Ageing without detailed balance: local scale invariance applied to two exactly solvable models

Florian Baumann\textsuperscript{a,b} 
\textsuperscript{a}Institut f"ur Theoretische Physik I, Universit"at Erlangen-N"urnberg, Staudtstra"asse 7B3, D – 91058 Erlangen, Germany 
\textsuperscript{b}Laboratoire de Physique des Matériaux,\textsuperscript{1} Université Henri Poincaré Nancy I, B.P. 239, F – 54506 Vandœuvre l`es Nancy Cedex, France

Abstract. I consider ageing behaviour in two exactly solvable reaction-diffusion systems. Ageing exponents and scaling functions are determined. I discuss in particular a case in which the equality of two critical exponents, known from systems with detailed balance, does not hold any more. Secondly it is shown that the form of the scaling functions can be understood by symmetry considerations.

1. Introduction
Ageing phenomena may occur in systems which are rapidly quenched into a region in parameter space with several competing stationary states. These phenomena have been studied quite extensively in systems with detailed balance such as simple magnetic systems, which are initially prepared in a high-temperature state and then quenched to or below the critical temperature $T_C$ \cite{1, 2, 3, 4, 5, 6}. One typically considers the autocorrelation and the autoresponse function, for which one expects scaling behaviour in the ageing regime, that is for $t,s$ and $t−s$ large compared to some microscopic timescale:

\begin{align}
C(t, s) &:= \langle \phi(x, t)\phi(x, s) \rangle \sim s^{-b}f_C(t/s) \\
R(t, s) &:= \frac{\delta\langle \phi(x, t)\phi(x+, s) \rangle}{\delta h(x, s)} \sim s^{-a-1}f_R(t/s)
\end{align}

(1)

where $\phi(x, t)$ is the order parameter describing the system and $h(x, s)$ is a small external perturbation, for instance a magnetic field. $a$ and $b$ are critical exponents and the scaling functions $f_C$ and $f_R$ behave for large arguments as

\begin{align}
f_C(y) &\sim y^{-\lambda_C/z} \\
f_R(y) &\sim y^{-\lambda_R/z}
\end{align}

(2)

where $\lambda_R$ and $\lambda_C$ are new exponents and $z$ is the dynamical critical exponent. For systems with decorrelated initial conditions $\lambda_R = \lambda_C$ has been found and the relation $a = b$ has been confirmed at criticality. Notice that for systems with detailed balance the latter condition is necessary because the fluctuation-dissipation ratio has to hold when the system reaches equilibrium.

\textsuperscript{1} Laboratoire associé au CNRS UMR 7556
The approach of local scale invariance (LSI) had been proposed in [7, 8] to understand the form of $C(t, s)$ and $R(t, s)$ on the basis of symmetry considerations. It turned out that for magnetic systems the form of $R(t, s)$ can be completely fixed by symmetries whereas $C(t, s)$ is fixed up to a scaling function. Monte Carlo simulations in the Ising model and the XY model showed excellent agreement with these predictions [9, 10, 11, 12], whereas renormalisation group computations yielded a correction at two loops [13], the origin of which is still not quite clear.

In the systems considered so far detailed balance holds. However many realistic systems do not possess this property. It is therefore interesting to see what happens if this condition is relaxed. Typical systems which lack detailed balance are reaction-diffusion systems, where particles undergo diffusion on a lattice and in addition, there are particle creation and annihilation processes. Numerical work has been done on the (fermionic) contact process [14, 15], showing that at criticality dynamical scaling holds for the response function and the connected correlator, but that opposed to the above mentioned magnetic systems $a \neq b$. Here, I shall ask the question: Are there exactly solvable systems without detailed balance where $a \neq b$, and if so, can the form of the scaling functions still be understood with the help of the theory of local scale invariance?

In this text I shall consider two specific models without detailed balance. On the one hand they allow for exact calculation of the ageing exponents and the scaling functions, on the other hand they can be described by a field-theoretical formalism and can therefore be attacked by the theory of local scale invariance. This paper is organised as follows: In section 2, I introduce the models and present the main results. In particular I show that in one case one indeed encounters $a \neq b$. In section 3, I look at the same models from the field-theoretical perspective and demonstrate that the form of the scaling functions can indeed be understood by considering the symmetries of the models.

2. The bosonic contact and pair-contact processes

2.1. The models

I consider a $d$-dimensional cubic lattice with one sort of particles $A$. It is important that the system is bosonic, which means that there is no restriction on the number of particles on one lattice site. In what follows I shall consider two different models:

- The bosonic contact process (BCPD): The particles undergo diffusion with diffusion constant $D$, that is, they may jump to a nearest-neighbor site with rate $D$. Furthermore single particles can disintegrate with rate $\lambda$ or produce offspring with rate $\mu$. The newly produced particles then sit on the same lattice site as the original particle.

\[
A \xrightarrow{D} A, \quad A \xrightarrow{\lambda} \emptyset, \quad A \xrightarrow{\mu} 2A
\]

This process has been used to model the clustering of biological organisms in [16].

- The bosonic pair-contact process (BPCPD): Also in this process there is diffusion with constant $D$. In addition two particles on one lattice site can coagulate with rate $\lambda$ or give birth to a third particle with rate $\mu$:

\[
A \xrightarrow{D} A, \quad 2A \xrightarrow{\lambda} A, \quad 2A \xrightarrow{\mu} 3A
\]

It has been shown by Paessens and Schütz [17] that this model can be solved analytically for any finite time. This is mainly due to a formal analogy to the spherical model, where the control parameter $\alpha$ defined below formally replaces the temperature.
The state of the system is characterised by the number of particles on each site. I denote this by \( \{ n \} = \{ \ldots, n_x, \ldots \} \), where the non-negative integer \( n_x \) gives the number of particles on site \( x \). The temporal evolution can be described by a master equation, which can be turned into a Schrödinger-type equation by standard techniques [18, 19, 20]. One then has annihilation and creation operators \( a(x) \) and \( a^\dagger(x) \) at each lattice site, the nonvanishing commutation relations of which are \( [a(x), a^\dagger(y)] = \delta_{x,y} \). Then two quantities are defined: the state vector \( |n⟩ := \prod_x (a^\dagger(x))^n |0⟩ \), where \( |0⟩ \) is the vacuum state representing the empty lattice, and the vector \( |P(t)⟩ := \sum_{\{n\}} P(\{n\},t) |n⟩ \), where \( P(\{n\},t) \) is the probability to find the system in the state \( \{ n \} \) at time \( t \). The latter quantity obeys the equation \( \partial_t |P(t)⟩ = -H|P(t)⟩ \), where the Hamiltonian \( H \) is given in terms of annihilation and creation operators and can be found in [17] for the cases at hand. Time-dependent observables are obtained by passing to the Heisenberg picture and the temporal evolution of an observable \( g(t) \) is then given by the Heisenberg equation of motion

\[
\partial_t g(t) = [H, g(t)]
\]

Finally, the average of \( g(t) \) is calculated as \( \langle g(t) \rangle := \langle s | g(t) | P(0) \rangle \), where \( \langle s | \) is a coherent state vector with the properties \( \langle s | a(\mathbf{x}) = \langle s | \) and \( \langle s | H = 0 \). The quantities of interest are:

- The local particle density

\[
\rho(\mathbf{x},t) := \langle a^\dagger(\mathbf{x},t) a(\mathbf{x},t) \rangle = \langle a(\mathbf{x},t) \rangle
\]

where the special property of the state \( \langle s | \) has been used in the last equality.

- The connected two-point correlator:

\[
G(\mathbf{x} - \mathbf{y}, t, s) := \langle a(\mathbf{x},t) a(\mathbf{y},s) \rangle - \langle a(\mathbf{x},t) \rangle \langle a(\mathbf{x},s) \rangle
\]

Here and in what follows I assume spatial translation invariance, so that two-point quantities depend only on the difference of the spatial coordinates. If scaling behaviour is found, the corresponding scaling function will be denoted by \( f_G \) in analogy to (1).

- The response function: To compute this quantity, one adds a small perturbation \( \sum_x h(\mathbf{x},t) a^\dagger(\mathbf{x}) \) to the Hamiltonian \( H \). This corresponds to spontaneous particle creation at an empty lattice site with rate \( h(\mathbf{x},t) \). Then the response function is simply defined as

\[
R(\mathbf{x} - \mathbf{y}, t, s) := \frac{\delta \langle a(\mathbf{x},t) \rangle}{\delta h(\mathbf{y},s)}
\]

The Heisenberg equation of motion (3) is used to derive differential equations for the quantities (4)-(6). For the particle density and the two-point correlator the following initial condition are chosen

\[
\rho(\mathbf{x},0) = \rho_0, \quad G(\mathbf{x},0,0) = 0
\]

whereas the response function is required to be a delta-peak at \( t = s \). All these equations can be solved by standard techniques.

2.2. Results

For the particle density, one finds the following results for both processes [16, 17]:

- For \( \mu > \lambda \) particle creation outweighs particle annihilation and \( \rho(\mathbf{x},t) \) diverges.

\[ \text{This ensures that the system is translationally invariant. Furthermore it can be shown that this choice can be realised with an initial Poisson distribution of the particle density at each lattice site.} \]
For $\mu < \lambda$ particle annihilation is stronger and the system runs into the empty lattice state.

Only if $\mu = \lambda$ creation and annihilation of particles are of equal strength. In this case one has $\rho(x,t) = \rho_0$ for all times.

In the sequel I shall only look at the most interesting case when $\lambda = \mu$. Then the creation and annihilation processes balance each other, one says that one is on the critical line. As the particle density remains constant, one needs to look at the variance $\sigma_\rho(t)$ of $\rho(x,t)$ for further insights. Notice that $\sigma_\rho(t)$ is equal to $G(0,t,t)$ up to a constant. One finds the following results [16, 17]

- For the BCPD the correlator $G(0,t,t)$ behaves as $t^{-\frac{d}{2}+1}$ (for $t \to \infty$) and one has a diverging variance $\sigma_\rho(t)$ if $d < 2^3$, otherwise the variance is bounded. This means that for $d \leq 2$ diffusion cannot spread particles evenly so that they accumulate on very few lattice sites - a clustering transition occurs.

- For the BPCPD the control parameter $\alpha$ is defined by

$$\alpha := \frac{\mu}{D}. \quad (8)$$

It measures the strength of the creation and annihilation processes in comparison to the diffusion process. It turns out that there is a critical value $\alpha_C > 0$ and the behaviour of $G(0,t,t)$ depends on whether $\alpha$ is larger, equal or smaller than $\alpha_C$. More precisely

$$\begin{array}{c|c|c|c|c}
\alpha < \alpha_C, d > 2 & \alpha = \alpha_C, d > 4 & \alpha = \alpha_C, 2 < d < 4 & \alpha > \alpha_C \text{ or } d \leq 2 \\
G(0,t,t) \overset{t \to \infty}{\sim} \text{const} & G(0,t,t) \overset{t \to \infty}{\sim} t & G(0,t,t) \overset{t \to \infty}{\sim} t^{\frac{d}{2}-1} & G(0,t,t) \overset{t \to \infty}{\sim} \exp(t/\tau)
\end{array} \quad (9)$$

For small $\alpha$, diffusion is dominant and the system stays more or less homogeneous. But if $\alpha$ is large enough, there is again a clustering transition.

Finally the two-time quantities [21] are considered. For the response function scaling behaviour is found in the ageing regime with the result

$$R(t,s) = r_0 s^{-\frac{d}{2}+1} \left( (t/s)^{-\frac{d}{2}+1} - 1 \right) \quad (10)$$

from which the critical exponents $a$ and $\lambda_R$ can be derived, as well as the scaling function $f_R(y)$.

For the connected two-time correlator, one also finds the scaling behaviour (2) in the BCPD, and in the BPCPD for the first three cases discussed in (9). The ageing exponents can be found in table (1), whereas the scaling function is given by the integral expression

$$f_G(y) = g_0 \int_0^1 d\theta \, \theta^{(a-b)}(y + 1 - 2\theta)^{-\frac{d}{2}} \quad (11)$$

Note that $a \neq b$ is found at criticality [21]. This entails in particular that there is no non-trivial analogue to the fluctuation-dissipation ratio known from magnetic system and that it is not possible to define in a straightforward way an effective temperature characterizing the system as suggested in [22].

\footnote{In the case $d = 2$ there is a logarithmic divergence.}
3. Local scale invariance

In this last section I consider the same processes from a field-theoretical point of view. First I define the fields \( \phi(x, t) := a(x, t) - \rho_0 \) and \( \tilde{\phi} := a^\dagger(x, t) - 1 \). In this way the correlator \( \langle \phi(x, t)\phi(x', s) \rangle \) equals the connected correlator (5). Then, by taking the continuum limit, the Hamiltonian \( H \) is turned into a field-theoretical action \( \Sigma[\phi, \tilde{\phi}] \) [8, 23, 24], from which \( n \)-point correlators can be computed via

\[
\langle \phi_1(x_1, t_1) \ldots \phi_n(x_n, t_n) \rangle = \int \mathcal{D}[\phi] \mathcal{D}[\tilde{\phi}] \phi_1(x_1, t_1) \ldots \phi_n(x_n, t_n) \exp(-\Sigma[\phi, \tilde{\phi}]).
\]

The action can be split up into two parts as \( \Sigma[\phi, \tilde{\phi}] = \Sigma_0[\phi, \tilde{\phi}] + \Sigma_{\text{noise}}[\phi, \tilde{\phi}] \), where the first part \( \Sigma_0[\phi, \tilde{\phi}] \) is given by

\[
\begin{align*}
\Sigma_0[\phi, \tilde{\phi}] &= \int \mathcal{D}R \mathcal{D}u \left[ \frac{1}{2} \left( \phi(2 \mathcal{M} \partial_u - \nabla^2) \phi \right) \right] \quad \text{for the BCPD} \\
\Sigma_0[\phi, \tilde{\phi}] &= \int \mathcal{D}R \mathcal{D}u \left[ \frac{1}{2} \left( \phi(2 \mathcal{M} \partial_u - \nabla^2) \phi - \alpha \phi^2 \phi^2 \right) \right] \quad \text{for the BPCPD.}
\end{align*}
\]

Both fields under the integral depend on the integration variables \( R \) and \( u \), but here and

| \begin{tabular}{|c|c|c|} 
| \hline
| \multicolumn{1}{|c|}{bosonic contact process} & \multicolumn{1}{|c|}{bosonic pair-contact process} \\
| \hline
| \alpha < \alpha_C & \alpha = \alpha_C \\
| \hline
| \( a \) & \( \frac{d}{2} - 1 \) & \( \frac{d}{2} - 1 \) \\
| \hline
| \( b \) & \( \frac{d}{2} - 1 \) & \( \frac{d}{2} - 1 \) \quad \text{if} \quad 2 < d < 4 \quad \text{if} \quad d > 4 \\
| \hline
\end{tabular} |

Table 1. Ageing exponents of the critical bosonic contact and pair-contact processes in the different regimes. \( \lambda_R = \lambda_C = d \) and \( z = 2 \) was found for all cases. The results for the bosonic contact process hold for an arbitrary dimension \( d \), but for the bosonic pair-contact process they only apply if \( d > 2 \), since \( \alpha_C = 0 \) for \( d \leq 2 \).

in what follows I shall suppress the arguments of the fields for space considerations. \( \mathcal{M} \) is a parameter related to the diffusion constant. The exact form of the second part is somewhat more involved and can be found in [23, 24]. Here it suffices to state that \( \Sigma_{\text{noise}}[\phi, \tilde{\phi}] \) has the form \( \Sigma_{\text{noise}}[\phi, \tilde{\phi}] = \sum_{n,m \leq 3} a_{nm} \int \mathcal{D}R \mathcal{D}u \phi^n \tilde{\phi}^m \), where the constants \( a_{nm} \) vanish whenever \( n \geq m \). The point of this split-up is the following: It can be shown that in both cases, \( \Sigma_0[\phi, \tilde{\phi}] \) has nontrivial symmetry properties. This is a well-known fact for the BCPD [8] as the corresponding evolution equation for \( \psi \) is a free Schrödinger equation, but has only been shown recently [25, 24] for the case of the BPCPD. From these symmetry properties the so-called Bargmann superselection rule can be inferred, stating that \( \langle \phi \ldots \phi \phi \ldots \phi \rangle_0 = 0 \) unless \( n = m^2 \). This entails that the response function is independent of \( \Sigma_{\text{noise}}[\phi, \tilde{\phi}] \) [8, 24] and is given by

\[
R(x - x', t, s) = \langle \phi(x, t)\tilde{\phi}(x', s) \rangle_0
\]

where \( \langle \ldots \rangle_0 \) denotes the average with respect to \( \Sigma_0[\phi, \tilde{\phi}] \). Further investigations reveal, that the connected correlator is essentially given by a sum of integrals over three- and four-point

4 For the BCPD and an even number of fields, this can also be inferred from causality combined with Wick’s theorem
functions, which have also been computed with respect to $\Sigma_0[\phi, \tilde{\phi}]$. Therefore, the main task is to determine the two-, three-, and four-point functions of the theory described by $\Sigma_0[\phi, \tilde{\phi}]$ only.

Let us consider the BCPD first. In this case the evolution equation for the field $\phi$ is a free Schrödinger equation. The symmetry group of this equation, i.e. the group of transformations carrying solutions to other solutions, is the well-known Schrödinger group [27]. An element $g$ of this group acts on space-time coordinates as $(x, t) \rightarrow (x', t') = g(x, t)$ with

$$t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad x \rightarrow x' = \frac{Rx + vt + \alpha}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1$$

(15)

where $R$ is a rotation matrix and $\alpha, \beta, \gamma, \delta$ and $v$ are parameters. A solution $\Psi(x, t)$ of the free Schrödinger equation is transformed as

$$\Psi(x, t) \rightarrow (T_g \Psi)(x, t) = f_g[g^{-1}(x, t)]\Psi[g^{-1}(x, t)]$$

(16)

where the companion function $f_g$ is known explicitly. A field transforming in this way is called *quasiprimary* and the important assumption is to identify the fields $\phi$ and $\tilde{\phi}$ as the appropriate quasiprimary fields of the theory. One can show that $n$-point functions built from quasiprimary fields satisfy certain linear partial differential equations involving the generators of the Schrödinger group. Solving these equations, one finds that the response function can be fixed completely with the result [24, 28]

$$R(x - x', t, s) = r_0(t - s)^{1/2} \Psi(\frac{t}{s}) \exp \left( -\frac{M}{2} \frac{(x - x')^2}{t - s} \right)$$

(17)

where $x_1$ and $x_2$ are free parameters which can be adjusted so that this expression is in line with the result (10). To deduce the exact functional form of $R(x - x', t, s)$ is the actual predictive power of this theory, providing new information which can for instance not be obtained by simple scaling arguments. Also three- and four-point functions can be fixed to a certain degree by the symmetries. There is, however, a degree of freedom that remains in the form of undetermined scaling functions. One can show that the result (11) can be reproduced by a suitable choice of these functions [24], but I shall not give the results here for limitations of space.

The procedure works similarly for the BPCPD. However, here the Schrödinger equation one has to consider is non-linear, which leads to a modification of the generators of the symmetry group, see [25, 26]. The result for the response function in this case is [24]

$$R(x - x', t, s) = r_0(t - s)^{1/2} \Psi(\frac{t}{s}) \exp \left( -\frac{M}{2} \frac{(x - x')^2}{t - s} \right)$$

(18)

with an arbitrary function $\Psi$ and another parameter $\hat{\gamma}$. Also this expression can be brought into agreement with (10). One can see that for the BPCPD the symmetries fix completely the form of the response function whereas for the BPCPD an arbitrary scaling function remains. We can argue however, that the universal scaling function $\Psi$ should not depend on the non-universal quantity $\alpha$, which leads to the statement $\Psi(\frac{t}{s}, \frac{t - s}{\alpha^{1/\hat{\gamma}}}, \frac{\alpha}{(t - s)^{\hat{\gamma}}}) = \tilde{\Psi}(t/s)$ for another undetermined function $\tilde{\Psi}$. This is in line with predictions obtained by renormalisation group arguments [6]. Finally one can show that in the BPCPD also the result (11) can be reproduced correctly by appropriately adjusting the free parameters of the theory [24]. Again I can not treat the latter point more explicitly here but have to defer the reader to the quoted references.
I conclude by summing up the main results of this paper. I have considered the bosonic contact and pair contact processes on the critical line. These are typical processes without detailed balance the absence of which leads to the violation of the relation $a = b$. However, also for these systems without detailed balance the approach of local scale invariance is still suitable to gain insight in the form of the scaling functions, even though the symmetries do not suffice, especially in the case of the pair-contact process, to fix completely the quantities of interest.

Acknowledgments

I acknowledge the support by the Deutsche Forschungsgemeinschaft through grant no. PL 323/2. This work was also supported by the Bayerisch-Französisches Hochschulzentrum (BFHZ). Furthermore I thank M. Henkel and M. Pleimling for useful comments.

[1] A.J. Bray, Adv. Phys. 43, 357 (1994).
[2] M.E. Cates and M.R. Evans (eds) Soft and fragile matter, IOP Press (Bristol 2000).
[3] L.F. Cugliandolo, in Slow Relaxation and non equilibrium dynamics in condensed matter, Les Houches Session 77 July 2002, J-L Barrat, J Dalibard, J Kurchan, M V Feigel’man eds (Springer, 2003); also available at cond-mat/0210312.
[4] C. Godrèche and J.M. Luck, J. Phys.: Condens. Matter 14, 1589 (2002).
[5] M. Henkel, Adv. Solid State Phys. 44, 389 (2004); see also cond-mat/0503739.
[6] P. Calabrese and A. Gambassi, J. Phys. A: Math. Gen. 38, R133 (2005).
[7] M. Henkel, Nucl. Phys. B641, 405 (2002).
[8] A. Picone and M. Henkel, Nucl. Phys. B688, 217 (2004).
[9] M. Henkel, M. Pleimling, C. Godrèche, and J.-M. Luck, Phys. Rev. Lett. 87, 265701 (2001).
[10] M. Henkel, M. Paessens, and M. Pleimling, Europhys. Lett. 62, 664 (2004); Phys. Rev. E 69, 056109 (2004).
[11] M. Henkel and M. Pleimling, Phys. Rev. E 68, 065101 (R) (2003).
[12] S. Abriet and D. Karevski, Eur. Phys. J. B 41, 79 (2004).
[13] P. Calabrese and A. Gambassi, Phys. Rev. B 65, 066120 (2002); Phys. Rev. E 66, 066101 (2002).
[14] T. Enss, M. Henkel, A. Picone and U. Schollwöck, J. Phys. A: Math. Gen. 37, 10479 (2004).
[15] J.J. Ramasco, M. Henkel, M.A. Santos et C.A. da Silva Santos, J. Phys. A: Math. Gen. 37 10497 (2004).
[16] B. Houchmandzadeh, Phys. Rev. E66, 052902 (2002).
[17] M. Paessens and G.M. Schütz, J. Phys. A: Math. Gen. 37, 4709 (2004).
[18] M. Doi, J. Phys. A: Math. Gen. 9, 1465 and 1479 (1976).
[19] L. Peliti, J. Phys. 46 1469.
[20] U.C.Täuber, M. Howard and B.P. Vollmayr-Lee, J. Phys. A: Math. Gen. 38 (2005), R79-R131.
[21] F. Baumann, M. Henkel, M. Pleimling and J. Richert, J. Phys. A: Math. Gen. 38 6623, (2005).
[22] F. Sastre, I. Dornic and H. Chaté, Phys. Rev. Lett. 91, 267205 (2003).
[23] M. Howard and U.C. Täuber, J. Phys. A: Math. Gen. 30, 7721 (1997).
[24] F.Baumann, S.Stoimenov and M.Henkel, cond-mat/0510778
[25] S.Stoimenov and M.Henkel, Nucl. Phys. B723, 205 (2005).
[26] F.Baumann, S.Stoimenov and M.Henkel, math-ph/0512025
[27] U. Niederer, Helv. Phys. Acta 45, 802 (1972).
[28] M.Henkel, J. Stat. Phys. 75, 1023 (1994).