SOME RESULTS ON CONTINUOUS DEFORMED FREE GROUP FACTORS

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Abstract. We construct a Fock space associated to a symmetric function \( Q : U \times U \to (-1, 1) \), where \( U \) is a nonempty open subset of \( \mathbb{R}^j \) for some \( j \). Namely, we will have operator-valued distributions \( a(x) \) and \( a^+(y) \) satisfying
\[
a(x)a^+(y) - Q(x, y)a^+(y)a(x) = \delta(x - y).
\]
Analogous to the \( q_j \)-Fock space of Bożejko and Speicher [3], we have field operators arising as the sum of the creation and annihilation operators. These operators generate a von Neumann algebra analogous to the free group factors, and are factors which do not have property \( \Gamma \). It was pointed out to us by an anonymous referee that this is a special case of a theorem of Krolak [9].

1. Introduction

In the study of operator algebras, much attention has been paid to the canonical commutation relations (CCR) and the canonical anti-commutations (CAR). Bożejko and Speicher [2] considered an interpolation between these relations. Specifically, for \( q \in [-1, 1] \), they constructed creation operators \( c^+(f) \) and annihilation operators \( c(f) \) on a \( q \)-twisted Fock space \( \mathcal{F}_q(\mathcal{H}) \) satisfying the relations
\[
c(f)c^+(g) - qc^+(g)c(f) = \langle f, g \rangle \cdot 1.
\]
In the \( q = 0 \) case, these are the creation and annihilation operators on the full Fock space.

It was shown by Voiculescu [10] that for a Hilbert space of dimension \( d \in \{1, 2, \ldots, \infty\} \), the Hermitian parts of the creation operators on the free Fock space generate von Neumann algebras isomorphic to the free group factor on \( d \) generators. Thus, we can view the algebras \( \Gamma_q(\mathcal{H}) := \{c(f) + c^+(f) : f \in \mathcal{H}\}' \) as \( q \)-deformations of the free group factors.

Various factoriality theorems have been proven for these algebras. First, Bożejko and Speicher [3] showed that these are factors when \( \dim \mathcal{H} \) is infinite. Śniady [9] subsequently showed that \( \Gamma_q(\mathcal{H}) \) is a factor for \( \dim \mathcal{H} \) sufficiently large but finite. Ricard [8] showed that in fact \( \Gamma_q(\mathcal{H}) \) is a factor for \( \dim \mathcal{H} \geq 2 \).

More general deformations of the free group factors have also been considered. For \( \mathcal{H} \) a Hilbert space with basis \( \{e_i\}_{i \in I} \), Bożejko and Speicher [3] constructed a solution to the \( q_{ij} \)-relations
\[
c(e_i)c^+(e_j) - q_{ij}c^+(e_j)c(e_i) = \delta_{ij},
\]
for \( q_{ij} \in [-1, 1] \) as well as a further generalization of the relations arising from a contraction \( T \in \mathcal{B}(\mathcal{H}) \) satisfying the braid relation (or Yang-Baxter relation) given by
\[
(1 \otimes T)(T \otimes 1)(1 \otimes T) = (T \otimes 1)(1 \otimes T)(T \otimes 1).
\]
Królak [6] proved that if \( \|T\| < 1 \), which in the \( q_{ij} \) case corresponds to the condition \( \sup\{|q_{ij}| : i, j \in I\} < 1 \), the resulting von Neumann algebra is a factor for \( \dim \mathcal{H} \) sufficiently large.

In another direction, Liguori and Mintchev [7] and Bożejko, Lytvynov, and Wysoczanski [1] have considered creation and annihilation operators on a Fock space arising from a continuous commutation relation associated with a Hermitian function \( Q \) from \( \mathbb{R}^j \times \mathbb{R}^j \) (or some more general space) to the unit circle. This construction also involves additional commutation relations on the creation operators, and includes the anyons as a special case.

Here we will consider a continuous \( Q \)-commutation relation arising from a function taking values in \((-1, 1)\). Before we state the problem more explicitly, we introduce some notations which will be used throughout the paper.

Notation 1. Let \( U \) be a nonempty open subset of \( \mathbb{R}^j \) for some integer \( j \geq 1 \). We also fix \( Q \in C(U \times U) \), the space of continuous functions on \( U \times U \). Further assume that \( q := \sup\{|Q(x, y)| : x, y \in U\} < 1 \) and that \( Q \) is a symmetric function, that is \( Q(x, y) = Q(y, x) \). Also define \( \mathcal{H} = L^2(U) \).
For points \( x, y \in U \), we wish to consider, at least heuristically, infinitesimal creation and annihilation operators on a \( Q \)-twisted Fock space satisfying the \( Q \)-commutation relation

\[
a(x)a^+(y) - Q(x, y)a^+(y)a(x) = \delta(x - y) \cdot 1,
\]

where \( \delta \) is the usual Dirac \( \delta \), whence

\[
\int \int \delta(x - y)f(x, y) \, dx \, dy = \int f(y, y) \, dy.
\]

Rigorously, this relation should be understood as a statement about operator-valued distributions, which makes sense upon smearing with a test function and considering the resulting quadratic forms. The meaning will be explained further in Section 2.

The operator-valued distributions \( a^+(x) \) and \( a(x) \) will give rise to creation and annihilation operators \( a(f) \) and \( a^+(f) \) on a \( Q \)-deformed Fock space \( F_Q(\mathcal{H}) \). We will use these to define a \( Q \)-deformed field operator \( \omega(f) = a(f) + a^+(f) \) and the von Neumann algebra \( \Gamma_Q(\mathcal{H}) \) generated by operators of this type.

This paper has four sections, not including this introduction. Section 2 will present the construction of a \( Q \)-Fock space with creation and annihilation operators realizing the \( Q \)-commutation relation. In Section 3, we will discuss basic properties of the von Neumann algebra generated by the field operators on this Fock space. In Section 4, we will show that the field operators arise as a limit in distribution of operators on discrete \( q_{ij} \)-Fock spaces considered by Bożejko and Speicher in [3]. In Section 5, we will show that the von Neumann algebra generated by these operators is a factor.

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2. The \( Q \)-Fock Space

We will construct our \( Q \)-Fock space by defining a deformed inner product on the algebraic Fock space. Fix \( n \) and define for \( 1 \leq i \leq n - 1 \) the operator \( T_{i}^{(n)} \) on \( \mathcal{H}^{\otimes n} \) by

\[
T_{i}^{(n)} f(x_1, \ldots, x_n) = Q(x_i, x_{i+1}) f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n).
\]

Evidently \( T_i \) is self-adjoint and bounded with norm at most \( q := \sup_{x,y} |Q(x, y)| < 1 \). It is easily verified that \( T_i^{(n)} \) is a decomposition of \( \pi \) into a minimal number of the \( \pi_i \) then we define

\[
\phi_n(\pi) = \phi_n(\pi_{i_1}) \cdots \phi_n(\pi_{i_k}) = T_{i_1}^{(n)} \cdots T_{i_k}^{(n)}.
\]

That this definition does not depend on our choice of minimal length decompositions for \( \pi \) is a consequence of the fact that the \( T_{i}^{(n)} \) satisfy (2). It follows from this definition that \( \phi_n(\sigma_1 \sigma_2) = \phi_n(\sigma_1) \phi_n(\sigma_2) \) whenever \( |\sigma_1| + |\sigma_2| = |\sigma_1 \sigma_2| \). Here \( |\sigma_k| \) denotes the number of inversions of the permutation \( \sigma_k \). That is, \( |\sigma_k| = |\{(i, j) : 1 \leq i < j \leq n, \sigma_k(i) > \sigma_k(j)\}| \).

Equivalently, \( |\sigma_k| \) is the length of the shortest word for \( \sigma_k \) as a product of the fundamental transpositions.

We now define the operator \( P_Q^{(n)} \in \mathcal{B}(\mathcal{H}^{\otimes n}) \) by

\[
P_Q^{(n)} = \sum_{\sigma \in S_n} \phi_n(\sigma).
\]
By Theorem 2.3 of [3], the operator $P_Q^{(n)}$ is strictly positive.

Let $\mathcal{F}_{\text{alg}}(\mathcal{H})$ be the algebraic Fock space on $\mathcal{H}$,
$$
\mathcal{F}_{\text{alg}}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},
$$
where the direct sum is the algebraic direct sum, so that only finite sums are permitted. Here $\mathcal{H}^{\otimes 0}$ is a one-dimensional vector space generated by a distinguished unit vector $\Omega$, which we call the vacuum vector.

The $Q$-inner products on the $\mathcal{H}^{\otimes n}$ naturally define a $Q$-inner product on $\mathcal{F}_{\text{alg}}(\mathcal{H})$ by sesquilinear extension of
$$
\langle f, g \rangle_Q = \begin{cases} 
\langle f, P_Q^{(n)} g \rangle_0, & m = n, \\
0, & m \neq n,
\end{cases}
$$
for $f \in \mathcal{H}^{\otimes n}$ and $g \in \mathcal{H}^{\otimes m}$. Here, $\langle \cdot, \cdot \rangle_0$ denotes the usual inner product on $\mathcal{H}^{\otimes n}$. We now define the $Q$-Fock space $\mathcal{F}_Q(\mathcal{H})$ as the completion of $\mathcal{F}_{\text{alg}}(\mathcal{H})$ with respect to the $Q$-inner product.

We are now almost ready to introduce the $Q$-creation and annihilation operators. We will define these in terms of the free creation and annihilation operators. For $f \in \mathcal{H}$, we define the free creation operator $l^+(h)$ on $\mathcal{F}_{\text{alg}}(\mathcal{H})$ by
$$
l^+(h)f = h \otimes f
$$
for $f \in \mathcal{H}^{\otimes n}$, where we adopt the convention for the $n = 0$ case that $h \otimes \Omega = h$. We define $l(h)$ to be the free annihilation operator, given by
$$
(l(h)f)(x_1, \ldots, x_{n-1}) := \int_{\mathcal{H}} h(y)f(y, x_1, \ldots, x_{n-1}) \, dy.
$$

We now define for $h \in \mathcal{H}$ the $Q$-creation operator $a^+(h)$ and the $Q$-annihilation operator $a(h)$ by
$$
a^+(h) := l^+(h) \quad \text{and} \quad a(h) := l(h)R_Q^{(n)}
$$
on $\mathcal{H}^{\otimes n}$ for $n > 0$, where
$$
R_Q^{(n)} := 1 + T_1^{(n)} + T_1^{(n)}T_2^{(n)} + \cdots + T_1^{(n)} \cdots T_{n-2}^{(n)}T_{n-1}^{(n)}.
$$
By writing each permutation $\sigma \in S_n$ as the product of an element of $S_1 \times S_{n-1}$ and the minimal length representative of the coset of $\sigma$, we can show that
$$
P_Q^{(n+1)} = (1 \otimes P_Q^{(n)})R_Q^{(n+1)}.
$$
One can analogously define $Q$-deformed right creation and annihilation operators. In general, we will state our results in terms of the left side versions, but analogous results hold on the right side with the same proofs, and we will occasionally need to make use of these analogs.

It was pointed out to us by an anonymous referee that the following is actually a special case of Theorem 3.1 of [5].

**Proposition 1.** For $h \in \mathcal{H}$, the operators $a(h)$ and $a^+(h)$ are adjoints with respect to the $Q$-norm. Furthermore, for $h \in \mathcal{H}$,
$$
\|a^+(h)\| \leq \|h\| \frac{1}{\sqrt{1-q}}.
$$
In particular, $a^+(h)$ and $a(h)$ extend to bounded operators on $\mathcal{F}_Q(\mathcal{H})$.

**Proof.** The proof of this theorem is very similar to that of Theorem 3.1 in [3]. We will first show that $a(h)$ and $a^+(h)$ are adjoints with respect to the $Q$ inner product. The definitions imply that
$$
l^+(h)T_i^{(n)} = T_i^{(n+1)}l^+(h),
$$
whence it follows that
$$
l^+(h)P_Q^{(n)} = (1 \otimes P_Q^{(n)})l^+(h) \quad \text{and} \quad P_Q^{(n)}l(h) = l(h)(1 \otimes P_Q^{(n)}).
$$
By applying (4), for \( f \in \mathcal{H}^{\otimes n} \)

\[
\langle a^+(h)f, g \rangle_Q = \langle a^+(h)f, P_Q^{(n+1)}g \rangle_Q \\
= \langle f, l(h)P_Q^{(n+1)}g \rangle_Q \\
= \langle f, l(h) (1 \otimes P_Q^{(n)}) R_Q^{(n+1)}g \rangle_Q \\
= \langle f, P_Q^{(n)}l(h)R_Q^{(n+1)}g \rangle_Q \\
= \langle f, P_Q^{(n)} a(h)g \rangle_Q \\
= \langle f, a(h)g \rangle_Q.
\]

This proves that \( a(h) \) and \( a^+(h) \) are adjoints with respect to the \( Q \)-inner product.

We now prove the bound on \( \|a^+(h)\| \). Since \( \|T_i^{(n)}\| \leq q \) for each \( i \),

\[
\left\| R_Q^{(n)} \right\| \leq 1 + q + q^2 + \cdots + q^{n-1} \leq \frac{1}{1-q}. 
\]

Thus,

\[
P_Q^{(n+1)} P_Q^{(n+1)} = P_Q^{(n+1)} \left( P_Q^{(n+1)} \right)^* \\
= \left( 1 \otimes P_Q^{(n)} \right) R_Q^{(n+1)} \left( R_Q^{(n+1)} \right)^* \left( 1 \otimes P_Q^{(n)} \right) \\
\leq \frac{1}{(1-q)^2} \left( 1 \otimes P_Q^{(n)} \right) \left( 1 \otimes P_Q^{(n)} \right).
\]

Since \( 1 \otimes P_Q^{(n)} \) and \( P_Q^{(n+1)} \) are positive operators, it follows that

\[
P_Q^{(n+1)} \leq \frac{1}{1-q} \left( 1 \otimes P_Q^{(n)} \right).
\]

Therefore, for \( f \in \mathcal{H}^{\otimes n} \),

\[
\left\| a^+(h)f \right\|^2 = \langle a^+(h)f, a^+(h)f \rangle_Q \\
= \langle h \otimes f, h \otimes f \rangle_Q \\
= \langle h \otimes f, P_Q^{(n+1)}(h \otimes f) \rangle_Q \\
\leq \frac{1}{1-q} \langle h \otimes f, 1 \otimes P_Q^{(n)}(h \otimes f) \rangle_Q \\
\leq \frac{1}{1-q} \langle h, h \rangle_Q \langle f, P_Q^{(n)}f \rangle_Q \\
\leq \frac{1}{1-q} \langle h, h \rangle \langle f, f \rangle_Q \\
\leq \frac{1}{1-q} \left\| h \right\|^2 \left\| f \right\|^2_Q.
\]

We can represent an element \( f \) of the Fock space \( \mathcal{F}_Q(\mathcal{H}) \) as a sequence of functions \( (f^{(0)}, f^{(1)}, \ldots) \), with \( f^{(n)} \in \mathcal{H}^{\otimes n} \) and

\[
\sum_{n=0}^{\infty} \left\| f^{(n)} \right\|^2_Q < \infty.
\]
We are now ready to define the operator-valued distributions $a(x)$ and $a^+(x)$. For $f \in \mathcal{H}^{\otimes n}$, we define these by

$$[a(x)f](x_1, \ldots, x_{n-1}) = \left(R_Q^{(n+1)} f^{(n+1)}\right)(x, x_1, \ldots, x_{n-1})$$

$$[a^+(x)f](x_1, \ldots, x_{n+1}) = \delta(x - x_1)f^{(n-1)}(x_2, \ldots, x_{n+1}).$$

These definitions, of course, make no sense as functions, but should be interpreted as distributions on $C^\infty_c(U)$. It is an immediate consequence of the definitions that

$$a(h) = \int_U \overline{h(x)} a(x) \, dx \quad \text{and} \quad a^+(h) = \int_U h(x) a^+(x) \, dx,$$

for functions $h \in C^\infty_c(U)$. These relations are understood rigorously in terms of the corresponding quadratic forms. That is, for $f \in \mathcal{H}^{\otimes n}$ and $g \in \mathcal{H}^{\otimes (n-1)}$,

$$\langle f, a^+(h)g \rangle_Q = \int_U h(x) \langle f, a^+(x)g \rangle_Q \, dx$$

$$= \int_U h(x) \int_{U^{n-1}} \left(P_Q^{(n)} f\right)(x_1, \ldots, x_n) \delta(x - x_1)g(x_2, \ldots, x_n) dx_1 \ldots dx_n dx,$$

and similarly for $a(h)$:

$$\langle g, a(h)f \rangle_Q = \int_U h(x) \langle g, a(x)f \rangle_Q \, dx$$

$$= \int_U h(x) \int_{U^{n-1}} \left(P_Q^{(n-1)} g\right)(x_1, \ldots, x_{n-1}) \overline{R_Q^{(n)} f}(x, x_1, \ldots, x_{n-1}) dx_1 \ldots dx_{n-1} dx.$$

It now follows from a simple computation that these operator-valued distributions satisfy the $Q$-commutation relations [1].

3. THE $Q$-DEFORMED FREE GROUP VON NEUMANN ALGEBRAS

We now define the main operators of interest, the field operators $w(h)$ by

$$w(h) := a^+(h) + a(h) \quad \text{for} \quad h \in \mathcal{H}.$$  

This allows us to define the $Q$-deformed free group von Neumann algebra by

$$\Gamma_Q(\mathcal{H}) := \{w(h) : h \in \mathcal{H}\}''.$$  

Before proving anything about these algebras, we will need some additional notation. We will sometimes let $a^-(h)$ denote $a(h)$ so that we can write $a^v(h)$ for $v \in \{-, +\}$ to denote either the annihilation or creation operator.

Given a finite ordered set $S$, we will denote the set of pairings of $S$ by $P(S)$. That is, $P(S) = \emptyset$ if $S$ has odd cardinality, and if $|S| = 2p$ then

$$P(S) = \{(a_1, z_1), \ldots, (a_p, z_p) | a_1 < z_1, \ldots, a_p < z_p, \{a_1, \ldots, a_p, z_1, \ldots, z_p\} = S\}.$$  

We will denote by $I(\mathcal{V})$ the set of crossings of a pairing $\mathcal{V}$, that is, for $\mathcal{V} = \{(a_1, z_1), \ldots, (a_p, z_p)\}$,

$$I(\mathcal{V}) = \{(k, l) \in \{1, \ldots, r\}^2 | a_k < a_l < z_k < z_l\},$$

where the inequalities are in the ordering given on $S$.

For a pairing $\mathcal{V} \in P(S)$ for $S \subset \{1, \ldots, n\}$, we define a function $Q^\mathcal{V}$ on $U^n$ by

$$Q^\mathcal{V}(x) = \prod_{(k,l) \in I(\mathcal{V})} Q(x_k, x_l).$$

We will simplify notation by writing

$$\delta^\mathcal{V}(x) = \prod_{(a,z) \in \mathcal{V}} \delta(x_a - x_z).$$

Note that the $\delta$ on the right side is the Dirac delta.
Proposition 2. Let \( f_1, \ldots, f_n \in \mathcal{H} \) and denote by \( S \) the set \( \{1, \ldots, n\} \). For \( v_1, \ldots, v_n \in \{-, +\} \)

\[
\langle a^{v_n}(f_n) \cdots a^{v_1}(f_1) \Omega, \Omega \rangle = \sum_{\mathcal{V} \in P(S)} D_{\mathcal{V}, \mathcal{V}} \int \cdots \int f_n(x_n) \cdots f_1(x_1) Q^v_\mathcal{V}(x) \delta^v_\mathcal{V}(x) dx_1 \cdots dx_n,
\]

where if \( n = 2p \), \( D_{\mathcal{V}, \mathcal{V}} \) is defined by

\[
D_{\mathcal{V}, \mathcal{V}} = \prod_{k=1}^p \delta_{v_k,-} \cdot \delta_{v_k,+}.
\]

In particular, \( \langle w(f_n) \cdots w(f_1) \Omega, \Omega \rangle = 0 \) when \( n \) is odd.

Proof. The proof of \( w \) by induction on \( N := \{\{(j, k) : j < k, v_j = +, v_k = -\}\} \). The claim is easily seen to be true in the case \( N = 0 \), so we proceed to assume that \( N > 0 \) and that the claim holds for \( N - 1 \). We will assume that \( f_1, \ldots, f_n \) lie in the dense subspace \( C_\infty(U) \) of \( \mathcal{H} \) and then use the \( Q \)-commutation relation \( [\mathcal{F}, \mathcal{G}] \). Since \( N > 0 \), we can choose \( j \) minimal to satisfy \( v_j = + \) and \( v_{j+1} = - \). Now applying \( [\mathcal{F}, \mathcal{G}] \),

\[
a^{v_n}(x_n) \cdots a^{v_1}(x_1) = a^{v_n}(x_n) \cdots a^{v_{j+2}}(x_{j+2}) a(x_{j+1}) a(x_{j+1}) a^{v_{j-1}}(x_{j-1}) \cdots a^{v_1}(x_1)
\]

\[
= a^{v_n}(x_n) \cdots a^{v_{j+2}}(x_{j+2}) Q(x_j, x_{j+1}) a^+(x_j) a(x_{j+1}) + \delta(x_j, x_{j+1}) a^{v_{j-1}}(x_{j-1}) \cdots a^{v_1}(x_1)
\]

\[
= Q(x_j, x_{j+1}) a^{v_n}(x_n) \cdots a^{v_{j+2}}(x_{j+2}) a^+(x_j) a(x_{j+1}) a^{v_{j-1}}(x_{j-1}) \cdots a^{v_1}(x_1)
\]

\[
+ \delta(x_j - x_{j+1}) a^{v_n}(x_n) \cdots a^{v_{j+3}}(x_{j+2}) a^{v_{j-1}}(x_{j-1}) \cdots a^{v_1}(x_1)
\]

(5)

We now consider the terms in the last line of (5) separately, denoting them by \( X_1 \) and \( X_2 \). For compactness of notation, we define \( S' = \{1, \ldots, j-1, j+1, j, j+2, \ldots, n\} \) (as an ordered set) and \( S = \{1, \ldots, j-1, j+2, \ldots, n\} \) and also write \( f(x) \) for the product \( f_n(x_n) \cdots f_1(x_1) \).

For the first term we have by the inductive hypothesis,

\[
\int \cdots \int f(x) (X_1 \Omega, \Omega) dx_1 \cdots dx_n = \sum_{\mathcal{V} \in P(S')} D_{\mathcal{V}, \mathcal{V}} \int \cdots \int f(x) Q(x_j, x_{j+1}) Q^v_\mathcal{V}(x) \delta^v_\mathcal{V}(x) dx_1 \cdots dx_n
\]

\[
= \sum_{\mathcal{V} \in P(S')} D_{\mathcal{V}, \mathcal{V}} \int \cdots \int f(x) Q^v_\mathcal{V}(x) \delta^v_\mathcal{V}(x) dx_1 \cdots dx_n,
\]

For the second term,

\[
\int \cdots \int f(x) (X_2 \Omega, \Omega) dx_1 \cdots dx_n = \sum_{\mathcal{V} \in P(S')} D_{\mathcal{V}, \mathcal{V}} \int \cdots \int \delta(x_j - x_{j+1}) f(x) Q^v_\mathcal{V}(x) \delta^v_\mathcal{V}(x) dx_1 \cdots dx_n
\]

\[
= \sum_{\mathcal{V} \in P(S')} D_{\mathcal{V}, \mathcal{V}} \int \cdots \int f(x) Q^v_\mathcal{V}(x) \delta^v_\mathcal{V}(x) dx_1 \cdots dx_n.
\]

The proposition now follows just by adding the results of the two computations just completed. \( \square \)

Corollary 1. Let \( f_1, \ldots, f_n \) and \( S \) be as in Proposition 2. Then

\[
\langle w(f_n) \cdots w(f_1) \Omega, \Omega \rangle = \sum_{\mathcal{V} \in P(S)} \int \cdots \int f_n(x_n) \cdots f_1(x_1) Q^v_\mathcal{V}(x) \delta^v_\mathcal{V}(x) dx_1 \cdots dx_n,
\]

Proof. Sum the formula of Proposition 2 over all choices of \( v_1, \ldots, v_n \). \( \square \)

Corollary 2. The vacuum state on \( \Gamma_Q(\mathcal{H}) \) is a trace.

Proof. The formula in Corollary 1 is invariant under cyclic permutations of the \( w(f_i) \). \( \square \)

Proposition 3. The vacuum vector \( \Omega \in \mathcal{F}_Q(\mathcal{H}) \) is cyclic and separating for \( \Gamma_Q(\mathcal{H}) \).
Proof. We first show that $\Omega$ is cyclic. It will suffice to show that an arbitrary $f \in \mathcal{H}^\otimes n$ is in the closure of $\Gamma_Q(\mathcal{H})\Omega$. The proof is by induction on $n$. The cases of $n = 0$ and $n = 1$ are obvious, so we assume $n > 1$ and $f \in L^2(U^n)$. If $\epsilon > 0$, we can choose $(f_{ij}) \in \mathcal{H}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, r$ such that

$$\left\| f - \sum_{j=1}^r f_{1j} \otimes \cdots \otimes f_{nj} \right\| < \epsilon/2.$$ 

But then

$$f - \sum_{j=1}^r w(f_{1j}) \cdots w(f_{nj})\Omega = \left( f - \sum_{j=1}^r f_{1j} \otimes \cdots \otimes f_{nj} \right) + g,$$

for some $g \in \bigoplus_{k=1}^{n-1} \mathcal{H}^\otimes n$. The claim now follows by applying the inductive hypothesis to $g$.

To show that $\Omega$ is separating for $\Gamma_Q(\mathcal{H})$, it will suffice to show that $\Omega$ is cyclic for $\Gamma_Q(\mathcal{H})'$. We define the anti-linear conjugation operator $J : F_Q(\mathcal{H}) \to F_Q(\mathcal{H})$ by $JX\Omega = X^*\Omega$ for $X \in \Gamma_Q(\mathcal{H})$. This operator is well-defined because by the tracial property $\|X\Omega\| = \|X^*\Omega\|$. Since $J\Gamma_Q(\mathcal{H})J$ commutes with $\Gamma_Q(\mathcal{H})$, and $\Omega$ is seen to be cyclic for $J\Gamma_Q(\mathcal{H})J$ in the same way as for $\Gamma_Q(\mathcal{H})$, the claim follows. \qed

4. THE DISCRETIZATION LEMMA

We will now show that the creation and annihilation operators $a^+ (h)$ and $a(h)$ can be realized as a limit in distribution of operators on a discrete Fock space arising from the discrete commutation relation as considered in [3]. Fix $\epsilon$ and let $U_\epsilon := U \cap \epsilon \mathbb{Z}$. We let $\mathcal{H}_\epsilon$ be a real Hilbert space with orthonormal basis \{ $e_x : x \in U_\epsilon$ \}. For $x, y \in U_\epsilon$, we define $q_{xy} = Q(x,y)$.

Bożejko and Speicher showed [3] that there is a $q_{xy}$-Fock space on $\mathcal{H}_\epsilon$ with vacuum vector $\Omega_\epsilon$, creation operators $a_n^+(f)$ and annihilation operators $a_n(e)$ for $e \in \mathcal{H}$ satisfying the discrete $q_{xy}$-commutation relation

$$a_n(e_x)a_n^+(e_y) - q_{ij}a_n^+(e_y)e_x = \delta_{xy}.1.$$ 

The creation operator $a_n^+(e_x)$ and the annihilation operator $a_n(e_x)$ are adjoints with respect to the deformed inner product on the Fock space. We will denote this Fock space by $\mathcal{F}_{Q,\epsilon}(\mathcal{H}_\epsilon)$, its vacuum vector by $\Omega_\epsilon$, and its inner product by $\langle \cdot, \cdot \rangle_{Q,\epsilon}$.

Now define $a_n(f)$ and $a_n^+(f)$ by

$$a_n(f) := \epsilon^{i/2} \sum_{x \in U_\epsilon} f(x) a_n(e_x) \quad \text{and} \quad a_n^+(f) := \epsilon^{i/2} \sum_{x \in U_\epsilon} f(x) a_n^+(e_x).$$

Evidently, $(a_n(f))^* = a_n^+(\overline{f})$.

To simplify notation, we define for a pairing $\mathcal{V}$,

$$D_{\mathcal{V}}^n(x) = \prod_{(a,z) \in \mathcal{V}} \delta_{xa,z},$$

where the $\delta_{xa,z}$ on the right side is a Kronecker delta.

Lemma 1. The family \{ $a_n(f) : f \in C_c^\infty(U)$ \} converges in joint $*$-distribution as $\epsilon \to 0$ to the family \{ $a(f) : f \in C_c^\infty(U)$ \} introduced in Section 3 where all of the distributions are with respect to the respective vacuum states.

Proof. We will use the fact, as shown by Bożejko and Speicher in [3], that for $v_1, \ldots, v_n \in \{ +, - \}$,

$$\langle a_n^{v_n}(x_n) \cdots a_n^{v_1}(x_1)\Omega_\epsilon, \Omega_\epsilon \rangle = \sum_{V \in P(S)} D_{\mathcal{V}}D_{\mathcal{V}}^n(x) \prod_{(k,j) \in I(\mathcal{V})} q_{x_k,x_{n_k}},$$
where $S = \{1, \ldots, n\}$ and $D_{\nu, V}$ is as in Proposition 2. Again writing $f(x)$ for the product $f_n(x_n) \cdots f_1(x_1)$, we have that

$$
\lim_{\epsilon \to 0} \langle a^{\nu_1}_e(f_n) \cdots a^{\nu_n}_e(f_1)\Omega, \Omega \rangle = \lim_{\epsilon \to 0} \epsilon^{n/2} \sum_{x \in U^\nu_n} \langle f(x) a_{\nu_1}^e(e_{x_n}) \cdots a_{\nu_1}^e(e_{x_1})\Omega, \Omega \rangle
$$

$$
= \lim_{\epsilon \to 0} \epsilon^{n/2} \sum_{x \in U^\nu_n} f(x) \sum_{v \in P(S)} D_{\nu, V} \tilde{D}_v^\nu(x) \prod_{(k, l) \in I(V)} q_{x_k, x_l}
$$

$$
= \int \cdots \int f(x) \sum_{v \in P(S)} D_{\nu, V} \tilde{D}_v^\nu(x) Q_v^\nu(x) dx_1 \cdots dx_n
$$

$$
= \langle a^{\nu_1}_e(f_n) \cdots a^{\nu_1}_e(f_1)\Omega, \Omega \rangle.
$$

We conclude this section by noting that the inner product on $F_{Q, \epsilon}(H)$ is defined using positive operators $P_{Q, \epsilon}^{(n)}$ on $H^\otimes n$ such that

$$
\langle \xi, \eta \rangle_{Q, \epsilon} = \langle \xi, P_{Q, \epsilon}^{(n)} \eta \rangle_{0, \epsilon},
$$

for $\xi, \eta \in H^\otimes n$, where $\langle \cdot, \cdot \rangle_{0, \epsilon}$ denotes the inner product of the Free fock space on $H$. Since we have assumed that $\sup_{x, y} |Q(x, y)| < 1$, there is an operator $P_{Q, \epsilon}^{(n)}$ of norm at most $(1 - q)^{-1}$ such that $P_{Q, \epsilon}^{(n+1)} = (1 \otimes P_{Q, \epsilon}^{(n)}) R_{Q, \epsilon}^{(n)}$. One can use this to show that $P_{Q, \epsilon}^{(n+1)} \leq (1 - q)^{-1} (1 \otimes P_{Q, \epsilon}^{(n)})$ for all $\epsilon$.

5. The factoriality result

To state our main theorem, we will need to introduce the right field operator $w_r(f)$ for $f \in H$. We define

$$
w_r(f) = Jw(f)J,
$$

where $J : F_Q(H) \to F_Q(H)$ is the canonical antilinear isometry defined by $J(X\Omega) = X^*\Omega$. Equivalently,

$$
w_r(f) = a_r(f) + a_r^+(f),
$$

where $a_r(f)$ and $a_r^+(f)$ are the right annihilation and right creation operators defined analogously to the left annihilation and left creation operators.

**Theorem 1.** Let $g_1, g_2, \ldots \in C^\infty_c(U)$ be real-valued functions with $g_i g_j = 0$ for $i \neq j$ and $\|g_i\|_2 = 1$. For each $d > 0$, define

$$
N_d = \sum_{i=1}^d (w(g_i) - w_r(g_i))^2.
$$

Then for $d$ sufficiently large, $\ker N_d = \mathbb{C}\Omega$ and $N_d > \epsilon l$ on $F_Q(H) \ominus \mathbb{C}\Omega$ for some $\epsilon > 0$.

In view of a theorem of Connes [4], this theorem will have the following consequence. It was pointed out to us by an anonymous referee that this follows immediately from the main theorem of Krolak in [6].

**Corollary 3.** The von Neumann algebra $\Gamma_Q(H)$ is a factor which does not have property $\Gamma$.

**Proof.** Choose $N_d$ large enough that $N_d > \epsilon l$ on the orthogonal complement of the vacuum subspace. If $X \in \Gamma_Q(H) \cap \Gamma_Q(H)'$ then $(w(g_i) - w_r(g_i))X = 0$ for $i = 1, \ldots, d$. Thus $X\Omega \in \ker N_d = \mathbb{C}\Omega$. Since $\Omega$ is separating, $X \in \mathbb{C}$. Thus, $\Gamma_Q(H)$ is a factor of Type $II_1$. By Theorem 2.1 of [4], $\Gamma_Q(H)$ does not have property $\Gamma$. □

Our method of proof of Theorem 4 will be similar to that used by Krolak [6] and will require some estimates.

**Proposition 4.** For each $n$, define operators

$$
\mathcal{L}_n : H \otimes H^\otimes (n-1) \to H^\otimes (n-2) \quad \text{and} \quad \mathcal{R}_n : H^\otimes (n-1) \otimes H \to H^\otimes (n-2)
$$

by

$$
\mathcal{L}_n(h \otimes f) = l(h)f \quad \text{and} \quad \mathcal{R}_n(f \otimes h) = r(h)f,
$$
where \( l(f) \) and \( r(f) \) are the free left and right annihilation operators, respectively acting on \( \mathcal{H}^{\otimes(n-1)} \) as a subspace of \( \mathcal{F}_Q(\mathcal{H}) \). Suppose that \( g \in \mathcal{H} \) with \( \|g\| = 1 \) and define \( D \) on \( \mathcal{H}^{\otimes n} \) by \( D(f) = g \otimes f \otimes g \). Then

\[
\left\| L_{n+2} T_2^{(n+2)} \cdots T_{n+1}^{(n+2)} D \right\|_Q \leq q^n \quad \text{and} \quad \left\| R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \right\|_Q \leq q^n.
\]

**Proof.** We will prove the second statement, and the first can be proven analogously. Our approach is similar to that of Lemma 7 in [6]. Namely, we will begin by showing that the operator \( R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \) commutes with \( \phi_Q^{(n)} \), the operator used to define the \( Q \)-inner product in Section 2. For this, it will suffice to show that \( R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \) commutes with \( \phi_n(\sigma) \) for each \( \sigma \in S_n \), where \( \phi_n : S_n \to \mathcal{H}^{\otimes n} \) is as in Section 2. By quasimultiplicativity of \( \phi_n \), we can further assume that \( \sigma \) is one of the fundamental transpositions \( \pi_k \). Using the relation [2], we have

\[
R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \phi_k(\pi_k) = R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) DT_k
\]

Therefore, \( R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \) commutes with \( \phi_Q^{(n)} = \sum_{\sigma \in S_n} \phi_n(\sigma) \). In particular, this means that

\[
\left\| R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \phi_n(\sigma) \right\|_Q = \left\| R_{n+2} (T_n^{(n+2)} \cdots T_1^{(n+2)}) D \phi_n(\sigma) \right\|_0
\]

\[
\leq \|R_{n+2}\|_0 \left\| (T_1^{(n+2)} \cdots T_n^{(n+2)}) \right\|_0 \|D \phi_n(\sigma)\|_0
\]

\[
\leq 1 \cdot q^n \cdot 1.
\]

In the last line, we have used the fact that \( D \) is an isometry in the 0-norm and \( R_{n+2} \) is a contraction when restricted to the subspace \( \mathcal{H}^{\otimes(n+1)} \otimes g \).

The next lemma provides an analog to parts of Lemma 8 of [6].

**Lemma 2.** There is a constant \( C \), depending only on \( Q \), such that all of the following estimates hold for any orthonormal vectors \( h_1, \ldots, h_d \in \mathcal{H} \):

1. \( \left\| \sum_{i=1}^d a^+(h_i) a_i^+(h_i) \right\|_Q \leq C \sqrt{d} \) and \( \left\| \sum_{i=1}^d a(h_i) a_r(h_i) \right\|_Q \leq C \sqrt{d} \)
2. \( \left\| \sum_{i=1}^d a^+(h_i) a_r(h_i) \right\|_Q \leq C \sqrt{d} \) and \( \left\| \sum_{i=1}^d a_i^+(h_i) a(h_i) \right\|_Q \leq C \sqrt{d} \)
3. \( \left\| \sum_{i=1}^d a(h_i) a(h_i) \right\|_Q \leq C \sqrt{d} \) and \( \left\| \sum_{i=1}^d a_r(h_i) a_r(h_i) \right\|_Q \leq C \sqrt{d} \)
4. \( \left\| \sum_{i=1}^d a^+(h_i) a(h_i) \right\|_Q \leq C \sqrt{d} \) and \( \left\| \sum_{i=1}^d a_i^+(h_i) a_r(h_i) \right\|_Q \leq C \sqrt{d} \)

**Proof.** We take \( C = \frac{1}{4} \), which is large enough so that \( P_Q^{(n+1)} \leq C (1 \otimes P_Q^{(n)}) \) for all \( n \) as is established in [3]. In general, to prove that an operator \( X \) has norm at most \( K \), it will be sufficient to prove that \( \|Xf\|^2 \leq K^2 \|f\|^2 \) for all of the form \( f \in \mathcal{H}^{\otimes n} \) where \( n \geq 0 \) is arbitrary. To prove the first bound in part 1
we have
\[ \left\| \sum_{i=1}^{d} a_i^+(h_i) a_i^+(h_i) f \right\|_{Q}^2 = \left\| \sum_{i=1}^{d} h_i \otimes f \otimes h_i \right\|_{Q}^2 \]
\[ = \left\langle \sum_{i=1}^{d} h_i \otimes f \otimes h_i, \sum_{j=1}^{d} P_Q^{(n+2)} h_j \otimes f \otimes h_j \right\rangle_Q \]
\[ \leq C^2 \left\langle \sum_{i=1}^{d} h_i \otimes f \otimes h_i, \sum_{j=1}^{d} (1 \otimes P_Q^{(n)} \otimes 1) h_j \otimes f \otimes h_j \right\rangle_0 \]
\[ = C^2 \sum_{i,j=1}^{d} (h_i, h_j) \langle f, h_i, h_j \rangle \left\langle f, P_Q^{(n)} f \right\rangle_0 \]
\[ = C^2 \sum_{i=1}^{d} \left\langle f, P_Q^{(n)} f \right\rangle_0 \]
\[ = dC^2 \| f \|_{Q}^2 \]

For the second bound in part 1, we have that
\[ \left\| \sum_{i=1}^{d} a(h_i) a_r(h_i) \right\|_{Q} = \left\| \left( \sum_{i=1}^{d} a_i^+(h_i) a_i^+(h_i) \right)^* \right\|_{Q} = \left\| \sum_{i=1}^{d} a_i^+(h_i) a_i^+(h_i) \right\|_{Q} \leq C \sqrt{d}, \]

where in the last line we have used the first bound in part 1.

The proof of the first bound in part 2 is similar:
\[ \left\| \sum_{i=1}^{d} a_i^+(h_i) a_r(h_i) f \right\|_{Q}^2 = \left\| \sum_{i=1}^{d} h_i \otimes a_r(h_i) f \right\|_{Q}^2 \]
\[ = \sum_{i,j=1}^{d} \left\langle P_Q^{(n)} (h_i \otimes a_r(h_i) f), h_j \otimes a_r(h_j) f \right\rangle_Q \]
\[ \leq C \sum_{i,j=1}^{d} \left\langle (1 \otimes P_Q^{(n-1)}) (h_i \otimes a_r(h_i) f), h_j \otimes a_r(h_j) f \right\rangle_Q \]
\[ \leq C \sum_{i=1}^{d} \left\langle P_Q^{(n-1)} a_r(h_i) f, a_r(h_i) f \right\rangle_Q \]
\[ \leq C \sum_{i=1}^{d} \| a_r(h_i) f \|_{Q}^2 \]
\[ \leq dC^2 \| f \|_{Q}^2. \]

The arguments used to prove the second inequality in part 2 and all the remaining estimates are similar to those cases just completed. \qed

We will need one additional bound, which is analogous to the last part of Lemma 8 of [6].

**Proposition 5.** If \( h_1, \ldots, h_d \in C_c^\infty(U) \) are such that \( \| h_i \|_2 = 1 \) and \( h_i h_j = 0 \) for \( i \neq 0 \) then there is a constant \( C \), depending only on \( Q \), such that
\[ \left\| \sum_{i=1}^{d} (a(h_i) a_i^+(h_i) - 1) \right\| \leq C q \sqrt{d} \quad \text{and} \quad \left\| \sum_{i=1}^{d} (a_r(h_i) a_i^+(h_i) - 1) \right\| \leq C q \sqrt{d}. \]
Proof. We will prove the first estimate; the proof of the second is analogous. It will suffice to show that for $f = \sum_{j \in J} f_{ij} \otimes \cdots \otimes f_{nj}$ with $f_{ij}, \cdots, f_{nj} \in C^\infty_c(U)$,

$$\left\| \sum_{i=1}^d (a(h_i) a^+(h_i) - 1) f \right\|_Q^2 \leq q^2 C^2 d \|f\|_{Q,e}^2.$$  

To prove this result, we will make use of Lemma 1, which implies that in the notation of Section 4

$$\left\| \sum_{i=1}^d (a(h_i) a^+(h_i) - 1) f \right\|_Q^2 = \lim_{\epsilon \to 0} \left\| \sum_{i=1}^d (a(h_i) a^+_e(h_i) - 1) \sum_{j \in J} a^+_e(f_{ij}) \cdots a^+_e(f_{nj}) \Omega_e \right\|_{Q,e}^2,$$

We again choose $C = \frac{1}{e_q}$. For this choice of the constant, we have $P^{(n)}_{Q,e} \geq C(1 \otimes P_{Q,e}^{(n-1)})$ and also $P_{Q,e}^{(n)} \geq C(P_{Q,e}^{(n-1)} \otimes 1)$. We define $f_e = \sum_{j \in J} a^+_e(f_{ij}) \cdots a^+_e(f_{nj}) \Omega_e$, fix $\epsilon > 0$, and denote $\sum_{i=1}^d (a(h_i) a^+_e(h_i) - 1)$ by $V_e$. Applying the discrete commutation relations and rearranging terms,

$$\|V_e f_e\|_{Q,e} = \left\| \sum_{i=1}^d \left( e^j \sum_{x_1, x_2 \in U_e} h_i(x_1) h_i(x_2) a_e(e_x) a^+_e(e_{x_2}) - 1 \right) f_e \right\|_{Q,e}$$

$$= \left\| \sum_{i=1}^d \sum_{x_1, x_2 \in U_e} e^j h_i(x_1) h_i(x_2) \left( Q(x_1, x_2) a^+_e(e_{x_2}) a_e(e_{x_1}) + \delta_{x_1, x_2} \right) f_e \right\|_{Q,e}$$

$$\leq \sum_{i=1}^d \sum_{x_1, x_2 \in U_e} e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) a^+_e(e_{x_2}) a_e(e_{x_1}) f_e + \sum_{i=1}^d \sum_{x_1 \in U_e} \|h_i(x_1)\|^2 f_e \right\|_{Q,e}.$$

Since $\|h_i\|^2 = 1$, the second term in the last line converges to 0 as $\epsilon \to 0$, whence we need only show that the first term has the needed bound in the limit. Denoting this term by $S_e$, we have

$$S_e^2 = \left\| \sum_{x_1, x_2 \in U_e} e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) a^+_e(e_{x_2}) a_e(e_{x_1}) f_e \right\|_{Q,e}^2$$

$$= \left\| \sum_{x_1, x_2 \in U_e} e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) e_{x_2} \otimes a_e(e_{x_1}) f_e \right\|_{Q,e}^2$$

$$\leq C \sum_{x_2 \in U_e} \left\| \sum_{x_1 \in U_e} e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) a_e(e_{x_1}) f_e \right\|_{Q,e}^2.$$

Here we have used the fact that $P_{Q,e}^{(n+1)} \leq C(1 \otimes P_{Q,e}^{(n)})$. To further simplify this bound, we use the fact that the adjoint map is an isometry and then make use of our choice of $C$ again:

$$S_e^2 \leq C \sum_{x_2 \in U_e} \left\| \sum_{x_1 \in U_e} e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) a^+_e(e_{x_1}) \right\|_{Q,e}^2 \|f_e\|_{Q,e}^2$$

$$\leq C \sum_{x_2 \in U_e} \left\| h_i(x_2) \sum_{x_1 \in U_e} e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) e_{x_1} \otimes g_{x_2} \right\|_{Q,e}^2 \|f_e\|_{Q,e}^2$$

$$\leq C^2 \sum_{x_1, x_2 \in U_e} \left\| \sum_{i=1}^d e^j h_i(x_1) h_i(x_2) Q(x_1, x_2) \right\|_{Q,e}^2 \|f_e\|_{Q,e}^2$$

$$\leq C^2 \sum_{x_1, x_2 \in U_e} \sum_{i=1}^d e^j h_i(x_1)^2 h_i(x_2)^2 |Q(x_1, x_2)|^2 \|f_e\|_{Q,e}^2.$$
Proof of Theorem 7] Expanding the definition of $N_d$ we have,

$$N_d = \sum_{i=1}^{d} \left( a^+(g_i) a^+(g_i) + a(g_i) a(g_i) + a^+(g_i) a(g_i) + a(g_i) a^+(g_i) \right)$$

$$+ \sum_{i=1}^{d} \left( a^+(g_i) a^+(g_i) + a_r(g_i) a_r(g_i) + a_r^+(g_i) a_r(g_i) + a_r(g_i) a_r^+(g_i) \right)$$

$$- \sum_{i=1}^{d} \left( 2a^+(g_i) a^+(g_i) + 2a(g_i) a_r(g_i) + a^+(g_i) a_r(g_i) + a(g_i) a_r^+(g_i) \right)$$

$$- \sum_{i=1}^{d} \left( a^+(g_i) a(g_i) + a_r(g_i) a^+(g_i) \right).$$

Here we have used the fact that $a^+(g_i) a^+(g_i) = a^+_r(g_i) a^+(g_i)$ and likewise for the left and right annihilation operators.

For each $i$, we denote by $D_i$ the map on $F_Q(\mathcal{H})$ given by linear extension of $f \mapsto g_i \otimes f \otimes g_i$ for $f \in \mathcal{H}^{\otimes n}$. By the definition of the left and right annihilation operators,

$$a(g_i) a^+_r(g_i) f = (a(g_i) f) \otimes g_i + \mathcal{L}_{n+2}(T_2^{(n+2)} \cdots T_{n+1}^{(n+2)}) D_i(f)$$

and

$$a_r(g_i) a^+(g_i) f = g_i \otimes (a_r(g_i) f) + \mathcal{R}_{n+2}(T_n^{(n+2)}) D_i(f),$$

for $f \in \mathcal{H}^{\otimes n}$, where $\mathcal{R}_{n+2}$ and $\mathcal{L}_{n+2}$ are as in Proposition 3. Now defining

$$B_1 := -2d + \sum_{i=1}^{d} (a(g_i) a^+_r(g_i) + a_r(g_i) a^+_r(g_i)),$$

we have by Proposition 5 that $\|B_1\| \leq 2Cq \sqrt{d}$ on $F_Q(\mathcal{H}) \otimes \mathbb{C}$. Define also

$$B_2 := \sum_{i=1}^{d} \left( \mathcal{R}_{n+2}(T_n^{(n+2)}) D_j(f) + \mathcal{L}_{n+2}(T_2^{(n+2)} \cdots T_{n+1}^{(n+2)}) D_j(f) \right).$$

By Proposition 4 we have $\|B_2\| \leq 2qd$. Finally letting

$$B_3 := N_d - 2d - B_1 + B_2,$$

we have by Lemma 2 that $\|B_3\| \leq 14C \sqrt{d}$. This yields an inequality of operators,

$$N_d |_{F_Q(\mathcal{H}) \otimes \mathbb{C}} \geq 2d(1 - q) - 2C \sqrt{q \cdot d} - 14C \sqrt{d}.$$

The expression on the right is positive for sufficiently large $d$. \qed

Remark 1. We have assumed throughout that $q := \sup_{x,y \in U} |Q(x,y)| < 1$. However, we can easily extend the construction to the case of $q = 1$. Write $U = \bigcup_{i \in I} B(x_i, r_i)$ where $B(x_i, r_i)$ denotes the open ball of radius $r_i$ centered at $x_i \in \mathbb{R}^d$. For each $N$, define $U_N := \bigcup_{i \in I} B \left( x_i, \frac{N}{N+1} r_i \right)$. Then

$$\sup_{x,y \in U_N} |Q(x,y)| \leq \sup_{x,y \in U_N} |Q(x,y)| < 1,$$

so we can define $\mathcal{H}_N := L^2(U_N)$ and apply the construction to get a factor $\Gamma_Q(\mathcal{H}_N)$. Moreover, we have a natural inclusion $\Gamma_Q(\mathcal{H}_N) \subseteq \Gamma_Q(\mathcal{H}_{N+1})$, so we can define $\bigcup_{N \in \mathbb{N}} \Gamma_Q(\mathcal{H}_N)$. The Fock space $F_Q(\mathcal{H})$ can be constructed by the GNS construction. Finally, by choosing the functions $g_1, g_2, \ldots$ in Theorem 1 to be
supported in some $U_N$, we see that we can construct an operator as in Theorem [1] so that Corollary [3] holds as well.

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