Space-time Supersymmetry in Asymmetric Orbifold Models

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Abstract

We study the condition for the appearance of space-time supercharges in twisted sectors of asymmetric orbifold models. We present a list of the asymmetric $Z_N$-orbifold models which satisfy a simple condition necessary for the appearance of space-time supercharges in twisted sectors. We investigate whether or not such asymmetric orbifold models possess $N = 1$ space-time supersymmetry and obtain a new class of $N = 1$ asymmetric orbifold models. It is pointed out that the result of space-time supersymmetry does not depend on any choice of a shift if the order of the left-moving degrees of freedom is preserved.
Orbifold compactification [1, 2] is one of the efficient methods to construct phenomenologically realistic string models and systematic construction of orbifold models has been performed by many groups [3]. Since the heterotic string [4] has left-right asymmetric nature, asymmetric orbifolds [5, 6] are natural extension of symmetric orbifolds and are expected to supply phenomenologically realistic string models. In the construction of four-dimensional string models, the preservation of $N = 1$ space-time supersymmetry seems to be the most reliable condition to select the phenomenologically viable string models. For symmetric orbifold models, the condition for the preservation of $N = 1$ space-time supersymmetry is given by the simple condition which is written in terms of the eigenvalues of the automorphism matrices as discussed in ref. [2]. However, this condition is not applicable to asymmetric orbifold models in general since there might appear gravitino states in some twisted sectors of asymmetric orbifolds\footnote{For symmetric orbifolds, gravitino states do not appear from twisted sectors.}. These gravitino states will correspond to the space-time supercharges from twisted sectors and as a result the space-time supersymmetry in such asymmetric orbifold models will be “enhanced” \footnote{An example of the asymmetric orbifold models which possesses the “enhanced” $N = 1$ space-time supersymmetry is discussed in ref. [8].}. In this paper, we shall study the condition for the occurrence of the space-time supersymmetry “enhancement” and investigate whether or not the asymmetric orbifold models which satisfy the necessary condition for the space-time supersymmetry “enhancement” possess $N = 1$ space-time supersymmetry\footnote{In this paper we use only the “weak” version [10, 11] of the bosonic string map.}.

In order to investigate the condition for the preservation of $N = 1$ space-time supersymmetry of the fermionic string models, it may be convenient to use the bosonic string map [3, 10, 11] and investigate the Kač-Moody algebras in the right-moving degrees of freedom of the corresponding bosonic string models\footnote{In this paper we use only the “weak” version [10, 11] of the bosonic string map.}. In the case of four-dimensional string models, the bosonic string map works as follows: The light-cone $SO(2)$ Kač-Moody algebra generated by the NSR fermions in the fermionic string theory is replaced by an $SO(10) \times E_8$ Kač-Moody algebra in the bosonic string theory. This is done in such a way that the $SO(2)$ Kač-Moody characters of the fermionic string theory are mapped to the $SO(10) \times E_8$ Kač-Moody characters of the bosonic string theory preserving the modular transformation properties of the Kač-Moody characters. Under this map, the
existence of a gravitino state (i.e. an $N = 1$ space-time supercharge) in the fermionic string theory corresponds to the existence of a set of operators of the left-right conformal weight $(0,1)$ transforming as a spinor of the $SO(10)$ in the bosonic string theory. Then it can be shown that, in the bosonic string theory, such a set of operators of the conformal weight $(0,1)$ should extend the $SO(10)$ Kač-Moody algebra to the $E_6$ Kač-Moody algebra \cite{12}.

Under the bosonic string map, the condition discussed in ref. \cite{2} for the preservation of $N = 1$ space-time supersymmetry of symmetric orbifold models becomes the condition that there exist the states of the conformal weight $(0,1)$ in the untwisted sector of the bosonic symmetric orbifold models and that the operators corresponding to them should extend the $SO(10)$ Kač-Moody algebra to the $E_6$ Kač-Moody algebra. The reason for this is that there is no state of the conformal weight $(0,1)$ in the twisted sectors of the bosonic symmetric orbifold models since the left- and right-conformal weights of the ground states of any twisted sector are both positive and equal in the case of the bosonic symmetric orbifold models. On the other hand, in the case of asymmetric orbifold models there is the possibility for the existence of the states of the conformal weight $(0,1)$ in some twisted sectors of the bosonic asymmetric orbifold models. The existence of such states implies a symmetry between the untwisted and twisted sectors and the operators corresponding to the $(0,1)$ states will enlarge the Kač-Moody algebra of the bosonic asymmetric orbifold models \cite{13, 14, 15, 7, 8, 16}. If these operators of the conformal weight $(0,1)$ in some twisted sectors enlarge the Kač-Moody algebra containing the $SO(10)$ Kač-Moody algebra of the bosonic asymmetric orbifold models, then the space-time supersymmetry of the asymmetric orbifold model will be “enhanced”. Therefore, the necessary condition for the space-time supersymmetry “enhancement” in asymmetric orbifold models is the existence of the states of the conformal weight $(0,1)$ in some twisted sectors of the corresponding bosonic asymmetric orbifold models. In the following, we will consider the asymmetric $\mathbb{Z}_N$-orbifold models which satisfy the necessary condition for the space-time supersymmetry “enhancement” and investigate whether or not such asymmetric orbifold models possess $N = 1$ space-time supersymmetry.

In the construction of an asymmetric orbifold model, we start with a toroidal compactification of the $E_8 \times E_8$ heterotic string theories which is specified by a $(22 + 6)$-
dimensional even self-dual lattice $\Gamma^{16,0} \oplus \Gamma^{6,6}$ [7], where $\Gamma^{16,0}$ is a root lattice of $E_8 \times E_8$. The left- and right-moving momentum $(p_i^L, p_i^R)$ ($I = 1, \ldots, 16, i = 1, \ldots, 6$) lies on the lattice $\Gamma^{16,0} \oplus \Gamma^{6,6}$. Let $g$ be a group element which generates a cyclic group $Z_N$. The $g$ is defined to act on the left- and right-moving string coordinate $(X_i^L, X_i^R)$ by

$$g : (X_i^L, X_i^R) \rightarrow (X_i^L + 2\pi v_i^L, U_{ij}^L X_j^L, U_{ij}^R X_j^R),$$

where $U_L$ and $U_R$ are rotation matrices which satisfy $U_N^L = U_N^R = 1$ and $v_i^L$ is a constant vector which satisfy $Nv_i^L \in \Gamma^{16,0}$. The $Z_N$-transformation must be an automorphism of the lattice $\Gamma^{6,6}$, i.e.,

$$(U_{ij}^L p_i^L, U_{ij}^R p_i^R) \in \Gamma^{6,6} \text{ for all } (p_i^L, p_i^R) \in \Gamma^{6,6}. \tag{2}$$

The action of the operator $g$ on the right-moving fermions is given by the $U_R$ rotation. We denote the eigenvalues of $U_L$ and $U_R$ by $\{e^{i2\pi \zeta^a_L}, e^{-i2\pi \zeta^a_L}; a = 1, 2, 3\}$ and $\{e^{i2\pi \zeta^a_R}, e^{-i2\pi \zeta^a_R}; a = 1, 2, 3\}$, respectively. The necessary condition for one-loop modular invariance is for $N$ odd

$$N \frac{1}{2} \sum_{a=1}^{3} \zeta^a_L (1 - \zeta^a_L) + N \frac{1}{2} (v_i^L)^2 = 0 \mod 1; \tag{3}$$

for $N$ even, in addition to the condition (3) we get the following conditions:

$$N \sum_{a=1}^{3} \zeta^a_R = 0 \mod 2, \tag{4}$$

$$p_i^L (U_N^L)^{ij} p_j^L - p_i^R (U_N^R)^{ij} p_j^R = 0 \mod 2 \tag{5}$$

for all $(p_i^L, p_i^R) \in \Gamma^{6,6}$. These are called the left-right level matching conditions and it has been proved that these are also sufficient conditions for one-loop modular invariance [18, 3].

Let $N_L$ be the order of the left-moving degrees of freedom, i.e. $U_N^{N_L} = 1$ and $N_L v_i^L \in \Gamma^{16,0}$. In the $g^\ell$-twisted sector where $\ell$ is the multiples of $N_L$, the left-moving conformal weight of the ground states of the corresponding bosonic asymmetric orbifold model is zero since $U_N^{N_L} = 1$ and $N_L v_i^L \in \Gamma^{16,0}$. Thus, there is the possibility for the existence of the states of the conformal weight $(0,1)$ in such $g^\ell$-twisted sectors of the corresponding bosonic asymmetric orbifold model. Therefore, in the case of the asymmetric $Z_N$-orbifold
models we obtain a simple condition $N_L \neq N$ which is necessary for the space-time supersymmetry “enhancement”, where $N_L$ is the order of the left-moving degrees of freedom. It should be noted that for our purpose it is sufficient to consider the “non-degenerate” element (i.e. $\zeta_R^a \neq 0 \ (a = 1, 2, 3)$) of the right-moving automorphism$^4$. The reason for this is that in the case of the “degenerate” element (i.e. $\zeta_R^a = 0$ for some $a$) of the automorphism the operators of the conformal weight $(0,1)$ in the untwisted sector of the bosonic asymmetric orbifold model will extend the $SO(10)$ Kač-Moody algebra to the Kač-Moody algebra containing the $SO(12)$ Kač-Moody algebra and that such an algebra containing the $SO(12)$ Kač-Moody algebra cannot be enlarged to the $E_6$ Kač-Moody algebra by the operators of the conformal weight $(0,1)$ in the twisted sectors of the bosonic asymmetric orbifold model. We also note that the result of the space-time supersymmetry “enhancement” does not depend on any choice of the shift $v^L_I$ as long as the order of the left-moving degrees of freedom $N_L$ is preserved since in the $g^\ell$-twisted sector where $\ell$ is the multiples of $N_L$ the physical states of the asymmetric orbifold model are determined irrespective of the choice of the shift $v^L_I$.

We will now discuss the asymmetric $Z_N$-orbifold models with inner automorphisms of the momentum lattices associated with simply-laced Lie algebras. We take the lattice $\Gamma^{6,6}$ of an asymmetric $Z_N$-orbifold model to be of the form:

$$\Gamma^{6,6} = \{ (p_L^i, p_R^i) | p_L^i, p_R^i \in \Lambda^*, p_L^i - p_R^i \in \Lambda \}, \tag{6}$$

where $\Lambda$ is a 6-dimensional lattice and $\Lambda^*$ is the dual lattice of $\Lambda$. It turns out that $\Gamma^{6,6}$ is Lorentzian even self-dual if $\Lambda$ is even integral. In the following, we will take $\Lambda$ in eq. (3) to be the products of root lattices of the simply-laced Lie algebras with the squared length of the root vectors normalized to two$^{[20]}$. The left- and right-rotation matrices of the $Z_N$-transformation in eq. (1) are taken from the Weyl group elements of the root lattices of the simply-laced Lie algebras. Then it is easy to see that such a $Z_N$-transformation is an automorphism of the lattice $\Gamma^{6,6}$ in eq. (3). For our purpose, it is sufficient to investigate the “non-degenerate” elements (i.e. $\zeta_R^a \neq 0 \ (a = 1, 2, 3)$) for the right-moving automorphisms. On the other hand, for the left-moving automorphisms

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$^4$The relation between the “non-degenerate” elements of the right-moving automorphisms and space-time chirality is discussed in ref. $^{[19]}$ in the context of the covariant lattice formulation of four-dimensional strings.
there is no reason for restricting our consideration to the “non-degenerate” elements (i.e. \( \zeta^a \neq 0 \) \((a = 1, 2, 3)\)) of the automorphisms and we will investigate all Weyl group elements including the “degenerate” elements (i.e. \( \zeta^a = 0 \) for some \( a \)) of the automorphisms in the left-moving degrees of freedom\(^5\). In our consideration the shift \( v^I_L \) is chosen as

\[
v^I_L = (\zeta^a_L, 0^5; 0^8).
\]

The choice (7) of the shift \( v^I_L \) simplifies the condition (3) for one-loop modular invariance: For \( N \) odd, the condition (3) is always satisfied; for \( N \) even, the condition (3) reduces to the following simple condition:

\[
N \sum_{a=1}^{3} \zeta^a_L = 0 \mod 2.
\]

The unified description of the Weyl groups of all simple Lie algebras have been discussed in refs. [22, 23, 14, 11, 24]. Here, we will briefly summarize the results. Let \( w \) be an element of a Weyl group \( W \). Any \( w \in W \) has the following expression [22]:

\[
w = w_{\alpha_1} \cdots w_{\alpha_k} w_{\alpha_{k+1}} \cdots w_{\alpha_{k+h}},
\]

where \( \alpha_1, \ldots, \alpha_{k+h} \) are linearly independent roots and \( \{\alpha_1, \ldots, \alpha_k\}, \{\alpha_{k+1}, \ldots, \alpha_{k+h}\} \) are each the set of mutually orthogonal roots. Corresponding to the decomposition (9), we shall define a Dynkin-like graph which is called the Carter diagram. If \( w \in W \) has a decomposition with a graph, any Weyl group element which is conjugate to \( w \) also has the decomposition with the same graph. Classification of the graphs associated with the decomposition (9) is discussed for any simple Lie algebras. Although the correspondence between the conjugacy classes and the graphs is not always one-to-one, the exceptions are fully discussed in ref. [22].

Although the left- and right-rotations of the \( Z_N \)-transformations in eq. (11) can be defined for arbitrary elements of the Weyl groups, the conditions for modular invariance put restrictions on the left- and right-rotations of the \( Z_N \)-transformations. All the models we have to consider are shown in table 1. The root lattices \( \Lambda \) associated with the

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\(^5\)Massless states of the four-dimensional string models arising from the covariant lattices which correspond to the “non-degenerate” Weyl group elements in both left- and right-moving degrees of freedom are investigated in ref. [21].
momentum lattices \( \Gamma^{6,6} \) of the asymmetric orbifold models are given in the first column of table 1. The left- and right-moving Carter diagrams \( C_L \) and \( C_R \) of the automorphisms of the momentum lattices \( \Gamma^{6,6} \) of the asymmetric orbifold models are given in the second and the third columns of table 1, respectively. The gauge groups \( G \) of the asymmetric orbifold models are calculated in the same way as in refs. \[15, 7\] by rewriting equivalently the automorphisms of the lattices \( \Gamma^{6,6} \) into shifts in the lattices \[2, 5, 25\] and the results are given in the fourth column of table 1. Using the bosonic string map, we investigate the space-time supersymmetry “enhancement” of the asymmetric orbifold models. The number of the space-time supercharges from the untwisted sector of the asymmetric orbifold models is given in the fifth column of table 1. The total number of the space-time supercharges from the untwisted and twisted sectors of the asymmetric orbifold models is given in the sixth column of table 1. We can easily check that, for all the asymmetric orbifold models with \( N_L \neq N \) investigated in this paper, there always exist the states with the conformal weight \( (0,1) \) in some twisted sectors of the corresponding bosonic asymmetric orbifold models. These operators of the conformal weight \( (0,1) \) in the twisted sectors often enlarge the Kač-Moody algebra containing the \( SO(10) \) Kač-Moody algebra and the space-time supersymmetry “enhancement” occurs in such asymmetric orbifold models. Accordingly, we obtain in table 1 a new class of asymmetric orbifold models with the “enhanced” \( N = 1 \) space-time supersymmetry. On the other hand, in some of the asymmetric orbifold models none of the operators of the conformal weight \( (0,1) \) in the twisted sectors enlarges the Kač-Moody algebra containing the \( SO(10) \) Kač-Moody algebra and no space-time supersymmetry “enhancement” is observed in such asymmetric orbifold models. Accordingly, we also obtain in table 1 a new class of \( N = 1 \) asymmetric orbifold models without the space-time supersymmetry “enhancement” which possess the operators of the conformal weight \( (0,1) \) in the twisted sectors of the corresponding bosonic asymmetric orbifold models.

In this paper we have studied the condition for the space-time supersymmetry “enhancement” in asymmetric orbifold models. We have presented in table 1 a list of the asymmetric \( \mathbb{Z}_N \)-orbifold models with inner automorphisms of the momentum lattices satisfying the simple condition \( N_L \neq N \) which is necessary for the space-time supersymmetry “enhancement”, where \( N_L \) is the order of the left-moving degrees of freedom.
In this list of the asymmetric orbifold models, we have obtained a new class of $N = 1$ asymmetric orbifold models with or without the space-time supersymmetry “enhancement”. It seems that whether or not the asymmetric orbifold models which satisfy the necessary condition for the space-time supersymmetry “enhancement” possess $N = 1$ space-time supersymmetry depends on the details of the momentum lattices and the automorphisms of the lattices. Although we have only considered the choice of the shift $v_L^I$ in eq. (7), the results in table 1 of space-time supersymmetry of the asymmetric $Z_N$-orbifold models are unchanged for the other choice of the shift $v_L^I$ as long as the order of the left-moving degrees of freedom $N_L$ is preserved. For example, it is easy to check that the previously discussed $E_8 \times E_8$ asymmetric orbifold models with the space-time supersymmetry “enhancement” are all understood by considering the $v_L^I = 0$ embedding (i.e. no embedding in the gauge degrees of freedom) of the asymmetric orbifold models discussed in this paper. (The orbifold models in ref. are to be regarded only as illustrative of the features of asymmetric orbifolds and not of a direct phenomenological relevance.) It appears that the choice of the shifts in eq. (7) and the other choice of the shifts which satisfy the condition (3) with the order of the left-moving degrees of freedom preserved are respectively the most natural generalization of the “standard” and “non-standard” embeddings of symmetric orbifolds. Classification of the shifts $v_L^I$ of asymmetric $Z_N$-orbifolds consistent with the condition (3) for one-loop modular invariance will be carried out in exactly the same way as has been done in refs. for symmetric $Z_N$-orbifolds. It would be of great interest to study the asymmetric orbifold models with such shifts $v_L^I$. We hope to get new phenomenologically interesting string models along this line.

The author would like to thank Dr. M. Sakamoto for reading the manuscript and useful comments.
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Table 1: List of asymmetric $Z_N$-orbifold models. $\Lambda$ denote the root lattices associated with the momentum lattices. $C_L$ and $C_R$ denote the left- and right-moving Carter diagrams of the automorphisms of the momentum lattices, respectively, where semicolons separate the direct products of the lattices. $G$ denote the gauge groups of the asymmetric orbifold models where the hidden $E_8$ gauge groups are omitted. $M$ denotes the number of the space-time supercharges from the untwisted sectors of the asymmetric orbifold models. $N$ denotes the total number of the space-time supercharges from the untwisted and twisted sectors of the asymmetric orbifold models.

| $\Lambda$ | $C_L$ | $C_R$ | $G$ | $M$ | $N$ |
|-----------|-------|-------|-----|-----|-----|
| $A_2 \times A_2 \times A_2$ | $(\varphi; \varphi; \varphi)$ | $(A_2; A_2; A_2)$ | $E_8 \times SU(3)^4$ | 1 | 2 |
| $A_4 \times A_2$ | $(\varphi; \varphi)$ | $(A_4; A_2)$ | $E_8 \times SU(5) \times SU(3)$ | 0 | 4 |
| $A_4 \times A_2$ | $(\varphi; A_2)$ | $(A_4; A_2)$ | $SO(14) \times SU(5) \times U(1)^3$ | 0 | 0 |
| $A_4 \times A_2$ | $(A_2; \varphi)$ | $(A_4; A_2)$ | $SO(14) \times SU(3)^2 \times U(1)^3$ | 0 | 0 |
| $A_4 \times A_2$ | $(A_2; A_2)$ | $(A_4; A_2)$ | $E_7 \times SU(3) \times U(1)^5$ | 0 | 0 |
| $A_4 \times A_2$ | $(A_4; A_2)$ | $(A_4; A_2)$ | $SO(12) \times SU(3) \times U(1)^6$ | 0 | 0 |
| $A_6$ | $(\varphi)$ | $(A_6)$ | $E_8 \times SU(7)$ | 1 | 4 |
| $A_6$ | $(A_2)$ | $(A_6)$ | $E_8 \times SU(7)$ | 1 | 4 |
| $A_6$ | $(A_2^2)$ | $(A_6)$ | $E_8 \times SU(7)$ | 1 | 4 |
| $A_6$ | $(A_4)$ | $(A_6)$ | $E_8 \times SU(7)$ | 1 | 4 |
| $D_4 \times A_2$ | $(\varphi; \varphi)$ | $(D_2^2; A_2)$ | $E_8 \times SO(8) \times SU(3)$ | 0 | 4 |
| $D_4 \times A_2$ | $(\varphi; A_2)$ | $(D_2^2; A_2)$ | $SO(14) \times SO(8) \times U(1)^3$ | 0 | 0 |
| $D_4 \times A_2$ | $(\varphi; \varphi)$ | $(D_4(a_1); A_2)$ | $E_8 \times SO(8) \times SU(3)$ | 0 | 4 |
| $D_4 \times A_2$ | $(\varphi; A_2)$ | $(D_4(a_1); A_2)$ | $SO(14) \times SO(8) \times U(1)^3$ | 0 | 0 |
| $D_4 \times A_2$ | $(\varphi; \varphi)$ | $(D_4(a_1); A_2)$ | $E_8 \times SO(8) \times SU(3)$ | 1 | 4 |
| $D_4 \times A_2$ | $(\varphi; A_2)$ | $(D_4(a_1); A_2)$ | $SO(14) \times SO(8) \times U(1)^3$ | 1 | 2 |
| $D_4 \times A_2$ | $(D_2^2; \varphi)$ | $(D_2^2; A_2)$ | $E_7 \times SU(3) \times SU(2)^5$ | 0 | 2 |
| $D_4 \times A_2$ | $(D_4(a_1); \varphi)$ | $(D_4(a_1); A_2)$ | $E_7 \times SU(3) \times SU(2)^5$ | 0 | 2 |
| $D_4 \times A_2$ | $(D_4(a_1); \varphi)$ | $(D_4(a_1); A_2)$ | $SO(10) \times SU(2)^6 \times U(1)^3$ | 0 | 0 |
| $D_4 \times A_2$ | $(A_2; \varphi)$ | $(D_2^2; A_2)$ | $SO(14) \times SU(3) \times SU(2)^3 \times U(1)^2$ | 0 | 0 |
| $D_4 \times A_2$ | $(A_2; A_2)$ | $(D_2^2; A_2)$ | $E_7 \times SU(2)^3 \times U(1)^4$ | 0 | 0 |
| $D_4 \times A_2$ | $(D_2^2; \varphi)$ | $(D_4; A_2)$ | $E_7 \times SU(3) \times SU(2)^5$ | 1 | 2 |
| $D_4 \times A_2$ | $(D_4(a_1); \varphi)$ | $(D_4(a_1); A_2)$ | $E_7 \times SU(3) \times SU(2)^3 \times U(1)^2$ | 0 | 0 |
| $D_4 \times A_2$ | $(A_2; \varphi)$ | $(D_4(a_1); A_2)$ | $E_7 \times SU(2)^3 \times U(1)^4$ | 0 | 0 |
| $D_4 \times A_2$ | $(D_4(a_1); \varphi)$ | $(D_4(a_1); A_2)$ | $E_7 \times SU(3) \times SU(2)^5$ | 1 | 2 |
| $D_4 \times A_2$ | $(D_4(a_1); \varphi)$ | $(D_4(a_1); A_2)$ | $SO(12) \times SU(3) \times U(1)^6$ | 0 | 0 |
| $D_4 \times A_2$ | $(D_4; \varphi)$ | $(D_4(a_1); A_2)$ | $E_6 \times U(1)^8$ | 0 | 0 |
| $D_4 \times A_2$ | $(A_2; \varphi)$ | $(D_4(a_1); A_2)$ | $SO(14) \times SU(3) \times SU(2)^3 \times U(1)^2$ | 1 | 2 |
| $D_4 \times A_2$ | $(A_2; A_2)$ | $(D_4(a_1); A_2)$ | $E_7 \times SU(2)^3 \times U(1)^4$ | 1 | 2 |
Table 1: (continued)

| Λ   | \( C_L \)       | \( C_R \)       | \( G \)                  | \( M \) | \( N \) |
|-----|-----------------|-----------------|--------------------------|--------|--------|
| \( D_6 \) | (\( φ \))     | (\( D_4(a_1)D_2 \)) | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( φ \))     | (\( D_6(a_1) \))   | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( D_2 \))   | (\( D_4(a_1)D_2 \)) | \( E_7 \times SO(8) \times SU(2)^3 \) | 1     | 2      |
| \( D_6 \) | (\( D_2 \))   | (\( D_6(a_1) \))   | \( E_7 \times SO(8) \times SU(2)^3 \) | 1     | 2      |
| \( D_6 \) | (\( A_2^2D_2 \)) | (\( D_4(a_1)D_2 \)) | \( E_7 \times SO(8) \times SU(2)^3 \) | 1     | 2      |
| \( D_6 \) | (\( A_2^2D_2 \)) | (\( D_6(a_1) \))   | \( E_7 \times SO(8) \times SU(2)^3 \) | 1     | 2      |
| \( D_6 \) | (\( D_4(a_1) \)) | (\( D_6(a_1) \))   | \( E_7 \times SU(4) \times SU(2) \times U(1)^3 \) | 1     | 1      |
| \( D_6 \) | (\( A_2 \))   | (\( D_4(a_1)D_2 \)) | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( A_2 \))   | (\( D_6(a_1) \))   | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( D_4 \))   | (\( D_4(a_1)D_2 \)) | \( E_7 \times SO(8) \times SU(2)^3 \) | 1     | 2      |
| \( D_6 \) | (\( A_2^2 \)) | (\( D_4(a_1)D_2 \)) | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( D_4(a_1)D_2 \)) | (\( D_6(a_1) \))   | \( E_6 \times SO(8) \times SU(2)^2 \times U(1)^2 \) | 1     | 1      |
| \( D_6 \) | (\( A_4 \))   | (\( D_4(a_1)D_2 \)) | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( A_2 \))   | (\( D_6(a_1) \))   | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( D_4 \))   | (\( D_6(a_1) \))   | \( E_7 \times SO(8) \times SU(2)^3 \) | 1     | 2      |
| \( D_6 \) | (\( A_2^2 \)) | (\( D_6(a_1) \))   | \( E_8 \times SO(12) \) | 1     | 4      |
| \( D_6 \) | (\( A_4 \))   | (\( D_6(a_1) \))   | \( E_8 \times SO(12) \) | 1     | 4      |
| \( E_6 \) | (\( φ \))     | (\( A^3_2 \))     | \( E_8 \times E_6 \) | 1     | 4      |
| \( E_6 \) | (\( φ \))     | (\( A_5A_1 \))    | \( E_8 \times E_6 \) | 1     | 4      |
| \( E_6 \) | (\( φ \))     | (\( E_6 \))       | \( E_8 \times E_6 \) | 1     | 4      |
| \( E_6 \) | (\( φ \))     | (\( E_6(a_1) \))  | \( E_8 \times E_6 \) | 0     | 4      |
| \( E_6 \) | (\( φ \))     | (\( E_6(a_2) \))  | \( E_8 \times E_6 \) | 1     | 4      |
| \( E_6 \) | (\( A_2 \))   | (\( A_5A_1 \))    | \( SO(14) \times SU(6) \times U(1)^2 \) | 1     | 1      |
| \( E_6 \) | (\( A_2 \))   | (\( E_6 \))       | \( SO(14) \times SU(6) \times U(1)^2 \) | 1     | 1      |
| \( E_6 \) | (\( A_2 \))   | (\( E_6(a_1) \))  | \( SO(14) \times SU(6) \times U(1)^2 \) | 0     | 1      |
| \( E_6 \) | (\( A_2 \))   | (\( E_6(a_2) \))  | \( SO(14) \times SU(6) \times U(1)^2 \) | 1     | 2      |
| \( E_6 \) | (\( A_1^2 \)) | (\( A^3_2 \))     | \( E_8 \times E_6 \) | 1     | 4      |
| \( E_6 \) | (\( A_1^2 \)) | (\( A_5A_1 \))    | \( E_7 \times SU(6) \times SU(2)^2 \) | 1     | 2      |
| \( E_6 \) | (\( A_1^2 \)) | (\( E_6 \))       | \( E_7 \times SU(6) \times SU(2)^2 \) | 1     | 2      |
| \( E_6 \) | (\( A_1^2 \)) | (\( E_6(a_1) \))  | \( E_8 \times E_6 \) | 0     | 4      |
| \( E_6 \) | (\( A_1^2 \)) | (\( E_6(a_2) \))  | \( E_7 \times SU(6) \times SU(2)^2 \) | 1     | 2      |
Table 1: (continued)

| A     | $C_L$    | $C_R$    | $G$                     | $M$ | $N$ |
|-------|----------|----------|-------------------------|-----|-----|
| $E_6$ | $(A_5^2)$ | $(A_5A_1)$ | $E_7 \times SO(8) \times U(1)^3$ | 1   | 1   |
| $E_6$ | $(A_2^2)$ | $(E_6)$   | $E_7 \times SO(8) \times U(1)^3$ | 1   | 1   |
| $E_6$ | $(A_2^3)$ | $(E_6(a_1))$ | $E_7 \times SO(8) \times U(1)^3$ | 0   | 1   |
| $E_6$ | $(A_2^2)$ | $(E_6(a_2))$ | $E_7 \times SO(8) \times U(1)^3$ | 1   | 2   |
| $E_6$ | $(A_1)$   | $(A_2)$   | $E_8 \times E_6$        | 1   | 4   |
| $E_6$ | $(A_4)$   | $(A_5A_1)$ | $E_8 \times E_6$        | 1   | 4   |
| $E_6$ | $(A_4)$   | $(E_6)$   | $E_8 \times E_6$        | 1   | 4   |
| $E_6$ | $(A_4)$   | $(E_6(a_1))$ | $E_8 \times E_6$        | 0   | 4   |
| $E_6$ | $(A_4)$   | $(E_6(a_2))$ | $E_8 \times E_6$        | 1   | 4   |
| $E_6$ | $(D_4)$   | $(E_6)$   | $SO(12) \times SU(3)^2 \times U(1)^4$ | 1   | 1   |
| $E_6$ | $(D_4)$   | $(E_6(a_1))$ | $SO(14) \times SU(6) \times U(1)^2$ | 0   | 1   |
| $E_6$ | $(D_4(a_1))$ | $(A_3^2)$ | $E_8 \times E_6$        | 1   | 4   |
| $E_6$ | $(D_4(a_1))$ | $(A_5A_1)$ | $E_7 \times SU(6) \times SU(2)^2$ | 1   | 2   |
| $E_6$ | $(D_4(a_1))$ | $(E_6)$   | $E_7 \times SU(3)^2 \times SU(2) \times U(1)^2$ | 1   | 2   |
| $E_6$ | $(D_4(a_1))$ | $(E_6(a_1))$ | $E_8 \times E_6$        | 0   | 4   |
| $E_6$ | $(D_4(a_1))$ | $(E_6(a_2))$ | $E_7 \times SU(6) \times SU(2)^2$ | 1   | 2   |
| $E_6$ | $(A_2^3)$ | $(A_5A_1)$ | $E_6 \times SU(3)^4$ | 1   | 1   |
| $E_6$ | $(A_2^3)$ | $(E_6)$   | $E_6 \times SU(3)^4$ | 1   | 1   |
| $E_6$ | $(A_2^3)$ | $(E_6(a_1))$ | $E_6 \times SU(3)^4$ | 0   | 1   |
| $E_6$ | $(A_2^3)$ | $(E_6(a_2))$ | $E_6 \times SU(3)^4$ | 1   | 2   |
| $E_6$ | $(A_5A_1)$ | $(E_6)$   | $E_6 \times SU(2)^4 \times U(1)^4$ | 1   | 1   |
| $E_6$ | $(E_6(a_1))$ | $(A_5A_1)$ | $E_6 \times SU(3)^4$ | 1   | 1   |
| $E_6$ | $(A_5A_1)$ | $(E_6(a_1))$ | $E_6 \times SU(3)^4$ | 0   | 1   |
| $E_6$ | $(E_6(a_1))$ | $(E_6)$   | $E_6 \times SU(3)^4$ | 1   | 1   |
| $E_6$ | $(E_6(a_1))$ | $(E_6(a_1))$ | $E_6 \times SU(3)^4$ | 0   | 1   |
| $E_6$ | $(E_6(a_2))$ | $(E_6)$   | $E_6 \times SU(2)^4 \times U(1)^4$ | 1   | 1   |
| $E_6$ | $(E_6(a_2))$ | $(E_6(a_1))$ | $E_7 \times SO(8) \times U(1)^3$ | 0   | 1   |
| $E_6$ | $(E_6(a_2))$ | $(E_6(a_2))$ | $E_6 \times SU(3)^4$ | 1   | 2   |