Research Article

Strongly Ad-Nilpotent Elements of the Lie Algebra of Upper Triangular Matrices

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In this paper, the strongly ad-nilpotent elements of the Lie algebra $\mathfrak{t}(n, \mathbb{C})$ of upper triangular complex matrices are studied. We prove that all the nilpotent matrices in $\mathfrak{t}(n, \mathbb{C})$ are strongly ad-nilpotent if and only if $n \leq 6$. Additionally, we prove that all the elements $\exp(\text{ad} \, x)$, $x$ strongly ad-nilpotent generate the inner automorphism group $\text{Int} \, \mathfrak{t}(n, \mathbb{C})$.

1. Introduction

In this paper, all the algebras are assumed to be finite dimensional. Let $L$ be a complex Lie algebra. Since $\text{ad} \, y$ is an endomorphism of $L$ for any $y \in L$, $L$ is the direct sum of all the generalized eigenspaces $L_\lambda(\text{ad} \, y) = \ker(\text{ad} \, y - \lambda \text{id})^m$, where $m$ is the multiplicity of $\lambda$ as a root of the characteristic polynomial of $\text{ad} \, y$. When $L_\lambda(\text{ad} \, y) = \ker(\text{ad} \, y - \lambda \text{id})$, it is the ordinary eigenspace, denoted by $E_\lambda(\text{ad} \, y)$ instead.

Definition 1. Call an element $x \in L$ strongly ad-nilpotent if there exists $y \in L$ and some non-zero eigenvalue $\lambda$ of $\text{ad} \, y$ such that $x \in L_\lambda(\text{ad} \, y)$.

By the definition, every element of the generalized eigenspaces $L_\lambda(\text{ad} \, y)$ associated with the non-zero eigenvalue $\lambda$ of $\text{ad} \, y$ is strongly ad-nilpotent. From the fact that $[L_\lambda(\text{ad} \, y), L_\mu(\text{ad} \, y)] \subseteq L_{\lambda + \mu}(\text{ad} \, y)$, we know that a strongly ad-nilpotent element must be ad-nilpotent.

We first introduce some notations used in this paper. Let $\text{Aut} \, L$ denote the group of all the automorphisms of $L$ and $\text{Int} \, L$ denote the subgroup of $\text{Aut} \, L$ generated by all $\exp(\text{ad} \, x)$, $x$ nilpotent. Denote by $\mathcal{N}(L)$ the set of all strongly ad-nilpotent elements of $L$ and by $\mathcal{B}(L)$ the subgroup of $\text{Int} \, L$ generated by all $\exp(\text{ad} \, x)$, $x \in \mathcal{N}(L)$.

The strongly ad-nilpotent elements and the group $\mathcal{B}(L)$ are important tools for the proof of the conjugacy of Cartan subalgebras; see [1]. In the case that $L$ is semisimple, for any ad-nilpotent $y \in L$, there exist $h, x \in L$ such that $\{h, x, y\}$ span a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Especially, we have $\text{ad} \, y(y) = -2y$, which shows $y$ is strongly ad-nilpotent. So, it is an equivalence between ad-nilpotency and strong ad-nilpotency in a semisimple Lie algebra.

In this paper, the strongly ad-nilpotent elements of the Lie algebra $\mathfrak{t}(n, \mathbb{C})$ of upper triangular complex matrices are studied. We prove that all the nilpotent matrices in $\mathfrak{t}(n, \mathbb{C})$ are strongly ad-nilpotent if and only if $n \leq 6$. Additionally, we prove that all the elements $\exp(\text{ad} \, x)$, $x$ strongly ad-nilpotent generate the inner automorphism group $\text{Int} \, \mathfrak{t}(n, \mathbb{C})$.

A Lie algebra $L$ is said to be solvable if the derived series

$$L = L^{(0)}L^{(1)} \cdots L^{(i)} \cdots ,$$

satisfies $L^{(i)} = 0$ for some $k \geq 1$, where $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, $i = 1, 2, \ldots$. In this paper, we consider the strongly ad-nilpotent elements of a special class of solvable Lie algebras, the linear Lie algebras of upper triangular complex matrices. Let $\mathfrak{t} = \mathfrak{t}(n, \mathbb{C})$ be the Lie algebra of upper triangular complex $n \times n$ matrices and $\mathfrak{n} = \mathfrak{n}(n, \mathbb{C})$ be the subalgebra of strictly upper triangular matrices.

The paper is organized as follows. Section 2 is devoted to introducing some results about the strongly ad-nilpotent elements in general Lie algebras. In Section 3, we present the Cartan decomposition of $\mathfrak{t}$ and the inner automorphism group $\text{Int} \, \mathfrak{t}$. In Section 4, we give a characterization of strongly ad-nilpotent elements of $\mathfrak{t}$. A simple sufficient condition is also given to determine a strongly ad-nilpotent
2. Preliminaries

In this section, we introduce some results about the strongly ad-nilpotent elements for general Lie algebras.

Proposition 1. Assume that $L$ is a Lie algebra which is not nilpotent. Then, $\mathcal{N}(L)$ has non-zero elements.

Proof. Since $L$ is not nilpotent, by Engel’s Theorem, there exists $x \in L$ such that $[x, x]$ is not nilpotent. So, there exists non-zero eigenvalue $\lambda$ of $\text{ad} x$ such that $L_1(\text{ad} x) \neq \{0\}$, which deduces the desired result.

Lemma 1. Let $L$ be a Lie algebra, and $x \in L$. Then, $x \in \mathcal{N}(L)$ if and only if there exists $y \in L$ such that $x \in L_1(\text{ad} y)$.

Proof. We only need to prove the necessity. Suppose that $x \in L_1(\text{ad} y)$ for some $y \in L$ and $\lambda \neq 0$. Then, $(\text{ad} y - \lambda \text{id})^k x = 0$ for some positive integer $k$. Thus, $(\text{ad} y - \lambda \text{id})^k x = 0$. It follows that $x \in L_1(\text{ad} y/\lambda)$, which completes the proof.

A Lie algebra $L$ is called decomposable if there exist ideals $L_1$ and $L_2$ of $L$ such that $L = L_1 \oplus L_2$. Otherwise, $L$ is called indecomposable. About the decomposable Lie algebra, we have the following.

Proposition 2. Let $L$ be a Lie algebra, and $x \in L$. If $L$ is the direct sum of its ideals $L_1$ and $L_2$, then $x = x_1 + x_2$ is (strongly) ad-nilpotent if and only if $x_1$ and $x_2$ are (strongly) ad-nilpotent elements of $L_1$ and $L_2$, respectively, where $x_i \in L_i$, $i = 1, 2$.

Proof. First suppose that $x_i \in \mathcal{N}(L_i)$ for $i = 1, 2$. By Lemma 1, there exists $y_i \in L_i$ such that $(\text{ad} y_i - \text{id})^k x_i = 0$ for some positive integer $k_i$. Then, we get

$$
(\text{ad} (y_1 + y_2) - \text{id})^{k_1 + k_2} (x_1 + x_2) = (\text{ad} (y_1 + y_2) - \text{id})^{k_1 + k_2} x_1 + (\text{ad} (y_1 + y_2) - \text{id})^{k_1 + k_2} x_2 = 0,
$$

from the fact $[L_1, L_2] = 0$. Therefore, $x_1 + x_2$ is a strongly ad-nilpotent element of $L$.

Conversely, let $x = x_1 + x_2$ be a strongly ad-nilpotent element of $L$. There exists $y \in L$ and some positive integer $k$ such that $x \in \ker (\text{ad} y - \lambda \text{id})^k$. Write $y = y_1 + y_2$, where $y_i \in L_i$, $i = 1, 2$. Then,

$$
0 = (\text{ad} (y_1 + y_2) - \text{id})^k (x_1 + x_2) = (\text{ad} y_1 - \text{id})^k x_1 + (\text{ad} y_2 - \text{id})^k x_2,
$$

Thus, for $i = 1, 2$, we have $(\text{ad} (y_i - i \cdot d)^k x_i = 0$, which shows $x_i \in \mathcal{N}(L_i)$.

Applying

$$
(\text{ad} (x_1 + x_2))^k (y_1 + y_2) = (\text{ad} x_1)^k y_1 + (\text{ad} x_2)^k y_2, \quad \forall y_1, y_2 \in L_1, y_2 \in L_2,
$$

by a similar argument as above, we get that $x = x_1 + x_2$ is ad-nilpotent if and only if $x_1$ and $x_2$ are ad-nilpotent elements of $L_1$ and $L_2$, respectively.

According to a theorem of Lan [2], a Lie algebra $L$ is solvable if and only if the set of all the ad-nilpotent elements of $L$ is a linear subspace of $L$. Furthermore, if $L$ is solvable, then the set $\{x \in L | \text{ad} x \text{ is nilpotent}\}$ is the nilpotent radical of $L$, the maximal nilpotent ideal of $L$; see [3].

3. The Cartan Decomposition of $\mathfrak{t}(n, \mathbb{C})$

Let

$$
\mathfrak{h} = \left\{ \text{diag}(h_1, h_2, \ldots, h_n) | \sum_{i=1}^n h_i = 0 \right\}.
$$

It is easily known that

$$
\mathfrak{h} = \mathfrak{h} + CI,
$$

where $I$ is the identity matrix, is a Cartan subalgebra of $\mathfrak{t}$, i.e., a nilpotent subalgebra with self-normalizer in $\mathfrak{t}$.

Furthermore, $\mathfrak{t}$ has the Cartan decomposition as follows:

$$
\mathfrak{t} = \mathfrak{h} \oplus \bigoplus_{a \in \Phi} \mathfrak{t}_a,
$$

where $\Phi \subset \mathfrak{h}^*$. Here, $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$. Set $a_i = e_i - e_{i+1}, 1 \leq i \leq n - 1$, where $e_i(h) = h_{ii}$ for all $h = \text{diag}(h_1, h_2, \ldots, h_n) \in \mathfrak{h}$. Then,

$$
\Phi = \left\{ \sum_{k=1}^n a_{kj} | 1 \leq j \leq n - 1 \right\},
$$

$$
\mathfrak{t}_{a_{k_1} + a_{k_2} + \cdots + a_{k_m}} = C e_{i_1, j_1}, \quad i \leq j.
$$

Here and thereafter, $e_{ij}$ denotes the matrix having one in the $(i, j)$ position and zeros elsewhere. It is well known that $\bigoplus_{a \in \Phi} \mathfrak{t}_a$ is a Borel subalgebra of $\mathfrak{sl}(n, \mathbb{C})$. In general, a Borel subalgebra of a Lie algebra $L$ is defined by the maximal solvable subalgebra of $L$.

Lemma 2. Let $x \in \mathfrak{t}$. Then, $x$ is ad-nilpotent if and only if $x = kI + x_0$, where $k \in \mathbb{C}$ and $x_0 \in \mathfrak{n}$.

Proof. If $x = kI + x_0$, then $\text{ad} x = \text{ad} x_0$. Since $x_0$ is nilpotent, $\text{ad} x_0$ is also nilpotent. On the other hand, set $x = (x_{ij})$. Assume $x_{ii} \neq x_{jj}$ for some $i < j$. We have

$$
\text{ad} x^m e_{ij} = (x_{ii} - x_{jj})^m e_{ij} + \left( \text{terms linearly independent to } e_{ij} \right),
$$

for $m = 1, 2, \ldots$. So, $\text{ad} x$ cannot be nilpotent. This is a contradiction.
Proposition 3. The inner automorphism group $\text{Int} \ t$ is given by

$$\text{Int} \ t = \{\exp(\text{ad}x) | x \in \mathfrak{n}\}. \quad (10)$$

Proof. By the above lemma, we know that $\text{Int} \ t$ is just the subgroup of $\text{Aut} \ t$ generated by $\exp(\text{ad}x)$, $x \in \mathfrak{n}$. Note that $\text{Aut} \ t$ is a linear Lie group with the Lie algebra $\text{Der} \ t$ consisting of all derivations of $t$. Since $\text{ad}_n \subset \text{Der} \ t$, let $G$ be the connected Lie subgroup of $\text{Aut} \ t$ whose Lie algebra is $\text{ad} \ n$. Since $\text{ad} \ n$ is a nilpotent algebra, the exponential mapping is a surjection. Hence, $G = \{\exp(\text{ad}x) | x \in \mathfrak{n}\}$. From $G \subset \text{Int} \ t$, by the definition of $\text{Int} \ t$, we get the desired result. □

Remark 1. It is easy to get that $\text{Int} \ t$ is isomorphic to the Lie group of the unipotent upper triangular complex matrices.

4. A Characterization of Strongly Ad-nilpotent Elements

We first give some equivalent conditions to depict strongly ad-nilpotent elements of $t$.

Theorem 1. Let $x \in \mathfrak{t}$. The following statements are equivalent:

1. $x$ is strongly ad-nilpotent.
2. There exists $y \in \mathfrak{t}$ such that $x \in L_1(\text{ad}y)$.
3. There exists $y \in \mathfrak{t}$ such that $x \in E_1(\text{ad}y)$, i.e., $[y, x] = x$.
4. There exists a semisimple element $y$ of $\mathfrak{t}$ such that $[y, x] = x$.

Proof. (1) $\Rightarrow$ (2) is followed from Lemma 1.

(2) $\Rightarrow$ (4) Let $y = y_s + y_n$ be the Jordan decomposition, i.e., $y_s$ is semisimple, $y_n$ is nilpotent, and $[y_s, y_n] = 0$. Since $y_s$ can be written as a polynomial in $y$ without constant term, we have $y_s \in \mathfrak{t}$. It is obvious that $\text{ad} y = \text{ad} y_s + \text{ad} y_n$ is also the Jordan decomposition of $\text{ad} y$. According to the fact $\text{ad} y_s$ and $\text{ad} y$ have the same generalized eigenspaces, we have $x \in L_1(\text{ad} y_s)$. For the reason that $\text{ad} y_s$ is diagonalizable, we get $L_1(\text{ad} y_s) = E_1(\text{ad} y_s)$.

(4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. □

Corollary 1. $\mathcal{N}(\mathfrak{t})$ is a subset of $\mathfrak{n}$.

Proof. For any $x \in \mathcal{N}(\mathfrak{t})$, there exists $y$ such that $[y, x] = x$. Thus, $x$ is nilpotent, and hence $x \in \mathfrak{n}$.

Recall the basic result about the automorphism and the strongly ad-nilpotent elements.

Lemma 3 (see [1]). Let $L$ be a Lie algebra. If $\phi \in \text{Aut}(L)$, then $\phi(x) \in \mathcal{N}(L)$ for all $x \in \mathcal{N}(L)$ and $\phi(L_1(\text{ad}y)) = L_1(\text{ad}\phi(y))$ for all $y \in L$.

From the above lemma, in order to determine which ones in $\mathfrak{n}$ are strongly ad-nilpotent, we only need to consider the equivalence classes under the acting of $\text{Aut}(\mathfrak{t})$. For $\text{Aut}(\mathfrak{t})$, the readers are referred to literatures [4, 5].

Denote by $t^*$ the set of invertible matrices of $t$. Then, $t^*$ is a Lie group with the Lie algebra $\mathfrak{t}$. For any $a \in t^*$, we can define an automorphism of $t$ by $\phi_a : x \to axa^{-1}$, for all $x \in t$. There is a natural group epimorphism from $t^*$ to $[\mathfrak{g}, a \in t^*]$. Thus, we can only consider the upper triangular nilpotent matrices under upper triangular similarity (shortly written as under $t^*$). It has been studied by many researchers (see [6] and the references therein); related research can be seen in [7–9]. Note that there are only finitely upper triangular similarity classes in the size 5 or less and infinitely classes in the size 6 or higher.

Proposition 4. Let $x = (x_{ij})$ and $y = (y_{ij})$ be two elements of $t(n, \mathbb{C})$.

1. If $x$ and $y$ are similar under $t^*$, then $x_{ii} = y_{ii}$ for $1 \leq i \leq n$.
2. If $x$ is semisimple, then $x$ is similar to $\text{diag}(x_{11}, x_{22}, \ldots, x_{nn})$ under $t^*$.

Proof. If $x = zyz^{-1}$ for an invertible upper triangular matrix $z = (z_{ij})$ with $z^{-1} = (z_{ij}^{-1})$, then the $(i, i)$ element yields

$$x_{ii} = \sum_{i \leq j \leq k} z_{ij} y_{jk} z_{ki} = a_{ii}^{z^{-1}} y_{ii} = y_{ii}. \quad (11)$$

Here, we have used the facts that $z^{-1}$ is upper triangular and $z^{t} = 1/z_{ii}$.

If $x$ is semisimple, then there is an invertible matrix $t$ and a diagonal matrix $d$ such that $x = t^{-1} d t$. By the QR decomposition, we have $t = u r$, where $u$ is a unitary matrix and $r$ is an upper triangular matrix. Then,

$$r x r^{-1} = u^{-1} d u. \quad (12)$$

From the above equation, the left is upper triangular and the right is normal. By the fact that an upper triangular normal matrix must be diagonal, we get that $x$ is similar to a diagonal matrix under $t^*$. According to the first result of the proposition, we get the second. □

For any $h \in \mathfrak{h}$, define

$$\Delta_h = \{\lambda \in \Phi | \lambda(h) = 1\}. \quad (13)$$

We call $h \in \mathfrak{h}$ a maximal element with one-eigenvalue if for any $h \in \mathfrak{h}$ with $E_i(\text{ad} h) \subset E_1(\text{ad} h)$, then $h = h$. Denote by $\mathfrak{D}$ the set of all the maximal elements with one-eigenvalue.

For any $x \in \mathfrak{n}$, we can write $x$ as $x = \sum_{i \in \Phi} x_i$. Define
\[ \Delta(x) = \{ \lambda \in \Phi | x^\lambda \neq 0 \}. \]  

**Lemma 4.** With the notations as above, the following claims hold:

1. Let \( h \in \mathfrak{h} \). If \( \text{span} \Delta_h \neq \mathfrak{h}^* \), then there exists \( h' \in \mathfrak{h} \) such that \( \Delta_h \subset \Delta_{h'} \) and \( \text{span} \Delta_{h'} = \mathfrak{h}^* \).
2. Let \( h, h' \in \mathfrak{h} \). If \( \text{span} \Delta_{h'} = \mathfrak{h}^* \) and \( \Delta_h \subset \Delta_{h'} \), then \( h = h' \).
3. Let \( h \in \mathfrak{h} \). Then, \( h \in \mathfrak{D} \) if and only if \( \text{span} \Delta_h = \mathfrak{h}^* \).

**Proof.**

(1) Take a maximal linearly independent subset \( \{ \lambda_1, \ldots, \lambda_s \} \) of \( \Delta_h \) and extend it to a basis \( \{ \lambda_1, \ldots, \lambda_{n-1} \} \subset \Phi \) for \( \mathfrak{h}^* \). It is apparent that there exists only one element \( h \in \mathfrak{h} \) such that \( \lambda_i(h) = 1 \) for all \( 1 \leq i \leq n-1 \). Obviously, \( \text{span} \Delta_h = \mathfrak{h}^* \). For any \( \lambda \in \Delta_h \), there exist \( a_1, \ldots, a_s \in \mathbb{C} \) such that \( \lambda = \sum_{i=1}^{s} a_i \lambda_i \). Thus,

\[ 1 = \lambda(h) = \sum_{i=1}^{s} a_i \lambda_i(h) = \sum_{i=1}^{s} a_i, \]  

(15)

Then,

\[ \lambda(h) = \sum_{i=1}^{s} a_i \lambda_i(h) = \sum_{i=1}^{s} a_i = 1. \]  

(16)

Therefore, \( \lambda \in \Delta_h \).

(2) Notice that both \( h \) and \( h' \) are the solution of \( \lambda(i) = 1 \) for all \( \lambda \in \Delta_h \). Since \( \text{span} \Delta_h = \mathfrak{h}^* \), by the uniqueness of solution, we know \( h = h' \).

(3) The "if" part is from (2) and the "only if" part is from (1). \( \square \)

From the above lemma, we know \( \mathfrak{D} \) is a finite set. It seems that it is an interesting question to determine the number of elements in \( \mathfrak{D} \).

**Proposition 5.** Let \( x \in \mathfrak{n} \). If the linear equations

\[ \lambda(h) = 1, \quad \text{for all } \lambda \in \Delta(x), \]  

(17)

have at least one solution \( h \) in \( \mathfrak{h} \), then \( x \in \mathcal{N}(\mathfrak{t}) \). Particularly, if \( \Delta(x) \) consists of linearly independent elements, then \( x \in \mathcal{N}(\mathfrak{t}) \).

**Proof.** From

\[ x \in \prod_{\lambda \in \Delta(x)} \mathfrak{t}_\lambda \subset \prod_{\lambda(h)=1} \mathfrak{t}_\lambda = E_1(\text{ad } h), \]  

(18)

we see \( x \) is strongly ad-nilpotent. \( \square \)

**Example 1.** Let \( k \neq 0 \). Consider

\[ \Delta(A) = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}. \]  

(20)

It is easy to obtain the unique solution

\[ h = \text{diag}(1.5, 0.5, 0.5, -0.5, -0.5, -1.5) \]  

(21)

satisfying \( \lambda(h) = 1 \) for all \( \lambda \in \Delta(A) \). So, \( A \) is a strongly ad-nilpotent element of \( \mathfrak{t}(6, \mathbb{C}) \).

**Theorem 2.** With the notations as above, we have

\[ \mathcal{N}(\mathfrak{t}) = \bigcup_{\mathfrak{t} \in \mathfrak{t}^*} \bigcup_{y \in \mathfrak{D}} t E_1(\text{ad } y) t^{-1}. \]  

(22)

**Proof.** For any \( x \in \mathcal{N}(\mathfrak{t}) \), by Theorem 1, we have

\[ [y, x] = x, \quad \text{for some semisimple element } y \text{ of } \mathfrak{t}. \]  

(23)

Without loss of generality, we can assume \( \text{tr}(y) = 0 \). By Proposition 4, there exists \( t \in \mathfrak{t}^* \) such that \( t^{-1}yt = h \) for some \( h \in \mathfrak{h} \). So, \( t^{-1}xt \in E_1(\text{ad } h) \). By Lemma 4, there is \( \tilde{h} \in \mathfrak{D} \) such that \( E_1(\text{ad } h) \subset E_1(\text{ad } \tilde{h}) \). Therefore, \( x \in t E_1(t \text{ad } \tilde{h}) t^{-1} \). \( \square \)

**5. The Main Results**

**Theorem 3.** For \( n \leq 6 \), we have

\[ \mathcal{N}(\mathfrak{t}(n, \mathbb{C})) = \mathfrak{n}(n, \mathbb{C}). \]  

(24)

**Proof.** The case \( n = 1 \) is trivial. By Corollary 1, it remains to prove \( \mathfrak{n} \subset \mathcal{N}(\mathfrak{t}) \). Since every upper triangular matrix is similar to a direct sum of indecomposable matrices under \( \mathfrak{t}^* \), we need only to consider the indecomposable elements. The canonical forms of all the indecomposable nilpotent matrices in \( \mathfrak{n}(n, \mathbb{C}) (2 \leq n \leq 6) \) are given by [6] (list in Theorem 2.1 and Theorem 2.2 in [6]). In the lists, the last matrix has been proved to be strongly ad-nilpotent in Example 1. Apart from the last matrix, it is easy to check that \( \Delta(x) \) consists of linearly independent elements for any \( x \) in the rest matrices. By Proposition 5, we know \( x \in \mathcal{N}(\mathfrak{t}) \), which completes the proof. \( \square \)

**Corollary 2.** If \( n \leq 6 \), then for any \( A \in \mathfrak{n}(n, \mathbb{C}) \), the matrix equation

\[ [x, A] = A, \]  

(25)

has at least one solution in \( \mathfrak{t}(n, \mathbb{C}) \).
Then, $A$ is not strongly ad-nilpotent in $\mathfrak{t}(7, \mathbb{C})$.

**Proof.** Assume that $A$ is strongly ad-nilpotent. By Theorem 1, there exists $x = (x_{ij}) \in \mathfrak{t}$ such that
\[
[x, A] = A. \tag{27}
\]

We can set $x_{77} = 0$; otherwise, replace $x$ by $x + cI$ for some suitable $c \in \mathbb{C}$. First, comparing the (1, 2), (3, 4), (6, 7) elements of (27), respectively, we have
\[
x_{11} - x_{22} = x_{33} - x_{44} = x_{66} - x_{77} = 1. \tag{28}
\]

Thus, $x_{66} = 1$. Next, from the (2, 5), (2, 6), (5, 7), (4, 6) elements of (27), we obtain
\[
x_{24} = 0, \quad x_{22} - x_{66} + x_{24} = 1, \quad x_{56} = 0, \quad x_{44} - x_{66} - x_{56} = 1, \tag{29}
\]
which deduces $x_{22} = x_{44} = 2$. By (28), we get $x_{33} = 3$. Then, from the (2, 4), (1, 3) elements of (27), we obtain
\[
x_{23} = 0, \quad x_{11} - x_{33} - x_{23} = 1, \tag{30}
\]
which forces $x_{11} = 4$. Therefore, by (28), we see $x_{22} = 3$, contrary to $x_{22} = 2$. \hfill \Box

From Theorem 3 and Proposition 6, we immediately get the following main result.

**Theorem 4.** Let $\mathfrak{t}(n, \mathbb{C})$ ($n \geq 2$) be the Lie algebra of upper triangular $n \times n$ complex matrices, $\mathfrak{n}(n, \mathbb{C})$ be the subalgebra of strictly upper triangular matrices, and $\mathcal{N}(\mathfrak{t}(n, \mathbb{C}))$ be the set of all strongly ad-nilpotent elements of $\mathfrak{t}(n, \mathbb{C})$. Then,
\[
\mathcal{N}(\mathfrak{t}(n, \mathbb{C})) \subset \mathfrak{n}(n, \mathbb{C}), \tag{31}
\]
with equality holds if and only if $n \leq 6$.

Although for $n > 6$, $\mathcal{N}(\mathfrak{t}(n, \mathbb{C}))$ is a proper subset of $\mathfrak{n}(n, \mathbb{C})$ as well as the set $\mathbb{C}I + \mathfrak{n}(n, \mathbb{C})$ of all ad-nilpotent elements of $\mathfrak{t}(n, \mathbb{C})$, we have the following result.

**Theorem 5.** For any $n \geq 1$, we have
\[
\mathcal{B}(\mathfrak{t}(n, \mathbb{C})) = \text{Int} \mathfrak{t}(n, \mathbb{C}). \tag{32}
\]

**Proof.** We suppose $n \geq 2$, the case $n = 1$ being trivial. Let $Z(\mathfrak{t})$ denote the center of $\mathfrak{t}$. Since $\ker \text{ad} = Z(\mathfrak{t}) \subset C\mathfrak{i}$, we have $(\ker \text{ad}) \cap \mathfrak{n} = 0$. Thus, $\text{ad}_n \mathfrak{n}$ is isomorphic to $\mathfrak{n}$. Let $U$ be the Lie group of the unipotent upper triangular complex matrices. Then, $U = \exp \mathfrak{n}$. Since $\mathfrak{n}$ is nilpotent and $U$ is simply connected, $\text{Int} \mathfrak{t}$ is isomorphic to $U$.

For any $i < j$, it is clear that $e_{ij} \in \mathcal{N}(\mathfrak{t})$ by Proposition 5. Then, we get
\[
\exp ke_{ij} = I + ke_{ij}, \quad \text{for all } k \in \mathbb{C}, i < j. \tag{34}
\]

It is well known that any element of $U$ can be written as a product of some matrices of the form $I + ke_{ij}$, $i < j$. Hence, the subgroup of $U$ generated by all $\exp x$, $x \in \mathcal{N}(\mathfrak{t})$, is just $U$ itself. Therefore, the subgroup of $\text{Int} \mathfrak{t}$ generated by all $\exp(\text{ad} x)$, $x \in \mathcal{N}(\mathfrak{t})$, is $\text{Int} \mathfrak{t}$ itself; that is, $\mathcal{B}(\mathfrak{t}) = \text{Int} \mathfrak{t}$. \hfill \Box

**Data Availability**

No data were used to support the findings of this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest.

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