HÖLDER ESTIMATES FOR THE NONCOMMUTATIVE MAZUR MAPS

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Abstract. For any von Neumann algebra $\mathcal{M}$, the noncommutative Mazur map $M_{p,q}$ from $L_p(\mathcal{M})$ to $L_q(\mathcal{M})$ with $1 \leq p, q < \infty$ is defined by $f \mapsto f^{\frac{p}{q}}$. In analogy with the commutative case, we gather estimates showing that $M_{p,q}$ is $\min\{\frac{p}{q}, 1\}$-Hölder on balls.

1. Introduction

In the integration theory, the Mazur map $M_{p,q}$ from $L_p(\Omega)$ to $L_q(\Omega)$ is defined by $f \mapsto f^{\frac{p}{q}}$. It is an easy exercise to check that it is $\min\{\frac{p}{q}, 1\}$-Hölder. These maps also make sense in the noncommutative $L_p$-setting for which one should expect a similar behavior. We refer to [7] for the definitions of $L_p$-spaces for semifinite von Neumann algebras or more general ones. Having a quantitative result on Mazur maps may be useful when dealing with the structure of noncommutative $L_p$-spaces (see also [9]). By the way, these maps are used implicitly in the definition of $L_p$. It is known that $M_{p,q}$ is locally uniformly continuous in full generality (Lemma 3.2 in [9]).

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The proofs provide a strange behaviour of the Hölder constants $c_{p,q}$ as $c_{p,q} \to \infty$ if $p < q \to 1$. The constant is also much worse than the commutative one when $p \to \infty$. This reflects the fact that the absolute value is not Lipschitz on $L_1$ or $L_\infty$.

We follow a basic approach, showing first the results for semifinite von Neumann algebras in section 2. We start by looking at positive elements and then use some commutator or anticommutator estimates. The ideas here are inspired by [2, 5]. In section 3, we explain briefly how the Haagerup reduction technique from [6] can be used to get the theorem in full generality.

2. Semifinite case

In this section $\mathcal{M}$ is assumed to be semifinite with a nsf trace $\tau$. We refer to [7] for definitions. We denote by $L_0(\mathcal{M}, \tau)$ the set of $\tau$-measurable operators, and

$$L_p(\mathcal{M}, \tau) = \left\{ f \in L_0(\mathcal{M}, \tau) \mid \|f\|_p^p = \tau(|f|^p) < \infty \right\}.$$ 

We drop the reference to $\tau$ in this section.

First we focus on the Mazur maps for positive elements using some basic inequalities. The first one can be found in [4] Lemma 1.2. An alternative proof can be obtained by adapting the arguments of [2] Theorem X.1.1 to semifinite von Neumann algebras.

Lemma 2.1. If $p \geq 1$, $0 < \theta \leq 1$, for any $x, y \in L^+_p(\mathcal{M})$, we have

$$\|x^{\theta} - y^{\theta}\|_p \leq \|x - y\|_{\theta p}^\theta.$$ 

Its proof relies on the fact that $s \mapsto s^\theta$ is operator monotone and has an integral representation

$$s^\theta = c_\theta \int_{\mathbb{R}_+} \frac{t^\theta s}{s + t} \frac{dt}{t},$$

with $c_\theta = \left( \int_{\mathbb{R}_+} \frac{u^\theta}{u(1 + u)} \, du \right)^{-1}$.
Lemma 2.2. If \( p \geq 1, 0 < \theta \leq 1 \), for any \( x, y \in L^\infty_{(1+\theta)p}(\mathcal{M}) \), we have:
\[
\|x^{1+\theta} - y^{1+\theta}\|_p \leq 3\|x - y\|_{(1+\theta)p} \max \left\{ \|x\|_{(1+\theta)p}, \|y\|_{(1+\theta)p} \right\}^\theta.
\]

Proof. By standard arguments, cutting \( x \) and \( y \) by some of their spectral projections, we may assume that \( \tau \) is finite \( x \) and \( y \) are bounded and invertible to avoid differentiability issues. We use
\[
s^{1+\theta} = c_0 \int_{\mathbb{R}^+} \frac{t^\theta s^2}{s + t} \frac{dt}{t}.
\]
On bounded and invertible elements the maps \( f_t : s \mapsto \frac{s}{s+t} = s(s+t)^{-1}s \) are differentiable and
\[
D_s f_t(\delta) = \delta(s+t)^{-1}s + s(s+t)^{-1}\delta - s(s+t)^{-1}\delta(s+t)^{-1}s.
\]
Hence putting \( \delta = x - y \), we get the integral representation
\[
x^{1+\theta} - y^{1+\theta} = c_0 \int_0^1 \int_{\mathbb{R}^+} \frac{t^\theta}{s + t} D_{y+u\delta} f_t(\delta) \frac{dt}{t} du.
\]
We get, letting \( g_t(s) = s(s+t)^{-1} \)
\[
x^{1+\theta} - y^{1+\theta} = \int_0^1 \left( (y + u\delta)^\theta \delta + \delta(y + u\delta)^\theta \right) du - c_0 \int_0^1 \int_{\mathbb{R}^+} t^\theta g_t(y + u\delta)\delta g_t(y + u\delta) \frac{dt}{t} du.
\]
The first term is easily handled by the Hölder inequality. When \( u \) is fixed, note that \( g_t(y + u\delta) \) is an invertible positive contraction. Put
\[
\gamma^2 = c_0 \int_{\mathbb{R}^+} t^\theta g_t(y + u\delta)^2 \frac{dt}{t} \leq (y + u\delta + t)^\theta,
\]
and write \( g_t(y + u\delta) = v_t \gamma \) so that \( v_t \) and \( y + u\delta \) commute and
\[
c_0 \int_{\mathbb{R}^+} t^\theta v_t^2 \frac{dt}{t} = 1.
\]
Therefore the map defined on \( \mathcal{M}, x \mapsto c_0 \int_{\mathbb{R}^+} t^\theta v_t x v_t^* \frac{dt}{t} = 1 \) is unital completely positive and trace preserving, hence it extends to a contraction on all \( L_q, 1 \leq q \leq \infty \) (see [6] for instance). Applying it to \( x = \gamma \delta \gamma \), we deduce
\[
\left\| c_0 \int_{\mathbb{R}^+} t^\theta g_t(y + u\delta)\delta g_t(y + u\delta) \frac{dt}{t} \right\|_p \leq \|\gamma \delta \gamma\|_p \leq \|\delta\|_{(1+\theta)p} \|\gamma\|_{(1+\theta)p} \|\delta\|_{(1+\theta)p} \|y + u\delta\|_{(1+\theta)p}.
\]
thanks to the Hölder inequality again, this is enough to get the conclusion. \( \square \)

Corollary 2.3. Let \( \alpha > 1, p \geq 1 \), for any \( x, y \in L^\infty_{\alpha p}(\mathcal{M}) \):
\[
\|x^\alpha - y^\alpha\|_p \leq 3\alpha \|x - y\|_{\alpha p} \max \left\{ \|x\|_{\alpha p}, \|y\|_{\alpha p} \right\}^{\alpha-1}.
\]

Proof. When \( \alpha = n \in \mathbb{N} \), the result is obvious with constant \( n \). For the general case, put \( n = [\alpha] \), so that \( \alpha = n(1 + \delta) \) with \( 0 \leq \delta < 1 \), then use the result for \( n \) and then Lemma 2.2. \( \square \)

Coming back to the Mazur map \( M_{p,q} \), Corollary 2.3 says that \( M_{p,q} \) is Lipschitz on the positive unit ball of \( L_p(\mathcal{M}) \) if \( q < p \). And the other hand Lemma 2.1 says that it is \( \frac{q}{q}\)-Hölder if \( q > p \). To release the positivity assumption, we will need the following result by Davies [5]:

Theorem 2.4. For \( 1 < p < \infty \), there is a constant \( c_p \) so that for any \( x, y \in L_p(\mathcal{M}) \):
\[
\|\|x| - |y|\|_p \leq c_p \|x - y\|_p.
\]

The original proof in only for the Schatten classes but the arguments remain valid for semifinite von Neumann algebras by a discretization argument that can be found in details in [10] (see also remark 2.4).

If \( x, y \in L_p(\mathcal{M}) \) are in the unit ball with polar decompositions \( x = u|x| \) and \( y = v|y| \), we want to prove that with \( \theta = \min\left\{ \frac{p}{q}, 1 \right\} \)
\[
\left\| u|x|^\frac{p}{q} - v|y|^\frac{p}{q}\right\|_q \leq c_{p,q} \|x - y\|_p \tag{1}
\]
(1)
First, we reduce the problem to $x$ and $y$ selfadjoint by a well known 2×2-trick. In $\mathbb{M}_2(\mathcal{M})$ equipped with the tensor trace, let
\[
\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}.
\]
They are selfadjoint with polar decompositions
\[
\tilde{x} = \tilde{u}|\tilde{x}| = \begin{pmatrix} 0 & u_x \\ u_x^* & 0 \end{pmatrix} \cdot \begin{pmatrix} u_x|x|u_x^* & 0 \\ 0 & |x| \end{pmatrix} \quad \text{and} \quad \tilde{y} = \tilde{v}|\tilde{y}| = \begin{pmatrix} 0 & v_y \\ v_y^* & 0 \end{pmatrix} \cdot \begin{pmatrix} v_y|y|v_y^* & 0 \\ 0 & |y| \end{pmatrix}.
\]
The estimates for $\tilde{x}$ and $\tilde{y}$ implies that for $x$ and $y$ as
\[
\tilde{u}|\tilde{x}|^\frac{p}{q} = \begin{pmatrix} 0 & u|\tilde{x}|^\frac{p}{q} \\ |\tilde{x}|^\frac{p}{q}u^* & 0 \end{pmatrix} \quad \text{and} \quad \tilde{v}|\tilde{y}|^\frac{p}{q} = \begin{pmatrix} 0 & v|\tilde{y}|^\frac{p}{q} \\ |\tilde{y}|^\frac{p}{q}v^* & 0 \end{pmatrix},
\]
we have
\[
\left\| \tilde{x} - \tilde{y} \right\|_p = 2^\frac{p}{q} \left\| x - y \right\|_p \quad \text{and} \quad \left\| \tilde{u}|\tilde{x}|^\frac{p}{q} - \tilde{v}|\tilde{y}|^\frac{p}{q} \right\|_q = 2^\frac{p}{q} \left\| u|\tilde{x}|^\frac{p}{q} - v|\tilde{y}|^\frac{p}{q} \right\|_q.
\]
Now we give a short argument to prove the theorem when $p \neq 1$. When $x$ is selfadjoint, we use the notation $x_+$ and $x_-$ for its positive and negative part. From Davies’ Theorem $\mathcal{M}$, we have
\[
\|x_+ - x_-\|_p = 2^\frac{p}{q} \left\| x - y \right\|_p \quad \text{and} \quad \left\| u|\tilde{x}|^\frac{p}{q} - v|\tilde{y}|^\frac{p}{q} \right\|_q = 2^\frac{p}{q} \left\| u|\tilde{x}|^\frac{p}{q} - v|\tilde{y}|^\frac{p}{q} \right\|_q.
\]
Clearly $u$ is unitary and functional calculus gives that $\|(1-u)^{-1}\|_\infty \leq \frac{1}{\sqrt{2}}$. We have, using Lemma 2.1
\[
\left\| [x^\theta, b] \right\|_p \leq 2\left\| (1-u)^{-1} \right\|_\infty \left\| x^\theta(1-u) - (1-u)x^\theta \right\|_p \leq \left\| u^*x^\theta u - x^\theta \right\|_p \leq \left\| xu - ux \right\|_p \leq \left\| (b+i)^{-1} \right\|_\infty \left\| x(b+i) - (b+i)x(b+i) \right\|_p \leq 2^\theta \left\| xb - bx \right\|_p.
\]
\[
\square
\]
\[
\text{Lemma 2.6. If } p \geq 1, 0 < \theta \leq 1, \text{ there is a constant } c_\theta \text{ so that for any } x, y \in L^+_p(\mathcal{M}) \text{ and } b \in \mathcal{M} \text{ then}
\]
\[
\left\| x^\theta b + by^\theta \right\|_p \leq c_\theta \left\| b \right\|_\infty \left\| xb + by \right\|_p.
\]
Proof. Using the 2 × 2-trick, we may assume \( x = y \).

Assume for the moment that \( x \) has full support (i.e. \( 1_{(0, \infty)}(x) = 1 \)), so that we can view
\( x \) as a density matrix. The result is then a particular case of the main theorem of [10]. The latter says the Banach spaces defined by norms \( \| b \|_{L_q(x^\alpha)} = \| x^\alpha b + bx^\alpha \|_q \) interpolate, so that \( L_\infty(x^\theta) = (L_\infty(x^0), L_p(x))_\theta \).

As a corollary, \[
\| x^\theta b + bx^\theta \|_p \leq c_\theta \| b \|_{-\frac{\theta}{\theta - 1}} \| xb + bx \|_p^\theta.
\]

To avoid the use of [10] we provide an alternate proof of the latter in equality with a better constant only when \( p = 1 \) and \( \theta \leq \frac{1}{2} \). Assuming \( \| b \|_\infty \leq 1 \), we use the Jensen’s inequality from [3] for the convex function \( x \mapsto x^\theta \) (for us it follows easily from the operator convexity of \( x^\alpha \) for \( \alpha \in [1, 2] \) and an iteration argument):
\[
\| x^\theta b + bx^\theta \|_p^\theta \leq 2 \theta \left( \| x^\theta b \|_p^\theta + \| bx^\theta \|_p^\theta \right) \leq 2 \theta \left( \| b^* x^{\theta 0} b \|_p^\theta + \| b x^{2 \theta} b^* \|_p^\theta \right) \leq 2 \theta \| b \|_2 \leq 2 \theta \| xb + bx \|_1.
\]

It remains to remove the assumption on the support of \( x \) but this is easy as if \( e = 1_{(0, \infty)}(x) \)
\[
\| x b + bx \|_p \sim \| x e b + e x b \|_p + \| x b (1 - e) \|_p + \| (1 - e) b x \|_p
\]
\[
\| x^\theta b + bx^\theta \|_p \sim \| x^\theta e b b e x^\theta \|_p + \| x^\theta (1 - e) \|_p + \| (1 - e) b x^\theta \|_p\]
\]
Each term of the second sum is controlled by the corresponding term in the first sum. Indeed for the first one this is the previous argument and for the other this is clear by interpolation as \( \| x^\theta b (1 - e) \|_p^\theta \leq \| x b (1 - e) \|_p^\theta \| b \|_{-\frac{\theta}{\theta - 1}}. \)

Remark 2.7. Actually the arguments of [10] show that for \( p > 1 \)
\[
\| x b + bx \|_p \sim_{c_p} \max \{ \| x b \|_p, \| b x \|_p \}.
\]

Lemma 2.8. If \( q > p \geq 1, \) and \( x \in L_p(M), \) \( x = x^* \) and \( b \in M \) then
\[
\| [M_{p,q}(x), b] \|_q \leq c_{p,q} \| b \|_{-1/2} \| x, b \|_p^{\gamma}. \]

Proof. Write \( e_+ = 1_{(0, \infty)}(x) \) and \( e_- = 1_{(-\infty, 0)}(x) \) and put \( b_{\pm, \pm} = e_{\pm} b e_{\pm} \).

So that
\[
[M_{p,q}(x), b] = [x^\pm, b_{+, +}] - [x^\pm, b_{-, -}] + (x^\pm b_{+, -} + b_{+, -} x^\mp) - (x^\pm b_{-, +} + b_{-, +} x^\mp).
\]

We can apply either Lemma 2.5 or 2.6 to each term. In any case, the upper bound we get is smaller than the right side of (2).

We can conclude to the proof of (1) by using the 2 × 2-trick with
\[
\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Indeed Lemma 2.8 gives (1) as
\[
\| [M_{p,q}(\tilde{x}), \tilde{b}] \|_q = \| M_{p,q}(x) - M_{p,q}(y) \|_q \quad \text{and} \quad \| \tilde{x}, \tilde{b} \|_p = \| x - y \|_p.
\]

Remark 2.9. The arguments of Lemma 2.5 and 2.6 actually show that (1) and (2) are equivalent for selfadjoint \( x \) and \( y \). This is also true if one replaces \( M_{p,q} \) by any function \( f : \mathbb{R} \to \mathbb{R} \). With such a general function \( f \), 2.8 boils down to the boundedness of some Schur multipliers on \( S_p(L_p(M)) \) (by the discretization from [10]), this is the argument of [5]. This also explains why the results of [5, 10, 8] remain true for semifinite von Neumann algebras.
3. General case

In the general case, we use the Haagerup definition of $L_p$-spaces \cite{11} and the Haagerup reduction technique from \cite{6} (see \cite{4} for extension from states to weights). As the construction is very technical, we only give a sketch to keep the paper short. Let $\mathcal{M}$ be a general von Neumann algebra with a fixed faithful normal semifinite weight $\varphi$ (we use the classical notation $n_\varphi$, $m_\varphi$, ..., for constructions associated to $\varphi$). As usual $\sigma^\varphi$ denotes the automorphisms group of $\varphi$. We let $\hat{\mathcal{M}} = M \rtimes_\varphi \mathbb{R}$ be the core of $\mathcal{M}$. It is a semifinite von Neumann algebra with a distinguished trace $\tau$ such that $\tau \circ \sigma_s = e^{-s\tau}$ where $\sigma$ is the dual action of $\mathbb{R}$ on $\hat{\mathcal{M}}$. The definition is then

$$L^\varphi_p(\mathcal{M}) = \left\{ f \in L_0(\hat{\mathcal{M}}, \tau) \mid \hat{\sigma}_s(x) = e^{-\frac{s}{\tau}x} \right\}.$$  

Then $L^\varphi_1(\mathcal{M})$ is order isometric to $M$, and the evaluation at 1 is denoted by tr. The $L^\varphi_p$ norm is given by $\|x\|_p^\varphi = \text{tr}|x|^p$. We also denote by $D_\varphi$ the Radon-Nykodym derivative of the dual weight $\hat{\varphi}$ with respect to $\tau$.

These $L^\varphi_p$ spaces are disjoint and the norm topology coincide with the measure topology of $L_0(\hat{\mathcal{M}}, \tau)$ (Proposition 26 in \cite{11}). The construction does not depend on the choice of $\varphi$ up to $*$-topological isomorphisms (see below) so that we may drop the superscript $\varphi$ when no confusion arise can.

The Haagerup reduction theorem is (see Theorem 2.1 in \cite{6} or Theorem 7.1 in \cite{4}):

**Theorem 3.1.** For any $(\mathcal{M}, \varphi)$ there is a bigger von Neumann algebra $(\mathcal{R}, \tilde{\varphi})$ where $\tilde{\varphi}$ a nfs weight extending $\varphi$, a family $a_n$ in the center of the centralizer of $\tilde{\varphi}$ so that

i) There is a conditional expectation $\mathcal{E} : \mathcal{R} \to \mathcal{M}$ such that $\varphi \circ \mathcal{E} = \tilde{\varphi}$ and $\mathcal{E} \circ \sigma^\varphi = \sigma^\tilde{\varphi} \circ \mathcal{E}$ for all $s \in \mathbb{R}$.

ii) The centralizer $\mathcal{R}_n$ of $\varphi_n(\cdot) = \tilde{\varphi}(e^{-a_n \cdot})$ is semifinite for all $n \geq 1$ (with trace $\varphi_n$).

iii) There exists conditional expectations $\mathcal{E}_n : \mathcal{R} \to \mathcal{R}_n$ such that $\varphi \circ \mathcal{E}_n = \tilde{\varphi}$ and $\mathcal{E}_n \circ \sigma^\varphi = \sigma^\tilde{\varphi} \circ \mathcal{E}_n$ for all $s \in \mathbb{R}$.

iv) $\mathcal{E}_n(x) \to x$ $\sigma$-strongly for $x \in a_n$ and $\bigcup_{n \geq 1} \mathcal{R}_n$ is $\sigma$-strongly dense in $\mathcal{R}$.

The modular conditions for the conditional expectations imply that we can view $L_p(\mathcal{M})$ and $L_p(\mathcal{R}_n)$ as subspaces of $L_p(\mathcal{R})$ and there are extensions;

$$\mathcal{E}^n_p : L_p(\mathcal{R}) \to L_p(\mathcal{M}) \quad \text{and} \quad \mathcal{E}^n : L_p(\mathcal{R}) \to L_p(\mathcal{R}_n).$$

Moreover from iv), for any $x \in L_p(\mathcal{R})$ ($1 \leq p < \infty$) we have (see Lemma 7.3 in \cite{4} for instance):

$$\lim_{n \to \infty} \left\| \mathcal{E}^n_p(x) - x \right\|_p = 0.$$

Now we make explicit the independence of $L_p(\mathcal{R}_n)$ relative the choice of the weight. Considering $\mathcal{R}_n$ with $\varphi_n$ or $\tilde{\varphi}_n$ gives two constructions, the corresponding spaces of measurable operators $\mathcal{N}_{\varphi_n} = L_0(\mathcal{R}_n \rtimes_\varphi \mathbb{R}, \varphi_n)$ and $N_{\tilde{\varphi}} = L_0(\mathcal{R}_n \rtimes_{\tilde{\varphi}} \mathbb{R}, \tau)$ in which the $L_p$-spaces live. By Corollary 38 in \cite{11}, there is a topological $*$-homomorphism $\kappa : N_{\tilde{\varphi}} \to N_{\varphi_n}$ so that $\kappa(L^\varphi_p(\mathcal{R}_n)) = L^{\tilde{\varphi}}_p(\mathcal{R}_n)$ and isometric on $L_p$.  

As $\varphi_n$ is a trace, we know that $\mathcal{R}_n \rtimes_{\varphi_n} \simeq \mathcal{R}_n \rtimes L_\infty(\mathbb{R})$ and the identification $\iota_p : L_p(\mathcal{R}_n, \varphi_n) \to L^{\tilde{\varphi}}_p(\mathcal{R}_n)$ is $\iota_p(x) = x \otimes e^{-\frac{p}{\tau}x}$. Hence we get isometric isomorphisms $\kappa_p = \iota_{p}^{-1} \circ \kappa : L_p(\mathcal{R}_n) \to L_p(\mathcal{R}_n, \varphi_n)$ that are compatible with left and right multiplications by elements of $\mathcal{R}_n$ and powers in the sense that for $1 \leq q, p < \infty$ and $x \in L^{\varphi}_p(\mathcal{R}_n)$

$$\kappa_p(x) \frac{p}{q} = \kappa_q(x \frac{p}{q}).$$

One can check that $\kappa_p$ is formally given by $\kappa_p(1_{\mathcal{R}_n} \otimes_\varphi x \otimes_\tilde{\varphi} n) = e^{-\frac{p}{\tau}x} x^{-\frac{p}{\tau}}$ for $x \in m_{\varphi_n}$.

Now we can conclude to the proof of the theorem in the general case. Take $x$ and $y$ in $L_p(\mathcal{M})$, then

$$\left\|x - y\right\|_p = \lim_{n \to \infty} \left\|\mathcal{E}_n(x) - \mathcal{E}_n(y)\right\|_{L_p(\mathcal{R})} = \lim_{n \to \infty} \left\|\kappa_p(\mathcal{E}_n(x)) - \kappa_p(\mathcal{E}_n(y))\right\|_{L_p(\mathcal{R}_n, \varphi_n)}.$$

By Lemma 3.2 in \cite{4}, the map $M_{p,q}$ is continuous on $N_2$, thus also $L_p \to L_q$, hence

$$\left\|M_{p,q}(x) - M_{p,q}(y)\right\|_q = \lim_{n \to \infty} \left\|\kappa_q(M_{p,q}(\mathcal{E}_n(x))) - \kappa_q(M_{p,q}(\mathcal{E}_n(y)))\right\|_{L_q(\mathcal{R}_n, \varphi_n)}.$$
But thanks to (3), \( \kappa_q(M_{p,q}(E_n(x))) = M_{p,q}(\kappa_p(E_n(x))) \), so that we can use the estimate for semifinite von Neumann algebras to conclude.

In the same way, all inequalities from section 2 can be extended to arbitrary von Neumann algebras (except Remark 2.9 as one can not make sense of \( f(x) \in L_q \) when \( x \in L^1 \) for general functions other than powers).

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