QUANTUM EQUATION OF MOTION AND TWO-LOOP CUTOFF RENORMALIZATION FOR $\phi^3$ MODEL

A. V. Ivanov* and N. V. Kharuk†

UDC 517.9

A two-loop renormalization of $\phi^3$ model effective action is presented by using the background field method and cutoff momentum regularization. A derivation of the quantum equation of motion and its application to the renormalization procedure are also studied. Bibliography: 23 titles.

1. Introduction

Renormalization theory (see [1]) plays a crucial role in the quantum field theory and largely depends on regularization. The present paper is devoted to the cutoff momentum regularization, which has its pros and cons. On the one hand, it breaks invariance and adds non-logarithmic divergences, but on the other hand, it is more physical and preserves dimension. As a rule, to study the properties of regularization and renormalization, we often choose the simplest theory (not necessarily physical), which clearly shows the process. We are going to work with a scalar $\phi^3$ model, which was used to study dimensional regularization in the four- (see [2]) and six-dimensional (see [3]) cases as well as for more complex versions of the theory [4–10].

In the present paper, we define a two-loop renormalization of the scalar $\phi^3$ theory with cutoff momentum regularization in 3, 4, and 5 dimensions (super-renormalizable cases), and in six dimensions (renormalizable case). We use the background field method (see [11–14]), obtain a quantum equation of motion, and explain its applications to the renormalization theory.

First, we need to introduce the Lagrangian density of the Euclidean $\phi^3$ model

$$L[\phi](x) = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} m^2 \phi^2(x) - \frac{g}{6} \phi^3(x), \quad x \in \mathbb{R}^n,$$

(1)

where $m$ is a mass parameter, $g$ is a coupling constant, and $n$ is the dimension. Then we can define an action of the theory as $S[\phi] = \int_{\mathbb{R}^n} d^n x L[\phi](x)$. Next, we assume that the scalar field $\phi$ decreases at infinity; therefore, one can integrate by parts and obtain the crucial property

$$S[\phi + B] = S[B] + (M, \phi) + \frac{1}{2} (N \phi, \phi) - \frac{g}{6} \int_{\mathbb{R}^n} d^n x \phi^3(x),$$

(2)

where the operator $N$ and field $M$ in the point $x \in \mathbb{R}^n$ are defined by the formulas

$$N(x) = -\partial_\mu \partial^\mu + m^2 - gB(x),$$

$$M(x) = -\partial_\mu \partial^\mu B(x) + m^2 B(x) - \frac{g}{2} B^2(x),$$

(3)

and where $B$ is a background field, which is defined below (see Sec. 5).

*St.Petersburg Department of the Steklov Mathematical Institute, St. Petersburg, Russia, e-mail: regul1@mail.ru.
†ITMO University, St.Petersburg, Russia, e-mail: natakharuk@mail.ru.

Published in Zapiski Nauchnykh Seminarov POMI, Vol. 487, 2019, pp. 151–166. Original article submitted November 10, 2019.
2. GREEN FUNCTION AND HEAT KERNEL

Let us introduce some extra definitions related to the operator $N(x)$. We denote by $G(x, y)$ and $K(x, y; \tau)$, respectively, the Green function and heat kernel which satisfy the problems

$$\begin{align*}
N(x)G(x, y) &= \delta(x, y), \\
\left(\frac{\partial}{\partial \tau} + N(x)\right)K(x, y; \tau) &= 0,
\end{align*}$$

(4)

for all $x, y \in \mathbb{R}^n$ and $\tau \in \mathbb{R}_+$. Under the above conditions, we have

$$\begin{align*}
\frac{\delta}{\delta B(z)}g(x, y) &= gG(x, z)G(z, y), \\
\frac{\delta}{\delta B(z)}K(x, y; \tau) &= g \int_0^\tau ds K(x, z; \tau - s)K(z, y; s).
\end{align*}$$

(5)

To prove the last formulas we need to apply the functional derivative satisfying the equality

$$\frac{\delta B(y)}{\delta B(x)} = \delta(x - y)$$

(6)

to problems (4) for the Green function and heat kernel. Then we introduce the logarithm of determinant of the operator $N$ as the following integral (see [15]):

$$\ln \det (N/N|_{B=0}) = -\int_{\mathbb{R}^n} d^n x \int_0^\infty \frac{d\tau}{\tau} [K(x, x; \tau) - e^{-m^2 \tau}].$$

(7)

Therefore, using the equality for the heat kernel

$$\int_{\mathbb{R}^n} d^n x K(z, x; \tau)K(x, y; s) = K(z, y; \tau + s),$$

(8)

one can find the first variation

$$\frac{\delta}{\delta B(x)} \ln \det (N) = -gG(x, x).$$

(9)

This equality makes sense for regularized objects. Additional properties one can find in Appendix A.

3. DIAGRAM TECHNIQUE

For clarity, it is convenient to introduce a diagram technique. We denote the Green function $G(x, y)$ by a line with two indices $x$ and $y$, and the integration by a dot. Let us give some examples of using the technique.

1) Let a functional $\rho(g, B)$ be equal to

$$\frac{\delta}{\delta B} \int_{\mathbb{R}^n} d^n x \left(\frac{\delta}{\delta \eta(x)}\right)^3 e^{\frac{1}{2}B(\eta, \eta)} \bigg|_{\eta=0}.$$

(10)

It is just a sum of connected vacuum diagrams (and their products). In Fig. 1, one can see the first two one-particle irreducible (1PI) terms of the expansion in powers of the coupling constant $g$. The next correction is multiplied by $g^4$.  

527
\[ \rho(g, B) = 1 + \frac{1}{12} g^2 + O(g^4). \]

Fig. 1. The main 1PI terms of \( \rho(g, B) \).

2) Let us define the extended Green function \( G(x, y) \) as a sum of such contributions to the functional

\[
\frac{\delta}{\delta \eta(x)} \frac{\delta}{\delta \eta(y)} e^{\phi^4 \int d^n x \left( \frac{\delta}{\delta \eta(x)} \right)^3 e^{\frac{g}{2} (G\eta, \eta)}}
\]

which would become 1PI if the free ends \( x \) and \( y \) were connected. Then it takes the form depicted in Fig. 2.

\[ G(x, y) = x + \frac{1}{4} g^2 x + O(g^4). \]

Fig. 2. The extended Green function with first correction.

**Lemma 1.** Under the above conditions, \( G(x, y) \) has the coefficient, \( \rho(g, B) \) in functional (11).

This statement can be proved using combinatorial methods and binomial coefficients.

4. **Background field method**

First, we need to introduce an effective action \( W \) as the path integral

\[
e^{-W} = \int_H D\phi e^{-S[\phi]},
\]

where \( H \) is a functional set, which is determined for physical reasons. Actually the effective action is a function of \( H \). Then, according to the background field method, we make a shift \( \phi \rightarrow \phi + B \). Thus, using formula (2), we get

\[
e^{-W[B]} = e^{-S[B]} \int_{H_0} D\phi e^{-(M, \phi - \frac{1}{2} (N\phi, \phi) + \frac{g}{2} \int d^n x \phi^3(x))},
\]

where \( H_0 = \{ \phi - B : \phi \in H \} \) is a new set of integration after the shift \( H \rightarrow H_0 \). We suppose that the dependence of \( W = W[B] \) on \( H \) is dictated by the background field \( B \), which is defined below by using the quantum equation of motion. Then we make one more shift \( \phi \rightarrow \phi + G\eta \), where \( G \) is the integration operator with kernel \( G(x, y) \), and \( \eta \) is a smooth auxiliary field. In this case, we have

\[
\int_{H_0} D\phi e^{-(M, \phi - \frac{1}{2} (N\phi, \phi) + \frac{g}{2} \int d^n x \phi^3(x))} = \det(N)^{-1/2} e^{-(M, \phi) + \frac{g}{2} \int d^n x \left( \frac{\delta}{\delta \eta(x)} \right)^3 e^{\frac{g}{2} (G\eta, \eta)}} \bigg|_{\eta=0},
\]

528
where we fixed the normalization property of the measure from formula (12) by using the condition
\[
\int_{H_0} \mathcal{D}\phi e^{-\frac{1}{4}(N\phi,\phi)} = [\det(N)]^{-1/2}.
\] (15)

5. QUANTUM EQUATION OF MOTION

Let us obtain the equation of motion. For this purpose, we need to find two kinds of contributions to the effective action \( W[B] \).

**Lemma 2.** The coefficient for \((GM,M)\) in (14), consisting of connected diagrams and their products is equal to \(\frac{1}{2}\rho(g,B)\).

**Lemma 3.** The construction from (14) contains terms which have one \(M\)-vertex. The sum of all such contributions is equal to \(-\frac{g}{2}\rho(g,B) \int d^nx GM(x)G(x,x)\).

**Proof.** To find the contribution, we need to consider the chain of equalities. The first one is
\[
-(M, \frac{\delta}{\delta \eta}) e^{\frac{g}{2} \int_{\mathbb{R}^n} d^n x \left( \frac{\delta}{\delta \eta(x)} \right)^3} e^{\frac{1}{2} \eta(GM,\eta)} \bigg|_{\eta=0} = e^{\frac{g}{2} \int_{\mathbb{R}^n} d^n x \left( \frac{\delta}{\delta \eta(x)} \right)^3} (GM,\eta) e^{\frac{1}{2} \eta(GM,\eta)} \bigg|_{\eta=0}. \] (16)

Then we need to use properties of the functional derivative in the form
\[
\left[ e^{\frac{g}{2} \int_{\mathbb{R}^n} d^n x \left( \frac{\delta}{\delta \eta(x)} \right)^3}, (GM,\eta) \right] = \frac{g}{2} e^{\frac{g}{2} \int_{\mathbb{R}^n} d^n x \left( \frac{\delta}{\delta \eta(x)} \right)^3} \left( GM, \frac{\delta^2}{\delta \eta^2} \right). \] (17)

Finally, the statement follows from Lemma 1. □

Thus we can give a definition of the quantum equation of motion. Using Lemmas 2 and 3, one can write down the equation in the form
\[
M(x) = \frac{g}{2} G(x,x), \] (18)

where \(x \in \mathbb{R}^n\). Of course, it contains the divergences, so we should understand it in the regularization sense. It is easily seen that the equation is nonlinear with respect to the background field. In particular case, after regularization, we can express the trace part of the Green function,
\[
G(x,x) = \frac{2}{g} M(x) + O(g^2). \] (19)

Now we can define the background field \(B\) as a solution of the problem which consists of the quantum equation of motion (18) and asymptotic behavior at infinity. The last condition is taken from the definition of \(H\).

**Theorem 1.** Under the above conditions, for all \(x \in \mathbb{R}^n\) we have
\[
\frac{\delta}{\delta B(x)} W[B] = M(x) - \frac{g}{2} G(x,x) \] (20)
in the regularization sense.

The last expression follows from formulas (3) and (9), and the definition of the function \(G(x,y)\). From equalities (13) and (14) one can express the effective action, which after using Theorem 1, has the form depicted in Fig. 3.

In particular, this means that diagrams such as “glasses” are canceled.
\[ W[B] = S[B] + \frac{1}{2} \ln \det(N) - \frac{1}{12} g^2 + O(g^4). \]

Fig. 3. The effective action with 1PI corrections.

6. Regularization

There are many ways to do regularization (dimensional, Pauli–Villars type, and others). We are going to use the cutoff momentum regularization in a special form. It should be noted that we are going to find infrared divergences in the coordinate representation. This means that one should regularize the Green function expansion when \( x \sim y \). The rules are as follows:

1. The factor \( r^{-k} \) with \( k \in \mathbb{N} \) tends to \( \chi_{r^A > 1} r^{-k} \),
2. The factor \( \ln r \) tends to \( \chi_{r^A > 1} \ln r - \chi_{r^A < 1} \ln \Lambda \),

where \( \chi_{(a,b)} \) is the characteristic function of \((a, b)\), and \( \Lambda \) is the parameter of regularization. This means that \( G^{\Lambda} \to G \) as \( \Lambda \to +\infty \) in the sense of generalized functions. In this case, one can write down the trace parts of the Green function for \( n \)-dimensional cases with \( n = 3, 4, 5, 6 \):

\[ G_3^{\Lambda}(y, y) = PS_3(y, y), \]
\[ G_4^{\Lambda}(y, y) = \frac{L}{8\pi^2} a_1(y, y) + PS_4(y, y), \]
\[ G_5^{\Lambda}(y, y) = PS_5(y, y), \]
\[ G_6^{\Lambda}(y, y) = \frac{L}{32\pi^3} a_2(y, y) + PS_6(y, y), \]

where the subscript corresponds to the dimension of the space and \( L = \ln(\Lambda/\mu) \). The last equalities do not violate the limit transition for the Green function \( G^{\Lambda}(x, y) \), they just redefine the value on the diagonal \( x = y \). Of course, after the cutoff regularization, the Green function has logarithmic \( L \) and powers \( \Lambda \) singularities. The second kind of them has a different nature (see, e.g., [16, 17]).

7. Renormalization

The renormalization process is based on redefining the model parameters \( m^2 \), \( \phi \), and \( g \). We are going to consider renormalizable case, when \( n = 6 \), and then super-renormalizable cases, when \( n = 3, 4, 5 \). For convenience, we introduce some extra types of the sign “=”.

The notation IR (\( \equiv \)) mean that both sides of the equality contain the same infrared singular contributions without consideration of the parts proportional to zero or the first degree of the background field \( B \). Note also that we use the logic and notations proposed in [18].

7.1. \( n=6 \) dimensional case. In the renormalizable case, we have an infinite number of divergences. Thus we need to find the renormalization constants \( Z \), \( Z_0 \), and \( Z_m \). Using the fact that the process of renormalization is equivalent to the transitions

\[ \phi \to \sqrt{Z} \phi, \quad g \to Z_0 Z^{-\frac{3}{2}} g, \quad m^2 \to Z_m Z^{-1} m^2, \]

which cancel the singularities, we plan to consider the two-loop renormalization. Using the Lagrangian density (1), one can conclude that only finitely many coefficients should be found:

\[ Z_0(g) = 1 - a_{12} g^2 L - a_{14} g^4 L - a_{24} g^4 L^2 + o(g^4), \]

530
\[ Z_n(g) = 1 - b_{12} g^2 L - b_{14} g^4 L - b_{24} g^4 L^2 + o(g^4), \]  
(27)  
\[ Z(g) = 1 - c_{12} g^2 L - c_{14} g^4 L - c_{24} g^4 L^2 + o(g^4). \]  
(28)  

First, we find the coefficients proportional to \( g^2 L \). To this end, we need to consider a singularity from the one-loop correction. From formulas (9) and (54), it follows that the singular logarithmic part has the form

\[ \ln \det(N) = -\frac{L}{32\pi^3} \int_{\mathbb{R}^6} d^6 x \, a_3(x, x). \]  
(29)  

Thereby, the contribution to the effective action has the form

\[ \frac{1}{2} \ln \det(N) = \frac{g^2 L}{6(4\pi)^3} \frac{2}{2} + \frac{m^2 g^2 L}{(4\pi)^3} \frac{2}{2} - \frac{g^3 L}{(4\pi)^3} \frac{B^2, B}{6}, \]  
(30)  

and the coefficients are

\[ c_{12} = \frac{1}{6(4\pi)^3}, \quad b_{12} = \frac{1}{(4\pi)^3}, \quad a_{12} = \frac{1}{(4\pi)^3}. \]  
(31)  

Let us find the coefficients proportional to \( g^4 L \). They come from the two-loop correction. Summing up all the terms from formulas (71)–(72) and (74)–(75), and using equalities (58), (13), and (14), one can obtain the contribution to the effective action as

\[ -\frac{11 g^4 L}{36(4\pi)^6} \frac{(\partial_{\mu} B, \partial_{\nu} B)}{2} + \frac{m^2 g^4 L}{6(4\pi)^6} \frac{(B, B)}{2} - \frac{g^5 L}{6(4\pi)^6} \frac{(B^2, B)}{6}. \]  
(32)  

This means that the coefficients are

\[ c_{14} = -\frac{11}{36(4\pi)^6}, \quad b_{14} = \frac{1}{6(4\pi)^6}, \quad a_{14} = \frac{1}{6(4\pi)^6}. \]  
(33)  

In the same way, using formulas (77)–(79), the contribution proportional to \( g^4 L^2 \) is given by

\[ \frac{5 g^4 L^2}{36(4\pi)^6} \frac{(\partial_{\mu} B, \partial_{\nu} B)}{2} + \frac{5 m^2 g^4 L^2}{4(4\pi)^6} \frac{(B, B)}{2} - \frac{g^5 g^2 L^2}{4(4\pi)^6} \frac{(B^2, B)}{6}. \]  
(34)  

Thus,

\[ c_{24} = \frac{5}{36(4\pi)^6}, \quad b_{24} = \frac{5}{4(4\pi)^6}, \quad a_{24} = \frac{5}{4(4\pi)^6}. \]  
(35)  

The coefficients obtained above are in complete agreement with the results obtained earlier (see [3]) in the case of dimensional regularization. We deliberately disregarded contributions of type (73). The sum of all such terms equals \(-\frac{5}{6(2\pi)^3} \int d^6 x \, v(x) \, P S_0(x, x)\); we consider it in Remark 1 of Sec. 7.3. It should also be noted that the two-loop correction contains a term of the form (see formulas (69) and (76))

\[ \frac{g^2 \Lambda^2}{2(4\pi)^3} \int_{\mathbb{R}^6} d^6 x \, \frac{\delta}{\delta B(x)} \ln \det(N) + \frac{g^2 \Lambda^2}{2(4\pi)^6} \int_{\mathbb{R}^6} d^6 x \, a_2(x, x). \]  
(36)  

It seems that the first term contains a high degree of the field \( B \), but it does not. One can use the expansion of the quantum equation of motion in the form (19). Therefore,

\[ \frac{\delta}{\delta B(x)} \ln \det(N) \sim -g B^2(x) + O(g^3), \]  
(37)
where the terms proportional to $B^1$ and $B^0$ are not taken into account. Further, using formula (56), we can rewrite the contribution as

$$\frac{-g^2 \Lambda^2}{(4\pi)^3} \left(1 - \frac{g^2}{2(4\pi)^3} \right) \frac{(B, B)}{2}.$$  \hfill (38)

Actually the singularity $\Lambda^2$ has a different nature and can be eliminated by redefining the regularized trace part of the Green function, or by a renormalization of the mass parameter.

### 7.2. n=5 dimensional case.

In the five-dimensional case, we have only a finite number of divergences. From formula (23), it follows that the one-loop correction does not have singularities. Thereby from equalities (62)–(66), we obtain the contribution to the effective action

$$-\frac{g^2 L}{12(4\pi)^4} \frac{(B, B)}{2} + \frac{g \Lambda}{6(4\pi)^2} \int d^5 x \frac{\delta}{\delta B(x)} \ln \det(N),$$  \hfill (39)

where the formulas $S^4 = \frac{8}{3} \pi^2$ and (9) were used. The second term in the last formula can also be considered by using the quantum equation of motion in the form (37). Therefore for renormalization, we need to shift only the mass parameter as follows:

$$m^2 \rightarrow m^2 + \frac{g^2}{12(4\pi)^4} L.$$  \hfill (40)

### 7.3. n=4 dimensional case.

The divergences in the effective action in the four-dimensional case follow from equalities (52) and (22), and formulas (59) and (60). Thus the contributions from first two loops have the form

$$-\frac{g^2 L}{(4\pi)^2} \frac{(B, B)}{2} - \frac{g^2 L}{2(4\pi)^2} \int d^4 x \frac{\delta}{\delta B(x)} \ln \det(N).$$  \hfill (41)

In this case, we have only logarithmic divergences. To renormalize the effective action, only the mass parameter should be shifted as follows:

$$m^2 \rightarrow m^2 + \frac{g^2}{(4\pi)^2} L.$$  \hfill (42)

The four-dimensional case is super-renormalizable. Let us see how the second singularity in formula (41) can be canceled. Let $\sigma$ be the finite part of $\ln \det(N)$ such that

$$g \frac{\delta \sigma}{\delta v(x)} = \frac{\delta \sigma}{\delta B(x)} = -g PS_4(x, x),$$  \hfill (43)

where $v(x) = -m^2 + gB(x)$. When using the shift (42), the effective action $W[B]$ after one-loop renormalization contains the term

$$\left. \frac{1}{2} \left( \sigma + \frac{g^2 L}{(4\pi)^2} \int d^4 x \frac{\delta \sigma}{\delta v(x)} \right) \right|_{m^2 \rightarrow m^2 + \frac{g^2}{(4\pi)^2} L}.$$  \hfill (44)

However all objects are constructed by using the Green function. This means that they are functionals of the field $v(x) = -m^2 + gB(x)$. At the same time, the operator

$$\exp \left( \frac{g^2 L}{(4\pi)^2} \int d^4 x \frac{\delta}{\delta v(x)} \right)$$  \hfill (45)
does a shift of the form

\[ v(x) \rightarrow v(x) + \frac{g^2 L}{(4\pi)^2}. \]  

(46)

So one can see that formula (44) is equal to \( \frac{1}{2} \sigma \) plus term which is canceled by the next high loop corrections. It is supposed that the same calculations can be done for a finite part of the two-loop correction, using the high loop contributions.

**Remark 1.** Let us go back to the case \( n = 6 \), where we noted that the term

\[ \frac{5}{6} \frac{g^2 L}{2(4\pi)^2} \int d^6 x v(x) PS_6(x, x) \]  

(47)

exists. By \( \sigma \) we denote the part of \( \ln \det(\mathcal{N}) \) such that \( \frac{\delta \sigma}{\delta v(x)} = -PS_6(x, x) \). By analogy with the case \( n = 4 \), we see that the term (47) is a part of exponential operator which transforms the potential in the \( \sigma \) from \( v \) to \( v + \frac{5}{6} \frac{g^2 L}{(4\pi)^3} v \). At the same time, after one-loop renormalization we have the shift

\[ v(x) = -m^2 + gB(x) \rightarrow -Z_m Z^{-1} m^2 + Z_0 Z^{-1} gB(x) = v - \frac{5}{6} \frac{g^2 L}{(4\pi)^3} v + \ldots \]  

(48)

This means that the shifts cancel each other. A similar procedure should work in the high loops.

### 8. Appendix A

It is well known (see [19]) that the heat kernel \( K(x, y; \tau) \) can be represented as a series in powers of a proper time \( \tau \). The coefficients \( a_k(x, y) \), \( k \in \mathbb{N} \), of the expansion satisfy the problem

\[
\begin{aligned}
  &a_0(x, y) = 1, \\
  &\left( k + (x - y)^\mu \partial_\mu \right) a_k(x, y) = (\partial_\mu \partial^\mu + v(x))a_{k-1}(x, y), \quad k > 0,
\end{aligned}
\]  

(49)

and are called the Seeley–DeWitt coefficients. They play an important role in physics. In particular, they give an asymptotic expansion for the Green function \( G_n(x, y) \) when \( x \sim y \). Let us introduce some notations:

\[(x - y)^{\mu_1 \ldots \mu_k} = (x - y)^{\mu_1} \ldots (x - y)^{\mu_k}, \quad \partial_{\mu_1} \ldots \partial_{\mu_k} = \partial_{\mu_1} \ldots \partial_{\mu_k}, \]  

(50)

where \( k \in \mathbb{N} \) and \( \mu_i \in \{1, \ldots, n\} \). So we can write down the expansions for \( n = 3, 4, 5, 6 \):

\[
G_3(x, y) = \frac{1}{4\pi r} - \frac{r}{8\pi} a_1(x, y) + PS_3(x, y) + o(r),
\]  

(51)

\[
G_4(x, y) = \frac{1}{4\pi^2 r^2} - \frac{\ln(r\mu)^2}{16\pi^2} a_1(x, y) + \frac{r^2 \ln(r\mu)^2}{64\pi^2} a_2(x, y) + PS_4(x, y) + o(r^2 \ln r^2),
\]  

(52)

\[
G_5(x, y) = \frac{1}{8\pi^2 r^3} + \frac{1}{16\pi^2 r} a_1(x, y) - \frac{r}{32\pi^2} a_2(x, y) + PS_5(x, y) + o(r),
\]  

(53)

\[
G_6(x, y) = \frac{1}{4\pi^3 r^4} + \frac{1}{16\pi^3 r^2} a_1(x, y) - \frac{\ln(r\mu)^2}{64\pi^3} a_2(x, y) + \frac{r^2 \ln(r\mu)^2}{256\pi^3} a_3(x, y) + PS_6(x, y) + o(r^2 \ln r^2),
\]  

(54)
where \( r = |x - y| \), \( PS_k(x, y) \) for \( k = 3, 4, 5, 6 \) are regular parts and depend on \( \mu \), although \( G(x, y) \) does not (see [20]). The first three coefficients have the forms (from [21–23]):

\[
a_1(x, y) = v(y) + \frac{1}{2} (x - y)^\mu \partial_\mu v(y) + \frac{1}{6} (x - y)^\mu\nu \partial_{\mu\nu} v(y) + \frac{1}{24} (x - y)^{\mu\nu\rho} \partial_{\mu\nu\rho} v(y) + \frac{1}{120} (x - y)^{\mu\nu\rho\sigma} \partial_{\mu\nu\rho\sigma} v(y) + o(r^4),
\]

\[
a_2(x, y) = \frac{1}{6} \partial_\mu v(y) + \frac{1}{2} v^2(y) + \frac{1}{12} (x - y)^\mu \partial_{\mu\nu} v(y) + \frac{1}{2} v(y)(x - y)^\mu \partial_\mu v(y) + \frac{1}{40} (x - y)^{\mu\nu} \partial_{\mu\nu\rho} v(y) + \frac{1}{8} ((x - y)^\mu \partial_\mu v(y))^2 + \frac{1}{6} v(y)(x - y)^\mu \partial_{\mu\nu} v(y) + o(r^2),
\]

\[
a_3(x, y) = \frac{1}{60} \partial_{\mu\nu\rho} v(y) + \frac{1}{6} v^3(y) + \frac{1}{12} \partial_\mu v(y) \partial_{\mu\nu} v(y) + \frac{1}{6} v(y) \partial_{\mu\nu} v(y).
\]

At the same time, after applying the operator \( N(x) \) to equality (54) and using the Green function definition, we have the following equality for \( n = 6 \):

\[
-\frac{a_3(y, y)}{16\pi^3} - v(y) PS_6(y, y) - \partial_\mu \partial^\mu PS_6(x, y) \bigg|_{x=y} = 0.
\]

### 9. Appendix B

#### 9.1. n=4:

In the four-dimensional case the two-loop diagram contains only two singularities which can be obtained by using formulas (52) and (55):

\[
3 \int d^4x d^4y \left( \frac{1}{4\pi^2 r^2} \right)^2 \left( \frac{-\ln(r\mu)^2}{16\pi^2} v(y) \right) \frac{IR}{28\pi^6} \int d^4y v(y) L^2,
\]

\[
3 \int d^4x d^4y \left( \frac{1}{4\pi^2 r^2} \right)^2 \frac{PS_4(x, y)}{24\pi^4} \frac{IR}{24\pi^4} \int d^4y PS_4(y, y) L.
\]

#### 9.2. n=5:

In the five-dimensional case, we have five terms with singularities among which there are not only logarithmic. So, to obtain them, we need to use expressions (53), (55), (56), and the equality of the form

\[
\int d^n x (x_j - y_j)^2 f(r) = \frac{1}{n} \int d^n x r^2 f(r), \quad j \in \{1, \ldots, n\}.
\]

So we have the contributions, which are proportional to \( \Lambda^2 \), \( \Lambda \), and \( L \):

\[
3 \int d^5x d^5y \left( \frac{1}{8\pi^2 r^3} \right)^2 \frac{v(y)}{16\pi^2 r} \frac{IR}{211\pi^6} \int d^5y v(y) \Lambda^2,
\]

\[
3 \int d^5x d^5y \left( \frac{1}{8\pi^2 r^3} \right)^2 \frac{PS_5(x, y)}{26\pi^4} \frac{IR}{26\pi^4} \int d^5y PS_5(y, y) \Lambda,
\]

\[
3 \int d^5x d^5y \left( \frac{1}{8\pi^2 r^3} \right)^2 \frac{v(y)}{16\pi^2 r} \frac{IR}{211\pi^6} \int d^5y v(y) L,
\]

\[
3 \int d^5x d^5y \left( \frac{1}{8\pi^2 r^3} \right)^2 \left( \frac{x - y)^{\mu\nu} \partial_{\mu\nu} v(y)}{6 \cdot 16\pi^2 r} \frac{IR}{211\pi^6} \int d^5y \partial_{\mu\nu} v(y) L,
\]

\[
3 \int d^5x d^5y \left( \frac{1}{8\pi^2 r^3} \right)^2 \left( \frac{r}{32\pi^2} a_2(x, y) \right) \frac{IR}{32\pi^4} \int d^5y \left( \frac{1}{3} \partial_{\mu\nu} v(y) + v^2(y) \right) L.
\]
9.3. n=6: In the six-dimensional case we have 13 contributions with singularities among which there are not only logarithmic. To calculate the divergences, we are going to use expressions (54), (55), (56), (57), and (61). Then we have

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{v(y)}{16\pi^3 r^2} \frac{IR}{3.5^5}{\pi^{10_5}} \int d^6y v(y) \Lambda^4, \tag{67} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{(x - y)_{\mu\nu} \partial_{\mu\nu} v(y)}{6 \cdot 16\pi^3 r^2} \frac{IR}{S^5}{\pi^{213_9}} \int d^6y \partial_{\mu\nu} v(y) \Lambda^2, \tag{68} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{PS_6(y, y)}{IR}{\pi^{25_6}} \int d^6y PS_6(y, y) \Lambda^2, \tag{69} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{(x - y)_{\mu\nu\rho\sigma} \partial_{\mu\nu\rho\sigma} v(y)}{120 \cdot 16\pi^3 r^2} \frac{IR}{215_6} \int d^6y \left( \sum_{\mu, \nu = 1}^6 \partial_\mu^2 \partial_\nu^2 v(y) \right) L, \tag{70} \]

\[ \int d^6x d^6y \left( \frac{v(y)}{16\pi^3 r^2} \right)^3 \frac{IR}{S^5}{\pi^{212_9}} \int d^6y v^3(y)L, \tag{71} \]

\[ 6 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{v(y)}{16\pi^3 r^2} \frac{IR}{S^5}{\pi^{213_9}} \int d^6y v(y) \partial_{\mu\nu} v(y) L, \tag{72} \]

\[ 6 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{v(y)}{16\pi^3 r^2} \frac{IR}{PS_6(x, y)}{\pi^{25_6}} \int d^6y v(y) PS_6(y, y) L, \tag{73} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{1}{2} (x - y)_{\mu\nu} \partial_{\mu\nu} PS_6(x, y) \frac{IR}{S^5}{\pi^{215_6}} \int d^6y \partial_{\mu\nu} PS_6(x, y) \bigg|_{x=y} L, \tag{74} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{(x - y)_{\mu\nu} \partial_{\mu\nu} v(y)}{2 \cdot 16\pi^3 r^2} \frac{IR}{S^5}{\pi^{213_9}} \int d^6y \partial_{\mu\nu} v(y) \partial_{\mu\nu} v(y) L, \tag{75} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{\ln(r \mu)^2 a_2(y, y)}{64\pi^3} \frac{IR}{S^5}{\pi^{213_9}} \int d^6y a_2(y, y) \left( \frac{1}{4} \Lambda^2 - \frac{1}{2} \Lambda^2 L \right), \tag{76} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{\ln(r \mu)^2 1}{64\pi^3} \frac{IR}{S^5}{\pi^{213_9}} \int d^6y a_2(x, y) \left( \frac{1}{2} \partial_{\mu\nu} a_2(x, y) \right) L^2, \tag{77} \]

\[ 3 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{x^2 \ln(r \mu)^2 a_3(y, y)}{256\pi^3} \frac{IR}{S^5}{\pi^{212_9}} \int d^6y a_3(y, y) L^2, \tag{78} \]

\[ 6 \int d^6x d^6y \left( \frac{1}{4\pi^3 r^4} \right)^2 \frac{v(y)}{16\pi^3 r^2} \frac{IR}{S^5}{\pi^{213_9}} \int d^6y v(y) a_2(y, y) L^2. \tag{79} \]

This research was supported by the Russian Science Foundation, Grant No. 19-11-00131.

A. V. Ivanov is a Young Russian Mathematics award winner and would like to thank its sponsors and jury.

REFERENCES

1. J. C. Collins, Renormalization: an Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion, Cambridge University Press, Cambridge (1984).

535
2. J. C. Collins, DAMPT preprint 73/38.
3. A. J. Macfarlane and G. Woo, “$\Phi^3$ theory in six dimensions and the renormalization
group,” Nucl. Phys. B, 77, 91–108 (1974).
4. J. L. Cardy, “High-energy behaviour in $\phi^3$ theory in six dimensions,” Nucl. Phys. B, 93, 525–546 (1975).
5. R. W. Brown, L. B. Gordon, T. F. Wong, and B. L. Young, “High-energy behavior of $\phi^3$
theory in six dimensions,” Phys. Rev. D, 11, 2209–2218 (1975).
6. A. J. McKane, D. J. Wallace, and R. K. P. Zia, “Models for strong interactions in $6 - \epsilon$
dimensions,” Phys. Lett. B, 65, 171–173 (1976).
7. S. J. Chang and Y. P. Yao, “Nonperturbative approach to infrared behavior for $(\phi^3)_6$
theory and a mechanism of confinement,” Phys. Rev. D, 16, 2948–2966 (1977).
8. R. Gass and M. Dresden, “Puzzling Aspect of Quantum Field Theory in Curved Space-
Time,” Phys. Rev. Lett., 54, 2281–2284 (1985).
9. L. Culumovic, D. G. C. McKeon, and T. N. Sherry, “Operator Regularization and the
Renormalization Group to Two-Loop Order in $\phi^3$,” Annals Phys., 197, 94–118 (1990).
10. J. A. Gracey, “Four loop renormalization of $\phi^3$ theory in six dimensions,” Phys. Rev. D, 92, 025012 (2015).
11. L. F. Abbott, “Introduction to the Background Field Method,” Acta Phys. Polon. B, 13, 33 (1982).
12. I. Y. Arefeva, A. A. Slavnov, and L. D. Faddeev, “Generating functional for the S matrix
in gauge-invariant theories,” Theor. Math. Phys., 21, 1165–1172 (1974).
13. L. D. Faddeev and A. A. Slavnov, “Gauge Fields. Introduction To Quantum Theory,”
Front. Phys., 50 (1980), [Front. Phys. 83 (1990)].
14. L. D. Faddeev, “Mass in Quantum Yang-Mills Theory: Comment on a Clay Millenium
problem,” arXiv:0911.1013 [math-ph] (2009).
15. V. Fock, “ Proper time in classical and quantum mechanics,” Phys. Z. Sowjetunion, 12, 404–425 (1937).
16. K. Hagiwara, S. Ishihara, R. Szalapski, and D. Zeppenfeld, “Low energy effects of new
interactions in the electroweak boson sector,” Phys. Rev. D, 48, 2182–2203 (1993).
17. M. Harada and K. Yamawaki, “Wilsonian matching of effective field theory with underlying
QCD,” Phys. Rev. D, 64, 014–023 (2001).
18. S. E. Derkachev, A. V. Ivanov, and L. D. Faddeev, “Renormalization scenario for the
quantum Yang Mills theory in four-dimensional space time,” Theor. Math. Phys., 192, No. 2, 1134–1140 (2017).
19. B. S. DeWitt, Dynamical Theory of Groups and Fields, Gordon and Breach, New York
(1965).
20. M. Lüscher, “Dimensional regularisation in the presence of large background fields,” Annals
Phys., 142, 359–392 (1982).
21. P. B. Gilkey, “The spectral geometry of a Riemannian manifold,” J. Diff. Geom., 10, 601–618 (1975).
22. A. V. Ivanov, “Diagram technique for the heat kernel of the covariant Laplace operator,”
Theor. Math. Phys., 198, No. 1, 100–117 (2019).
23. A. V. Ivanov and N. V. Kharuk, “Heat kernel: proper time method, Fock-Schwinger gauge,
path integral representation, and Wilson line,” arXiv:1906.04019 [hep-th] (2019).