38,39] for the \( P_{\gamma} \) transition form factors in the real photon limit (\( Q^2 = 0 \)) in terms of simple interpolation formulas between (21) and (33). This issue has been resolved in [34] where also the proper interpolation formulas are given.3

Sample Feynman graphs contributing to the form factor at NLO are shown in Fig. 5. Here, \( K \) is a known function evaluated from the NLO graphs. To NLO there is also a contribution from the leading-twist two-gluon distribution amplitudes of \( \eta \) and \( \eta' \). These two-gluon distribution amplitudes mix with the quark singlet distributions under evolution [37]. Their size is not yet clear [16,35], but in the case of the transition form factors, where the gluonic contributions only enters at NLO, they seem to be negligible. This is consistent with the neglect of OZI-rule violating contributions. It is important to realize that the \( U_A(1) \) anomaly also contributes to the two-gluon Fock states but

\[ \frac{\text{Fig. 4. The decay } J/\psi \to \gamma\eta(\eta')} \text{through the } U_A(1) \text{ anomaly (indicated by the grey blob).} \]

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it generates higher-twist distribution amplitudes \([28]\) which are suppressed by inverse powers of the large scale, \(Q^2\). Thus, despite the fact that anomaly matrix elements \(\langle 0|GG|P(p)\rangle \propto a^2\) are large, the gluonic content of the \(\eta\) and \(\eta'\) mesons are hardly perceptible in the transition form factors at large momentum transfer. For low scales, on the other hand, the suppression of higher-twist contributions is not operative and the \(U_A(1)\) anomaly plays an important role as we discussed in this article. In cases where the leading-twist contributions are strongly suppressed for one or the other reason the higher-twist gluonic contribution may be dominant even at large scales. An example is set by the OZI-rule forbidden \(J^c = c\!\!\! - g\eta, g\eta'\) decays which, although taking place at the large scale \(M^2 = c\!\!\! -\) are anomaly-controlled (see Section 4.1).

One can show \([33]\) that for \(\omega \lesssim 0.6\) and \(Q \gtrsim 2\) GeV the form factors become independent of the shape of the distribution amplitudes to a high degree of accuracy which leads to

\[
F_{\gamma'P} \xrightarrow{\omega \lesssim 0.6} \frac{\sqrt{2} \kappa_{P}}{3Q^2} \left(1 - \frac{5\alpha_s}{3\pi}\right) + \mathcal{O}(\omega^2).
\]  

(35)

In this kinematical region we thus have a parameter-free prediction from QCD to leading-twist accuracy. It has a status comparable to the famous expression of the cross section ratio \(R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)\) or to that of the Bjorken sum rule. Hence, (35) well deserves experimental verification.

In the limit of one real photon \(Q' = 0\) while the other one is highly virtual photon \(Q^2 \rightarrow \infty\) the form factors become independent of the shape of the distribution amplitude, too, since any distribution amplitude evolves into the asymptotic form \(\phi_P \rightarrow \phi_{AS} = 3/2(1-\xi^2)\) and \(\phi_P' \rightarrow 0\). In this limit one finds

\[
F_{\gamma P} \xrightarrow{Q^2 \rightarrow \infty} \frac{\sqrt{2} \kappa_{P}}{Q^2} \left(1 - \frac{5\alpha_s}{3\pi}\right).
\]  

(36)

In Fig. 6 the data for the light pseudoscalar meson-photon transition form factors \([38,39]\), scaled by the leading-order (LO) asymptotic results, are shown. We see that the data for \(\pi, \eta\) and \(\eta'\) form factors are universal within the errors and lie about 20% below the LO asymptotic value. There are many attempts to explain this difference \([16,31-33]\). The \(\alpha_s\) corrections, choosing \(\bar{Q} = Q/\sqrt{2}\) as the renormalization scale, account for about a half of the deviation. Utilizing a distribution amplitude that is somewhat narrower than the asymptotic one (in terms of the Gegenbauer

![Fig. 5. Sample Feynman graphs contributing to meson-photon transition form factor.](image)

![Fig. 6. The \(P_T\) transition form factors scaled by the LO asymptotic result. Data are taken from [38,39,41], the theoretical results from [16] (solid lines) and [40] (dashed-dotted line).](image)
The GT relation (37) also provides for the form factor would exhibit universality, too.

The four nucleon-nucleon forward scattering amplitudes but also the mixing behaviour of the $\eta$ and $\eta'$ decay constants [16].

For comparison we also show in Fig. 6 the $\eta,\gamma$ transition form factor. Obviously, this form factor behaves differently. The reason is clear -- there is a second large scale in this process, namely the $\eta_c$ mass which leads to the observed suppression [40]. If the measured values of $Q^2$ were so large that the $\eta_c$ mass could be neglected as compared to it, this form factor would exhibit universality, too.

4.3. The $\eta, \eta'$-nucleon coupling constants

The coupling constants of the pseudoscalar mesons with nucleons are important ingredients in many analyses of hadronic reactions at low energies. In order to obtain an estimate of these coupling constants, we use generalized Goldberger-Treiman (GT) relations, and apply again the quark-flavor scheme [10]

$$2M_NG_3^d = f_\pi g_{NNN},$$

$$2M_NG_4^d = \sum_{p=\eta,\eta'} f_{\pi q}^a g_{PNN}, \quad a = 0, 8. \tag{37}$$

The $G_i^d$ are the axial vector coupling constants and $M_N$ is the nucleon mass. The GT relations differ from the Shore-Veneziano ansatz [42] where an additional direct coupling of the Veneziano ghost to the nucleon has been allowed for. The axial-vector couplings are known from phenomenology:

$$G_3^d = 0.900 \pm 0.002,$$

$$G_4^d = 0.24 \pm 0.01,$$

$$G_0^d = 0.16 \pm 0.10,$$

from neutron $\beta$-decay,

from hyperon $\beta$-decay, \tag{38}

from Bjorken sum rule.

The latter result holds at a scale of 5 GeV$^2$ [43]. For the case of the pion the phenomenological results for the axial-vector couplings lead to $g_{\pi NN}^2/(4\pi) = 13.2$, a value that is somewhat smaller than that one obtained from dispersion theory [44] ($= 14.2$). Potential models [45] and nucleon-nucleon scattering phase shift analysis [46] provide values which are closer to the GT value. The origin of this little discrepancy between the GT result and phenomenology is not yet clear. The GT relation (37) also provides

$$s_{\rho NN}^2/(4\pi) = 0.92 \pm 0.3,$$

$$s_{\pi NN}^2/(4\pi) = 0.2 \pm 0.2. \tag{39}$$

A recent measurement of near-threshold $\eta'$ production in proton-proton collisions [47] provided the bound $g_{\eta' NN}^2/(4\pi) < 0.5$. This admittedly model-dependent result is in agreement with the GT value (39). A dispersion analysis of the six nucleon-nucleon forward scattering amplitudes [48] yielded $g_{\pi NN}^2/(4\pi) < 1.0$ in agreement with the GT relation, too. The $\eta$ and $\eta'$ couplings could not be disentangled from each other in this analysis. One-boson exchange potentials [45], on the other hand, provide much larger values ($g_{\pi NN}^2/(4\pi) = 3.7$, $g_{\eta NN}^2/(4\pi) = 4.2$) which are in conflict with (39). Attention must be paid to the fact that in the OBE potential models only meson exchange in a non-relativistic reduction is taken into account while contributions from multi-meson exchanges are ignored. In the dispersion analysis [48], on the other hand, evidence for contributions from the $2\pi$ and $3\pi$ continuum has been found. With regard to this one should take coupling constants obtained in potential models with some care. They are rather effective parameters than fundamental quantities.

4.4. Isospin-singlet admixtures to the pion

Isospin violation in the pseudoscalar meson sector can be viewed as $\eta$ and $\eta'$ admixtures to the pion:

$|u^0\rangle = \Phi_3 + \epsilon |\eta\rangle + \epsilon' |\eta'\rangle \tag{40}$

where $\Phi_3$ denotes the pure isospin-triplet state. A straightforward generalization of the quark-flavor mixing scheme by treating the $u$- and the $d$-quark separately, allows for a determination of the parameters $\epsilon$ and $\epsilon'$ [9]

$$\epsilon = \cos \phi \frac{m_{ud}^2 - m_{uu}^2}{2(M_\eta^2 - M_{2\eta}^2)}, \quad \epsilon' = \sin \phi \frac{m_{dd}^2 - m_{uu}^2}{2(M_\eta^2 - M_{2\eta}^2)}. \tag{41}$$

The quark mass difference $m_{ud}^2 - m_{uu}^2$ can be estimated from $2(M_{2\eta}^2 - M_{\eta}^2)$ and $2(M_{2\eta}^2 - M_{\eta}^2)$ in which, according to Dashen [49], masses of electromagnetic origin are expected to cancel to a large extent. A possible difference in the $u$ and $d$ quark decay constants is ignored in the derivation of (41). With the phenomenological value of 39.3° for the mixing angle one obtains

$$\epsilon = 0.014, \quad \epsilon' = 0.0037. \tag{42}$$

An extraction of $\epsilon$ from the anomaly dominated decays $\Psi' \to J/\psi \pi^0, J/\psi \eta$

$$\Gamma(\Psi' \to J/\psi \pi^0) = \left(\cos^2 \phi \right)^2 \left(\sin^2 \phi \right)^2 \left(\frac{k_{J/\psi \pi^0}}{k_{J/\psi \eta}}\right)^3, \tag{43}$$

On the other hand, yields $\epsilon = 0.026 \pm 0.003$. The observed violation of charge symmetry in the cross sections for $\pi^+d \to p\eta n$ and $\pi^-d \to n\eta p$ [50] lead to the same value ($\epsilon = 0.026 \pm 0.007$) within a slightly model-dependent analysis. Thus, it seems that there is a discrepancy of about a factor of 2 whose origin is not yet clear. Is the $u$-$d$ mass difference underestimated or are higher order electromagnetic or other corrections lacking? The neglected OZI-rule violating effects are of the same order as $\epsilon$, namely they amount to a few percent. One may therefore suspect that the quark-flavor mixing scheme, defined by (2) and (7), has reached its limits of accuracy for effects of that size. It is possible that at that level of precision three mixing angles, $\phi_\eta$, $\phi_\pi$, and $\phi_\sigma$, are required for an adequate description of all aspects of $\eta$-$\eta'$ mixing.

Isospin violation plays an important role in the analysis of the decays $\eta(\eta') \to 3\pi$ [51,52] as well as in the investigation...
of CP-violation in $B \to \pi \pi$ decays [53]. They break the isospin triangle relation
\[ M(\bar{B}^0 \to \pi^0 \pi^0) = M(\bar{B}^0 \to \pi^+ \pi^-)/\sqrt{2} + M(\bar{B}^- \to \pi^0 \pi^-), \]
and may therefore affect the determination of the CKM-angle $\alpha$.

One may wonder whether the $\eta$ and $\eta'$ admixtures to the $\pi^0$ do not affect the interpretation of the $J/\psi \to \gamma \eta^0$ decay. This is, however, not the case as a quick estimate reveals. The anomaly contribution to this process through these admixtures is
\[ \frac{\Gamma(J/\psi \to \gamma \pi^0)}{\Gamma(J/\psi \to \gamma \eta)} = |z - \cot \theta_c z'|^2 \left( \frac{k_{\pi^0}}{k_{\eta'}} \right)^3, \]
which provides a value of only $6.1 \cdot 10^{-4}$ for the ratio of decay widths while the experimental value is $(4.5 \pm 1.6) \cdot 10^{-2}$ [20]. Thus, we can conclude – the radiative decay of the $J/\psi$ into the $\pi^0$ is mediated by $c \bar{c} \to \gamma \to q \bar{q}$, a contribution that is proportional to the $\pi^0$ axial-vector transition form factor, see Section 4.2, and by a vector meson dominance contribution $J/\psi \to \rho^0 \pi^0 \to \pi^0 \gamma^0$ and not by the $U_A(1)$ anomaly as the corresponding decays into $\eta$ or $\eta'$.

5. Conclusion

We have discussed $\eta$--$\eta'$ mixing in the octet-singlet and in the quark-flavor basis. We have shown that for a complete and consistent understanding it is not sufficient to consider only state mixing. We therefore carefully considered the mixing behaviour of the decay constants and of the matrix elements of the $U_A(1)$ anomaly operator. In this context, the quark-flavor mixing scheme, defined through Eqs (2) and (7), appears to be favored since it only requires three basic mixing parameters ($f, f_q, f_s$) to describe all aspects of $\eta$--$\eta'$ mixing to an accuracy of about a few percent, i.e. at a level at which OZI-rule violations become noticeable.

The three basic mixing parameters can be determined with the help of the divergences of the axial-vector currents, which embody the $U_A(1)$ anomaly, and first order SU(3)$_F$ symmetry relations. Alternatively, the mixing parameters can be determined by using various phenomenological input. The most important processes to verify the consistency of quark-flavor mixing scheme are the radiative decays of S-wave quarkonia into $\eta$ or $\eta'$, and the transition form factors between photons and pseudoscalar mesons.

On the other hand, the simple octet-singlet mixing scheme, defined by Eqs (1) and (5), is obsolete and in clear conflict with phenomenology. Of course, one may still use the octet-singlet basis but, if decay constants or matrix elements of the anomaly operator are considered, one has to allow for two additional mixing angles $\theta_3$ and $\theta_2$, (see (6), (28)) with substantially different values.

The reason for the preference of the quark-flavor scheme is the smallness of OZI-rule violations, which amount to only a few percent as can be seen, for instance, from the difference between the angles $\theta_3$ and $\theta_2$, as determined in the analysis of the $\eta\gamma$ and $\eta'$--$\gamma$ transition form factors [16]. On the other hand, SU(3)$_F$ symmetry is broken at the level of 10--20%, if one takes the difference between the pseudoscalar decay constants as a relevant measure, and cannot be neglected, in contrast to OZI-rule violations. Via Eq. (23) this immediately rules out the simple ansatz (5).

We finally remark that a simple one-angle description of $\eta$--$\eta'$ mixing can, at most, hold in one basis except in the trivial case of perfect SU(3)$_c$ symmetry. In any other basis, mixing is unavoidably more complicated as can easily be shown by calculations analogous to the one sketched in Section 2.

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