Research Article

Finite-Time Stability Criteria for a Class of High-Order Fractional Cohen–Grossberg Neural Networks with Delay

Zhanying Yang, Jie Zhang, Junhao Hu, and Jun Mei

School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, Hubei, China

Correspondence should be addressed to Jun Mei; meij0000@163.com

Received 13 July 2020; Revised 10 August 2020; Accepted 18 August 2020; Published 9 September 2020

Copyright © 2020 Zhanying Yang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper focuses on a class of delayed fractional Cohen–Grossberg neural networks with the fractional order between 1 and 2. Two kinds of criteria are developed to guarantee the finite-time stability of networks based on some analytical techniques. This method is different from those in some earlier works. Moreover, the obtained criteria are expressed as some algebraic inequalities independent of the Mittag–Leffler functions, and thus, the calculation is relatively simple in both theoretical analysis and practical applications. Finally, the feasibility and validity of obtained results are supported by the analysis of numerical simulations.

1. Introduction

Neural networks have been paid much attention owing to the powerful applications in diverse fields. With the increasing requirements in practical applications, many researchers have made great efforts to develop various types of neural networks, such as Cohen–Grossberg neural networks (CGNNs) [1], cellular neural networks, bidirectional associative memory neural networks, and recurrent neural networks. As a kind of special recurrent neural network, CGNNs were firstly proposed by Cohen and Grossberg in [1]. CGNN is quite general since it includes some well-known types of neural networks, such as Hopfield neural network, cellular neural network, and shunting neural network. Nowadays, CGNNs have gained more and more interests due to their promising applications in classification, parallel computation and optimization, etc. Many researchers have made great contribution to the research on CGNNs; see [2–11] and the references therein.

Nowadays, the fractional calculus has achieved significant progress in both theoretical research and practical applications. Compared with the integer-order derivative, the fractional-order derivative has some distinctive features, such as infinite memory and great freedom. Consequently, the fractional-order derivative can better characterize many systems in the real world [12–14]. In order to more accurately model the dynamics of neurons, various fractional-order neural networks (FONNs) have been generated based on the integration of the fractional calculus and neural networks. In the recent decades, the research on FONNs has undergone a prosperous development, and there have been numerous works (see [10, 15–20] and the references therein). As we all know, the successful applications of FONNs are closely associated to the dynamics of networks, among which stability has been an active topic. There have been substantial works on various types of stability, such as asymptotical stability [10], finite-time stability [21], exponential stability [22], Mittag–Leffler stability [8], and Lagrange stability [11]. Many sufficient conditions have been established to achieve the stability of systems; see [16, 18–20, 23, 24] and the references therein.

Among various types of stability, finite-time stability (FTS) has aroused more interests in many fields, since many systems always operate over a limited period of time or it is necessary to focus on the behavior of systems within a limited period of time. In the existing works, there are
mainly two concepts of FTS. One means that the error of any two state variables tends to zero in a limited time interval, which is also regarded as a special case of Lyapunov asymptotic stability. The other is also called finite-time boundedness, which describes that the quantity related to state does not exceed a prescribed threshold in a limited time interval for a given bound on the initial value [25, 26]. It is obvious that “boundedness” is a distinctive feature of this FTS. As revealed in [25, 26], it is essentially different from the classical Lyapunov asymptotic stability. –D he other is also called finite-
interval, which is also regarded as a special case of Lyapunov asymptotic stability. –D herefore, it is significant to follow through the problems some Lyapunov methods [10] and LMI method [33], some classical methods for the case science (see, for instance, 29–32). On the other hand, some generalizations on fractional order in-
properties related to the fractional calculus. For the FONNs in the aforementioned works, notice that the fractional order α is between 0 and 1. In the real world, the fractional systems with high fractional order can appropriately describe many phenomena and have been successfully applied in physics, biology, and information science (see, for instance, [29–32]). On the other hand, some classical methods for the case α ∈ (0, 1), such as some Lyapunov methods [10] and LMI method [33], could not be directly extended to the high-order cases. Therefore, it is significant to follow through the problems on high-order FONNs.

For FONNs with the fractional order α ∈ (1, 2), there have been many excellent works on the finite-time stability or finite-time synchronization [15–17, 34–37]. The analysis is mainly based on the Laplace transform, the inverse Laplace transform, and the generalized Gronwall inequality related to the Mittag–Leffler functions. However, this method cannot be directly used to deal with the FTS for FOCGNNs with α ∈ (1, 2) owing to the technical reason. In order to solve this problem, it is desired to investigate a kind of different method.

In this paper, we are devoted to the FTS for a class of delayed FOCGNNs with α ∈ (1, 2). The main contributions are summarized as follows: (i) The fractional order of system considered in this paper is between 1 and 2. A criterion is derived to achieve the FTS of system. Moreover, a criterion is established to ensure the FTS for the equilibrium point of system. (ii) The proofs are based on some analytical techniques, such as the Cauchy–Schwarz inequality, the generalized Gronwall inequality, and some properties of the Caputo derivative. This method is completely different from those in some earlier works [9, 15–17, 24, 34–37]. In particular, the obtained results are expressed as some algebraic inequalities and hence, the calculation is relatively easy in practical applications.

2. Preliminaries

This section starts with recalling some necessary definitions and properties related to the Caputo derivative.

Definition 1 (see [38]). Let μ ∈ ℜ+, m ∈ ℜ+ and m − 1 < μ < m. For θ(t) ∈ ℜm([t0, +∞)). The Caputo derivative with fractional order μ of θ is defined by

\[ C^\mu D^\mu_t \theta(t) = \frac{1}{\Gamma(m - \mu)} \int_{t_0}^{t} (t - r)^{m - \mu - 1} \theta^{(m)}(r) dr, \quad t \geq t_0, \]

where Γ(·) denotes the Gamma function, i.e.,

\[ \Gamma(r) = \int_{0}^{\infty} t^{r-1} e^{-t} dt. \]

Definition 2 (see [38]). For μ ∈ ℜ+ the fractional order integral with order μ of a function θ(t) is defined by

\[ D_{t_0}^\mu \theta(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} (t - r)^{\mu - 1} \theta(r) dr. \]

Proposition 1 (see [39]). Let μ ∈ ℜ+, m ∈ ℜ+, and m − 1 < μ < m. If \( \theta(t) \in C^m([t_0, +\infty), \mathbb{R}) \), then

\[ D_{t_0}^\mu C^\mu D^\mu_t \theta(t) = \theta(t) - \sum_{k=0}^{m-1} \frac{\theta^{(k)}(t_0)}{k!} t^k. \]

Next, we list two inequalities, which will play a key role in the proofs of main results.

Proposition 2 (generalized Gronwall inequality [40]). Let \( u(t) \), \( v(t) \) and \( w(t) \) be nonnegative \( L_q \) functions on the interval \([0, T]\) for \( q \in [1, +\infty) \) if

\[ u(t) \leq v(t) + w(t) \left( \int_{0}^{t} u^q(r) dr \right)^{1/q}, \quad t \in [0, T]. \]

Then,

\[ \int_{0}^{t} u^q(r) dr \leq \left[ 1 - (1 - \Phi(t))^{1/q} \right]^{q} \int_{0}^{t} v^q(r) \Phi(r) dr, \]

where \( \Phi(t) = \exp(-\int_{0}^{t} w^q(r) dr) \).
Proposition 3 (generalized Bernoulli inequality [41]). Let \( t < 1 \) and \( t \neq 0 \). For \( 0 < \alpha < 1 \), we have \((1-t)^\alpha < 1 - dt\). Moreover, \((1 - (1-t)^\alpha)^{-1} < (dt)^{-1}\).

In what follows, we introduce a class of delayed FOCGNNs, which can be described as

\[
\frac{d^\alpha}{dt^\alpha}x_i(t) = -p_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} g_j(x_i(t - \tau)) - I_i, \quad i = 1, 2, \ldots, n, \tag{7}
\]

where \( 1 < \alpha < 2 \), \( x_i(t) \) is the state of the \( i \)-th neuron at time \( t \). \( p_i(\cdot) \) stands for the amplification function and \( q_i(\cdot) \) corresponds to the behaved function. The constant \( \tau > 0 \) denotes the time delay. \( a_{ij} \) and \( b_{ij} \) are the connection weights. \( f_j \) and \( g_j \) represent the activation functions. \( I_i \) stands for the constant external input.

For \( h(t) \in C([-\tau, 0], \mathbb{R}^m) \), the norm is defined as \( \|h\| = \sup_{t \in [-\tau, 0]} \sum_{i=1}^{m} |h_i(t)| \). Let \( x(t) \) and \( y(t) \) stand for two arbitrary solutions for network (7), and let \( z(t) = x(t) - y(t) \). The initial condition is given as follows:

\[
\begin{align*}
z(t) &= \varphi(t), \\
z'(t) &= \psi(t), \\
t &\in [-\tau, 0],
\end{align*}
\]  

where \( \varphi(t), \psi(t) \in C([-\tau, 0], \mathbb{R}^n) \).

Definition 3. Let \( 0 < \delta < \epsilon \). For \( 1 < \alpha < 2 \), if \( \|\varphi\|, \|\psi\| < \delta \) implies

\[
\|z(t)\| < \epsilon, \quad \forall t \in [0, T],
\]

where \( \|z(t)\| = \sum_{i=1}^{n} |z_i(t)| \), then network (7) can achieve the finite-time stability w.r.t. \( [\delta, \epsilon, T] \).

3. Main Results

In this section, we are devoted to two kinds of finite-time stability criteria for network (7) based on some properties related to the Caputo derivative and some inequalities.

3.1. Stability Criterion I for Network (7). Let us first introduce some further assumptions on the parameters of network (7) and some necessary notation.

(A1) The function \( p_i \) \( (i = 1, 2, \ldots, n) \) is a continuous and bounded function such that

\[
0 < p_i \leq p_i(x) \leq \overline{p}_i, \\
|p_i(x) - p_i(y)| \leq p^*|x - y|, \tag{10}
\]

for \( x, y \in \mathbb{R} \), where \( p^*, \overline{p}_i, \) and \( p^* \) are some positive constants. 

(A2) For the functions \( p_i \) and \( q_i \) \( (i = 1, 2, \ldots, n) \), there exists \( \theta > 0 \) such that

\[
|p_i(x)q_i(x) - p_i(y)q_i(y)| \leq \theta|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{11}
\]

(A3) The functions \( f_j \) and \( g_j \) \( (j = 1, 2, \ldots, n) \) are bounded and satisfy the Lipschitz conditions, namely,

\[
\begin{align*}
|f_j(x)| &\leq F_j, \\
|g_j(x)| &\leq G_j, \\
\end{align*}
\]

\[
\begin{align*}
|f_j(x) - f_j(y)| &\leq \xi_j |x - y|, \\
|g_j(x) - g_j(y)| &\leq \eta_j |x - y|, \\
\forall x, y &\in \mathbb{R},
\end{align*}
\]

where \( F_j, G_j, \xi_j \) and \( \eta_j \) are positive constants.

Let

\[
\lambda_1 = \max_{1 \leq i \leq n} \left\{ \theta_i + p^* |I_i| + p^* \sum_{j=1}^{n} \left( |a_{ij}| F_j + |b_{ij}| G_j \right) \right\} \\
+ \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \xi_j |\overline{\rho}_j| a_{ij} \right),
\]

\[
\lambda_2 = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \eta_j |\overline{\rho}_j| b_{ij} \right).
\]

Theorem 1. Let \( 0 < \delta < \epsilon \). Under the assumptions (A1)-(A3), network (7) can achieve the FTS w.r.t. \( [\delta, \epsilon, T] \) if

\[
(1 + t)^{\frac{1}{2}}  \left( 1 + 2e^{(\lambda \alpha - 1)T} \right) \left( 1 - e^{-\lambda \alpha T} \right)^{1/2} \leq \frac{\epsilon}{\delta}, \quad \forall t \in [0, T],
\]

where \( \lambda = (\lambda_1 + \lambda_2)e^{-\alpha T}/2\Gamma(2\alpha - 1)/2e^{\alpha T}(\alpha) \).

Proof. Let \( x(t), y(t), \) and \( z(t) \) be defined as in Section 2. By Proposition 1, we obtain
\[ x_i(t) - y_i(t) = x_i(0) - y_i(0) + (x'_i(0) - y'_i(0))t \]

\[ -\frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} (p_i(x_i(r))q_i(x_i(r)) - p_i(y_i(r))q_i(y_i(r))) \, dr \]

\[ + \frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} \left( p_i(x_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) - p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(y_j(r)) \right) \, dr \]

\[ + \frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} \left( p_i(x_i(r)) \sum_{j=1}^n b_{ij} (g_j x_j(r - \tau)) - p_i(y_i(r)) \sum_{j=1}^n b_{ij} g_j y_j(r - \tau) \right) \, dr \]

\[ + \frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} I_i [p_i(x_i(r)) - p_i(y_i(r))] \, dr. \]  

(15)

Using the assumptions (A1)-(A2), we have

\[ |x_i(t) - y_i(t)| \leq |x_i(0) - y_i(0)| + |x'_i(0) - y'_i(0)|t \]

\[ + \frac{\theta_i + p^* |I_i|}{\Gamma(a)} \int_0^t (t-r)^{a-1} |x_i(r) - y_i(r)| \, dr \]

\[ + \frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} \left| p_i(x_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) - p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(y_j(r)) \right| \, dr \]

\[ + \frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} \left| p_i(x_i(r)) \sum_{j=1}^n b_{ij} (g_j x_j(r - \tau)) - p_i(y_i(r)) \sum_{j=1}^n b_{ij} g_j y_j(r - \tau) \right| \, dr. \]  

(16)

For the term \( |p_i(x_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) - p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(y_j(r))| \), the assumptions (A1) and (A3) lead to

\[ p_i(x_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) - p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(y_j(r)) \]

\[ = p_i(x_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) - p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) \]

\[ + p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(x_j(r)) - p_i(y_i(r)) \sum_{j=1}^n a_{ij} f_j(y_j(r)) \]

\[ \leq \sum_{j=1}^n p^* |a_{ij}| |f_j(x_j(r) - y_j(r))| + \sum_{j=1}^n p^* |a_{ij}| |x_j(r) - y_j(r)|. \]

(17)

In the same way, we obtain

\[ p_i(x_i(r)) \sum_{j=1}^n b_{ij} (g_j x_j(r - \tau)) - p_i(y_i(r)) \sum_{j=1}^n b_{ij} g_j y_j(r - \tau) \]

\[ \leq \sum_{j=1}^n p^* |b_{ij}| |G_j(x_j(r) - y_j(r))| + \sum_{j=1}^n p^* |b_{ij}| |x_j(r - \tau) - y_j(r - \tau)|. \]

(18)
Substituting (17) and (18) into (16), we obtain

\[ |z_i(t)| \leq |z_i(0)| + |z'_i(0)| t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left[ (\theta_i + p^* |I_i|) + p^* \sum_{j=1}^n \xi_j F_j \right] |z_i(r)| dr + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left[ \sum_{j=1}^n \eta_j F_j |b_{ij}| |z_j(r-r)| \right] dr.

Consequently,

\[ \|z(t)\| \leq \|z(0)\| + \|z'(0)\| t + \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \|z(r)\| dr + \frac{\lambda_2}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \|z(r-r)\| dr.

With the Cauchy–Schwartz inequality, we obtain

\[ \|z_i(t)\| \leq \|z_i(0)\| + \|z'_i(0)\| t + \frac{\lambda_1}{\Gamma(\alpha)} \left( \int_0^t (t-r)^{2(\alpha-1)} e^{2r} dr \right)^{1/2} \left( \int_0^t e^{-2r} \|z(r)\|^2 dr \right)^{1/2} + \frac{\lambda_2}{\Gamma(\alpha)} \left( \int_0^t (t-r)^{2(\alpha-1)} e^{2r} dr \right)^{1/2} \left( \int_0^t e^{-2r} \|z(r-r)\|^2 dr \right)^{1/2}.

In view of \( \int_0^t (t-r)^{2(\alpha-1)} e^{2r} dr < (2e^{2t}/4\alpha) t^{(2\alpha-1)} \), we derive

\[ \|z(t)\| e^{-t} \leq \|z(0)\| e^{-t} + \|z'(0)\| e^{-t} + \frac{\sqrt{2\Gamma(2\alpha-1)}}{2\Gamma(\alpha)} \left\{ \lambda_1 \left( \int_0^t e^{-2r} \|z(r)\|^2 dr \right)^{1/2} + \lambda_2 e^{-t} \left( \int_0^t e^{-(2r(r-r))} \|z(r-r)\|^2 dr \right)^{1/2} \right\}.

Let \( \omega(t) = \sup_{r \in [0,t]} \|z(r)\| \), then, for any \( r \in [0,t] \), we have

\[ e^{-r} \|z(r)\| \leq \omega(r), \quad e^{-(r-r)} \|z(r-r)\| \leq \omega(r).

Thus, inequality (22) gives

\[ \omega(t) \leq \delta e^{-t} (1+t) + \alpha \left( \int_0^t \omega^2(r) dr \right)^{1/2}.

With Proposition 2, this yields

\[ \left( \int_0^t \omega^2(s) ds \right)^{1/2} \leq \delta \left[ 1 - \left( 1 - e^{-\lambda t} \right)^{1/2} \right]^{-1} \left( \int_0^t (1+r)^{2} e^{-2r} e^{-\Lambda r} dr \right)^{1/2}.

Substituting this into inequality (24), we obtain

\[ \omega(t) \leq \delta e^{-t} (1+t) + \delta \Lambda \left[ 1 - \left( 1 - e^{-\lambda t} \right)^{1/2} \right]^{-1} \left( \int_0^t (1+r)^{2} e^{-2r} e^{-\Lambda r} dr \right)^{1/2}.

By virtue of Proposition 3, it follows that

\[ \omega(t) \leq \delta (1+t) e^{-t} + 2\delta e^{\lambda t} (1+t) \left( 1 - e^{-\lambda t} \right)^{1/2}.

Thus,

\[ \|z(t)\| \leq \delta (1+t) \left[ 1 + 2e^{\lambda (t+1)} \right] \left( 1 - e^{-\lambda t} \right)^{1/2}.

With (14), this gives \( \|z(t)\| < \epsilon \) for \( t \in [0,T] \), which shows that network (7) achieves the FTS w.r.t. \( \{\delta, \epsilon, T\} \). The proof is finished.

3.2 Stability Criterion II for Network (7). In this section, we discuss the finite-time stability of equilibrium point for network (7). For the parameters of network (7), some further hypotheses [9] are given as follows:

(H1) The function \( p_i(\cdot) (i = 1, 2, \cdots, n) \) is a continuous and bounded function such that \( 0 < p_i \leq p_i(\cdot) \leq \overline{p_i} \) on \( \mathbb{R} \), where \( \overline{p_i} \) and \( \overline{p_i} \) are two positive constants.

(H2) The function \( q_i(\cdot) (i = 1, 2, \cdots, n) \) is a monotonic differentiable function such that \( 0 < q_i \leq q_i(\cdot) \leq \overline{q_i} \) on \( \mathbb{R} \), where \( \overline{q_i} \) and \( \overline{q_i} \) are two positive constants.

(H3) The functions \( f_j \) and \( g_j (j = 1, 2, \cdots, n) \) satisfy the Lipschitz conditions:

\[ \|f_j(x) - f_j(y)\| \leq \xi_j |x - y|, \quad |g_j(x) - g_j(y)| \leq \eta_j |x - y|.

∀x, y \in \mathbb{R},

where \( \xi_j \) and \( \eta_j \) are two positive constants.

(H4) For \( i, j = 1, 2, \cdots, n, a_{ij}, b_{ij}, \xi_j, \xi_j \) and \( \eta_j \) satisfy the following condition:
where $a^* = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$, $b^* = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |b_{ij}|$.

By an argument similar to that in [9, 24], the assumptions (H1)–(H4) can guarantee the existence and uniqueness of equilibrium point for network (7). In what follows, we will concentrate on the finite-time stability for the equilibrium point $x^*$. Now, we introduce some notation. Let

$\rho_1 = \max_{1 \leq j \leq n} \{ \overline{p_j} \} + \max_{1 \leq j \leq n} \left( a^* \xi_j + b^* \eta_j \right)$,

$\rho_2 = \max_{1 \leq j \leq n} \left( b^* \eta_j \sum_{i=1}^{n} \overline{p_i} \right)$,

$\varrho = \left( \rho_1 + \rho_2 e^{-r} \right) \sqrt{2T(2\alpha - 1)} / 2\alpha \Gamma (\alpha)$.

For network (7), let $x(t)$ represent an arbitrary solution with the initial conditions: $x(t) = \phi^{(0)}(t), x'(t) (t \in [-\tau, 0])$,

$$x_i(t) - x_i^* = x_i(0) - x_i^* + x_i'(0)t - \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \left( p_i(x_i)(q_i(x_i(r)) - q_i(x_i^*)) \right) dr$$
$$+ \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \left( p_i(x_i) \sum_{j=1}^{n} a_{ij}(f_j(x_j(r)) - f_j(x_j^*)) \right) dr$$
$$+ \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \left( p_i(x_i) \sum_{j=1}^{n} b_{ij}(g_j(x_j(r - \tau)) - g_j(x_j^*)) \right) dr.$$

For the term $q_i(x_i(r)) - q_i(x_i^*)$, applying Lagrange’s mean value theorem, it follows that

$$|q_i(x_i(r)) - q_i(x_i^*)| \leq \overline{q_i}|x_i(r) - x_i^*|.$$  (35)

Let $u_i(t) = x_i(t) - x_i^*$. Obviously, equation (34) leads to

$$|u_i(t)| \leq |u_i(0)| + |u_i(0)| t + \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \left( \overline{p_i} \overline{q_i} |u_i(r)| \right)$$
$$+ \sum_{j=1}^{n} \overline{p_j} |a_{ij}| \xi_j |u_j(r)| + \sum_{j=1}^{n} \overline{p_j} |b_{ij}| \eta_j |u_j(r - \tau)| dr.$$  (36)

Following the treatment similar to that of (20), we can obtain inequality (32).

**Remark 1.** When $p_i(x_i(t)) = 1$ and $q_i(x_i(t)) = c_i x_i(t)$ ($c_i > 0$), network (7) is reduced to that in [34]. The corresponding results can be easily derived from those in this paper.

**Theorem 2.** Let $0 < \delta < \epsilon$. Under the assumptions (H1)–(H4), the unique equilibrium point $x^*$ of network (7) achieves the FTS w.r.t. $[\delta, \epsilon, T]$, if $\| \phi - x^* \| < \delta$ and

$$(1 + t) \left\{ 1 + 2e^{(\theta + 1)t} \left( 1 - e^{-\theta t} \right)^{1/2} \right\} < \epsilon, \quad \forall t \in [0, T].$$  (32)

**Proof.** Since $x^*$ is the equilibrium point for system (7), we have

$$q_i(x_i^*) - \sum_{j=1}^{n} a_{ij} f_j(x_j^*) - \sum_{j=1}^{n} b_{ij} g_j(x_j^*) - I_i = 0,$$

$$i = 1, 2, \cdots, n.$$  (33)

Based on (7) and (33), we use Proposition 1 to obtain

$$x_i(t) - x_i^* = x_i(0) - x_i^* + x_i'(0)t$$

where $\phi^{(0)}(t), \phi^{(1)}(t) \in C([-\tau, 0], \mathbb{R}^n)$. Let $\| \phi - x^* \| = \max \{ \| \phi^{(0)}(t) - x^* \|, \| \phi^{(1)}(t) - (x^*)' \| \}$.

**Remark 2.** When $\alpha = 1$, network (7) is reduced to an integer-order one. The corresponding finite-time criteria can be easily obtained by repeating the Proofs of Theorems 1 and 2.

**Remark 3.** For $\alpha \in (0, 1)$, Ke and Miao [9] studied the FTS of equilibrium point for a class of delayed FOCCGNNs based on the generalized Bellman–Gronwall inequality; Rajivganthi et al. [24] considered the FTS for a class of BAM FOCCGNNs with delay by resorting to some inequalities; Zheng et al. [23] reported the FTS for a class of memristor-based FOCCGNNs with delay based on a kind of Gronwall’s inequality. It seems to us that these methods can not be directly extended to the case of $\alpha \in (1, 2)$.

**Remark 4.** In the literature, there have been many works [15–17, 34–37] on the finite-time stability or finite-time synchronization for FONNs with $\alpha \in (1, 2)$. The obtained sufficient conditions are some inequalities involving the Mittag–Leffler functions. The proofs are mainly based on the
Laplace transform, the inverse Laplace transform, and the generalized Gronwall–Bellman inequalities related to the Mittag–Leffler functions. However, this method is not applicable to network (7) owing to the technical reason. In the present paper, a kind of different method was used to discuss the FTS of network (7). More precisely, two kinds of finite-time criteria were obtained based on some properties of the Caputo derivative and some inequalities. Especially, these two criteria are expressed as some algebraic inequalities independent of the Mittag–Leffler functions. Therefore, the verification is relatively easy in practical applications.

4. Numerical Simulations

In this section, two examples are presented to illustrate the effectiveness of two criteria.

Example 1. Consider the following FOCGNN model:

\[ \begin{align*}
\frac{C_0D_t^{1.8}}{t} x_i(t) &= -p_i(x_i(t)) \left[ q_i(x_i(t)) - \sum_{j=1}^{3} a_{ij} f_j(x_j(t)) \right] \\
&\quad - \sum_{j=1}^{3} b_{ij} g_j(x_j(t - 0.1) - I_j),
\end{align*} \]

(37)

Here, \( p_i(x_i(t)) = 0.2 \sin(x_i(t)) + 0.3 \) and \( q_i(x_i(t)) = 0.8 \cos(x_i(t)) \) for \( i = 1, 2, 3 \), \( a_{11} = 0.027, a_{12} = 0.008, a_{13} = 0.029, a_{21} = 0.018, a_{22} = 0.017, a_{23} = 0.005, a_{31} = 0.003, a_{32} = 0.029, a_{33} = 0.029, b_{11} = 0.029, b_{12} = 0.004, b_{13} = 0.024, b_{21} = 0.015, b_{22} = 0.013, b_{23} = 0.029, b_{31} = 0.024, b_{32} = 0.028, b_{33} = 0.02, f_j(x_j(t)) = 0.05(|x_j(t) + 1| - |x_j(t) - 1|), \)
\( g_j(x_j(t - 0.1)) = 0.05(|x_j(t - 0.1) + 1| - |x_j(t - 0.1) - 1|), \)
\( I_1 = 0.0478, I_2 = -0.014, \) and \( I_3 = 0.081. \)

Obviously, \( 0.1 \leq p_1(x_1(t)) \leq 0.5, 0.1 \leq p_2(x_2(t)) \leq 0.5, 0.1 \leq p_3(x_3(t)) \leq 0.5, p^* = 0.2, \)
\( \theta_1 = \theta_2 = \theta_3 = 0.56, F_j = G_j = \xi_j = \eta_j = 0.1. \) Moreover, we obtain \( \lambda_1 = 0.5815, \lambda_2 = 0.011, \) and \( \lambda = 0.218. \)

Let \( x(t) \) and \( y(t) \) be two solutions of network (37) with the initial conditions:

\[ \begin{align*}
x(t) &= (1.12 - 0.007t, 1.56 + 0.002t, -1.53 + 0.01t)^T, \\
x'(t) &= (-0.007, 0.002, 0.01)^T, \\
y(t) &= (1.125 - 0.001t, 1.566 + 0.006t, -1.525 + 0.02t)^T, \\
y'(t) &= (-0.001, 0.006, 0.02)^T,
\end{align*} \]

(38)

for \( t \in [-0.1, 0]. \) The time curves for \( x(t) \) and \( y(t) \) are shown in Figure 1.

Based on the initial conditions, \( \delta \) is taken as \( \delta = 0.02. \) Let \( \epsilon = 1. \) Inequality (14) gives the settling time \( T_s = 2.7424. \) The time response of \( \|x(t) - y(t)\| \) is depicted in Figure 2. Obviously, \( \|x(t) - y(t)\| < 1 \) holds for any \( t \in [0, 2.7424] \) which coincides with the result of Theorem 1.

Example 2. Consider the following FOCGNN model:

\[ \begin{align*}
\frac{C_0D_t^{1.8}}{t} x_i(t) &= -p_i(x_i(t)) \left[ q_i(x_i(t)) - \sum_{j=1}^{3} a_{ij} f_j(x_j(t)) \right] \\
&\quad - \sum_{j=1}^{3} b_{ij} g_j(x_j(t - 0.1) - I_j),
\end{align*} \]

(39)

for \( i = 1, 2, 3, \) where \( p_i(x_i(t)) = 0.1 \cos(x_i(t)) + 0.2 \) \( (i = 1, 2, 3), q_i(x_i(t)) = 0.25 x_i(t), q_2(x_2(t)) = 0.4 x_2(t), \)
\( q_3(x_3(t)) = 0.5 x_3(t), a_{11} = 0.04, a_{12} = 0.05, a_{13} = -0.01, a_{21} = 0.04, a_{22} = -0.03, a_{23} = -0.03, a_{31} = -0.06, a_{32} = 0.05, a_{33} = 0.03, b_{11} = -0.05, b_{12} = 0.05, b_{13} = 0.09, b_{21} = 0.08, b_{22} = -0.024, b_{23} = -0.02, b_{31} = 0.05, b_{32} = -0.04, b_{33} = 0.04, f_j(x_j(t)) = 0.025(|x_j(t) + 1| - |x_j(t - 1)|), \)
\( g_j(x_j(t - 0.1)) = 0.025(|x_j(t - 0.1) + 1| - |x_j(t - 0.1) - 1|), I_1 = 0.042, I_2 = 0.061, \) and \( I_3 = 0.039. \)

Obviously,

\[ \begin{align*}
0.1 &\leq p_1(x_1(t)) \leq 0.3, \\
0.1 &\leq p_2(x_2(t)) \leq 0.3, \\
0.1 &\leq p_3(x_3(t)) \leq 0.3,
\end{align*} \]

(40)

\[ \begin{align*}
\frac{dq_1(x_1)}{dx_1} &= 0.25, \\
\frac{dq_2(x_2)}{dx_2} &= 0.4, \\
\frac{dq_3(x_3)}{dx_3} &= 0.5.
\end{align*} \]

(41)

From the above data, we take \( \xi_j = \eta_j = 0.05 \) \( (j = 1, 2, 3), a^* = 0.14, \) and \( b^* = 0.18. \) Moreover,

\[ \sum_{i=1}^{3} \frac{1}{d_j} \max \left( a^* \xi_j + b^* \eta_j \right) = 0.136 < 1. \]

This indicates that network (39) has a unique equilibrium point \( x^*. \) Based on (33), we have

\[ \begin{align*}
0.25 x_1^* + 0.01 f_1(x_1^*) - 0.1 f_2(x_2^*) - 0.08 f_3(x_3^*) - 0.042 &= 0, \\
0.4 x_2^* - 0.12 f_1(x_1^*) + 0.054 f_2(x_2^*) + 0.05 f_3(x_3^*) - 0.061 &= 0, \\
0.5 x_3^* + 0.01 f_1(x_1^*) - 0.01 f_2(x_2^*) - 0.09 f_3(x_3^*) - 0.039 &= 0.
\end{align*} \]

(42)
Let $x(t)$ and $y(t)$ be two solutions with the initial conditions:

$$x(t) = (0.15 + 0.0015t, 0.16 - 0.001t, 0.1 - 0.001t)^T,$$

$$y(t) = (0.18 - 0.001t, 0.13 + 0.0012t, 0.007 + 0.0012t)^T,$$

for any $t \in [-0.1, 0]$. The time curves are depicted in Figure 3. Moreover, the time evolution for $\|x(t) - x^*\|$ and $\|y(t) - x^*\|$ is shown in Figure 4.
Figure 3: The time response for $x(t)$ and $y(t)$.

Figure 4: The time response for $\|x(t) - x^*\|$ and $\|y(t) - x^*\|$.
Based on the above data, it follows that $\rho_1 = 0.1563$, $\rho_2 = 0.0081$ and $\theta = 0.0897$. We take $\delta = 0.05$. Let $\epsilon = 1$. The condition (32) gives the settling time $T_s = 2.694$. From Figure 4, it can be checked that $\|x(t) - x^*\| < 1$ and $\|y(t) - x^*\| < 1$ hold for $t \in [0, 2.694]$. This fact is consistent with Theorem 2.

5. Conclusions

In the recent decade, many efforts have been made to the research on FONNs with the fractional order between 1 and 2. The methods are mainly based on the Laplace transform, the inverse Laplace transform, and the generalized Gronwall inequality related to the Mittag–Leffler functions. However, these methods do not work well for the considered FOGNNs owing to the technical reason. In this paper, the finite-time stability criteria were derived based on the analytic techniques and some inequalities. This kind of method is completely different from the above ones. In particular, the obtained criteria are expressed as some algebraic inequalities independent of the Mittag–Leffler functions and thus, they can be easily verified in practical applications. In the future work, we will investigate the finite-time guaranteed cost control for FONNs with high fractional order. It seems to us that some new techniques would be developed to deal with this problem.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant nos. 61773220, 61876192, and 61907021), the Natural Science Foundation of Hubei Province of China, the Fundamental Research Funds for the Central Universities of South-Central University for Nationalities (Grant nos. KTT20051, CZT20020, and CZT20022), and School Talent Funds (No. YZZ19004).

References

[1] M. A. Cohen and S. Grossberg, “Absolute stability of global pattern formation and parallel memory storage by competitive neural networks,” *IEEE Transactions on Systems, Man, and Cybernetics*, vol. SMC-13, no. 5, pp. 815–826, 1983.

[2] Q. Zhu, J. Cao, and R. Rakkiyappan, “Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays,” *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1085–1098, 2015.

[3] X. Nie, W. X. Zheng, and J. Cao, “Multistability of memristive Cohen-Grossberg neural networks with non-monotonic piecewise linear activation functions and time-varying delays,” *Neural Networks*, vol. 71, pp. 27–36, 2015.

[4] M. Şaylı and E. Yilmaz, “State-dependent impulse Cohen-Grossberg neural networks with time-varying delays,” *Neurocomputing*, vol. 171, pp. 1375–1386, 2016.

[5] A. Abdurahman, H. Jiang, and K. Rahman, “Function projective synchronization of memristor-based Cohen-Grossberg neural networks with time-varying delays,” *Cognitive Neurodynamics*, vol. 9, no. 6, pp. 603–613, 2015.

[6] Z. Chen, D. Zhao, and J. Ruan, “Dynamic analysis of high-order Cohen-Grossberg neural networks with time delay,” *Chaos, Solitons & Fractals*, vol. 32, no. 4, pp. 1538–1546, 2007.

[7] Y. Shi and P. Zhu, “Asymptotic stability analysis of stochastic reaction-diffusion Cohen-Grossberg neural networks with mixed time delays,” *Applied Mathematics and Computation*, vol. 242, pp. 159–167, 2014.

[8] L. G. Wan and A. L. Wu, “Mittag-Leffler stability analysis of fractional-order fuzzy Cohen-Grossberg neural networks with deviating argument,” *Advances in Difference Equations*, vol. 2017, no. 1, 2017.

[9] Y. Ke and C. Miao, “Stability analysis of fractional-order Cohen-Grossberg neural networks with time delay,” *International Journal of Computer Mathematics*, vol. 92, no. 6, pp. 1102–1113, 2015.

[10] A. Pratap, R. Raja, J. Cao, C. P. Lim, and O. Bagdasar, “Stability and pinning synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous activations,” *Applied Mathematics and Computation*, vol. 359, pp. 241–260, 2019.

[11] Y.-J. Huang, X.-Y. Yuan, X.-H. Yang, H.-X. Long, and J. Xiao, “Multiple Lagrange stability and Lyapunov asymptotical stability of delayed fractional-order Cohen-Grossberg neural networks,” *Chinese Physics B*, vol. 29, no. 2. Article ID 020703, 2020.

[12] R. C. Koeller, “Application of fractional calculus to the theory of viscoelasticity,” *Journal of Applied Mechanics*, vol. 51, no. 2, pp. 294–298, 1984.

[13] R. L. Bagley and R. A. Calico, “Fractional order state equations for the control of viscoelastically damped structures,” *Journal of Guidance, Control, and Dynamics*, vol. 14, no. 2, pp. 304–311, 1991.

[14] T. T. Hartley and C. F. Lorenzo, “Dynamics and control of initialized fractional-order systems,” *Nonlinear Dynamics*, vol. 29, no. 1–4., pp. 201–233, 2002.

[15] C. Rajivganthi, F. A. Rihan, S. Lakshmanan, R. Rakkiyappan, and P. Muthukumar, “Synchronization of memristor-based delayed BAM neural networks with fractional-order derivatives,” *Complexity*, vol. 21, no. 2. pp. 412–426, 2016.

[16] R. Rakkiyappan, G. Velamurugan, and J. Cao, “Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays,” *Nonlinear Dynamics*, vol. 78, no. 4, pp. 2823–2836, 2014.

[17] C. Y. Chen, S. Zhu, Y. C. Wei, and C. Y. Yang, “Finite-time stability of delayed memristor-based fractional-order neural networks,” *IEEE Transactions on Cybernetics*, vol. 50, no. 4, pp. 1607–1616, 2018.

[18] R. Wu, Y. Lu, and L. Chen, “Finite-time stability of fractional delayed neural networks,” *Neurocomputing*, vol. 149, pp. 700–707, 2015.

[19] L. Chen, J. Cao, R. Wu, J. A. Tenreiro Machado, A. M. Lopes, and H. Yang, “Stability and synchronization of fractional-order memristive neural networks with multiple delays,” *Neural Networks*, vol. 94, pp. 76–85, 2017.

[20] Z. Y. Yang and J. Zhang, “Stability analysis of fractional-order bidirectional associative memory neural networks with mixed
time-varying delays,” Complexity, vol. 2019, Article ID 2363707, 22 pages, 2019.

[21] J. Hu, G. Sui, X. Lv, and X. Li, “Fixed-time control of delayed neural networks with impulsive perturbations,” Nonlinear Analysis: Modelling and Control, vol. 23, no. 6, pp. 904–920, 2018.

[22] X. Li, D. O’Regan, and H. Akca, “Global exponential stabilization of impulsive neural networks with unbounded continuously distributed delays,” IMA Journal of Applied Mathematics, vol. 80, no. 1, pp. 85–99, 2015.

[23] M. Zheng, L. Li, H. Peng, J. Xiao, Y. Yang, and H. Zhao, “Finite-time stability and synchronization for memristor-based fractional-order Cohen-Grossberg neural network,” The European Physical Journal B, vol. 89, p. 204, 2016.

[24] C. Rajivganthi, F. A. Rihan, S. Lakshmanan, and P. Muthukumar, “Finite-time stability analysis for fractional-order Cohen-Grossberg BAM neural networks with time delays,” Neural Computing and Applications, vol. 29, no. 12, pp. 1308–1320, 2016.

[25] X. Yang, X. Li, and J. Cao, “Robust finite-time stability of singular nonlinear systems with interval time-varying delay,” Journal of the Franklin Institute, vol. 355, no. 3, pp. 1241–1258, 2018.

[26] X. Li, X. Yang, and S. Song, “Lyapunov conditions for finite-time stability of time-varying time-delay systems,” Automatica, vol. 103, pp. 135–140, 2019.

[27] M. Hui, C. Wei, J. Zhang et al., “Finite-time synchronization of memristor-based fractional order Cohen-Grossberg neural networks,” IEEE Access, vol. 8, pp. 73698–73713, 2020.

[28] P. Wan and J. Jian, “Global mittag-leffler boundedness for fractional-order complex-valued cohen-grossberg neural networks,” Neural Processing Letters, vol. 49, no. 1, pp. 121–139, 2019.

[29] F. Mainardi, “The fundamental solutions for the fractional diffusion-wave equation,” Applied Mathematics Letters, vol. 9, no. 6, pp. 23–28, 1996.

[30] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, and P. Paradisi, "Fractional diffusion: probability distributions and random walk models," Physica A: Statistical Mechanics and Its Applications, vol. 305, no. 1-2, pp. 106–112, 2002.

[31] W. Zhang, X. Cai, and S. Holm, “Time-fractional heat equations and negative absolute temperatures,” Computers & Mathematics with Applications, vol. 67, no. 1, pp. 164–171, 2014.

[32] W. Yu, Y. Li, G. Wen, X. Yu, and J. Cao, "Observer design for tracking consensus in second-order multi-agent systems: fractional order less than two," IEEE Transactions on Automatic Control, vol. 62, no. 2, pp. 894–900, 2017.

[33] X. Li, J. Shen, H. Akca, and R. Rakkiyappan, “LMI-based stability for singularly perturbed nonlinear impulsive differential systems with delays of small parameter,” Applied Mathematics and Computation, vol. 250, pp. 798–804, 2015.

[34] R.-C. Wu, X.-D. Hei, and L.-P. Chen, “Finite-time stability of fractional-order neural networks with delay,” Communications in Theoretical Physics, vol. 60, no. 2, pp. 189–193, 2013.

[35] R. Rakkiyappan, G. Velmurugan, and J. D. Cao, "Finite-time synchronization of fractional-order memristor-based neural networks with time delays," Neural Networks, vol. 73, pp. 36–46, 2016.

[36] J. Xiao, S. Zhong, Y. Li, and F. Xu, “Finite-time Mittag-Leffler synchronization of fractional-order memristive BAM neural networks with time delays,” Neurocomputing, vol. 219, pp. 431–439, 2017.

[37] Y. P. Cao and C. Z. Bai, “Finite-time stability of fractional-order BAM neural networks with distributed delay,” Abstract and Applied Analysis, vol. 2014, Article ID 634803, 8 pages, 2014.

[38] I. Podlubny, Fractional Differential Equations, Academic Press, New York, NY, USA, 1999.

[39] C. Li and W. Deng, “Remarks on fractional derivatives,” Applied Mathematics and Computation, vol. 187, no. 2, pp. 777–784, 2007.

[40] D. Willett, "Nonlinear vector integral equations as contraction mappings," Archive for Rational Mechanics and Analysis, vol. 15, no. 1, pp. 79–86, 1964.

[41] D. Mitrovic, Analytic Inequalities, Springer, Berlin, Germany, 1970.