Abstract

In our last work, we formulate a Fourier transformation on the infinite-dimensional space of functionals. Here we first calculate the Fourier transformation of infinite-dimensional Gaussian distribution \( \exp \left(-\pi \xi \int_{-\infty}^{\infty} \alpha^2(t) dt \right) \) for \( \xi \in \mathbb{C} \) with \( \text{Re}(\xi) > 0 \), \( \alpha \in L^2(\mathbb{R}) \), using our formulated Feynman path integral. Secondly we develop the Poisson summation formula for the space of functionals, and define a functional \( Z_s, s \in \mathbb{C} \), the Feynman path integral of that corresponds to the Riemann zeta function in the case \( \text{Re}(s) > 1 \).

0. Introduction

Feynman([F-H]) used the concept of his path integral for physical quantizations. The word "physical quantizations" has two meanings: one is for quantum mechanics and the other is for quantum field theory. We usually use the same word "Feynman path integral". However the meanings included in "Feynman path integral" are two sides, according to the above. One is of quantum mechanics and the other is of quantum field theory. The first Feynman path integral corresponds to a study of functional analysis on the space of functions. For functional analysis, there exist many works from standard analysis and nonstandard analysis. However an approach has been hard from standard analysis or nonstandard analysis to study the space of "functionals" associated with the second Feynman path integral.

In our last paper([N-O2]), we defined a delta functional \( \delta \) and an infinitesimal Fourier transformation \( F \) in the space of functionals as one of generalizations for Kinoshita's infinitesimal Fourier transformation in the space of functions. Historically, in 1962, Gaishi Takeuchi([T]) introduced an infinitesimal \( \delta \)-function for the space of functions under nonstandard analysis. In 1988, 1990, Kinoshita([K1],[K2]) defined an infinitesimal Fourier transformation for the space of functions. Nitta and Okada([N-O1],[N-O2]) defined, for functionals, an infinitesimal Fourier transformation, using a concept of double infinitesimals, and calculated the infinitesimal Fourier transformation for two typical examples. The main idea is to use the concept of double infinitesimals and putting standard parts twice \( \text{st}(\text{st}(\ . \ )) \). In our theory, the infinitesimal Fourier transformation of \( \delta, \delta^2, \ldots \), and \( \sqrt{\delta}, \ldots \) can be calculated as constant functionals, 1, infinite, \( \ldots \), and infinitesimal, \( \ldots \).

Now let \( H \) be an even infinite number in \( \ast \mathbb{R} \), and \( L \) be an infinitesimal lattice.
Let $L := \{ \varepsilon z \mid z \in \mathbb{Z}, -\frac{H}{2} \leq \varepsilon z < \frac{H}{2} \}$, where $\varepsilon = \frac{1}{H}$, and let $H'$ be an even infinite number in $(* \mathbb{R})$, and $L'$ be an infinitesimal lattice

$$L' := \{ \varepsilon' z' \mid z' \in (* \mathbb{Z}), -\frac{H'}{2} \leq \varepsilon' z' < \frac{H'}{2} \},$$

where $\varepsilon' = \frac{1}{H'}$. Then we extend the calculation in our previous work to the case of

$$g_\xi(a) = \exp \left( -\pi \xi^* \varepsilon \sum_{n \in L} a(\varepsilon n)^2 \right)$$

for $\xi \in \mathbb{C}$ with $\text{Re}(\xi) > 0$.

If there exists $\alpha \in L^2$ so that $a = *\alpha$, then $\text{st} \left( \exp \left( -\pi \xi^* \varepsilon \sum_{n \in L} a(\varepsilon n)^2 \right) \right)$ is equal to $\exp \left( -\pi \xi \int_{-\infty}^{\infty} \alpha^2(t)dt \right)$. The standard part of the functional

$$\exp \left( -\pi \xi^* \varepsilon \sum_{n \in L} a(\varepsilon n)^2 \right)$$

corresponds naturally to $\exp \left( -\pi \xi \int_{-\infty}^{\infty} \alpha^2(t)dt \right)$ in the standard meaning. Our Fourier transformation of $\exp \left( -\pi \xi^* \varepsilon \sum_{n \in L} a(\varepsilon n)^2 \right)$ is $C_\xi(b) \exp \left( -\pi \xi^{-1}^* \varepsilon \sum_{n \in L} b(\varepsilon n)^2 \right)$ where $\text{st}(C_\xi(b)) = \left( * \left( \frac{1}{\sqrt{\xi}} \right) \right)^H \in (* \mathbb{R})$. In the calculation, we assume that the real part $\xi$ is positive. Even if $\xi$ is $i$ or $-i$, the coefficient is equal to $\left( * \left( \frac{1}{\sqrt{\xi}} \right) \right)^H$ shown in the previous paper. Furthermore, let $m$ be an integer so that $\text{Re}(\xi) > 0$, and

$$g_{im}(a) = \exp \left( -i \pi m^* \varepsilon \sum_{k \in L} a^2(k) \right)$$

that associates with $\exp \left( -im \pi \int_{-\infty}^{\infty} \alpha^2(t)dt \right)$ in the standard meaning. Then we calculate our Fourier transformation for $g_{im} : (Fg_{im})(b) = C_{im}(b)g_{im}(b)$.

We show that $C_{im}(b) = \left( \sqrt{\frac{m}{2}} \frac{2^{*H \mu'}}{1+i} \right)^H$ for positive $m$ and $C_{im}(b) = \left( \sqrt{-\frac{m}{2}} \frac{1+(-)^m2^{*H \mu'}}{1-i} \right)^H$ for negative $m$ if $m(b(k))$ for arbitrary $k$ in $L$.

Furthermore Using the second infinitesimal and the lattice, we extend the Poisson summation formula of finite group to infinitesimal Fourier transformations for the space of functions and also for the space of functionals. For an example, we apply the Poisson summation formula to the above functional $g_\xi$. If the groups are special, it appears the $H^2$-th product of $\theta$-functions, or the constant $\left( * \left( \frac{1}{\sqrt{\xi}} \right) \right)^H$ . We also apply it to the functional $g_{im}$. Finally we define a functional that associates to the Riemann zeta function. Using our Poisson summation formula for functionals, we study a relationship between the functional and the Riemann zeta function.

1. Preliminaries

1-1. Infinitesimal Fourier transformations by Kinoshita (cf. [Ki],[N-O1],[N-O2])

Let $\Lambda$ be an infinite set. Let $F$ be an ultrafilter on $\Lambda$. For each $\lambda \in \Lambda$, let $S_\lambda$ be a set. We put an equivalence relation $\sim$ induced from $F$ on $\prod_{\lambda \in \Lambda} S_\lambda$. For $\alpha = (\alpha_\lambda)$, $\beta = (\beta_\lambda)$ ($\lambda \in \Lambda$),

$$\alpha \sim \beta \iff \{ \lambda \in \Lambda \mid \alpha_\lambda = \beta_\lambda \} \in F.$$
The set of equivalence classes is called ultraproduct of $S_\lambda$ for $F$ with respect to $\sim$. If $S_\lambda = S$ for $\lambda \in \Lambda$, then it is called ultraproduct of $S$ for $F$ and it is written as $^*S$. The set $S$ is naturally embedded in $^*S$ by the following mapping:

$$s \in S \mapsto [(s_\lambda = s), \lambda \in \Lambda] \in ^*S,$$

where $[ \ ]$ denotes the equivalence class with respect to the ultrafilter $F$. We write the mapping as $*$, and call it naturally elementary embedding. From now on, we identify the image $^*(S)$ as $S$.

Let $H \in ^*\mathbb{Z}$ be an infinite even number. The infinite number $H$ is even, when for $H = \{(H_\lambda), \lambda \in \Lambda\}, \{(\lambda \in \Lambda| H_\lambda \text{ is even}\} \in F$. We denote $\frac{1}{\Pi}$ by $\varepsilon$. We define an infinitesimal lattice space $L$, an infinitesimal lattice subspace $L$ and a space of functions $R(L)$ on $L$ as follows:

$$L := \varepsilon \, ^* \mathbb{Z} = \{ \varepsilon z | z \in ^* \mathbb{Z}\},$$

$$L := \{ \varepsilon z | z \in ^* \mathbb{Z}, -\frac{H}{2} \leq \varepsilon z < \frac{H}{2}\} \subset L,$$

$$R(L) := \{ \varphi | \varphi \text{ is an internal function from } L \text{ to } ^*\mathbb{C}\}.$$

We extend $R(L)$ to the space of periodic functions on $L$ with period $H$. We write the same notation $R(L)$ for the space of periodic functions.

Gaishi Takeuchi([T]) introduced an infinitesimal $\delta$ function. Furthermore Moto-o Kinoshita ([Kij]) constructed an infinitesimal Fourier transformation theory on $R(L)$.

We explain it briefly. For $\varphi, \psi \in R(L)$, the infinitesimal $\delta$ function, the infinitesimal Fourier transformation $F\varphi \in R(L)$, the inverse infinitesimal Fourier transformation $\overline{F}\varphi \in R(L)$ and the convolution $\varphi * \psi \in R(L)$ are defined as follows:

$$\delta \in R(L), \quad \delta(x) := \begin{cases} H & (x = 0), \\ 0 & (x \neq 0), \end{cases}$$

$$(F\varphi)(p) := \sum_{x \in L} \varepsilon \exp(-2\pi ipx) \varphi(x),$$

$$(\overline{F}\varphi)(p) := \sum_{x \in L} \varepsilon \exp(2\pi ipx) \varphi(x),$$

$$(\varphi * \psi)(x) := \sum_{y \in L} \varepsilon \varphi(x - y) \psi(y).$$

1-2. Formulation of infinitesimal Fourier transformation on the space of functionals (cf. [N-O1],[N-O2])

To treat a $^*$-unbounded functional $f$ in the nonstandard analysis, we need a second nonstandardization. Let $F_2 := F$ be a nonprincipal ultrafilter on an infinite set $\Lambda_2 := \Lambda$ as above. Denote the ultraproduct of a set $S$ with respect to $F_2$ by $^*S$ as above. Let $F_1$ be another nonprincipal ultrafilter on an infinite set $\Lambda_1$. Take the $^*$-ultrafilter $^*F_1$ on $^*\Lambda_1$. For an internal set $S$ in the sense of $^*$-nonstandardization, let $^*S$ be the $^*$-ultraproduct of $S$ with respect to $^*F_1$. Thus, we define a double ultraproduct $^*(^*R), ^*(^*Z)$, etc for the set $R, Z$, etc. It is shown easily that

$$^*(^*S) = S^{\Lambda_1 \times \Lambda_2}/F_1^{F_2},$$

where $F_1^{F_2}$ denotes the ultrafilter on $\Lambda_1 \times \Lambda_2$ such that for any $A \subset \Lambda_1 \times \Lambda_2$, $A \in F_1^{F_2}$ if and only if
\{ \lambda \in \Lambda_1 \mid \{ \mu \in \Lambda_2 \mid (\lambda, \mu) \in A \} \in F_2 \} \in F_1.

We always work with this double nonstandardization. The natural embedding \(*S\) of an internal element \(S\) which is not considered as a set in \(*\)-nonstandardization is often denoted simply by \(S\).

An infinite number in \(*\mathbb{R}\) is defined to be greater than any element in \(*\mathbb{R}\). We remark that an infinite number in \(*\mathbb{R}\) is not infinite in \(*\mathbb{R}\), that is, the word "an infinite number in \(*\mathbb{R}\)" has a double meaning. An infinitesimal number in \(*\mathbb{R}\) is also defined to be nonzero and whose absolute value is less than each positive number in \(*\mathbb{R}\).

**Definition 1.1.** Let \(H(\in \mathbb{Z}), H'(\in \mathcal{Z})\) be even positive numbers such that \(H'\) is larger than any element in \(*\mathbb{Z}\), and let \(\varepsilon(\in \mathbb{Z}), \varepsilon'(\in \mathcal{Z})\) be infinitesimals satisfying \(\varepsilon H = 1, \varepsilon' H' = 1\). We define as follows:

\[
\begin{align*}
L & := \varepsilon \mathbb{Z} = \{ \varepsilon z \mid z \in \mathbb{Z} \}, \\
L' & := \varepsilon' \mathcal{Z} = \{ \varepsilon' z' \mid z' \in \mathcal{Z} \}, \\
L & := \{ \varepsilon z \mid z \in \mathbb{Z}, -\frac{H}{2} \leq \varepsilon z < \frac{H}{2} \} (\subset L), \\
L' & := \{ \varepsilon' z' \mid z' \in \mathcal{Z}, -\frac{H'}{2} \leq \varepsilon' z' < \frac{H'}{2} \} (\subset L').
\end{align*}
\]

We define a lattice space of functions \(X\) as follows,

\[X := \{ a \mid a \text{ is an internal function with a double meaning, from } \ast (L) \text{ to } L' \}\]

We define three equivalence relations \(\sim_H, \sim_{\ast(H)}\) and \(\sim_{H'}\) on \(L, \ast(L)\) and \(L'\):

\[
x \sim_H y \iff x - y \in H \ast \mathbb{Z}, \quad x \sim_{\ast(H)} y \iff x - y \in \ast(H) \ast \mathcal{Z}, \\
x \sim_{H'} y \iff x - y \in H' \ast \mathcal{Z}.
\]

Then we identify \(L/ \sim_H, \ast(L)/ \sim_{\ast(H)}\) and \(L'/ \sim_{H'}\) as \(L, \ast(L)\) and \(L'\). Since \(\ast(L)\) is identified with \(L\), the set \(\ast(L)/ \sim_{\ast(H)}\) is identified with \(L/ \sim_H\). Furthermore we represent \(X\) as the following internal set:

\[\{ a \mid a \text{ is an internal function with a double meaning, from } \ast(L)/ \sim_{\ast(H)} \text{ to } L'/ \sim_{H'} \}.\]

We use the same notation as a function from \(\ast(L)\) to \(L'\) to represent a function in the above internal set. We define the space \(A\) of functionals as follows:

\[A := \{ f \mid f \text{ is an internal function with a double meaning, from } X \text{ to } \mathbb{C} \}.\]

We define an infinitesimal delta function \(\delta(a)(\in A)\), an infinitesimal Fourier transformation of \(f(\in A)\), an inverse infinitesimal Fourier transformation of \(f\) and a convolution of \(f, g(\in A)\), by the following:

**Definition 2.**

\[
\delta(a) := \begin{cases} (H')^{(\ast H)^2} & (a = 0), \\ 0 & (a \neq 0), \end{cases}
\]

\[
\varepsilon_0 := (H')^{-(\ast H)^2} \in \ast(\mathbb{R}),
\]

\[
(F f)(b) := \sum_{a \in X} \varepsilon_0 \exp \left( -2\pi i \sum_{k \in L} a(k)b(k) \right) f(a),
\]

\[
(\overline{F} f)(b) := \sum_{a \in X} \varepsilon_0 \exp \left( 2\pi i \sum_{k \in L} a(k)b(k) \right) f(a),
\]

\[
(f * g)(a) := \sum_{a' \in X} \varepsilon_0 f(a - a')g(a').
\]

We define an inner product on \(A\):
functions

by the following, we shall define another type of infinitesimal Fourier transforma-
tion. The different point is only the definition of an inner product of the space
of functions \(X\). In Definition 1.2, the inner product of \(a, b(\in X)\) is \(\sum_{k \in L} a(k)b(k)\),
and in the following definition, it is \(*\varepsilon \sum_{k \in L} a(k)b(k)\).

**Definition 1.3.**

\[ L' := \{ \varepsilon' z' \mid z' \in *\{*X\}, -\frac{H'H}{2} \leq \varepsilon' z' < \frac{H'H}{2} \}, \]

\[ \delta(a) := \begin{cases} \left(\frac{H'}{H}\right)^{\frac{H'H}{2}} (H'\cdot H^2) & (a = 0), \\ 0 & (a \neq 0), \end{cases} \]

\[ \varepsilon_0 := \left(\frac{H'}{H}\right)^{\frac{H'H}{2}} (H'-\cdot H^2) \]

\[ (Ff)(b) := \sum_{a \in X} \varepsilon_0 \exp \left(-2\pi i z \sum_{k \in L} a(k)b(k)\right) f(a), \]

\[ (\mathcal{F}f)(b) := \sum_{a \in X} \varepsilon_0 \exp \left(2\pi i z \sum_{k \in L} a(k)b(k)\right) f(a). \]

Then we obtain the following theorem:

**Theorem 1.4 ([N-O2]).** If \(l \in \mathbb{R}^+\), then \(F^l = (H')^{(l-1)(*H)^2} \).

If there exists \(a, b \in L^2(\mathbb{R})\) so that \(a = \alpha |_{\mathbb{R}}, b = \beta |_{\mathbb{R}}, \) that is, \(a(k) = *\alpha(k), b(k) = *\beta(k)\), then \(\text{st}(\varepsilon \sum_{k \in L} a(k)b(k)) = \int_{-\infty}^{\infty} a(x)b(x)dx\). Definition 1.3 is easier understanding than Definition 1.2 for a standard meaning. For the reason, we consider mainly Definition 1.3 about several examples.

2. **Examples of the infinitesimal Fourier transformation on the space of functions**

We calculate the infinitesimal Fourier transformations of \(\varphi_\xi, \varphi_{im} \in R(L)\):

1. \(\varphi_\xi(x) = \exp(-\xi \pi x^2), \) where \(\xi \in \mathbb{C}, \text{Re}(\xi) > 0, \)
2. \(\varphi_{im}(x) = \exp(-im\pi x^2), \) where \(m \in \mathbb{Z}. \)

For \(\varphi_\xi, \) we obtain:

**Proposition 2.1.**

\[(F\varphi_\xi)(p) = c_\xi(p) \varphi_\xi(\frac{p}{\xi}), \) where \(c_\xi(p) = \sum_{x \in L} \varepsilon \exp(-\xi \pi (x + \frac{p}{\xi})^2) \).

If \(p\) is finite, then \(\text{st}(c_\xi(p)) = \frac{1}{\sqrt{\xi}}.\)

**Proof.** The infinitesimal Fourier transformations of \(\varphi_\xi\) is:

\[(F\varphi_\xi)(p) = \sum_{x \in L} \varepsilon \exp(-2\pi ipx) \exp(-\xi \pi x^2) \]
\[= \sum_{x \in L} \varepsilon \exp(-\xi \pi (x + \frac{p}{\xi})^2 - \pi \frac{1}{\xi} p^2) \]
\[
= \left( \sum_{x \in L} \varepsilon \exp \left( -\xi \pi (x + \frac{i}{\xi} p) \right) \right) \exp \left( -\pi \frac{1}{\xi} p^2 \right) = c_\xi(p) \varphi_\xi \left( \frac{p}{\xi} \right),
\]
where \( c_\xi(p) = \sum_{x \in L} \varepsilon \exp \left( -\xi \pi (x + \frac{i}{\xi} p) \right) \). If \( p \) is finite, then \( \text{st}(c_\xi(p)) \)
\[
= \int_{-\infty}^{\infty} \exp \left( -\xi \pi \left( t + \frac{i}{\xi} \text{st}(p) \right)^2 \right) \, dt = \frac{1}{\sqrt{\xi}}.
\]
Using Theorem 1.4(8), we obtain for \( c_\xi \):

**Proposition 2.2.** \( \varphi_\xi(x') = \left( \mathcal{F}c_\xi(p) \ast \left( c_\frac{1}{\xi}(-x)\varphi_\xi(x) \right) \right)(x') \).

**Proof.** We obtain : \( (F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi \left( \frac{p}{\xi} \right) \), and put \( \mathcal{F} \) to the above :
\[
(\mathcal{F}(F\varphi_\xi))(x) = \left( \mathcal{F}(c_\xi(p)\varphi_\xi \left( \frac{p}{\xi} \right)) \right)(x)
\]
Now \( (\mathcal{F}\varphi_\xi \left( \frac{p}{\xi} \right))(x) = \sum_{p \in L} \varepsilon \exp \left( -\frac{\pi}{\xi} \left( p^2 - 2\pi i x \right) \right) \exp \left( -\frac{\xi}{\pi} \left( p - i x \right)^2 \right) \), that is, \( \varphi_\xi(x) = \left( \mathcal{F}c_\xi(p) \ast \mathcal{F}\varphi_\xi \left( \frac{p}{\xi} \right) \right)(x) \).

By the definition : \( c_\xi(p) = \sum_{x \in L} \varepsilon \exp \left( -\xi \pi (x + \frac{i}{\xi} p) \right) \), the summation \( \sum_{x \in L} \varepsilon \exp \left( -\frac{\pi}{\xi} \left( p - i x \right)^2 \right) \) is \( c_\xi(x) \). Hence \( \varphi_\xi(x') = \left( \mathcal{F}c_\xi(p) \ast \left( c_\frac{1}{\xi}(-x)\varphi_\xi(x) \right) \right)(x') \).

For the following proposition 2.3, we recall the Gauss sum(cf.[R]) :

For \( z \in \mathbb{N} \), Gauss sum \( \sum_{l=0}^{z-1} \exp \left( -\frac{2\pi i z}{x} \right) \) is equal to \( \sqrt{z} \frac{1+(-i)^{z}}{1+i} \).

**Proposition 2.3.** If \( m \mid 2H^2 \) and \( m \perp_{\xi} \), then \( (F\varphi_{im})(p) = c_{im}(p) \exp(i\pi \frac{1+(-i)^{2H^2}}{1+i}) \),
where \( c_{im}(p) = \sqrt{\frac{m(2+(-i)^{2H^2})}{1+i}} \) for positive \( m \) and \( c_{im}(p) = \sqrt{-\frac{m(2+(-i)^{2H^2})}{1+i}} \) for negative \( m \).

**Proof.** \( (F\varphi_{im})(p) = \sum_{x \in L} \varepsilon \exp \left( -i m \pi x^2 \right) \exp \left( -2\pi i x p \right) \)
\[
= \sum_{x \in L} \varepsilon \exp \left( -i m \pi \left( x + \frac{\xi}{m} p \right)^2 \right) \exp \left( i\pi \frac{1}{m} p^2 \right),
\]
where \( c_{im}(p) = \sum_{x \in L} \varepsilon \exp \left( -i m \pi \left( x + \frac{\xi}{m} \right)^2 \right) \).

Since \( m \perp_{\xi} \), the element \( \frac{\xi}{m} \) is in \( L \). We remark that \( \exp \left( -i \pi m x^2 \right) = \exp \left( -i \pi m (x + H^2) \right) \). For positive \( m \),
\[
c_{im}(p) = \sum_{x \in L} \varepsilon \exp \left( -i m \pi x^2 \right) = \sum_{0 \leq n < H^2} \varepsilon \exp \left( -i 2\pi \frac{m}{2H^2} n^2 \right)
\]
\[
= \frac{m}{H^2} \sum_{0 \leq n < H^2} \varepsilon \exp \left( -i 2\pi \frac{m}{2H^2} n^2 \right) = \frac{m}{H^2} \left( \varepsilon \sqrt{\frac{2H^2}{m} \frac{1+(-i)^{2H^2}}{1+i}} \right),
\]
by the above Gauss sum. Hence \( c_{im}(p) = \sqrt{\frac{m(2+(-i)^{2H^2})}{1+i}} \). For negative \( m \), the proof is as same as the above.

3. Examples of the infinitesimal Fourier transformation for the space of functionals

We define an equivalence relation \( \sim_{HH'} \) in \( L' \) by \( x \sim_{HH'} y \Leftrightarrow x - y \in *HH' \ast \{*Z\} \). We identify \( L'/\sim_{HH'} \) with \( L' \). Let
\[
X_{HH'} := \{a' \mid a' \text{ is an internal function with a double meaning, from } *\ast(L)/\sim_{HH'} \to L'/\sim_{*HH'} \},
\]
and let $\textbf{e}$ be a mapping from $X$ to $X_H, \bullet_{HH'}$, defined by $(\textbf{e}(a)([k]) = [a(k)])$, where $[\ ]$ in left hand side represents the equivalence class for the equivalence relation $\sim_{\bullet}$ in $\bullet(L)$, $k$ is a representative in $\bullet(L)$ satisfying $k \sim_{\bullet} k'$, and $[\ ]$ in right hand side represents the equivalence class for the equivalence relation $\sim_{HH'}$ in $L'$. Furthermore let $\textbf{e}^*(f)(a')$ be defined by $f(\textbf{e}^{-1}(a'))$.

3-1. The infinitesimal Fourier transformation of $g_\xi(a) = \exp(-\pi^* \varepsilon \xi \sum_{k \in L} a^2(k))$ with $\xi \in \mathbb{C}$, $\Re(\xi) > 0$

We calculate the infinitesimal Fourier transformation of $g_\xi(a) = \exp(-\pi^* \varepsilon \xi \sum_{k \in L} a^2(k))$, where $\xi \in \mathbb{C}$, $\Re(\xi) > 0$,
in the space $A$ of functionals, for Definition 1.3. We identify $*(\xi) \in \mathbb{C}$ with $\xi \in \mathbb{C}$.

**Theorem 3.1.** $(F(\textbf{e}^*(g_\xi)))(b) = C_\xi(b)g_\xi(b)$, where $b \in X$ and

$$C_\xi(b) = \sum_{a \in X} \varepsilon_0 \exp\left(-\pi^* \varepsilon \xi \sum_{k \in L} (a(k) + i\frac{1}{\xi} b(k))^2\right).$$

**Proof.** We do the infinitesimal Fourier transformation of $\textbf{e}^*(g_\xi)(a)$.

$$(F(\textbf{e}^*(g_\xi)))(b) = F\left(\exp(-\pi^* \varepsilon \xi \sum_{k \in L} a^2(k))\right)(b)$$

$$= \sum_{a \in X} \varepsilon_0 \exp\left(-2i\pi^* \varepsilon \xi \sum_{k \in L} a(k)b(k)\right) \exp\left(-\pi^* \varepsilon \xi \sum_{k \in L} a^2(k)\right)$$

$$= \sum_{a \in X} \varepsilon_0 \exp\left(-\pi^* \varepsilon \xi \sum_{k \in L} a^2(k) + 2i\frac{1}{\xi} \varepsilon \xi \sum_{k \in L} a(k)b(k)\right)$$

$$= \sum_{a \in X} \varepsilon_0 \exp\left(-\pi^* \varepsilon \xi \sum_{k \in L} \left(a^2(k) + 2i\frac{1}{\xi} \varepsilon \xi a(k)b(k) + \frac{1}{\xi^2} b^2(k)\right)\right)$$

$$= \left(\sum_{a \in X} \varepsilon_0 \exp\left(-\pi^* \varepsilon \xi \sum_{k \in L} a(k) + i\frac{1}{\xi} b(k)\right)\right) \exp\left(-\pi^* \varepsilon \xi \sum_{k \in L} b^2(k)\right)$$

$$= C_\xi(b)g_\xi(b).$$

Let $\star \circ \star : \mathbb{R} \rightarrow *(\mathbb{R})$ be the natural elementary embedding and let $\textbf{st}(c)$ for $c \in *(\mathbb{R})$ be the standard part of $c$ with respect to the natural elementary embedding $\star \circ \star$.

**Theorem 3.2.** If the image of $b \in X$ is bounded by a finite value of $*(\mathbb{R})$, that is, $\exists b_0 \in *(\mathbb{R})$ s.t. $k \in L \Rightarrow |b(k)| \leq *(b_0)$, then

$$\textbf{st}(C_\xi(b)) = \left(\star \left(\frac{1}{\sqrt{\xi}}\right)^{H^2}\right) \in *(\mathbb{R}) \quad \text{and} \quad \textbf{st}\left(\frac{C_\xi(b)}{\star\left(\left(\star(\frac{1}{\sqrt{\xi}})^{H^2}\right)\right)^{-1}}\right) = 1.$$
We assume that the image of $b \in X$ is bounded by a finite value of $^* \mathbf{R}$, that is, $\exists b_0 \in \ast \mathbf{R}$ s.t. $k \in L \Rightarrow |b(k)| \leq \ast(b_0)$. The $\lambda \mu$-component of
\[
\frac{\sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2) - \frac{1}{\sqrt{\mu^2/2}} \exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2)}{\sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2)}
\]
equals \ast \left(\sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2)\right)

We write $\sqrt{\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2}$ as $a_\lambda$, $b_\lambda$ for simplicity, and
\[
B_{\lambda \mu}(k_\mu) := \sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2)\]

It is equal to
\[
- \int_{-\infty}^{\infty} \exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2) dx + \int_{-\infty}^{\infty} \exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2) dx,
\]

Then the above is equal to
\[
\prod_{k_\mu \in L \mu} \frac{\sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2)}{\sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2)} \ast \left(\sum_{a \in X} \exp(-\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2)\right)
\]

We show that $[(B_{\lambda \mu}(k_\mu))]$ is infinitesimal in $^* \mathbf{C}$ with respect to $^* \mathbf{C}$. It implies that $\left(\frac{1}{B_{\lambda \mu}(k_\mu)}\right)$ is infinite in $^* \mathbf{C}$. Since $b_\mu$ is finite and $\left(\sqrt{\pi \xi b_\mu^2}\right)$ is infinitesimal in $^* \mathbf{R}$ with respect to $^* \mathbf{R}$, the first and second terms of $(\ast)$, that is,
\[
\left[\int_{-\infty}^{\infty} \exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2) dx\right] \text{ and } \left[\int_{-\infty}^{\infty} \exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2) dx\right]
\]
is infinitesimal in $^* \mathbf{C}$ with respect to $^* \mathbf{C}$. In order to show that $[(B_{\lambda \mu}(k_\mu))]$ is infinitesimal in $^* \mathbf{C}$, we consider the third and fourth terms in $(\ast)$, and we prove that it represents an infinitesimal number.

First we calculate
\[
\exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2) - \exp(-\pi \xi (\sqrt{\pi \xi a \int_k (a(k) + i \frac{1}{\xi} b(k))^2})^2).
\]

Since
\[
\exp(-\pi \xi (x + i \frac{1}{\xi} b_\lambda)^2)
\]
\[
= \exp(-\pi (\alpha x^2 - \frac{\beta^2}{\alpha + \beta}) \exp(-i \pi (\beta x^2 + 2 b_\lambda x + \frac{\beta^2}{\alpha + \beta}))
\]
\[
= \exp(-\pi (\alpha x^2 - \frac{\beta^2}{\alpha + \beta}) \cos(\pi \beta x^2 + 2 b_\lambda x + \frac{\beta^2}{\alpha + \beta})
\]
\[
- i \exp(-\pi (\alpha x^2 - \frac{\beta^2}{\alpha + \beta})) \sin(\pi \beta x^2 + 2 b_\lambda x + \frac{\beta^2}{\alpha + \beta}),
\]
the above is
exp(−πξ(x + \frac{i}{\xi}b_{\lambda\mu})) - exp(−πξ(a_{\lambda\mu} + \frac{i}{\xi}b_{\lambda\mu}))
= \{\exp(−\pi(\alpha x^2 - \frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\cos(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))
- \exp(−\pi(\alpha a_{\lambda\mu}^2 - \frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\cos(\pi(\beta a_{\lambda\mu}^2 + 2b_{\lambda\mu}a_{\lambda\mu} + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\}\n- i\{\exp(−\pi(\alpha x^2 - \frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\sin(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))
- \exp(−\pi(\alpha a_{\lambda\mu}^2 - \frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\sin(\pi(\beta a_{\lambda\mu}^2 + 2b_{\lambda\mu}a_{\lambda\mu} + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\}\. \cdots (*)

We consider the first term of (\ast 3). Then
\exp(−\pi(\alpha x^2 - \frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))\cos(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))
is equal to
\exp(\pi \frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2})\exp(−\pi \alpha x^2)\cos(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2})).

We put
f(x) = \exp(−\pi \alpha x^2)\cos(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2})).

We assume that 0 ≤ b_{\lambda\mu}.

f′(x) = −2\pi \alpha x \exp(−\pi \alpha x^2)\cos(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))
- \exp(−\pi \alpha x^2)(2\beta x + 2\pi b_{\lambda\mu})\sin(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}))
= −2\pi \sqrt{(\alpha x)^2 + (\beta x + b_{\lambda\mu})^2}\exp(−\pi \alpha x^2)\cos(\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2})) + \alpha x),

where
\cos \alpha_x = \frac{\alpha x}{\sqrt{(\alpha x)^2 + (\beta x + b_{\lambda\mu})^2}}, \quad -\sin \alpha_x = \frac{\beta x + b_{\lambda\mu}}{\sqrt{(\alpha x)^2 + (\beta x + b_{\lambda\mu})^2}}.

Since 0 < \alpha, if 0 ≤ \beta and 0 ≤ x, then 0 ≤ \cos \alpha_x and \sin \alpha_x ≤ 0. Hence
−\frac{\pi}{2} ≤ \alpha_x < 0. There is a unique maximum of |f(x)| in
\{x \in \mathbb{R} \mid \frac{\pi}{2}(2m - 1) ≤ \pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}) < \frac{\pi}{2}(2m + 1)\}
for each m ∈ \mathbb{Z} (-1 ≤ m), that is, x satisfies
f′(x) = 0, \quad \frac{\pi}{2}(2m - 1) ≤ \pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}) < \frac{\pi}{2}(2m + 1)
\iff \pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}) + \alpha_x = \frac{\pi}{2}(2m - 1). \quad \cdots (\ast 4)

We write the value of x having the maximum of |f(x)| in the interval as (A_{2m})_{\lambda\mu}.
On the other hand, we denote the value \alpha_x at x = (A_{2m})_{\lambda\mu} by \alpha_{A_{2m}}. Then
(A_{2m})_{\lambda\mu} = \frac{-b_{\lambda\mu} + \sqrt{\frac{a b_{\lambda\mu}^2}{\alpha^2 + \beta^2} + \frac{\alpha^2}{\alpha^2 + \beta^2}(2m - 1) - \alpha a_{\lambda\mu}}}{\beta}.

The maximum of f(x) is f((A_{2m})_{\lambda\mu}) = \exp(−\pi \alpha (A_{2m})_{\lambda\mu}^2)\cos(\pi m - \frac{\pi}{2} - \alpha_{A_{2m}}).\)
Since \lim_{m \to \infty} (A_{2m})_{\lambda\mu}/\sqrt{\frac{\beta(2m - 1)}{\beta + \beta^2}} = 1, there exists m such that \sqrt{\frac{2m - 1}{\beta \beta + \beta^2}} < (A_{2m})_{\lambda\mu}. Hence there exists m such that
|f((A_{2m})_{\lambda\mu})| ≤ \exp(−\pi (A_{2m})_{\lambda\mu}^2) ≤ \exp(−\pi \frac{2m - 1}{\beta \beta + \beta^2}).

We denote the value of x at f(x) = 0, that is,
\pi(\beta x^2 + 2b_{\lambda\mu}x + \frac{\beta b_{\lambda\mu}^2}{\alpha^2 + \beta^2}) = \frac{\pi}{2}(2m + 1)
by \((A_{2m+1})_{\lambda \mu}\). Then \((A_{2m+1})_{\lambda \mu} = \frac{-b_{\lambda \mu} + \sqrt{\frac{\alpha^2 b_{\lambda \mu}^*}{\alpha^2 + \beta^2} + \frac{\beta}{2}(2m-1)}}{\beta}\). We consider
\[
\left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(-\pi(\sqrt{\varepsilon_{\lambda \mu}^a} z_{\lambda \mu}^a)^2 - (\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a)^2)) \cos(2\pi \varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a z_{\lambda \mu}^a) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} \exp(-\pi(x^2 - b_{\lambda \mu}^2)) \cos(2\pi b_{\lambda \mu} x) dx \right|.
\]
It is equal to
\[
\exp(\pi b_{\lambda \mu}^2) \left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(-\pi(\sqrt{\varepsilon_{\lambda \mu}^a} z_{\lambda \mu}^a)^2) \cos(2\pi \varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a z_{\lambda \mu}^a) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} f(x) dx \right|
= \exp(\pi b_{\lambda \mu}^2) \left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(\varepsilon_{\lambda \mu}^a f(\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a)^2) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} f(x) dx \right|.
\]
We show that the following term is infinitesimal in \(*R\) with respect to \(*R\) :
\[
\left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(\varepsilon_{\lambda \mu}^a f(\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a)^2) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} f(x) dx \right|.
\]
Now
\[
(A_{2(m+1)+1})_{\lambda \mu} - (A_{2m+1})_{\lambda \mu}
= -b_{\lambda \mu} + \frac{\sqrt{\frac{\alpha^2 b_{\lambda \mu}^*}{\alpha^2 + \beta^2} \frac{\beta}{2}(2m+1)}}{\beta} - b_{\lambda \mu} + \frac{\sqrt{\frac{\alpha^2 b_{\lambda \mu}^*}{\alpha^2 + \beta^2} + \frac{\beta}{2}(2m+1)}}{\beta}
= \frac{\sqrt{\frac{\alpha^2 b_{\lambda \mu}^*}{\alpha^2 + \beta^2} + \frac{\beta}{2}(2m+1)}}{\beta}
\]
\[
\left( \frac{\alpha^2 b_{\lambda \mu}^*}{\alpha^2 + \beta^2} + \frac{\beta}{2}(2m+1) \right) \cdot \left( \frac{\alpha^2 b_{\lambda \mu}^*}{\alpha^2 + \beta^2} + \frac{\beta}{2}(2m+1) \right) \cdots (\ast_5).
\]
Since
\[-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*} \leq x \leq \sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*} \cdots (\ast_6)\]
and the image of \(b \in X\) is bounded by a finite value of \(*R\), that is, \(\exists b_0 \in *R\) s.t. \(k \in L \Rightarrow |b(k)| \leq *b_0\), the above \((\ast_5)\) is greater than the following value : \(\frac{1}{4\sqrt{|b_0|^2 + \frac{\beta}{2}(2m+1)}}\). The value \(\pi(2m+1)\) satisfies \((\ast_6)\), \(\pi(2m+1) \leq \sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}\), that is, \(2m+1 \leq \frac{1}{4\sqrt{|b_0|^2 + \frac{\beta}{2}(2m+1)}}\), and \(\frac{1}{4\sqrt{|b_0|^2 + \frac{\beta}{2}(2m+1)}} \geq \sqrt{\varepsilon_{\lambda \mu}^a} \cdot \sqrt{\varepsilon_{\lambda \mu}^a}\).

Furthermore \(\frac{1}{4\sqrt{|b_0|^2 + \frac{\beta}{2}(2m+1)}} \geq \frac{1}{4\sqrt{|b_0|^2 + \frac{\beta}{2}(2m+1)}} \geq \sqrt{\varepsilon_{\lambda \mu}^a}\).

Hence
\[
\{ \lambda | \{ \mu \mid (A_{2(m+1)+1})_{\lambda \mu} - (A_{2m+1})_{\lambda \mu} > \varepsilon_{\lambda \mu}^a \} \in F_2 \} \in F_1 .
\]
We consider the \(\lambda \mu\)-component satisfying the above. Now
\[
\left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(\varepsilon_{\lambda \mu}^a f(\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a)^2) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} f(x) dx \right|
\leq \left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(\varepsilon_{\lambda \mu}^a f(\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a)^2) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} f(x) dx \right|
+ \left| \sum_{\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a \in L_{\lambda \mu}} \sqrt{\varepsilon_{\lambda \mu}^a} \exp(\varepsilon_{\lambda \mu}^a f(\varepsilon_{\lambda \mu}^a z_{\lambda \mu}^a)^2) - f^{\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}}_{-\sqrt{\varepsilon_{\mu} H_{\lambda \mu}^*}} f(x) dx \right|.
\]
We consider the difference
\[
\sum_{\varepsilon \in \mathcal{X}_\mu} \varepsilon \mathcal{A}_\mu \in \mathcal{L}_\mu, z_{\varepsilon \mu} \geq 0 \sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu f(\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu z_{\varepsilon \mu})^2) - \int_0^{\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu} f(x)dx.
\]
We devide the interval \((-\infty, \infty)\) into a sum of intervals where the function \(f\) is monoton increasing or monoton decreasing. The absolute value of the difference is bounded to the sum of the absolute values of the difference whose integral areas are restricted to these intervals. Each difference is bounde d to the product \(\varepsilon \varepsilon\) of \(\mathcal{A}_\mu\). Hence
\[
\left| \sum_{\varepsilon \in \mathcal{X}_\mu} \varepsilon \mathcal{A}_\mu \in \mathcal{L}_\mu, z_{\varepsilon \mu} \geq 0 \sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu f(\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu z_{\varepsilon \mu})^2) - \int_0^{\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu} f(x)dx \right|
\leq \sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu 2 \sum_{m=0}^{\infty} |f(A_m)|
\leq \sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu 2 \sum_{m=0}^{\infty} \exp\left(-\pi \left(\frac{2m-1}{4\beta}\right)\right).
\]
Since the value \(2 \sum_{m=0}^{\infty} \exp\left(-\pi \left(\frac{2m-1}{4\beta}\right)\right)\) is finite, the following value
\[
\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu 2 \sum_{m=0}^{\infty} \exp\left(-\pi \left(\frac{2m-1}{4\beta}\right)\right)
\]
is infinitesimal in \(* (\mathbb{R})\) with respect to \(* \mathbb{R}\).

The same argument implies in the case \(x < 0\)
\[
\left| \sum_{\varepsilon \in \mathcal{X}_\mu} \varepsilon \mathcal{A}_\mu \in \mathcal{L}_\mu, z_{\varepsilon \mu} \leq 0 \sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu f(\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu z_{\varepsilon \mu})^2) - \int_0^{\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu} f(x)dx \right|
\]
is infinitesimal in \(* (\mathbb{R})\) with respect to \(* \mathbb{R}\) also.

Hence
\[
\left| \sum_{\varepsilon \in \mathcal{X}_\mu} \varepsilon \mathcal{A}_\mu \in \mathcal{L}_\mu \sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu f(\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu z_{\varepsilon \mu})^2) - \int_0^{\sqrt{\varepsilon \varepsilon} \mathcal{L}_\mu} f(x)dx \right|
\]
is infinitesimal in \(* (\mathbb{R})\) with respect to \(* \mathbb{R}\).

If \(b_{\lambda \mu} < 0\), the argument is parallel, and also, for the term of sin in \((*)_3\), though sin is not an even function, the same argument holds. Hence \([B_{\lambda \mu}(k)]\) is infinitesimal in \(* (\mathbb{C})\) with respect to \(* \mathbb{C}\). Hence
\[
\text{st} \left( \sum_{a \in \mathcal{H}_\mu} \varepsilon_0 \exp\left(-\pi \varepsilon_0 \sum_{a \in \mathcal{L}} (a(k)+i\beta b(k))^2\right) \right)
\]
\[
= \prod_{a \mu \in \mathcal{A}_a} \left( 1 + \frac{B_{\lambda \mu}(k)}{\int_{-\infty}^{\infty} \exp(-\pi \xi x^2)dx} \right)
\]
\[
= \prod_{a \mu \in \mathcal{A}_a} \left( 1 + \frac{B_{\lambda \mu}(k)}{\int_{-\infty}^{\infty} \exp(-\pi \xi x^2)dx} \right)
\]
\[
= 1.
\]
Since \(\int_{-\infty}^{\infty} \exp(-\pi \xi x^2)dx = \frac{1}{\sqrt{\xi}},\) then
\[
\text{st} \left( \sum_{a \in \mathcal{H}_\mu} \varepsilon_0 \exp\left(-\pi \varepsilon_0 \sum_{a \in \mathcal{L}} (a(k)+i\beta b(k))^2\right) \right)
\]
\[
= 1, \text{ that is, } \text{st} \left( \frac{C_{\xi}(b)}{\left(\mathcal{H}(\mathcal{X})\right)} \right) = 1.
\]
Furthermore
\[
\text{st} \left( \frac{C_{\xi}(b)}{\star \left( \sqrt{\frac{1}{\xi}} \right)^2} \right) = \text{st} \left( \text{st} \left( \frac{C_{\xi}(b)}{\star \left( \sqrt{\frac{1}{\xi}} \right)^2} \right) \right) = 1.
\]

The argument is same about the infinitesimal Fourier transformation of \( g_\xi'(a) = \exp(-\pi \xi \sum_{k \in L} a^2(k)) \), for Definition 1.2, as the above.

**Theorem 3.3.** \((F(e^t(g_\xi')))(b) = B_\xi(b)g_\xi'( \frac{t}{\xi})\), where \( b \in X \) and \( B_\xi(b) = \sum_{a \in X} \varepsilon_0 \exp \left( -\pi \xi \sum_{k \in L} (a(k) + \frac{1}{m} b(k))^2 \right) \). Furthermore, if the image of \( b \) (\( b \in X \)) is bounded by a finite value of \( ^* B \), that is, \( \exists \varepsilon_0 \in ^* B \) s.t. \( k \in L \Rightarrow |b(k)| \leq \varepsilon_0 b_0 \), then

\[
\text{st}(B_\xi(b)) = \left( \frac{1}{\sqrt{\xi}} \right)^2 \quad (\varepsilon_0 \in ^* B) \quad \text{and} \quad \text{st} \left( \frac{B_\xi(b)}{\star \left( \sqrt{\frac{1}{\xi}} \right)^2} \right) = 1.
\]

### 3-2. The infinitesimal Fourier transformation of \( g_{im} = \exp(-i \pi m \varepsilon \sum_{k \in L} a^2(k)) \) with \( m \in \mathbb{Z} \)

We calculate the infinitesimal Fourier transformation of

\[
g_{im}(a) = \exp(-i \pi m \varepsilon \sum_{k \in L} a^2(k)), \quad \text{where} \ m \in \mathbb{Z},
\]

for Definition 1.3.

**Proposition 3.4.** \((F(e^t(g_{im}')))(b) = C_{im}(b)g_{im}'(b)\).

If \( m \varepsilon \sum_{k \in L} a^2(k) \) and \( m \varepsilon \sum_{k \in L} b^2(k) \) for an arbitrary \( k \in L \), then \((F(e^t(g_{im}')))(b) = C_{im}(b)g_{im}'(b)\),

where \( C_{im}(b) = \left( \sqrt{\frac{m}{2} + \frac{1}{2} \sum_{k \in L} b^2(k)} \right) \) for a positive \( m \) and

\[
C_{im}(b) = \left( \sqrt{\frac{m}{2} - \frac{1}{2} \sum_{k \in L} b^2(k)} \right) \quad \text{for a negative} \ m.
\]

**Proof.** \((F(e^t(g_{im}')))(b) = \sum_{a \in X} \varepsilon_0 \exp(-2 \pi i \varepsilon \sum_{k \in L} a(k)b(k)) \exp(-i \pi m \varepsilon \sum_{k \in L} a^2(k)) = \sum_{a \in X} \varepsilon_0 \exp(-i \pi m \varepsilon \sum_{k \in L} a^2(k)) \exp(-i \pi m \varepsilon \sum_{k \in L} b^2(k)) = C_{im}(b)g_{im}'(b)\), where \( C_{im}(b) = \sum_{a \in X} \varepsilon_0 \exp(-i \pi m \varepsilon \sum_{k \in L} a^2(k)) \).

When we denote \( a(k), b(k) \) by \( \varepsilon' n', \varepsilon'' n' \),

\[
\sum_{-\pi m \varepsilon \geq a(k) < \pi m \varepsilon} \exp(-i \pi m \varepsilon \sum_{k \in L} (a(k) + \frac{1}{m} b(k))^2) = \sum_{-\pi m \varepsilon \geq a(k) < \pi m \varepsilon} \exp(-i \pi m \varepsilon \sum_{k \in L} (\varepsilon' n' + \varepsilon'' n')).
\]

Since \( m \varepsilon \sum_{k \in L} b^2(k) \), it is equal to

\[
\sum_{-\pi m \varepsilon \geq a(k) < \pi m \varepsilon} \exp(-i \pi m \varepsilon (\varepsilon' n')) = \sum_{-\pi m \varepsilon \geq a(k) < \pi m \varepsilon} \exp(-2 \pi i \frac{m}{2} \sum_{k \in L} b^2(k)) = \frac{m}{2} \sqrt{\frac{2 \pi m \varepsilon}{1+1+1} \sum_{k \in L} b^2(k)}.
\]

by Lemma 2.3. Hence \( C_{im} = \left( \sqrt{\frac{m}{2} + \frac{1}{2} \sum_{k \in L} b^2(k)} \right) \) for a positive \( m \). For a negative \( m \), the proof is as same as the above.

The argument is same about the infinitesimal Fourier transformation of
\[ g'_{im}(a) = \exp(-i\pi m \sum_{k \in L} a^2(k)) \], where \( m \in \mathbb{Z} \), for Definition 1.2, as the above.

**Proposition 3.5.** If \( m|2^*HH^2 \) and \( m|\frac{b(k)}{e} \) for an arbitrary \( k \) in \( L \), then
\[
(F(e^\#(g'_{im}))(b) = B_{im}(b)g_{im}^\dagger(b), \text{ where } B_{im}(b) = \left( \sqrt{\frac{m^2}{2} + \frac{2HH^2}{1-i}} \right)^{(\star)H^2} \text{ for a positive } m \text{ and } B_{im}(b) = \left( \sqrt{-\frac{m^2}{2} - \frac{2HH^2}{1+i}} \right)^{(\star)H^2} \text{ for a negative } m.
\]

4. Poisson summation formula for infinitesimal Fourier transformation by Kinoshita

We extend Poisson summation formula of finite group to Kinoshita’s infinitesimal Fourier transformation.

**4-1. Formulation**

**Theorem 4.1.** Let \( S \) be an internal subgroup of \( L \). Then we obtain, for \( \varphi \in R(L) \),
\[
|S^\perp|^{-\frac{1}{2}} \sum_{p \in S^\perp} (F\varphi)(p) = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi(x),
\]
where \( S^\perp := \{ p \in L \mid \exp(2\pi ipx) = 1 \text{ for } \forall x \in S \} \).

Since \( L \) is an internal cyclic group, \( S \) is also an internal cyclic group. The generator of \( L \) is \( \varepsilon \). The generator of \( S \) is written as \( \varepsilon s \) (\( s \in *\mathbb{Z} \)). Since the order of \( L \) is \( HH^2 \), so \( s \) is a factor of \( H^2 \).

We prepare the following lemma for the proof of Theorem 4.1.

**Lemma 4.2.** \( S^\perp = < \varepsilon \frac{H^2}{s} > \).

**Proof of Lemma 4.2.** For \( p \in S^\perp \), we write \( p = \varepsilon t \). Then we obtain the following :
\[
\exp(2\pi ip\varepsilon s) = 1 \iff \exp(2\pi i\varepsilon t\varepsilon s) = 1 \iff \exp(2\pi t\frac{s}{HH^2}) = 1 \iff t\frac{s}{HH^2} \in *\mathbb{Z}.
\]
Hence the generator of \( S^\perp \) is \( \varepsilon \frac{H^2}{s} \).

**Proof of Theorem 4.1.** By Lemma 4.2, \( |S| = \frac{H^2}{s} \) and \( |S^\perp| = s \). If \( x \notin S \), then \( \varepsilon \frac{H^2}{s} xs = \varepsilon H^2 x \in *\mathbb{Z} \), \( \left( \exp\left(2\pi is\frac{H^2}{s}x\right) \right)^s = 1. \) For \( x \in L, \)
\[
\sum_{p \in S^\perp} \exp(2\pi ipx) = \begin{cases} 
\frac{\exp(2\pi(\frac{H}{s})x)(1-(\exp(2\pi is\frac{H^2}{s}x))^s)}{1-\exp(2\pi is\frac{H^2}{s}x)} & (x \notin S) \\
\sum_{p \in S^\perp} 1 & (x \in S)
\end{cases}
\]
Hence
\[
\sum_{p \in S^\perp} (F\varphi)(p) = \sum_{p \in S^\perp} \varepsilon (\sum_{x \in L} \varphi(x) \exp(2\pi ipx)) = \varepsilon (\sum_{x \in L} \varphi(x)(\sum_{p \in S^\perp} \exp(2\pi ipx)))
\]
Thus
\[
\varepsilon \sum_{x \in \mathbb{Z}} \varphi(x)s = \frac{H}{s} \sum_{x \in \mathbb{Z}} \varphi(x).
\]
\[ \frac{1}{\sqrt{p}} \sum_{p \in S^\perp} (F \varphi)(p) = \frac{1}{\sqrt{|H|}} \sum_{x \in S} \varphi(x) = \sqrt{\frac{|H|}{p}} \sum_{x \in S} \varphi(x) \quad \cdots (1). \]

| \[ S^\perp |^{-\frac{1}{2}} \sum_{p \in S^\perp} (F \varphi)(p) = \frac{1}{|S|^{\frac{1}{2}}} \sum_{x \in S} \varphi(x). \]

**Proposition 4.3** Especially if \( s \) is equal to \( H \), then (1) implies that
\[ \sum_{p \in S^\perp} (F \varphi)(p) = \sum_{x \in S} \varphi(x). \]

The standard part of the above is
\[ \text{st}(\sum_{p \in S^\perp} (F \varphi)(p)) = \text{st}(\sum_{x \in S} \varphi(x)). \]

If there exists a standard function \( \varphi : \mathbb{R} \to \mathbb{C} \) so that \( \varphi = \ast \varphi \rvert_L \), then the right hand side is equal to \( \sum_{-\infty < x < \infty} \varphi'(x) \), that is, \( \sum_{-\infty < x < \infty} \text{st}(\varphi)(x) \). Furthermore if \( \varepsilon s \) is infinitesimal and \( \varphi' \) is integrable on \( \mathbb{R} \), then
\[ \text{st}(\varepsilon s \sum_{x \in S} \varphi(x)) = \int_{-\infty}^{\infty} \varphi'(x) \, dx. \]

Since (1) implies that
\[ \sum_{p \in S^\perp} (F \varphi)(p) = \varepsilon s \sum_{x \in S} \varphi(x), \]
we obtain \( \text{st}(\sum_{p \in S^\perp} (F \varphi)(p)) = \int_{-\infty}^{\infty} \varphi'(x) \, dx \), that is, \( \int_{-\infty}^{\infty} \text{st}(\varphi)(x) \, dx \).

We decompose \( H \) to prime factors \( H = p_1^{l_1} p_2^{l_2} \cdot \cdot \cdot p_m^{l_m} \), where \( p_1 = 2 \), \( p_1 < p_2 < \cdot \cdot \cdot < p_m \), each \( p_i \) is a prime number, \( 0 < l_i \). Since \( S \) is a subgroup of \( L \), the number \( s \) is a factor of \( H^2 \). When we write \( s \) as \( p_1^{k_1} p_2^{k_2} \cdot \cdot \cdot p_m^{k_m} \), the order of \( S \) is equal to \( p_1^{2l_1-k_1} p_2^{2l_2-k_2} \cdot \cdot \cdot p_m^{2l_m-k_m} \) and the order of \( S^\perp \) is \( p_1^{k_1} p_2^{k_2} \cdot \cdot \cdot p_m^{k_m} \). Hence (1) is
\[ (p_1^{k_1} p_2^{k_2} \cdot \cdot \cdot p_m^{k_m})^{-\frac{1}{2}} \sum_{p \in S^\perp} (F \varphi)(p)) = (p_1^{2l_1-k_1} p_2^{2l_2-k_2} \cdot \cdot \cdot p_m^{2l_m-k_m})^{-\frac{1}{2}} \sum_{x \in S} \varphi(x). \]

**4.2. Examples**

We apply Theorem 4.1 to the following two functions :
1. \( \varphi_1(x) = \exp(-i \pi x^2) \),
2. \( \varphi_2(x) = \exp(-\xi \pi x^2) \).

The infinitesimal Fourier transformations of the functions are :
1. \( (F \varphi_1)(p) = \exp(-i \frac{\pi}{4} \varphi_1(p)) \),
2. \( (F \varphi_2)(p) = c_\xi(p) \varphi_2\left( \frac{p}{\xi} \right) \),

where \( \text{st}(c_\xi(p)) = \frac{1}{\sqrt{\xi}} \), if \( p \) is finite. Hence we obtain :
1. \( |S^\perp|^{-\frac{1}{4}} \exp(-i \frac{\pi}{4}) \sum_{p \in S^\perp} \varphi_1(p) = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi_1(x) \),
2. \( |S|^{-\frac{1}{4}} \sum_{p \in S^\perp} c_\xi(p) \varphi_2\left( \frac{p}{\xi} \right) = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi_2(x) \).

When the generator of \( S \) is \( \varepsilon s \), we write this as the following, explicitly :
1. \( H \exp(-i \frac{\pi}{4}) \sum_{p \in S^\perp} \exp(i p \pi p^2) = s \sum_{x \in S} \exp(-i \pi x^2) \),
2. \( H \sum_{p \in S^\perp} c_\xi(p) \exp(-\frac{\pi}{4} x^2) = s \sum_{x \in S} \exp(-\xi \pi x^2) \).

We obtain the following proposition :

**Proposition 4.4**

(i) If \( s = H \), then the generator of \( S \) is 1 and \( S = S^\perp = L \cap * \mathbb{Z} \). Hence
1. \( \exp(-i \frac{\pi}{4}) \sum_{p \in L \cap * \mathbb{Z}} \exp(i p \pi p^2) = \sum_{x \in L \cap \mathbb{Z}} \exp(-i \pi x^2) \),
2. \( \sum_{p \in L \cap \mathbf{Z}} c(p) \exp\left( -\frac{1}{\xi} p^2 \right) = \sum_{x \in L \cap \mathbf{Z}} \exp(-\xi \pi x^2) \).

We put the standard part of the above, we obtain:

1. \( \exp\left( -i\xi_\mathbf{Z} \right) \sum_{-\infty < p < \infty} \exp(i\pi p^2) = \sum_{-\infty < x < \infty} \exp(-i\pi x^2) \),
2. \( \text{st} \left( \sum_{p \in L \cap \mathbf{Z}} c(p) \exp\left( -\frac{1}{\xi} p^2 \right) \right) = \text{st} \left( \sum_{x \in L \cap \mathbf{Z}} \exp(-\xi \pi x^2) \right) = \sum_{-\infty < n < \infty} \exp(-\xi \pi n^2) = \theta(i\xi) \), where \( \theta(z) \) is the \( \theta \)-function.

(ii) If \( \varepsilon s \) is infinitesimal, then the equation:

2. \( H \sum_{p \in S_{\perp}} c(p) \exp\left( -\frac{1}{\xi} p^2 \right) = s \sum_{x \in S} \exp(-\xi \pi x^2) \) implies the following:

\[ \text{st} \left( \sum_{p \in S_{\perp}} c(p) \exp\left( -\frac{1}{\xi} p^2 \right) \right) = \varepsilon s \sum_{x \in S} \exp(-\xi \pi x^2) \]

\[ = \int_{-\infty}^{\infty} \exp(-\xi \pi x^2) dx = \frac{1}{\sqrt{\xi}}. \]

It is known that \( \text{st} \left( c(p) \right) = \frac{1}{\sqrt{\xi}} \), and \( \sum_{-\infty < x < \infty} \exp(-\xi \pi x^2) \) in 2 of (i) is equal to \( \frac{1}{\sqrt{\xi}} \sum_{-\infty < x < \infty} \exp(-\xi \pi x^2) \) by the standard Poisson summation formula. Hence, by 2 of (i), \( \text{st} \left( \sum_{p \in S_{\perp}} c(p) \exp\left( -\frac{1}{\xi} p^2 \right) \right) = \sum_{-\infty < p < \infty} \text{st} \left( c(p) \exp(-\xi \pi p^2) \right) \).

We extend the above formulation of \( \varphi_i(x) \) to \( \varphi_{im}(x) = \exp(-im\pi x^2) \), for an integer \( m \) so that \( m|2H^2 \). If \( m|2 \), we recall

\[ (F\varphi_{im})(p) = c_{im}(p) \exp(im\pi p^2), \]

where \( c_{im}(p) = \sqrt{\frac{m+1}{2} \frac{2m^2}{1+i}} \) for a positive \( m \) and \( c_{im}(p) = \sqrt{-\frac{m+1}{2} \frac{2m^2}{1-i}} \) for a negative \( m \).

Hence \( |S_{\perp}|^{-\frac{1}{2}} \sum_{p \in S_{\perp}} c_{im}(p) \varphi_{im}(p) = |S_{\perp}|^{-\frac{1}{2}} \sum_{x \in S} \varphi_{im}(x) \). When the generator \( \varepsilon s' \) of \( S_{\perp} \) satifies \( m|s' \), that is, the generator \( \varepsilon s \) of \( S \) satifies \( m|H^2 \), it reduces to the following:

\[ H \sqrt{\frac{m+1}{2} \frac{2m^2}{1+i}} \sum_{p \in S_{\perp}} \exp(im\pi p^2) = s \sum_{x \in S} \exp(-im\pi x^2) \] for a positive \( m \),

\[ H \sqrt{-\frac{m+1}{2} \frac{2m^2}{1-i}} \sum_{p \in S_{\perp}} \exp(im\pi p^2) = s \sum_{x \in S} \exp(-im\pi x^2) \] for a negative \( m \).

If \( s = H \) and \( m|H \), then

\[ \sqrt{\frac{m+1}{2} \frac{2m^2}{1+i}} \sum_{-\infty < p < \infty} \exp(im\pi p^2) = \sum_{-\infty < x < \infty} \exp(-im\pi x^2) \]

for a positive \( m \),

\[ \sqrt{-\frac{m+1}{2} \frac{2m^2}{1-i}} \sum_{-\infty < p < \infty} \exp(im\pi p^2) = \sum_{-\infty < x < \infty} \exp(-im\pi x^2) \]

for a negative \( m \), that is,

\[ \sqrt{m} \exp\left( i\frac{\pi}{2} \right) \sum_{-\infty < p < \infty} \exp(im\pi p^2) = \sum_{-\infty < x < \infty} \exp(-im\pi x^2) \] for a positive \( m \),

\[ \sqrt{-m} \exp\left( i\frac{\pi}{2} \right) \sum_{-\infty < p < \infty} \exp(im\pi p^2) = \sum_{-\infty < x < \infty} \exp(-im\pi x^2) \] for a negative \( m \).

We remark that it does not coincide with the formula \( \exp\left( i\frac{\pi}{2} \right) \frac{1}{\sqrt{-\xi}} \theta(-\frac{1}{\xi}) = \theta(z) \) for \( \theta \)-function of \( \text{Im}(z) > 0 \). The reason is that the above nonstandard calculation implies an \( m \) multiple of the domain for the function \( \exp(-im\pi x^2) \).

5. Poisson summation formula for Definition 1.2 on the space of functionals
We extend Poisson summation formula of finite group to our infinitesimal Fourier transformation, Definition 1.2, on the space of functionals originally defined in [N-O1].

5-1. Formulation

**Theorem 5.1.** Let \( Y \) be an internal subgroup of \( X \). Then we obtain, for \( f \in A \),

\[
|Y^\perp|^{-\frac{i}{2}} \sum_{b \in Y^\perp} (Ff)(b) = |Y|^{-\frac{i}{2}} \sum_{a \in Y} f(a),
\]

where \( Y^\perp := \{ b \in X \mid \exp(2\pi i < a, b>) = 1 \text{ for } \forall a \in X \} \) and \( < a, b > := \sum_{k \in L} a(k)b(k) \).

**Lemma 5.2.** \( |Y^\perp| = \frac{|Y|}{|X|} \).

**Proof of Lemma 5.2.** For \( k \in L \), we denote \( Y_k := \{ a(k) \in L' \mid a \in Y \} \).

\[
b \in Y^\perp \iff \forall a \in Y, \exp(2\pi i \sum_{k \in L} a(k)b(k)) = 1
\]
\[
\iff \forall a \in Y, \prod_{k \in L} \exp(2\pi i a(k)b(k)) = 1
\]
\[
\iff \forall k \in L, \forall a(k) \in Y_k, \exp(2\pi i a(k)b(k)) = 1
\]
\[
\iff \forall k \in L, b(k) \in Y_k^\perp
\]
\[
\iff b : L \to L', \forall k \in L, b(k) \in Y_k^\perp.
\]

Hence \( |Y^\perp| = \prod_{k \in L} |Y_k^\perp| \). Theorem 3.1 implies \( |Y_k^\perp| = \frac{|L|}{|Y_k|} \). Thus

\[
|Y^\perp| = \prod_{k \in L} \left( \frac{|L|}{|Y_k|} \right) = \frac{|L|}{\prod_{k \in L} |Y_k|} = \frac{|X|}{|Y|}.
\]

**Proof of Theorem 5.1.**

\[
|Y^\perp|^{-\frac{i}{2}} \sum_{b \in Y^\perp} (Ff)(b)
\]
\[
= |Y^\perp|^{-\frac{i}{2}} \sum_{b \in Y^\perp} \sum_{a \in X} \varepsilon_0 \exp(-2\pi i < a, b>) f(a)
\]
\[
= |Y^\perp|^{-\frac{i}{2}} \sum_{a \in X} \varepsilon_0 (\sum_{b \in Y^\perp} \exp(-2\pi i < a, b>)) f(a).
\]

Since \( \sum_{b \in Y^\perp} \exp(-2\pi i < a, b>) = \begin{cases} 0 & (a \notin Y) \\ |Y^\perp| & (a \in Y) \end{cases} \), the above is equal to

\[
|Y^\perp|^{-\frac{i}{2}} \varepsilon_0 |Y^\perp| \sum_{a \in Y} f(a) = |Y^\perp|^{-\frac{i}{2}} (H')^{-\ast} H^2 \sum_{a \in Y} f(a) = |Y^\perp|^{-\frac{i}{2}} \sum_{a \in Y} f(a).
\]

Especially if \( f \) is written as \( \prod_{k \in L} f_k \), that is, \( f(a) = \prod_{k \in L} f_k(a(k)) \), then \( (Ff)(b) \) is \( \sum_{a \in X} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} a(k)b(k)) \prod_{k \in L} f_k(a(k)) \). It is calculated to \( \prod_{k \in L} (\sum_{a(k) \in L} \varepsilon \exp(-2\pi i a(k)b(k)) f_k(a(k)) \), that represents an infinite product of infinitesimal Fourier transformation defined by Kinoshita. In general, since \( f \) is not written as \( \prod_{k \in L} f_k \), our infinitesimal Fourier transformation is not represented as an product of infinitesimal Fourier transformation defined by Kinoshita.

We summarize the argument, we obtain: \( Ff = \prod_{k \in L} F_k f_k \), where \( F_k \) is an infinitesimal Fourier transformation for each \( k \in L \). We apply Proposition 5.3 to each \( F_k \).

**Corollary 5.3**

(i) If each generator of \( Y_k \) is equal to 1, \( f \) is written as \( \prod_{k \in L} f_k \), \( f_k = \ast (\text{st}(f_k))|_{L'} \), and \( \sum_{-\infty < n < \infty} \text{st}(f_k)(n) \) converges, then

\[
\text{st}(\sum_{b \in Y^\perp} (Ff)(b)) = \prod_{k \in L} (\sum_{-\infty < n < \infty} \text{st}(f_k)(n)).
\]
(ii) If each generator of $Y_k$ is infinitesimal, $f$ is written as $\prod_{k \in L} f_k$, $f_k = *(\text{st}(f_k))|_{L'}$ and $\text{st}(f_k)$ is $L_1$-integrable on $\mathbf{R}$, then

\[
\text{st}(\sum_{b \in Y^+} (Ff)(b)) = \prod_{k \in L} \int_{-\infty < t < \infty} \text{st}(f_k)(t) dt.
\]

5-2. Examples

From now on the infinitesimal Fourier transformation $F(e^{a}(f))$ for a functional $f \in A$ is often denoted simply $Ff$. We apply Theorem 5.2 to the following two functionals:

1. $f_t(a) = \exp(-i\pi \sum_{k \in L} a(k)^2)$,
2. $f\xi(a) = \exp(-\xi \pi \sum_{k \in L} a(k)^2)$, where $\xi \in \mathbf{C}$, Re($\xi$) > 0.

The infinitesimal Fourier transformations of the functionals are:

1. $(Ff_t)(b) = (-1)^{\frac{H}{2}} f_t(b)$,
2. $(Ff\xi)(b) = B\xi(b) f\xi(\frac{b}{\xi})$,

hence we obtain:

1. $|Y^+|^{-\frac{1}{2}} (-1)^{\frac{H}{2}} \sum_{b \in Y^+} f_t(b) = |Y^+|^{-\frac{1}{2}} \sum_{a \in Y} f_t(a)$,
2. $|Y^+|^{-\frac{1}{2}} \sum_{b \in Y^+} B\xi(b) f\xi(\frac{b}{\xi}) = |Y^+|^{-\frac{1}{2}} \sum_{a \in Y} f\xi(a)$.

We write this as the following, explicitly:

1. $|Y^+|^{-\frac{1}{2}} (-1)^{\frac{H}{2}} \sum_{b \in Y^+} \exp(-i\pi \sum_{k \in L} b(k)^2) = |Y^+|^{-\frac{1}{2}} \sum_{a \in Y} \exp(-i\pi \sum_{k \in L} a(k)^2)$,
2. $|Y^+|^{-\frac{1}{2}} \sum_{b \in Y^+} B\xi(b) \exp(-\xi \pi \sum_{k \in L} b(k)^2) = |Y^+|^{-\frac{1}{2}} \sum_{a \in Y} \exp(-\xi \pi \sum_{k \in L} a(k)^2)$.

Corollary 5.3 implies the following proposition 5.4.

Proposition 5.4

(i) If each generator of $Y_k$ is equal to 1, then

1. $(-1)^{\frac{H}{2}} \text{st}(\sum_{b \in Y^+} \exp(-i\pi \prod_{k \in L} b(k)^2)) = (\sum_{-\infty < n < \infty} \exp(-i\pi n^2))^H^2$,
2. $\text{st}(\sum_{b \in Y^+} B\xi(b) \exp(-\xi \pi \sum_{k \in L} b(k)^2)) = (\sum_{-\infty < n < \infty} \exp(-\xi \pi n^2))^H^2$

$= (\theta(i\xi))^H^2$.

(ii) If each generator of $Y_k$ is equal to a natural number $m_k$, then

1. $(-1)^{\frac{H}{2}} \text{st}(\sum_{b \in Y^+} \exp(-i\pi \prod_{k \in L} b(k)^2)) = \prod_{k \in L} (m_k \sum_{-\infty < n < \infty} \exp(-i\pi m_k n^2))$,
2. $\text{st}(\sum_{b \in Y^+} B\xi(b) \exp(-\xi \pi \sum_{k \in L} b(k)^2)) = \prod_{k \in L} (m_k \sum_{-\infty < n < \infty} \exp(-\xi \pi m_k n^2))$

$= \prod_{k \in L} (m_k \theta(im_k \xi))$.

(iii) If each generator of $Y_k$ is infinitesimal, then

2. $\text{st}(\sum_{b \in Y^+} B\xi(b) \exp(-\xi \pi \sum_{k \in L} b(k)^2)) = (\int_{-\infty}^{\infty} \exp(-\xi \pi t^2) dt)^H^2$

$= \left( \star \left( \frac{1}{\sqrt{\pi}} \right) \right)^H^2$.

We extend the above formulation of $g_t(a)$ to $g_m(a) = \exp(-im\pi \sum_{k \in L} a^2(k))$, for an integer $m$ so that $m|2H^2$. If $m\frac{(b(k)}{\xi}$, we recall

\[(Fg_m)(b) = B_{im}(b) g_{im}^{\frac{1}{im}}(b), \text{ where } B_{im}(b) = \left(\frac{\sqrt{\frac{m}{2}} + \frac{2H^2}{2m}}{1 + i} \right)^{(*)H^2} \text{ for a positive } m \text{ and } B_{im}(b) = \left(\sqrt{\frac{-m}{2}} + \frac{2H^2}{2m} \right)^{(*)H^2} \text{ for a negative } m.
\]
Hence \(|Y^\perp|^{-\frac{1}{2}} \sum_{b \in Y^\perp} B_{im}(b)g_{im}(b) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} g_{im}(a)\). When each generator \(\varepsilon' s_k\) of \(Y_k\) satisfies \(m|s_k'\), that is, each generator \(\varepsilon' s_k\) of \(Y_k\) satisfies \(m|H^2_{s_k}\), it reduces to the following:

\[
H^t(\ast H)^2 \left( \frac{m}{1+(-1)^{2m}} \right)^{\ast H} \sum_{b \in Y^\perp} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \prod_{k \in L} s_k \sum_{a \in Y} \exp(-im \pi \sum_{k \in L} a(k)^2) \quad \text{for a positive } m, \text{ and }
\]

\[
H^t(\ast H)^2 \left( \frac{m}{1+(-1)^{2m}} \right)^{\ast H} \sum_{b \in Y^\perp} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \prod_{k \in L} s_k \sum_{a \in Y} \exp(-im \pi \sum_{k \in L} a(k)^2) \quad \text{for a negative } m.
\]

If \(s_k = H'\) and \(m|H'\), then

\[
\left( \frac{m}{1+(-1)^{2m}} \right)^{\ast H} \sum_{b \in Y^\perp} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \sum_{a \in Y} \exp(-im \pi \sum_{k \in L} a(k)^2) \quad \text{for a positive } m, \text{ and }
\]

\[
\left( \frac{m}{1+(-1)^{2m}} \right)^{\ast H} \sum_{b \in Y^\perp} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \sum_{a \in Y} \exp(-im \pi \sum_{k \in L} a(k)^2) \quad \text{for a negative } m, \text{ that is,}
\]

\[
\left( \sqrt{m} \exp(-i\pi \frac{1}{m}) \right)^{\ast H} \sum_{b \in Y^\perp} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \sum_{a \in Y} \exp(-im \pi \sum_{k \in L} a(k)^2) \quad \text{for a positive } m, \text{ and }
\]

\[
\left( \sqrt{m} \exp(i\pi \frac{1}{m}) \right)^{\ast H} \sum_{b \in Y^\perp} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \sum_{a \in Y} \exp(-im \pi \sum_{k \in L} a(k)^2) \quad \text{for a negative } m.
\]

6. Poisson summation formula for Definition 1.3 on the space of functionals

We extend Poisson summation formula of finite group to our infinitesimal Fourier transformation, Definition 1.3, on the space of functionals originally defined in [N-O1].

6-1. Formulation

We obtain the following theorem for Definition 1.3 as the above argument.

**Theorem 6.1.** Let \(Y\) be an internal subgroup of \(X\). Then we obtain, for \(f \in A\),

\[
|Y^\perp|^{-\frac{1}{2}} \sum_{b \in Y^\perp} (Ff)(b) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} f(a),
\]

where \(Y^\perp := \{b \in X \mid \exp(2\pi i \varepsilon < a, b >_\varepsilon) = 1\ \forall a \in X\} \text{ and } <a, b>_\varepsilon := *\varepsilon \sum_{k \in L} a(k)b(k)\).

**Lemma 6.2.** \(|Y^\perp| = \frac{|X|}{|Y|}\).

**Proof of Lemma 6.2.** For \(k \in L\), we denote \(Y_k := \{a(k) \in L' \mid a \in Y\}\).

\(b \in Y^\perp \iff \forall a \in Y, \exp(2\pi i * \varepsilon \sum_{k \in L} a(k)b(k)) = 1\)

\(\iff \forall a \in Y, \prod_{k \in L} (\exp(2\pi i * \varepsilon a(k)b(k))) = 1\)

\(\iff \forall k \in L, \forall a(k) \in Y_k, \exp(2\pi i * \varepsilon a(k)b(k)) = 1\)

\(\iff \forall k \in L, * \varepsilon b(k) \in Y_k^\perp\)

\(\iff b : L \to L', \forall k \in L, * \varepsilon b(k) \in Y_k^\perp\).
For $k \in L$, we write $m, n$ as generators defined by:

$$Y_k = \langle \varepsilon' m \rangle, \{b(k) \in L' \mid \ast \varepsilon b(k) \in Y_k^\perp\} = \langle \varepsilon' n \rangle.$$ 

Now

$$\exp(2\pi i \ast \varepsilon' m \varepsilon' n) = 1 \iff \ast \varepsilon' m \varepsilon' n \in \ast (\ast \mathbb{Z})$$

$$\iff \ast \varepsilon' m \varepsilon' n = 1 \cdots (1).$$

We write $Y_k^{\perp L} := \{b(k) \in L' \mid \ast b(k) \in Y_k^\perp\}$. Then $|Y_k^{\perp L}| = |L'| = \frac{\ast H \pi^2}{\pi^2} = \frac{1}{\varepsilon \varepsilon' n} = m$. This is equal to $\frac{\ast H \pi^2}{\pi^2} = |L'|/|Y|$. Hence

$$|Y_1^{\perp L}| = \prod_{k \in L} |Y_k^{\perp L}| = \prod_{k \in L} |L'| = \prod_{k \in L} |Y|,$$

**Proof of Theorem 6.1.**

$$|Y_1^{\perp L}| = \frac{1}{\pi} \sum_{b \in Y_1^{\perp L}} (Ff)(b)$$

$$= |Y_1^{\perp L}| - \frac{1}{\pi} \sum_{b \in Y_1^{\perp L}} \sum_{a \in X} \varepsilon_0 \exp(-2\pi i < a, b >) f(a)$$

$$= |Y_1^{\perp L}| - \frac{1}{\pi} \sum_{a \in X} \varepsilon_0 (\sum_{b \in Y_1^{\perp L}} \exp(-2\pi i < a, b >)) f(a).$$

Since $\sum_{b \in Y_1^{\perp L}} \exp(-2\pi i < a, b >) = \begin{cases} 0 & (a \notin Y) \\ |Y_1^{\perp L}| & (a \in Y) \end{cases}$, the above is equal to

$$|Y_1^{\perp L}| - \frac{1}{\pi} \varepsilon_0 |Y_1^{\perp L}| \sum_{a \in Y} f(a) = |Y_1^{\perp L}| - \frac{1}{\pi} (H^\prime - \ast H^2) \sum_{a \in Y} f(a) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} f(a).$$

We obtain the following:

**Corollary 6.3**

(i) If each generator of $Y_k$ is equal to 1, $f$ is written as $\prod_{k \in L} f_k$, $f_k = \ast (\ast st(f_k))(n)$ converges, then

$$H \frac{\pi^2}{\pi^2} \text{st} \left( \sum_{b \in Y_1^{\perp L}} (Ff)(b) \right) = \prod_{k \in L} \left( \sum_{-\infty < h < \infty} \text{st}(f_k)(n) \right).$$

(ii) If each generator of $Y_k$ is infinitesimal, $f$ is written as $\prod_{k \in L} f_k$, $f_k = \ast (\ast st(f_k))(n)$, and $\text{st}(f_k)$ is $L_1$-integrable on $\mathbb{R}$, then

$$H \frac{\pi^2}{\pi^2} \text{st} \left( \sum_{b \in Y_1^{\perp L}} (Ff)(b) \right) = \prod_{k \in L} \int_{-\infty}^{\infty} \text{st}(f_k)(t) dt.$$

6-2. Examples

We apply Theorem 3.3 to the following two functionals:

1. $g_i(a) = \exp(-i\pi \ast \varepsilon \sum_{k \in L} a(k)^2)$,
2. $g_\xi(a) = \exp(-\xi \pi \ast \varepsilon \sum_{k \in L} a(k)^2)$.

The infinitesimal Fourier transformations of the functionals are:

1. $(Fg_i)(b) = (-1)^\frac{\pi^2}{\pi^2} g_i(b)$,
2. $(Fg_\xi)(b) = C_\xi(b) g_\xi(\frac{b}{\xi})$,

hence we obtain:

1. $|Y_1^{\perp L}| \frac{\pi^2}{\pi^2} \sum_{b \in Y_1^{\perp L}} g_i(b) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} g_i(a)$,
2. $|Y_1^{\perp L}| \frac{\pi^2}{\pi^2} \sum_{b \in Y_1^{\perp L}} C_\xi(b) g_\xi(\frac{b}{\xi}) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} g_\xi(a)$.

We write this as the following, explicitly:

1. $|Y_1^{\perp L}| \frac{\pi^2}{\pi^2} \sum_{b \in Y_1^{\perp L}} \exp(-i\pi \ast \varepsilon \sum_{k \in L} b(k)^2) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} \exp(-i\pi \ast \varepsilon \sum_{k \in L} a(k)^2)$,
2. $|Y^{\perp}\varepsilon|^{-\frac{1}{2}} \sum_{b \in Y^{\perp}\varepsilon} C_\varepsilon(b) \exp(-\frac{1}{\xi} \pi^* \varepsilon \sum_{k \in L} a(k)^2) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} \exp(-\xi \pi^* \varepsilon \sum_{k \in L} a(k)^2)$. Corollary 5.3 implies the following proposition 5.8.

**Proposition 6.4**

(i) If each generator of $Y_k$ is equal to 1, then the standard parts are:

1. $H^{H^2}_{H}(-1) \frac{H}{2} \pi \exp(-i \pi \varepsilon \sum_{k \in L} b(k)^2) = (\sum_{-\infty < n < \infty} \exp(-i \pi \varepsilon n^2))^{H^2}$,
2. $H^{H^2}_{H} \pi (\sum_{b \in Y^{\perp}\varepsilon} C_\varepsilon(b) \exp(-\frac{1}{\xi} \pi^* \varepsilon \sum_{k \in L} b(k)^2)) = (\sum_{-\infty < n < \infty} \exp(-\xi \pi^* \varepsilon n^2))^{H^2}$

(ii) If each generator of $Y_k$ is equal to a natural number $m_k$, then

1. $H^{H^2}_{H}(-1) \frac{H}{2} \pi \exp(-i \pi \varepsilon \sum_{k \in L} b(k)^2) = \prod_{k \in L} (m_k \sum_{-\infty < n < \infty} \exp(-i \pi \varepsilon m_k^2 n^2))$,
2. $H^{H^2}_{H} \pi (\sum_{b \in Y^{\perp}\varepsilon} C_\varepsilon(b) \exp(-\frac{1}{\xi} \pi^* \varepsilon \sum_{k \in L} b(k)^2)) = \prod_{k \in L} (m_k \sum_{-\infty < n < \infty} \exp(-\xi \pi^* \varepsilon m_k^2 n^2))$

(iii) If each generator of $Y_k$ is infinitesimal, then

2. $\pi (\sum_{b \in Y^{\perp}\varepsilon} C_\varepsilon(b) \exp(-\frac{1}{\xi} \pi^* \varepsilon \sum_{k \in L} b(k)^2)) = (f_{-\infty}^{\sum} \exp(-\xi \pi^* t^2) dt)^{H^2}$

We extend the above formulation of $g_i(a)$ to $g_{im}(a) = \exp(-im \pi^* \varepsilon \sum_{k \in L} a(k)^2)$, for an integer $m$ so that $m|2*HH^2$. If $m|\frac{|b(k)|}{\varepsilon^2}$ and $k \in L$, we recall

$$(Fg_{im})(b) = C_{im}(b)g_{\frac{1}{m}}(b),$$

where $C_{im}(b) = \left(\sqrt{\frac{m}{2} \frac{1+(-1)^{2*HH^2}}{1+i}} \right)^{H^2}$ for a positive $m$ and $C_{im}(b) = \left(\sqrt{-\frac{m}{2} \frac{1+(-1)^{2*HH^2}}{1-i}} \right)^{H^2}$ for a negative $m$.

Hence $|Y^{\perp}\varepsilon|^{-\frac{1}{2}} \sum_{b \in Y^{\perp}\varepsilon} C_{im}(b)g_{\frac{1}{m}}(b) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} g_{im}(a)$. When each generator $\varepsilon^*s_k$ of $Y_k^{\perp}\varepsilon$ satisfies $m|s_k$, that is, each generator $\varepsilon^*s_k$ of $Y_k$ satisfies $m|\frac{s_k}{\varepsilon}$, it reduces to the following:

$$H^{H^2}_{H} \pi (\sum_{b \in Y^{\perp}\varepsilon} C_{im}(b)g_{\frac{1}{m}}(b)) = (\sum_{-\infty < n < \infty} \exp(-\xi \pi m^2 n^2))^{H^2}$$

for a positive $m$, and

$$H^{H^2}_{H} \pi (\sum_{b \in Y^{\perp}\varepsilon} C_{im}(b)g_{\frac{1}{m}}(b)) = \prod_{k \in L} (m_k \sum_{-\infty < n < \infty} \exp(-\xi \pi m_k^2 n^2))^{H^2}$$

for a negative $m$. If $s_k = H'$ and $m|H'$, then

$$H^{H^2}_{H} \pi (\sum_{b \in Y^{\perp}\varepsilon} C_{im}(b)g_{\frac{1}{m}}(b)) = \sum_{a \in Y} \exp(-i \pi \varepsilon \sum_{k \in L} a(k)^2)$$

for a positive $m$, and

$$H^{H^2}_{H} \pi (\sum_{b \in Y^{\perp}\varepsilon} C_{im}(b)g_{\frac{1}{m}}(b)) = \sum_{a \in Y} \exp(-i \pi \varepsilon \sum_{k \in L} a(k)^2)$$

for a negative $m$, that is, $H^{H^2}_{H} \pi (\sum_{a \in Y} \exp(-i \pi \varepsilon \sum_{k \in L} a(k)^2))^{H^2}$.
7. The infinitesimal Fourier transformation of a functional \( Z_s(a) \)

In this section, we define a functional on \( X \), and study a relationship between the functional and the Riemann zeta function. We order all prime numbers as \( p(1) = 2, p(2) = 3, \ldots, p(n) < p(n+1), \ldots \), that is, \( p \) is a mapping from \( \mathbb{N} \) to the set \{prime number\}. \( p : \mathbb{N} \rightarrow \{\text{prime number}\} \). The nonstandard extension \(*p : *\mathbb{N} \rightarrow *\{\text{prime number}\}\) is written as \(*p([l]) = [p(l)]\), and we define a mapping \( \tilde{p} : *\mathbb{N} \rightarrow *\{\{\text{prime number}\}\} \) as \( \tilde{p}([l]) = [p(l)] \). For \( s \in \mathbb{C} \), we define \( Z_s(\epsilon) \) as the following:

\[
Z_s(a) := \prod_{k \in \mathbb{L}} \tilde{p}(H(k + \frac{H}{2}) + 1)^{-s(a(k) + \mu')}.
\]

now \( H(k + \frac{H}{2}) + 1 \) is an element of \(*\mathbb{N}\) and \( a(k) + \mu' \) is an element of \(*\{\text{prime number}\}\). Then \( Z_s(a) \) is calculated as \( \exp(-s \sum_{k \in \mathbb{L}} \log(\tilde{p}(H(k + \frac{H}{2}) + 1)a(k)) \prod_{k \in \mathbb{L}} \tilde{p}(H(k + \frac{H}{2}) + 1)^{-s\mu'}) \). We obtain the following theorem for the Fourier transformation of \( \mathbb{E}'(Z_s) \) for Definition 1.2:

**Theorem 7.1.** \((F(\mathbb{E}'(Z_s)))(b) = \left(\prod_{k \in \mathbb{L}} \tilde{p}(H(k + \frac{H}{2}) + 1)^{-s\mu'} \right) \cdot \prod_{k \in \mathbb{L}} \epsilon' \cdot \exp(-s \sum_{k \in \mathbb{L}} \log(\tilde{p}(H(k + \frac{H}{2}) + 1)a(k)) \sum_{k \in \mathbb{L}} a(k)b(k)) \]
\[
\left( \text{st} \left( \prod_{k \in L} \frac{1}{1 - \tilde{p}(H(k + \frac{H}{2}) + 1)^{-s}} \right) \right) = \zeta(s).
\]
Furthermore, Corollary 5.3.(1) and Theorem 7.2 imply the following:

Corollary 7.3. \( \text{st}(\sum_{b \in Y} (F(e^s(Z_s))(b))) = \text{st}(\prod_{k \in L} \frac{1}{1 - \tilde{p}(H(k + \frac{H}{2}) + 1)^{-s} H'}) \).

Hence we obtain: \( \text{st}(\sum_{b \in Y} (F(e^s(Z_s))(b))) \rangle = \zeta(s) \) for \( \text{Re}(s) > 1 \).

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