GROWTH ESTIMATES FOR A CLASS OF SUBHARMONIC FUNCTIONS IN A HALF PLANE

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Abstract. A class of subharmonic functions represented by the modified kernels are proved to have the growth estimates \( u(z) = o(y^{1-\alpha}|z|^{m+\alpha}) \) at infinity in the upper half plane \( \mathbb{C}_+ \), which generalizes the growth properties of analytic functions and harmonic functions.

1. Introduction and Main Theorem

Let \( \mathbb{C} \) denote the complex plane with points \( z = x + iy \), where \( x, y \in \mathbb{R} \). The boundary and closure of an open \( \Omega \) of \( \mathbb{C} \) are denoted by \( \partial \Omega \) and \( \overline{\Omega} \) respectively. The upper half-plane \( \mathbb{C}_+ \) is the set \( \{z = x + iy \in \mathbb{C} : y > 0\} \), whose boundary is \( \partial \mathbb{C}_+ \). We write \( B(z, \rho) \) and \( \partial B(z, \rho) \) for the open ball and the sphere of radius \( \rho \) centered at \( z \) in \( \mathbb{C} \). We identify \( \partial \mathbb{C}_+ \) with \( \mathbb{R} \).

For \( z \in \mathbb{C}\setminus\{0\} \), let (\[3\])

\[
E(z) = (2\pi)^{-1} \log |z|
\]

where \( |z| \) is the Euclidean norm. We know that \( E \) is locally integrable in \( \mathbb{C} \).

We define the Green function \( G(z, \zeta) \) for the upper half plane \( \mathbb{C}_+ \) by (\[3\])

\[
G(z, \zeta) = E(z - \zeta) - E(z - \overline{\zeta}) \quad z, \zeta \in \overline{\mathbb{C}_+}, \ z \neq \zeta. \tag{1.1}
\]

We define the Poisson kernel \( P(z, \xi) \) when \( z \in \mathbb{C}_+ \) and \( \xi \in \partial \mathbb{C}_+ \) by

\[
P(z, \xi) = -\left. \frac{\partial G(z, \zeta)}{\partial \eta} \right|_{\eta=0} = \frac{y}{\pi |z - \xi|^2}.
\]

The Dirichlet problem of upper half plane is to find a function \( u \) satisfying

\[
\mathbb{C}^2(\mathbb{C}_+),
\]

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\[ \Delta u = 0, \quad z \in \mathbb{C}_+, \]

\[ \lim_{z \to x} u(z) = f(x) \text{ non-tangentially a.e. } x \in \partial \mathbb{C}_+, \]

where \( f \) is a measurable function of \( \mathbb{R} \). The Poisson integral of the upper half plane is defined by

\[ v(z) = P[f](z) = \int_{\mathbb{R}} P(z, \xi) f(\xi) d\xi. \tag{1.2} \]

We have know that, the Poisson integral \( P[f] \) exists if

\[ \int_{\mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^2} d\xi < \infty. \tag{1.3} \]

(see [4] and [5]) In this paper, we will consider measurable functions \( f \) in \( \mathbb{R} \) satisfying

\[ \int_{\mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi < \infty, \tag{1.4} \]

where \( m \) is a natural number. To obtain a solution of Dirichlet problem for the boundary date \( f \), we use the following modified functions defined by

\[ E_m(z - \zeta) = \begin{cases} E(z - \zeta) & \text{when } |\zeta| \leq 1, \\ E(z - \zeta) - \frac{1}{2\pi} \Re(\log \zeta - \sum_{k=1}^{m-1} \frac{z^k}{k\zeta^k}) & \text{when } |\zeta| > 1. \end{cases} \]

Then we can define modified Green function \( G_m(z, \zeta) \) and the modified Poisson kernel \( P_m(z, \xi) \) by

\[ G_m(z, \zeta) = E_{m+1}(z - \zeta) - E_{m+1}(z - \overline{\zeta}), \quad z, \zeta \in \overline{\mathbb{C}_+}, \; z \neq \zeta; \tag{1.5} \]

\[ P_m(z, \xi) = \begin{cases} P(z, \xi) & \text{when } |\xi| \leq 1, \\ P(z, \xi) - \frac{1}{\pi} \Im \sum_{k=0}^{m} \frac{z^k}{\xi^{k+1}} & \text{when } |\xi| > 1. \end{cases} \tag{1.6} \]

where \( z = x + iy, \zeta = \xi + i\eta \).

Hayman([1]) has proved the following result:

**Theorem A** Let \( f \) be a measurable function in \( \mathbb{R} \) satisfying (1.3), let \( \mu \) be a Borel positive measure satisfying

\[ \int_{\mathbb{C}_+} \frac{\eta}{1 + |\zeta|^2} d\mu(\zeta) < \infty. \]

Write the subharmonic function

\[ u(z) = v(z) + h(z), \quad z \in \mathbb{C}_+ \]

where \( v(z) \) be the harmonic function defined by (1.2), \( h(z) \) is defined by

\[ h(z) = \int_{\mathbb{C}_+} G(z, \zeta) d\mu(\zeta) \]
and \( G(z, \zeta) \) is defined by (1.1). Then there exists \( z_j \in \mathbb{C}_+, \rho_j > 0 \), such that
\[
\sum_{j=1}^\infty \frac{\rho_j}{|z_j|} < \infty
\]
holds and
\[
u(z) = o(|z|) \quad \text{as} \quad |z| \to \infty
\]
holds in \( \mathbb{C}_+ - G \), where \( G = \bigcup_{j=1}^\infty B(z_j, \rho_j) \).

Our aim in this paper is to establish the following theorems.

**Theorem 1** Let \( f \) be a measurable function in \( \mathbb{R} \) satisfying (1.4), and \( 0 < \alpha \leq 2 \). Let \( v(z) \) be the harmonic function defined by
\[
v(z) = \int_{\mathbb{R}} P_m(z, \xi) f(\xi) d\xi \quad z \in \mathbb{C}_+ \quad (1.7)
\]
where \( P_m(z, \xi) \) is defined by (1.6). Then there exists \( z_j \in \mathbb{C}_+, \rho_j > 0 \), such that
\[
\sum_{j=1}^\infty \frac{\rho_j^{2-\alpha}}{|z_j|^{2-\alpha}} < \infty
\]
(1.8) holds and
\[
v(z) = o(y^{1-\alpha}|z|^{m+\alpha}) \quad \text{as} \quad |z| \to \infty
\]
holds in \( \mathbb{C}_+ - G \), where \( G = \bigcup_{j=1}^\infty B(z_j, \rho_j) \).

**Remark 1** If \( \alpha = 2 \), then (1.8) is a finite sum, the set \( G \) is a bounded set, so (1.9) holds in \( \mathbb{C}_+ \).

Next, we will generalize Theorem 1 to subharmonic functions.

**Theorem 2** Let \( f \) be a measurable function in \( \mathbb{R} \) satisfying (1.4), let \( \mu \) be a Borel positive measure satisfying
\[
\int_{\mathbb{C}_+} \frac{\eta}{1 + |\zeta|^{2+m}} d\mu(\zeta) < \infty.
\]
Write the subharmonic function
\[
u(z) = v(z) + h(z), \quad z \in \mathbb{C}_+
\]
where \( v(z) \) be the harmonic function defined by (1.7), \( h(z) \) is defined by
\[
h(z) = \int_{\mathbb{C}_+} G_m(z, \zeta) d\mu(\zeta)
\]
and \( G_m(z, \zeta) \) is defined by (1.5). Then there exists \( z_j \in \mathbb{C}_+, \rho_j > 0 \), such that (1.8) holds and
\[
u(z) = o(y^{1-\alpha}|z|^{m+\alpha}) \quad \text{as} \quad |z| \to \infty
\]
(1.10) holds in \( \mathbb{C}_+ - G \), where \( G = \bigcup_{j=1}^\infty B(z_j, \rho_j) \) and \( 0 < \alpha < 2 \).

**Remark 2** If \( \alpha = 1, m = 0 \), this is just the result of Hamman, so our result (1.10) is the generalization of Theorem A.
2. Proof of Theorem

Let $\mu$ be a positive Borel measure in $\mathbb{C}$, $\beta \geq 0$, the maximal function $M(d\mu)(z)$ of order $\beta$ is defined by

$$M(d\mu)(z) = \sup_{0<r<\infty} \frac{\mu(B(z, r))}{r^\beta},$$

then the maximal function $M(d\mu)(z) : \mathbb{C} \to [0, \infty)$ is semicontinuous, hence measurable. To see this, $\forall \lambda > 0$, let $D(\lambda) = \{z \in \mathbb{C} : M(d\mu)(z) > \lambda\}$. Fix $z \in D(\lambda)$, then $\exists r > 0$ such that $\mu(B(z, r)) > tr^\beta$ for some $t > \lambda$, and $\exists \delta > 0$ satisfying $(r + \delta)^\beta < \frac{t \mu(z)}{\lambda}$. If $|z - \zeta| < \delta$, then $B(\zeta, r + \delta) \subset B(z, r)$, therefore $\mu(B(\zeta, r + \delta)) \geq tr^\beta = t(\frac{\zeta}{r + \delta})^\beta \lambda > \lambda(\zeta^\beta + \delta)$. Thus $B(z, \delta) \subset D(\lambda)$. This proves that $D(\lambda)$ is open for each $\lambda > 0$.

In order to obtain the result, we need these lemmas below:

**Lemma 1** Let $\mu$ be a positive Borel measure in $\mathbb{C}$, $\beta \geq 0$, $\mu(\mathbb{C}) < \infty$, $\forall \lambda \geq 5^\beta \mu(\mathbb{C})$, set

$$E(\lambda) = \{z \in \mathbb{C} : |z| \geq 2, M(d\mu)(z) > \frac{\lambda}{|z|^\beta}\}$$

then $\exists z_j \in E(\lambda)$, $\rho_j > 0$, $j = 1, 2, \cdots$, such that

$$E(\lambda) \subset \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \quad (2.1)$$

and

$$\sum_{j=1}^{\infty} \frac{\rho_j^\beta}{|z_j|^\beta} \leq \frac{3\mu(\mathbb{C})5^\beta}{\lambda}. \quad (2.2)$$

Proof: Let $E_k(\lambda) = \{z \in E(\lambda) : 2^k \leq |z| < 2^{k+1}\}$, then $\forall z \in E_k(\lambda), \exists r(z) > 0$, such that $\mu(B(z, r(z))) > \lambda(\frac{r(z)}{|z|})^\beta$, therefore $r(z) \leq 2^{k-1}$. Since $E_k(\lambda)$ can be covered by the union of a family of balls $\{B(z, r(z)) : z \in E_k(\lambda)\}$, by the Vitali Lemma([2]), $\exists \Lambda_k \subset E_k(\lambda)$, $\Lambda_k$ is at most countable, such that $\{B(z, r(z)) : z \in \Lambda_k\}$ are disjoint and

$$E_k(\lambda) \subset \bigcup_{z \in \Lambda_k} B(z, 5r(z)),$$

so

$$E(\lambda) = \bigcup_{k=1}^{\infty} E_k(\lambda) \subset \bigcup_{k=1}^{\infty} \bigcup_{z \in \Lambda_k} B(z, 5r(z)). \quad (2.3)$$

On the other hand, note that $\bigcup_{z \in \Lambda_k} B(z, r(z)) \subset \{z : 2^{k-1} \leq |z| < 2^{k+2}\}$, so that

$$\sum_{z \in \Lambda_k} \frac{(5r(z))^\beta}{|z|^\beta} \leq 5^\beta \sum_{z \in \Lambda_k} \frac{\mu(B(z, r(z)))}{\lambda} \leq \frac{5^\beta}{\lambda} \mu\{z : 2^{k-1} \leq |z| < 2^{k+2}\}. \quad (2.4)$$
Hence we obtain
\[
\sum_{k=1}^{\infty} \sum_{z \in \Lambda_k} \frac{(5r(z))^\beta}{|z|^\beta} \leq \sum_{k=1}^{\infty} \frac{5^\beta}{\lambda} \mu\{z : 2^{k-1} \leq |z| < 2^{k+2}\} \leq \frac{3\mu(C)5^\beta}{\lambda}. \tag{2.4}
\]

Rearrange \( \{z : z \in \Lambda_k, k = 1, 2, \ldots\} \) and \( \{5r(z) : z \in \Lambda_k, k = 1, 2, \ldots\} \), we get \( \{z_j\} \) and \( \{\rho_j\} \) such that (2.1) and (2.2) hold.

**Lemma 2**

1. \(|\sum_{k=0}^{m} \frac{z^k}{\xi^{k+\nu}}| \leq \sum_{k=0}^{m-1} 2^k y |z|^k, \tag{3}

2. \(|\sum_{k=0}^{\infty} \frac{z^{k+1}}{\xi^{k+\nu}}| \leq 2^m y |z|^m, \tag{4}

3. \(|G_m(z, \xi) - G(z, \xi)| \leq \frac{1}{\pi} \sum_{k=1}^{m} \frac{k y |z|^{k-1}}{|\xi|^{k+\nu}} \tag{5}

4. \(|G_m(z, \xi)| \leq \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{k y |z|^{k-1}}{|\xi|^{k+\nu}} \tag{6}

Now we are ready to prove Theorems.

Throughout the proof, \( A \) denote various positive constants.

**Proof of Theorem 1**

Define the measure \( dm(\xi) \) and the kernel \( K(z, \xi) \) by

\[
dm(\xi) = \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi, \quad K(z, \xi) = P_m(z, \xi)(1 + |\xi|^{2+m}).
\]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that

\[
\int_{|\xi| \geq R_\varepsilon} dm(\xi) \leq \frac{\varepsilon}{5^{2-\alpha}}.
\]

For every Lebesgue measurable set \( E \subset \mathbb{R} \), the measure \( m^{(e)} \) defined by \( m^{(e)}(E) = m(E \cap \{x \in \mathbb{R} : |x| \geq R_\varepsilon\}) \) satisfies \( m^{(e)}(\mathbb{R}) \leq \frac{\varepsilon}{5^{2-\alpha}} \), write

\[
v_1(z) = \int_{|\xi - z| \leq 3|z|} P(z, \xi)(1 + |\xi|^{2+m}) dm^{(e)}(\xi),
\]

\[
v_2(z) = \int_{|\xi - z| \leq 3|z|} (P_m(z, \xi) - P(z, \xi))(1 + |\xi|^{2+m}) dm^{(e)}(\xi),
\]

\[
v_3(z) = \int_{|\xi - z| > 3|z|} K(z, \xi) dm^{(e)}(\xi),
\]

\[
v_4(z) = \int_{1 < |\xi| < R_\varepsilon} K(z, \xi) dm(\xi),
\]

\[
v_5(z) = \int_{|\xi| \leq 1} K(z, \xi) dm(\xi).
\]

then

\[
|v(z)| \leq |v_1(z)| + |v_2(z)| + |v_3(z)| + |v_4(z)| + |v_5(z)|. \tag{2.5}
\]
Let $E_1(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists \varepsilon > 0, m^{(\varepsilon)}(B(z, t) \cap \mathbb{R}) > \lambda(\frac{t}{|z|})^{2-\alpha} \}$, when $|z| \geq 2R_\varepsilon$, $z \notin E_1(\lambda)$, then

$$\forall t > 0, \ m^{(\varepsilon)}(B(z, t) \cap \mathbb{R}) \leq \lambda(\frac{t}{|z|})^{2-\alpha}.$$  

So we have

$$|v_1(z)| \leq \frac{y}{\pi |z - \xi|^2} 2|\xi|^{2+m} m^{(\varepsilon)}(\xi) \leq \frac{2^{2m+5}}{\pi} |z|^{2+m} \int_{y \leq |\xi - z| \leq 3|z|} \frac{1}{|z - \xi|^2} m^{(\varepsilon)}(\xi) = \frac{2^{2m+5}}{\pi} |z|^{m+2} \int_{y \leq |\xi - z| \leq 3|z|} \frac{1}{t^2} m^{(\varepsilon)}(t).$$

where $m^{(\varepsilon)}(t) = \int_{|\xi - z| \leq t} dm^{(\varepsilon)}(\xi)$, since for $z \notin E_1(\lambda)$,

$$\int_{y \leq |\xi - z| \leq 3|z|} \frac{1}{t^2} m^{(\varepsilon)}(t) \leq \frac{m^{(\varepsilon)}(3|z|)}{(3|z|)^2} + 2 \int_{y \leq |\xi - z| \leq 3|z|} \frac{m^{(\varepsilon)}(t)}{t^3} dt \leq \frac{3^\alpha |z|^2}{3^\alpha |z|^2} + 2 \int_{y \leq |\xi - z| \leq 3|z|} \frac{t^{2-\alpha}}{t^3} dt \leq \frac{\lambda}{|z|^2} \left[ \frac{1}{3^\alpha} + \frac{2 |z|^\alpha}{\alpha y^\alpha} \right],$$

so that

$$|v_1(z)| \leq \frac{2^{2m+5}}{\pi} \left( \frac{1}{3^\alpha} + \frac{2}{\alpha} \right) \lambda y^{1-\alpha} |z|^{m+\alpha}. \quad (2.6)$$

By (1) of Lemma 2, we obtain

$$|v_2(z)| \leq \int_{y \leq |\xi - z| \leq 3|z|} \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{2^k y |z|^k}{|\xi|^{2+k}} \cdot 2|\xi|^{2+m} m^{(\varepsilon)}(\xi) \leq \int_{y \leq |\xi - z| \leq 3|z|} \sum_{k=0}^{m-1} \frac{2^{k+1} y |z|^k}{\pi} \frac{(4|z|)^{m-k}}{|z|^{m-k}} m^{(\varepsilon)}(\xi) \leq \frac{2^{2m+1}}{\pi} \sum_{k=0}^{m-1} \frac{1}{2^k 5^{2-\alpha}} \varepsilon y |z|^m \leq \frac{4^{m-1+\alpha}}{\pi} \varepsilon y |z|^m. \quad (2.7)$$
By (2) of Lemma 2, we see that

\[ |v_3(z)| \leq \int_{|z-\xi|>3|z|} \left| \sum_{k=0}^{\infty} \frac{z^{k+1}}{z^{2+k}} \right| \cdot 2|\xi|^{2+m} dm(\xi) \]

\[ = \int_{|z-\xi|>3|z|} \frac{2}{\pi} \left| \sum_{k=0}^{\infty} \frac{z^{k+m+1}}{\xi_k} \right| dm(\xi) \]

\[ \leq \frac{2^{m+2}}{\pi} \frac{\varepsilon}{5^{2-\alpha}} |z|^m \]

\[ \leq \frac{2^{m-2+2\alpha}}{\pi} \varepsilon |z|^m. \quad (2.8) \]

Write

\[ v_4(z) = \int_{1<|\xi|<R} \left| P(z, \xi) - \frac{1}{\pi} \sum_{k=0}^{m} \frac{z^k}{\xi^{1+k}} \right| (1 + |\xi|^{2+m}) dm(\xi) \]

\[ = v_{41}(z) - v_{42}(z), \]

then

\[ |v_{41}(z)| \leq \int_{1<|\xi|<R} \frac{y}{\pi |z-\xi|^2} 2|\xi|^{2+m} dm(\xi) \]

\[ \leq \frac{2R_y^{2+m} y}{\pi} \int_{1<|\xi|<R} \frac{1}{(|z|^2)^2} dm(\xi) \]

\[ \leq \frac{2^3 R_y^{2+m} m(R)}{\pi} \frac{y}{|z|^2}. \quad (2.9) \]

by (1) of Lemma 2, we obtain

\[ |v_{42}(z)| \leq \int_{1<|\xi|<R} \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{2^k y |z|^k}{|\xi|^{2+k}} \cdot 2|\xi|^{2+m} dm(\xi) \]

\[ \leq \sum_{k=0}^{m-1} \frac{2^{k+1}}{\pi} y |z|^k R_y^{m-k} m(R) \]

\[ \leq \frac{2^{m+1} R_y^{m} m(R)}{\pi} y |z|^{m-1}. \quad (2.10) \]

In case \(|\xi| \leq 1\), note that

\[ K(z, \xi) = P_m(z, \xi)(1 + |\xi|^{2+m}) \leq \frac{2y}{\pi |z-\xi|^2}, \]

so that

\[ |v_5(z)| \leq \int_{|\xi|<1} \frac{2y}{\pi (|\xi|^2)^2} dm(\xi) \leq \frac{2^3 m(R)}{\pi} \frac{y}{|z|^2}. \quad (2.11) \]
Thus, by collecting (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11), there exists a positive constant $A$ independent of $\varepsilon$, such that if $|z| \geq 2R_\varepsilon$ and $z \notin E_1(\varepsilon)$, we have

$$|v(z)| \leq A\varepsilon y^{1-\alpha}|z|^{m+\alpha}.$$  

Let $\mu_\varepsilon$ be a measure in $C$ defined by $\mu_\varepsilon(E) = m^{(\varepsilon)}(E \cap R)$ for every measurable set $E$ in $C$. Take $\varepsilon = \varepsilon_p = \frac{1}{2p+2}$, $p = 1, 2, 3, \ldots$, then there exists a sequence $\{R_p\}$: $1 = R_0 < R_1 < R_2 < \cdots$ such that

$$\mu_\varepsilon(\varepsilon) = \int_{|\xi| \geq R_p} dm(\xi) < \frac{\varepsilon_p}{5^{2-\alpha}}.$$  

Take $\lambda = 3 \cdot 5^{2-\alpha} \cdot 2^{p} \mu_\varepsilon(\varepsilon)$ in Lemma 1, then $\exists z_{j,p}$ and $\rho_{j,p}$, where $R_{p-1} \leq |z_{j,p}| < R_p$ such that

$$\sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2-\alpha} \leq \frac{1}{2^p}.$$  

So if $R_{p-1} \leq |z| < R_p$, $z \notin \cup_{j=1}^{\infty} B(z_{j,p}, \rho_{j,p})$, we have

$$|v(z)| \leq A\varepsilon_p y^{1-\alpha}|z|^{m+\alpha},$$  

thereby

$$\sum_{p=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2-\alpha} \leq \sum_{p=1}^{\infty} \frac{1}{2^p} = 1 < \infty.$$  

Set $G = \cup_{p=1}^{\infty} G_p$, then Theorem 1 holds.

**Proof of Theorem 2**

Define the measure $dn(\zeta)$ and the kernel $L(z, \zeta)$ by

$$dn(\zeta) = \frac{\eta d\mu(\zeta)}{1 + |\zeta|^{2+m}}, \quad L(z, \zeta) = G_m(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta}.$$  

then the function $h(z)$ can be written as

$$h(z) = \int_{C_+} L(z, \zeta)dn(\zeta).$$  

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|\zeta| \geq R_\varepsilon} dn(\zeta) < \frac{\varepsilon}{5^{2-\alpha}}.$$  

For every Lebesgue measurable set $E \subset C$, the measure $n^{(\varepsilon)}$ defined by $n^{(\varepsilon)}(E) = n(E \cap \{ \zeta \in C_+ : |\zeta| \geq R_\varepsilon \})$ satisfies $n^{(\varepsilon)}(C_+) \leq \frac{\varepsilon}{5^{2-\alpha}}$,.
write

\[ h_1(z) = \int_{|\zeta - z| \leq \frac{\eta}{2}} G(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(e)}(\zeta), \]

\[ h_2(z) = \int_{\frac{\eta}{2} < |\zeta - z| \leq |z|} G(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(e)}(\zeta), \]

\[ h_3(z) = \int_{|\zeta - z| \leq |z|} (G_m(z, \zeta) - G(z, \zeta)) \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(e)}(\zeta), \]

\[ h_4(z) = \int_{|\zeta - z| > |z|} L(z, \zeta) dn^{(e)}(\zeta), \]

\[ h_5(z) = \int_{1 < |\zeta| < R} L(z, \zeta) dn(\zeta), \]

\[ h_6(z) = \int_{|\zeta| \leq 1} L(z, \zeta) dn(\zeta). \]

then

\[ h(z) = h_1(z) + h_2(z) + h_3(z) + h_4(z) + h_5(z) + h_6(z). \quad (2.12) \]

Let \( E_2(\lambda) = \{ z \in C : |z| \geq 2, \forall t > 0, n^{(e)}(B(z, t) \cap C_+) > \lambda \left( \frac{t}{|z|} \right)^{2-\alpha} \} \),

when \(|z| \geq 2R_\varepsilon\), \( z \notin E_2(\lambda) \), then

\[ \forall t > 0, n^{(e)}(B(z, t) \cap C_+) \leq \lambda \left( \frac{t}{|z|} \right)^{2-\alpha}. \]

So we have

\[ |h_1(z)| \leq \int_{|\zeta - z| \leq \frac{\eta}{2}} \frac{1}{2\pi} \log \left| \frac{\zeta - \pi}{\zeta - z} \right| \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(e)}(\zeta) \]

\[ \leq \int_{|\zeta - z| \leq \frac{\eta}{2}} \frac{1}{2\pi} \log \left| \frac{3y}{|\zeta - z|} \right| \frac{2|\zeta|^{2+m}}{y} dn^{(e)}(\zeta) \]

\[ \leq \frac{2 \times (3/2)^{2+m}}{\pi} \frac{|z|^{2+m}}{y} \int_{|\zeta - z| \leq \frac{\eta}{2}} \log \left| \frac{3y}{|\zeta - z|} \right| dn^{(e)}(\zeta) \]

\[ = \frac{2 \times (3/2)^{2+m}}{\pi} \frac{|z|^{2+m}}{y} \int_{0}^{y/2} \log \left| \frac{3y}{t} \right| dn^{(e)}(t) \]

\[ \leq \frac{2 \times (3/2)^{2+m}}{\pi} \left[ \log 6 \frac{y}{2^{2-\alpha}} + \frac{1}{(2 - \alpha)2^{2-\alpha}} \right] \lambda y^{1-\alpha} |z|^{m+\alpha}. \quad (2.13) \]

where \( n^{(e)}_z(t) = \int_{|\zeta - z| \leq t} dn^{(e)}(\zeta) \).

Note that

\[ |G(z, \zeta)| = |E(z - \zeta) - E(z - \zeta)| \leq \frac{\eta y}{\pi |z - \zeta|^2} \quad (2.14) \]
then by (2.14), we have

\[
|h_2(z)| \leq \int_{\frac{y}{R} < |\zeta - z| \leq 3|z|} \frac{y|\zeta|^{2+m}}{\pi|z-\zeta|^2} d\eta(\zeta)
\]

\[
\leq \frac{2^{m+5}}{\pi} y|z|^{2+m} \int_{\frac{y}{R} < |\zeta - z| \leq 3|z|} \frac{1}{|z-\zeta|^2} d\eta(\zeta)
\]

\[
= \frac{2^{m+5}}{\pi} y|z|^{2+m} \int_{\frac{y}{R}}^{3|z|} \frac{1}{t^2} d\eta(t)
\]

\[
\leq \frac{2^{m+5}}{\pi} y|z|^{2+m} \lambda \left( \frac{1}{3^\alpha} + \frac{2^{\alpha+1}|z^\alpha}{\alpha y^\alpha} \right)
\]

\[
\leq \frac{2^{m+5}}{\pi} \left( \frac{1}{3^\alpha} + \frac{2^{\alpha+1}}{\alpha} \right) \lambda y^{-\alpha} |z|^{m+\alpha}. \quad (2.15)
\]

By (3) of Lemma 2, we obtain

\[
|h_3(z)| \leq \int_{|\zeta - z| \leq 3|z|} \frac{1}{\pi} \sum_{k=1}^{m} k \eta|\zeta|^{k-1} \frac{2|\zeta|^{2+m}}{\eta} d\eta(\zeta)
\]

\[
\leq \int_{|\zeta - z| \leq 3|z|} \frac{2}{\pi} \sum_{k=1}^{m} k \eta|\zeta|^{k-1} (4|z|)^{m-k+1} d\eta(\zeta)
\]

\[
\leq \frac{2^{m+1}}{\pi} \sum_{k=1}^{m} \frac{k}{4^{k-1} 5^{2-\alpha}} \eta|\zeta|^m
\]

\[
\leq \frac{2^{m+2\alpha+1}}{9\pi} \eta|\zeta|^m. \quad (2.16)
\]

By (4) of Lemma 2, we see that

\[
|h_4(z)| \leq \int_{|\zeta - z| > 3|z|} \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{k \eta|\zeta|^{k-1} 2|\zeta|^{2+m}}{\eta} d\eta(\zeta)
\]

\[
\leq \int_{|\zeta - z| > 3|z|} \frac{2}{\pi} \sum_{k=m+1}^{\infty} k \eta|\zeta|^{k-1} (2|z|)^{k-m-1} d\eta(\zeta)
\]

\[
\leq \frac{2^{m+2}}{\pi} \sum_{k=m+1}^{\infty} \frac{k}{2^{k} 5^{2-\alpha}} \eta|\zeta|^m
\]

\[
\leq \frac{4^{\alpha-1} (m+2)}{\pi} \eta|\zeta|^m. \quad (2.17)
\]

Write

\[
h_5(z) = \int_{1 < |\zeta| < R_z} \left[ G(z, \zeta) + (G_m(z, \zeta) - G(z, \zeta)) \right] \frac{1 + |\zeta|^{2+m}}{\eta} d\eta(\zeta)
\]

\[
= h_{51}(z) + h_{52}(z),
\]
then we obtain by (2.14)
\[
|h_{51}(z)| \leq \int_{1<|\zeta|<R_\varepsilon} \frac{y\eta}{\pi|z-\zeta|^2} \frac{2|\zeta|^{2+m}}{\eta} \, dn(\zeta)
\]
\[
\leq \frac{2R^{2+m}_\varepsilon y}{\pi} \int_{1<|\zeta|<R_\varepsilon} \frac{1}{(\frac{|\zeta|}{2})^2} \, dn(\zeta)
\]
\[
\leq \frac{2^3 R^{2+m}_\varepsilon n(C_+) \, y}{\pi |z|^2}. \tag{2.18}
\]
by (3) of Lemma 2, we obtain
\[
|h_{52}(z)| \leq \int_{1<|\zeta|<R_\varepsilon} \frac{1}{\pi} \sum_{k=1}^{m} \frac{k y n|z|^{k-1} 2|\zeta|^{2+m}}{|\zeta|^{1+k} \eta} \, dn(\zeta)
\]
\[
\leq \frac{2}{\pi} \sum_{k=1}^{m} k y|z|^{k-1} R^{m-k+1}_\varepsilon n(C_+)
\]
\[
\leq \frac{m(m+1) R^{m}_\varepsilon n(C_+)}{\pi} y |z|^{m-1}. \tag{2.19}
\]
In case $|\zeta| \leq 1$, by (2.14), we have
\[
|L(z, \zeta)| \leq \frac{y\eta}{\pi|z-\zeta|^2} \frac{2 \eta}{\pi|z-\zeta|^2},
\]
so that
\[
|h_6(z)| \leq \int_{|\zeta| \leq 1} \frac{2 y}{\pi|z-\zeta|^2} \, dn(\zeta) \leq \frac{2^3 n(C_+)}{\pi} \frac{y}{|z|^2}. \tag{2.20}
\]
Thus, by collecting (2.12), (2.13), (2.15), (2.16), (2.17), (2.18), (2.19) and (2.20), there exists a positive constant $A$ independent of $\varepsilon$, such that if $|z| \geq 2R_\varepsilon$ and $z \notin E_2(\varepsilon)$, we have
\[
|h(z)| \leq A \varepsilon y^{1-\alpha} |z|^{m+\alpha}.
\]
Similarly, if $z \notin G$, we have
\[
h(z) = o(y^{1-\alpha} |z|^{m+\alpha}) \quad \text{as } |z| \to \infty. \tag{2.21}
\]
by (1.9) and (2.21), we obtain
\[
u(z) = v(z) + h(z) = o(y^{1-\alpha} |z|^{m+\alpha}) \quad \text{as } |z| \to \infty
\]
hold in $C_+ - G$.

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