A new MUSIC electromagnetic imaging method with enhanced resolution for small inclusions

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Abstract.
This paper investigates the influence of test dipole on the resolution of the multiple signal classification (MUSIC) imaging method applied to the electromagnetic inverse scattering problem of determining the locations of a collection of small objects embedded in a known background medium. Based on the analysis of the induced electric dipoles in eigenstates, an algorithm is proposed to determine the test dipole that generates a pseudo-spectrum with enhanced resolution. The amplitudes in three directions of the optimal test dipole are not necessarily in phase, i.e., the optimal test dipole may not correspond to a physical direction in the real three-dimensional space. In addition, the proposed test-dipole-searching algorithm is able to deal with some special scenarios, due to the shapes and materials of objects, to which the standard MUSIC doesn’t apply.

1. Introduction
The multiple signal classification (MUSIC) algorithm has been of great interest in the inverse scattering community since it was proposed to locate point-like scatterers in 2000 [1]. Positions of the small objects are retrieved from the multistatic response (MSR) matrix generated by an array of transceivers [2–6]. MUSIC imaging method was first applied to acoustic imaging, where scalar field is involved. The test function used to generate the MUSIC pseudo-spectrum is the Green’s function of the background medium associated with a monopole source [3,5,7]. Recently, MUSIC algorithm was generalized to electromagnetic imaging of small three-dimensional targets [6,8]. In case of electromagnetic scattering, the induced sources inside small objects are electric dipoles and/or magnetic dipoles, not monopoles any more. The test function that is used to generate the MUSIC pseudo-spectrum is chosen to be the Green’s function of the background medium associated with an electric or magnetic dipole source with an arbitrary orientation [6,8].

This paper focuses on two phenomena in electromagnetic MUSIC imaging that are different from the ones in acoustic MUSIC imaging. The first is regarding the spatial resolution in the presence of noise. In acoustic imaging, the test function used to generate MUSIC pseudo-spectrum is the Green’s function of the background medium associated with a monopole source. In electromagnetic imaging, the test function that is used to generate the MUSIC pseudo-spectrum is chosen to be the Green’s function of the background medium associated with an electric or magnetic dipole. Although the test dipole can be oriented in any direction in noise free case for non-degenerate scatterers, the MUSIC pseudo-spectrum depends noticeably on the orientation of the test dipole in noisy scenarios. The second phenomenon is regarding degenerate
scatterers in which only one or two independent components of an electric or magnetic dipole are induced inside some small scatterers due to special shapes or composing materials of the scatterers. For example, a needle-like or disk-like small object may present only one or two dominant components of induced electric dipoles. For an anisotropic small sphere, when some components in the principal axes of its permittivity tensor are equal to the permittivity of the background medium, the number of independent electric dipole components are less than three. In degenerate cases, standard MUSIC algorithms [6,8] do not work because an arbitrarily chosen direction of test dipole is not necessarily located in the space spanned by actually induced independent dipole components.

The choice of the test dipole direction has been investigated in [9,10] in two and three dimensional cases, respectively, to deal with degenerate scatterers. However, to the best of our knowledge, the effect of the choosing test dipole direction on the resolution of imaging has not been investigated. In this paper, we propose an algorithm to obtain the direction of the test dipole that yields enhanced resolution, and it can also deal with degenerate cases. Compared with the previous MUSIC algorithms [6,8–10] that search for the test dipole direction so that the corresponding Green’s function vector is orthogonal to the noise space that is the orthogonal complement to the range of the MSR matrix, the proposed algorithm determines the test dipole direction so that the corresponding Green’s function vector is in the space spanned by the dominant eigenvectors of the MSR matrix. Analysis of the induced electric dipoles in eigenstates provides the physical insight of the proposed method. Theoretical and numerical results show that three components of the optimal test dipole are not necessarily in phase, i.e., the optimal test dipole may not correspond to a physical direction in the real three-dimensional space. The proposed algorithm was tested through numerical simulations and was found to not only provide better resolution than the standard MUSIC algorithm but also work well in the presence of degenerate objects.

2. Forward scattering problem

Consider $M$ three-dimensional objects that are illuminated by time-harmonic electromagnetic waves radiated by an array of $N$ antenna units. The antenna units are located at $\mathbf{r}_{i1}, \mathbf{r}_{i2}, \ldots, \mathbf{r}_{iN}$, and each consists of 3 small dipole antennas oriented in the $x$, $y$ and $z$ direction with the same length $l$ and driving current $I_{ix}$, $I_{iy}$, $I_{iz}$, respectively, $i = 1, 2, \ldots, N$. The $M$ scatterers can be of any shape, but we consider only spherical or ellipsoidal objects in this paper for ease of presenting. The size of each scatterer is much smaller than the wavelength so that Rayleigh scattering occurs. The centers of the scatterers are located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M$. The scatterers are made of isotropic or anisotropic materials. The shape and composing material of each small scatterer determine its polarization tensor $\tilde{\xi}_j$ [11,12], which relates the induced electric current dipole $\mathbf{I}(\mathbf{r}_j)$ inside the object to the total incident electric field $\mathbf{E}^\text{in}(\mathbf{r}_j)$ by $\mathbf{I}(\mathbf{r}_j) = \tilde{\xi}_j \cdot \mathbf{E}^\text{in}(\mathbf{r}_j)$, $j = 1, 2, \ldots, M$. Expressions for $\tilde{\xi}_j$ for isotropic and anisotropic spheres and ellipsoids can be found in Ref. [11]. In this paper, we assume all scatterers are non-magnetic and the background medium is free space.

By using the Foldy-Lax equation, the multi-static response (MSR) matrix that relates scattered fields to driving current dipoles is given by [8,10]

$$
\tilde{K} = \tilde{R} \cdot \tilde{\Lambda} \cdot (\tilde{I}_{3M} - \tilde{\Phi} \cdot \tilde{\Lambda})^{-1} \cdot \tilde{T},
$$

where $\tilde{R}(i,j) = i\mu_0 \tilde{G}_0(\mathbf{r}_i', \mathbf{r}_j)$, $\tilde{T} = \tilde{T}^T$, $\tilde{\Lambda} = \text{diag}[\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_M]$, $\tilde{I}_{3M}$ is a $3M$-dimensional identity matrix, $\mu_0$ is the permeability of free space, and $\tilde{\Phi}(j,j')$ is null for $j = j'$ or is $i\mu_0 \tilde{G}_0(\mathbf{r}_j, \mathbf{r}_j')$ for $j \neq j'$ with $\tilde{G}_0(\mathbf{r}, \mathbf{r}')$ being the dyadic Green’s function in free space [13]. The MSR matrix is characteristic of the collection of scatterers for given sets of transceivers at the frequency of operation.
3. The MUSIC algorithm

3.1. Standard MUSIC algorithm

The MSR matrix $\tilde{K}$ maps $C^{3N}$, the vector space of complex 3N-tuples, to its range $S_r \subseteq C^{3N}$. From the singular value decomposition (SVD) analysis [14], the MSR matrix could be represented as $\tilde{K} \cdot \tilde{v}_p = \sigma_p \tilde{u}_p$ and $\tilde{K}^* \cdot \tilde{u}_p = \sigma_p \tilde{v}_p$, $p = 1, 2, \ldots, 3N$, where the superscript * denotes the Hermitian. The vector space $C^{3N}$ can be decomposed into the direct sum of the range $S_r = \text{span}\{\tilde{u}_p, \sigma_p > 0\}$ and the orthogonal complement space $S_n = \text{span}\{\tilde{u}_p, \sigma_p = 0\}$ that is referred to as noise space. Consider non-degenerate scatterers in the absence of noise, three independent electric current dipole components are induced in each scatterer, and the scattered field $\tilde{E}^s$ is in the space $S_0$ spanned by the background Green’s function vectors associated with the $x, y,$ and $z$ components of electric dipoles evaluated at the position of each scatterer, i.e., $\tilde{E}^s \in S_0 = \text{span}\{\tilde{G}_x(r_j), \tilde{G}_y(r_j), \tilde{G}_z(r_j); j = 1, 2, \ldots, M\}$, where $\tilde{G}_x(r_j)$, $\tilde{G}_y(r_j)$ and $\tilde{G}_z(r_j)$ are the $[3(j-1)+1]^{th}$, $[3(j-1)+2]^{th}$, and $[3(j-1)+3]^{th}$ column of matrix $\tilde{R}$, respectively. Green’s function vector, $\tilde{G}_l(r)$, $l = x, y, z$, evaluated at an arbitrary position $r$ can be defined similarly. In this case, it is easy to conclude that two subspaces $S_r$ and $S_0$ are identical [2,3,8].

Due to the orthogonality between the range $S_r$ and the noise space $S_n$, we have $|\tilde{u}_p^* \tilde{G}_l(r_m)| = 0$ and for $\sigma_p = 0$, $m = 1, 2, \ldots, M$ and $l = x, y, z$. The standard MUSIC algorithm [6,8] defines the following pseudo-spectrum

$$\Phi(r) = \frac{1}{\sum_{\sigma_p=0} |\tilde{u}_p^* \tilde{f}(r)|^2},$$

where test function $\tilde{f}(r)$ can be any linear combination of $\tilde{G}_x(r)$, $\tilde{G}_y(r)$, and $\tilde{G}_z(r)$. This pseudo-spectrum becomes infinite at the position of every scatterer.

3.2. MUSIC algorithm with the optimal test dipole direction

In degenerate cases, however, since the independent electric dipoles induced in a degenerate scatterer is less than three, an arbitrarily chosen direction of test dipole is not necessarily located in the space spanned by actually induced independent dipole components. Thus the standard MUSIC algorithm may fail to detect the degenerate scatterer [10]. In addition, even if there is no degenerate scatterers, when scattered fields are noise contaminated, the performance of the MUSIC algorithm is found to noticeably depend on the orientation of the test dipole.

To find the optimal test dipole direction, it is equivalent to determine $\mathbf{a} \in C^3$ subject to $||\mathbf{a}|| = 1$, so that the solution $\tilde{x}$ to the equation

$$\tilde{K} \cdot \tilde{x} = \tilde{G}(r) \cdot \mathbf{a}$$

is most robust in the presence of noise, where $\tilde{G}(r) = [\tilde{G}_x(r), \tilde{G}_y(r), \tilde{G}_z(r)]$. The SVD of $\tilde{K}$ is given by

$$\tilde{K} = \sum_{i=1}^{3N} \tilde{u}_i \sigma_i \tilde{v}_i^*.$$  

Assume eigenvalues are in non-increasing order, $\sigma_1 \geq \sigma_2 \geq \ldots, \geq \sigma_{3N} \geq 0$. The least squares solution of $\tilde{x}$ is given by

$$\tilde{x} = \sum_{i=1}^{3N} \frac{\tilde{u}_i^* \cdot \tilde{G}(r) \cdot \mathbf{a}}{\sigma_i} \tilde{v}_i.$$  

Note that the value of $\frac{1}{\sigma_i}$ is large for a small $\sigma_i$. To obtain a stable solution $\tilde{x}$, we should find $\mathbf{a}$ so that $\tilde{u}_i^* \cdot \tilde{G}(r) \cdot \mathbf{a}$ is non-zero for only the first few items. Due to the orthogonality of $\tilde{u}_i$, we
need to find $\mathbf{a}$ so that $\tilde{\mathbf{G}}(\mathbf{r}) \cdot \mathbf{a}$ is a linear combination of the first few $\bar{u}_i$.

$$\sum_{i=1}^{L} \lambda_i \bar{u}_i = \tilde{\mathbf{G}}(\mathbf{r}) \cdot \mathbf{a}. \quad (6)$$

The proposed MUSIC algorithm is based on the analysis of the induced electric current dipoles in the eigenstate, which is referred to as the eigen-dipole hereafter. Assume that the current in the $j$th scatterer in the $i$th eigen-state is equal to $\mathbf{J}^{(i)}_j$. We have

$$\bar{u}_i = \sum_{j=1}^{M} \tilde{\mathbf{G}}(\mathbf{r}_j) \cdot \mathbf{J}^{(i)}_j, \quad i = 1, 2, \ldots, L. \quad (7)$$

In the absence of noise, (7) is exactly hold, whereas in the presence of noise, the quantities in the left is approximately equal to that of the right and their difference is equal to the noise. The substitution of (7) into (6) yields

$$\sum_{j=1}^{M} \tilde{\mathbf{G}}(\mathbf{r}_j) \cdot \sum_{i=1}^{L} \lambda_i \mathbf{J}^{(i)}_j = \tilde{\mathbf{G}}(\mathbf{r}) \cdot \mathbf{a}. \quad (8)$$

There are two cases to be considered: (a) when the test position $\mathbf{r}$ is not at any of the scatterers, $\mathbf{r}_j$, and (b) $\mathbf{r}$ is at one of the scatterers.

Note that the Green’s function vector $\tilde{\mathbf{G}}(\mathbf{r})$ are linearly independent to each other [5, 6]. When the test position $\mathbf{r}$ is not at any of the scatterers, (8) is hold only when $\mathbf{a} = 0$ and $\lambda_i = 0$. Therefore, for any dipole direction $\mathbf{a}$, which satisfies $||\mathbf{a}|| = 1$, $\bar{u}_i^* \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \mathbf{a}$ is not equal to zero for all $3N$ left singular vectors $\bar{u}_i$. In this case, the solution $\bar{x}$ is the linear combination of all $3N$ right eigenvectors $\bar{v}_i$ as shown in (5). The norm of $\bar{x}$ is large and it is not stable due to the presence of small $\sigma_i$.

When $\mathbf{r}$ is at one of the scatterers, for example, $\mathbf{r} = \mathbf{r}_1$, (8) requires

$$\sum_{i=1}^{L} \mathbf{J}^{(i)}_1 \lambda_i = \mathbf{a}, \quad (9a)$$

$$\sum_{i=1}^{L} \mathbf{J}^{(i)}_j \lambda_i = 0, \quad j = 2, 3, \ldots, M. \quad (9b)$$

Eq. (9b) amounts to determining the minimum value of $L$ so that $\mathbf{J}^{(1)}$, $\mathbf{J}^{(2)}$, $\ldots$, $\mathbf{J}^{(L)}$ are linearly dependent, where $\mathbf{J}^{(i)}$ is a column vector of length $3(M - 1)$ consisting of $\mathbf{J}^{(i)}_j$, $j = 2, 3, \ldots, M$. Therefor, when the test point $\mathbf{r}$ is at one of the scatterers, the value of minimum $L$ is equal to one plus the total number of independent dipoles induced in other scatterers. For example, for $M$ isotropic spheres, the value of $L$ equals to $3M - 2$. It is stressed that the algorithm also applies to degenerate cases. When the nontrivial $\lambda_i$, $i = 1, 2, \ldots, L$ obtained from (9b) are plugged into (9a), the resulting $\mathbf{a}$ is generally a complex value. When we force the vector $\mathbf{a}$ to be real, more eigen-states are needed to solve (9b). Thus from (5), we know that the solution is not as robust as the one obtained from the previous complex $\mathbf{a}$.

An empirical approach to determine the approximate value of $L$ is to find the total number of dominant singular values from the spectrum. The test dipole direction is determined by finding $\mathbf{a} \in \mathbb{C}^3$ subject to $||\mathbf{a}|| = 1$, so that $\tilde{\mathbf{G}}(\mathbf{r}) \cdot \mathbf{a}$ is close to the space spanned by the first $L$ dominant
eigenvectors $\vec{u}_i$, i.e., we aim at a minimum projection angle between the vector $\vec{G}(r) \cdot \mathbf{a}$ and the space spanned by the eigenvectors $\vec{u}_i$, $i = 1, 2, \ldots, L$:

$$\mathbf{a} = \arg \max_{\mathbf{a}} \frac{\sum_{i=1}^{L} |\vec{u}_i^* \cdot \vec{G}(r) \cdot \mathbf{a}|^2}{|\vec{G}(r) \cdot \mathbf{a}|^2}$$  \hspace{1cm} (10)$$

From the general eigenvalue decomposition, we obtained the solution $\mathbf{a}$ that is given by the eigenvector corresponding to the maximum eigenvalue of the matrix 

$$\left( \vec{G}(r)^* \cdot \vec{G}(r) \right)^{-1} \left( \vec{U} \cdot \vec{G}(r) \right)^* \left( \vec{U} \cdot \vec{G}(r) \right),$$

where $\vec{U} = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_L]^*$. Then, the pseudospectrum can be defined as

$$\Phi(r) = \frac{1}{1 - \sum_{i=1}^{L} |\vec{u}_i^* \cdot \vec{G}(r) \cdot \mathbf{a}_{\text{max}}|^2} |\vec{G}(r) \cdot \mathbf{a}_{\text{max}}|^2,$$  \hspace{1cm} (11)$$

where $\mathbf{a}_{\text{max}}$ is the optimal direction obtained by (10).

4. Numerical simulation

The inversion method proposed in previous section is tested through numerical simulations in two scenarios, noise free case and noise-contaminated case.

We assume that three small spheres are located at $\mathbf{r}_1 = (0.084\lambda, 0.196\lambda, 0.084\lambda)$, $\mathbf{r}_2 = (-0.168\lambda, -0.056\lambda, -0.112\lambda)$ and $\mathbf{r}_3 = (-0.196\lambda, -0.084\lambda, 0.140\lambda)$, the first two of which are isotropic spheres with permittivity $\epsilon_1 = \epsilon_2 = 2\epsilon_0$, while the third is a rotated anisotropic sphere with permittivity tensor $\vec{\epsilon}_3 = \text{diag}[\epsilon_0, 3\epsilon_0, 9\epsilon_0]$ and rotation Euler angles $[\psi, \phi, \theta] = (\pi/4, \pi/3, 3\pi/8)$. Here, $\epsilon_0$ is the permittivity of the background free space. These three spheres are electrically small with the same radius $a = \lambda/30$. Note that the smallest distance between

Figure 1. Singular values and pseudospectrum obtained by the standard MUSIC algorithm in noise free case. (a) The 10 base logarithm of the singular values of the MSR matrix $(j = 1, 2, \ldots, 48)$. (b), (c) and (d) are the 10 base logarithm of the pseudospectrum in $y = x + 0.112\lambda$ plane obtained by the standard MUSIC algorithm with test dipoles in $x$, $y$ and $z$ directions, respectively.

Figure 2. Pseudospectrum obtained by the proposed MUSIC algorithm in noise free case. (a), (b), (c) and (d) are the 10 base logarithm of the pseudospectrum in $y = x + 0.112\lambda$ plane obtained by the proposed MUSIC algorithm corresponding to the $L = 4, 5, 6$ and 7 cases, respectively.
the centers of spheres is 0.255\(\lambda\) (the distance between the second and the third one), and, for the convenience of depiction of the test results, all three spheres are chosen to locate in the 
\(y = x + 0.112\lambda\) plane. Note that, from the constitutive parameters of the scatterers, there are up to eight independent secondary sources induced inside the three scatterers.

There are 16 antenna units employed in this simulation, half of which are aligned along the \(y\) axis while the other half aligned along the \(z\) axis in the \(x = -13\lambda\) plane. The two linear arrays are centered at \((-13\lambda, -9\lambda, 11\lambda)\) with \(5\lambda\) separation distance between neighboring units.

For the noise free case, the MSR matrix is calculated by (1) under the aforementioned circumstance. The singular values of the MSR matrix are shown in Fig.1(a), in which we see that the first eight singular values are much larger than the rest, since they are corresponding to the eight singular vectors spanning the signal space. Fig.1(b), Fig.1(c) and Fig.1(d) are the pseudo-spectrum in \(y = x + 0.112\lambda\) plane obtained by the standard MUSIC method using \(x-, y-\) and \(z\)-oriented test dipole, respectively. Not surprisingly, the standard MUSIC algorithm can only find the first two isotropic spheres and fail to locate the third degenerate anisotropic target. Here, since the pseudo-spectrum value is too large at the positions of the scatterers, we plot the base 10 logarithm of it, and the horizontal and vertical axes in Fig.1(b), Fig.1(c) and Fig.1(d) are the \(x\) and \(z\) coordinate of spatial points in \(y = x + 0.112\lambda\) plane, so do the cases hereafter.

By using (11), the pseudo-spectrum obtained by the proposed MUSIC algorithm are shown in Fig.2 with \(L = 4, 5, 6\) and \(7\). From these results, we see that, to locate the first two isotropic spheres, we only need \(L = 6\), but, to locate the third degenerate anisotropic sphere, we need

Figure 3. Pseudo-spectrum obtained by the proposed MUSIC algorithm in noise free case when the test dipole is constrained to be real. (a), (b), (c), (d), (e) and (f) are the 10 base logarithm of the pseudo-spectrum in \(y = x + 0.112\lambda\) plane obtained by the proposed MUSIC algorithm corresponding to the \(L = 4, 5, 6, 7, 8\) and 9 cases, respectively.

Figure 4. Singular values and pseudo-spectrum obtained by the standard MUSIC algorithm in noise-contaminated case (30dB). (a) The 10 base logarithm of the singular values of the MSR matrix \((j = 1, 2, \ldots, 48)\). (b), (c) and (d) are the pseudo-spectrum in \(y = x + 0.112\lambda\) plane obtained by the standard MUSIC algorithm with test dipoles in \(x, y\) and \(z\) directions, respectively.
$L = 7$. This is due to the reason that when locating one of the first two isotropic spheres, the rest two spheres have only five independent induced dipoles, which means that $L = 6$ is sufficient for (9b) to have exact solutions; but, if we want to locate the third degenerate sphere, the rest two isotropic spheres have totally six independent induced dipoles, thus only when $L = 7$ can we solve (9b). For the $L = 4$ and 5 cases, since the $L$ is not large enough to solve (9b), none of the three scatterers can be located precisely. If $L$ is further increased to 8 and 9, the result will be almost the same as the one in $L = 7$ case, which are not presented here. If we constrain the test dipole to be real, i.e. $\mathbf{a}$ in (10) and (11) is real, the pseudo-spectrum is shown in Fig.3. We see that $L$ needs to be larger in order to locate the scatterers, i.e., to locate the first two isotropic spheres $L$ need to be at least 7, while to locate the third degenerate sphere $L$ at least 8. Such a phenomenon was expected based on the analysis in the previous section.

For the noise-contaminated case, we add additive white Gaussian noise to the MSR matrix. The noise level is quantified by the signal to noise ratio (SNR) in dB defined as $20 \log_{10} \frac{\| \tilde{\kappa} \|}{\| \kappa \|}$, where $\tilde{\kappa}$ is the additive white Gaussian noise and $\| \cdot \|$ denotes the Frobenius norm of a matrix [2] [8]. In this simulation, 30dB white Gaussian noise is added. Fig.4(a) shows the singular values of the noise-contaminated MSR matrix, in which the singular values corresponding to the noise space are much larger than those in the noise free case. In such a case, if we apply the standard MUSIC algorithm to locate the scatterers, the pseudo-spectrum obtained by the test dipoles in $x$, $y$ and $z$ direction are shown in Fig.4(b), Fig.4(c) and Fig.4(d), respectively, which show that all the three test dipole directions fail to locate any of the three scatterers. By using the

![Figure 5](image1.png)  
**Figure 5.** Pseudo-spectrum obtained by the proposed MUSIC algorithm in noise-contaminated case (30dB). (a), (b), (c), (d), (e) and (f) are the pseudo-spectrum in $y = x + 0.112\lambda$ plane obtained by the proposed MUSIC algorithm corresponding to the $L = 4, 5, 6, 7, 8$ and 9 cases, respectively.

![Figure 6](image2.png)  
**Figure 6.** Pseudo-spectrum obtained by the proposed MUSIC algorithm in noise-contaminated case (30dB) when the test dipole is constrained to be real. (a), (b), (c), (d), (e) and (f) are the pseudo-spectrum in $y = x + 0.112\lambda$ plane obtained by the proposed MUSIC algorithm corresponding to the $L = 4, 5, 6, 7, 8$ and 9 cases, respectively.
proposed MUSIC algorithm, the pseudo-spectrum are drawn in Fig.5. In Fig.5, for the $L = 4, 5, 6$ and 7 cases, similar results can be observed as in Fig.2, however, for the $L = 8$ and 9 cases, some unwanted disturbance appear in between the second and the third spheres, which shows that the singular vector corresponding to the eighth singular value is contaminated by the noise to an extent so that it cannot be regarded as in the signal space anymore. We further constrain the test dipole to be real and obtain the results shown in Fig.6, from which we clearly see that the third sphere cannot be located precisely. This result could also be predicted from the results shown in Fig.3 and Fig.5. From Fig.3 we know that to locate the third degenerate case we need the information from the singular vector corresponding to the eighth singular value which however has be moderately contaminated by noise according to the result shown in Fig.5.

5. Conclusion
A MUSIC algorithm with choosing the optimal test dipole direction has been proposed in this paper, which not only obtains a better resolution than the standard MUSIC algorithm does but also is able to deal with the degenerate scatterers. Based on the analysis of the eigenstate of the MSR matrix, the proposed MUSIC algorithm determines the test dipole direction so that the corresponding Green's function vector is in the space spanned by the dominant eigenvectors of the MSR matrix. In addition, the three components of the optimal test dipole are not necessarily in phase, i.e., the optimal test dipole may not correspond to a physical direction in real three-dimensional space. The proposed algorithm was tested through numerical simulations. Such an algorithm is a good candidate imaging method in situations where partial-aspected low SNR data are collected.

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