Orders of elements and zeros and heights of characters in a finite group*

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1 Introduction

If \( G \) is a finite group and \( g \in G \) and \( \chi \in \text{Irr}(G) \) are such that \( \chi(g) \neq 0 \), then by a fundamental result on projective irreducible characters (e.g. [7, Theorem 8.17]), the squarefree part of the order of \( g \) divides \( |G|/\chi(1) \). We conjecture that the same holds for the order of \( g \) itself:

**Conjecture 1.1.** Let \( G \) be a finite group. Let \( \chi \in \text{Irr}(G) \) and \( g \in G \) and suppose that \( \chi(g) \neq 0 \). Then the order of \( g \) divides \( |G|/\chi(1) \).

It is convenient to say that Conjecture 1.1 holds at \( p \) for a given prime \( p \), if we are able to prove that \( \chi(g) \neq 0 \) implies that the \( p \)-part of the order of \( g \) divides \( |G|/\chi(1) \). For example, if \( \chi \) has height zero in its \( p \)-block, then Conjecture 1.1 holds at \( p \). To see this, let \( D \) be a defect group of the \( p \)-block containing \( \chi \). If \( \chi(g) \neq 0 \) then by e.g. [7, Corollary 15.49], \( g_p \) is conjugate to an element of \( D \). Hence the order of \( g_p \) divides \( |D| \). But by definition, \( \chi \) having height zero is equivalent to \( |D| = (|G|/\chi(1))_p \). Therefore \( g_p \) divides \( |G|/\chi(1) \), as was to be shown. This example suggests a connection between Conjecture 1.1 and character heights; in fact we are able to exploit this to obtain some results on heights (see Section 3).

We briefly discuss evidence for Conjecture 1.1. It holds if \( G \) is solvable, as we show in Section 4. It also holds for the symmetric groups, as a consequence of the hook length formula and Murnaghan-Nakayama rule [21 §4.12 and §4.45]. It is consistent with a theorem of Feit [11, Theorem 1], which states that the values of \( \chi \) are contained in the field of \( |G|/\chi(1)^{th} \) roots of unity. Empirically, we have verified it for all groups in the GAP(2005) character table library [5]; the library includes the ATLAS groups.

In this note, we present some results on Conjecture 1.1 together with related results and questions. In Section 2 we prove a partial result for any finite group and in Section 3 we give an application to character heights. In Section 4 we prove Conjecture 1.1 for solvable groups. In Section 5 we conjecture a congruence for central character values which implies Conjecture 1.1 and is perhaps of some independent interest. We prove this second conjecture in some special cases, including for solvable groups, and give some consequences.

2 Results for arbitrary \( G \)

In this section, we prove the following partial result for arbitrary finite groups \( G \):

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Theorem 2.1. Let $G$ be a finite group and suppose $g \in G$ and $\chi \in \text{Irr}(G)$ are such that $\chi(g) \neq 0$. Let $n$ be the order of $g$, and let $n_0 = \prod_{p|n} p$ be the squarefree part of $n$. Then

\[(i) \ n_0 \text{ divides } \left(\frac{|G|}{\chi(1)}\right)^2 \text{ and (ii) } n^3 \text{ divides } \frac{|G|^3}{\chi(1)^2}.
\]

Theorem 2.1(i) generalizes the classical fact that $n_0$ divides $|G|/\chi(1)$ mentioned at the outset. Theorem 2.1 implies Conjecture 1.1 when $|G|$ is not divisible by any fifth power. More precisely:

Corollary 2.2. In the situation of Theorem 2.1 if $p$ is a prime and $n_p \nmid |G|/\chi(1)$, then $p^5$ divides $n, p^3$ divides $\chi(1)$, and $p^5$ divides $|G|$.

Proof of Corollary. The first two statements are immediate from Theorem 2.1(i), (ii) respectively. Hence $p^{15}$ divides $n^3\chi(1)^2$, so $p^5$ divides $|G|$ by a second application of Theorem 2.1(ii). □

Our proof of Theorem 2.1 is along similar lines to the proof of [7, Theorem 8.17]. We use the following notation: Two elements of $G$ are rationally conjugate if they generate conjugate cyclic subgroups of $G$. If $x$ is a $p$-element of $G$, let $S_p(x)$ be the $p$-section of $x$, that is the set of elements of $G$ whose $p$-part is conjugate to $x$, and write $T_p(x)$ for the rational $p$-section of $x$, that is the set of elements of $G$ whose $p$-part is rationally conjugate to $x$. Let $R(G)$ be the ring of virtual characters of $G$ and let $A$ be the ring of algebraic integers. For a subset $X \subseteq G$, we write $\delta_X$ for the indicator function of $X$, defined by $\delta_X(g) = 1$ if $g \in X$ and $\delta_X(g) = 0$ otherwise. The first two lemmas are standard results.

Lemma 2.3. Let $x$ be a $p$-element of the finite group $G$. If $S_p(x)$ and $T_p(x)$ are respectively the $p$-section and rational $p$-section of $x$, then $|C_G(x)|_p \delta_{S_p(x)} \in A \otimes R(G)$ and $|C_G(x)|_p \delta_{T_p(x)} \in R(G)$.

Proof. By Brauer’s characterization of characters, it suffices to show that $|C_G(x)|_p \delta_{S_p(x)}|_H \in A \otimes R(H)$ for each nilpotent subgroup $H \subseteq G$. But this is a linear combination of $|C_H(x')|_p \delta_{S_p(x')}$ where $x'$ runs over a set of representatives for the $H$-conjugacy classes contained in $x'^G \cap H$. Hence we may assume $G = H$ is nilpotent. Writing $G = P \times Q$ where $P$ is the Sylow $p$-subgroup of $G$, then $\delta_{S_p(x)} = \delta_{x^P} \times 1_Q$, and the result follows as $C_P(x) \delta_{x^P} \in A \otimes R(P)$. The second assertion follows because $T_p(x)$ is a disjoint union of $p$-sections, and if $\chi \in \text{Irr}(G)$ then $[\chi, \delta_{T_p(x)}]$ is a rational number. □

Lemma 2.4. If $u$ is an integer of the cyclotomic field $E = \mathbb{Q}(\zeta_n)$ and $n_0$ is the squarefree part of $n$, then $tr_{E/\mathbb{Q}}(u)$ is divisible by $n/n_0$.

Proof. Since the ring of integers of $E$ is $\mathbb{Z}[\zeta_n]$, we may assume $u = \zeta_m$ is a primitive $m^{th}$ root of unity for some $m$ dividing $n$. Then

\[tr_{E/\mathbb{Q}}(u) = \sum_{0 \leq r < n, (r,n) = 1} \zeta_m^r = \frac{\varphi(n)}{\varphi(m)} \mu(m)\]

where $\varphi$ is Euler’s function and $\mu$ is the M"obius function. However $\mu(m)$ is zero unless $m$ is square free, in which case the right hand side is divisible by $\varphi(n)/\varphi(n_0) = n/n_0$. □

If $H$ is a subgroup of $G$ and $g \in G$ is an element of order $n$, it is convenient to write $\text{Aut}_H(g)$ for the subgroup of $(\mathbb{Z}/n\mathbb{Z})^\ast$ consisting of automorphisms induced on $\langle g \rangle$ by $N_H\langle g \rangle$. Thus

\[\text{Aut}_H(g) = \{ r \in \mathbb{Z}/n\mathbb{Z}, g^r \in g^H \}\]

Clearly $\text{Aut}_H(g)$ is isomorphic to $\frac{N_H\langle g \rangle}{C_H(g)}$. 2
Lemma 2.5. Suppose $H$ is a subgroup of $G$. Let $g \in H$ have order $n$, and let $X_g \subseteq H$ be the rational conjugacy class of $g$ in $H$. If $\chi \in \text{Irr}(G)$ and $\varphi \in \text{Irr}(H)$ then

$$ \frac{1}{|H|} \sum_{h \in X_g} \chi(h) \bar{\chi}(h) \bar{\varphi}(h) \in \left[ \frac{|C_G(g)| n/n_0}{(|G|/|\chi(1)|)^2 |\text{Aut}_H(g)|} \right] \mathbb{Z}. $$

Proof. The rational conjugacy class $X_g$ is the union of equivalence classes under the relation $h_1 \sim h_2$ if $h_1$ and $h_2$ generate the same cyclic subgroup of $H$. The number of these classes is $|H : N_H(g)|$ and for any one such equivalence class $Y \subseteq X_g$, it follows from the definition of the trace that

$$ \sum_{h \in Y} \chi(h) \bar{\chi}(h) \bar{\varphi}(h) = \text{tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\chi(g) \bar{\chi}(g) \bar{\varphi}(g)). $$

Therefore

$$ \frac{1}{|H|} \sum_{h \in X_g} \chi(h) \bar{\chi}(h) \bar{\varphi}(h) = \frac{1}{|H|} \frac{|H|}{|N_H(g)|} \text{tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\chi(g) \bar{\chi}(g) \bar{\varphi}(g)) $$

$$ = \frac{1}{|N_H(g)|} \frac{|C_G(g)|^2}{(|G|/|\chi(1)|)^2} \text{tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\omega_\chi(g) \bar{\omega}_\chi(g) \bar{\varphi}(g)) $$

$$ = \frac{|C_G(g)|}{|C_H(g)|} \frac{|C_G(g)|}{(|G|/|\chi(1)|)^2 |\text{Aut}_H(g)|} \text{tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\omega_\chi(g) \bar{\omega}_\chi(g) \bar{\varphi}(g)) $$

$$ \in \left[ \frac{|C_G(g)| n/n_0}{(|G|/|\chi(1)|)^2 |\text{Aut}_H(g)|} \right] \mathbb{Z}, $$

where $\omega_\chi(g) = \frac{h_\chi(g)}{|\chi(1)|}$ is the value of the central character associated with $\chi$, and the last step follows by Lemma 2.4 because $\omega_\chi(g) \bar{\omega}_\chi(g) \bar{\varphi}(g)$ is an algebraic integer. \hfill \Box

If $g \in G$, let $o(g)$ denote the order of $g$ and let $\nu_p(n)$ be the power of $p$ which divides the integer $n$. Theorem 2.4 will follow easily once we have proved the next result.

Theorem 2.6. Let $\chi \in \text{Irr}(G)$. Suppose $g \in G$ has $\chi(g) \neq 0$. Then for any prime $p$,

$$ 2\nu_p \left( \frac{|G|}{|\chi(1)|} \right) + \nu_p(|\text{Aut}_G(g)|) \geq 2\nu_p(o(g)). $$

Proof. Let $x = g_p$ and define a class function $\theta$ on $G$ by $\theta = \chi \bar{\delta}_{T_p(x)}$. By hypothesis, $\chi(g) \neq 0$, so $\theta$ is not identically zero. We may clearly suppose $x \neq 1$, so that $1 \notin T_p(x)$. Hence

$$ 0 < [\theta, 1_G] = \frac{1}{|G|} \sum_{h \in T_p(x)} |\chi(h)|^2 < [\chi, \chi] = 1, $$

so $\theta$ is not a virtual character of $G$. By Lemma 2.3, $|G| \theta \in R(G)$, so we must have $m\theta \notin R(G)$ whenever $m$ is prime to $p$. By Brauer’s characterization of characters (local form), there exists a $p$-elementary subgroup $H \subseteq G$ and a character $\varphi \in \text{Irr}(H)$ such that $|H|/|\theta, \varphi|_H \notin \mathbb{Z}$. Then $T_p(x) \cap H$ cannot be empty, and is a union of rational conjugacy classes of $H$, and for one of these, say $X_g' \subseteq H$, $\frac{|G|}{|\chi(1)|} \sum_{h \in X_g'} |\chi(h)|^2 \notin \mathbb{Z}$. By Lemma 2.5, we must have

$$ 2\nu_p \left( \frac{|G|}{|\chi(1)|} \right) + \nu_p(|\text{Aut}_H(g')|) > \nu_p(|\text{Aut}_G(g')|) + \nu_p(o(g')) - 1. $$

Since $H$ is $p$-elementary, $|\text{Aut}_H(g')| = |\text{Aut}_H(g'_p)|$, and since $g'_p$ and $g_p$ are rationally conjugate, we have $\nu_p(o(g')) = \nu_p(o(g))$, $\nu_p(|C_G(g')|) \geq \nu_p(o(g))$ and $|\text{Aut}_H(g'_p)|$ divides $|\text{Aut}_G(g_p)|$. The desired inequality follows. \hfill \Box
Proof of Theorem 2.1. If \( g \) has order \( n \) and \( k = \nu_p(n) \geq 1 \) then \( \text{Aut}_G(g_p) \) is a subgroup of \((\mathbb{Z}/p^k\mathbb{Z})^*\), which has order \( p^{k-1}(p-1) \). Therefore \( \nu_p(|\text{Aut}_G(g_p)|) \leq \nu_p(n) - 1 \). Hence by Theorem 2.6, if \( \chi(g) \neq 0 \) then \( 2\nu_p(|G|/\chi(1)) \geq \nu_p(n) + 1 \). This holds for each prime dividing \( n \), so \( nn_0 \) divides \((|G|/\chi(1))^2 \), which is Theorem 2.1 (i). On the other hand, \( \nu_p(|\text{Aut}_G(g_p)|) \leq \nu_p(|G|) - \nu_p(n) \). Hence by Theorem 2.6 if \( \chi(g) \neq 0 \) then \( 2\nu_p(|G|/\chi(1)) + \nu_p(|G|) \geq 3\nu_p(n) \), which similarly implies Theorem 2.1 (ii). \( \square \)

In the proof of Theorem 2.1 the term \( \nu_p(|\text{Aut}_G(g_p)|) \) from Theorem 2.6 is the obstruction to a proof of Conjecture 1.1. If for a given prime \( p \), the Sylow \( p \)-subgroups of \( G \) are abelian, this term vanishes, and the proof of Theorem 2.1 establishes that \( n_p \) divides \(|G|/\chi(1)| \), so Conjecture 1.1 holds at \( p \). Using the following theorem of Willems, we can obtain the same conclusion if a defect group of the \( p \)-block containing \( \chi \) is abelian.

**Theorem 2.7.** ([11 Corollary 3.5]) Let \( \chi \in \text{Irr}(B) \) where \( B \) is a \( p \)-block of the finite group \( G \). Let \( D \) be a defect group of \( B \). Then there exist \( a_i \in \mathbb{Z} \) and elementary subgroups \( H_i \subseteq G \) such that

\[
\chi = \sum a_i \lambda_i^G,
\]

where \( \lambda_i \) is a linear character of \( H_i \) and the Sylow \( p \)-subgroup of each \( H_i \) is contained in \( D \).

**Theorem 2.8.** In the situation of Theorem 2.1 let \( p \) be a prime dividing \( n \) and let \( B \) be the block of \( G \) in characteristic \( p \) which contains \( \chi \). If the defect groups of \( B \) are abelian, then \( n_p \) divides \(|G|/\chi(1)| \). In other words, Conjecture 1.1 holds for \( \chi \) at \( p \).

**Proof.** Let \( D \) be a defect group of \( B \) and let \( \chi = \sum a_i \lambda_i^G \), be the sum in Theorem 2.7. Let \( \theta = \chi\bar{\delta}_{T_p(x)} \) be the class function defined in the proof of Theorem 2.6 and also write \( \theta_1 = \chi\delta_{T_p(x)} \).

As in the proof of Theorem 2.6, \( m\theta \notin R(G) \) for any \( m \) prime to \( p \). Since

\[
\theta = \bar{\chi}\theta_1 = \sum a_i (\bar{\lambda}_i\theta_1|_{H_i})^G,
\]

it follows that for some \( i \) there exists a \( p \)-elementary subgroup \( H \) with \( H \subseteq H_i \) and a character \( \varphi \in \text{Irr}(H) \) such that \( |H|p \theta_1|_{H}\varphi|_H \notin \mathbb{Z} \). Again \( T_p(x) \cap H \) cannot be empty, and is a union of rational conjugacy classes of \( H \), and for one of these, say \( x_1 \subseteq H \), \( \frac{1}{|H|} \sum_{h \in X} \chi(h)\bar{\varphi}(h) \notin \mathbb{Z} \). This is just as in the proof of Theorem 2.6 except that we have \( \chi \) in place of \( \chi\bar{\chi} \). The proof of Lemma 2.10 with \( \chi \) in place of \( \chi\bar{\chi} \) shows that

\[
\frac{1}{|H|_p} \sum_{h \in X} \chi(h)\bar{\varphi}(h) \in \frac{n'/n_0'}{|G|/\chi(1)|\text{Aut}_H(g')|} \mathbb{Z},
\]

where \( n' = o(g') \). But \( H \) is \( p \)-elementary, and its Sylow \( p \)-subgroup is contained in \( D \), so in fact \( H \) is abelian and \( \text{Aut}_H(g') = 1 \). We conclude that \( \nu_p(|G|/\chi(1)|) \geq \nu_p(n') + \nu_p(n) \), as required. \( \square \)

If \( D \) is abelian, then according to Brauer’s height zero conjecture, \( \chi \) in Theorem 2.8 should have height zero, from which it would follow that Conjecture 1.1 holds at \( p \), as remarked in the introduction. The above proof does not rely on the height zero conjecture; moreover, the fact that \( D \) is abelian was only used to ensure \( \text{Aut}_H(g') = 1 \). The proof therefore goes through unchanged provided that in \( D \), each cyclic subgroup is in the center of its normalizer. This holds, for example, if \( D \) is the direct product of an abelian group and a group of exponent \( p \).
3 An application to heights of characters

The results in Section 2 can be used to give some control over heights of characters. If \( B \) is a \( p \)-block of \( G \), we write \( ht_p(\chi) \) for the height of \( \chi \) in \( B \), so that if \( |G|_p = p^a \) and \( B \) has defect \( d \) then \( \chi(1)_p = p^{a-d+ht_p(\chi)} \). For any group \( G \), write \( e(G) \) for the exponent of \( G \).

**Theorem 3.1.** Let \( B \) be a \( p \)-block of the finite group \( G \) and let \( D \) be a defect group of \( B \). Let \( |G|_p = p^a \) and \( |D| = p^d \). Let \( \chi \in \text{Irr}(B) \).

(i) If Conjecture 1.1 holds for \( \chi \) at \( p \), then \( ht_p(\chi) \leq d - \nu_p(e(Z(D))) \). In particular, this holds if \( D \) is abelian or \( G \) is solvable.

(ii) Without assuming Conjecture 1.1 we have \( ht_p(\chi) \leq \frac{a+d}{2} - \nu_p(e(Z(D))) \). If \( a-d \leq 1 \) then (i) follows without Conjecture 1.1. In particular, (i) holds if \( B \) is the principal block of any finite group.

**Proof.** Let \( D \) be a defect group of \( B \), and let \( y \) be such that \( y^G \) is a defect class of \( B \) and \( D \) is a Sylow \( p \)-subgroup of \( C_G(y) \). Let \( x \in Z(D) \) and let \( g = xy \). We claim that \( \chi(g) \neq 0 \). This follows from \([9\text{ Lemma 5.15(b)}]\), but we sketch the short argument. First, \( D \) is a Sylow \( p \)-subgroup of \( C_G(g) \), so if \( \psi \in \text{Irr}(B) \) is a character of height zero, then \( h_g/\psi(1) \) is an integer prime to \( p \). Also if \( p \) is a prime lying over \( p \) in a splitting field for \( G \), then \( \psi(g) = \psi(y) \neq 0 \) mod \( p \), since \( y \) is a defect of \( B \). Hence

\[
\omega_\psi(g) = \frac{h_g\psi(g)}{\psi(1)} \neq 0 \mod p.
\]

Since \( \chi \) and \( \psi \) lie in the same block, also \( \omega_\chi(g) \neq 0 \mod p \), so certainly \( \chi(g) \neq 0 \) as required. Now assuming Conjecture 1.1 at \( p \), we conclude that \( o(x) \) divides \( (|G|/\chi(1))_p = p^{d-ht_p(\chi)} \). Since \( x \in Z(D) \) was arbitrary, (i) follows. Conjecture 1.1 is proved at Theorem 2.8 for \( D \) abelian, and at Corollary 1.3 below for \( G \) solvable.

Next, we do not assume Conjecture 1.1 and appeal instead to Theorem 2.6. Since \( x \in Z(D) \), we have \( \nu_p(|Aut_G(x)|) \leq a - d \), so by Theorem 2.6

\[
2(d - ht_p(\chi)) + a - d \geq 2\nu_p(o(x)).
\]

As \( x \in Z(D) \) was arbitrary, this gives (ii). If \( a-d \leq 1 \) then \( \frac{a+d}{2} \leq \frac{d+1}{2} \), so (ii) implies (i) in this case. \( \square \)

The solvable case of Theorem 3.1(i) is really redundant, since the better inequality \( ht_p(\chi) \leq \nu_p(|D:Z(D)|) \) is true for \( p \)-solvable groups by a well known result of Fong ([3\text{ Theorem 3C}, or see [9\text{ Theorem 10.21}]]. However, if \( G \) is not solvable, then Theorem 3.1 seems in some cases to give a better bound than known results. For example, the bound in [2\text{ Corollary 9.11}] is \( ht_p(\chi) < d - \nu_p(|Z(D)|) + \frac{1}{2}r(r+1) \) where \( r = \nu_p(|D: \Phi(D)|) \). If say \( D = C_{p^r} \times (C_p)^{r-1} \) then this gives \( ht_p(\chi) < \frac{1}{2}r(r+1) \), while Theorem 3.1 gives \( ht_p(\chi) < r \).

4 Solvable groups

Conjecture 1.1 is true for solvable groups. More generally, if \( G \) is \( p \)-solvable then Conjecture 1.1 holds at \( p \) for \( G \), in the sense mentioned in the introduction. This will follow easily from:

**Theorem 4.1.** Let \( G \) be a finite \( p \)-solvable group. Suppose \( \chi \in \text{Irr}(G) \) is primitive. If \( Q \) is an abelian \( p \)-subgroup of \( G \) then \( \chi(1) \) divides \( |G:Q| \).
Theorem 4.1 does not hold for all groups. For example, the alternating group $A_5$ has a primitive character of degree 4.

Corollary 4.2. In the situation of Theorem 4.1 if $G$ is solvable and $A \subseteq G$ has abelian Sylow subgroups, then $\chi(1)$ divides $|G : A|$

Proof. Immediate from Theorem 4.1.

Corollary 4.2 appears as [6] Theorem 5.15b] under the (unnecessary) hypothesis that $A$ is normal in $G$. Our proof above is essentially the same as the proof given there.

Corollary 4.3. Conjecture 1.4 holds for solvable groups.

Proof. Let $\chi \in \text{Irr}(G)$ where $G$ is solvable, and let $\varphi \in \text{Irr}(H)$ be a primitive character of $H \subseteq G$ such that $\chi = \varphi^G$. If $g \in G$ and $\chi(g) \neq 0$ then a conjugate of $g$ lies in $H$. Then Corollary 4.2 implies that the order of $g$ divides $|H|/\varphi(1) = |G|/\chi(1)$.

It would be interesting if Theorem 4.1 could be used along the lines of Theorem 3.1 to prove Fong’s theorem that $ht_p(\chi) \leq \nu_p(|D : Z(D)|)$ for a character in a $p$-block of a $p$-solvable group with defect group $D$. We show this can be done in the easiest case, when $B$ is the principal block of $G$. We fix a prime $p$ and write $G^0$ for the set of $p$-regular elements of $G$ and $[\chi, \psi]^0$ for $\frac{1}{|G|} \sum_{g \in G^0} \chi(g^{-1})\psi(g)$.

Corollary 4.4. Let $G$ be a finite $p$-solvable group and suppose $\chi \in \text{Irr}(G)$ belongs to the principal $p$-block of $G$. Let $P$ be a Sylow $p$-subgroup of $G$. Then $ht_p(\chi) \leq \nu_p(|P : Z(P)|)$.

Proof. As is well known, $\chi$ is lifted from a character in the principal block of $G/O_p(G)$, so by induction we assume $O_p(G) = 1$. We have to prove that $\chi(1_p) \leq |P : Z(D)|$. This follows from Theorem 4.1 if $\chi$ is primitive, so we may assume that $\chi = \varphi^G$ for some maximal subgroup $H$ of $G$ and $\varphi \in \text{Irr}(H)$. Since $\chi$ is in the principal block, $[\chi, 1_G]^0_G \neq 0$, so $[\varphi, 1_H]^0_H \neq 0$ by Frobenius reciprocity; therefore $\varphi$ belongs to the principal block of $H$ (or use the Third Main Theorem). By induction, $\varphi(1)$ divides $|H : Z(Q)|$ where $Q$ is a Sylow $p$-subgroup of $H$. Hence, it suffices to show that $Z(P) \subseteq H$. Suppose the contrary and let $x \in Z(P) \setminus H$. Since $O_p(G) = 1$, the Hall-Higman Lemma implies $x \in Z(O_p(G))$. In particular, $O_p(G) \not\subseteq H$, and as $H$ is maximal we have $G = HO_p(G)$. Then since $x \in Z(O_p(G))$, we have $x^G = x^H$, so $x^G$ is disjoint from $H$ and we conclude that $\chi(x) = 0$. As $\chi$ is in the principal block, $0 = \omega_\chi(x) = |h_x|$ mod $p$, where $p$ is a prime lying over $p$ in a splitting field for $G$. Hence $p \mid h_x$, contradicting the fact that $x \in Z(P)$. This contradiction shows that $Z(P) \subseteq H$ and the proof is complete.

We mention a relative of Corollary 4.2 in which, using projective representations, we replace the character degree $\chi(1)$ by the ramification index of $\chi$ with respect to a normal subgroup.

Theorem 4.5. Let $G$ be a finite group and suppose $\chi \in \text{Irr}(G)$ is primitive. Let $N \lhd G$ be a normal subgroup of $G$ and suppose $G/N$ is solvable. Let $\varphi \in \text{Irr}(N)$ be the character of $N$ lying under $\chi$. Let $e(G/N)$ be the exponent of $G/N$. Then $\chi(1)/\varphi(1)$ divides $|G : N|/e(G/N)$. 

Proof. Since $\chi$ is primitive, $\chi_N = f \varphi$ where $f$ is the ramification index. By the theory of projective representations, there exists a central extension

$$1 \to Z \to \Gamma \xrightarrow{\pi} G \to 1$$

such that $\pi^{-1}(N) = \tilde{N} \times Z$ for some normal subgroup $\tilde{N} \triangleleft \Gamma$, and $\Gamma$ has irreducible characters $\alpha, \beta \in \text{Irr}(\Gamma)$ with $\beta(1) = f, \tilde{N} \subseteq \ker \beta$, and $Z \subseteq \ker(\alpha \beta)$ and $\alpha \beta = \chi$ viewed as a character of $G$. We claim that $\beta$ is primitive. For suppose not, so that $\beta = \eta^\Gamma$ where $\Sigma \subset \Gamma$ is a proper subgroup of $\Gamma$ and $\eta \in \text{Irr}(\Sigma)$. Then $\alpha \beta = \alpha \eta^\Gamma = (\alpha \Sigma \eta)^\Gamma$. Also $\ker(\alpha \beta) = core_G(\ker(\alpha \Sigma \eta))$ by Lemma 5.1], so $Z \subseteq \Sigma$ and $\alpha \Sigma \eta$ is the lift of some $\psi \in \text{Irr}(H)$, where $H$ is the image of $\Sigma$ in $G$. But then $H$ is a proper subgroup of $G$ and $\chi = \psi^G$, contrary to hypothesis. ($\alpha$ is primitive for the same reason but we do not use this.)

Now the sequence above gives rise to the deflated sequence

$$1 \to Z \to \Gamma/\tilde{N} \xrightarrow{\bar{\pi}} G/N \to 1.$$ 

If $C \subseteq G/N$ is any cyclic subgroup then $\bar{\pi}^{-1}(C)$ is abelian. $\beta$ is the inflation of a character of $\Gamma/\tilde{N}$, which must be primitive, so by Corollary 4.2, $\beta(1)$ divides $|\Gamma/\tilde{N} : \bar{\pi}^{-1}(C)| = |G/N : C|$, and the result follows.

For solvable $G$, Corollary 4.2 is an essentially stronger result than Conjecture 1.1. More is also known in a slightly different direction. The following is the main result of [10]:

**Theorem 4.6.** ([10], Main Theorem) Let $G$ be a finite solvable group and suppose $\chi \in \text{Irr}(G)$. Then there exist subgroups $H_i \subseteq G$ such that $\chi(1)$ divides $|G : H_i|$ for each $i$, and a sum

$$\chi = \sum_i a_i \lambda_i^G$$

where for each $i$, $a_i$ is an integer and $\lambda_i$ is a linear character of $H_i$.

Conjecture 1.1 follows from Theorem 4.6 since if $g \in G$ and $\chi(g) \neq 0$ then a conjugate of $g$ lies in some $H_i$, and then the order of $g$ divides $|H_i|$ which divides $|G : \chi(1)|$. Unlike Theorem 4.2, it seems possible that Theorem 4.6 holds for all finite groups. We remark that, for any $G$ and $\chi \in \text{Irr}(G)$, if $\chi$ has height zero in its $p$-block, then the sum in Theorem 2.4 has $\chi(1)_p$ dividing $|G : H_i|$, so that in the obvious sense, Theorem 4.6 holds for $\chi$ at $p$.

Lastly in this section, we remark that the conclusion of Theorem 4.6 also holds for any group $G$ when $|G|/\chi(1)$ is a prime power. In fact, then $\chi$ is monomial:

**Theorem 4.7.** Let $\chi \in \text{Irr}(G)$, where $G$ is a finite group, and suppose that $|G|/\chi(1) = p^k$ where $p$ is prime. Then $\chi$ is monomial.

Proof. Let $P$ be a Sylow $p$-subgroup of $G$ and let $|P| = p^m$. Then $\chi(1) = p^{m-k} |G|_{p'}$. Since $p^{m-k+1} | \chi(1)$, some irreducible component $\psi$ of $\chi|_{p'}$ has degree $\psi(1) \leq p^{m-k}$. But then $\psi^G(1) \leq p^{m-k} |G|_{p'} \leq \chi(1)$, so since $[\chi, \psi^G] \neq 0$, equality holds and $\chi = \psi^G$. The result follows since $\psi$ is monomial. 

\[\square\]
5 Central character values

In our proof of Theorem 2.1, the term \( \nu_p(\lvert \text{Aut}_G(g) \rvert) \) from Theorem 2.6 prevents us from establishing Conjecture 1.1. This term would be removable if in Lemma 2.5 \( \omega_\chi(g) \) were divisible by \( \lvert \text{Aut}_H(g) \rvert \). However, it is not true in general that \( \lvert \text{Aut}_G(g) \rvert \) divides \( \omega_\chi(g) \) for any \( \chi \in \text{Irr}(G) \) and \( g \in G \). For example, let \( \chi \) be the permutation character of degree 4 of the symmetric group \( S_5 \) and let \( g \) be a 5-cycle. Then \( \omega_\chi(g) = -6 \) is not divisible by \( \lvert \text{Aut}_G(g) \rvert = 4 \). Solvable examples also exist.

However, there is some evidence that \( \omega_\chi(g) \) is divisible by the order of a certain large subgroup \( \text{Aut}_G^0(g) \subseteq \text{Aut}_G(g) \). If \( H \) is a subgroup of \( G \) and \( g \in G \) has order \( n \), then we define \( \text{Aut}_H^0(g) \subseteq \text{Aut}_H(g) \) as follows. For each prime \( p \) dividing \( n \), set
\[
\text{Aut}_H^0(g) = \{ m \in \text{Aut}_H(g); m = 1 \mod pn_p' \}
\]
if \( p \) is odd or \( p = 2 \) and \( 4 \notdivides n \), and
\[
\text{Aut}_H^0(g) = \{ m \in \text{Aut}_H(g); m = 1 \mod 4n_2' \}
\]
if \( p = 2 \) and \( 4 \divides n \). Finally, set
\[
\text{Aut}_H^0(g) = \prod_{p \mid n} \text{Aut}_H^0(g).
\]

If \( p \) is odd then \( \text{Aut}_H^0(g) \) is naturally isomorphic to the Sylow \( p \)-subgroup of \( \frac{N_H(g) \cap C_H(g_p)}{C_H(g_p)} \). If \( p = 2 \) and \( 4 \) divides \( n \) then \( \text{Aut}_H^0(g) \) is naturally isomorphic to the subgroup of index at most 2 in this Sylow \( p \)-subgroup, consisting of elements that centralize the unique subgroup of order 4 in \( \langle g \rangle \). Finally, note that if \( H \subseteq G \) is nilpotent and \( g \in H \) then \( \text{Aut}_H^0(g) \subseteq \text{Aut}_H(g) \) has index at most 2.

**Conjecture 5.1.** Let \( \chi \in \text{Irr}(G) \) and for \( g \in G \), let \( \omega_\chi(g) = \frac{h_\chi(g)}{\chi(1)} \) be the central character value. Then \( \omega_\chi(g) \equiv 0 \mod \lvert \text{Aut}_G^0(g) \rvert \).

As with Conjecture 1.1 it is convenient to have a notion of Conjecture 5.1 holding at \( p \) for a prime \( p \). Here, this means that for all \( g \in G \), \( \omega_\chi(g) \) is divisible by \( \lvert \text{Aut}_G^0(g) \rvert = \lvert \text{Aut}_G(g) \rvert_p \).

We will prove in Corollary 5.8 that Conjecture 5.1 is true when \( G \) is solvable, and is true at \( p \) whenever \( \chi \) has height zero in its \( p \)-block. We first show that Conjecture 5.1 implies Conjecture 2.3. Specifically:

**Theorem 5.2.** Let \( G \) be a finite group and suppose \( \chi \in \text{Irr}(G) \). If Conjecture 5.1 holds for \( \chi \) then Conjecture 1.1 also holds for \( \chi \).

**Proof.** Assume that \( \chi \in \text{Irr}(G) \) and \( g \in G \) are such that \( \chi(g) \neq 0 \), and let \( x \) be the \( p \)-part of \( g \) for a fixed prime \( p \). Following the proof of Theorem 2.6, there is a \( p \)-elementary subgroup \( H \subseteq G \) and a rational conjugacy class \( X_g' \subseteq \text{Rep}(x) \cap H \), such that \( \frac{1}{|H|} \sum_{h \in X_g'} \chi(h)\bar{\chi}(h) \bar{\phi}(h) \neq 0 \).

Assuming Conjecture 5.1, we may strengthen Lemma 2.5 to
\[
\frac{1}{|H|} \sum_{h \in X_g} \chi(h)\bar{\chi}(h) \bar{\phi}(h) \in \frac{|C_G(g)|n/n_0}{\lvert \text{Aut}_G^0(g) \rvert^2} \frac{|G/\chi(1)|}{|\text{Aut}_H(g)|} \mathbb{Z}.
\]
Hence as in the proof of Theorem 2.6 we now obtain:
\[
2\nu_p\left( \frac{|G|}{\chi(1)} \right) + \nu_p(\lvert \text{Aut}_H(g') \rvert) - 2\nu_p(\lvert \text{Aut}_G^0(g') \rvert) > \nu_p(\lvert C_G(g') \rvert) + \nu_p(o(g')) - 1.
\]
Since $H$ is nilpotent, $\text{Aut}_H^0(g) \subseteq \text{Aut}_H(g)$ has index at most 2, so $\nu_p(|\text{Aut}_H(g')|) \leq \nu_p(|\text{Aut}_G^0(g')|) + 1$ (with the 1 only needed when $p = 2$) and

$$2\nu_p\left(\frac{|G|}{\chi(1)}\right) + 1 - \nu_p(|\text{Aut}_G^0(g')|) \geq \nu_p(|C_G(g')|) + \nu_p(o(g')).$$

Hence certainly

$$2\nu_p\left(\frac{|G|}{\chi(1)}\right) + 1 \geq \nu_p(|C_G(g')|) + \nu_p(o(g')),$$

and arguing as in the proof of Theorem 2.6 we obtain $2\nu_p\left(\frac{|G|}{\chi(1)}\right) + 1 \geq 2\nu_p(o(g))$. The 1 may be discarded, and Conjecture 1.1 follows. □

In the penultimate equation of the above proof, we replaced $\nu_p(|\text{Aut}_G^0(g')|)$ with zero. We cannot make better use of this term in general because we do not have control over the $p'$-part of $g'$. If $G$ is rational, however, then we have $|\text{Aut}_G^0(g)| = n/2n_0$ if $n = o(g)$ is divisible by 4, and $|\text{Aut}_G^0(g)| = n/n_0$ otherwise. Since the $p$-parts of $g$ and $g'$ are rationally conjugate, we know in this case that $\nu_p(|\text{Aut}_G^0(g')|) \geq \nu_p(n) - 2$ for $p = 2$, and $\geq \nu_p(n) - 1$ for odd $p$. The penultimate equation in the proof above is therefore improved to

$$2\nu_p\left(\frac{|G|}{\chi(1)}\right) + 1 - \nu_p(o(g)) + 2 \geq 2\nu_p(o(g)),$$

where the 1 and 2 can be replaced with 0 and 1 respectively for $p$ odd. We have shown:

**Theorem 5.3.** Let $G$ be a finite rational group, let $\chi \in \text{Irr}(G)$ and suppose Conjecture 5.1 holds for $G$ and $\chi$. If $g \in G$ has order $n$ and $\chi(g) \neq 0$, then $n^3/n_0$ divides $4(|G|/\chi(1))^2$.

In particular, Theorem 5.3 will hold when $G$ is a rational solvable group, by Theorem 5.8. Another result depending on Conjecture 5.1 and so true for solvable groups $G$ is the following.

**Theorem 5.4.** Let $G$ be a finite group. Suppose $\chi \in \text{Irr}(G)$ is a faithful irreducible character with $\chi(1)$ a power of an odd prime $p$. Suppose further that Conjecture 5.1 holds for $G$ and $\chi$. If $P$ is a Sylow $p$-subgroup of $G$ and $g \in P$ is such that $\langle g \rangle \triangleleft P$, then either $g \in Z(G)$ or $\chi(g) = 0$.

**Proof.** Since $\text{Aut}_P(g) = P/C_P(g)$ and $C_P(g)$ is a Sylow subgroup of $C_G(g)$, the $p$-part of $h_g$ equals $|\text{Aut}_P(g)|$. Also $\text{Aut}_P(g) \subseteq \text{Aut}_G^0(g)$, so Conjecture 5.1 implies that $\chi(1)$ divides $\chi(g)$. Hence by the standard argument of Burnside, either $\chi(g) = 0$ or $g \in Z(G/\ker \chi) = Z(G)$. □

We turn to the proof of Conjecture 5.1 for solvable groups. The following well known lemma is a $p$-local form of Lemma 2.4.

**Lemma 5.5.** Suppose $n = p^kn_0'$ where $p$ is odd and $k \geq 2$ and suppose $E$ and $F$ are fields with $\mathbb{Q}(\zeta_{pn_0'}) \subseteq F \subseteq E \subseteq \mathbb{Q}(\zeta_n)$. If $p = 2$, suppose additionally that $i = \sqrt{-1} \in F$. If $u \in E$ is an algebraic integer then $tr_{E/F}(u)$ is divisible by $|E : F|$.

**Proof.** First suppose $p$ is odd. Then $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{pn_0'})$ is a cyclic Galois extension of prime power degree having the unique chain of subfields

$$\mathbb{Q}(\zeta_{pn_0'}) = \zeta_n^{p^{k-1}} \subset \ldots \subset \mathbb{Q}(\zeta_n^{p^k}) \subset \mathbb{Q}(\zeta_n).$$
$E$ is one of these fields, so it is no loss to assume that $E = \mathbb{Q}(\zeta_n)$ and $F = \mathbb{Q}(\zeta_n')$ for some $r$ with $0 \leq r \leq k - 1$. Then $[E : F] = p^r$, and the Galois group of $E/F$ is $\frac{1 + p^{k-r}m'n\mathbb{Z}}{m\mathbb{Z}}$. Since the ring of integers of $E$ is $\mathbb{Z}[\zeta_n]$, we may assume that $u = \zeta_n^m$ is a root of unity. Then

$$tr_{E/F}(u) = \sum_{s=0}^{p^r-1} (\zeta_n^m)^{1+p^{k-r}n\bar{s}} = \zeta_n^m \sum_{s=0}^{p^r-1} \bar{s},$$

where $\omega = (\zeta_n^m)^{p^{k-r}n\bar{s}}$. If $\omega = 1$ then the sum is divisible by $p^r = [E : F]$; otherwise the sum is $\zeta_n^m(\omega^{p^r} - 1)/(\omega - 1) = 0$. This completes the proof for odd $p$.

Finally, if $p = 2$ then our hypothesis ensures $\mathbb{Q}(\zeta_{4n^2}) \subseteq F$. Now $\mathbb{Q}(4n^2)/\mathbb{Q}(\zeta_{4n^2})$ is a cyclic Galois extension of 2-power degree, and the rest of the proof goes through as above. 

In the following lemma, we use the notation $\omega_\theta(g) = \frac{h_\theta(g)}{\theta(1)}$ where $\theta$ is a possibly reducible character of $G$. As is well known, $\omega_\theta$ is an algebraic integer provided $\theta$ is induced from an irreducible character of a subgroup of $G$.

**Lemma 5.6.** Suppose $\theta = \psi^G$ where $\psi \in \text{Irr}(H)$ for a subgroup $H \subseteq G$. If $\omega_\psi(h)$ is divisible by $|\text{Aut}_{H}^0(h)|$ for all $h \in H$, then $\omega_\theta(g)$ is divisible by $|\text{Aut}_{G}^0(g)|$ for all $g \in G$.

**Proof.** Let $g \in G$, and let $g_1, ..., g_r$ be a set of representatives of the $H$-conjugacy classes in $g^G \cap H$. Then $\omega_\theta(g) = \sum_k \omega_\psi(g_k)$. (This shows that $\omega_\theta$ is an algebraic integer, as mentioned above.) Let $n$ be the order of $g$, and fix a prime $p$ dividing $n$. We define a field $F$ and extension fields $E_k$ for $1 \leq k \leq r$ by

$$F = \mathbb{Q}(\zeta_n)^{\text{Aut}_{G,p}^0(g)} \quad \text{and} \quad E_k = \mathbb{Q}(\zeta_n)^{\text{Aut}_{H,p}^0(g_k)}.$$ 

Note that $\mathbb{Q}(\zeta_{pn^2}, \psi(g_k)) \subseteq E_k \subseteq \mathbb{Q}(\zeta_n)$, and if $p = 2$ and $4 \mid n$, then also $i \in E_k$. Now $\text{Aut}_{G,p}^0(g)$ permutes the set of $H$-classes in $g^G \cap H$ and the stabilizer of the class containing $g_k$ is exactly $\text{Aut}_{H,p}^0(g_k)$. Hence, if $X$ is a subset of $\{1, ..., r\}$ such that the classes $g_k^H$ for $k \in X$ are representatives for this permutation action, then

$$\omega_\theta(g) = \sum_{k \in X} tr_{E_k/F}(\omega_\psi(g_k)).$$

By hypothesis, $\omega_\psi(g_k)$ is divisible by $|\text{Aut}_{H,p}^0(g_k)|$ for each $k$. Hence by Lemma 5.5, $tr_{E_k/F}(\omega_\psi(g_k))$ is divisible by $|\text{Aut}_{H,p}^0(g_k)| [E_k : F] = |\text{Aut}_{G,p}^0(g)|$. The prime $p$ was any prime dividing $n$, so the result follows.

**Theorem 5.7.** Let $G$ be a finite group and let $p$ be a prime. Let $\chi \in \text{Irr}(G)$ and suppose there exists an expression

$$\chi = \sum_i a_i \chi_i^G,$$

where $H_i \subseteq G$ are subgroups with $\chi(1)_p$ dividing $|G : H_i|$ and for each $i$, $\lambda_i$ is a linear character of $H_i$ and $a_i \in \mathbb{Z}$. Then Conjecture 5.7 holds for $\chi$ at $p$.

**Proof.** Let $g \in G$. We must prove that $\omega_\chi(g)$ is divisible by $|\text{Aut}_{G,p}^0(g)|$. This is equivalent to

$$\frac{h_g \chi(g)}{|\text{Aut}_{G,p}^0(g)|} = 0 \mod \chi(1)_p,$$
where the left hand side is an integer because $\text{Aut}_G^0(g)$ is a subgroup of $N_G(g)/C_G(g)$. We have

$$\frac{h_g}{|\text{Aut}_G^0(g)|} \chi(g) = \sum_i a_i \frac{h_g}{|\text{Aut}_G^0(g)|} \lambda_i^G(g).$$

Since $\lambda_i(1) = 1$, the conjecture fails for $\omega_\lambda$, so by Lemma 5.6, $\omega_\lambda^G(g)$ is divisible by $|\text{Aut}_G^0(g)|$, or equivalently, $\frac{h_g}{|\text{Aut}_G^0(g)|} \lambda_i^G(g)$ is divisible by $\lambda_i^G(1)$. But $\lambda_i^G(1) = |G : H_i|$ is divisible by $|\chi(1)|_p$, and the result follows.

**Corollary 5.8.** Let $G$ be any finite group and let $\chi \in \text{Irr}(G)$.

(i) If $G$ is solvable, then Conjecture 5.1 holds for $\chi$.

(ii) If $G$ is arbitrary but $\chi$ has height zero at a prime $p$, then Conjecture 5.1 holds for $\chi$ at $p$.

**Proof.** Immediate by Theorem 5.7 and Theorem 4.6 when $G$ is solvable, or Theorem 2.7 when $\chi$ has height zero at $p$. □

Corollary 5.8 provides some evidence for Conjecture 5.1, and we hope it has some independent interest, but it does not help us to establish Conjecture 1.1. Indeed, the hypothesis of Theorem 5.7 involves requiring that Conjecture 1.1 holds for $G$ and $\chi$ at $p$, since (as we already remarked below Theorem 4.6), if $\chi$ has an expression of the form $\chi = \sum_i a_i \lambda_i^G$ with $\chi(1)_p$ dividing each $|G : H_i|$, then $\chi(g) = 0$ implies that $g$ is conjugate to an element of some $H_i$, and then the order of $g$ divides $|H_i|$ which divides $|G|/\chi(1)_p$.

An additional case where it is easy to prove Conjecture 5.1 is where $\chi$ is the nontrivial irreducible component of a doubly transitive permutation character:

**Theorem 5.9.** Suppose $G$ acts doubly transitively on a set $\Omega$ with permutation character $\pi$. Let $\chi = \pi - 1_G$, so that $\chi$ is an irreducible character of $G$. Then Conjecture 5.1 holds for $\chi$.

**Proof.** Fix $g \in G$ and a prime $p$. Let $Q$ be a Sylow $p$-subgroup of $N_G(g_p) \cap C_G(g_p')$. It suffices to show that $|Q|$ divides $|G|/\chi(1)$. This is equivalent to Conjecture 5.1 except for $p = 2$, where it is stronger than Conjecture 5.1 as we do not need to take a subgroup of index 2 in this case. If $H$ is the stabilizer of a point of $\Omega$, and $s \in G \setminus H$ is any element then

$$\frac{|G|}{\chi(1)} = \frac{|G|}{|\Omega| - 1} = \frac{|G|}{|G : H| - 1} = |G : H| |H^s \cap H|.$$

We may assume that $p$ divides $\chi(1)$ or the result is obvious. Hence $|\Omega|$ is prime to $p$, so $Q$ has a fixed point $x \in \Omega$. Let $H$ be the stabilizer of $x$, so $Q \subseteq H$. If $g \notin H$ then as $g$ normalizes $Q$, we find $Q \subseteq H^g \cap H$ and the result follows from the above formula with $s = g$. Otherwise $g \in H$ and $\chi(g) = (\Omega \setminus x)^g$. Now $Q$ normalizes $g$, and so acts on the set $(\Omega \setminus x)^g$. If $s \notin H$ then the stabilizer in $Q$ of $y = sx$ is $Q \cap H^s \cap H$, so $|H^s \cap H| |y^Q| = |H^s \cap H| |Q| / |Q \cap H^s \cap H| = 0 \mod |Q|$. Since $|H^s \cap H|$ is independent of $s$ provided $s \notin H$, this shows that $|H^s \cap H| \chi(g)$ is divisible by $|Q|$, and the result follows by the formula for $|G|/\chi(1)$ above. □

Note that by Theorem 5.2 Conjecture 1.1 also holds for $\chi$ in this case. Of course this is also easy to verify directly.
References

[1] Feit, W. (1982), Some properties of characters of finite groups, Bulletin of the London Mathematical Society, Vol. 14.

[2] Feit, W. (1982), The Representation Theory of Finite Groups, North-Holland Mathematical Library.

[3] Fong, P. (1961), On the characters of p-solvable groups, Transactions of the American Mathematical Society, 98, 263-284.

[4] Fulton, W. and Harris, J. (1991), Representation Theory, A First Course, Springer.

[5] GAP (2005), The GAP Group, Groups, Algorithms and Programming, Version 4.6. http://www.gap-system.org.

[6] Gow, R., Huppert, B., Knörr, R., Mainz, O., Staszewski, R. and Willems, W. (1987), Clifford Theory and Applications, Lecture notes, Trento, 1987.

[7] Isaacs, I.M. (1976), Character Theory of Finite Groups, Academic Press.

[8] Manz O. and Wolf T.R. (1993), Representations of Solvable Groups; London Mathematical Society Lecture Note Series, Vol. 185.

[9] Navarro, G. (1998), Characters and Blocks of Finite Groups; London Mathematical Society Lecture Note Series, Vol. 250.

[10] Wilde T., A note on Brauer’s Induction Theorem in a soluble group, Communications in Algebra, forthcoming.

[11] Willems, W. (1979), A note on Brauer’s Induction Theorem, Journal of Algebra, 58, 523-526.