Generalized double-logarithmic large-x resummation in inclusive deep-inelastic scattering

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Abstract

We present all-order results for the highest three large-x logarithms of the splitting functions $P_{qg}$ and $P_{gq}$ and of the coefficient functions $C_{\phi,q}$, $C_{2,g}$ and $C_{L,g}$ for structure functions in Higgs- and gauge-boson exchange DIS in massless perturbative QCD. The corresponding coefficients have been derived by studying the unfactorized partonic structure functions in dimensional regularization independently in terms of their iterative structure and in terms of the constraints imposed by the functional forms of the real- and virtual-emission contributions together with their Kinoshita–Lee-Nauenberg cancellations required by the mass-factorization theorem. The numerical resummation corrections are small for the splitting functions, but partly very large for the coefficient functions. The highest two (three for $C_{L,g}$) logarithms can be resummed in a closed form in terms of new special functions recently introduced in the context of the resummation of the leading logarithms.
1 Introduction

The splitting functions governing the scale dependence of the parton densities of hadrons and the coefficient functions for inclusive deep-inelastic scattering (DIS) are benchmark quantities of perturbative QCD \[1\]. At this point these are the only quantities depending on a dimensionless scaling variable, the parton momentum fraction and the Bjorken variable (both usually denoted by \(x\)), for which third-order corrections in the strong coupling constant \(\alpha_s\) are fully known. The corresponding three-loop calculations started with DIS sum rules \[2–4\] and proceeded via a series of low integer moments of the splitting functions and the coefficient functions for the most important structure functions \[5–7\] to the corresponding complete calculations of Refs. \[8–12\].

Such higher-order calculations do not only improve the numerical accuracy of the predictions of perturbative QCD but also help to uncover general structures, for example in the soft-gluon limit \(x \to 1\). Writing the expansion of the splitting functions in the \(\overline{\text{MS}}\) scheme as

\[
P_{ik}(x, \alpha_s) = \sum_{n=0}^{\infty} a_s^{n+1} P_{ik}^{(n)}(x) \quad \text{with} \quad a_s \equiv \frac{\alpha_s}{4\pi},
\]

the diagonal (quark-quark and gluon-gluon) splitting functions take the form

\[
P_{kk}^{(n-1)}(x) = A_k^{(n)}(1-x)^{-1} + B_k^{(n)} \delta(1-x) + C_k^{(n)} \ln(1-x) + O(1)
\]

with the \(n\)-loop quark and gluon cusp anomalous dimensions related by \(A_g^{(n)}/A_q^{(n)} = C_A/C_F\) at \(n \leq 3\), where \(C_A\) and \(C_F\) are the usual SU(N) colour factors with \(C_A = 3\) and \(C_F = 4/3\) in QCD \[13\]. It was not known before Ref. \[14\], inspired partly by observations made for the three-loop results in Refs. \[8, 9\], that the third term in Eq. \(1.2\) is linear in \(\ln(1-x)\) at all orders \(n\), and that its coefficients \(C_k^{(n)}\) (with \(C_k^{(1)} = 0\)) are simple functions of lower-order cusp anomalous dimensions.

The form of the off-diagonal (quark-gluon and gluon-quark) splitting functions, on the other hand, is not stable under higher-order corrections but shows a double-logarithmic enhancement,

\[
P_{i \neq k}^{(n)}(x) = \sum_{\ell=0}^{2n-1} D_{ik}^{(n,\ell)} \ln^{2n-\ell}(1-x) + O(1).
\]

The terms with \(\ell = 0\) form the leading-logarithmic (LL) large-\(x\) approximation, those with \(\ell = 1\) the next-to-leading-logarithmic (NLL) approximation etc. Recently an all-order resummation of the former contributions to Eq. \(1.3\) has been presented \[15\]. A main purpose of this article is to extend those results to the \(\ell = 2\) next-to-next-to-leading logarithmic (NNLL) terms.

The dominant \((1-x)^{-1}\) large-\(x\) contributions to the quark coefficient functions for gauge-boson exchange structure functions in DIS such as \(F_2\) and \(F_3\) (and the gluon coefficient function for Higgs-exchange structure function \(F_6\) in the heavy-top limit \[16, 17\]) also show a double logarithmic enhancement. These terms are resummed to all orders by the soft-gluon exponentiation \[18–25\] which presently fixes the coefficients of the highest six logarithms analytically, and the seventh term for all numerical purposes since the effect to the presently unknown four-loop
cusp anomalous dimension can be neglected in this context \([25-27]\). A resummation of the highest three \((1-x)^0\) logarithms has been inferred in Ref. \([28]\) from the properties of the corresponding flavour non-singlet physical evolution kernels. While those subleading contributions to the ‘diagonal’ \(O(\alpha_s^0)\) coefficient functions are not the main topic of this article, we will be able to verify the DIS part of those results and fix the only missing coefficient for the fourth \((N^3L)\) logarithms.

The ‘off-diagonal’ \(O(\alpha_s^1)\) coefficient functions, such as

\[
C_{a,k}^{(n)}(x, \alpha_s) = \sum_{n=1}^{\infty} a_s^n c_{a,k}^{(n)}(x) \quad \text{for} \quad a, k = 2, g \text{ or } \phi, q ,
\]

receive a double-logarithmic higher-order enhancement as \(x \to 1\) as well,

\[
c_{a,k}^{(n)}(x) = \sum_{\ell=0}^{2n-2} D_{a,k}^{(n,\ell)} \ln^{2n-1-\ell} (1-x) + O(1) .
\]

Also here the \(\ell = 0\) LL coefficients have been determined at all orders \(n\) in Ref. \([15]\), and also here we will extend those results by deriving the corresponding NLL and NNLL results.

Finally we will also address the coefficient functions \(C_{L,q}\) and \(C_{L,g}\) for the longitudinal structure function \(F_L\). These quantities have a perturbative expansion of the form (1.4), and are given by

\[
c_{L,k}^{(n)}(x) = (1-x)^{\delta_{kq}} \left( \sum_{\ell=0}^{2n-3} D_{L,k}^{(n,\ell)} \ln^{2n-2-\ell} (1-x) + O(1) \right) , \quad k = q, g ,
\]

at large \(x\), i.e., they are suppressed by one power of \((1-x)\) and \(\ln(1-x)\) with respect to their counterparts for the structure function \(F_2\). The coefficients \(D_{L,q}^{(n,\ell)}\) for \(\ell = 0, 1, 2\) have been obtained already in Ref. \([29]\), again from physical-kernel considerations. For the gluon coefficient function, however, only the LL coefficients \(D_{L,g}^{(n,0)}\) have been determined completely in that article. Below we will verify those results and extend also the resummation of \(C_{L,g}\) to the NNLL terms.

The remainder of this article is organized as follows: In Section 2 we derive all-order expressions for the (dimensionally regulated) Mellin-\(N\) space transition functions as far as required for the mass-factorization of the structure functions at the level of the dominant and sub-dominant contributions discussed above. The NNLL resummations of the unfactorized partonic structure functions \(T_{a,k}\) are then derived in Section 3 for the cases (1.4), where to NLL accuracy we employ two different methods to drive the same results, and in Section 4 for \(F_L\). The partly rather lengthy results of these three sections are then combined, and in Sections 5 and 6 we present and discuss the respective resummed expressions for the moments of the off-diagonal splitting functions – which receive rather small resummation corrections at relevant values of \(N\) – and the coefficient functions (1.4) and \(C_{L,g}\) for which these corrections are (very) large. We summarize our findings in Section 7 where we also present a brief outlook to future extensions and applications of some of the results. Closed expressions have not been found so far for the third logarithms in Eqs. (1.3) and (1.5). Numerical and symbolic tables of NNLL coefficients to high orders are therefore finally presented in Appendix A for the splitting functions and Appendix B for the coefficient functions.
2 Large-\(x\) / large-\(N\) mass factorization to all orders

The main part of our calculations is performed after transformation to Mellin-\(N\) space,

\[
 f(N) = \int_0^1 dx \, x^{N-1} f(x) \quad \text{or} \quad f(N) = \int_0^1 dx \left( x^{N-1} - 1 \right) f(x) ,
\]

(2.1)

where the ubiquitous \(x\)-space Mellin convolutions are reduced to simple products. To the accuracy required below, the relations between the large-\(x\) logarithms and their moment-space counterparts are given by

\[
(-1)^k \left( \frac{\ln^{k-1}(1-x)}{1-x} \right)_{+}^{M} = \frac{1}{k} \left( [S_{1-}(N)]^k + \frac{1}{2} k(k-1) \zeta_2 [S_{1-}(N)]^{k-2} + \frac{1}{6} k(k-1)(k-2) \zeta_3 [S_{1-}(N)]^{k-3} + O([S_{1-}(N)]^{k-4}) \right) ,
\]

(2.2)

\[
(-1)^k \ln^k (1-x) = \frac{1}{N} \left( \ln^k \tilde{N} + \frac{1}{2} k(k-1) \zeta_2 \ln^{k-2} \tilde{N} + \frac{1}{6} k(k-1)(k-2) \zeta_3 \ln^{k-3} \tilde{N} + O\left( \ln^{k-4} \tilde{N} \right) \right) ,
\]

(2.3)

\[
(-1)^k (1-x) \ln^k (1-x) = \frac{1}{N^2} \left( \ln^k \tilde{N} - k \ln^{k-1} \tilde{N} + \frac{1}{2} k(k-1) \zeta_2 \ln^{k-2} \tilde{N} + O\left( \ln^{k-3} \tilde{N} \right) \right) ,
\]

with \(S_{1-}(N) = \ln \tilde{N} - 1/(2N) + O(1/N^2)\) and \(\tilde{N} = N e^{\gamma_E}\), i.e., \(\ln \tilde{N} = \ln N + \gamma_E\) with \(\gamma_E \approx 0.577216\). Here \(M\) indicates that the right-hand-side is the Mellin transform (2.1) of the previous expression.

The primary objects of our resummations are the dimensionally regulated unfactorized partonic structure functions or forward Compton amplitudes \(T_{a,k}\) for the combinations of \(a\) and \(k\) of Eqs. (1.4) and (1.6). For brevity suppressing all functional dependences on \(N\), \(\alpha_s\) and the dimensional offset \(\varepsilon\) with \(D = 4 - 2\varepsilon\), these quantities can be factorized as

\[
T_{a,k} = \tilde{C}_{a,i} Z_{ik} .
\]

Here the process-dependent \(D\)-dimensional coefficient functions (Wilson coefficients) \(\tilde{C}_{a,i}\) include contributions with all non-negative powers of \(\varepsilon\). The universal transition functions (renormalization constants) \(Z_{ik}\) collect all negative powers of \(\varepsilon\) and are related to the splitting functions in Eq. (1.1) (or the anomalous dimensions \(\gamma\)) by

\[
- \gamma = P = \frac{dZ}{d\ln Q^2} Z^{-1} .
\]

(2.4)

Here and below we identify, as already in the introduction, the renormalization and factorization scale with the physical hard scale \(Q^2\) without loss of information. Using the \(D\)-dimensional evolution of the coupling,

\[
\frac{d\alpha_s}{d\ln Q^2} = -\varepsilon \alpha_s + \beta(\alpha_s) ,
\]

(2.5)

where \(\beta(\alpha_s)\) denotes the usual four-dimensional beta function of QCD, \(\beta(\alpha_s) = -\beta_0 \alpha_s^2 + \ldots\) with \(\beta_0 = 11/3 C_A - 2/3 n_f\), Eq. (2.4) can be solved for \(Z\) order by order in \(\alpha_s\).
The general elements of $Z^{(n)}$ become extremely lengthy at very high powers $n$ of $a_s$. Here, however, we are interested only in the LL, NLL and NNLL contributions at order $N^{-1}$ for $Z_{gg}$ and $Z_{gq}$ and $N^0$ for $Z_{qq}$ and $Z_{gg}$ (required for Eq. (2.3) also in the off-diagonal cases). Consequently there can be at most one off-diagonal $N^{-1}$ factor per term. Moreover $P^{(1)}_{kk}$ in Eq. (1.1) can enter only (once) at NLL accuracy, and higher-order coefficients $P^{(n\geq 2)}_{kk}$ in Eq. (1.1) are not at all relevant at this level. Finally $\beta_0$ in Eq. (2.5) and $\beta_0^2$ only contribute from the NLL and NNLL terms, respectively, and $\beta_{n\geq 1}$ would enter only at the next logarithmic accuracy. All this can be easily read off already from the well-known third-order expression for $Z$,

$$
Z = 1 + a_s \frac{1}{\epsilon} \gamma^{(0)} + a_s^2 \left( \frac{1}{2\epsilon^2} \left( \gamma^{(0)} - \beta_0 \right) \gamma^{(0)} + \frac{1}{2\epsilon} \gamma^{(1)} \right)
+ a_s^3 \left( \frac{1}{6\epsilon^3} \left( \gamma^{(0)} - \beta_0 \right) \left( \gamma^{(0)} - 2\beta_0 \right) \gamma^{(0)} \right.
+ \frac{1}{6\epsilon^2} \left[ \left( \gamma^{(0)} - 2\beta_0 \right) \gamma^{(1)} + \left( \gamma^{(1)} - \beta_1 \right) 2\gamma^{(0)} \right] + \frac{1}{6\epsilon} \gamma^{(2)} + \ldots \tag{2.6}
$$

(2.6) together with Eqs. (1.2), (1.3) and (2.2) above.

The terms that do contribute to the $a \neq b$ off-diagonal entries of $Z$ at the present accuracy can be grouped as follows:

$$
Z^{(k)}_{ab} = Z^{(k)}_{ab} \bigg|_0 + Z^{(k)}_{ab} \bigg|_{\beta_0} + Z^{(k)}_{ab} \bigg|_{\beta_0^2} + Z^{(k)}_{ab} \bigg|_{\gamma^{(1)}\gamma^{(1)}} + Z^{(k)}_{ab} \bigg|_{\gamma^{(1)\gamma^{(1)}}} \tag{2.7}
$$

The first term on the right-hand-side collects all contributions with at most one higher-order anomalous dimension $\gamma^{(i\geq 1)}$ but no contribution from the beta function. It starts at $k = 1$ and reads

$$
Z^{(k)}_{ab} \bigg|_0 = \frac{1}{k!\epsilon^k} \left\{ \sum_{i=0}^{k-1} \epsilon^i \sum_{j=0}^{k-1-i} \frac{(j+i)!}{j!} \left( \gamma^{(0)}_{aa} \right)^{k-1-i-j} \gamma^{(i)}_{ab} \left( \gamma^{(0)}_{bb} \right)^j 
+ \epsilon \sum_{j=0}^{k-3} \frac{1}{2} (k-j-2)(k-j-1) \left( \gamma^{(0)}_{aa} \right)^j \gamma^{(0)}_{ab} \gamma^{(1)}_{bb} \left( \gamma^{(0)}_{bb} \right)^{k-j-3} 
+ \epsilon \sum_{j=0}^{k-3} \frac{1}{2} (k-j-2)(k+j+1) \left( \gamma^{(0)}_{aa} \right)^{k-j-3} \gamma^{(1)}_{aa} \gamma^{(0)}_{ab} \left( \gamma^{(0)}_{bb} \right)^j \right\} \tag{2.8}
$$

The first line includes, for $i = 0$, the LL expression used in Ref. [15]. The contributions linear in $\beta_0$ contribute from $k = 2$ and NLL accuracy and are given by

$$
Z^{(k)}_{ab} \bigg|_{\beta_0} = -\frac{\beta_0}{2} \frac{1}{k!\epsilon^k} \sum_{i=0}^{k-2} \epsilon^i \sum_{j=0}^{k-2-i} \frac{(i+j)!}{j!} \left[ k(k-1) - i(i+j+1) \right] \left( \gamma^{(0)}_{aa} \right)^{k-2-i-j} \gamma^{(i)}_{ab} \left( \gamma^{(0)}_{bb} \right)^j, \tag{2.9}
$$

while the corresponding NNLL $\beta_0^2$ term in Eq. (2.7) for $k \geq 3$ is

$$
Z^{(k)}_{ab} \bigg|_{\beta_0^2} = \frac{\beta_0^2}{24} \frac{1}{k!\epsilon^k} \sum_{i=1}^{k-3} \epsilon^i \sum_{j=0}^{k-3-i} \frac{(j+i)!}{j!} \left[ k(k-1)(k-2)(3k-1) - 6i(i+j+1)k(k-1) 
+ i(3i+1)(i+j+1)(i+j+2) \right] \left( \gamma^{(0)}_{aa} \right)^{k-3-i-j} \gamma^{(i)}_{ab} \left( \gamma^{(0)}_{bb} \right)^j. \tag{2.10}
$$
Finally we distinguish NNLL contributions with $\gamma^{(1)} \gamma^{(1)}$, where one of the factors is the off-diagonal $O(N^{-1})$ entry, and contributions with $\gamma^{(1)} \gamma^{(\ell+1)}$, where the latter has to be off-diagonal at the present level of accuracy (recall that every term includes one off-diagonal anomalous dimension $\gamma_{a\neq b}$). The former terms contribute from order $\alpha_s^4$ and are given by

$$Z^{(k)}_{ab} \big|_{\gamma^{(1)} \gamma^{(1)}} = \frac{1}{k!} e^{-k+1} \sum_{i=0}^{k-4} \sum_{j=0}^{k-4-i} [(k-i-2)^2 - 1 - j(k-i-1)] (\gamma^{(0)}_{aa})^i (\gamma^{(1)}_{bb})^j (\gamma^{(0)}_{bb})^{k-i-j-4}$$

and the corresponding final contribution to Eq. (2.7) at $k \geq 5$ reads

$$Z^{(k)}_{ab} \big|_{\gamma^{(1)} \gamma^{(\ell)}} = \frac{1}{k!} e^{-k} \sum_{\ell=2}^{k-3} \sum_{i=0}^{k-\ell-3} \sum_{j=0}^{k-3-i-\ell} [(k-i-1)] (\gamma^{(0)}_{aa})^i (\gamma^{(0)}_{ab})^j (\gamma^{(0)}_{bb})^{k-i-j-\ell-3}$$

The coefficients Eqs. (2.8) – (2.12) have been inferred by analyzing the respective first five to seven non-trivial orders $k$ and then verified to ‘all’ orders using, as for a large part of our symbolic manipulations, the programs FORM and TForm [30][31].

The corresponding result for the diagonal entries of the $Z$-matrices are much simpler due to Eq. (1.2). Including also terms which contribute to the next-to-next-to-next-to-leading logarithmic (N$^3$LL) terms suppressed by one power of $1/N$, the coefficients at order $\alpha_s^k$ are given by

$$Z^{(k)}_{aa} = \frac{e^{-k}}{k!} (\gamma^{(0)}_{aa})^k + \frac{e^{-k+1}}{2(k-2)!} (\gamma^{(0)}_{aa})^{k-2} (\gamma^{(0)}_{aa}) - \frac{\beta_0}{2} \frac{e^{-k}}{(k-2)!} (\gamma^{(0)}_{aa})^{k-1}$$

Only the first four terms contribute to the $N^0$ NNLL expression entering the off-diagonal $N^{-1}$ mass factorization (2.3). Note that, unlike at order $N^0$, $P^{(1)}_{aa}$ and $P^{(2)}_{aa}$ are NLL and N$^3$LL quantities at order $N^{-1}$ due to $C^{(1)}_k = 0$ and $C^{(n\geq 2)}_k \neq 0$ in Eq. (1.2).
The unfactorized structure functions (2.3) are given by these results multiplied by
\[ \widetilde{C}_{a,i} = \delta_{\alpha\gamma} \delta_{iq} + \delta_{\alpha\phi} \delta_{ig} + \sum_{n=1}^{\infty} a_s^n \sum_{k=0}^{\infty} \varepsilon^k c_a^{(n,k)} . \] (2.14)

The index \( \gamma \) generically represents the gauge-boson exchange structure functions (except for \( F_L \)). \( c_a^{(n,0)} = c_a^{(n)} \) are the \( n \)-th order coefficient functions in Eq. (1.4) for any combination of \( a \) and \( i \).

The quantities \( c_a^{(n,k)} \) – usually denoted by \( a_a^{(n)} \) and \( b_a^{(n)} \) etc in fixed-order calculations – are enhanced by factors \( \ln^k N \) with respect to those four-dimensional coefficient functions.

The calculation of \( T_a \) to order \( \alpha_s^{\ell \leq n+1} \) and \( \varepsilon^{n-\ell} \) (for \( F_L \): \( \varepsilon^{n-\ell+1} \)) provides the \( N^n \) LO (leading-order, next-to-leading-order etc) renormalization-group improved fixed-order approximation to the structure functions \( F_a \). It is obvious from Eqs. (2.6) – (2.14) that a full \( N^n \) LO result completely fixes the highest \( n+1 \) powers of \( 1/\varepsilon \) to all orders in \( \alpha_s \). An all-order resummation of the splitting functions and coefficient functions requires, at the logarithmic accuracy under consideration, an extension of these results to all powers of \( \varepsilon \). The flavour-singlet structure functions considered here are fully known at \( N^2 \) LO from Refs. [8, 10, 16, 17] and the earlier coefficient-function calculations of Refs. [32, 35]. Hence a double-logarithmic resummation based on these results can be expected to predict up to the highest three logarithms at all higher orders, including the corresponding contributions to the three-loop coefficient functions for \( F_2 \) and \( F_\phi \) exactly computed in Refs. [11, 17] and the large-x predictions of the four-loop splitting functions in the latter article.

3 Resummation of the unfactorized expressions for \( F_2 \) and \( F_\phi \)

In this section we derive all-order expressions for the leading \( N^{-1} \) contributions to the off-diagonal amplitudes or unfactorized structure functions \( T_{2,g} \) and \( T_{\phi,g} \) at NNLL accuracy. We will apply two approaches: first an iteration of amplitudes generalizing the leading logarithmic results of Ref. [15], and then an apparently new and more rigorous treatment which makes use of only the \( D \)-dimensional structure of the unfactorized structure functions in the large-x limit and the KLN cancellations [36, 37] between its real- and virtual-emission contributions.

Both calculations require the corresponding expressions for the \( N^0 \) parts of \( Z_{qq}, Z_{gg}, \widetilde{C}_{2,q} \) and \( \widetilde{C}_{\phi,g} \) which can be determined from the diagonal amplitudes \( T_{2,q} \) and \( T_{\phi,g} \) in the limit governed by the soft-gluon exponentiation. These quantities are given by
\[ T_{a,k} = \exp \left( \hat{a}_s \tilde{T}^{(1)}_{a,k} + \hat{a}_s^2 \tilde{T}^{(2)}_{a,k} + \hat{a}_s^3 \tilde{T}^{(3)}_{a,k} + \ldots \right) \] (3.1)
with
\[ \tilde{T}^{(n)}_{a,k} = \sum_{\ell = -n-1}^{\infty} \varepsilon^\ell \left( R^{(n,\ell)}_{a,k} \exp(n \varepsilon \ln N) - V^{(n,\ell)}_{a,k} \right) . \] (3.2)

For the quark case the coefficients entering the highest four logarithms at all orders in \( \alpha_s \) and \( \varepsilon \) read
\[ R^{(1,-2)}_{2,q} = 4 , \quad R^{(1,-1)}_{2,q} = 3 , \quad R^{(1,0)}_{2,q} = 7 - 4 \zeta_2 , \quad R^{(1,1)}_{2,q} = 14 - 3 \zeta_2 - 8 \zeta_3 , \]
\[ V^{(1,-2)}_{2,q} = 4 , \quad V^{(1,-1)}_{2,q} = 6 , \quad V^{(1,0)}_{2,q} = 16 + 2 \zeta_2 , \quad V^{(1,1)}_{2,q} = 32 - 3 \zeta_2 - \frac{28}{3} \zeta_3 , \] (3.3)
The corresponding gluonic coefficients are required only to NNLL accuracy here and read

\[ R_{2,q}^{(2,-3)} = V_{2,q}^{(2,-3)} = \beta_0, \]
\[ R_{2,q}^{(2,-2)} = \left( \frac{4}{3} - 2\zeta_2 \right) C_A + \frac{19}{6} \beta_0, \quad V_{2,q}^{(2,-2)} = R_{2,q}^{(2,-2)} - \frac{3}{2} \beta_0, \]
\[ R_{2,q}^{(2,-1)} = \left( \frac{3}{4} - 6\zeta_2 + 12\zeta_3 \right) C_F + \left( \frac{73}{18} - 20\zeta_3 \right) C_A + \left( \frac{373}{36} - 3\zeta_2 \right) \beta_0, \]
\[ V_{2,q}^{(2,-1)} = \left( \frac{3}{2} - 12\zeta_2 + 24\zeta_3 \right) C_F - \left( \frac{41}{9} + 26\zeta_3 \right) C_A - \left( \frac{353}{18} + 2\zeta_2 \right) \beta_0, \quad (3.4) \]
\[ R_{2,q}^{(3,-4)} = V_{2,q}^{(3,-4)} = \frac{4}{9} \beta_0^2, \quad V_{2,q}^{(3,-3)} = R_{2,q}^{(3,-3)} + \beta_0^2, \]
\[ R_{2,q}^{(3,-3)} = -\frac{14}{9} C_A^2 - \frac{22}{9} C_F C_A + \frac{2}{3} C_F \beta_0 + \left( \frac{62}{27} - \frac{16}{9} \zeta_2 \right) C_A \beta_0 + \frac{67}{27} \beta_0^2 \quad (3.5) \]

and

\[ R_{2,q}^{(4,-5)} = V_{2,q}^{(4,-5)} = \frac{1}{4} \beta_0^3, \quad (3.6) \]

where we have suppressed an obvious overall factor of \( C_F \) and expressed the dependence of the number \( n_f \) of effectively massless flavours in terms of \( \beta_0 \). To \( N^3LL \) accuracy these results are converted to the renormalized coupling used elsewhere in this article via

\[ \hat{a}_s = a_s - \frac{\beta_0}{\epsilon} a_s^2 + \left( \frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) a_s^3 + \frac{\beta_0^3}{\epsilon^3} a_s^4 + \ldots. \quad (3.7) \]

The corresponding gluonic coefficients are required only to NNLL accuracy here and read

\[ \begin{align*} R_{\phi,g}^{(1,-2)} &= 4C_A, \quad R_{\phi,g}^{(1,-1)} = \beta_0, \quad R_{\phi,g}^{(1,0)} = \left( \frac{4}{3} - 4\zeta_2 \right) C_A + \frac{5}{3} \beta_0, \\
V_{\phi,g}^{(1,-2)} &= 4C_A, \quad V_{\phi,g}^{(1,-1)} = 0, \quad V_{\phi,g}^{(1,0)} = 2\zeta_2 C_A, \\
R_{\phi,g}^{(2,-3)} &= V_{\phi,g}^{(2,-3)} = C_A \beta_0, \\
R_{\phi,g}^{(2,-2)} &= \left( \frac{4}{3} - 2\zeta_2 \right) C_A^2 + \frac{5}{3} C_A \beta_0 + \frac{1}{2} \beta_0^2, \quad V_{\phi,g}^{(2,-2)} = R_{\phi,g}^{(2,-2)} - \frac{1}{2} \beta_0^2, \\
R_{\phi,g}^{(3,-4)} &= V_{\phi,g}^{(3,-4)} = \frac{4}{9} C_A \beta_0^2. \quad (3.10) \end{align*} \]

After the transformation to the renormalized coupling, Eqs. (3.1) and (3.2) for \( T_{\phi,g} \) need to be multiplied by the renormalization constant of \( G^{\mu\nu} G_{\mu\nu} \) \([38,39]\),

\[ 1 - 2\beta_0 \epsilon^{-1} a_s^2 + 3\beta_0^2 \epsilon^{-2} a_s^3 + \ldots. \quad (3.11) \]

Note that there is no new physics content in Eqs. (3.1) – (3.10), which represent the Mellin transforms of the decomposition of \( T_{2,q} \) and \( T_{\phi,g} \) used in Refs. \([40,41]\) and its obvious higher-order generalizations, cf. Ref. \([42]\), recast in an exponential all-order form. Recall also, from the same references, that of the \( \ell \) coefficients \( R^{(1,\ell-3)} \), \( R^{(2,\ell-5)} \) \ldots \( R^{(n,\ell-2n-1)} \) relevant for the \( \ell \)-th logarithm at order \( n \) only one combination is not fixed by lower-order information, and that the same holds for their virtual-correction counterparts \( V^{(1,\ell-3)} \), \( V^{(2,\ell-5)} \) \ldots \( V^{(n,\ell-2n-1)} \).
We are now ready to address the resummation of the off-diagonal amplitudes. Using the particularly simple colour structure of the identical leading-logarithmic contributions to $T_{\phi q}/C_F$ and $T_{2g}/n_f$, their LL resummation has been inferred in Ref. [15] to which the reader is referred for a detailed discussion. The result can be written as

$$T_{a,k}^{(n)} \equiv \frac{1}{n} T_{a,k}^{(1)} \sum_{i=0}^{n-1} \binom{n-1}{i} \phi^{-1} T_{\phi g}^{(i)} T_{2q}^{(n-i-1)}$$

(3.12)

where, of course, only the respective first terms of

$$T_{\phi q}^{(1)} = -\frac{C_F}{N} \exp(\epsilon \ln N) \left( 2 - \epsilon - (3 - 2\zeta_2) \epsilon^2 + \ldots \right),$$

$$T_{2g}^{(1)} = -\frac{n_f}{N} \exp(\epsilon \ln N) \left( 2 - 2\epsilon - (6 - 2\zeta_2) \epsilon^2 + \ldots \right)$$

(3.13)

are required. The corresponding expressions for $T_{2q}$ and $T_{\phi g}$ can be read off from Eqs. (3.1), (3.2), (3.3) and (3.8) above.

Eq. (3.12) can be generalized to next-to-leading logarithmic accuracy, where running-coupling effects enter for the first time, using the natural ansatz

$$T_{a,k}^{(n)\text{NL}} \equiv \frac{1}{n} T_{a,k}^{(1)} \left\{ \sum_{i=0}^{n-1} f_{a,k}(n, i, \epsilon) T_{\phi g}^{(i)} T_{2q}^{(n-i-1)} - \frac{\beta_0}{\epsilon} \sum_{i=0}^{n-2} g_{a,k}(n, i) T_{\phi g}^{(i)} T_{2q}^{(n-i-2)} \right\}$$

(3.14)

where $f_{a,k}(n, i, \epsilon)$ are linear functions of $\epsilon$. It is not a priori clear that such a simple ansatz is compatible with the infinite number of constraints provided by the NLL coefficients of the four highest powers of $1/\epsilon$ at all orders $n$ in $\alpha_s$ which are provided by the results of Refs. [8,9,11,17] including the large-$x$ results for the four-loop splitting functions $P_{qg}^{(3)}$ and $P_{gq}^{(3)}$ in Eq. (1.1). However, all these constraints can indeed be fulfilled, and the resulting coefficients are given by

$$f_{\phi q}(n, i, \epsilon) = \binom{n-1}{i}^{-1} \left[ 1 + \epsilon \left( \frac{\beta_0}{8C_A} (i+1)(n-i) \theta_{i1} - \frac{3}{2} (1-n\delta_{i0}) \right) \right],$$

$$f_{2g}(n, i, \epsilon) = \binom{n-1}{i}^{-1} \left[ 1 + \epsilon \left( \frac{\beta_0}{8C_A} (n-i-3) + \frac{1}{2} (3i+1-n\delta_{i1}) \right) \right]$$

(3.15)

and

$$g_{\phi q}(n, i) = g_{2q}(n, i) = \binom{n}{i+1}^{-1}$$

(3.16)

with $\theta_{k,j} = 1$ for $k \geq j$ and $\theta_{k,j} = 0$ else. Eqs. (3.14) – (3.16) together with their diagonal counterparts (3.1), keeping the $\epsilon^{-2}$ and $\epsilon^{-1}$ contributions to $T_{a,k}^{(1)}$ and the $\epsilon^{-3}$ terms of $T_{a,k}^{(2)}$ in Eq. (3.2), facilitate the extension of the all-order mass factorization to the next-to-leading logarithms.

An extension of Eq. (3.14) to the the third logarithms can be expected to become much more cumbersome, requiring at least $\epsilon^2$ corrections to Eqs. (3.15), $\epsilon$ corrections to Eqs. (3.16) and a new $\beta_0^2$ contribution, but presumably also terms respectively involving $T_{2q}^{(2)}$ or $T_{\phi q}^{(2)}$. Instead of pursuing this approach, we now switch to our second method for the resummation of $T_{2g}$ and $T_{\phi q}$. 

8
For this purpose we consider the calculation of $T_{a,k}$ via suitably projected gauge-boson parton cross sections as performed at two loops in Refs. \cite{32,34}. The maximal ($2 \to n + 1$ particles) phase space for these processes at order $\alpha_s^n$ is \cite{4,3,41,42}

\[(1-x)^{-1-n\epsilon} \int_0^1 d(3n-2 \text{ other variables}) f(x, \ldots), \quad (3.17)\]

where one trivial azimuthal integration has not been counted. The integrals for the $n$-th order purely real (tree graph) contributions $T^{(n)R}_{a,j}$ do not lead to any further factors $(1-x)^{-\epsilon}$, hence their expansion around $x = 1$ can be written as

\[T^{(n)R}_{a,j} = (1-x)^{-1-n\epsilon} \sum_{\xi=0}^\infty (1-x)^\xi \frac{1}{\epsilon^{2n-1}} \left\{ R^{(n)LL}_{a,j,\xi} + \epsilon R^{(n)NLL}_{a,j,\xi} + \epsilon^2 R^{(n)NNL}_{a,j,\xi} + \ldots \right\}. \quad (3.18)\]

The mixed contributions ($2 \to r + 1$ particles with $n - r \geq 1$ loops) include up to $n - r$ additional factors of $(1-x)^{-\epsilon}$ from the loop integrals on top the phase-space factor, leading to

\[T^{(n)M}_{a,j} = \sum_{\ell=r}^n (1-x)^{-1-\ell\epsilon} \sum_{\xi=0}^\infty (1-x)^\xi \frac{1}{\epsilon^{2n-1}} \cdot \left\{ M^{(n)LL}_{a,j,\ell,\xi} + \epsilon M^{(n)NLL}_{a,j,\ell,\xi} + \epsilon^2 M^{(n)NNL}_{a,j,\ell,\xi} + \ldots \right\}. \quad (3.19)\]

Finally the diagonal cases, where terms with $\xi = 0$ are present in Eqs. \eqref{3.18} and \eqref{3.19}, also receive purely virtual contributions $T^{(n)V}_{a,j}$ given by the $\gamma^*qq$ and $\phi gg$ form factors which are known to an amply sufficient accuracy \cite{40,41,45,48}

\[T^{(n)V}_{a,j} = \delta(1-x) \frac{1}{\epsilon^{2\ell}} \left\{ V^{(n)LL}_{a,j} + \epsilon V^{(n)NLL}_{a,j} + \epsilon^2 V^{(n)NNL}_{a,j} + \ldots \right\}. \quad (3.20)\]

Note that for $\xi = 0$ the $(1-x)$ factors in Eqs. \eqref{3.18} and \eqref{3.19} are $D$-dimensional $+$-distributions which include a factor $\epsilon^{-1} \delta(1-x)$ after expansion in $\epsilon$.

The partonic cross sections $T^{(n)}_{a,k}$ at order $\alpha_s^n$ do not include any poles higher than $\epsilon^{-n}$, i.e.,

\[T^{(n)}_{a,k} = T^{(n)R}_{a,k} + T^{(n)M}_{a,k} + (\delta_{aT} \delta_{iq} + \delta_{a\phi} \delta_{ig}) T^{(n)V}_{a,k} \]

\[= \frac{1}{\epsilon^n} \left\{ T^{(n)0}_{a,j} + \epsilon T^{(n)1}_{a,j} + \epsilon^2 T^{(n)2}_{a,j} + \ldots \right\}. \quad (3.21)\]

(recall Eq. \eqref{2.14} concerning the index $\gamma$). Hence there are $n - 1 - m$ KLN relations between the $n$ $N^m$LL coefficients in Eqs. \eqref{3.18} and \eqref{3.19}. Since a full $N^k$LO calculation provides, as discussed above, the first $k + 1$ non-trivial powers of $\epsilon^{-1}$, it leads to a total of $n - m + k$ relations. Consequently the coefficients up to the $N^k$LL terms are fixed (and those for $m < k$ over-constrained) in terms of the $N^k$LO results by nothing but the above $D$-dimensional structure and the mass-factorization formula \eqref{2.3} which is guaranteed for structure functions in DIS by the operator-product expansion \cite{49}, see also, e.g., Refs. \cite{50,51}.
Note that this resummation is far less predictive than the soft-gluon exponentiation \[^{18–22}\] of the \((1-x)^{-1}/N^0\) terms of \(C_{2,q}\) and \(C_{\phi,q}\) which involves an additional factorization in the threshold limit. Therefore, as mentioned below Eq. (3.11), only one of the \(n\) \(N^{n}\)LL coefficients for \(\xi = 0\) is actually independent. Hence a \(N^{n}\)LO calculation in this case implies a \(N^{n}\)LL exponentiation which fixes the highest \(2n+1\) logarithms at all orders in \(\alpha_s\).

Now we switch back to Mellin space and apply the above resultsto \(T_{2,g}\) and \(T_{\Phi,q}\). The \(n\)-th order contributions to these quantities can be written as

\[
T_{a,k}^{(n)}(N) = \frac{1}{N} \sum_{i=0}^{n-1} \left( A_{a,k}^{(n,i)} + \varepsilon B_{a,k}^{(n,i)} + \varepsilon^2 C_{a,k}^{(n,i)} + \ldots \right) \exp(\varepsilon (n-i) \ln N). \tag{3.22}
\]

As discussed below Eq. (3.21), only one of the LL coefficients \(A_{a,k}^{(n,i)}\) is independent for each value of \(n\) and \(a, k\). Choosing \(i = 0\) for that, the KLN constraints on the other coefficients read

\[
A_{a,k}^{(n,i)} = (-1)^i \binom{n-1}{i} A_{a,k}^{(n,0)}. \tag{3.23}
\]

The \(i = 0\) coefficients are found to be

\[
\frac{1}{n_f} A_{2,g}^{(n,0)} = \frac{1}{C_F} A_{\Phi,q}^{(n,0)} = -2^{n-1} \frac{1}{n_f} \sum_{\ell=0}^{n-1} C_\ell C_\ell A^{-\ell-1}. \tag{3.24}
\]

At NLL level two coefficients in the sum in Eq. (3.22) are not fixed by the KLN cancellations. In terms of our choice \(i = 0\) and \(i = 1\) the remaining coefficients are given by

\[
B_{a,k}^{(n,i+1)} = (-1)^i \left[ \binom{n-2}{i} B_{a,k}^{(n,1)} + i \binom{n-1}{i+1} B_{a,k}^{(n,0)} \right]. \tag{3.25}
\]

The all-order results for the respective two coefficients determined by the NLO results are

\[
B_{2,g}^{(n,0)} = -\frac{4^{n-2}}{6n!} n_f \left\{ \sum_{\ell=1}^{n-2} C_\ell C_\ell A^{-\ell-1} \left( (n+1)^2 + 39n + 22 - 14\ell \right) + 36 C_F^{n-1}(n+1) \theta_{n2} + C_A^{n-1}(11n^2 + 15n + 22) - 2n_f \sum_{\ell=0}^{n-2} C_\ell C_\ell A^{-\ell-2} (n^2 - 3n + 2(\ell + 1)) \right\}, \tag{3.26}
\]

\[
B_{2,g}^{(n,1)} = \frac{4^{n-2}}{6n!} n_f \left\{ \sum_{\ell=1}^{n-2} C_\ell C_\ell A^{-\ell-1} \left( (n+2)^2 + 50n^2 - (9 + 14(\ell - 1))n - 66 + 6\ell \right) + 36 C_F^{n-1}(n+2)(n-1) + C_A^{n-1}(11n^3 + 26n^2 + 29n - 66) - 2n_f \sum_{\ell=0}^{n-2} C_\ell C_\ell A^{-\ell-2} (n^3 - 2n^2 + (7 + 2\ell)n - 6(\ell + 1)) \right\}. \tag{3.27}
\]
The coefficients for the three independent coefficient at rather lengthy. However, especially
eq i, + \sum_{\ell=0}^{n-2} C_F^n C_A^{n-\ell-2} n^2 - (3 + 2\ell)n - 2(\ell + 1) \right) \right) .

(3.29)

Similar to Eqs. (2.8) – (2.12), we have first inferred these results by analyzing a couple of or-
ders (enough to over-constrain the numerator polynomials) and then verified them to ‘all’ orders.
Eqs. (3.22) – (3.29) lead to the same results as Eqs. (3.13) – (3.16) above.

At NNLL accuracy, finally, all but three coefficients (chosen as \( i = 0, 1, 2 \)) in the sum in
Eq. (3.22) are specified by the vanishing of poles higher that \( \varepsilon^{-n} \) in \( T_{2g}^{(n)} \) and \( T_{\phi,q}^{(n)} \). These co-
efficients can be written as

\[
C_{a,k}^{(n,i+2)} = (-1)^i \left[ \binom{n-3}{i} C_{a,k}^{(n,2)} + i \binom{n-2}{i+1} C_{a,k}^{(n,1)} + \frac{1}{2} i(i+1) \binom{n-1}{i+2} C_{a,k}^{(n,0)} \right].
\]

(3.30)

The general expressions for the three independent coefficient at rather lengthy. However, especially
since we will not be able to express the NNLL mass-factorized results in a closed form, they are
presented here nevertheless in order to assist future research by others. The coefficients for \( T_{2g} \) are

\[
C_{2,g}^{(n,0)} = - \frac{4^{n-2}}{6n!} n_f \left\{ C_A^{n-1} \left( 76n^2 + 60n + 8 \right) - C_A^{n-1} \zeta_2 \left( 30n^2 + 30n - 12 \right) + 300 C_F n \zeta_2 n \theta_n + \beta_0 C_A^{n-2} \left( 3n^3 - 4n^2 - 9n + 10 \right) \theta_n^2 + C_F n \left( \frac{27}{2} n^2 + \frac{195}{2} n + 111 \right) \theta_n^2 + C_F C_A^{n-2} \left( 28n^2 + 132n - 68 \right) \theta_n^3 + \sum_{\ell=0}^{n-3} \beta_0 C_F C_A^{n-\ell-3} \left[ \frac{3}{32} n^4 - \frac{29}{48} n^3 + \left( \frac{3}{8} \ell + \frac{41}{32} \right) n^2 - \left( \frac{5}{8} \ell + \frac{49}{48} \right) n + \frac{1}{8} (\ell^2 + 3\ell + 2) \right] \theta_n^3 + \sum_{\ell=1}^{n-2} \beta_0 C_F C_A^{n-\ell-2} \left[ \frac{9}{2} n^3 - \left( \frac{9}{4} \ell + \frac{17}{2} \right) n^2 + \left( \frac{45}{4} \ell - 6 \right) n - \frac{9}{4} \ell^2 + \frac{31}{4} \ell + 10 \right] \theta_n^3 + \sum_{\ell=1}^{n-2} C_F C_A^{n-\ell-1} \zeta_2 \left[ 6n^2 + 54n - 12(\ell + 1) \right] \theta_n^3 + \sum_{\ell=2}^{n-2} C_F C_A^{n-\ell-1} \left[ 58n^2 - (54\ell - 156)n + \frac{27}{2} \ell^2 - \frac{179}{2} \ell + 8 \right] \theta_n^4 \right\},
\]

(3.31)
\[ C^{(n,1)}_{2,g} = \frac{4^{n-2}}{6n!} n_f \left\{ C_A^{n-1} \left(76n^3 - 24n^2 - 52n\right) - C_A^{n-1} \zeta_2 \left(30n^3 - 12n^2 - 6n - 12\right) \\
+ 456 C_F \delta n_2 - C_F^{n-1} \zeta_2 \left(48n^3 - 72n - 24\right) \theta n_2 + \beta_0 C_A^{n-2} \left(3n^4 - n^3 - 21n^2 + 49n - 30\right) \theta n_2 \\
+ C_F^{n-1} \left(\frac{27}{2} n^2 + 111n^2 + \frac{189}{2} n - 225\right) \theta n_3 + C_F C_A^{n-2} \left(28n^3 + 96n^2 - 152n + 144\right) \theta n_3 \\
+ \sum_{\ell=0}^{n-3} \beta_0^2 C_F^\ell C_A^{n-\ell-3} \left[\frac{3}{32} n^5 - \frac{31}{96} n^4 + \left(\frac{3}{8} \ell + \frac{13}{96}\right) n^3 - \left(\ell + \frac{113}{96}\right) n^2 \\
+ \left(\frac{\ell^2 + 4\ell + \frac{217}{48}}{8}\right) n - \frac{13}{8} (\ell^2 + 3\ell + 2)\right] \theta n_3 \\
+ \sum_{\ell=1}^{n-2} \beta_0 C_F^\ell C_A^{n-\ell-2} \left[\frac{9}{2} n^4 - \left(\frac{9}{4} \ell + 4\right) n^3 + \left(\frac{45}{4} \ell - \frac{33}{2}\right) n^2 - \left(\frac{9}{4} \ell^2 + \frac{113}{4} \ell - 46\right) n \\
+ \frac{27}{2} \ell^2 - \frac{33}{2} \ell - 30\right] \theta n_3 \\
+ \sum_{\ell=2}^{n-3} C_F^\ell C_A^{n-\ell-1} \left[58n^3 - (54\ell - 90)n^2 + \left(\frac{27}{2} \ell^2 + \frac{37}{2} \ell - 154\right) n - \frac{81}{2} \ell^2 + \frac{369}{2} \ell\right] \theta n_4 \\
- \sum_{\ell=1}^{n-2} C_F^\ell C_A^{n-\ell-1} \zeta_2 \left[6n^3 + 36n^2 - (12\ell + 78)n + 12(\ell - 1)\right] \theta n_3 \right\}, \tag{3.32} \]

\[ C^{(n,2)}_{2,g} = -\frac{4^{n-2}}{6n!} n_f \left\{ C_A^{n-1} \left(38n^4 - 92n^3 + 6n^2 + 56n - 8\right) \\
- C_A^{n-1} \zeta_2 \left(15n^4 - 42n^3 + 39n^2 - 48n + 36\right) - C_F^{n-1} \zeta_2 \left(24n^3 - 96n^2 + 72n + 48\right) \theta n_2 \\
+ \beta_0 C_A^{n-2} \left[\frac{3}{2} n^5 - \frac{1}{2} n^4 - \frac{47}{2} n^3 + \frac{153}{2} n^2 - 104n + 50\right] \theta n_2 + C_F^{n-1} \left[\frac{27}{4} n^4 + \frac{111}{2} n^3 \\
- \frac{147}{4} n^2 - \frac{771}{2} n + 366\right] \theta n_3 + C_F C_A^{n-2} \left[14n^4 + 16n^3 - 140n^2 + 214n - 220\right] \theta n_3 \\
+ \sum_{\ell=0}^{n-3} \beta_0^2 C_F^\ell C_A^{n-\ell-3} \left[\frac{3}{64} n^6 - \frac{13}{192} n^5 + \left(\frac{3}{16} \ell - \frac{31}{64}\right) n^4 - \left(\frac{7}{8} \ell - \frac{353}{192}\right) n^3 \\
+ \left(\frac{1}{16} \ell^2 + \frac{33}{16}\right) n^2 - \left(\frac{27}{16} \ell^2 + \frac{211}{16} \ell + \frac{607}{48}\right) n + \frac{37}{8} (\ell^2 + 3\ell + 2)\right] \theta n_3 \\
+ \sum_{\ell=1}^{n-2} \beta_0 C_F^\ell C_A^{n-\ell-2} \left[\frac{9}{4} n^5 - \left(\frac{9}{8} \ell + 2\right) n^4 + \left(\frac{27}{4} \ell^2 - \frac{91}{4}\right) n^3 \\
- \left(\frac{9}{8} \ell^2 + 31\ell - \frac{147}{2}\right) n^2 + \left(\frac{117}{8} \ell^2 + \frac{359}{8} \ell - 101\right) n - \frac{135}{4} \ell^2 + \frac{65}{4} \ell + 50\right] \theta n_3 \\
- \sum_{\ell=1}^{n-2} C_F^\ell C_A^{n-\ell-1} \zeta_2 \left[3n^4 + 6n^3 - (6\ell + 69)n^2 + (18\ell + 72)n - 12(\ell - 3)\right] \theta n_3 \\
+ \sum_{\ell=2}^{n-2} C_F^\ell C_A^{n-\ell-1} \left[29n^4 - (27\ell + 17)n^3 + \left(\frac{27}{4} \ell^2 + \frac{361}{4} \ell - 162\right) n^2 \\
- \left(\frac{189}{4} \ell^2 - \frac{269}{4} \ell - 164\right) n + 81 \ell^2 - 293 \ell - 8\right] \theta n_4 \right\}. \tag{3.33} \]
The corresponding results for $T_{q,q}$ read

$$
C_{\phi,q}^{(n,0)} = - \frac{4n^{-2}}{6n!} C_F \left\{ C_F^{n-1} \left( \frac{183}{2} n^3 + \frac{3}{2} n^2 - 21 \right) + C_F^{n-1} \zeta_2 \left( 48n^2 - 144n + 48 \right)
- C_F^{n-2} C_A \left( \frac{13}{2} n^2 - \frac{17}{2} n + 7 \right) \theta_{n2} - C_F^{n-2} C_A^{n} \zeta_2 \left( 54n^2 + 6n - 60 \right) \theta_{n2}
+ C_F^{n-2} \beta_0 \left( \frac{27}{4} n^3 + \frac{13}{2} n^2 - \frac{55}{4} n - \frac{1}{2} \right) \theta_{n2} + \sum_{\ell=0}^{n-3} C_F^{\ell} C_A^{n-\ell-1} \left[ 4n^2 + 12n + \frac{27}{2} \ell^2 \\
+ \frac{53}{2} \ell - 8 \right] \theta_{n3} - \sum_{\ell=0}^{n-3} C_F^{\ell} C_A^{n-\ell-1} \zeta_2 \left[ 6n^2 + 42n + 12\ell - 36 \right] \theta_{n3}
+ \sum_{\ell=0}^{n-3} \beta_0 C_F^{\ell} C_A^{n-\ell-2} \left[ \left( \frac{9}{4} \ell + 2 \right) n^2 + \left( \frac{27}{4} \ell + 6 \right) n - \frac{9}{4} \ell^2 - \frac{25}{4} \ell - 4 \right] \theta_{n3}
+ \sum_{\ell=0}^{n-3} \beta_0^2 C_F^{\ell} C_A^{n-\ell-3} \left[ \frac{3}{32} n^4 + \frac{25}{48} n^3 - \frac{1}{32} (12\ell - 13)n^2 - \frac{1}{48} (54\ell + 121)n \\
+ \frac{1}{8} (\ell^2 + 3\ell + 2) \right] \theta_{n3} \right\},
$$

(3.34)

$$
C_{\phi,q}^{(n,1)} = \frac{4n^{-2}}{6n!} C_F \left\{ C_F^{n-1} \left( \frac{183}{2} n^3 + \frac{3}{2} n^2 - \frac{171}{2} n + 3 \right) + C_F^{n-1} \zeta_2 \left( 48n^3 - 192n^2 + 216n - 72 \right)
- C_F^{n-2} C_A \left( \frac{13}{2} n^3 - 34n^2 + \frac{253}{2} n - 65 \right) \theta_{n2} - C_F^{n-2} C_A^{n} \zeta_2 \left( 54n^3 - 60n^2 - 78n + 84 \right) \theta_{n2}
+ C_F^{n-2} \beta_0 \left( \frac{27}{4} n^4 + \frac{31}{2} n^3 - \frac{343}{4} n^2 + \frac{91}{2} n - 32 \right) \theta_{n2}
+ \sum_{\ell=0}^{n-3} C_F^{\ell} C_A^{n-\ell-1} \left[ 4n^3 + \left( \frac{27}{2} \ell^2 + \frac{53}{2} \ell - 28 \right) n + \frac{27}{2} \ell^2 - \frac{43}{2} \ell - 32 \right] \theta_{n3}
- \sum_{\ell=0}^{n-3} C_F^{\ell} C_A^{n-\ell-1} \zeta_2 \left[ 6n^3 + 24n^2 + (12\ell - 90)n - 12\ell + 60 \right] \theta_{n3}
+ \sum_{\ell=0}^{n-3} \beta_0 C_F^{\ell} C_A^{n-\ell-2} \left[ \left( \frac{9}{4} \ell + 2 \right) n^3 + \left( \frac{45}{4} \ell - 12 \right) n^2 - \left( \frac{9}{4} \ell^2 + \frac{7}{4} \ell + 26 \right) n \\
- 18\ell^2 - 22\ell - 4 \right] \theta_{n3}
+ \sum_{\ell=0}^{n-3} \beta_0^2 C_F^{\ell} C_A^{n-\ell-3} \left[ \frac{3}{32} n^5 + \frac{77}{96} n^4 - \frac{1}{36} (36\ell - 37)n^3 - \frac{1}{96} (216\ell + 533)n^2 \\
+ \frac{1}{4} (\ell - 6\ell + \frac{1}{12}) n^3 + \frac{1}{8} (23\ell^2 + 69\ell + 46) \right] \theta_{n3} \right\},
$$

(3.35)

and

$$
C_{\phi,q}^{(n,2)} = - \frac{4n^{-2}}{6n!} C_F \left\{ C_F^{n-1} \left( \frac{183}{4} n^4 - \frac{111}{2} n^3 - \frac{531}{4} n^2 + \frac{201}{2} n + 42 \right)
+ C_F^{n-1} \zeta_2 \left( 24n^4 - 144n^3 + 312n^2 - 288n + 96 \right) - C_F^{n-2} C_A \left( \frac{13}{4} n^4 - 33n^3 
$$
4 Resummation of the unfactorized expressions for $F_L$

Up to one additional power of $\varepsilon$ the moments of the unfactorized longitudinal structure functions $T_{L,q}$ and $T_{L,g}$ are built up from $D$-dimensional exponentials in the same way as $T_{2,g}$ and $T_{\phi,q}$,

$$T_{L,k}^{(n)}(N) = \frac{1}{N^{1+\delta_{kq}2n-2}} \sum_{i=0}^{n-1} \left( A_{L,k}^{(n,i)} + \varepsilon B_{L,k}^{(n,i)} + \varepsilon^2 C_{L,k}^{(n,i)} + \ldots \right) \exp(\varepsilon(n-i)\ln N). \quad (4.1)$$

In these cases also the $\varepsilon^{-n}$ poles vanish at order $\alpha_s^n$ which compensates the absence of $\varepsilon^{-2n+1}$ contributions. Consequently Eqs. (3.23), (3.25) and (3.30) are valid also for $F_L$. As discussed in the introduction, our main objective for $F_L$ is the resummation of $C_{L,g}$. As in the case of $F_2$ the corresponding $D$-dimensional quark coefficient function, and thus $T_{L,q}$, is also required for this.

Also here the LL coefficients $A_{L,k}^{(n,0)}$ are very simple and closely related,

$$A_{L,q}^{(n,0)} = \frac{2^{2n}}{(n-1)!} C_F^n, \quad A_{L,g}^{(n,0)} = \frac{2^{2n+1}}{(n-1)!} C_A^{n-1} n_f. \quad (4.2)$$

The corresponding NLL contributions to Eq. (4.1) are given by

$$B_{L,q}^{(n,0)} = \frac{4^{n-2}}{(n-1)!} C_F^{n-1} \left\{ \beta_0(n-1)(n-2) + 16 C_A(n-1)(1-\zeta_2) - 4 C_F(9n-13) + 32 C_F^2 \zeta_2(n-1) \right\}, \quad (4.3)$$
\[ B_{L,q}^{(n,1)} = - \frac{4^{n-2}}{(n-2)!} C_F^{n-1} \left\{ \beta_0(n^2 + 3n - 6) + 16C_A(n-1)(1 - \zeta_2) - 4C_F(9n - 16) + 32C_F\zeta_2(n-1) \right\} \] (4.4)\\
and\\n\[ B_{L,g}^{(n,0)} = \frac{4^{n-2}}{(n-1)!} n_f C_A^{n-2} \left\{ 2\beta_0(n-1)(n-2)\theta_{n2} - 8C_F(2n-1)\theta_{n2} + 64C_An \right\} - 2 \frac{4^{n-1}}{(n-1)!} n_f \sum_{\ell=2}^{n-1} C_F C_A^{n-\ell-1} \theta_{n3} , \] (4.5)\\n\[ B_{L,g}^{(n,1)} = - \frac{4^{n-2}}{(n-1)!} n_f C_A^{n-2} \left\{ 2\beta_0(n^2 - n + 2)(n-1)\theta_{n2} - 8C_F(n-3)\theta_{n2} + 64C_A(n-1) \right\} + 2 \frac{4^{n-1}}{(n-1)!} n_f \sum_{\ell=2}^{n-1} C_F C_A^{n-\ell-1}(n-2) \theta_{n3} . \] (4.6)\\
The NNLL terms for \( T_{L,q} \) read\\n\[ C_{L,q}^{(n,0)} = \frac{4^{n-2}}{12(n-1)!} C_F^{n-2} \left\{ \beta_0^2 \left( \frac{3}{8} n^4 + \frac{7}{12} n^3 - \frac{7}{8} n^2 - \frac{85}{12} n + 7 \right) + C_A^2 \left( 96n^2 - 288n + 192 \right) (\zeta_2 - \zeta_3) + \beta_0 C_A \left( 12n^3 - 84n + 72 \right) (1 - \zeta_2) - \beta_0 C_F \left[ 3(9 - 8\zeta_2)n^3 - 32n^2 - (293 - 168\zeta_2)n + 2(149 - 72\zeta_2) \right] + 6C_F^2 \left[ (17 + 48\zeta_2 - 64\zeta_3)n^2 - (463 + 112\zeta_2 - 544\zeta_3)n + 510 + 32\zeta_2 - 480\zeta_3 \right] - 8C_F C_A \left[ (4 + 45\zeta_2 - 48\zeta_3)n^2 - (134 + 111\zeta_2 - 276\zeta_3)n + 130 + 66\zeta_2 - 228\zeta_3 \right] \right\} , \] (4.7)\\n\[ C_{L,q}^{(n,1)} = - \frac{4^{n-2}}{12(n-1)!} C_F^{n-2} \left\{ \beta_0^2 \left( \frac{3}{8} n^5 + \frac{41}{24} n^4 - \frac{203}{24} n^3 - \frac{41}{24} n^2 + \frac{325}{12} n - 19 \right) + C_A^2 \left( 96n^3 - 384n^2 + 480n - 192 \right) (\zeta_2 - \zeta_3) + \beta_0 C_A \left( 12n^4 + 12n^3 - 228n^2 + 420n - 216 \right) (1 - \zeta_2) - \beta_0 C_F \left[ 3(9 - 8\zeta_2)n^4 - 2(7 + 12\zeta_2)n^3 - 19(29 - 24\zeta_2)n^2 + 2(649 - 420\zeta_2)n \right] - 8(95 - 54\zeta_2) + 6C_F^2 \left[ (17 + 48\zeta_2 - 64\zeta_3)n^3 - (534 + 112\zeta_2 - 608\zeta_3)n^2 + (1227 + 80\zeta_2 - 1120\zeta_3)n - 710 - 16\zeta_2 + 576\zeta_3 \right] - 8C_F C_A \left[ (4 + 45\zeta_2 - 48\zeta_3)n^3 - (152 + 144\zeta_2 - 324\zeta_3)n^2 + (308 + 165\zeta_2 - 540\zeta_3)n - 160 - 66\zeta_2 + 264\zeta_3 \right] \right\} \] (4.8)
The corresponding (and more complicated) coefficients for the gluon case are

\[
C_{L,q}^{(n,2)} = \frac{4^{n-2}}{12(n-1)!} C_F^{n-2} \left\{ \beta_0^2 \left( \frac{3}{16} n^6 + \frac{59}{48} n^5 - \frac{151}{16} n^4 + \frac{281}{48} n^3 + \frac{183}{4} n^2 - \frac{1039}{12} n + 43 \right) + C_A^2 (48n^4 - 288n^3 + 624n^2 - 576n + 192) (\zeta_2 - \zeta_3) + \beta_0 C_A \left( 6n^5 + 6n^4 - 222n^3 + 714n^2 - 864n + 360 \right) (1 - \zeta_2) \right. \\
- \beta_0 C_F \left[ \frac{3}{2} (9 - 8\zeta_2)n^5 - \frac{1}{2} (23 + 24\zeta_2)n^4 - \frac{3}{2} (313 - 296\zeta_2)n^3 + \frac{3}{2} (1289 - 952\zeta_2)n^2 \\
- 24(115 - 72\zeta_2)n + 2(647 - 360\zeta_2) \right] \\
+ 3 C_F^2 \left[ (17 + 48\zeta_2 - 64\zeta_3)n^4 - 2(311 + 80\zeta_2 - 368\zeta_3)n^3 + (2675 + 144\zeta_2 - 2432\zeta_3)n^2 \\
- 2(1963 + 16\zeta_2 - 1552\zeta_3)n + 64(29 - 21\zeta_3) \right] \\
- 4 C_F C_A \left[ (4 + 45\zeta_2 - 48\zeta_3)n^4 - 6(29 + 37\zeta_2 - 70\zeta_3)n^3 + 3(228 + 139\zeta_2 - 408\zeta_3)n^2 \\
- 6(149 + 62\zeta_2 - 242\zeta_3)n + 4(95 + 33\zeta_2 - 150\zeta_3) \right] \right\} . \tag{4.9}
\]

The corresponding (and more complicated) coefficients for the gluon case are

\[
C_{L,g}^{(n,0)} = \frac{4^{n-2}}{(n-1)!} n_f \left\{ C_A^{n-1} \left( \frac{4}{3} (77 - 15\zeta_2)n^2 - 4(5 + \zeta_2)n + \frac{8}{3} (17 - 3\zeta_2) \right) \\
- 8 C_F \delta n_2 - 2(4 C_F \beta_0 - 31 C_F^2 + 16 C_F^2 \zeta_2) \delta n_3 + C_A^{n-2} \beta_0 \left( 4n^3 - \frac{26}{3} n^2 - 2n + \frac{20}{3} \right) \theta n_2 \\
- C_A^{n-2} C_F \left[ 16(5 - \zeta_2)n^2 - 8(29 - 5\zeta_2)n + 8(20 - 3\zeta_2) \right] \theta n_3 \\
- C_A^{n-3} d^{abc} d_{abc} f l^g_{11} (32n^2 - 96n + 64) (11 + 2\zeta_2 - 12\zeta_3) \theta n_3 \\
+ C_A^{n-3} \beta_0 \left( \frac{1}{16} n^4 - \frac{29}{72} n^3 + \frac{41}{48} n^2 - \frac{49}{72} n + \frac{1}{6} \right) \theta n_3 \\
+ C_F^{n-1} [2(9 - 8\zeta_2)n - 16(2 - \zeta_2)] \theta n_4 - C_A^{n-2} \beta_0 \left( \frac{1}{2} n^2 + \frac{1}{2} n - 2 \right) \theta n_4 \\
+ C_A^{n-3} C_F^2 \left[ 20n^2 - 8(7 + \zeta_2)n + 2(17 + 4\zeta_2) \right] \theta n_4 \\
- C_A^{n-3} C_F \beta_0 \left( n^3 - \frac{7}{2} n^2 + \frac{9}{2} n - 1 \right) \theta n_4 - \sum_{\ell=2}^{n-3} C_F C_A^{n-\ell-2} \beta_0 \left( \frac{1}{2} n^2 - \frac{1}{2} n + \ell \right) \theta n_4 \\
+ \sum_{\ell=3}^{n-2} C_F C_A^{n-\ell-1} [4(1 - 2\zeta_2)n + 6\ell - 2(9 - 4\zeta_2)] \theta n_4 \right\} . \tag{4.10}
\]

\[
C_{L,g}^{(n,1)} = - \frac{4^{n-2}}{(n-1)!} n_f \left\{ 16 C_F \zeta_2 \delta n_2 - 8(3 C_F \beta_0 - 12 C_F^2 + 4 C_F^2 \zeta_2) \delta n_3 \\
+ C_A^{n-1} \left( \frac{4}{3} (77 - 15\zeta_2)n^3 - 8(16 - 3\zeta_2)n^2 + \frac{4}{3} (61 - 21\zeta_2)n - 8(7 - 3\zeta_2) \right) \\
+ C_A^{n-2} \beta_0 \left( 4n^4 - \frac{14}{3} n^3 - 24n^2 + \frac{182}{3} n - 36 \right) \theta n_2 \\
- C_A^{n-2} C_F \left[ 16(5 - \zeta_2)n^3 - 8(39 - 7\zeta_2)n^2 + 24(16 - 3\zeta_2)n - 32(5 - \zeta_2) \right] \theta n_3
\]

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\[ -C_A^{n-3} \frac{d^{abc} d_{abc}}{n_a} f l_1^g (32n^3 - 128n^2 + 160n - 64)(11 + 2\zeta_2 - 12\zeta_3)\theta_{n3} \]

\[ + C_A^{n-3} \beta_0^2 \left( \frac{1}{16} n^5 - \frac{31}{144} n^4 - \frac{59}{144} n^3 + \frac{319}{144} n^2 - \frac{179}{72} n + \frac{5}{6} \right) \theta_{n3} \]

\[ + C_F^{n-1} \left[ 2(9 - 8\zeta_2)n^2 - 2(37 - 24\zeta_2)n + 4(19 - 8\zeta_2) \right] \theta_{n4} \]

\[ - C_A^{n-3} C_F \beta_0 \left( n^4 - \frac{5}{2} n^3 + \frac{1}{2} n^2 + 4n - 6 \right) \theta_{n4} \]

\[ + C_A^{n-3} C_F^2 \left[ 20n^3 - 4(19 + 2\zeta_2)n^2 + 2(43 + 12\zeta_2)n - 2(17 + 8\zeta_2) \right] \theta_{n4} \]

\[ - C_F^{n-2} \beta_0 \left( \frac{1}{2} n^3 + \frac{1}{2} n^2 - 6n + 8 \right) \theta_{n4} - \sum_{\ell=2}^{n-3} C_F^\ell C_A^{n-\ell-2} \beta_0 \left( \frac{1}{2} n^3 - \frac{1}{2} n^2 + \ell n - 4\ell \right) \theta_{n4} \]

\[ + \sum_{\ell=3}^{n-2} C_F^\ell C_A^{n-\ell-1} \left[ 4(1 - 2\zeta_2)n^2 + (6\ell - 2(13 - 12\zeta_2))n - 18\ell + 2(21 - 8\zeta_2) \right] \theta_{n4} \]  (4.11)

and finally

\[ C_{L,g}^{(n,2)} = \frac{4^{n-2}}{(n - 1)!} n_f \left\{ C_A^{n-2} \beta_0 \left( 2n^5 - \frac{7}{3} n^4 - \frac{92}{3} n^3 + 111n^2 - \frac{436}{3} n + \frac{196}{3} \right) \theta_{n2} \right. \]

\[ - 4(3C_F\beta_0 - 10C_F^2) d_{n3} + C_A^{n-1} \left( \frac{2}{3} (77 - 15\zeta_2)n^4 - \frac{4}{3} (127 - 27\zeta_2)n^3 \right. \]

\[ + 2(91 - 29\zeta_2)n^2 - \frac{8}{3} (49 - 27\zeta_2)n + \frac{40}{3} (5 - 3\zeta_2) \right) \]

\[ - C_A^{n-2} C_F \left[ 8(5 - \zeta_2)n^4 - 4(59 - 11\zeta_2)n^3 + 4(125 - 24\zeta_2)n^2 \right. \]

\[ - 4(114 - 25\zeta_2)n + 40(4 - \zeta_2) \right) \theta_{n3} \]

\[ - C_A^{n-3} \frac{d^{abc} d_{abc}}{n_a} f l_1^g (16n^4 - 96n^3 + 208n^2 - 192n + 64)(11 + 2\zeta_2 - 12\zeta_3)\theta_{n3} \]

\[ + C_A^{n-3} \beta_0^2 \left( \frac{1}{32} n^6 - \frac{13}{288} n^5 - \frac{79}{96} n^4 + \frac{1217}{288} n^3 - \frac{65}{8} n^2 + \frac{473}{72} n - \frac{11}{6} \right) \theta_{n3} \]

\[ + C_F^{n-1} \left[ (9 - 8\zeta_2)n^3 - (67 - 48\zeta_2)n^2 + 4(41 - 22\zeta_2)n - 12(11 - 4\zeta_2) \right] \theta_{n4} \]

\[ + C_A^{n-3} C_F^2 \left[ 10n^4 - 2(29 + 2\zeta_2)n^3 + 3(39 + 8\zeta_2)n^2 - 11(9 + 4\zeta_2)n + 8(5 + 3\zeta_2) \right] \theta_{n4} \]

\[ - C_A^{n-3} C_F \beta_0 \left( \frac{1}{4} n^5 - \frac{5}{4} n^4 - 3n^3 + \frac{51}{4} n^2 - 19n + 15 \right) \theta_{n4} \]

\[ - C_F^{n-2} \beta_0 \left( \frac{1}{4} n^4 - \frac{27}{4} n^3 + \frac{39}{2} n^2 - 18 \right) \theta_{n4} \]

\[ - \sum_{\ell=2}^{n-3} C_F^\ell C_A^{n-\ell-2} \beta_0 \left( \frac{1}{4} n^4 - \frac{1}{2} n^3 + \frac{1}{4} (2\ell - 5)n^2 - \frac{3}{2} (3\ell - 1)n + 9\ell \right) \theta_{n4} \]

\[ + \sum_{\ell=3}^{n-2} C_F^\ell C_A^{n-\ell-1} \left[ 2(1 - 2\zeta_2)n^3 + (3\ell - 19 + 24\zeta_2)n^2 - (21\ell - 63 + 44\zeta_2)n \right. \]

\[ + 36\ell - 24(3 - \zeta_2) \right] \theta_{n4} \} \right. \]  (4.12)
5 NNLL resummation of the off-diagonal splitting functions

Together with the mass-factorization relations of Section 2, the results of the previous two sections facilitate the iterative determination of the respective coefficients \( D_{ik}^{(n,\ell \leq 2)} \) and \( D_{a,k}^{(n,\ell \leq 2)} \) in Eqs. (1.3), Eqs. (1.5) and Eqs. (1.6) to any order \( \alpha_s^n \). In this section we present the resulting expressions for the splitting functions in Mellin-\( N \) space, from which the \( x \)-space coefficients \( D_{qg}^{(n,\ell \leq 2)} \) and \( D_{gq}^{(n,\ell \leq 2)} \) can readily be obtained by inverting the second relation in (2.2). The corresponding results for the coefficient functions are discussed in the next section.

The LL and NLL contributions to \( P_{qg} \) and \( P_{gq} \) can be expressed in a closed form in terms of a new class of functions with Taylor expansions in terms of Bernoulli numbers. Extending the definitions of Ref. [15] to \( k = 2, 3, \ldots \), these functions are given by

\[
\bar{b}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!} x^n, \quad \bar{b}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!} x^n. \quad (5.1)
\]

\( B_n \) are the Bernoulli numbers in the standard normalization of Ref. [52]: \( B_{2n+1} = 0 \) for \( n \geq 1 \) and

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad \ldots, \quad B_{12} = -\frac{691}{2730}, \quad \ldots. \quad (5.2)
\]

The functions \( \bar{b}_{\pm k} \) for \( k > 0 \) are related to \( \bar{b}_0 \), which can also be written as

\[
\bar{b}_0(x) = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} B_{2n} x^{2n} = 1 - \frac{x}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \zeta_{2n} \left( \frac{x}{2\pi} \right)^{2n}, \quad (5.3)
\]

by

\[
\frac{d^k}{dx^k} (x^k \bar{b}_k) = \bar{b}_0, \quad \frac{d^k}{dx^k} \bar{b}_0 = \frac{1}{x^k} \bar{b}_{-k}. \quad (5.4)
\]

Due to \( \zeta_{2n} \to 1 \) for \( n \to \infty \) the series (5.1) converge absolutely for all values on \( x \).

The numerical behaviour of the functions \( \bar{b}_k(x) \) with \( k = 0, \pm 1, -2 \) which enter our results below is illustrated in Fig. 1. Similar to \( \bar{b}_0(x) \) – where this oscillation has been shown to continue, albeit in a much more irregular fashion, to much larger values of \( x \) [53] – \( \bar{b}_{k \geq 0}(x) \) oscillates around \( y = 0 \) for positive \( x \) and \( y = -x/(k+1)! \) for negative \( x \). On the other hand, \( \bar{b}_{k < 0}(x) \) also oscillates around \( y = 0 \) for positive \( x \) but around \( y = -x \) for negative \( x \). As can be seen from the figures, the amplitude of these oscillations increases very rapidly with decreasing \( k \).

The resummed gluon-quark splitting function at large \( N \) can now be written as

\[
NP_{ag}(N, \alpha_s) = 2\alpha_s n_f \bar{b}_0(\bar{a}_s) + \alpha_s^2 \ln \bar{N} n_f \left[ (12C_F - 2\beta_0) \frac{1}{\bar{a}_s} \bar{b}_{-1}(\bar{a}_s) + \beta_0 \frac{1}{\bar{a}_s} \bar{b}_{-2}(\bar{a}_s) + (6C_F - \beta_0) \frac{1}{\bar{a}_s} \bar{b}_1(\bar{a}_s) \right] + \text{NNLL contributions} + \ldots. \quad (5.5)
\]
Figure 1: The functions $B_k(x)$, $k = 0, \pm 1, 2$, evaluated using their defining Taylor expansions (5.1).
Here and below we use (recall $\ln \tilde{N} = \ln N + \gamma_e$)

$$\tilde{a}_s \equiv 4a_s(C_A - C_F) \ln^2 \tilde{N}. \tag{5.6}$$

The corresponding result for the quark-gluon splitting reads

$$NP_{gq}(N, \alpha_s) = 2a_s C_F B_0(-\tilde{a}_s)$$
$$+ a_s^2 \ln \tilde{N} C_F \left[ (12C_F - 6\beta_0) \frac{1}{\tilde{a}_s} B_{-1}(-\tilde{a}_s) - \frac{\beta_0}{\tilde{a}_s} B_{-2}(-\tilde{a}_s) \right.$$
$$+ (14C_F - 8C_A - \beta_0) B_1(-\tilde{a}_s) \left. \right]$$
$$+ \text{NNLL contributions} + \ldots. \tag{5.7}$$

The first lines of Eqs. (5.5) and (5.7) are the respective LL results, derived in Ref. [15] from relations equivalent to Eq. (3.12). Unlike for the NLL corrections, we have not been able to find closed relations for all colour factors contributing to the NNLL terms. We therefore provide these results in the form of tables to order $\alpha_s^{18}$ which can be found in Appendix A.

Our results for the fourth-order ($N^3\text{LO}$) splitting functions agree with the predictions of Ref. [17] derived from the conjectured single-logarithmic large-$x$ enhancement of the physical evolution kernels for the system $(F_2, F_\phi)$ of flavour-singlet structure functions. Furthermore they show the expected extension of the colour-factor pattern seen in those results to all orders in $\alpha_s$: the LL terms for both $P_{qg}^{(n)}$ and $P_{gq}^{(n)}$ [recall Eq. (1.1)] are proportional to $(C_F - C_A)^n$, the NLL terms include at most two colour factors other than $(C_F - C_A)$, and the NNLL terms of Appendix A involve $(C_F - C_A)^{n-2}$ or higher powers of $(C_F - C_A)$. It appears that generally all double-logarithmic contributions, $\ln^k \tilde{N}$ with $n+1 \leq k \leq 2n$, vanish for $C_F = C_A$ which is part of the colour-factor choice leading to an $\mathcal{N} = 1$ supersymmetric theory.

It is clear already from the discussion of the fourth-order results in Ref. [17] (Figs. 12 and 13) that the highest three logarithms of $P_{qg}^{(n>2)}$ and $P_{gq}^{(n>2)}$ can provide no more than a very rough indication of the size of the corrections beyond NNLO (note that the expansion in Ref. [17] is in terms of $\ln \tilde{N}$ instead of $\ln N$). Nevertheless it is useful to take a brief look at the numerical size and convergence of the resummation corrections to these quantities. This is done in Figs. 2 and 3 at the reference point

$$\alpha_s(Q^2) = 0.2, \quad n_f = 4 \tag{5.8}$$

used before, e.g., in Refs. [8–12]. Depending on the precise value of $\alpha_s$ at the $Z$-boson mass, this choice corresponds to a scale $Q^2 \approx 25 \ldots 50$ GeV$^2$ typical for measurements of DIS [1].

The figures show that the resummation corrections to the NNLO splitting functions are dominated by the third (NNL) logarithms which completely overwhelm the LL and NLL contributions (except, of course, for huge but practically irrelevant values of $\tilde{N}$). The relative corrections to the already small large-$N$ off-diagonal splitting functions are rather small, amounting to less than 2% and about 3% at $\tilde{N} = 20$ for for $P_{qg}$ and $P_{gq}$ respectively. At least at the present level of logarithmic accuracy contributions beyond order $\alpha_s^4$ ($N^3\text{LO}$) are negligible.
\[
\alpha_s = 0.2, \ n_f = 4
\]

Figure 2: The Mellin-\(N\) splitting function \(P_{qg}\) and \(P_{gq}\), multiplied by \(N\) for display purposes. Shown are the LL, NLL and NNLL large-\(N\) resummation corrections to the complete NNLO results [9].

Figure 3: The contributions of the various orders in \(\alpha_s\) to the resummation corrections at \(N = 20\). The LL, NLL and NNLL terms of order \(\alpha_s^n\) are added at the corresponding values of the abscissa.
6 NNLL resummation of the coefficient functions

The resummed results for the coefficient functions \(C_{2,g}\) and \(C_{\phi,q}\) are more complicated than those for the splitting functions already at the leading-log level. Hence it is not surprising that we are not able to give closed NNLL expressions for these quantities either at this point. Postponing the NNLL contributions to Appendix B, the results are

\[
NC_{2,g}(N, \alpha_s) = \frac{1}{2\ln N} \frac{n_f}{C_A - C_F} \left[ \exp(2a_sC_F \ln^2 \tilde{N}) \beta_0(\bar{\alpha}_s) - \exp(2a_sC_A \ln^2 \tilde{N}) \right] \\
- \frac{1}{8\ln^2 N} \frac{n_f(3C_F - \beta_0)}{(C_A - C_F)^2} \left[ \exp(2a_sC_F \ln^2 \tilde{N}) \beta_0(\bar{\alpha}_s) - \exp(2a_sC_A \ln^2 \tilde{N}) \right] \\
- \frac{a_s}{4} \frac{n_f}{C_A - C_F} \exp(2a_sC_A \ln^2 \tilde{N}) (8C_A + 4C_F - \beta_0) \\
- \frac{a_s}{4} \frac{n_f}{C_A - C_F} \exp(2a_sC_F \ln^2 \tilde{N}) \left[ -6C_F \beta_0(\bar{\alpha}_s) - (6C_F - \beta_0) \beta_1(\bar{\alpha}_s) \\
- (12C_F - 4\beta_0) \frac{1}{\bar{\alpha}_s} \beta_{-1}(\bar{\alpha}_s) - \frac{\beta_0}{\bar{\alpha}_s} \beta_{-2}(\bar{\alpha}_s) \right] \\
- \frac{a_s^2}{3} \beta_0 \ln^2 \tilde{N} \frac{n_f}{C_A - C_F} \left[ C_A \exp(2a_sC_A \ln^2 \tilde{N}) - C_F \exp(2a_sC_F \ln^2 \tilde{N}) \beta_0(\bar{\alpha}_s) \right] \\
+ \text{NNLL contributions} + \ldots \tag{6.1}
\]

and

\[
NC_{\phi,q}(N, \alpha_s) = \frac{1}{2\ln N} \frac{C_F}{C_F - C_A} \left[ \exp(2a_sC_A \ln^2 \tilde{N}) \beta_0(-\bar{\alpha}_s) - \exp(2a_sC_F \ln^2 \tilde{N}) \right] \\
+ \frac{1}{8\ln^2 N} \frac{C_F(3C_F - \beta_0)}{(C_F - C_A)^2} \left[ \exp(2a_sC_A \ln^2 \tilde{N}) \beta_0(-\bar{\alpha}_s) - \exp(2a_sC_F \ln^2 \tilde{N}) \right] \\
+ \frac{a_s}{4} \frac{C_F}{C_F - C_A} \exp(2a_sC_F \ln^2 \tilde{N}) (12C_A - 18C_F - \beta_0) \\
+ \frac{a_s}{4} \frac{C_F}{C_F - C_A} \exp(2a_sC_A \ln^2 \tilde{N}) \left[ 2\beta_0 \beta_0(-\bar{\alpha}_s) - (\beta_0 - 6C_F + 8C_{AF}) \beta_1(-\bar{\alpha}_s) \\
- (4\beta_0 - 12C_F) \frac{1}{\bar{\alpha}_s} B_{-1}(-\bar{\alpha}_s) - \frac{\beta_0}{\bar{\alpha}_s} B_{-2}(-\bar{\alpha}_s) \right] \\
+ \frac{a_s^2}{3} \beta_0 \ln^2 \tilde{N} \frac{C_F}{C_F - C_A} \left[ C_A \exp(2a_sC_A \ln^2 \tilde{N}) \beta_0(-\bar{\alpha}_s) - C_F \exp(2a_sC_F \ln^2 \tilde{N}) \right] \\
+ \text{NNLL contributions} + \ldots \tag{6.2}
\]

Here the first lines are the respective LL results of Ref. [13]. As in these terms, the \((C_A - C_F)\) denominators are generally cancelled by corresponding numerator factors as can be seen by expanding all functions in powers of \(\alpha_s\). Unlike for the splitting functions the double-logarithmic contributions to the coefficient functions do not vanish for \(C_F = C_A\). However, they can be expressed in terms of exponentials in this case since all \(B\)-functions have the argument \((5.6)\).
Our results for the quark coefficient function for $F_L$ completely agree with those of Ref. [29]. The complexity of the N$^2$LL resummed expression for the gluon coefficient function $C_{L,g}$ is similar to that of $C_{2,g}$ at $N^{n-1}$LL level. Hence it can be written down in a closed form at NNLL accuracy,

$$N^2 C_{L,g}(N, \alpha_s) = 8a_s n_f \exp(2a_s C_A \ln^2 \tilde{N}) + 4a_s C_F N C_{2,g}^{LL}(N, \alpha_s)$$

$$+ 16a_s^2 \ln \tilde{N} n_f \exp(2a_s C_A \ln^2 \tilde{N}) \left[(4C_A - C_F) + \frac{1}{3} a_s \ln^2 \tilde{N} C_A \beta_0 \right]$$

$$+ 4a_s^2 \ln \tilde{N} C_F \left[\beta_0 + 4C_A(1 - \zeta_2) - 4C_F(3 - 2\zeta_2)\right] N C_{2,g}^{LL}(N, \alpha_s)$$

$$+ 4a_s C_F N C_{2,g}^{NLL}(N, \alpha_s) + 8a_s^2 n_f \exp(2a_s C_A \ln^2 \tilde{N}) \left[C_F(1 - 2\zeta_2) + \frac{1}{3} a_s \ln^2 \tilde{N} \left(\beta_0(22C_A - 3C_F) + 2C_A^2(79 - 18\zeta_2) + 30C_F^2 \right) 

- 24C_A C_F(5 - \zeta_2) - 48 \frac{d^{abc} d_{abc}}{n_a} f^{a} \left(11 + 2\zeta_2 - 12\zeta_3\right) \right] + \ldots \quad (6.3)$$

Here the first term is the LL result also obtained already in Ref. [29]. The next term, where $C_{2,g}^{LL}$ stands for the first line of Eq. (6.1), and the second line form the NLL contribution. The remainder of Eq. (6.3) collects the NNL logarithms, where $C_{2,g}^{NLL}$ represents the second to sixth line of Eq. (6.1). Furthermore this is the only NNLL contribution including the charge factor

$$f^{a} = \langle e \rangle^2 / \langle e^2 \rangle \quad \text{with} \quad \langle e^k \rangle = n_f^{-1} \sum_{i=1}^{n_f} e_i^k$$

(6.4)

(where $e_i$ is the charge of the $i$-th effectively massless flavour in units of the proton charge) arising from diagrams with the colour factor $d^{abc} d_{abc} / n_a = 5/48 n_f^2$ in QCD where the two (neutral) gauge bosons couple to different quark loops of the gauge-bosons gluon forward amplitude [6][10][11].

The NLL and NNLL contributions to Eq. (6.3) for $C_F = 0$ and $f^{a} = 0$ agree with the previous all-order result [8] in this gluonic ‘non-singlet’ limit to which $C_{2,g}^{(N)LL}$ and hence the special functions (5.1) do not contribute. The complete fourth-order coefficient function (for $W$-exchange, i.e., without the $f^{a} = 0$ part) has been predicted in Ref. [54] from the physical evolution kernel for the system $(F_2, F_L)$ of flavour-singlet structure functions together with the four-loop splitting-function results of Ref. [17]. The present results are in full agreement also with that prediction.

Eqs. (5.5) and (5.7) for the resummed splitting functions together with their counterparts (6.1) – (6.3) for the coefficient functions and the corresponding simpler results for $C_{2,q}$, $C_{\phi,q}$ and $C_{L,q}$ facilitate an assumption-free NNLL calculation of the physical kernels for $(F_2, F_\phi)$ and $(F_2, F_L)$ to any order in $\alpha_s$. It turns out that their highest double logarithms, as far as they can be determined from available fixed-order results now, do indeed vanish. Hence the conjectures of Refs. [17][28][29][54] are proven by our present calculations for the leading $N^{-1}$ large-$N$ contributions.

The numerical size and $\alpha_s$-convergence of the LL, NLL and NNLL resummation corrections to the respective third-order results are illustrated in Fig. 4 for $C_{2,g}$ and Figs. 5 and 6 for the coefficient functions of $F_L$. For brevity we do not show the corresponding results for $C_{\phi,q}$ which is of theoretical but not phenomenological interest. The corrections are dominated by the NNLL
Contributions, suggesting that they underestimate the impact of at least very high orders. Already the known terms, e.g., at $N = 20$ for the reference point (5.8), are sizeable for $C_{L,q}$ with about 15%, large for $C_{2,g}$ (about 35%) and huge for $C_{L,g}$ with about 100%. The gluonic quantities receive significant contributions from the fourth to sixth order in $\alpha_s$. With values of 10.2 and 49.5 the forth-order coefficients are comparable to the corresponding (averaged) Padé estimates.

\begin{align}
- C_{2,g}(N = 20) &= 0.127 \alpha_s + 0.642 \alpha_s^2 + 2.76 \alpha_s^3 + 12 \text{Padé } \alpha_s^4 + \ldots, \\
NC_{L,g}(N = 20) &= 0.110 \alpha_s + 1.240 \alpha_s^2 + 9.51 \alpha_s^3 + 65 \text{Padé } \alpha_s^4 + \ldots.
\end{align}

It is obvious from Eqs. (3.18) – (3.21) that our method can be applied as well to non-leading terms in the expansion in powers of $(1-x)^0$ or $1/N$ contributions to diagonal coefficient function such as $C_{2,q}$ and $C_{\phi,g}$. Due to the stable form (1.2) of the diagonal splitting functions, the corresponding $N^3$LO corrections to the non-singlet structure functions are known except for the single- and non-logarithmic contributions. Consequently, as discussed below Eq. (3.21), we are able to resum highest four $N^{-1}$ logarithms.

The resummation of the $N^{-1}$ non-singlet coefficient functions for $F_{1,2,3}$ has been inferred in Ref. [28] from the behaviour of the physical evolution kernels, obtaining complete NNLL results and the $N^3$LL corrections up to one undetermined number called $\xi_{\text{DIS}_4}$. Hence it was not necessary to perform a very cumbersome all-order $N^3$LL calculation analogous to Section 3. Instead we have verified those previous results to order $\alpha_s^7$ and determined the hitherto missing parameter,

\begin{equation}
\xi_{\text{DIS}_4} = 100/3 \quad \text{in Eqs. (5.24) – (5.26) of Ref. [28].}
\end{equation}
\[ \alpha_s = 0.2, \ n_f = 4 \]

Figure 5: The Mellin-N space quark (left) and gluon (right, multiplied by \( N \)) coefficient functions for \( F_L \) at the reference point (5.8). Shown are the cumulative LL, NLL and NNLL large-\( N \) resummation corrections to the third-order (NNLO) results.

\[ \alpha_s = 0.2, \ n_f = 4 \]

Figure 6: As Fig. 3, but for the coefficient functions \( C_{L,q} \) and \( C_{L,g} \) shown in Fig. 5 above.
7 Summary and outlook

Unlike the case of the dominant \((1-x)^{-1}/N^0\) large-x/large-N terms, only a small amount of (published) research has been devoted until recently to the all-order structure of \(1/N\)-suppressed threshold contributions to the coefficient functions for the structure functions in deep-inelastic scattering (DIS) and crossing-related (semi-)inclusive quantities in perturbative QCD. See Ref. [55] for a well-known leading-log conjecture, and Refs. [56, 57] for early studies of the longitudinal structure function \(F_L\). The study of these contributions is however not only theoretically interesting but also phenomenologically relevant, e.g., for assessing the kinematic region, different for different processes, see Ref. [42], in which the \(N^0\) terms and their soft-gluon exponentiation can be used as a quantitative substitute for the full coefficient functions.

Consequently several groups have addressed this issue with various approaches in the past few years, for work by others see Refs. [58–61] and [62–64]. However, to the best of our knowledge, this research has not yet led to explicit all-order predictions for complete and exact coefficients of the next-to-leading and next-to-next-to-leading logarithms, or even the leading logarithms to quantities such as the gluon contribution to most important structure function \(F_2\).

In Refs. [28, 29] a resummation of the highest three \(1/N\)-suppressed logarithms (actually the \(1/N\) behaviour is more formal than real for practically relevant values of \(N\)) for various quark coefficient functions has been obtained by studying the large-\(N\) behaviour of non-singlet physical evolution kernels. These quantities express the scaling violations of a physical quantity in terms of the same physical quantity and hence do not depend on the scheme for the factorization of the mass singularities. It turns out that the non-singlet physical kernels show an only single-logarithmic large-\(N\) enhancement also beyond the dominant \(N^0\) contributions (where this feature is a simple consequence of the soft-gluon exponentiation, see Ref. [65]) up to at least the next-to-next-to-leading or next-to-next-to-next-to-leading orders (NNLO or \(N^3\)LO). The results of Refs. [28, 29] are based on the conjecture that this behaviour continues to all orders in the strong coupling \(\alpha_s\).

Completely analogous observations were made [17, 54] for the physical-kernel matrices of the systems \((F_2, F_\phi)\) and \((F_2, F_L)\) of flavour-singlet structure functions [66–68]. Unlike in the non-singlet case, recall Eq. (12), the singlet kernels receive double-logarithmic contributions from both the splitting functions (13) and the coefficient functions in Eqs. (1.5) and (1.6). Consequently the single-logarithmic enhancement of the physical kernels can only provide one all-order constraint between two quantities. Definite fourth-order predictions of the highest three logarithms are possible though: first for the \(N^3\)LO splitting functions [17] using the diagram calculations of the corresponding three-loop coefficient functions [10, 11, 17], and then, using these predictions, for the \(N^3\)LO (fourth-order) coefficient functions for \(F_L\) [54].

While we expect the results of Refs. [17, 28, 29, 54] to be the final word on the predicted coefficients, clearly more work is required to put them on a firmer theoretical footing and to extend them to all orders also for the off-diagonal splitting functions and flavour-singlet coefficient functions. At the leading-logarithmic level this was done in Ref. [15] by finding the now expected all-order iterative structure of the unfactorized partonic structure functions (forward-Compton amplitudes).
In the present paper we have extended these results to the next-to-leading and next-to-next-to-leading (NLL and NNLL) logarithms. Note that this is the counting of a resummation, not that of a more powerful exponentiation [18–22]: the present predictive power in terms of higher-order coefficients corresponds to that of a next-to-leading logarithmic exponentiation, see, e.g., Ref. [25].

In order to achieve this, we have employed two independent methods. The first is a direct generalization of the amplitude iteration of Ref. [15] to higher logarithmic accuracy, presented at NLL accuracy in Eqs. (3.14) – (3.16) above. The second, worked out to NNLL accuracy in the rest of Section 3 and Section 4, is conceptually extremely simple and revealing but not iterative, since all coefficients are determined order-by-order in $\alpha_s$ without any explicit reference to complete lower-order amplitudes. This method is based on only the form of the $D$-dimensional phase space for inclusive DIS and the all-order mass-factorization formula (guaranteed by the operator-product expansion) for the unfactorized structure functions. In this way we have been able to verify all DIS predictions in Refs. [17, 28, 29, 54], extend them to the fourth logarithms for $F_{2, ns}$ and to all orders in $\alpha_s$ for the highest three logarithms of the off-diagonal splitting functions and the coefficient functions $C_{2,g}, C_{L,g}$ and $C_{\phi,q}$. Our results prove the non-singlet conjecture of Refs. [28, 29] and show that also the above singlet physical evolution kernels are single-log enhanced to all orders.

The resulting all-order off-diagonal splitting functions and the above coefficient functions can be written down in a closed all-order form to NLL (NNLL for $C_{L,g}$) accuracy in terms of the apparently new special functions introduced in Ref. [15]. We did not find (so far) similar relations for all colour factors of $P_{qg}$ and $P_{gq}$ and for $C_{2,g}$ and $C_{\phi,q}$ at the NNLL level, but can analytically determine their coefficients to ‘any’ order in $\alpha_s$. The splitting-function results show a particular colour structure with all double logarithmic contributions, $\alpha_s^n \ln^{\ell} (1-x)$ with $\ell \geq n$, vanishing in the supersymmetric limit $C_F = C_A$. This pattern does not hold for the more complicated coefficient functions which however can be expressed in terms of exponentials in the limit.

Numerically the resummation corrections to the $\alpha_s^3$ splitting functions appear to be rather small and quickly converging: at the present level of accuracy only the fourth-order terms have an impact. The corresponding corrections for $C_{L,q}$ are larger and receive relevant fourth- and fifth-order contributions, but they are still small compared to those up to order $\alpha_s^6$ for $C_{2,g}$ and especially $C_{L,g}$. Taking into account that the highest three logarithms most likely underestimate the corrections at high orders of $\alpha_s$, these results reinforce the third-order findings of Ref. [11] which indicated that the perturbative expansion of $C_{L,g}$ is not well-behaved even at moderately large $N$ in the important region of scales $Q^2 \approx 25 \ldots 50$ GeV$^2$ corresponding to $\alpha_s(Q^2) \approx 0.2$.

Beyond first estimates of the numerical impact of higher orders at large $N$, our results will prove very useful once the fixed-moment calculations of Refs. [5–7] have been extended to the fourth order in $\alpha_s$. First results have already been presented on four-loop second moments and sum rules [26, 69], hence this extension can be expected in the foreseeable future. The main way to use a couple of moments of, e.g., the N$^3$LO splitting functions will be effective $x$-space parametrizations analogous to those of Refs. [70, 71] at NNLO. Already knowing the coefficient of $\ln^{\ell} (1-x)$ for $\ell = 4, 5, 6$ will then considerably assist achieving a decent accuracy from a limited number of moments. The situation is analogous for the coefficient functions.
There appears to be no reason why our determination of the highest double logarithms cannot be extended beyond the $N^{-1}$ ($N^{-2}$ for $C_{L,g}$) terms. An all-order calculation would obviously be very cumbersome given the form of some of the intermediate expression in the present limit, but a fourth-order calculation would definitely be feasible. We would expect to recover the results of Refs. [17, 28] for the coefficient of the highest three logarithms at all-powers of $(1-x)$ of $c_{a,q}^{(4)}$ with $a = 2, 3, L, P_{qg}^{(3)}$ and $P_{gq}^{(3)}$ in this manner, and to derive corresponding new results for $c_{a,g}^{(4)}$ ($a = 2, L$) and $c_{g,q}^{(4)}$. The main application of such results would be as a check of an all-$x$ fourth-order calculation of the splitting functions and DIS coefficient functions which, however, we do not expect for the near future.

Due to the similar phase-space integrations [72, 73], the present approach can be carried over directly to the case of semi-inclusive $e^+e^-$ annihilation [1] (but not, unfortunately, to the Drell-Yan process and inclusive Higgs production in pp collisions). Only the diagonal NNLO ‘timelike’ splitting functions for parton fragmentation have been determined up to now [74, 75], hence a NNLL resummation is not yet possible in this case as the third logarithms of the NNLO splitting functions are a necessary input. Therefore this case will be addressed in a future publication after the determination of the off-diagonal timelike splitting functions [76].

Let us finally stress again that the present double-logarithmic resummation is not relying on any specific large-$N$ structure beyond the general form of the phase space integrations. One may hope that, as in the case of the soft-gluon exponentiation of the $N^0$ coefficient functions, such additional structures can be found, e.g., using the new approach of Refs. [59, 60] or by improved applications of the soft-collinear effective theory to DIS, see, e.g., Refs. [77–80]. It should then become possible to resum also lower logarithms analogous to the standard threshold resummation.

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Appendix A: Large-$N$ splitting functions at NNLL accuracy

Here we provide the expansion coefficients of the highest three logarithms of the off-diagonal splitting functions, defined via

\[
N\tilde{P}_{ij}^{(n)}(N) = C_{AF}^n \ln^{2n}\tilde{N} \frac{4^n}{(n!)^2} D^{(n)}_{LL} + C_{AF}^{n-1} \ln^{2n-1}\tilde{N} \frac{4^{n-1}}{n!(n-1)!} \left[ C_F D_{NLL,2}^{(n)} + \beta_0 D_{NLL,3}^{(n)} \right] \\
+ C_{AF}^{n} \ln^{2n-2}\tilde{N} \frac{4^{n-2}}{n!(n-2)!} \left[ C_F^2 D_{NLL,4}^{(n)} + C_F \beta_0 D_{NLL,5}^{(n)} + \beta_0^2 D_{NLL,6}^{(n)} \right] \\
+ O(\ln^{2n-3}\tilde{N}) \quad (A.1)
\]

with \( \tilde{P}_{gq}(N) = P_{gq}/n_f \), \( \tilde{P}_{gq}(N) = P_{gq}/C_F \) and \( C_{AF} = C_A - C_F \). It is understood that the coefficients on the r.h.s. depend on the splitting function under consideration. \( D_{NLL,1} \) vanishes for \( P_{gq} \), but not for \( P_{gq} \). The coefficients of the leading and next-to-leading logarithms, for which closed expressions have been written down in Eqs. (5.5) and (5.7), are given up to order \( \alpha_s^{18} \) in Table 1. Their counterparts for the third logarithms are provided in Table 2 for \( P_{gq} \) and Table 3 for \( P_{gq} \). The closed expression leading to most of these coefficients is not known yet.

Appendix B: Large-$N$ coefficient functions at NNLL accuracy

Finally we turn to the NNLL contributions to the off-diagonal coefficient functions \( C_{2,g} \) and \( C_{\phi,q} \). The more complicated colour structure, with \( 3n - 3 \) contributions at order \( n \), precludes a representation in the form of Tables 2 and 3. Instead we provide the resulting expression at order \( \alpha_s^4 \) and \( \alpha_s^5 \) for general SU(N) colour factors, and then the six-figure values for QCD to order \( \alpha_s^{12} \) which are more than sufficient for numerical applications. In all cases we include also the LL and NLL terms, for which closed expression have been given in Eqs. (6.1) and Eqs. (6.2) above.

The fourth- and fifth-order contributions to the gluon coefficient function for \( F_2 \) are given by

\[
Nc_{2,g}^{(4)}(N) = -n_f \ln^7\tilde{N} \left[ \frac{8}{3} C_F^3 + \frac{4}{3} C_{AF} C_F^2 + \frac{4}{3} C_{AF}^2 C_F + \frac{46}{135} C_{AF}^3 \right] \\
- n_f \ln^6\tilde{N} \left[ \frac{8}{3} C_F^2 \beta_0 + \frac{62}{3} C_F^3 + C_{AF} \left( \frac{11}{9} C_F \beta_0 + \frac{49}{3} C_F^2 \right) \right] \\
+ C_{AF}^2 \left( \frac{56}{135} \beta_0 + \frac{1061}{90} C_F \right) + \frac{8}{3} C_{AF}^3 \]
\]
Table 1: The coefficients of the leading and next-to-leading large-\(N\) logarithms of the off-diagonal splitting functions \(P_{gq}\) and \(P_{qg}\) as defined in Eq. (A.1) up to the 18-th order \(\alpha_s = \alpha_s/(4\pi)\). Note the respective appearance of the numerators 691 and 3617 from \(n = 12\) and \(n = 16\) which clearly signals the presence of the Bernoulli numbers \((5.2)\) also in the NLL coefficients.
Table 2: The coefficients of the next-to-next-to-leading large-$N$ logarithms of the off-diagonal splitting function $P_{qg}$, as defined in Eq. (A.1), up to the 18-th order $a_s = \alpha_s/(4\pi)$.
| $n$ | $D_{\text{NNL.1}}^{(n)}$ | $D_{\text{NNL.2}}^{(n)}$ | $D_{\text{NNL.3}}^{(n)}$ | $D_{\text{NNL.4}}^{(n)}$ | $D_{\text{NNL.5}}^{(n)}$ | $D_{\text{NNL.6}}^{(n)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | $\frac{2}{3}$ | $-\frac{44}{3} - 4\zeta_2$ | $\frac{32}{3}$ | 0 | 0 | 0 |
| 2 | $\frac{4}{3}$ | $\frac{4}{3} - 8\zeta_2$ | $\frac{31}{3}$ | 54 | $-36$ | 6 |
| 3 | $\frac{1}{6} + \zeta_2$ | $\frac{56}{9} - 6\zeta_2$ | $\frac{52}{9}$ | 75 | $\frac{41}{3}$ | 4 |
| 4 | $-\frac{1}{3}(2 - 9\zeta_2)$ | $\frac{1}{3} - 10(2 - 9\zeta_2)$ | $\frac{1}{3}$ | 8 | 8 | $-32$ |
| 5 | $\frac{5}{6} + 3\zeta_2$ | $-\frac{2}{3} + 2\zeta_2$ | $\frac{44}{15}$ | 81 | 81 | $-32$ |
| 6 | $\frac{1}{6}(2 - 9\zeta_2)$ | $-\frac{1}{9}(10 - 3\zeta_2)$ | $\frac{1}{9}$ | 12 | 57 | 937 |
| 7 | $\frac{1}{18}(31 - 99\zeta_2)$ | $\frac{80}{63} - \frac{8}{3} \zeta_2$ | $\frac{34}{9}$ | 41 | 45 | 61 |
| 8 | $-\frac{2}{9}(2 - 9\zeta_2)$ | $\frac{1}{9}(10 - 3\zeta_2)$ | $\frac{1}{9}$ | 16 | 19 | $-45$ |
| 9 | $\frac{1}{6} \zeta_2 + \zeta_2$ | $\frac{146}{45} + 6\zeta_2$ | $\frac{376}{45}$ | 33 | 197 | $-8$ |
| 10 | $\frac{1}{2}(2 - 9\zeta_2)$ | $-\frac{1}{5}(10 - 3\zeta_2)$ | $\frac{1}{5}$ | 376 | 1171 | 25517 |
| 11 | $\frac{5}{6}(29 - 75\zeta_2)$ | $\frac{380}{33} - 20\zeta_2$ | $\frac{910}{33}$ | 207 | 267 | 41 |
| 12 | $-\frac{5}{3}(2 - 9\zeta_2)$ | $\frac{5}{9}(10 - 3\zeta_2)$ | $\frac{5}{9}$ | 3714 | 133953 | 1652771 |
| 13 | $-\frac{691}{630}(127 - 306\zeta_2)$ | $-\frac{1382}{4095}(163 - 273\zeta_2)$ | $-\frac{516868}{4095}$ | $-\frac{57353}{3640}$ | $-\frac{15893}{728}$ | $-2764$ |
| 14 | $-\frac{691}{90}(2 - 9\zeta_2)$ | $-\frac{691}{315}(10 - 3\zeta_2)$ | $\frac{691}{315}$ | 17772 | 62497 | 1725089 |
| 15 | $\frac{245}{6}(25 - 57\zeta_2)$ | $\frac{3080}{9} - 56\zeta_2$ | $\frac{6874}{9}$ | 291 | 431 | 67 |
| 16 | $-\frac{140}{3}(2 - 9\zeta_2)$ | $\frac{35}{3}(10 - 3\zeta_2)$ | $\frac{35}{3}$ | 103202 | 390957 | 2985994 |
| 17 | $-\frac{25319}{30}(11 - 24\zeta_2)$ | $-\frac{7234}{255}(95 - 153\zeta_2)$ | $-\frac{1504672}{255}$ | $-\frac{1204461}{2720}$ | $-\frac{379785}{544}$ | $-148297$ |

Table 3: As Table 2, but for the splitting function $P_{gq}$. 

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\[-n_f \ln^5 \tilde{N} \left[ \frac{2}{3} C_F \beta_0^2 + 30 C_F^2 \beta_0 + \frac{230}{3} C_F^3 - 40 \xi_2 C_F^3 \right]
+ C_{AF} \left( \frac{23}{180} \beta_0^2 + \frac{15761}{1080} C_F \beta_0 + \frac{122647}{1080} C_F^2 - \frac{164}{3} \xi_2 C_F^2 \right)
+ C_{AF}^2 \left( \frac{259}{54} \beta_0 + \frac{2423}{27} C_F - 33 \xi_2 C_F \right) + C_{AF}^3 \left( \frac{730}{27} - \frac{97}{9} \xi_2 \right) \right]
+ \mathcal{O}(\ln^4 \tilde{N}) \quad \text{(B.1)}

\[
N_{c_{2,g}}^{(5)}(N) = -n_f \ln^9 \tilde{N} \left[ \frac{4}{3} C_F^4 + \frac{8}{9} C_{AF} C_F^3 + \frac{4}{3} C_{AF}^2 C_F^2 + \frac{92}{135} C_{AF}^3 C_F + \frac{2}{15} C_{AF}^4 \right]
- n_f \ln^8 \tilde{N} \left[ \frac{8}{3} C_F^2 \beta_0 + \frac{40}{3} C_F^4 + C_{AF} \left( \frac{5}{3} C_F^2 \beta_0 + \frac{41}{3} C_F^3 \right) \right]
+ C_{AF}^2 \left( \frac{172}{135} C_F \beta_0 + \frac{701}{45} C_F^2 \right) + C_{AF}^3 \left( \frac{419}{1350} \beta_0 + \frac{363}{50} C_F \right) + \frac{4}{3} C_{AF}^4 \right]
- n_f \ln^7 \tilde{N} \left[ \frac{16}{9} C_F^2 \beta_0^2 + 36 C_F^3 \beta_0 + \frac{206}{3} C_F^4 - 32 \xi_2 C_F^4 \right]
+ C_{AF} \left( \frac{199}{270} C_F \beta_0^2 + \frac{14221}{540} C_F^2 \beta_0 + \frac{64007}{540} C_F^3 - \frac{160}{3} \xi_2 C_F^3 \right)
+ C_{AF}^2 \left( \frac{659}{2700} \beta_0 + \frac{25097}{1350} C_F \beta_0 + \frac{123451}{900} C_F^2 - \frac{466}{9} \xi_2 C_F^2 \right)
+ C_{AF}^3 \left( \frac{1768}{405} \beta_0 + \frac{157504}{2025} C_F - \frac{494}{15} \xi_2 C_F \right) + C_{AF}^4 \left( \frac{36484}{2025} - \frac{4976}{675} \xi_2 \right) \right]
+ \mathcal{O}(\ln^6 \tilde{N}) \quad \text{(B.2)}

The corresponding results for the quark coefficient function for $F_0$ read

\[
N_{c_{4,q}}^{(4)}(N) = -C_F \ln^7 \tilde{N} \left[ \frac{8}{3} C_F^3 + \frac{20}{3} C_{AF} C_F^2 + \frac{20}{3} C_{AF}^2 C_F + \frac{314}{135} C_{AF}^3 \right]
- C_F \ln^6 \tilde{N} \left[ \frac{14}{3} C_F^2 \beta_0 + \frac{40}{3} C_F^3 + C_{AF} \left( \frac{67}{9} C_F \beta_0 + \frac{17}{3} C_F^2 \right) \right]
+ C_{AF}^2 \left( \frac{502}{135} \beta_0 - \frac{811}{90} C_F \right) - \frac{68}{9} C_{AF}^3 \right]
- C_F \ln^5 \tilde{N} \left[ 4 C_F \beta_0^2 + \frac{136}{3} C_F^2 \beta_0 + \frac{142}{3} C_F^3 - 32 \xi_2 C_F^3 \right]
+ C_{AF} \left( \frac{187}{60} \beta_0^2 + \frac{40427}{1080} C_F \beta_0 + \frac{60853}{1080} C_F^2 - \frac{268}{3} \xi_2 C_F^2 \right)
+ C_{AF}^2 \left( \frac{301}{27} \beta_0 + \frac{452}{9} C_F - \frac{497}{9} \xi_2 C_F \right) + C_{AF}^3 \left( \frac{398}{27} - \frac{109}{9} \xi_2 \right) \right]
+ \mathcal{O}(\ln^4 \tilde{N}) \quad \text{(B.3)}

\]
$$N_{C_{\phi,q}}^{(5)}(N) = -C_F \ln^3 N \left[ \frac{4}{3} C_F^4 + \frac{40}{9} C_{AF} C_F^3 + \frac{20}{3} C_{\bar{A}F} C_F^2 + \frac{628}{135} C_{\bar{A}F} C_F + \frac{166}{135} C_{\bar{A}F}^4 \right]$$

$$- C_F \ln^2 N \left[ 4 C_F^3 \beta_0 + \frac{26}{3} C_F^4 + C_{AF} \left( \frac{29}{3} C_F^2 \beta_0 + \frac{29}{3} C_F^3 \right) \right]$$

$$+ C_{\bar{A}F}^2 \left( \frac{3104}{135} C_F \beta_0 - \frac{17}{3} C_F^4 \right) + C_{\bar{A}F}^3 \left( \frac{4601}{1350} \beta_0 - \frac{1519}{150} C_F - \frac{3296}{675} C_{\bar{A}F} \right)$$

$$- C_F \ln N \left[ \frac{52}{9} C_F^2 \beta_0 + \frac{386}{9} C_F^3 \beta_0 + \frac{376}{9} C_F^4 - \frac{80}{3} \zeta_2 C_F^4 \right]$$

$$+ C_{AF} \left( \frac{2453}{270} C_F \beta_0 + \frac{31207}{540} C_F^2 \beta_0 + \frac{41573}{540} C_F^3 - \frac{326}{3} \zeta_2 C_F^3 \right)$$

$$+ C_{\bar{A}F}^2 \left( \frac{2393}{540} \beta_0 + \frac{33149}{900} C_F \beta_0 + \frac{12904}{135} C_F^2 - \frac{326}{3} \zeta_2 C_F^2 \right)$$

$$+ C_{\bar{A}F}^3 \left( \frac{4562}{675} \beta_0 + \frac{12254}{225} C_F - \frac{916}{15} \zeta_2 C_F \right) + C_{\bar{A}F}^4 \left( \frac{1648}{135} - \frac{2638}{225} \zeta_2 \right)$$

$$+ O \left( \ln^6 N \right). \quad (B.4)$$

Our notation for the Tables 4-6, where we include $C_{L,g}$ for the convenience of the reader, is

$$N_{C_{\phi,q}}^{(n)}(N) = - \ln^{2n-3} N D^{(n)}_{a,LL} - \ln^{2n-2} N \left[ D^{(n)1}_{a,NLL} - D^{(n)2}_{a,NLL} n_f \right]$$

$$- \ln^{2n-3} N \left[ D^{(n)1}_{a,NNL} - D^{(n)2}_{a,NNL} n_f + D^{(n)3}_{a,NNL} n_f^2 \right] + O \left( \ln^{2n-4} N \right), \quad (B.5)$$

$$n_f^{-1} N^2 c_{L,g}^{(n)}(N) = + \ln^{2n-3} N D^{(n)}_{L,LL} + \ln^{2n-3} N \left[ D^{(n)1}_{L,NLL} - D^{(n)2}_{L,NLL} n_f \right]$$

$$+ \ln^{2n-4} N \left[ D^{(n)1}_{L,NNL} - D^{(n)2}_{L,NNL} n_f + D^{(n)3}_{L,NNL} n_f^2 + D^{(n)4}_{L,NNL} n_f^3 \right]$$

$$+ O \left( \ln^{2n-5} N \right) \quad (B.6)$$

with $\tilde{C}_{2,g} = C_{2,g} / n_f$ and $\tilde{C}_{\phi,q} = C_{\phi,q} / C_F$. $f l^g_{11}$ has been defined in Eq. (6.4).

Finally it is worthwhile to note that a closed (if presumably rather lengthy) expression for the NNLL contributions to $C_{2,g}$ and $C_{\phi,q}$ can be derived once such an expression has been obtained for the corresponding contributions to the splitting functions in Appendix A. At that point all quantities but $\tilde{C}_{2,g}$ and $\tilde{C}_{\phi,q}$ entering the vanishing off-diagonal NNLL elements of the physical evolution kernel

$$\frac{dF}{d\ln Q^2} = \left( \beta_0(s) \frac{dC}{ds} C^{-1} + CPC^{-1} \right) F \equiv KF = 0_{NNLL} \quad (B.7)$$

with the standard matrix $P$ of the singlet splitting functions and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,q} & C_{2,g} \\ C_{\phi,q} & C_{\phi,g} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi2} & K_{\phi\phi} \end{pmatrix} \quad (B.8)$$

will be known, and Eqs. (B.7) can be solved for the NNLL parts of $C_{2,g}$ and $C_{\phi,q}$. In fact, we have applied the analogous NLL procedure to find the closed expression given in Eqs. (6.1) and (6.2).
Table 4: The LL, NLL and NNLL coefficients of $C_{2,g}$ in QCD, as defined in Eq. (B.5), to the 12-th order in $\alpha_s = \alpha_s/(4\pi)$. All these coefficients further decrease at higher orders and tend to zero in the infinite-order limit.

| n  | $D_{2,\text{LL}}^{(n)}$ | $D_{2,\text{NL}}^{(n)}$ | $D_{2,\text{NLL}}^{(n)}$ | $D_{2,\text{NNLL}}^{(n)}$ | $D_{2,\text{NNLL}}^{(n)}$ | $D_{2,\text{NNLL}}^{(n)}$ |
|----|------------------|------------------|------------------|------------------|------------------|------------------|
| 1  | 2                | 2                | 0                | 0                | 0                | 0                |
| 2  | 6.444444        | 24               | 0                | -3.43495         | 0                | 0                |
| 3  | 11.9259         | 103.9444         | 1.61728          | 285.481          | 13.7942          | 0                |
| 4  | 16.7874         | 248.0911         | 5.73937          | 1493.50          | 82.2181          | 0.489712         |
| 5  | 19.5455         | 419.7268         | 11.6100          | 4059.62          | 248.604          | 2.43393          |
| 6  | 19.3641         | 561.6677         | 17.4564          | 7676.98          | 508.882          | 6.25137          |
| 7  | 16.5861         | 624.6211         | 21.2105          | 11287.10         | 797.203          | 11.2984          |
| 8  | 12.4589         | 592.8111         | 21.5942          | 13619.90         | 1014.87          | 15.9871          |
| 9  | 8.31829         | 489.1240         | 18.8489          | 13915.50         | 1085.50          | 18.5769          |
| 10 | 4.99558         | 356.1780         | 14.3660          | 12296.60         | 997.538          | 18.2401          |
| 11 | 2.72610         | 231.8488         | 9.70788          | 9552.90          | 801.374          | 15.4587          |
| 12 | 1.36329         | 136.3690         | 5.89075          | 70891.30         | 8453.50          | 244.331          |

Table 5: As Table 4, but for the coefficient function $C_{\phi,q}$.

| n  | $D_{\phi,\text{LL}}^{(n)}$ | $D_{\phi,\text{NL}}^{(n)}$ | $D_{\phi,\text{NLL}}^{(n)}$ | $D_{\phi,\text{NNLL}}^{(n)}$ | $D_{\phi,\text{NNLL}}^{(n)}$ | $D_{\phi,\text{NNLL}}^{(n)}$ |
|----|------------------|------------------|------------------|------------------|------------------|------------------|
| 1  | 2.66667          | 1.33333          | 0                | 0                | 0                | 0                |
| 2  | 14.5185          | 15.5556          | 0.888889         | 104.051          | 8.59259          | 0                |
| 3  | 41.5802          | 121.539          | 7.81893          | 897.918          | 79.0343          | 0.790123         |
| 4  | 82.0448          | 489.1960         | 31.2611          | 3947.46          | 378.981          | 6.23868          |
| 5  | 123.863          | 1261.66          | 79.7128          | 11928.5          | 1228.05          | 25.3455          |
| 6  | 151.299          | 2363.55          | 148.1560         | 27287.3          | 2950.46          | 68.9743          |
| 7  | 154.905          | 3455.21          | 215.4755         | 49475.2          | 5530.83          | 139.478          |
| 8  | 136.254          | 4124.46          | 256.338          | 73284.1          | 8381.24          | 222.015          |
| 9  | 104.882          | 4147.64          | 257.174          | 90802.9          | 10551.2          | 289.008          |
| 10 | 71.6724          | 3594.93          | 222.527          | 95974.4          | 11279.5          | 316.347          |
| 11 | 43.9873          | 2732.85          | 168.950          | 87965.5          | 10424.5          | 297.446          |
| 12 | 24.4770          | 1847.41          | 114.098          | 70891.3          | 8453.50          | 244.331          |
| $n$ | $D_{L,LL}^{(n)}$ | $D_{L,NL}^{(n)1}$ | $D_{L,NL}^{(n)2}$ | $D_{L,NNL}^{(n)1}$ | $D_{L,NNL}^{(n)2}$ | $D_{L,NNL}^{(n)3}$ | $D_{L,NNL}^{(n)4}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 8              | 0             | 0             | 0             | 0             | 0             | 0             |
| 2   | 48             | 160           | 0             | -35.0919      | 0             | 0             | 0             |
| 3   | 144            | 1165.63       | 10.6667       | 3146.95       | 103.111       | 4.79341       | 0             |
| 4   | 288            | 4064.40       | 64            | 24400.0       | 974.683       | 28.7605       | 3.55556       |
| 5   | 432            | 9222.47       | 192           | 90273.6       | 4214.99       | 86.2814       | 28.4444       |
| 6   | 518.4          | 15447.8       | 384           | 217656.       | 11462.4       | 172.563       | 106.667       |
| 7   | 518.4          | 20459.9       | 576           | 387314.       | 22429.4       | 258.844       | 256           |
| 8   | 444.343        | 22375.5       | 691.2         | 544345.       | 34032.4       | 310.613       | 448           |
| 9   | 333.257        | 20817.7       | 691.2         | 630665.       | 41989.1       | 310.613       | 614.4         |
| 10  | 222.171        | 16840.7       | 592.457       | 620424.       | 43532.4       | 266.24        | 691.2         |
| 11  | 133.303        | 12044.7       | 444.343       | 529638.       | 38847.9       | 199.68        | 658.286       |
| 12  | 72.7106        | 7717.03       | 296.229       | 398936.       | 30392.6       | 133.12        | 543.086       |

Table 6: The LL, NLL and NNLL coefficients of $C_{L,g}$ in QCD, as defined in Eq. (B.6), to the 12-th order in $\alpha_s = \alpha_s/(4\pi)$. Also these coefficients further decrease at higher orders and tend to zero in the infinite-order limit. The fifth column represents the $f_{11}^L$ term absent in charged-current DIS.

References

[1] K. Nakamura et al. [Particle Data Group], J. Phys. G37 (2010) 075021
[2] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, Phys. Rev. Lett. 66 (1991) 862
[3] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345
[4] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, Phys. Lett. B272 (1991) 121
[5] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. B427 (1994) 41
[6] S.A. Larin, P. Nogueira, T. van Ritbergen and J. Vermaseren, Nucl. Phys. B492 (1997) 338, hep-ph/9605317
[7] A. Retey and J.A.M. Vermaseren, Nucl. Phys. B604 (2001) 281, hep-ph/0007294
[8] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192
[9] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111
[10] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B606 (2005) 123, hep-ph/0411112
[11] J.A.M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B724 (2005) 3, hep-ph/0504242
[12] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B813 (2009) 220, arXiv:0812.4517v8 [hep-ph]
[13] G.P. Korchemsky, Mod. Phys. Lett. A4 (1989) 1257
[14] Y.L. Dokshitzer, G. Marchesini and G.P. Salam, Phys. Lett. B634 (2006) 504, hep-ph/0511302

36
[15] A. Vogt, Phys. Lett. B691 (2010) 77, [arXiv:1005.1606 [hep-ph]]
[16] A. Daleo, A. Gehrmann-De Ridder, T. Gehrmann and G. Luisoni, JHEP 1001 (2010) 118, [arXiv:0912.0374]
[17] G. Soar, S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B832 (2010) 152, [arXiv:0912.0369 [hep-ph]]
[18] G. Sterman, Nucl. Phys. B281 (1987) 310
[19] L. Magnea, Nucl. Phys. B349 (1991) 703
[20] S. Catani and L. Trentadue, Nucl. Phys. B327 (1989) 323; B353 (1991) 183
[21] H. Contopanagos, E. Laenen, and G. Sterman, Nucl. Phys. B484 (1997) 303, [hep-ph/9604313]
[22] S. Catani, M.L. Mangano, P. Nason and L. Trentadue, Nucl. Phys. B478 (1996) 273, [hep-ph/9604351]
[23] A. Vogt, Phys. Lett. B497 (2001) 228, [hep-ph/0010146]
[24] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B646 (2002) 181, [hep-ph/0209100]
[25] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B726 (2005) 317, [hep-ph/0506288v2]
[26] P.A. Baikov and K.G. Chetyrkin, Nucl. Phys. (Proc. Suppl.) 160 (2006) 76
[27] A. Vogt, [arXiv:0707.4106 [hep-ph]], proceedings of DIS07, Munich (Germany), April 2007
[28] S. Moch and A. Vogt, JHEP 11 (2009) 099, [arXiv:0909.2124 [hep-ph]]
[29] S. Moch and A. Vogt, JHEP 04 (2009) 081, [arXiv:0902.2342 [hep-ph]]
[30] J.A.M. Vermaseren, New features of FORM, [math-ph/0010025]
[31] M. Tentyukov and J.A.M. Vermaseren, Comput. Phys. Commun. 181 (2010) 1419, [arXiv:hep-ph/0702279]
[32] W.L. van Neerven and E.B. Zijlstra, Phys. Lett. B272 (1991) 127
[33] E.B. Zijlstra and W.L. van Neerven, Phys. Lett. B273 (1991) 476
[34] E.B. Zijlstra and W.L. van Neerven, Nucl. Phys. B383 (1992) 525
[35] S. Moch and J.A.M. Vermaseren, Nucl. Phys. B573 (2000) 853, [hep-ph/9912355]
[36] T. Kinoshita, J. Math Phys. 3 (1962) 650
[37] T.D. Lee and M. Nauenberg, Phys. Rev. B133 (1964) 1549
[38] H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D12 (1975) 467
[39] J.C. Collins, A. Duncan and S.D. Joglekar, Phys. Rev. D16 (1977) 438
[40] S. Moch, J.A.M. Vermaseren and A. Vogt, JHEP 08 (2005) 049, [hep-ph/0507039]
[41] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B625 (2005) 245, [hep-ph/0508055]
[42] S. Moch and A. Vogt, Phys. Lett. B680 (2009) 239, [arXiv:0908.2746 [hep-ph]]
[43] T. Matsuura and W.L. van Neerven, Z. Phys. C38 (1988) 623
[44] T. Matsuura, S.C. van der Marck and W.L. van Neerven, Nucl. Phys. B319 (1989) 570
[45] P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, Phys. Rev. Lett. 102 (2009) 212002, [arXiv:0902.3519 [hep-ph]]
[46] R.N. Lee, A.V. Smirnov and V.A. Smirnov, JHEP 1004 (2010) 020, [arXiv:1001.2887 [hep-ph]]
