Self-Adjoint Extension of Symmetric Maps

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Abstract

A densely-defined symmetric linear map from/to a real Hilbert space extends to a self-adjoint map. Extension is expressed via Riesz representation. For a case including Friedrichs extension of a strongly monotone map, self-adjoint extension is unique, and equals closure of the given map.

Let \( \{ A : X \supseteq \mathcal{D}(A) \to X \} \) be a densely-defined symmetric linear map. Recall that if Hilbert-space \( X \) is complex, then \( A \) may lack self-adjoint extension (see e.g. [R]). In contrast, self-adjoint extension must exist if our Hilbert-space is real, as will be shown here.

To prepare, we express well-known material in a form convenient for the present purpose. For \( x \in X \), let \( (x \mid A) \) denote the linear function \( \{ \mathcal{D}(A) \ni y \mapsto (x \mid Ay) \} \); we use the convention that scalar-product is linear in the second entry, conjugate-linear in the first. Observe the adjoint domain \( \mathcal{D}(A^*) \) equals \( \{ x \in X : (x \mid A) \text{ continuous} \} \). Recall: \( \mathcal{D}(A) \subseteq \mathcal{D}(A^*) \); \( A \) is self-adjoint iff \( \mathcal{D}(A) = \mathcal{D}(A^*) \). Let \( J \) denote the duality-map on \( X \), which maps \( x \) to function \( (x \mid \cdot) \) in dual-space \( X^* \); write \( J^{-1} = R \), Riesz-representation. Extend Riesz-map \( R \) so as to act on densely-defined (continuous linear) functions, such as \( (x \mid A) \) if \( x \in \mathcal{D}(A^*) \).

Note. Let \( A \) have symmetric extension \( B \). Then

(i) \( \mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq \mathcal{D}(B^*) \subseteq \mathcal{D}(A^*) \).

(ii) \( B(x \mid B) = R(x \mid A) \), if \( x \in \mathcal{D}(B^*) \subseteq \mathcal{D}(A^*) \).

(iii) \( Bx = R(x \mid A) \) if \( x \in \mathcal{D}(B) \).

Proof. (i) is known. For \( y \in \mathcal{D}(A) \), see \( (R(x \mid B) \mid y) = (x \mid By) = (x \mid Ay) = (R(x \mid A) \mid y) \); density of \( \mathcal{D}(A) \) gives (ii). For \( x \in \mathcal{D}(B) \) and \( y \in \mathcal{D}(A) \), see \( (x \mid A) \) is continuous, and \( (Bx \mid y) = (x \mid By) = (x \mid Ay) = (R(x \mid A) \mid y) \); density of \( \mathcal{D}(A) \) gives (iii). Done.

Denote by \( \Lambda \) the linear map \( \{ \mathcal{D}(A^*) \ni x \mapsto R(x \mid A) \} \). Note (iii) (above) says \( A \) has at-most-one
symmetric extension to a given subspace \( Y \), with \( \mathcal{D}(A) \subseteq Y \subseteq \mathcal{D}(A^*) \); if such extension exists, then it equals the restriction \( \Lambda|_Y \).

**Theorem.** Every symmetric map from/to a real Hilbert space has self-adjoint extension.

**Proof.** Let \( E \) denote the order-set of linear subspaces \( Y \), with \( \mathcal{D}(A) \subseteq Y \subseteq \mathcal{D}(A^*) \), for which restriction \( \Lambda|_Y \) is symmetric; order by inclusion. (\( E \ni \mathcal{D}(A) \).) A chain \( C \in E \) is bound above by the union of subspaces in \( C \); so Zorn’s lemma ensures \( E \) has a maximal member, \( Z \). \( \Lambda|_Z \) is a maximal symmetric extension of \( A \).

Write \( \Lambda|_Z = M \). We claim \( \mathcal{D}(M) = \mathcal{D}(M^*) \); if true, then \( M \) would be self-adjoint, concluding the proof. It is enough to show \( \mathcal{D}(M^*) \subseteq \mathcal{D}(M) \); suppose not, seek a contradiction. Fix \( p \in \mathcal{D}(M^*) \setminus \mathcal{D}(M) \). On the subspace \( \mathcal{D}(M) \oplus \mathbb{R}p \), define a map \( T \):

\[
T(x + ap) = Mx + aR(p|M) \quad \text{if} \quad x \in \mathcal{D}(M), \ a \in \mathbb{R}.
\]

See \( T \) is linear, and \( T \) properly extends \( M \). To show symmetry of \( T \), let \( \{x, y\} \subset \mathcal{D}(M) \) and \( \{a, b\} \subset \mathbb{R} \); note \( \langle x | R(p|M) \rangle = \langle p | Mx \rangle \), \( \langle R(p|M) | y \rangle = \langle p | My \rangle \); compute:

\[
\begin{align*}
(T(x + ap) | y + bp) &= (Mx + aR(p|M) | y + bp) = \\
(Mx | y) + b(Mx | p) + a(R(p|M) | y) + ab(R(p|M) | p) = \\
(x | My) + b(x | R(p|M)) + a(p | My) + ab(p | R(p|M)) = \\
(x + ap | My + bR(p|M)) = (x + ap | T(y + bp)).
\end{align*}
\]

\( M \) has symmetric proper extension \( T \), so \( M \) is not a maximal symmetric extension of \( A \); contra. Done.

So, self-adjoint extension exists; now treat uniqueness. Fortunately, extension is unique for some cases of interest; sometimes we may even express extension simply, as closure of the given map. To prepare to show this, recall \( A \) has symmetric closure \( \tilde{A} \subseteq M \). Here, as before, \( \{A : X \supseteq \mathcal{D}(A) \to X\} \) is symmetric, with self-adjoint extension \( M \), from/to a Hilbert space \( X \), now assumed real. We also need the following two facts.

**Note 1.** If \( A \) has dense image and continuous inverse, then \( \tilde{A} \) is the unique self-adjoint extension of \( A \); \( M = \tilde{A} \). \( \tilde{A} \) maps onto \( X \), and has continuous self-adjoint inverse.

**Proof.** \( \tilde{A} \) has dense image (since \( A \) does); recall a symmetric map (\( \tilde{A} \)) with dense image has symmetric inverse; \( \tilde{A}^{-1} \) is also closed, since \( \tilde{A} \) is so. \( \tilde{A}^{-1} \) equals closure of a continuous map (\( A^{-1} \)), hence \( \tilde{A}^{-1} \) is continuous. Since \( \tilde{A}^{-1} \) is closed, continuous, and has dense domain (including \( \text{Im}(A) \) ), we have \( \mathcal{D}(\tilde{A}^{-1}) = X \). A continuous symmetric map (\( \tilde{A}^{-1} \)) on the whole Hilbert space is self-adjoint. Recall a self-adjoint map (\( \tilde{A}^{-1} \)) with dense image (including \( \mathcal{D}(A) \)) has self-adjoint inverse (\( \tilde{A} \)). Hence \( \{\tilde{A}, M\} \) are self-adjoint extensions of \( A \), with \( \tilde{A} \subseteq M \); this forces \( \tilde{A} = M \), because a self-adjoint map is maximal-symmetric. Done.

**Note 2.** A (densely-defined) closed 1:1 symmetric map has dense image.
Proof. It is enough to show $p = 0$, if $p \in \mathcal{Im}^\perp(A)$ (orthogonal complement of image). Since $\mathcal{Do}(A)$ is dense, it has a sequence $\{u_n\}$ converging to $p$. If $x \in \mathcal{Do}(A)$, then
\[0 = (p|Ax) = \lim(u_n|Ax) = \lim(Au_n|x).\]
Density of $\mathcal{Do}(A)$ forces $\lim Au_n = 0$. $A$ is closed; $(\lim u_n = p)$ and $(\lim Au_n = 0)$; hence $p \in \mathcal{Do}(A)$, $Ap = 0$. Since $A$ is 1:1, we have $p = 0$. Done.

Recall (e.g. [Z]) that if our map $A$ is strongly monotone, then it has Friedrichs extension, which is self-adjoint, 1:1, onto, with continuous self-adjoint inverse.

**Theorem.** If $A$ is strongly monotone, then closure $\bar{A}$ is the unique self-adjoint extension of $A$; $\bar{A}$ equals Friedrichs extension.

**Proof.** Let $\hat{A}$ denote Friedrichs extension; $\hat{A} \supseteq \bar{A}$. Since $\hat{A}$ is 1:1 with continuous inverse, so is its restriction $\bar{A}$. By Note 2, closed symmetric 1:1 map $\bar{A}$ has dense image; then Note 1 makes $\bar{A}$ the unique self-adjoint extension of itself, and of $A$. $\hat{A}$ is a self-adjoint extension of $A$, hence $\hat{A} = \bar{A}$.

Done.

Construction of the Friedrichs extension is complicated; how nice to express it simply (as closure), and to know it is the only self-adjoint extension.

**References**

[R] Rudin, W. *Functional Analysis*. McGraw-Hill, 1991.

[Z] Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics*. Springer, 1995.