The Continuous Galerkin Method for an Integro-Differential Equation Modeling Dynamic Fractional Order Viscoelasticity

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Abstract. We consider a fractional order integro-differential equation with a weakly singular convolution kernel. The equation with homogeneous Dirichlet boundary conditions is reformulated as an abstract Cauchy problem, and well-posedness is verified in the context of linear semigroup theory. Then we formulate a continuous Galerkin method for the problem, and we prove stability estimates. These are then used to prove a priori error estimates. The theory is illustrated by a numerical example.

1. Introduction

R. L. Bagley and P. J. Torvik [5] have proved that fractional order operators (integrals and derivatives) are very suitable for modeling viscoelastic materials. Basic equations of the viscoelastic dynamic problem, with surface loads, can be written in the strong form,

$$\rho \ddot{u}(x, t) - \nabla \cdot \sigma_0(u; x, t) + \int_0^t \beta(t - s) \nabla \cdot \sigma_1(u; x, s) \, ds = f(x, t) \quad \text{in } \Omega \times I,$$

$$u(x, t) = 0 \quad \text{on } \Gamma_D \times I,$$

$$\sigma(u; x, t) \cdot n(x) = g(x, t) \quad \text{on } \Gamma_N \times I,$$

$$u(x, 0) = u^0(x) \quad \text{in } \Omega,$$

$$\dot{u}(x, 0) = v^0(x) \quad \text{in } \Omega,$$

(throughout this text we use \( \cdot \) to denote \( \frac{\partial}{\partial t} \)) where \( u \) is the displacement vector, \( \rho \) is the (constant) mass density, \( f \) and \( g \) represent, respectively, the volume and surface loads, \( \sigma_0 \) and \( \sigma_1 \) are the stresses according to

$$\sigma(t) = \sigma_0(t) - \int_0^t e(t - s) \sigma_1(s) \, ds,$$

$$\sigma_0(t) = 2\mu_0 \epsilon(t) + \lambda_0 \text{tr} \epsilon(t) I,$$

$$\sigma_1(t) = 2\mu_1 \epsilon(t) + \lambda_1 \text{tr} \epsilon(t) I,$$

where \( \lambda_0 > \lambda_1 > 0 \) and \( \mu_0 > \mu_1 > 0 \) are elastic constants of Lamé type, \( \epsilon \) is the strain which is defined through the usual linear kinematic relation \( \epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \), and \( \epsilon \) is the convolution kernel

$$e(t) = -\frac{d}{dt} \left( E_\alpha \left( -\frac{t}{\tau} \right)^\alpha \right) = \frac{\alpha}{\tau} \left( \frac{t}{\tau} \right)^{\alpha-1} E_\alpha \left( -\frac{t}{\tau} \right)^\alpha$$

$$\approx C t^{-1+\alpha}, \quad t \rightarrow 0.$$
Here $\tau > 0$ is the relaxation time and $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}$ is the Mittag-Leffler function of order $\alpha \in (0,1)$, and $\gamma$ is introduced to be $\gamma = \frac{\mu_2}{\mu_0} = \frac{\delta}{\zeta} < 1$, so we have $\sigma_1 = \gamma \sigma_0$ and we define $\beta(t) = \gamma e(t)$. The convolution term is weakly singular and $\beta \in L_1(0,\infty)$ with $\int_0^\infty \beta(t) \, dt = \gamma$. And we introduce the function
\begin{equation}
\xi(t) = \gamma - \int_0^t \beta(s) \, ds = \int_t^\infty \beta(s) \, ds,
\end{equation}
which is decreasing with $\xi(0) = \gamma$, $\lim_{t \to \infty} \xi(t) = 0$, so that $0 < \xi(t) \leq \gamma$.

We let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$ where $\Gamma_D$ and $\Gamma_N$ are disjoint and $\text{meas}(\Gamma_D) \neq 0$. We introduce the function spaces $H = L_2(\Omega)^d$, $H^{\Gamma_N} = L_2(\Gamma_N)^d$ and $V = \{ v \in H^1(\Omega)^d : v|_{\Gamma_D} = 0 \}$. We denote the norms in $H$ and $H^{\Gamma_N}$ by $\| \cdot \|$ and $\| \cdot \|_{\Gamma_N}$, respectively, and we equip $V$ with the inner product $a(\cdot, \cdot)$ and norm $\| v \|_V^2 = a(v, v)$. We also define a bilinear form (with the usual summation convention)
\begin{equation}
a(v, w) = \int_\Omega \left( 2 \mu_0 \epsilon_{ij}(v) \epsilon_{ij}(w) + \lambda_0 \epsilon_{ii}(v) \epsilon_{jj}(w) \right) \, dx, \quad v, w \in V,
\end{equation}
which is coercive on $V$. Setting $A u = -\nabla \cdot \sigma_0(u)$ with dom($A$) = $H^2(\Omega)^d \cap V$ such that $a(u, v) = (A u, v)$ for sufficiently smooth $u, v \in V$, we can write the weak form of the equation of motion (1.1) as: Find $u(t) \in V$ such that $u(0) = u^0$, $\dot{u}(0) = v^0$ and,
\begin{equation}
\rho(\dot{u}(t), v) + a(u(t), v) - \int_0^t \beta(t-s) a(u(s), v) \, ds
\end{equation}
\begin{equation}
= (f(t), v) + (g(t), v)_{\Gamma_N}, \quad \forall v \in V,
\end{equation}
with $(g(t), v)_{\Gamma_N} = \int_{\Gamma_N} g(t) \cdot v \, dS$. For more details see [4], [1], [2], [3] and references therein.

We define $u_1 = u$ and $u_2 = \dot{u}$, and henceforth we set $f_2 = f$. Then we can write the weak form (1.6) as: Find $u_1(t), u_2(t) \in V$ such that $u_1(0) = u^0$, $u_2(0) = v^0$ and,
\begin{equation}
a(u_1(t), v_1) - a(u_2(t), v_1) = 0,
\end{equation}
\begin{equation}
\rho(\dot{u}_2(t), v_2) + a(u_1(t), v_2) - \int_0^t \beta(t-s) a(u_1(s), v_2) \, ds
\end{equation}
\begin{equation}
= (f_2(t), v_2) + (g(t), v_2)_{\Gamma_N}, \quad \forall v_1, v_2 \in V.
\end{equation}

In the next section, using (1.6) with $g = 0$ $(\Gamma = \Gamma_D)$, we reformulate the problem as an abstract Cauchy problem and prove well-posedness. We also discuss the regularity properties and we obtain some regularity estimates. In §3 we use (1.7) to formulate a continuous Galerkin method based on polynomials of degree at most $q$ in time, and polynomials of degree at most $p$ in space. Then in §4 we show stability estimates for the continuous Galerkin method, and in §5 we use them to prove a priori error estimates, for the linear case $p = q = 1$, that are optimal in $L_\infty(L_2)$ and $L_\infty(H^1)$. Finally, in §6, we illustrate the theory for the linear case by computing the approximated solutions of (1.1) in a simple but realistic numerical example. In this paper we only study the original form of the numerical method and we do not discuss fast or adaptive strategies such as sparse quadrature or adaptive strategy based on a posteriori error estimates, e.g., [8], [1] and [2]. We postpone this to the forthcoming work. We also do not discuss adaptive, fast and oblivious convolution...
quadrature [10] and [7], to speed up the performance and decreasing the necessary memory.

2. Existence and uniqueness

In this section, using the theory of linear operator semigroups, we show that there is a unique solution of (1.6) for \( t \geq 0 \), when \( g = 0 \) (\( \Gamma = \tilde{\Gamma}_D \)), provided the data is regular enough. The techniques are adapted from [6].

We consider the strong form of (1.6), for any fixed \( T > 0 \), that is

\[
\rho \ddot{u}(t) + A u(t) - \int_0^t \beta(t-s)A u(s) \, ds = f(t), \quad 0 < t < T,
\]

with the initial data

\[
u(0) = u^0, \quad \dot{u}(0) = v^0.
\]

We extend \( u \) by \( u(t) = h(t) \) for \( t < 0 \) with \( h \) to be chosen. Then adding 

\[-\int_{-\infty}^0 \beta(t-s)A h(s) \, ds\]

to both sides of (2.1), changing the variables in the convolution terms and defining \( w(t, s) = u(t) - u(t - s) \), we get

\[
\rho \ddot{u}(t) + \tilde{\gamma} A u(t) + \int_0^\infty \beta(s)A w(t, s) \, ds = f(t) - \int_0^\infty \beta(s)A h(t - s) \, ds,
\]

where \( \tilde{\gamma} = 1 - \gamma = 1 - \int_0^\infty \beta(s) \, ds \).

2.1. An abstract Cauchy problem. We choose \( h(\cdot) = u^0 \) in (2.3), so that

\[
\rho \ddot{u}(t) + \tilde{\gamma} A u(t) + \int_0^\infty \beta(s)A w(t, s) \, ds = \tilde{f}(t),
\]

where, in view of (1.4),

\[
\tilde{f}(t) = f(t) - A u^0 \xi(t).
\]

Then we reformulate the equation (2.4) as an abstract Cauchy problem and prove well-posedness.

We set \( v = \rho \dot{u} \) and define the Hilbert spaces

\[
W = L^2_0(0, \infty; V) = \{ w : (0, \infty) \rightarrow V : \|w\|_W^2 = \rho \int_0^\infty \beta(s) \|w(s)\|_V^2 \, ds < \infty \},
\]

\[
Z = V \times H \times W = \{ z = (u, v, w) : \|z\|_Z^2 = \tilde{\gamma} \rho \|u\|_V^2 + \|v\|^2 + \|w\|^2_W < \infty \}.
\]

So the inner products in \( W \) and \( Z \) are, respectively, \( \langle \cdot, \cdot \rangle_W = \rho \int_0^\infty \beta(s)u(\cdot, \cdot) \, ds \) and \( \langle \cdot, \cdot, \cdot \rangle_Z = \tilde{\gamma} \rho a(\cdot, \cdot) + \langle \cdot, \cdot \rangle + \langle \cdot, \cdot \rangle_W \).

We also define the linear operator \( A : \text{dom}(A) \rightarrow Z \) such that, for \( z = (u, v, w) \)

\[
Az = \left( \frac{1}{\rho} v, -A(\tilde{\gamma} u + \int_0^\infty \beta(s)w(s) \, ds), \frac{1}{\rho} v - D w \right),
\]

with

\[
\text{dom}(A) = \left\{ (u, v, w) \in Z : v \in V, \tilde{\gamma} u + \int_0^\infty \beta(s)w(s) \, ds \in \text{dom}(A), \quad D w \in W, w(0) = 0 \right\},
\]

where

\[
D \phi = \frac{\partial}{\partial s} \phi \quad \text{with} \quad \text{dom}(D) = \{ \phi \in W : D \phi \in W \text{ and } \phi(0) = 0 \}.
\]
Then (2.4), with the initial values (2.2), can be written as an abstract Cauchy problem

\[ \dot{z}(t) = Az(t) + F(t), \quad 0 < t < T, \]
\[ z(0) = z^0, \]

where \( F(t) = (0, \mathbf{f}(t), 0) \) and \( z^0 = (u^0, v^0, w^0(\cdot)) \) with

\[ w^0(\cdot) = w(0, \cdot) = 0, \]

since \( w(0, s) = u(0) - u(-s) = u(0) - h(-s) = u^0 - u^0 = 0 \). We also note that

\[ w(t, 0) = u(t) - u(t) = 0. \]

A function \( z \) which is differentiable a.e. on \([0,T]\) such that \( \dot{z} \in L_1([0,T]; Z) \) is called a strong solution of the initial value problem (2.6) if \( z(0) = z^0, \ z(t) \in \text{dom}(\mathcal{A}), \) and \( \dot{z} = \mathcal{A}z(t) + F(t) \) a.e. on \([0,T]\).

**Remark 1.** By a solution of (1.6) we mean a weak solution in the way that is defined in (1.6), and by a solution of (2.1), that is often called strong solution in the literature, we mean a function \( u \) such that

\[ u(t) \in \text{dom}(\mathcal{A}), \quad \mathbf{u}(t) \in H \quad \text{and} \quad \mathbf{A}u \in L_1([0,T]; H), \]

and also satisfies (2.1) a.e. on \([0,T]\) and the initial conditions (2.2). Henceforth, to avoid confusion, we call a weak solution of (1.6) and a strong solution of (2.1) just a solution of the relevant problem. We note that a solution \( u \) of (2.1) is also a solution of (1.6), when \( \Gamma = \Gamma_D \).

**Lemma 1.** Let \( z = (u, v, w) \) be a strong solution of (2.6). Then \( u \) is a solution of (2.1) with initial data (2.2).

**Proof.** For a given strong solution \( z \) of (2.6), considering (2.7), we get \( u(0) = u^0 \) and \( v(0) = v^0 \), which are the initial conditions (2.2). Indeed for \( u \in V, \ v \in V \) (since \( z \in \text{dom}(\mathcal{A}) \)) and \( w \in W \) the components of the strong solution \( z \) of (2.6), we have

\[ \mathbf{u}(t) = \frac{1}{\rho} \mathbf{v}, \]
\[ \mathbf{v}(t) = -A(\gamma \mathbf{u}(t) + \int_0^\infty \beta(s)w(t, s)ds) + \mathbf{f}(t), \]
\[ \mathbf{w} = \frac{1}{\rho} \mathbf{v} - D\mathbf{w}. \]

The first and the third equation with initial value (2.7) imply that \( w(t, s) = u(t) - u(t-s) \). This and the fact that (2.4) is obtained from the first two equations, imply that \( u \) satisfies (2.1) a.e. on \([0,T]\) by backward calculations from (2.3). From the definition of the operator \( \mathcal{A} \) and its domain we deduce (2.8), and this completes the proof.

**Theorem 1.** There is a unique solution \( u = u(t) \) of (2.1)-(2.2) for all \( u^0 \in \text{dom}(\mathcal{A}) \) and \( v^0 \in V \), if \( f : [0,T] \to H \) is Lipschitz continuous. Moreover, we have the regularity estimate

\[ ||u||_V + ||\mathbf{u}|| \leq C \left(||u^0||_{H^2(\Omega)^\mathbf{x}} + ||v^0|| + \int_0^t ||f|| ds\right). \]
Proof. For any \( \mathbf{u}^0 \in \text{dom}(A) \) and \( \mathbf{v}^0 \in V \), considering (2.7), we have \( \mathbf{z}^0 = (\mathbf{u}^0, \mathbf{v}^0, \mathbf{w}^0(\cdot)) \in \text{dom}(A) \). We first show that \( F \) in (2.6) is differentiable a.e. on \([0, T]\) and \( \tilde{F} \in L_1([0, T]; Z) \). Then we show that the linear operator \( A \) is an infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) on \( Z \). These prove that there is a unique strong solution of (2.6) by \([9]\), Corollary 4.2.10, and the proof of the first part is complete by Lemma 1.

1. By assumption \( f \) is Lipschitz continuous on \([0, T]\). Hence \( f \) is differentiable a.e. on \([0, T]\) and \( \tilde{f} \in L_1([0, T]; H) \), since \( H \) is a Hilbert space. Since \( \xi(t) = -\beta(t) \) by (1.4), from (2.5) we get

\[
\dot{\mathbf{f}}(t) = \dot{f}(t) + A \mathbf{u}^0(\beta(t)),
\]

which shows that \( \tilde{f} \) is differentiable a.e. on \([0, T]\). Thus \( F \) is differentiable a.e. on \([0, T]\) and \( \tilde{F} \in L_1([0, T]; Z) \).

2. We use the Lumer-Phillips Theorem \([9]\) to show that \( A \) generates a \( C_0 \) semigroup on \( Z \) (in fact, \( A \) generates a \( C_0 \) semigroup of contractions on \( Z \)). To this end we first justify that \( A \) is dissipative. For \( z = (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \text{dom}(A) \) we have

\[
\langle A z, z \rangle_Z = \tilde{\gamma}(\mathbf{u}, \mathbf{v}) - (A(\tilde{\gamma} \mathbf{u}) + \int_0^\infty \beta(s)\mathbf{w}(s) ds, \mathbf{v}) + \left( \frac{1}{2} \mathbf{v} - D \mathbf{w}, \mathbf{w} \right)_W
\]

\[
= -\rho \int_0^\infty \beta(s)\mathbf{v}(D \mathbf{w}(s), \mathbf{w}(s)) ds = -\frac{1}{2} \rho \int_0^\infty \beta(s)D\mathbf{w}(s)\mathbf{w}(s) ds.
\]

To prove that the last term is non-positive, and hence \( A \) is dissipative, we consider for \( \epsilon > 0 \),

\[
\int_{\epsilon}^{\infty} \beta(s)D\|\mathbf{w}(s)\|_V^2 ds = \lim_{M \to \infty} \int_{\epsilon}^{M} \beta(s)D\|\mathbf{w}(s)\|_V^2 ds
\]

\[
= \lim_{M \to \infty} \beta(M)\|\mathbf{w}(M)\|_V^2 - \beta(\epsilon)\|\mathbf{w}(\epsilon)\|_V^2
\]

\[
- \int_{\epsilon}^{\infty} \beta'(s)\|\mathbf{w}(s)\|_V^2 ds
\]

\[
\geq -\beta(\epsilon)\|\mathbf{w}(\epsilon)\|_V^2,
\]

because \( \beta'(s) < 0 \) and \( \lim_{M \to \infty} \beta(M)\|\mathbf{w}(M)\|_V^2 = 0 \), since \( \int_{0}^{\infty} \beta(s)\|\mathbf{w}(s)\|_V^2 ds < \infty \). Since \( \mathbf{w}(\epsilon) = \int_{0}^{\epsilon} D \mathbf{w}(s) ds \), by the Cauchy-Schwarz inequality we have

\[
\|\mathbf{w}(\epsilon)\|_V^2 \leq \left( \int_{0}^{\epsilon} \|D \mathbf{w}(s)\|_V ds \right)^2 \leq \int_{0}^{\epsilon} \frac{1}{\beta(\epsilon)} ds \int_{0}^{\epsilon} \beta(s)\|D \mathbf{w}(s)\|_V^2 ds,
\]

and consequently we get

\[
\int_{\epsilon}^{\infty} \beta(s)D\|\mathbf{w}(s)\|_V^2 ds \geq -\frac{\beta(\epsilon)}{\beta(s)} \int_{0}^{\epsilon} \beta(s)\|D \mathbf{w}(s)\|_V^2 ds.
\]

But \( \frac{\beta(\epsilon)}{\beta(s)} \leq 1 \), which yields \( \int_{0}^{\epsilon} \frac{\beta(\epsilon)}{\beta(s)} ds \leq \int_{0}^{\epsilon} ds = \epsilon \), so that

\[
\int_{\epsilon}^{\infty} \beta(s)D\|\mathbf{w}(s)\|_V^2 ds \geq -\epsilon \int_{0}^{\epsilon} \beta(s)\|D \mathbf{w}(s)\|_V^2 ds.
\]

Since \( (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \text{dom}(A) \) implies \( D \mathbf{w} \in W \), i.e., \( \int_{0}^{\epsilon} \beta(s)\|D \mathbf{w}(s)\|_V^2 ds < \infty \). Therefore

\[
\langle A z, z \rangle_Z \leq -\frac{1}{2} \rho \lim_{\epsilon \to 0} \epsilon \int_{0}^{\epsilon} \beta(s)\|D \mathbf{w}\|_V^2 ds = 0,
\]

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and $\mathcal{A}$ is dissipative.

Next we show that $R(I - \mathcal{A}) = Z$. To see this, for an arbitrary $(\phi, \psi, \omega) \in Z$ we must find $(u, v, w) \in \text{dom}(\mathcal{A})$ such that $(I - \mathcal{A})(u, v, w) = (\phi, \psi, \omega)$, that is,

$$
u = \frac{1}{\rho} v = \phi,$$

$$v + A\left(\vec{\gamma}u + \int_0^\infty \beta(s)w(s)\,ds\right) = \psi,$$

$$w - \frac{1}{\rho} v + Dw = \omega.$$

From the first and third equations and $w(0) = 0$ we get

$$v = \frac{1}{\rho}(u - \phi),$$

$$w(s) = \int_0^s e^{-s} \left(\frac{1}{\rho} v + \omega(r)\right)\,dr.$$

Substituting these into the second equation of (2.10), we get

$$\frac{1}{\rho}(u - \phi) + A\left(\vec{\gamma}u + \int_0^\infty \beta(s)\int_0^s e^{-s}(u - \phi + \omega(r))\,dr\,ds\right) = \psi,$$

and hence

$$u + \kappa Au = \phi + \frac{1}{\rho} \left(\psi + \int_0^\infty \beta(s)e^{-s} \int_0^s e^r A(\phi - \omega(r))\,dr\,ds\right),$$

where $\kappa = \frac{1}{\rho} \left(1 - \int_0^\infty \beta(s)e^{-s}\,ds\right)$. Now we need to show that this equation has a solution. We define

$$\Delta = I + \kappa A.$$

Consider the bilinear form

$$(u, v)_\Delta = (u, v) + \kappa a(u, v) \quad \text{for } u, v \in V,$$

and the linear form

$$L(v) = (\phi, v) + \frac{1}{\rho} (\psi, v) + \frac{1}{\rho} \int_0^\infty \beta(s)e^{-s} \int_0^s e^r a(\phi - \omega(r), v)\,dr\,ds.$$

Then for some positive constants $C_1, C_2$ and $C_3$ by the boundedness and coercivity of $a(\cdot, \cdot)$

$$|(u, v)_\Delta| \leq C_1 ||u||_V ||v||_V \quad \text{for } u, v \in V,$$

$$(u, u)_\Delta \geq C_2 ||u||_V^2 \quad \text{for } u \in V,$$

$$|L(v)| \leq C_3 ||v||_V \quad \text{for } v \in V.$$

Therefore by Riesz representation theorem, there is a unique solution of the problem: find $u \in V$ such that,

$$(u, v)_\Delta = L(v) \quad \forall v \in V,$$

that implies there is a unique solution of the problem (2.11). Hence $R(I - \mathcal{A}) = Z$.

Since $Z$ is a Hilbert space, it follows from [9], Theorem 1.4.6, that $\text{dom}(\mathcal{A}) = Z$. So we have verified all the hypotheses of the Lumer-Philips theorem to complete the first part of the proof.

3. Now we have the unique strong solution of (2.6), i.e.,

$$z(t) = T(t)z^0 + \int_0^t T(t - s)F(s)\,ds,$$
and $\|T(t)\|_Z \leq 1$, since $A$ generates a $C_0$ semigroup of contractions. Therefore we have

$$\|z\|_Z \leq \|T(t)\|_Z \|z^0\|_Z + \int_0^t \|T(t-s)F(s)\|_Z \, ds \leq \|z^0\|_Z + \int_0^t \|F(s)\|_Z \, ds.$$ 

Then considering $v = \rho \dot{u}$, $z^0 = (u^0, v^0, 0)$ and $\|F(s)\|_Z = \|\tilde{f}(s)\| = \|f(s) - Au^0 \xi(s)\|$, we have

$$\left(\dot{\tilde{\gamma}} \rho \|u\|_V^2 + \rho^2 \|\dot{u}\|^2 + \rho \int_0^\infty \beta(s) \|w(s)\|^2 \|\dot{v}\| \, ds \right)^{1/2} \leq \left(\dot{\tilde{\gamma}} \rho \|u^0\|_V^2 + \|v^0\|^2 \right)^{1/2} + \int_0^t (\|f(s)\| + \|u^0\|_{H^2(\Omega)\ast \xi(s)}) \, ds.$$ 

Consequently, we have the estimate (2.9) with $C = C(\tilde{\gamma}, \rho, T)$. \hfill \Box

**Remark 2.** Due to singularity of the kernel $\beta$ at the origin, $\xi = \xi(t)$ in (1.4) is not Lipschitz continuous. With a smoother kernel $\tilde{\beta}$, $\xi = \xi(t)$ would be Lipschitz continuous so that we could get a unique strong solution of (2.6) by [9], Corollary 4.2.11, instead of [8], Corollary 4.2.10, when $f$ is Lipschitz continuous on $[0, T]$.

### 2.2. Regularity

By Theorem 1 there is a unique solution of (1.6), if the data are smooth enough. To find sufficient conditions on the data for more regularity, we assume that the data are smooth enough to justify the following calculations.

We first choose $h(t) = tv^0$ in (2.3), so that

$$\rho \ddot{u}(t) + \tilde{\gamma} Au(t) + \int_0^\infty \tilde{\beta}(s) Aw(t, s) \, ds = \tilde{f}(t),$$

where

$$\tilde{f}(t) = f(t) - A v^0 \int_0^\infty (t-s) \beta(s) \, ds.$$ 

Then differentiating the equation (2.12) in time we get

$$\rho \dot{\ddot{u}}(t) + \tilde{\gamma} A \dot{u}(t) + \int_0^\infty \beta(s) A \dot{w}(t, s) \, ds = \ddot{f}(t),$$

which, with an underline instead of one time derivative, can be written as

$$\rho \ddot{u}(t) + \tilde{\gamma} A u(t) + \int_0^\infty \beta(s) A w(t, s) \, ds = \ddot{f}(t),$$

with the initial values

$$\ddot{u}(0) = \dot{u}^0 = v^0, \quad \dot{u}(0) = v^0 = \frac{1}{\rho} (f(0) - Au^0),$$

and

$$\ddot{f}(t) = \dot{f}(t) = f(t) - A v^0 \xi(t),$$

and $w = \dot{w}(t, \cdot) = \dot{u}(t) - \dot{u}(t - \cdot) = u(t) - u(t - \cdot)$, so that $w(t, 0) = 0$.

Then, in the same way as in § 2.1 with $u = \rho \ddot{u}$, we can reformulate (2.15)-(2.16) as the abstract Cauchy problem

$$\ddot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{\ddot{f}}(t), \quad 0 < t < T,$$

$$\ddot{\tilde{x}}(0) = \ddot{\tilde{x}}^0,$$
where $\tilde{z}(t) = (0, \tilde{f}(t), 0)$ and $\tilde{z}^0 = (u^0, v^0, w^0(\cdot))$ with
\begin{equation}
(2.19)
\begin{align*}
w^0(\cdot) &= w(0, \cdot) = 0,
\end{align*}
\end{equation}
since $w(0, s) = \tilde{u}(0) - u(t - s) = \tilde{u}(0) - \frac{d}{dt} h(t - s) |_{s=0} = v^0 - v^0 = 0.$

**Lemma 2.** Let $\tilde{z} = (u, v, w)$ be a strong solution of $(2.18)$. Then $u(t) = u^0 + \int_0^t u(s) \, ds$ is a solution of $(2.1)$ with initial data $(2.2)$.

**Proof.** For a given strong solution $\tilde{z}$ of $(2.18)$, considering $(2.16)$ and $(2.19)$, we have

\begin{equation}
(2.20)
\begin{align*}
\rho \tilde{u}(t) - \rho \tilde{u}(0) + \tilde{\gamma} A u(t) - \tilde{\gamma} A u(0) + \int_0^t \int_0^\infty \beta(s) A D_r w(r, s) \, ds \, dr
&= \tilde{f}(t) - \tilde{f}(0).
\end{align*}
\end{equation}

From $(2.12)$ for $t = 0$, we have

\begin{equation}
(2.21)
\begin{align*}
\rho \tilde{u}(0) + \tilde{\gamma} A u(0) + \int_0^\infty \beta(s) A w(0, s) \, ds = \tilde{f}(0).
\end{align*}
\end{equation}

And the integral in $(2.20)$ is

\begin{equation}
(2.22)
\begin{align*}
\int_0^t \int_0^\infty \beta(s) A D_r w(r, s) \, ds \, dr &= \int_0^\infty \beta(s) \int_0^t A D_r w(r, s) \, dr \, ds
\end{align*}
\end{equation}

Hence $(2.20)$, considering $(2.21)$, $(2.22)$ and $w(t, s) = u(t) - u(t - s)$, gives $(2.12)$, that implies $u$ satisfies $(2.1)$ a.e. on $[0, T]$ by backward calculations from $(2.3)$. Finally from the definition of the operator $A$ and its domain we deduce $(2.8)$, and this complete the proof. 

In the next theorem we find the circumstances under which, there is a unique solution of $(2.1)$ with more regularity.
Theorem 2. There is a unique solution \( u = u(t) \) of (2.1)-(2.2) for all \( v^0 \in \text{dom}(A), A v^0 \in V \) and \( f(0) \in V \), if \( \tilde{f} : [0,T] \to H \) is Lipschitz continuous. Moreover, we have the regularity estimate

\[
(2.23) \quad \| \tilde{u} \|_V + \| \tilde{\tilde{u}} \| \leq C \left( \| u^0 \|_{H^2(\Omega)} + \| v^0 \|_{H^2(\Omega)} + \| f(0) \| + \int_0^t \| \tilde{f} \| \, ds \right).
\]

Proof. 1. From the assumptions on \( u^0, v^0 \) and \( f(0) \) and recalling (2.16), we have

\[
\tilde{z}^0 = \left( \tilde{u}^0, \tilde{v}^0, \tilde{u}^0(\cdot) \right) \in \text{dom}(A).
\]

Also considering \( \tilde{f} = f(t) + A v^0(0) \),

obtained from (2.17), and the assumptions on \( v^0, \tilde{f}, \tilde{E} \) is differentiable a.e. on \([0,T]\) and \( \tilde{E} \in L_1([0,T]; Z) \). Then, since the linear operator \( A \) is an infinitesimal generator of a \( C_0 \)-semigroup of contractions \( T(t) \) on \( Z \) by the second step of the proof of Theorem 1, there is a unique strong solution of (2.18) by [9], Corollary 4.2.10. Hence the first part of the proof is complete by Lemma 2.

2. We have the unique strong solution of (2.18), i.e.

\[
\tilde{z}(t) = T(t) \tilde{z}^0 + \int_0^t T(t-s) \tilde{E}(s) \, ds,
\]

with \( \| T(t) \|_Z \leq 1 \), since \( A \) generates a \( C_0 \)-semigroup of contractions. Then we have

\[
\| \tilde{z} \|_Z \leq \| \tilde{z}^0 \|_Z + \int_0^t \| \tilde{E}(s) \|_Z \, ds.
\]

Therefore considering \( \tilde{v} = \rho \tilde{u}, \tilde{z}^0 = (\tilde{u}^0, \tilde{v}^0, 0) \) and \( \| \tilde{E}(s) \|_Z = \| \tilde{f}(s) \| = \| f(s) - A v^0 \tilde{\xi}(s) \|, \)

we have

\[
\left( \tilde{\gamma} \rho \| \tilde{u} \|_V^2 + \rho^2 \| \tilde{u} \|^2 + \rho \int_0^\infty \beta(s) \| \tilde{w}(s) \|_V^2 \, ds \right)^{1/2}
\]

\[
\leq \left( \tilde{\gamma} \rho \| v^0 \|_V^2 + \frac{1}{\rho^2} \| f(0) - A v^0 \|_V^2 \right)^{1/2} + \int_0^t \left( \| \tilde{f}(s) \| + \| v^0 \|_{H^2(\Omega)} \| \tilde{\xi}(s) \| \right) \, ds.
\]

Consequently, for some \( C = C(\tilde{\gamma}, \rho,T), \) we get (2.23).  \( \square \)

2.3. Higher regularity. We want to generalize the procedure in §2.1 and §2.2 to obtain higher regularity. To this end we choose, for \( n \geq 2, \)

\[
h(t) = \frac{\rho^n}{n!} u_n(0),
\]

in (2.3) to get

\[
\rho \tilde{u}(t) + \tilde{\gamma} A u(t) + \int_0^\infty \beta(s) A w(t,s) \, ds = f(t) - A u_n(0) \int_0^\infty (t-s)^n \frac{\beta(s)}{n!} \, ds,
\]

where by \( \phi_n(t) \) we mean \( \frac{\partial^n}{\partial t^n} \phi(t), \) considering the trivial cases \( \phi_0 = \phi \) and \( \phi_1 = \dot{\phi}. \)

The cases \( n = 0 \) and \( n = 1 \) have already been discussed in the previous sections.

Then differentiating \( n \) times with respect to \( t, \) we have

\[
(2.24) \quad \rho \tilde{u}_n(t) + \tilde{\gamma} A u_n(t) + \int_0^\infty \beta(s) A w_n(t,s) \, ds = f_n(t) - A u_n(0) \tilde{\xi}(t),
\]

with initial values

\[
(2.25) \quad u_n(0) = u_n^0 \quad \text{and} \quad u_{n+1}(0) = \tilde{u}_n(0) = v_n^0,
\]
where
\[ u_m(0) = \frac{1}{\rho} (f_{m-2}(0) - Au_{m-2}(0)), \quad \text{for} \ 2 \leq m \leq n+1, \]
is obtained from (2.1) and (2.2) by recursion, and implies that for \( n = 2k (k = 1, 2, \ldots) \)

\[ u_n(0) = \frac{A^k}{\rho^k} u^0 + \sum_{j=1}^{k} (-1)^{j-1} \frac{A^{j-1}}{\rho^j} f_{2k-2j}(0), \tag{2.26} \]
and for \( n = 2k + 1 (k = 1, 2, \ldots) \)

\[ u_n(0) = \frac{A^k}{\rho^k} v^0 + \sum_{j=1}^{k} (-1)^{j-1} \frac{A^{j-1}}{\rho^j} f_{2k-2j+1}(0). \tag{2.27} \]

Similar to §2.1 and §2.2 we can reformulate (2.24)–(2.25) as an abstract Cauchy problem

\[ \dot{z}_n(t) = \mathcal{A}z_n(t) + F_n(t), \quad 0 < t < T, \]
\[ z_n(0) = z^0_n, \]
where \( F_n(t) = (0, f_n(t) - Au_n(0)\xi(t), 0) \) and \( z^0_n = (u^0_n, v^0_n, w^0_n(\cdot)) \) with

\[ w^0_n(\cdot) = w_n(0, \cdot) = 0, \tag{2.29} \]

since \( w_n(0, s) = u_n(0) - u_n(-s) = u_n(0) - \frac{\partial^n}{\partial s^n} \Phi(t - s) \big|_{s=0} = u^0_n - u^0_n = 0. \) We also note that \( w_n(t, 0) = u_n(t) - u_n(0) = 0. \)

**Theorem 3.** Let \( \frac{\partial^m}{\partial s^m} f = f_n : [0, T] \to H \) be Lipschitz continuous. Recalling \( \text{dom}(\mathcal{A}) = H^2(\Omega)^d \cap V \), we also assume that:

for \( n = 2k (k = 1, 2, \ldots) \)

\[ A^k u^0 \in \text{dom}(\mathcal{A}), \quad A^k v^0 \in V, \tag{2.30} \]

\[ f_{2k-2j}(0) \in H^2(\Omega)^d \cap V, \quad f_{2k-2j+1}(0) \in H^{2j-1}(\Omega)^d \cap V, \]
for \( j = 1, \ldots, k, \)

and for \( n = 2k + 1 (k = 1, 2, \ldots) \)

\[ A^k u^0 \in \text{dom}(\mathcal{A}), \quad A^k v^0 \in V, \tag{2.31} \]

\[ f_{2k-2j+1}(0) \in H^2(\Omega)^d \cap V, \quad f_{2k-2j+2}(0) \in H^{2j-1}(\Omega)^d \cap V, \]
for \( j = 1, \ldots, k+1, \)

where \( H^m(\Omega)^d \) is the standard Sobolev space with the standard norm denoted by \( \| \cdot \|_m \).

Then there is a unique solution of (2.1)–(2.2). Moreover, for some \( C = C(\bar{\gamma}, \rho, T) \):

\[ \| u_n \|_V + \| u_{n+1} \| \leq C \left( \| u^0 \|_{2k+2} + \| v^0 \|_{2k} + \sum_{j=1}^{k} \| f_{2k-2j}(0) \|_{2j} \right) \]
\[ + \sum_{j=1}^{k} \| f_{2k-2j+1}(0) \|_{2j-2} + \int_0^T \| f_n(s) \| \, ds \), \tag{2.32} \]
for \( n = 2k \) \((k = 1, 2, \cdots)\), and

\[
\|u_n\|_V + \|u_{n+1}\| \leq C \left( \|u^0\|_{2k+2} + \|v^0\|_{2k+2} + \sum_{j=1}^{k} \|f_{2k-2j+1}(0)\|_{2j} + \sum_{j=1}^{k+1} \|f_{2k-2j+2}(0)\|_{2j-2} + \int_0^t \|f_n(s)\| \, ds \right),
\]

(2.33)

for \( n = 2k+1 \) \((k = 1, 2, \cdots)\).

**Proof.** The assumptions in Theorem 1 and Theorem 2 are fulfilled from the given hypothesis here, respectively, for even and odd \( n \). Therefore existence and uniqueness of the solution \( u \) of (2.1)–(2.2) is proved.

2. To prove the regularity estimates (2.32) and (2.33) we need to find a strong solution of (2.28). To this end, we note that \( z_n = (u^0_n, v^0_n) \) is a \( \mathcal{A} \)-semigroup of contractions in Hilbert space \( H \), \( f_n \) is Lipschitz continuous a.e. on \([0, T]\) from (2.29) and (2.30) and recalling (2.26) and (2.27).

The next step is to show that \( F_n = (0, f_n - Au^0_n \xi(t), 0) \) is differentiable a.e. on \([0, T]\) and \( F_n \in L_1([0, T]; Z) \). Indeed \( \xi(t) \) is differentiable a.e. on \([0, T]\) and \( \xi(t) = -\beta(t) \) from (1.4), and we also have \( Au^0_n \in H \) by assumptions (2.30) and (2.31) and (2.26) and (2.27). These give us the desired fact about \( F_n \).

Finally, considering the fact that the linear operator \( \mathcal{A} \) is an infinitesimal generator of \( \mathcal{C}_0 \)-semigroup (of contractions) \( T(t) \) on \( Z \) by Theorem 1, there is a unique strong solution \( z_n(t) = (u_n(t), v_n(t), w_n(t, \cdot)) \in \text{dom}(\mathcal{A}) \) of (2.28), by [9], Corollary 4.2.10, so that

\[
z_n(t) = T(t)z^0_n + \int_0^t T(t-s)F_n(s) \, ds,
\]

with \( \|T(t)\|_Z \leq 1 \). Then taking \( Z \)-norm \( \|\cdot\|_Z \) of both sides the equation and recalling \( v_n = \rho u_n \), \( z^0_n = (u^0_n, v^0_n, 0) \) and \( \|F_n(s)\|_Z = \|f_n(s) - Au^0_n \xi(s)\| \), we have

\[
(\bar{\gamma} \rho) \|u_n\|_V^2 + \|v_n\|^2 + \rho \frac{1}{\beta} \int_0^\infty \beta(s) \|w_n(s)\|^2 \, ds^{1/2}
\]

\[
\leq (\bar{\gamma} \rho) \|u^0_n\|_V^2 + \|v^0_n\|^2 + \frac{1}{2} \int_0^t \|f_n(s)\| + \|Au^0_n\| \, ds.
\]

Hence for some \( C = C(\bar{\gamma}, \rho, T) \) we get

\[
\|u_n\|_V + \|u_{n+1}\| \leq C \left( \|u^0_n\|_2 + \|v^0_n\| + \int_0^t \|f_n(s)\| \, ds \right),
\]

that implies the desired estimates (2.32) and (2.33), considering (2.25), (2.26) and (2.27) and the assumptions (2.30) and (2.31).

**Remark 3.** Inspired by Lemma 1 and Lemma 2, one may prove (by induction) that for any strong solution \( z_n = (u_n, v_n, w_n(\cdot)) \), \( u \) is a solution of (2.1)–(2.2). This, of course, gives an alternative to prove existence and uniqueness of \( u \) in Theorem 3.
3. The continuous Galerkin method

Recalling the function spaces $H = L_2(\Omega)^d$, $H^{Γ_N} = L_2(Γ_N)^d$ and $V = \{v ∈ H^1(\Omega)^d : v|_{Γ_D} = 0\}$ ($d = 2,3$), we provide some definitions which will be used in the forthcoming discussions.

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$ be a partition of the time interval $I = [0,T]$. To each time subinterval $I_n = (t_{n-1}, t_n)$ of length $k_n = t_n - t_{n-1}$, associate a triangulation $T_{h,n}$ of $Ω$ with meshsize function $h_n$ defined by

$$h_n(x) = \text{diam}(K), \quad \text{where } K ∈ T_{h,n} \text{ and } x ∈ K,$$

for all $x ∈ Ω$, and for $p ≥ 1$ corresponding finite element space $V^{(p)}_{h,n}$ of vector-valued continuous piecewise polynomials in $Ω$ of degree at most $p$, that vanish on $Γ_D$. (This requires that the mesh is adjusted to fit $Γ_D$). We also define the spaces, for $q ≥ 0$,

$$W^{(q,p)} = \left\{ w : w|_{Ω×I_n} = w^n ∈ W_Γ(q), n = 1, \ldots, N \right\},$$

where, with $P^d_q$ the set of all vector-valued polynomials of degree at most $q$,

$$W^{(q,p)}_{h,n} = \left\{ w : w(x, t) ∈ P^d_q(I_n), w(·, t) ∈ V^{(p)}_{h,n}, (x, t) ∈ Ω × I_n, \right\}.$$

Note that $w ∈ W^{(q,p)}$ may be discontinuous at $t = t_n$, and $w ∈ W^{(0,p)}$ is piecewise constant in time. In the sequel we write $W^q = W^{(q,p)}$ for short.

The Ritz (elliptic) and orthogonal projections $R_{h,n} : V → V^{(p)}_{h,n}$, $P_{h,n} : H → V^{(p)}_{h,n}$ and $P_{k,n} : L_2(I_n)^d → P^d-q-1(I_n)$ are defined, respectively, by

$$a(R_{h,n} v - w, χ) = 0, \quad ∀ v ∈ V \text{ and } χ ∈ V^{(p)}_{h,n},$$

$$a(P_{h,n} v - w, χ) = 0, \quad ∀ v ∈ H \text{ and } χ ∈ V^{(p)}_{h,n},$$

$$\int_{I_n} (P_{k,n} v - w) · ψ \, dt = 0, \quad ∀ v ∈ L_2(I_n)^d \text{ and } ψ ∈ P^d-q-1(I_n).$$

Correspondingly, we define $R_h v$ and $P_h v$ for $t ∈ I_n (n = 1, \ldots, N)$, by $(R_h v)(t) = R_{h,n} v(t)$ and $(P_h v)(t) = P_{h,n} v(t)$, and also $P_h v ∈ \prod_{n=1}^N P^d_{q-1}(I_n)$ by $P_h v = P_{k,n} v|_{I_n}$ on $I_n$. We also define the orthogonal projections, $R_n : L_2(I_n, V) → W^{(q-1)}_n$, $P_n : L_2(I_n, H) → W^{(q-1)}_n$ and $P^Γ_n : L_2(I_n, H^{Γ_N}) → W^{(q-1)}_n$, such that

$$\int_{I_n} a(R_n u - w, ψ) \, dt = 0, \quad ∀ ψ ∈ W^{(q-1)}_n, u ∈ L_2(I_n, V),$$

$$\int_{I_n} (P_n u - w, ψ) \, dt = 0, \quad ∀ ψ ∈ W^{(q-1)}_n, u ∈ L_2(I_n, H),$$

$$\int_{I_n} (P^Γ_n u, ψ) \, dt = \int_{I_n} (u, ψ)_{Γ_n} \, dt, \quad ∀ ψ ∈ W^{(q-1)}_n, u ∈ L_2(I_n, H^{Γ_N}).$$

Correspondingly, we define $R : L_2(I, V) → W^{(q-1)}$, $P : L_2(I, H) → W^{(q-1)}$ and $P^Γ : L_2(I, H^{Γ_N}) → W^{(q-1)}$ in the obvious way.

One can easily show that

$$R = R_h P_k = P_k R_h, \quad P = P_h P_k = P_k P_h,$$

and $∀ u ∈ W^{(q)}_n$, $v ∈ W^{(q-1)}_n$,

$$\int_{I_n} (u, v) \, dt = \int_{I_n} (P_{k,n} u, v) \, dt,$$
We introduce the linear operator \( A_{h,n,r} : V_{h,r}^{(p)} \rightarrow V_{h,n}^{(p)} \) by

\[
a(v_r, w_n) = (A_{h,n,r}v_r, w_n) \quad \forall v_r \in V_{h,r}^{(p)}, \ w_n \in V_{h,n}^{(p)}.
\]

We set \( A_{h,n} = A_{h,n,r} \), with discrete norms

\[
|v_n|_{h,l} = \| A_{h,n}^{l/2} v_n \| = \sqrt{(v_n, A_{h,n}^{l} v_n)}, \quad v_n \in V_{h,n}^{(p)} \text{ and } l \in \mathbb{R},
\]

and \( A_h \) so that \( A_h \mathbf{v} = A_{h,n} \mathbf{v} \) for \( \mathbf{v} \in V_{h,n}^{(p)} \). For later use in our error analysis we note that

\[
\mathcal{P}_h A = A_h \mathcal{R}_h.
\]

We define the bilinear form \( B : W \times V \rightarrow \mathbb{R} \), and the linear forms \( L, \dot{L} : V \rightarrow \mathbb{R} \) by

\[
B(u, v) = \sum_{n=1}^{N} \int_{I_n} -a(u_2, v_1) + a(u_1, v_1) + \rho(\dot{u}_2, v_2) + a(u_2, v_2) \, dt
\]

\[\quad - \sum_{n=1}^{N} \int_{I_n} \int_{0}^{t} \beta(t-s)a(u_1(s), v_2(t)) \, ds \, dt,
\]

\[
L(w) = \sum_{n=1}^{N} \int_{I_n} (f_2, w_2) + (g, w_2) \Gamma_N \, dt,
\]

\[
\dot{L}(w) = \sum_{n=1}^{N} \int_{I_n} a(f_1, w_1) + (f_2, w_2) + (g, w_2) \Gamma_N \, dt,
\]

where \( W \) is the space of pairs of vector-valued functions \( \mathbf{u} = (u_1, u_2) \) that are piecewise smooth with respect to the temporal mesh. We may note that \( (W^{(q)})^2 \subset W \) for \( q \geq 0 \).

The continuous Galerkin method of degree \((q,p)\) is based on the variational formulation (1.7) and reads: Find \( U = (U_1, U_2) \in (W^{(q)})^2 \) such that, for \( n = 1, \ldots, N, \)

\[
\int_{I_n} a(\ddot{U}_1, V_1) - a(U_2, V_1) \, dt = 0,
\]

(3.7)

\[
\int_{I_n} (\rho(\ddot{U}_2, V_2) + a(U_1, V_2) - \int_{0}^{t} \beta(t-s)a(U_1(s), V_2(t)) \, ds) \, dt
\]

\[\quad = \int_{I_n} (f_2, V_2) \, dt + \int_{I_n} (g, V_2) \Gamma_N \, dt, \quad \forall (V_1, V_2) \in \left(W^{(q-1)}\right)^2,
\]

\[
U_{1,n-1}^+ = \mathcal{R}_{h,n} U_{1,n-1}^- , \quad U_{2,n-1}^+ = \mathcal{P}_{h,n} U_{2,n-1}^- ,
\]

where \( U_{1,0}^+ = u_0, U_{2,0}^- = v_0 \). Then \( U \in (W^{(q)})^2 \), which was defined in (3.7), satisfies:

\[
B(U, \mathcal{P}_k V) = L(\mathcal{P}_k V), \quad \forall V \in \left(W^{(q)}\right)^2,
\]

\[
U_{1,n-1}^+ = \mathcal{R}_{h,n} U_{1,n-1}^- , \quad U_{2,n-1}^+ = \mathcal{P}_{h,n} U_{2,n-1}^- ,
\]

\[
U_{1,0}^+ = u_0, \quad U_{2,0}^- = v_0.
\]
where $\mathcal{P}_k V = (\mathcal{P}_0 V_1, \mathcal{P}_k V_2)$.

Since the variational form (1.7) can be written as: Find $u \in W$ such that

$$B(u, v) = L(v), \quad \forall v \in W,$$

we may, for later reference, note the Galerkin orthogonality

$$B(U - u, \mathcal{P}_k V) = 0, \quad \forall V \in \left(W^{(q)}\right)^2. \quad (3.8)$$

For $q = p = 1$, considering the fact that functions in $W_n^{(0,p)}$ are constant with respect to time, we can write (3.7) as

$$A_{h,n}(U_{1,n}^+ - U_{1,n-1}^-) - \frac{k_n}{2} A_{h,n}(U_{2,n}^+ + U_{2,n-1}^-) = 0,$$

$$A_{h,n}\left(\frac{k_n}{2} - \gamma \omega_{n,n}^-\right) U_{1,n}^- + \left(\frac{k_n}{2} - \gamma \omega_{n,n-1}^+\right) U_{1,n-1}^+ + \rho (U_{2,n}^- - U_{2,n-1}^+) = H_n + b_n,$$

where

$$b_n = k_n (P_n f_2 + P_n^{T \times} g),$$

$$H_n = \frac{k_n}{2} \sum_{r=1}^{n-1} k_r A_{h,n,r}(\omega_{r-r_1}^- + \omega_{r-r_1}^+),$$

$$\omega_{n,r}^- = \int_{I_n} \int_{t_{r-1}}^{t_r, t} \beta(t - s) \tilde{\psi}_{r}^-(s) ds \, dt, \quad t_r \wedge t = \min(t_r, t),$$

$$\omega_{n,r-1}^+ = \int_{I_n} \int_{t_{r-1}}^{t_r, t} \beta(t - s) \tilde{\psi}_{r-1}^+(s) ds \, dt,$$

and $\tilde{\psi}_n^-, \tilde{\psi}_n^+$ are the linear Lagrange basis functions on $I_n$, so that, for $i = 1, 2,$

$$U_i(x, t) |_{a \times I_n} = \psi_{i-1}^+(t) U_{1,n-1}^+(x) + \psi_n^-(t) U_{1,n}^-(x).$$

If we do not change the mesh, or just refine the mesh from a time step to the next, that is $V_{h,n}^{(p)} \subseteq V_{h,n}^{(p)}$, then $\mathcal{R}_{h,n}$ and $\mathcal{P}_{h,n}$ reduce to the identity, i.e.,

$$U_{i,n} = U_{1,n}^- = U_{1,n}^+, \quad (n = 0, 1, \ldots, N), \quad i = 1, 2.$$

From now on, we assume that $V_{h,n-1}^{(p)} \subseteq V_{h,n}^{(p)}$, $n = 2, \ldots, N$. So we have defined the initial values of the discrete form to be $U_1(\cdot, 0) = \mathbf{u}_h^0 = \mathcal{R}_{h,1} \mathbf{u}^0$ and $U_2(\cdot, 0) = \mathbf{v}_h^0 = \mathcal{P}_{h,1} \mathbf{v}^0$. In this case $U$ in continuous with respect to $t$.

We also consider a modified problem by adding an extra load function, say $f_1 = f_1(t)$, to the first equation of (3.7). This kind of problem will occur in our error analysis below. Then the continuous Galerkin method of order $(q, p)$ is: Find $U \in (W^{(q)})^2$ such that, for $n = 1, \ldots, N,$

$$\int_{I_n} a(U_1, V_1) - a(U_2, V_1) \, dt = \int_{I_n} a(f_1, V_1) \, dt,$$

$$\int_{I_n} \left(\rho (U_2, V_2) + a(U_1, V_2) - \int_0^t \beta(t - s) a(U_1(s), V_2(t)) ds\right) \, dt$$

$$= \int_{I_n} (f_2, V_2) \, dt + \int_{I_n} (g, V_2) \mathcal{P}_n \, dt, \quad \forall (V_1, V_2) \in \left(W_n^{(q-1)}\right)^2,$$

$U_1, U_2$ continuous at $t_{n-1},$

$U_1(\cdot, 0) = \mathbf{u}_h^0 = \mathcal{R}_{h,1} \mathbf{u}^0, \quad U_2(\cdot, 0) = \mathbf{v}_h^0 = \mathcal{P}_{h,1} \mathbf{v}^0.$
Then $U$ satisfies:

$$B(U, \mathcal{P}_k V) = \mathcal{L}(\mathcal{P}_k V), \quad \forall V \in (W^{(\alpha)})^2,$$

(3.10)

$U_1, U_2$ continuous at $t_{n-1}$,

$U_1(., 0) = u_0, \quad U_2(., 0) = v_0^0.$

4. Stability estimates

In the next theorem we prove an energy identity for problem (3.9) which will be used for proving the error estimates in the next section.

**Theorem 4.** Let $U = (U_1, U_2)$ be the solution of (3.9). Then for any $l \in \mathbb{R}$, $T > 0$, we have the equality

$$\rho|U_2,N|_{h,l}^2 + \bar{\xi}(T)|U_1,N|_{h,l+1}^2 + \int_0^T \beta|U_1|_{h,l+1}^2 \, dt$$

$$+ \int_0^T \int_0^t \beta(t-s)D t_{W_1(t,s)}^2 \, ds \, dt$$

$$= \rho|u_0^0|_{h,l}^2 + |u_0^0|_{h,l+1}^2$$

$$+ 2 \int_0^T (P f_2, A_k^1 U_2) \, dt + 2 \int_0^T (P^{\tau} \cdot g, A_k^1 U_2) \, dt$$

$$+ 2 \int_0^T \bar{\xi}(R f_1, A_k^1 U_1) \, dt$$

$$+ 2 \int_0^T \beta(t-s)\alpha(R f_1(t), A_k^i W_1(t, s)) \, ds \, dt,$$

(4.1)

where $W_1(t, s) = U_1(t) - U_1(s)$ and, recalling (1.4),

$$\bar{\xi}(t) = \xi(t) + 1 - \gamma,$$

(4.2)

with $0 < 1 - \gamma < \bar{\xi}(t) \leq 1$. All terms on the left hand side are non-negative.

**Proof.** Throughout the proof we take into account that $U_i$ ($i = 1, 2$) are continuous, hence piecewise differentiable, so that $\dot{U}_i$ exist a.e. in $[0, T]$. We organize our proof in 6 steps.

1. Expressing $U_2$ in terms of $U_1$. For any $n = 1, \cdots, N$ the first equation of (3.9) may be written as,

$$\int_{I_n} a(U_2, V_1) \, dt = \int_{I_n} a(\dot{U}_1 - R_n f_1, V_1) \, dt, \quad \forall V_1 \in W^{(q-1)}_n.$$

Then by (3.6)

$$\int_{I_n} a(\mathcal{P}_k U_2, V_1) \, dt = \int_{I_n} a(\dot{\mathcal{U}}_1 - R_n f_1, V_1) \, dt, \quad \forall V_1 \in W^{(q-1)}_n.$$

Therefore, we get

$$\mathcal{P}_k U_2(t) = \dot{\mathcal{U}}_1(t) - R f_1(t), \quad t \in I.$$
2. Recalling the definitions of the orthogonal projections \( P \) and \( P^{T, \xi} \) in (3.3) and the functions \( W_1 \) and \( \xi \), we can write the second equation of (3.9) in the form

\[
\int_0^T \left( \rho(U_2(t), V_2) + \tilde{\xi}(t) a(U_1(t), V_2) + \int_0^t \beta(t-s) a(W_1(t, s), V_2(t)) \, ds \right) \, dt
= \int_0^T \left( (P_n f, V_2) + (P^{T, \xi} g, V_2) \right) \, dt, \quad \forall V_2 \in W^{(q-1)}.
\]

Then choosing \( V_2 = A^h_{k} \mathcal{P}_h U_2 \) gives us

\[
\int_0^T \rho(U_2, A^h_{k} \mathcal{P}_h U_2) \, dt + \int_0^T \tilde{\xi}(t) a(U_1, A^h_{k} \mathcal{P}_h U_2) \, dt
\]
(4.4)

\[
+ \int_0^T \int_0^t \beta(t-s) a(W_1(t, s), A^h_{k} \mathcal{P}_h U_2(t)) \, ds \, dt
= \int_0^T \left( (P f, A^h_{k} U_2) + (P^{T, \xi} g, A^h_{k} U_2) \right) \, dt.
\]

There are three terms in the left hand side of the above equation.

3. Using (3.5) and \( \bar{U}_2(t) \in W^{(q-1)} \), we can write the first term of the left hand side of (4.4) as

\[
\int_0^T \rho(U_2, A^h_{k} \mathcal{P}_h U_2) \, dt = \rho \int_0^T \rho(U_2, A^h_{k} U_2) \, dt = \rho \int_0^T D_1 |U_2|_{h, t} \, dt
= \rho \sum_{k=1}^N \left( |U_{2,n}^0|_{h, t}^2 - |U_{2,n-1}^0|_{h, t}^2 \right) = \rho \sum_{k=1}^N \left( |U_{2,n}^0|_{h, t}^2 - |v_{h,n}^0|_{h, t}^2 \right),
\]

where in the last equality we have used the continuity of \( U_2 \) in time, due to the assumption \( V_{h,n-1}^{(p)} \subset V_{h,n}^{(p)} \).

4. With (4.3), we can write the second term as

\[
\int_0^T \tilde{\xi} a(U_1, A^h_{k} \mathcal{P}_h U_2) \, dt = \frac{\rho}{2} \sum_{n=1}^N \int_{I_n} \tilde{\xi} D_1 |U_1|_{h, t}^2 \, dt - \int_0^T \tilde{\xi} a(U_1, A^h_{k} R f_1) \, dt.
\]

Then we integrate by parts in the first term of the right hand side and use the facts that \( \tilde{\xi}(t) = -\beta(t) \) and \( \tilde{\xi}(0) = 1 \), to get

\[
\int_0^T \tilde{\xi} a(U_1, A^h_{k} \mathcal{P}_h U_2) \, dt = \frac{\rho}{2} \sum_{n=1}^N \left( \tilde{\xi}(t_n)|U_{1,n}^0|_{h, t}^2 - \tilde{\xi}(t_{n-1})|U_{1,n-1}^0|_{h, t}^2 \right)
\]

\[
- \frac{\rho}{2} \sum_{n=1}^N \int_{I_n} \tilde{\xi} |U_1(t)|_{h, t}^2 \, dt - \int_0^T \tilde{\xi} a(U_1, A^h_{k} R f_1) \, dt
= \frac{1}{2} \left( \tilde{\xi}(T)|U_{1,N}^0|_{h, t}^2 - |w_{h,N}^0|_{h, t}^2 \right)
\]

\[
+ \frac{1}{2} \int_0^T \beta |U_1(t)|_{h, t}^2 \, dt - \int_0^T \tilde{\xi} a(R f_1, A^h_{k} U_1) \, dt,
\]

where again we used the continuity of \( U_1 \).
5. Consider now the third term in the left hand side of (4.4). Using (4.3) and the fact that $W_i(0) = U_i(0)$ we have

$$
\int_0^T \int_0^t \beta(t - s) R(t, s) \, ds \, dt
$$

and it follows that

$$
\int_0^T \int_0^t \beta(t - s) \beta(t - s) \, ds \, dt
$$

The first term of the right hand side is non-negative. To prove this, for a fixed mesh, take $\varepsilon \in (0, 1)$. Then

$$
\int_0^T \int_0^{t - \varepsilon} \beta(t - s) D_1(t, s) |W_1(t, s)|^2_{h, l+1} \, ds \, dt
$$

$$
= \int_0^{T - \varepsilon} \int_0^T \beta(t - s) D_1(t, s) |W_1(t, s)|^2_{h, l+1} \, dt \, ds
$$

$$
\leq \int_0^{T - \varepsilon} \int_0^T \beta(t - s) |W_1(t, s)|^2_{h, l+1} \, dt \, ds
$$

$$
- \int_0^{T - \varepsilon} \int_0^T \beta(t - s) |W_1(t, s)|^2_{h, l+1} \, dt \, ds
$$

$$
\geq -\beta(\varepsilon) \int_0^{T - \varepsilon} |W_1(t, s)|^2_{h, l+1} \, ds,
$$

where we changed the order of the integrals in the first equation, we integrated by parts in the next one, and we considered the facts that $\beta(t) \leq 0$ and $\beta(t) \geq 0$ for the last inequality. Then using $W_1(t, s + \varepsilon) = \int_s^{t + \varepsilon} D_1(t, s) \, dt$ and the Cauchy-Schwarz inequality we get

$$
|W_1(s + \varepsilon, s)|^2_{h, l+1} \leq \left( \int_s^{s + \varepsilon} |D_1(t, s)|_{h, l+1} \, dt \right)^2,
$$

$$
\leq \int_s^{s + \varepsilon} \frac{dt}{\beta(t - s)} \int_s^{s + \varepsilon} \beta(t - s) |D_1(t, s)|^2_{h, l+1} \, dt.
$$

So (4.5) can be written as

$$
\int_0^T \int_0^{t - \varepsilon} \beta(t - s) D_1(t, s) |W_1(t, s)|^2_{h, l+1} \, ds \, dt
$$

$$
\geq -\beta(\varepsilon) \int_0^{T - \varepsilon} \int_0^{s + \varepsilon} \beta(t - s) |D_1(t, s)|^2_{h, l+1} \, dt \, ds
$$

$$
\geq -\varepsilon \int_0^{T - \varepsilon} \int_s^{s + \varepsilon} \beta(t - s) |D_1(t, s)|^2_{h, l+1} \, dt \, ds,
$$

since $\frac{\beta(\varepsilon)}{\beta(t - s)} \leq 1$. Obviously, for a fixed mesh, the last integral in the right hand side is bounded by

$$
\int_0^T \int_0^T \beta(t - s) |D_1(t, s)|^2_{h, l+1} \, dt \, ds < \infty.
Therefore, letting $\epsilon \to 0$ we get
\[
\int_0^T \int_0^T \beta(t-s) D_{ij} |W_i(t,s)|^2_{h,l+1} \, ds \, dt \geq 0.
\]

6. Putting the results from steps 3, 4 and 5 into (4.4) completes the proof. \[\square\]

**Remark 4.** In [4] the auxiliary function $w(t,s) = u(t) - u(t-s)$, was used the same as in our \$2, to obtain stability estimates for the spatially semidiscrete finite element method. This does not work here because $U_1(t) - U_1(t-s)$ does not belong to $W^{(q)}$ if the temporal mesh is non-uniform.

From now on we specialize to the case $p = q = 1$. We use (4.1) to obtain a stability estimate to be used in the error analysis. To this end, from (4.1) with $g = 0$, we have
\[
\rho |U_{2,N}|_{h,l}^2 + \tilde{\xi}(T) |U_{1,N}|_{h,l+1}^2 \leq \rho |v_{0,h}|_{h,l}^2 + |u_{0,h}|_{h,l+1}^2 + 2 \int_0^T (A_h^I P f_2, U_2) \, dt
\]
\[
+ 2 \int_0^T a(A_h^I R f_1, U_1) \, dt
\]
\[
+ 2 \int_0^T \int_0^t \beta(t-s) a(A_h^I R f_1(t), W_1(t,s)) \, ds \, dt.
\]

Therefore using (3.4), $1 - \gamma < \tilde{\xi}(t) \leq 1$ and $\int_0^t \beta(s) \, ds \leq \gamma$, we get
\[
\rho |U_{2,N}|_{h,l}^2 + (1 - \gamma) |U_{1,N}|_{h,l+1}^2
\]
\[
\leq \rho |v_{0,h}|_{h,l}^2 + |u_{0,h}|_{h,l+1}^2 + 2 \int_0^T (A_h^{1/2} p_h f_2, A_h^{1/2} U_2) \, dt
\]
\[
+ 2 \int_0^T a(A_h^{1/2} p_h R f_1, A_h^{1/2} U_1) \, dt
\]
\[
+ 2 \int_0^T \int_0^t \beta(t-s) a(A_h^{1/2} p_h f_1(t), A_h^{1/2} W_1(t,s)) \, ds \, dt
\]
\[
\leq \rho |v_{0,h}|_{h,l}^2 + |u_{0,h}|_{h,l+1}^2 + 2 \int_0^T [p_h p_h f_2 |h,l| U_2]_{h,l} \, dt
\]
\[
+ 2 \int_0^T [p_h R f_1 |h,l+1| U_1 |h,l+1] \, dt
\]
\[
+ 2 \gamma \int_0^T [p_h R f_1 |h,l+1| \max_{0 \leq s \leq T} |W_1(t,s)|_{h,l+1} \, dt
\]
\[
\leq \rho |v_{0,h}|_{h,l}^2 + |u_{0,h}|_{h,l+1}^2 + \frac{1}{2} \rho \max_{0 \leq s \leq T} |U_2|_{h,l}^2 + C \left( \int_0^T [p_h p_h f_2 |h,l] \, dt \right)^2
\]
\[
+ \frac{1}{2} (1 - \gamma) \max_{0 \leq s \leq T} |U_1|_{h,l+1}^2 + C \left( \int_0^T [p_h R f_1 |h,l+1|] \, dt \right)^2
\]
where $C = C(\rho, \gamma)$. Using that, for $q = 1$, we have
\[
\max_{[0,T]} |U_i| \leq \max_{[0,T]} |U_{i,n}|,
\]
and
\[
\int_0^T |p_h f| \, dt \leq \int_0^T |f| \, dt,
\]
and that the above inequality holds for arbitrary $N$, we conclude in a standard way

$$
|U_{2,N}|_{h,l} + |U_{1,N}|_{h,l+1}
$$

(4.6)

$$
\leq C \left( |v^0_{h}|_{h,l} + |u^0_{h}|_{h,l+1} + \int_0^T \left( |R_h f_1|_{h,l+1} + |P_h f_2|_{h,l} \right) dt \right),
$$

with $C = C(\rho, \gamma)$.

5. A priori error estimates

To simplify the notation we denote the Sobolev norms $\| \cdot \|_{H^q(\Omega)}$ by $\| \cdot \|_q$. We define the standard interpolant $I_k v \in W(1)$ by

$$(5.1) \quad I_k v(t_n) = v(t_n), \quad n = 0, 1, \cdots, N.$$ 

By standard arguments in approximation theory we see that, for $q = 0, 1$,

$$(5.2) \quad \int_0^T \| I_k v - v \|_q \, dt \leq C k^{q+1} \int_0^T \| D^{q+1}_t v \|_q \, dt, \quad \text{for } i = 0, 1, 2,$$

where

$$
k = \max_{1 \leq n \leq N} k_n.
$$

We assume the elliptic regularity estimate $\| v \|_2 \leq C \| A v \|$, $\forall v \in \text{dom}(A)$, so that the following error estimates for the Ritz projection (3.2), hold true

$$(5.3) \quad \| R_h v - v \| \leq C h^s \| v \|_s, \quad \forall v \in H^s \cap V, \quad s = 1, 2.$$ 

**Theorem 5.** Assume $q = p = 1$ and let $u$ and $U$ be the solutions of (1.7) and (3.9). Then, with $e = U - u$ and $C = C(\rho, \gamma)$, we have

$$(5.4) \quad e = \theta + \eta + \rho = (U - \pi u) + (\pi u - J u) + (J u - u),$$

for some suitable operators $\pi$ and $J$ which will be specified in terms of the time interpolant $I_k$ in (5.1) and projectors $R_h$ and $P_h$ in (3.2), so that $\pi u \in W(1)$ and $\eta$ and $\rho$ will correspond to the temporal and spatial errors, respectively. Due to (5.2)–(5.3) we just need to estimate $\theta$. To this end, using the Galerkin orthogonality
(3.8) and the definition of $\theta$, we get

$$B(\theta, P_k V) = -B(\eta, P_k V) - B(\rho, P_k V)$$

$$= \int_0^T a(\eta_2, P_k V_1) - a(\eta_1, P_k V_1) - \rho(\eta_2, P_k V_2) - a(\eta_1, P_k V_2) dt$$

$$+ \int_0^T \int_0^t \beta(t-s) a(\eta_1(s), P_k V_2(t)) ds dt$$

$$+ \int_0^T a(\rho_2, P_k V_1) - a(\rho_1, P_k V_1) - \rho(\rho_2, P_k V_2) - a(\rho_1, P_k V_2) dt$$

$$+ \int_0^T \int_0^t \beta(t-s) a(\rho_1(s), P_k V_2(t)) ds dt$$

$$= \sum_{j=1}^{10} E_j, \quad \forall V \in (W^{(1)})^2.$$  

We consider two different choices of the operators $\pi$ and $J$. In order to prove the first two error estimates we set, for $i = 1, 2,$

$$\theta_i = U_i - I_k \mathcal{R}_h u_i, \quad \eta_i = (I_k - I)\mathcal{R}_h u_i, \quad \rho_i = (\mathcal{R}_h - I)u_i.$$  

Integrating by parts in $E_2$ and $E_3$ with respect to time and using (5.1) we have for both cases

$$E_2 = E_3 = 0.$$  

Moreover, by the definitions of $\eta$ and $\rho$, we have

$$E_6 = E_7 = E_9 = E_{10} = 0.$$  

Therefore,

$$B(\theta, P_k V) = \int_0^T a(\eta_2, P_k V_1) dt$$

$$+ \int_0^T \left( a(-\eta_1 + \int_0^t \beta(t-s)\eta_1(s) ds, P_k V_2) - \rho(\rho_2, P_k V_2) \right) dt$$

$$= \tilde{L}(P_k V), \quad \forall V \in (W^{(1)})^2,$$

which is of the form (3.10) with $f_1 = \eta_2, f_2 = A_h (-\eta_1 + \int_0^t \beta(t-s)\eta_1(s) ds) - \rho \tilde{P}_2$ and $g = 0$.

Applying the stability inequality (4.6) with $t = 0$, and considering the fact that $|\cdot|_{0,h} = \|\cdot\|$ and $|\cdot|_{h,1} = |\cdot|_1$, we have

$$|\theta_{2,N}| + |\theta_{1,N}| \leq C \left( \|\theta_2(0)\| + |\theta_1(0)|_1 \right) + C \int_0^T |\mathcal{R}_h \eta_2|_1 dt$$

$$+ C \int_0^T \left( \|P_h A_h \eta_1\| + \|P_h A_h \int_{0}^{t} \beta(t-s) \eta_1(s) ds\| + \rho \|P_h \tilde{P}_2\| \right) dt,$$

where $\theta_1(0) = 0$, since $U_1(0) = \mathcal{R}_h u^0$. Since $|\mathcal{R}_h v|_1 \leq \|v\|_1, \|P_h v\| \leq \|v\|, \forall v \in V$ and $\mathcal{R}_h A_h = P_h A$, we have

$$|\mathcal{R}_h \eta_2|_1 \leq \|(I_k - I) \mathcal{R}_h u_2\|_1 \leq \|(I_k - I) u_2\|_1,$$

$$\|P_h A_h \eta_1\| = \|A_h \eta_1\| = \|(I_k - I) A_h \mathcal{R}_h u_1\| = \|(I_k - I) P_h A u_1\|$$

$$\leq \|(I_k - I) A u_1\| \leq C \|(I_k - I) u_1\|_2,$$
and
\[
\int_0^T \left\| \mathcal{P}_h A_h \int_0^t \beta(t-s) \eta_1(s) \, ds \right\| \, dt \leq \int_0^T \left\| A_h \int_0^t \beta(t-s) \eta_1(s) \, ds \right\| \, dt \\
\leq C \int_0^T \left\| \int_0^t \beta(t-s) \| (I_k - I) \mu_1(s) \|_2 \, ds \right\| \, dt \\
\leq C \int_0^T \beta(t) \, dt \int_0^T \| (I_k - I) \mu_1 \|_2 \, dt \\
\leq C \gamma \int_0^T \| (I_k - I) \mu_1 \|_2 \, dt.
\]

Therefore by \( \theta = e - \eta - \nu, \eta(t_n) = 0 \) and \( \theta_1(0) = 0 \), we get
\[
\| e_{2,N} \| \leq \| \rho_{2,N} \| + C \theta_2(0) \\
+ C \int_0^T \left( \| (I_k - I) \mu_2 \|_1 + \| (I_k - I) \mu_1 \|_2 + \| (\mathcal{R}_h - I) \mu_2 \|_1 \right) \, dt,
\]
which implies the first two estimates by (5.2) and (5.3).

Finally, we choose
\[
\theta_1 = U_1 - I_k \mathcal{R}_h u_1, \quad \nu_1 = (I_k - I) \mathcal{R}_h u_1, \quad \rho_1 = (\mathcal{R}_h - I) u_1, \\
\theta_2 = U_2 - I_k \mathcal{P}_h u_2, \quad \nu_2 = (I_k - I) \mathcal{P}_h u_2, \quad \rho_2 = (\mathcal{P}_h - I) u_2.
\]

By the definitions of \( \mathcal{R}_h \) and \( \mathcal{P}_h \) in (3.2), this implies
\[
E_7 = E_8 = E_9 = E_{10} = 0,
\]
and we still have (5.6). Therefore, (5.5) becomes
\[
B(\theta, \mathcal{P}_h V) = \int_0^T a(\nu_2 + \rho_2, \mathcal{P}_h V_1) \, dt \\
+ \int_0^T a(- \nu_1 + \int_0^t \beta(t-s) \eta_1(s) \, ds, \mathcal{P}_h V_2) \, dt \\
= \hat{L}(\mathcal{P}_h V), \quad \forall V \in (W^{1})^2,
\]
which is of the form (3.10) with \( f_1 = \eta_2 + \rho_2, \quad f_2 = A_h (- \nu_1 + \int_0^t \beta(t-s) \eta_1(s) \, ds) \)
and \( g = 0 \).

Again applying the stability inequality (4.6), this time with \( l = -1 \), and using \( \|\cdot\|_{h,0} = \|\cdot\| \), we have
\[
\| \theta_{1,N} \| \leq C \int_0^T \left( \| \mathcal{R}_h \eta_2 \| + \| \mathcal{R}_h \rho_2 \| \right) \, dt \\
+ C \int_0^T \left( \| \mathcal{P}_h A_h \eta_1 \|_{h,-1} + \| \mathcal{P}_h A_h \int_0^t \beta(t-s) \eta_1(s) \, ds \|_{h,-1} \right) \, dt,
\]
where we used that \( \theta(0) = 0 \), since \( U_1(0) = \mathcal{R}_{h,1} u^0 \) and \( U_2(0) = \mathcal{P}_{h,1} v^0 \). Then, since

\[
\|\mathcal{R}_h u_1\| = \|(I - I)\mathcal{P}_h u_2\| \leq \|(I - I) u_2\|,
\]

\[
\|\mathcal{R}_h \rho_1\| = \|\mathcal{P}_h (I - \mathcal{R}_h) u_2\| \leq \|\mathcal{P}_h (I - I) u_2\|,
\]

\[
|\mathcal{P}_h A_h \eta_1|_{h,-1} = |A_h \mathcal{R}_h (I - I) u_1|_{h,-1} = |\mathcal{R}_h (I - I) u_1|_{h,1} \leq |(I - I) u_1|_{h,1},
\]

and

\[
\int_0^T |\mathcal{P}_h A_h \int_0^t \beta(t-s) \eta_1(s) ds|_{h,-1} dt \leq \int_0^T \int_0^t \beta(t-s) |(I - I) u_1(s)|_{h,1} ds dt \\
\leq \gamma \int_0^T |(I - I) u_1|_{h,1} dt,
\]

we conclude

\[
\|e_1,N\| \leq \|\rho_{1,N}\| + C \int_0^T \left( \|(I - I) u_2\| + \|\mathcal{R}_h (I - I) u_2\| + |(I - I) u_1|_{h,1} \right) dt,
\]

which implies the last estimate by (5.2) and (5.3).

\[
\square
\]

6. Numerical example

In this section we demonstrate the numerical method by solving a simple but realistic example for a two dimensional structure, see Figure 1 (a), using piecewise linear polynomials, i.e., \( q = p = 1 \).

We consider the initial conditions: \( u(x,0) = 0 \) m, \( \dot{u}(x,0) = 0 \) m/s, the boundary conditions: \( u = 0 \) at \( x = 0 \), \( g = (0,-1) \) Pa at \( x = 1.5 \) and zero on the rest of the boundary. The volume load is assumed to be \( f = 0 \) N/m\(^3\). And the model parameters are: \( \gamma = 0.5, \tau = 0.25, \nu = 0.3, E = 5MPa \) and \( \rho = 7000 \) kg/m\(^3\). The deformed mesh at \( t/\tau = 9 \) for \( \alpha = 1/2 \) is displayed in Figure 1 (b), with the displacement magnified by the factor 10\(^5\), and the computed vertical displacement at the point (1.5,1.5) for different \( \alpha \) are shown in Figure 2.

![Figure 1](image-url)
Figure 2. Vertical displacement for different $\alpha$.

References

1. K. Adolfsson, M. Enelund, and S. Larsson, Adaptive discretization of an integro-differential equation with a weakly singular convolution kernel, Comput. Methods Appl. Mech. Engrg. 192 (2003), 5285–5304.
2. , Adaptive discretization of fractional order viscoelasticity using sparse time history, Comput. Methods Appl. Mech. Engrg. 193 (2004), 4567–4590.
3. , Space-time discretization of an integro-differential equation modeling quasi-static fractional order viscoelasticity, J. Vibration Control (2008), to appear.
4. K. Adolfsson, M. Enelund, S. Larsson, and M. Racheva, Discretization of integro-differential equations modeling dynamic fractional order viscoelasticity, LNCS 3743 (2006), 76–83.
5. R. L. Bagley and P. J. Torvik, Fractional calculus—a different approach to the analysis of viscoelastically damped structures, AIAA J. 21 (1983), 741–748.
6. R. H. Fabiano and K. Ito, Semigroup theory and numerical approximation for equations in linear viscoelasticity, SIAM J. Math. Anal. 21 (1990), 374–393.
7. M. Lopez-Fernandez, C. Lubich, and A. Schädle, Adaptive, fast and oblivious convolution in evolution equations with memory, Preprint (2006).
8. W. McLean and V. Thomée, Numerical solution of an evolution equation with a positive-type memory term, J. Austral. Math. Soc. Ser. B 35 (1993), 23–70.
9. A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.
10. A. Schädle, M. López-Fernández, and Ch. Lubich, Fast and oblivious convolution quadrature, SIAM J. Sci. Comput. 28 (2006), 421–438.