Nonlinear Perpendicular Diffusion in Strong Turbulent Electromagnetic Fields

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Abstract

A nonlinear description for perpendicular particle diffusion in strong electromagnetic fluctuations is developed by using the fundamental Newton-Lorentz equation. Although not based on the same approach, the recently presented nonlinear guiding center (NLGC) theory is recovered as a special case. The approach used here is rather based on the argument that a well defined particle gyromotion does not exist in strong fluctuations than on the assumption that the particle gyrocenter follows magnetic field lines which themselves separate diffusively. The assumption of a guiding center motion and the diffusive separation of magnetic field lines is absolutely central to the NLGC theory. It is argued that the NLGC result should provide most accurate results for strong fluctuations. Furthermore, as a direct consequence of the particle equation of motion, it is shown that particle diffusion in one perpendicular direction is governed by the fluctuations in the other normal direction. This results contradicts the NLGC result, where perpendicular diffusion is triggered by fluctuations in the same direction. This is of particular interest for anisotropic perpendicular diffusion in a non-axisymmetric turbulence. Future numerical simulation results for non-axisymmetric magnetic turbulence and their comparison with the approach presented here and the NLGC theory have to provide an answer whether particle diffusion in one perpendicular direction is governed by the fluctuations in the other normal direction or by fluctuations in the same direction.

Key words: Cosmic Rays, Diffusion, Turbulence

1 Introduction and Motivation

Diffusive particle motion in random fields plays a key role in space physics, astrophysics and the physics of fusion devices. Turbulence properties are es-
sential for understanding the three-dimensional (anisotropic) diffusive particle transport in a collisionless, turbulent and magnetized plasma such as the solar wind or the interstellar medium. Of particular interest are the transport coefficients describing particle diffusion perpendicular to an ambient magnetic field.

In spite of its long-standing importance in space and astrophysics, perpendicular diffusion has been an unsolved puzzle during many decades and a variety of studies have been carried out to achieve closure and to pin it down at a theoretical level. Models have been proposed for hard-sphere scattering in a magnetized plasma (Gleeson, 1969) and for related extensions developed on the basis of the Boltzmann equation (Jones, 1990), but they seem to be inapplicable to space plasmas where (electro)magnetic fluctuations trigger particle scattering (see Bieber and Matthaeus, 1997).

Other models are based on the field line random walk (FLRW) limit emerging, for slab turbulence geometry, from quasilinear theory (QLT Jokipii, 1966; Jokipii and Parker, 1969), which has been considered and used in subsequent studies (e.g. Forman et al., 1974; Forman, 1977; Bieber and Matthaeus, 1997). Although FLRW provides for a physically appealing picture, it has been shown recently that its applicability is questionable for particle transport in certain turbulence geometries, particularly those with at least one ignorable coordinate being the case for slab geometry (Jokipii et al., 1993). Furthermore, numerical simulations, taking into account the magnetic nature of the turbulence, have shown that FLRW fails to explain perpendicular transport of low-energy particles (Giacalone and Jokipii, 1999; Mace et al., 2000), since pitch-angle scattering is neglected in the FLRW limit (e.g., Qin et al., 2002a).

It has been argued for some time that diffusion along a background magnetic field can reduce perpendicular diffusion to subdiffusive levels if the turbulence reveals slab geometry only (see, e.g., Urch, 1977; Kóta and Jokipii, 2000; Qin et al., 2002b). However, when the turbulent magnetic field has sufficient structure normal to the mean magnetic field, subdiffusion as seen in pure slab turbulence can be overcome and diffusion is recovered (see Qin et al., 2002a).

Recently, Matthaeus et al. (2003) proposed the so-called nonlinear guiding center (NLGC) theory. For their approach, they use the following two key assumptions: First, perpendicular transport is governed by the velocity of particle gyrocenters that follow magnetic field lines. Second, the magnetic field lines themselves separate diffusively. Based on these two assumptions, Matthaeus et al. obtain for the velocity of the particle gyrocenter in the $x$-direction the expression

$$v_x = av_z \frac{\delta B_x}{B_0},$$

(1)
where $\delta B_x$ is the $x$-component of the turbulent magnetic field, $B_0$ is the strength of the background magnetic field. The constant $a$ has to be determined after the fact. Consequently, in the NLGC theory, particle diffusion in normal direction is solely triggered by the diffusive separation of the underlying magnetic field lines. Employing the so-called Taylor-Green-Kubo (TGK) formulation (see, e.g., Bieber and Matthaeus, 1997; Matthaeus et al., 2003, for more details), Matthaeus et al. derive, on the basis of equation (1), the following nonlinear integral equation for the perpendicular diffusion coefficient:

$$
\kappa_{xx} = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \frac{S_{xx}(k)}{v/\lambda_{zz} + \kappa_{xx} k_\perp^2 + \kappa_{zz} k_z^2 + \gamma(k)}. \tag{2}
$$

Here, $\kappa_{zz} = v \lambda_{zz}/3$ is the parallel diffusion coefficient, $v$ is the particle speed and $\lambda_{zz}$ is the parallel mean free path. The wavevector components parallel and perpendicular to the mean magnetic field are given by $k_z$ and $k_\perp^2 = k_x^2 + k_y^2$, respectively, and $S_{xx}$ is the spectral amplitude of magnetic fluctuations in the $x$-direction. The decorrelation rate $\gamma(k)$ allows to include dynamical effects due to the decay of turbulent energy (see, e.g., Bieber et al., 1994, for more details concerning dynamical magnetic turbulence). Since the NLGC theory is based on the TGK formalism, equation (2) only admits Markovian diffusion; anomalous transport processes such as subdiffusion and superdiffusion can not be accommodated by the NLGC model (see also Zank et al., 2004, for further discussion).

Considering the particular case of a static ($\gamma(k) = 0$) turbulence, Matthaeus et al. (2003) use the NLGC approach for weak ($\delta B^2/B_0^2 = 0.04$) and stronger ($\delta B^2/B_0^2 = 1.0$) turbulence and compare the results with numerical simulations. The numerical simulation results used for the comparison are obtained by applying a fourth-order Runge-Kutta method with subsequent integration to the equation of motion of a charged test particle (see Qin et al., 2002a,b; Matthaeus et al., 2003). The finding of the comparison is that the NLGC theory seems to be consistent with the numerical simulations for the case $a = 1/\sqrt{3}$. Since the NLGC approach has been fruitful for a better understanding of perpendicular diffusion, it was recently applied to cosmic ray modulation and shock acceleration (Zank et al., 2004).

Although the NLGC approach seems to be in agreement with the numerical simulation results, several questions arise. Among those are the following two: (1) Does a well defined particle gyromotion and, therefore, a gyrocenter actually exists for fluctuations being that strong that the background magnetic field has no control over the particle motion? Wandering in the particle pitch-angle and gyrophase are then very rapid, and the concept of a gyrocenter velocity following magnetic field lines is expected to fail for strong fluctuations. (2) By using the fundamental particle equation of motion itself, is it actually
possible to derive spatial diffusion coefficients being in agreement with the
NLGC result? A closer inspection of the ansatz (1) shows that the NLGC ap-
proach is not developed on the basis of the Newton-Lorentz equation, which
describes particle diffusion in turbulent electromagnetic fields. The NLGC re-
sult, equation (2), is rather based on the diffusivity of the underlying magnetic
field lines. The diffusivity of magnetic field lines is given by the magnetic field
streamline equation (see, e.g., Jokipii and Parker, 1969; Ruffolo et a l., 2004),
which apparently was employed to result in equation (1).

In view of the two questions given above, a nonlinear description for per-
pendicular diffusion in strong electromagnetic fluctuations is presen-
ted which explicitly excludes an “ordered” particle gyromotion. The approach used here
is based on the particle equation of motion and contradicts the above men-
tioned key assumptions made by Matthaeus et al. for the NLGC theory.

2 Basic Equations and Formulation

For a random, diffusive particle motion and statistically homogeneous and
stationary fluctuations, $\kappa_{xx}$ is given by the Taylor-Green-Kubo (TGK) formula
(Kubo, 1957) which can be expressed as

$$\kappa_{xx} = \Re \int_0^\infty d\xi \langle v_x(0)v_x^*(\xi) \rangle,$$  (3)

where $v_x$ is the $x$-component of the particle velocity. An analogous expression is
given for the diffusion coefficient in the $y$-direction, $\kappa_{yy}$. For a large turbulence
coherence time $\xi$, the second-order velocity correlation function $\langle v_x(0)v_x^*(\xi) \rangle$
must go to zero, and the integral in equation (3) approaches a constant value
for $t \rightarrow \infty$. This is characteristic for Markovian diffusion.

The use of the TGK formula is somewhat critical for situations where particle
transport reveals rather an anomalous diffusion process than normal Marko-
vian diffusion. In this case, the standard Fokker-Planck coefficient (anomalous
transport law) $\kappa_{xx} = \langle \Delta x^2 \rangle / 2t$ applies, where the mean square displacement
scales more general as $\langle \Delta x^2 \rangle \propto t^s$. Depending on the exponent $s$, different
diffusion processes can be taken into account: $s = 1$ for the diffusive regime,
corresponding to a Gaussian random walk of particles; $s = 2$ in the superdif-
fusive regime, i.e. strictly scatter-free propagation of the particles (see, e.g.,
McComb, 1990), and $s < 1$ (e.g. $s = 1/2$, as shown by Qin et al. (2002b)
and Kóta and Jokipii (2000)) in the case of particle trapping (subdiffusion or
compound diffusion). For $s = 1$ (normal Markovian diffusion), the standard
Fokker-Planck coefficient is equivalent to the TGK formula (3) for the large $t$
2.1 Equations of Motion

Generally, the random walk of a particle with mass $m$, charge $q$ and velocity $v(t) = \dot{x}(t)$ in fluctuating fields is governed by its equation of motion, i.e. the Newton-Lorentz equation

$$\dot{p} = \frac{d}{dt}(\gamma m \dot{v}(t)) = qE[x(t), t] + \frac{q}{c}v(t) \times B[x(t), t],$$

where $\gamma$ is the Lorentz factor. The magnetic field $B = B_0 + \delta B$ consists of a uniform, steady background magnetic field $B_0 = B_0 e_z$ to which is added a broad band spectrum of magnetic fluctuations $\delta B[x(t), t]$. Analogously, the electric field $E = E_0 + \delta E$ consists of a vanishing background electric field $\langle E \rangle = E_0 = 0$ with superimposed electric fluctuations $\delta E[x(t), t]$. Equation (4) can formally be integrated in time, to obtain for the perpendicular velocity components the expressions

$$v_x(t) = \frac{q}{\gamma mc} \int_0^t dt' (c\delta E_x[x(t'), t'] + v_y(t')\delta B_z[x(t'), t'] - v_z(t')\delta B_y[x(t'), t'])$$

$$+ \Omega \int_0^t dt' v_y(t'),$$

and

$$v_y(t) = \frac{q}{\gamma mc} \int_0^t dt' (c\delta E_y[x(t'), t'] + v_z(t')\delta B_x[x(t'), t'] - v_x(t')\delta B_z[x(t'), t'])$$

$$- \Omega \int_0^t dt' v_x(t'),$$

where $\Omega = qB_0/\gamma mc$ is the relativistic gyrofrequency.

The first integrals in equations (5) and (6) describe the random velocity components of the particle in the $x$ and $y$-direction, respectively, due to the fluctuating fields. They enter equations (5) and (6) solely by these contributions and represent a stochastic force acting on the particle and forcing it to deviate from its unperturbed orbit given by the second integrals. In a uniform and ordered background magnetic field, each unperturbed orbit is a perfect helix along which the particle moves at constant speed $v_z$, while the transverse velocity components $v_x$ and $v_y$ oscillate at angular frequency $\Omega$ around a magnetic line of force. Now consider the effects of fluctuations on the unperturbed particle orbit. For weak ($\delta B^2/B_0^2 \ll 1$) turbulence conditions, the fluctuations
cause the particle to wander in its pitch-angle and gyrophase. The force induced by $B_0$ dominates the particle transport and stochastic contributions resulting from the first integrals are comparably small. Consequently, a particle “remembers” its helical orbit and performs a diffusive but ordered motion along a preferred direction defined by $B_0$. The deviation from its helical orbit in perpendicular directions is then marginal, and diffusion is dominated by parallel scattering.

In the opposite case of strong ($\delta B^2/B_0^2 \geq 1$) turbulence, the ordered force induced by $B_0$ is small and particle transport is dominated by the “chaotic” contributions given by the first terms in equations (5) and (6). Then, a particle moves barely on a perfect trajectory and “forgets” its ideal orbit very quickly. Decorrelation from the helical orbit then occurs very rapidly, the wandering in pitch-angle and gyrophase is then purely chaotic, and a well defined particle gyromotion does not exist. Whether the first or second term in equations (5) and (6) dominate depends then solely on the magnitude of the ratio $\delta B^2/B_0^2$.

The decisive step taken here is to assume strong turbulence conditions, that the particle has no memory of its formerly helical trajectory. To do so, the contribution representing the helical orbit is dropped. This does not imply that $B_0 = 0$, it is rather assumed that the random velocity components (first integral) overwhelm the unperturbed particle orbit (second integral). Proceeding, the irregular fields are expressed by Fourier transforms and the first integrals are, for simplicity and the reason of overview, approximated by their integrands. This yields

$$v_x(t) = \frac{q\tau_x}{\gamma mc} \int d^3k e^{ik \cdot x(t)} [c\delta E_x(k,t) + v_y(t)\delta B_z(k,t) - v_z(t)\delta B_y(k,t)] ,$$

$$v_y(t) = \frac{q\tau_y}{\gamma mc} \int d^3k e^{ik \cdot x(t)} [c\delta E_y(k,t) + v_z(t)\delta B_x(k,t) - v_x(t)\delta B_z(k,t)] .$$

Here, $\tau_x$ and $\tau_y$ denote timescales related to the deviation of the particle position from the unperturbed orbit in the $x$ and $y$-direction, respectively. The timescales $\tau_x$ and $\tau_y$ might depend on particle rigidity and turbulence properties, but they are here introduced for the reason of dimensionality. A future study requires, of course, in any case a detailed treatment of the time integrations. This is far beyond of the scope of the current paper, since the evaluation is a study on its own. In particular, the TGK formula (3) requires the evaluation of $v_x$ at time zero, which is a result of the time translation invariance property usually employed for the derivation of the TGK formula (see, e.g., McComas, 1990; Kótai and Jokipii, 2000). Obviously, the substitution of equations (5) and (6) into the TGK formula would then yield zero for $t = 0$. Actually, the approach used here by employing directly the particle equation of motion requires the more general Fokker-Planck definition $\kappa_{xx} = \langle \Delta x^2 \rangle / 2t$. 

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and this is currently under investigation. The purpose of this paper is rather to demonstrate that the same mathematical structure on diffusion coefficients as those of the NLGC model can be obtained if the Newton-Lorentz equation is used for the assumption of electromagnetic fluctuations overwhelming effects related to the background magnetic field. As becomes clear below, the timescales $\tau_x$ and $\tau_y$ are then related to the “numerical” constant $a$ given in equation (1).

2.2 Velocity Correlation Functions

The substitution of equations (7) and (8) into the TGK formula (3) converts the integrands into a sum of nine individual contributions involving random particle velocity components and/or turbulent field components at two different wavevectors. Since the particle velocity components are random quantities, i.e. $\langle v_i \rangle = 0$, four contributions vanish. Proceeding, it is assumed that the particle velocity components are uncorrelated with the local magnetic field vector (see also Matthaeus et al., 2003). Furthermore, supposing that the probability distribution of the particle displacements and that of the Eulerian velocity field are statistically independent, which is also referred to as Corrsin’s independence hypothesis (Corrsin, 1959; Matthaeus et al., 2003; McComb, 1990), one obtains for the velocity correlation function in $x$-direction the expression

$$\langle v_x(0)v_x^*(\xi) \rangle = \frac{q\tau_x}{\gamma mc} \int d^3k \int d^3k' \langle e^{-i k' \cdot x(\xi)} \rangle \left[ c^2 R_{xx}(k, k', \xi) \right.$$

$$+ \langle v_z(0)v_z^*(\xi) \rangle S_{yy}(k, k', \xi) + \langle v_y(0)v_y^*(\xi) \rangle S_{zz}(k, k', \xi)$$

$$- \langle v_y(0)v_z^*(\xi) \rangle S_{zy}(k, k', \xi) - \langle v_z(0)v_y^*(\xi) \rangle S_{yz}(k, k', \xi) \right]$$

and an analogous expression for $\langle v_y(0)v_y^*(\xi) \rangle$. Here, the spectral amplitudes

$$S_{nm}(k, k', \xi) = \langle \delta B_n(k, 0) \delta B_m^*(k', \xi) \rangle$$

$$R_{nm}(k, k', \xi) = \langle \delta E_n(k, 0) \delta E_m^*(k', \xi) \rangle$$

were introduced, where the subscripts refer to the Cartesian coordinates. Furthermore, without loss of generality, the initial condition $x(0) = 0$ was used. For statistically homogeneous turbulence, it is convenient to employ the relations $S_{nm}(k, k', \xi) = \delta(k-k')S_{nm}(k, \xi)$ and $R_{nm}(k, k', \xi) = \delta(k-k')R_{nm}(k, \xi)$, and it is assumed that electromagnetic fluctuations can be represented by a
superposition of $N$ individual plasma wave modes, i.e.

$$\delta \mathbf{B}(\mathbf{k}, t) = \sum_{j=1}^{N} \delta \mathbf{B}^j(\mathbf{k}) \exp(-i\omega_j t)$$  \hspace{1cm} (12)$$

$$\delta \mathbf{E}(\mathbf{k}, t) = \sum_{j=1}^{N} \delta \mathbf{E}^j(\mathbf{k}) \exp(-i\omega_j t).$$  \hspace{1cm} (13)

Here, $\omega_j(\mathbf{k}) = \omega_{j,R}(\mathbf{k}) + i\Gamma_j(\mathbf{k})$ is a complex dispersion relation of wave mode $j$, where $\omega_{j,R}(\mathbf{k})$ is the real frequency of the mode. The imaginary part, $\Gamma_j(\mathbf{k}) \leq 0$, represents dissipation of turbulent energy due to plasma wave damping. Note that such an ansatz for the time variation is usually used in quasilinear theory (see, e.g., Gary, 1993; Schlickeiser, 2002), where fluctuations are supposed to be very weak. Here, however, it is assumed that the same approach holds for fluctuations dominating the influence of the background magnetic field on particle motion.

Restricting the considerations to transverse fluctuations, i.e. $\delta \mathbf{E}^j \cdot \mathbf{k} = 0$, and using Faraday’s law, the turbulent electric field can easily be expressed by the corresponding magnetic counterparts, yielding

$$\delta E_x^j = \frac{\omega_j}{ck^2} \left( \delta B_y^j k_y - \delta B_x^j k_x \right)$$

$$\delta E_y^j = \frac{\omega_j}{ck^2} \left( \delta B_x^j k_x - \delta B_y^j k_y \right)$$  \hspace{1cm} (14)

and, subsequently, the relation

$$R_{xx} = \frac{1}{e^2 k^4} \left[ S_{yy}^j k_y^2 + S_{zz}^j k_y^2 - k_x k_y \left( S_{zy}^j + S_{yz}^j \right) \right].$$  \hspace{1cm} (15)

For simplicity, it is assumed that $S_{zy}^j(\mathbf{k}) = S_{yz}^j(\mathbf{k}) = 0$ for following calculations. In what follows, one arrives at

$$\langle v_x(0)v_x^*(\xi) \rangle = \left( \frac{q\tau_x}{\gamma mc} \right)^2 \sum_{j=1}^{N} \int d^3k \left[ e^{-i\mathbf{k} \cdot \mathbf{x(\xi)}} e^{i\omega_{j,R}(\mathbf{k})(\xi) + \Gamma_j(\mathbf{k})\xi} \right]$$

$$\times \left[ \frac{|\omega_j|^2}{k^4} \left( k_x^2 S_{yy}^j(\mathbf{k}) + k_y^2 S_{zz}^j(\mathbf{k}) \right) \right]$$

$$+ \langle v_z(0)v_z^*(\xi) \rangle S_{yy}^j(\mathbf{k}) + \langle v_y(0)v_y^*(\xi) \rangle S_{zz}^j(\mathbf{k})$$  \hspace{1cm} (16)

and obtains an analogous expression for $\langle v_y(0)v_y^*(\xi) \rangle$. 

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2.3 Nonlinear Diffusion Coefficients

One can now proceed to derive expressions for $\kappa_{xx}$ as well as $\kappa_{yy}$. To do so, the time-integration in equation (3) has to be performed, and further progress requires the variations of the individual contributions in the coherence time $\xi$. First, the velocity correlation functions in equation (16) can be written as (see also Matthaeus et al., 2003)

$$\langle v_n(0)v^*_n(\xi) \rangle = (v^2/3) \exp(-v\xi/\lambda_{nn}),$$

(17)

where $\kappa_{nn} = v\lambda_{nn}/3$ is used, being in agreement with the TGK definition (3). Here, $\lambda_{nn}$ is the mean free path in the direction of one of the Cartesian coordinates. Second, the ensemble-averaged exponential expression can be expressed as

$$\langle \exp [-i\mathbf{k} \cdot \mathbf{x}(\xi)] \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz P(x, y, z) e^{-i k_x x - i k_y y - i k_z z},$$

(18)

where the distribution function $P(x, y, z)$ represents displacements of the particle position from the unperturbed orbit in Cartesian space due to the decorrelation of the trajectory. The decisive step is to assume a Gaussian distribution, i.e.

$$P(x, y, z) = \frac{1}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_z} \exp \left[ -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2} \right],$$

(19)

where the variances $\sigma_x^2 = \langle \Delta x^2 \rangle$, $\sigma_y^2 = \langle \Delta y^2 \rangle$ and $\sigma_z^2 = \langle \Delta z^2 \rangle$ are assumed to be diffusive in the sense that $\langle \Delta x^2 \rangle = 2\kappa_{xx} \xi$, $\langle \Delta y^2 \rangle = 2\kappa_{yy} \xi$ and $\langle \Delta z^2 \rangle = 2\kappa_{zz} \xi$. Upon substituting the Gaussian distribution function (19) into equation (18), one arrives, after elementary integrations, at

$$\langle \exp [-i\mathbf{k} \cdot \mathbf{x}(\xi)] \rangle = \exp[-(\kappa_{xx} k_x^2 + \kappa_{yy} k_y^2 + \kappa_{zz} k_z^2)\xi].$$

(20)

Using these two entries, the $\xi$-integration in equation (3) is elementary. Upon extending in equation (16) the factor in front of the wavevector integral by $B_0$ (not assumed to be zero), one finally obtains for $\kappa_{xx}$ and $\kappa_{yy}$ the following expressions:

$$\kappa_{xx} = \frac{a^2 v^2}{3 B_0^2} \sum_{j=1}^{N} \int d^3 k \left[ \frac{3|\omega_j|^2}{v^2 k^4} \left( k^2 S^j_y(k) + k^2 S^j_z(k) \right) \right] \frac{A}{A^2 + \omega_j^2},$$

(21)
$$+ S_{yy}^i(k) \frac{A + v/\lambda_{zz}}{(A + v/\lambda_{zz})^2 + \omega_{j,R}^2} + S_{zz}^i(k) \frac{A + v/\lambda_{yy}}{(A + v/\lambda_{yy})^2 + \omega_{j,R}^2}$$

$$\kappa_{yy} = \frac{a_x^2 v_x^2}{3B_0^2} \sum_{j=1}^{N} \int d^3k \left[ \frac{3|\omega_j|^2}{v^2 k^4} \left( k^2 S_{xx}^j(k) + k_z^2 S_{zz}^j(k) \right) \right] \frac{A}{A^2 + \omega_{j,R}^2}$$

$$+ S_{xx}^j(k) \frac{A + v/\lambda_{zz}}{(A + v/\lambda_{zz})^2 + \omega_{j,R}^2} + S_{zz}^i(k) \frac{A + v/\lambda_{xx}}{(A + v/\lambda_{xx})^2 + \omega_{j,R}^2}.$$ (22)

Here, \( A = \kappa_{xx} k_x^2 + \kappa_{yy} k_y^2 + \kappa_{zz} k_z^2 - \Gamma_j(k) \) and \( a_x = \tau_x \Omega \) as well as \( a_y = \tau_y \Omega \) are introduced. Equations (21) and (22), the main results of this paper, form a set of two nonlinear and coupled equations for the two diffusion coefficients in the two distinguished perpendicular directions. Each diffusion coefficient depends not only on \( \kappa_{zz} \), but also on the perpendicular transport parameter describing (diffusive) particle transport in the other normal direction.

An interpretation of the individual contributions can be given by the “right-hand” rule known from electrodynamics. The scattering of a particle in a certain Cartesian direction can be considered as the result of a stochastic force acting on the particle and forcing it to move diffusively in this direction. For instance, consider the contributions appearing in \( \kappa_{xx} \). According to the Lorentz-force term in equation (4), a “chaotic” force in \( x \)-direction requires a random particle speed along the \( z \)-axis. In the second contribution in equation (21), we see that there is a stochastic velocity component \( v_z \) present, through \( \lambda_{zz} \). This random speed, together with the irregular magnetic fluctuation in the \( y \)-direction, via \( S_{yy}^i \), results in the diffusion in the \( x \)-direction. Particle diffusion in one perpendicular direction is, therefore, governed by the fluctuations in the other normal direction. The third contribution results from the turbulent field component aligned along the weak background magnetic field. According to the Lorentz-force term, a random particle speed in \( y \)-direction is then required or, equivalently, the mean free path \( \lambda_{yy} \). Finally, the first term in equation (21) results from Faraday’s law by which the fluctuating electric field in \( x \)-direction is expressed by the corresponding magnetic field components.

It is instructive to compare the general diffusion coefficient (21) with the NLGC result by Matthaeus et al. (2003) in equation (2). For their derivation, they assume that the fluctuations are purely magnetic. The limit of vanishing random electric fields can be achieved by choosing \( \omega_{j,R}(k) = 0 \) in equation (21), initially derived for the plasma wave viewpoint. The first term in equation (21) results from Faraday’s law and, therefore, would not occur for purely magnetic fluctuations. Consequently, it is dropped for the comparison. Since the concept of a superposition of individual wave modes does not apply, \( N = 1 \), and the corresponding \( j = 1 \)-sub(super)scripts are omitted. For the dynamical behavior of purely magnetic fluctuations, Matthaeus et al. introduced the quantity \( \gamma(k) \). To take this into account for the comparison, the plasma wave
dissipation rate $\Gamma_j(k) \leq 0$ has to be replaced by $-\gamma(k)$. For purely magnetic fluctuations, one then obtains the general relations

\[
\kappa_{xx} = \frac{a_x^2 v^2}{3B_0^2} \int d^3 k \left[ \frac{S_{yy}(k)}{B + v/\lambda_{zz} + \gamma(k)} + \frac{S_{zz}(k)}{B + v/\lambda_{yy} + \gamma(k)} \right],
\]

(23)

\[
\kappa_{yy} = \frac{a_y^2 v^2}{3B_0^2} \int d^3 k \left[ \frac{S_{xx}(k)}{B + v/\lambda_{zz} + \gamma(k)} + \frac{S_{zz}(k)}{B + v/\lambda_{xx} + \gamma(k)} \right],
\]

(24)

where $B = \kappa_{xx}k_x^2 + \kappa_{yy}k_y^2 + \kappa_{zz}k_z^2$. For their approach, Matthaeus et al. neglect fluctuations along the background magnetic field, implying here $S_{zz} = 0$. Their assumption of an axisymmetric turbulence can be achieved by using $\tau_x = \tau_y$, $S_{xx} = S_{yy}$ and $\kappa_{xx} = \kappa_{yy}$. The diffusion coefficient (23) then reduces to the NLGC result. The only difference is the term in front of the integral, and a comparison leads to the important result that $a = \tau_x \Omega$. Matthaeus et al. argue that their factor $a$ is related to the gyrocenter velocity. Numerically, they found $a = 1/\sqrt{3}$, implying here $\tau_x \Omega = 1/\sqrt{3}$. But this is characteristic of strong turbulence, where an "ordered" and well defined gyromotion of charged particles is not expected to be present anymore. Equations (21) and (22) and their simplified versions (23) and (24), respectively, are, therefore, referred to as the strong nonlinear (SNL) theory. The considerations presented above and the formal agreement of the simplified SNL equations with the NLGC result indicate that the latter will probably provide for most accurate results if strong fluctuations are assumed.

3 Summary and Conclusions

The focus of this paper is nonlinear perpendicular diffusion of charged particles in irregular electromagnetic fields. Using the fundamental Newton-Lorentz equation for the random motion of a test particle and assuming that a well defined particle gyromotion does not exist in strong turbulence, nonlinear integral equations are derived representing particle diffusion in the two distinct perpendicular directions. The two new nonlinear diffusion coefficients are valid for a plasma wave dispersion relation, i.e. real frequency of a plasma wave mode and its decay due to dissipation, depending arbitrarily on wavevector. Both transport parameters are not only coupled with the diffusion coefficient for parallel particle transport, but they are also coupled among one another. The new set of nonlinear perpendicular diffusion coefficients is referred to as the strong nonlinear (SNL) theory, and the NLGC theory by Matthaeus et al. (2003) is recovered for the case of purely magnetic fluctuations. Although the SNL theory generalizes the NLGC approach and, furthermore, is based
on another physically appealing approach, namely the Newton-Lorentz equation, a complete theory requires a prescription for determining the input parameters \(a_x = \tau_x \Omega\) and \(a_y = \tau_y \Omega\) or, better, the detailed evaluation of the time-integrations in equations (5) and (6). Of particular interest is the observation that particle diffusion in one perpendicular direction is governed by the fluctuations in the other normal direction, e.g., \(\kappa_{xx} \propto S_{yy}\). This is a direct consequence of the Lorentz-term appearing in the Newton-Lorentz equation (4). On the other hand, the NLGC result, equation (2), predicts \(\kappa_{xx} \propto S_{xx}\). This difference might be of interest for perpendicular diffusion in non-axisymmetric turbulence, since \(S_{xx}\) and \(S_{yy}\) are then not equal (e.g., [Ruffolo et al., 2001]). Future numerical simulation results for non-axisymmetric magnetic turbulence and their comparison with equations (23) and (24) as well as the NLGC theory should then provide for an answer whether particle diffusion in one perpendicular direction is governed by the fluctuations in the other normal direction or by fluctuations in the same direction.

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