On signed generators of groups and algebras
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Abstract
Operators acting on the discrete random chaos yield signed multiplicative systems, extending the notion of spin matrices and quaternions. We investigate signed groups through the associated sign matrices, focusing on generators and their replacements. Of particular interest are anticommutative generators leading to a complete classification of the generated groups. The classification of finite groups of mixed commutativity is also obtained.

Keywords
signed groups and algebras, sign matrices, binary matrices, anticommutativity, signatures, quantum operators, dyadic orthogonality, Pauli spin matrices, quaternions

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1 Introduction

Studies of anticommuting matrices, with additional properties such as antisymmetry, orthogonality, roots of identity, etc., can be traced to [1], if not to older times, and have been attracted wide interest among both mathematicians and physicists. Even most recent works still refer to Hurwitz-Radon theorem and their consequences as well as to new approaches (cf. e.g., [2]). Anticommutativity $AB = -BA$ is equivalent to the ‘Pythagorean formula’

\[(A + B)^2 = A^2 + B^2,\]

and hence it suggests a specific notion of orthogonality. For example, if $A$ and $B$ are complex matrices, and one of them (say, $B$) is orthogonal, then (1) implies that $A$ and $B$ are orthogonal in the classical sense ($AB = -BA \Rightarrow B^*A = -AB^* \Rightarrow \text{tr} (AB^*) = 0$). Otherwise, the new type of orthogonality, imposed by (1) (or anticommutativity), may defy intuition (see Example 2.1 below).

These ideas also emerge in the following context. The Hilbert space $L^2[0,1]$ admits a representation by Walsh series which are products of Rademacher functions and form an orthonormal basis. Analytic aspects lead to discrete Fourier analysis (cf. [3]) while connections to quantum probability and physics entail so called ‘toy Fock spaces’, introduced by K.R. Parthasarathy [4] and further investigated by P-A. Meyer [5] (for more recent developments see [6]). Three operators on the toy Fock space: conservation or number, creation, and annihilation, have become the core of the theory. Yet, their commutative structure is very complicated, although the involved calculus reveals a great deal of orderly behavior (cf. [4, 5, 7]). On the other hand, just two simple symmetries, akin to rotations, exhibit a simple commutativity and anticommutativity pattern and span all possible compositions of the quantum operators. And, yes, they were also considered by Hurwitz. This particular feature led to the notion of a signed multiplicative system [7] which captures the simplest aspects of these and also previously considered objects and actions. On the grounds of the group theory, the definition follows.

A **signed group** $S$, in addition to the unit 1, contains an element denoted by $-1$, commuting with all other elements; the group’s members either commute or anticommutate; and every member $e$ admits a **signature** $e^2 \in \{-1, 1\}$. Subsets of $S$ or sequences of

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elements of $G$ will be also called signed. Members $e \in G$ are called positive or negative according to their signatures $e^2$, and the attribute extends to sets or sequences. We call a set or sequence pure, if it is either positive or negative, otherwise we call it mixed.

Our intention is to examine such simple axiomatic system on its own, without referring to deeper and more complicated theories that may lie behind it, as far as it is possible. Consider finitary 0-1 sequences $p = (p_n)$, i.e., $p_n = 0$ eventually. A signed sequence $e = (e_n)$ generates the subgroup $\mathcal{G}(e)$ of powers

$$e^p = \prod_n e_n^{p_n} \quad \text{(by convention, } e^0 = 1 \text{ and } e^1 = e).$$

A set $F \subset G$ is called basic and its elements are said to be independent if no product of the set's elements yields ±1. We will always assume that generators $e$ spanning $\mathcal{G}(e)$ are basic. To reiterate: 'generator' = 'basic generator'. Then each member of $\mathcal{G}(e)$ appears as the unique power. A generator of size $n$ entails the group of order $2^{n+1}$. Conversely, every finite signed group of order greater than 2 admits a maximal generator, and therefore the group must have order $2^{n+1}$ for some $n \geq 0$. The case $n = 0$ corresponds to $\mathcal{G}_0 = \{ \pm 1 \}$. A generator is called a replacement of another generator if they generate the same group. The signatures and commutativity may vary among replacements. We identify generators that can be replaced trivially by change of signs or by permutations. For example, we identify the sequences $(\pm e_1, \pm e_2, \pm e_3)$ and the set $\{ e_1, e_2, e_3 \}$. The reason is that the main use of such generators (e.g., consisting of functions, matrices, operators, etc.) is to span the real or complex vector algebra of polynomials

$$\left\{ \sum_p a_p e^p : a_p = 0 \text{ for all but finitely many indices} \right\}$$

(such vector algebra also will be called signed), where signs or permutations are irrelevant. Let us emphasize that signs should not be confused with signatures.

In Section 2, we standardize the overall notation, and list necessary facts about signed groups. We indicate connections to Pauli spin matrices, quaternions, and Clifford algebras.

Section 3 contains a basic dyadic linear algebra that is needed to analyze the distribution of signatures and signs reflecting commutativity and anticommutativity. The invariance of anticommutativity under replacements is examined, leading to the notion of dyadic orthogonality. The presence of at least two anticommuting elements in a signed group entails possibly complex sign arrangements. Yet, desirably, one generator might be replaced by a 'better' one that would ensure transparency and order.

In Section 4 we focus on an anticommutative generator and classify groups with respect to the distribution of signatures. Up to an isomorphism and/or replacement, there are three types of odd sized generators, two types of even sized generators, and only one of an infinite (countable) generator.

In the first part of Section 5 we set aside the signatures in a generator as they have no effect on the commutativity matrix and find replacements and their partitions that yield orderly patterns of the generated group. We show that every finite signed group is proper, that is, it admits a generator $K \cup M$, where $K$ and $M$ commute, $K$ is anticommutative, and $M$ is commutative. Additional patterns are examined, in particular, involving a partition of the anticommutating generator $K$ into a union of two types of doubletons, Pauli-like and quaternion-like. We classify all finite signed groups, also with respect to signatures.

## 2 Notation and basic facts

Sequences or vectors will be denoted by a bold sanserif font. As long as the operations on elements are well defined, all operations between sequences are understood componentwise,
e.g., $pq = (p_nq_n)$. The only exception is the power $e^p = e^{p_1}e^{p_2} \cdots$. For the sequence of powers we will use the left superscript $e^p = (e^{p_1}, e^{p_2}, \ldots)$. If we can add components, then we write $p = \langle p \rangle = \sum p_n n$.

Consider the set $\mathbb{D} = \{ p = (p_n) \in \{0, 1\}^\mathbb{N} : d_n = 0 \text{ eventually} \}$ of finitary 0-1 sequences. The set is equipped with the linear lexicographic order that coincides with the natural order of $\mathbb{N}$ through the binary representation $p \leftrightarrow n = \sum_{j=1}^{\infty} p_j 2^{j-1}$, as well as with the partial componentwise order that coincides with the inclusion relation between finite subsets of $\mathbb{N}$, $K \leftrightarrow p$ with $p_n = \mathbb{1}_K(n)$.

We often toggle between the multiplicative and additive notation:

$$\{ -1, 1 \} \ni c \leftrightarrow d = \frac{1-c}{2} \in \{0, 1\}, \text{ and also } c = (-1)^d.$$  

(2)

(The function $c \rightarrow d$ may be perceived as the main branch of ‘the logarithm to base $-1$’ on $\{ \pm 1 \}$.) Additional notation will be introduced as needed, typically at the beginning of each section, and a local notation may appear right before the corresponding statement.

Let us explain how the concept emerged (cf. [7]). The real Hilbert space $\mathbb{H} = L^2[0,1]$, equipped with the Walsh orthonormal basis, has been often viewed in the literature as the closure of ‘discrete random chaos’, a linearly dense commutative algebra spanned by random signs. The random signs can be modeled by Rademacher functions that stem from the restriction of the condensed square wave, i.e., the periodic odd extension $r(t)$ of the indicator function $\mathbb{1}_{[0,1]}$:

$$r_n(t) = r(2^n t)\big|_{[0,1]}.$$  

A finitary 0-1 sequence $p$ and the Rademacher sequence $r = (r_n)$ yield the corresponding Walsh functions

$$w(p) = r^p = r_1^{p_1}r_2^{p_2} \cdots \leftrightarrow w_n.$$  

Bounded or even unbounded linear operators can be defined directly on the basis $r^p$. P.-A. Meyer [3] introduced and investigated ‘quantum operators’, calling the Rademacher chaos a ‘toy Fock space’. Meyer’s original notation was set-theoretic rather than algebraic. The number or conservation operator $N_j r^p = p_j r^p$ records the occurrences of the variable, the annihilation operator $D_j r^p = p_j r_j r^p$ removes the variable when present in a term or the entire term lacking that variable, the creation operator $1D_j r^p = (1-p_j) r_j r^p$ adds the variable when missed or puts aside the term containing the variable.

The generated algebra is isomorphic to $B(\mathbb{C}^2)$, associating our operators via the usual symbols of quantum mechanics (cf. [3]):

$$b^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leftrightarrow D, \quad b^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \leftrightarrow 1D, \quad n = b^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow N.$$  

(The subscript $j$ is fixed and then suppressed). The composition structure of the sequences $\mathbb{N}, \mathbb{D}, \mathbb{D}$ is rather complex (they are just members of the standard yet sometimes unfriendly basis). In contrast, the simple symmetries,

$$R_j r^p = r_j r^p \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_j = S_j r^p = (1-2p_j) r^p \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  

(together with the identity and their product $A = RS = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$) form a real anticommutative basis of $B(\mathbb{C}^2)$, although with mixed signatures: $R^2 = S^2 = 1, A^2 = -1$. One can recognize the first and the third of the three Pauli spin matrices (they were already present and put to work in Hurwitz’ paper [3]). The products $\pm R^p S^p$ form a signed group, as defined in Introduction:

$$R^p S^p = (-1)^{p} S^p R^p \Rightarrow \left( R^p S^p \right) \left( R^q S^q \right) = (-1)^{pq} R^{p+1} S^{p+q}$$  

(3)
In [7], after suppressing letters and leaving plain indices, the assignment \((0_1) = R, (1_0) = S\) entailed a signed group of double 0-1 finitary sequences \((s_p) \in \mathbb{R}^s \mathbb{S}^p\) that was called a ‘double logic’. This group is universal in the sense that it possesses a copy of every finite or countable signed system. As the original operators \(R\) and \(S\) can be viewed as simple \(2 \times 2\) matrices entailing the asymmetry \(A = RS = -SR = - (1_1)\), so they generate a doubling (or rather quadrupling) procedure. That is, the repetitive algorithm generates the quadruple offspring out of an element \(e\):

\[
\begin{align*}
(0_0) e &= (e \ 0 \ e), & (0_1) e &= (0 \ e \ 0), & (1_0) e &= (e \ 0 \ -e), & (1_1) e &= (0 \ -e \ 0).
\end{align*}
\]

This yields the representation of the double sequence \((s_p)\) of length \(n\) as the \(2^n \times 2^n\) matrix that corresponds to the iteration starting with \(e = [1]\). Although such matrices seem to be incompatible because of different dimensions for different \(n\) yet the described doubling of matrices, or equivalently, concatenation of double sequences, enables one to bring matrices of different dyadic dimensions up to par. For a fixed length \(n\), the simple count proves that the linearly independent matrices \((s_p)\) form a basis in the vector space of matrices of size \(2^n \times 2^n\).

The matrix representation includes generalizations of Pauli spin matrices and quaternions. For example, two components are needed to ‘complexify’ the group. That is, focusing on sequences of length 2, the negative element \((1_1) = (01) = S_1 R_1\) serves as ‘the imaginary number’ \(i\), i.e., the unique element besides \(\pm 1 = \pm (00)\) that commutes with the Pauli basis \(\sigma_1 = (00) = R_2, \sigma_2 = -(11) = R_1S_1S_2R_2, \sigma_3 = (10) = S_2\), or the quaternion basis \(i = i_3 = (10) = S_1R_1S_2, j = i_2 = (01) = S_2R_2, k = i_1 = (00) = S_1R_1R_2\).

Note that this specific matrix representations of quaternions, one of many possible, as well as their brief ‘double logic’ codes are the consequence of the assumed doubling algorithm. As seen above, in contrast to doubly coded implicit compositions, the explicit compositions lack transparency. Yet, the direct action on the algebraic span of the first two Rademacher functions \(r_1, r_2\) (or any pair) is clear. The full matrix representations with respect to the basis \((1, r_1, r_2, r_1r_2)\) are, as expected,

\[
\begin{align*}
i &\leftrightarrow \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}, & j &\leftrightarrow \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot \end{pmatrix}, & k &\leftrightarrow \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{pmatrix},
\end{align*}
\]

In addition, for doubles \(D, D' \in \{ (0_0), (0_1), (1_0), (1_1) \}\), we have

\[
De \cdot D' f = (DD') ef, \quad De \cdot D f = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ef.
\]

In particular, doubles of the same kind, or if one is \((0_0)\), preserve commutativity or anticommutativity while the doubles of distinct kinds, different from \((0_0)\), switch it. To produce more examples of anticommutating matrices we need to start somewhere.

**Example 2.1** We will illustrate a possible advantage of using a basis alternative to the standard, Pauli, or quaternion bases.

Let us solve the problem (cf. [8]): given a \(2 \times 2\) complex matrix \(M\) (or with entries from a commutative ring) find all anticommuting matrices \(X\), i.e., \(MX + XM = 0\). Equivalently, the ‘Pythagorean formula’ \((1)\) holds. To this end, let us choose the real basis \(\{ I, R, S, A = RS \}\), so we can easily pass to real matrices (which would be little tedious if we selected the Pauli or quaternion basis). We will see that the diagonal matrix
\( G = \text{diag}(1, 1, -1) \) of signatures determines a notion of orthogonality in \( \mathbb{C}^3 \) with the help of the bilinear form
\[
G(v, w) = v^t G w = ax + by - cz, \quad v = (a, b, c)^T, \; w = (x, y, z)^T \in \mathbb{C}^3.
\]

We write \( M = \gamma + v \), where \( v = aR + bS + cA \). This yields the G-conjugate or G-transpose \( M^c = \gamma - v \). For complex coefficients we check that
\[
(\gamma + aR + bS + cA)^2 = (\gamma^2 + a^2 + b^2 - c^2) + 2\gamma(aR + bS + cA),
\]
or \( (\gamma + v)^2 = \gamma^2 + G(v, v) + 2\gamma v \). In particular, we obtain a quadratic form on \( \mathbb{C}^4 \):
\[
MM^c = (\gamma + v)(\gamma - v) = \gamma^2 - G(v, v) = \gamma^2 - a^2 - b^2 + c^2.
\]

Also, with the help of little algebra the ‘Pythagorean formula’ reads
\[
\gamma t + ax + by - cz = 0, \quad \gamma x = -ta, \quad \gamma y = -tb, \quad \gamma z = -tc.
\]

Thus we arrive at two cases of anticommuting matrices:

1. scalar-free \( v \) and \( w \), where \( G(v, w) = 0 \),
2. \( M = \gamma + v \) and \( M^c \), where \( MM^c = 0 \), i.e., \( G(v, v) = a^2 + b^2 - c^2 = \gamma^2 \).

It is easy to switch to the standard basis:
\[
\gamma + aR + bS + cA = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \iff \gamma + b = p, \quad \gamma - b = s, \quad a - c = q, \quad a + c = r.
\]

Hence the restriction in Case (2) is equivalent to the singularity of the matrix, \( ps = qr \).

The case of commuting matrices is even simpler: a matrix \( M = \gamma + v \) commutes only with matrices \( X = \gamma' + a\mathbf{v} \).

Both groups of matrices and of operators extend to algebras, akin to Clifford algebras, and can be placed within the category of rigged Hilbert spaces \([7, 9]\). In particular, any generator yields orthonormal powers and thus powers become an algebraic basis of the spanned algebra. We are not aware of simple proofs of the latter property, beyond generators of small size, without referring to the representation.

Staying merely with the definition of a signed group or its generator, a pure anticommuting doubleton \( \{ e_1, e_2 \} \) will be called either Pauli-like if it is positive, or quaternion-like if it is negative. Besides this semantic link to the physical framework, from now on we will not use any matrix or operator representation of signed groups, only the basic definition.

### 3 Replacements

#### 3.1 The AC (anti-commutativity) matrix

The signatures of independent elements define the sequence \( \sigma = \sigma(e) \) and the symmetric commutativity function on \( S \times S \) defines the symmetric matrix \( C = [c(e, e')] \), often called the AC-matrix:
\[
c(e, e') = e \circ e' = \begin{cases} 1, & \text{if } \; e e' = e' e, \\
-1, & \text{if } \; e e' = -e' e. \end{cases}
\]

The sequence \( \sigma \) has no effect on the AC-matrix but the former depends on the latter.

For a square matrix \( C = [c_{jk}] \), denote by \( C^U = [c_{jk} \mathbb{I}_{\{k>j\}}] \) the upper triangular cut of \( C \) (Matlab’s `triu(C,1)`).
Lemma 3.1 Let \( e \) be a basic sequence in \( S \) with an AC-matrix \( C = [c_{jk}] = [c(e_j, e_k)] \) or its 0-1 equivalent \( D \) (cf. (2)). Then
\[
e^p e^q = (-1)^{\langle D^p, p, q \rangle} e^{p+q}, \quad p, q \in \mathbb{D},
\]
that is,
\[
\langle D^p, p, q \rangle = \sum_{j \geq 1} \sum_{k>j} d_{jk} p_k q_j = \frac{1}{2} \sum_{j \geq 1} \sum_{k>j} (1 - c_{jk}) p_k q_j.
\]
In particular, we observe the following properties:

1. \( e^p \circ e^q = (-1)^{\langle (C^p) - pq \rangle/2} \); 
2. for an anticommutative sequence,
\[
e^p \circ e^q = (-1)^{\langle p \rangle - pq};
\]
3. The only group \( G \subset S \) with entirely anticommutative subset \( G \setminus G_0 \) must be of order 8, i.e., must be generated by just two elements (in other words, it must be isomorphic either to the Pauli group or to the quaternion group);
4. Denote \( p = \langle p \rangle \). The signature \( \sigma(e^p) = e^{2p}(-1)^{p(p - 1)/2} \). That is, for pure sequences,
   (a) If \( e \) is positive then \( \sigma(e^p) = -1 \) iff \( p = 2 \) (mod 4) or \( p = 3 \) (mod 4);
   (b) If \( e \) is negative then \( \sigma(e^p) = -1 \) iff \( p = 2 \) (mod 4) or \( p = 1 \) (mod 4).
5. Let \( s_+ \) and \( s_- \) denote the counts of negative elements in \( \{ e^p : p \in \mathbb{D}_n \} \) when \( e \) is positive or negative, respectively. Then
\[
s_+ = b_2 + b_3 \quad \text{and} \quad s_- = b_1 + b_2, \quad \text{where} \quad b_q = \sum_j \left( \begin{array}{c} n \\ q + 4j \end{array} \right) \quad \text{(i.e.,} \quad q + 4j \leq n). \] (5)

Proof. First, we note the tautology for \( i, j \in S \) and \( p, q \in \{0, 1\} \):
\[
i^p j^q = \begin{cases} 
   i^p j^q = i^{p+q}, & \text{if } i = j; \\
   c(i, j) j^q i^p, & \text{if } i \neq j.
\end{cases}
\]
Consider
\[
e_1^{p_1} \cdots e_n^{p_n} e_1^{q_1} \cdots e_n^{q_n}
\]
While moving the factor \( e_i^{p_i} \) to the left until it meets \( e_i^{q_i} \), the tautology records consecutive swaps and produces the exponent \( (d_{2,1} p_2 + \cdots + d_{n,1} p_n)q_1 \). The repetition of the process for each next factor entails the quadratic form in the exponent of \(-1\). The remaining formulas follow immediately. Particular forms of the exponents are due to the identities \((-1)^a = (-1)^{-a} \) and \((-1)^p = (-1)^{\langle p \rangle}\).

(3) Three anticommuting independent elements \( e_1, e_2, e_3 \) would yield \( e_1(e_2 e_3) = (e_2 e_3) e_1 \).
(4) The classification simply relies on the parity of the quadratic functions \( p(p - 1)/2 \) and \( p(p + 1)/2 \).
(5) The identities follow directly from statement (4).

Remark 3.2
1. The exponents \( p \) and \( q \) correspond uniquely to the finite subsets of \( \mathbb{N} \), and hence for an anticommutative sequence the sign in (4) is determined by the parity of the off-diagonal elements in the Cartesian product:
\[
I_P = \sum_j p_j I_{(j)}, \quad I_Q = \sum_j q_j I_{(j)} \quad \Rightarrow \quad \langle p \rangle \langle q \rangle - pq = |P \times Q| - |P \cap Q|.
\]
2. The order restriction, addressed in Statement 3, is a consequence of the imposed condition: anticommutativity of all basic elements of the induced algebra. In particular, there appears a quick proof of an analogous statement [10, Prop. 4.5]. However, the formal simplicity, bordering on triviality, comes at a cost: the argument for Statement 3 is devoid of a deeper physical meaning. If the requirement of total anticommutativity of the induced products is relaxed then the dimension size (but not its numerical structure, though) is no longer limited.

We will take a closer look at the replacements of generators of fixed length, conforming to notational conventions of linear algebra. In the literature devoted to algebra such topics rarely appear on their own, being rather a margin of algebra of finite fields. Properties in this particular case of fields of characteristic 2 are usually derived from the general theory as much as it is possible. Often the beautiful general theory fails to deliver because its tools, e.g., quadratic forms and polynomials, break at the boundary of characteristic 2. Let us illustrate the situation by quoting Robert Wilson [11, Chap. 3.8]:

In characteristic 2 everything is different. The quadratic form has a different definition, the canonical forms are different, there are no reflections, the determinant tells us nothing, and there is no spinor norm.

Another area that deals specifically with dyadic matrices is the coding theory, equipped with its own specialized terminology (e.g., ‘code’ is the synonym of a dyadic vector space) and its own topics of interest. We would rather refer to the common linear algebra and basic arithmetics, keeping the simple - simple. The practice of dyadic vector spaces is like the classical practice, but often not quite the same.

Thus, first we refresh and adjust our old notation. Our default form for a vector \( \mathbf{v} \) of size \( n \) is a vertical \( n \times 1 \) matrix, and \( \mathbf{v} = (\mathbf{v}) = \sum_i v_i \). If \( \mathbf{v} \) has 0-1 components then \( (\mathbf{v}) \) simply counts ones, so it may be called the \textit{mass} of \( \mathbf{v} \). For a matrix \( V = [v^j_i] \), \( v^j_i \) denote its columns while \( \mathbf{v}^i \) denote its rows (horizontal vectors), entailing \( \mathbf{v} = [v_1, \ldots, v_n]^\top \) and \( \mathbf{v'} = [v^1, \ldots, v^n]^\top \), and also \( \mathbf{v} = (\mathbf{v}) = \sum_i v^i \). The zero vector is denoted by \( \mathbf{0} \) and \( \mathbf{1} \) is the vector of ones, while \( \mathbf{1}_j \) has all zeros except for the solitary 1 at the \( j \)-th position. The zero matrix is denoted by \( \mathbf{O} \), and we denote a block of ones by \( \mathbf{I} \equiv 11^\top \) (or \( \text{ONES}(n,k) \) in the Matlab’s notation).

Two integers \( m, n \) are called \textit{congruent}, which is written as \( m \equiv n \), when \( m = n \pmod{2} \). For \( c \in \{0,1\} \), we denote its complement by \( \overline{c} = 1 - c \). The relations are extended to integer matrices. That is, \( P \equiv Q \) if \( P = Q + 2R \) for some integer matrix \( R \) and \( \overline{P} = \overline{Q} - P \). Even though we focus on dyadic matrices yet occasionally we still wish to keep their ‘memory’ of being integer matrices.

A binary or integer matrix \( P \) with columns \( \mathbf{p}_1, \mathbf{p}_2, \ldots \) transforms a signed anticommutative basic sequence \( \mathbf{e} \) to the sequence \( P\mathbf{e} \equiv \langle e^{P}_1, e^{P}_2, \ldots \rangle \). We use the left superscript because the right superscript would yield a power (cf. the notation section). Note that \((P\mathbf{e}) = (\pm e^{q_1^P}, \pm e^{q_2^P}, \ldots)\), associated with but not exactly equal to \( QP\mathbf{e} \). By convention, we identify both sequences.

Formula (4) provides the characterizing condition for anticommutativity of the transformed sequence. In the language of matrices the condition reads

\[
P^\top \mathbf{T} P \cong \mathbf{T}.
\]

We simply call such \( P \) anticommutative. The set of anticommutative integer matrices \( P \) is closed under the product, i.e., it is a semigroup. We will examine anticommutative \( P \) such that \( P\mathbf{e} \) replaces the original generator, preserving the independence.

Clearly, anticommutativity is preserved by permutations of columns or rows of \( P \), and the resulting matrix \( Q \) will be deemed \textit{indistinguishable} from \( P, Q \leftrightarrow P \) in short.
3.2 Dyadic inverse

Let us note a crude multiplication table for rectangular matrices, where \( n \) is the inner dimension:

\[
\begin{array}{c|cc}
\text{if } n \text{ is even:} & I & II \\
& II & O \\
\text{if } n \text{ is odd:} & I & I I \\
& II & O I \\
& II & O II \\
\end{array}
\]

Hence \( TTTT \cong T \) regardless of parity. Elementary computations stand behind these congruence formulas, e.g., for square \( n \times n \) matrices,

\[
II^2 = nII, \quad T^2 = I + (n-2)II, \quad TI = II = I + (n-1)II, \quad \text{etc.}
\]

Notice that \( I \) is nonsingular but \( I^{-1} = (n-1)I \) is not an integer matrix. In view of this observation, tracing the idea of replacement, we say that \( P \) is \textit{dyadically invertible}, DI in short, if there is an integer matrix \( A \) such that \( AP \cong I \).

**Proposition 3.3** Let \( P \) be a square integer matrix.

1. If \( P \cong I \) then \( \det P \neq 0 \).

   Indeed, using the Levi-Civita symbols \( \epsilon_j, n! \) \( \det P = \sum_{j} \epsilon_j p_{1j} \cdots p_{nj} n! \), where the sum runs over \( n! \) permutations of \( (1, \ldots, n) \). If \( P \cong I \) then with the solitary exception of an odd integer \( p_{11} \cdots p_{nn} \) all products are even, so the sum is odd.

2. If \( Q \) is DI and \( P \cong Q \), then \( P \) is also DI.

   Indeed, \( AQ \cong I \) and \( P = Q + 2R \) entails \( AP = AQ + 2AR \cong I \).

3. The following conditions are equivalent:

   (a) \( P \) is DI;

   (b) \( Pc \cong 0 \Rightarrow c \cong 0 \);

   (c) \( \det P \cong 1 \) and \( (\det P) \cdot P^{-1} \) is an integer matrix;

   (d) There is an integer matrix \( B \) such that \( PB \cong I \).

   (a) \( \Rightarrow \) (b): Let \( AP = I + 2R \). If \( P \cong 2r \) then \( c + 2Rc = APc = 2Ar \), i.e., \( c \cong 0 \).

   (b) \( \Rightarrow \) (c): (b) ensures no even column. In fact, by (2) we may consider just a binary 0-1 matrix with no zero column. Therefore a Gauss row reduction based only on permutations and subtraction of rows yields a triangular matrix with ones on the diagonal. The other formula just uses the adjugate matrix.

   (c) \( \Rightarrow \) (d): choose \( B = \text{adj}(P) = (\det P) P^{-1} \) in lieu of the actual inverse.

   (d) \( \Rightarrow \) (a). Assume (d), i.e. \( PB = I + 2R \) for an integer matrix \( R \). Hence \( b = \det B \cong 1 \) and \( bB^{-1} \) is integer by (a) \( \Rightarrow \) (c). Hence \( bP = bB^{-1} + 2bRB^{-1} \). Put \( A = B \), so \( AP \cong bBP = bI + 2bBRB^{-1} \cong I + 2R' \).

**Remark 3.4** Let \( P \) be a square integer matrix.

1. As integer matrices the left dyadic inverse \( A \) and the right dyadic inverse \( B \) may differ but they are congruent. In particular, in defining \( P \) as \textit{dyadically orthogonal} (or ‘D-orthogonal’, in short) by the relation \( P^T P \cong I \), the condition is equivalent to \( PP^T \cong I \).
2. $P^{-1}$ is integer iff $|\det P| = 1$.

3. Let $P \cong \mathbf{I}$. If $n$ is even, then $P$ is D-orthogonal, so $P$ is DI and thus nonsingular. If $n$ is odd then $\det P \cong 0$, so $P$ is not DI and may be singular. However, the actual $\mathbf{I}$ is nonsingular.

The first statement is the particular case of Remark 1 above. Consider an odd $n$. Let us mark the dimension, $P = P_n$. We do not know a quick argument, so a quite tedious row reduction followed by equally tedious column reduction will reduce $\det P_n$ to $\det P_{n-2}$, where $P_{n-2} \cong \mathbf{I}_{n-2}$. Repeating, we reduce the case to dimension $n = 3$, and then to dimension 1 with $P_1 = [p]$ for some even number $p$, possibly 0. E.g., $P_3 = [0, 1, 1; -1, 0, 1; -1, -1, 0] \mapsto P_1 = [0]$.

The inverse of the actual $\mathbf{I}$ was shown right above the proposition.

4. While every DI matrix is nonsingular, the inverse implication fails.

Indeed, the inverse is just a rational matrix, not necessarily an integer matrix. A counterexample may involve just even rows, for an odd $n$ we take $\mathbf{I}$ while for an even $n$, say, $n = 4$:

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow P \mathbf{1} \cong \mathbf{0}, \quad \det P \neq 0.$$  

5. If $Q$ is another square integer matrix and $PQ$ is DI, then both $P$ and $Q$ are DI. In particular, if an anticommutative $P$ is DI, then its dyadic inverse is also anticommutative.

Indeed, consider integer determinants: the odd product requires odd factors.

6. The set $\mathcal{D}$ of DI matrices forms a transpose-invariant group.

The columns of a fixed DI matrix form a basis in the vector space of all DI matrices. We need to decide whether or not we identify DI matrices modified by permutations $P \leftrightarrow \pi^T \pi P$, i.e., whether we perceive an ordered or unordered basis. The number of ordered bases (cf. [12, Lemma 1.17] or [11, (3.1)]) can be found exactly as well as its asymptotics:

$$\prod_{i=1}^{n-1} (2^n - 2^i) = d_n 2^{n^2}, \quad \text{where } d_n = \prod_{i=1}^{n-1} (1 - 2^{-i}) \to \phi(1/2) \approx 0.288788095...,$$

and $\phi(z)$ is the Euler function. That is, almost 30% of large dyadic matrices are DI. The division by $n!$ yields the number of unordered bases or permutation equivalent DI matrices.

An anticommutative matrix admits yet another characterization:

$$C = P^T P \cong \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix}. \quad (8)$$

Factually, the matrix on the right is $\pi^T C \pi$ but we use the inverse permutations for the more transparent display, which will be called canonical from now on.

Example 3.5
1. Consider the matrix of the generator’s transformation ‘multiply by an element $e_i$’, say, by the 1st element $e_1$. That is, the first column is added to all other columns. The corresponding matrix $C_1 = \begin{pmatrix} 1 & 1 \\ 0 & I \end{pmatrix}$ is anticommutative:

$$PC_1 = \begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix} + \begin{pmatrix} p_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & I \end{pmatrix}.$$

However, $C_1^T$ (deemed as the left action $P \mapsto C_1^T P$, adding the first row to all other rows) is anticommutative iff $n$ is even, because $C_1 C_1^T \sim = \begin{pmatrix} n & 1 \\ 1 & 1 \end{pmatrix}$.

Factually, this example involves the replacement $F(f) = \{ f \} \cup f(F \setminus \{ f \}) = \{ f \} \cup \{ ff' : f' \neq f \}, \ f \in F$. (9)

We observe that the transformation preserves anticommutativity.

2. Consider the ‘add one row to another’ matrix $R (P \mapsto RP)$, say, the 1st row to the 2nd row:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I \end{pmatrix} \Rightarrow R^T R \sim = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Since such matrices do not anticommute, the Gauss row reduction does not preserve anticommutativity.

The identity $I$ occupies the entire matrix (8) iff $P$ is D-orthogonal. The other extreme occurs when $C = P^T P = I$, so we are tempted to call such $P$ ‘antiorthogonal’.

**Proposition 3.6** Let $P$ be anticommutative of dimension $n$.

Consider the matrix $C$ from (8) and let $n = m + k$, where $m$ is the dimension of the NW (North-West) block and $k$ is the dimension of the SE block. Then, with the only exception of even $m$ and odd $k$ (making $n$ odd), $P$ is DI.

**Proof.** We will examine $C^2$, going to $C^3$ if necessary. It will be convenient to mark the connecting dimensions of essence by ‘0’ (even) or ‘1’ (odd) in the multiplication table (7):

$$
\begin{array}{c|cc}
I_0 & 0^T & 0^T \\
II_0 & I & II \\
\end{array} \quad \text{and} \quad 
\begin{array}{c|cc}
I_3 & 1^T & 1^T \\
II_3 & I & O \\
\end{array}.

$$

(10)

Depending on the parity of $m$ and $k$, in $C^2$ we encounter the following four patterns:

\[
k:
\begin{array}{c|c}
0 & 1 \\
1 & I_0 \\
\end{array}
\]

\[
m:
\begin{array}{c|c}
0 & 1 \\
1 & I_0 \\
\end{array}
\]

where $D = \begin{pmatrix} I_1 & II_1 \\ II_1 & I_1 \end{pmatrix}$.

We verify that $C^3 \neq CD \neq I$. Then, the statement follows from Remark 3.4.5.

**Remark 3.7** Let $P$ be anticommutative with $n = m + k$ defined above.

1. Necessarily, $m \geq 1$, i.e., there is no antiorthogonal $P$ of even dimension. Further, an antiorthogonal matrix is never DI.

The analysis of parity of $m$ admits $m = 0$, which does not exclude DI when $k$ is even. However, an antiorthogonal matrix $P$ cannot have a dyadic inverse. Indeed, suppose by contrary that $PQ \equiv I$ for some integer matrix $Q$. Then $I \equiv (PQ)^T(PQ) = Q^T(P^T P)Q \equiv Q^TIQ = Q^TIIQ - Q^TQ = qq^T - Q^TQ$. This would yield the congruence of diagonals, $1 \equiv 0$, a contradiction.
2. The complement $P \leftrightarrow \overline{P}$ preserves both extremes of dyadic-orthogonality when $n$ is even, and switches them if $n$ is odd.

Indeed, $\overline{P}P \cong nI + P^T P$.

3.3 Orthogonal replacements

We will examine some properties of D-orthogonal matrices, $P^TP \cong I$, including generating algorithms. We begin by counting.

Denote the group of D-orthogonal matrices by $\mathcal{O}$. Writing $P = I + A$, the D-orthogonality means exactly that $A + A^T + A^T A \cong O$ (for symmetric matrices: $A^TA \cong 0$).

Row and column permutations $P \leftrightarrow \pi P \sigma$ scatter the elements, preserving the orthogonality but changing the appearance. Although the result may be deemed indistinguishable from the original but the mapping is not a homomorphism in $\mathcal{O}$, $P \leftrightarrow P'$, $Q \leftrightarrow Q' \not\Rightarrow P Q \leftrightarrow P' Q'$, with the exception of a fixed $\pi$ and $\sigma = \pi^T$.

The order of $\mathcal{O}$ is well known and elementary calculations can be found, e.g., in [13]. Two DI matrices $M$ and $N$ are called orthogonally equivalent, $M \sim N$ if $NM^{-1} \in \mathcal{O}$, i.e., $N \in \mathcal{O}M$. There is a one-to-one correspondence between the coset $\mathcal{O}M$ and the set of DI symmetric matrices $M^T M$, so this set and $\mathcal{O}$ are equal in size. Then the author’s [13] count of symmetric matrices relies on a classical 1938 result [14] stating that the representation $M^T M$ of a symmetric matrix is possible if and only if the diagonal of this symmetric matrix has nonzero mass. The count, including also arbitrary ranks, is performed in general context of a finite field $GF(q)$, $q = 2^p$, $p$ a prime. In our case $q = 2$ the number of DI symmetric matrices of dimension $n = 2m$ or $n = 2m + 1$ equals

$$N_m = \prod_{i=1}^{m} 2^{2i} (2^{2i-1} - 1) = c_m 2^{2m^2 + m},$$

where $c_m \rightarrow \frac{\phi(1/2)}{\phi(1/4)} \approx 0.41942244...$, using the Euler function $\phi$ again. At the same time the number of DI symmetric matrices with 0 diagonal is $2^{-2m}N_m$ when $n$ is even and there is none when $n$ is odd. Hence, the number of D-orthogonal matrices is still in the range of $2^{2m^2}$. Therefore, even when pooling together permutation modified matrices (involving the division by a number not exceeding $(n!)^2$, the order does not change significantly (which is subject to discussion about the meaning of ‘significance’, involving the degree of smallness among big numbers). Thus, we obtain a crude estimate of the order $\approx 2^{n^2/2}$.

3.3.1 A modified Gram-Schmidt algorithm

First, we will discuss the analog of the Gram-Schmidt orthogonalization which turns out to be almost a copy of the classical one but with a twist. It might be tempting to dub two dyadic vectors $p, q$ D-orthogonal if $\langle pq \rangle = 0$, or even $C$-orthogonal with the help of the bilinear form $p^T C q = 0$, but soon that would lead to a sort of havoc, or a distraction, or an ambiguity at least. Such situations are discussed in detail in [11, 3.4-3.8].

However, here D-orthogonal matrices popped up naturally prior to defining D-orthogonal vectors. For any dyadic matrix $C$, $C^T C$ contains the masses of columns of $C$ on the diagonal and the masses of their intersections (or of entry-wise products) off the diagonal. So, a D-orthogonal matrix by definition has odd columns (i.e., of odd masses) with even mutual intersections. So, it seems natural to say that vectors are D-orthogonal if they are columns of a D-orthogonal matrix, i.e. $p \perp q$ if $\langle pq \rangle \cong 0$ and $\langle p \rangle \cong \langle q \rangle \cong 1$, which entails the notion...
of the orthogonal complement of an odd dyadic vector. At the same time, an even vector would have no D-orthogonal complement by default.

The vector \(1\), a ‘flatline’, whether even or odd, has no orthogonal complement, nor does any vector subspace containing it. The classical Gram-Schmidt process seems to be useless. Even low dimension examples show that a subspace may have not even a pair of orthogonal vectors, or just a few. It may contain the ‘flatline’ or not at all. Yet, the Gram-Schmidt process can be used to construct orthogonal matrices, i.e., orthogonal bases, starting with a single vector, which we may also call ‘D-orthogonal’ although it is solitary.

Given D-orthogonal vectors \(p_1, \ldots, p_k\) of length \(n\) such that
\[
1 \not\in \text{span} \{p_1, \ldots, p_k\}
\] (11)
we put
\[
p = p_1 + \cdots + p_k.
\]
Necessarily, \(k < n\). Next, consider the projection of a vector \(u\):
\[
E[u|p_1, \ldots, p_k] = \sum_{j=1}^{k} \langle up_j \rangle p_j \quad \text{and} \quad u' = u + E[u|p_1, \ldots, p_k].
\] (12)
We check that
\[
\langle u' p_j \rangle \cong 0 \quad \text{and} \quad \langle u' \rangle = \langle u \rangle + \langle u' p \rangle.
\]
The vector \(u'\) is odd when exactly one of the summands is odd. Therefore and since \(p \neq 1\) we have at least two choices of such \(u\) (say, odd or even). Because of two choices we can augment our orthogonal set by \(p_{k+1} = u'\) to avoid the ‘flatline’, i.e., \(p_1 + \cdots + p_{k+1} \neq 1\) while \(k+1 < n\). Then the algorithm continues until it is complete when \(k+1 = n\).

### 3.3.2 Quick generators

To obtain a quick example of a D-orthogonal matrix consider
\[
P \cong \begin{pmatrix} I & O \\ O & T \end{pmatrix},
\] (13)
where the lower block has even dimension. Given two 0-1 sequences \(u, v\), the scattered block of ones can be written with the help of the classical tensor notation \(u^I v = uv^T = u \otimes v\).

In particular, \(\overline{u^I v} = \overline{v} \otimes \overline{v}\), hence the old multiplication table (7) now takes the following form:
\[
\begin{align*}
    u \otimes v \cdot r \otimes s &= \langle vr \rangle u \otimes s \\
    u \otimes v \cdot \overline{v} \otimes \overline{v} &= v u \otimes v \\
    \overline{v} \otimes \overline{v} \cdot \overline{v} \otimes \overline{v} &= (\overline{v} \otimes \overline{v})^v.
\end{align*}
\] (14)
(recall that \(a^0 = 1\) and \(a^1 = a\)). More general formulas are available but we will not need them here. So, \(I + u \otimes u\), where \(u\) is even, is permutation equivalent to (13). Let us examine a slight extension \(P = I + \sum_i u_i \otimes v_i\), where \(\langle u_i, u_j \rangle \cong 0\) (even intersections, including self). First, we check when \(P\) is dyadically orthogonal, \(I \cong P^T P = \)
\[
I + \sum_i (v_i \otimes u_i + u_i \otimes v_i) + \sum_i \sum_j \langle u_i u_j \rangle v_i \otimes v_j \cong I + \sum_i (v_i \otimes u_i + u_i \otimes v_i).
\]
We see that necessarily \(u_i \cong v_i\), i.e.,
\[
P \cong I + A = I + \sum_i u_i \otimes u_i.
\] (15)

When the vectors \(u_i\) are factually disjoint, a permutation equivalent form of \(A\) will have quadratic blocks of ones along the diagonal. However, in general it is harder to visualize its form even with the help of permutations.
The family of D-orthogonal matrices \([15]\), denoted by \(\mathcal{K}\), is not closed under the product \(PQ\) (first of all, the symmetry is not product invariant):

\[
\left( I + \sum_i u_i \otimes u_i \right) \left( I + \sum_i v_i \otimes v_i \right) = I + \sum_i (u_i \otimes u_i + v_i \otimes v_i) + \sum_i \sum_j (u_i \otimes v_j) u_i \otimes v_j = I + \sum_i (u_i \otimes (u_i + t_i) + v_i \otimes v_i), \quad \text{where} \quad t_i = \sum_j (u_i \otimes v_j) v_j.
\]

For example, \(P = I + A, Q = I + B\), where

\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow A + B + AB = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\]

However, here \(P, Q\) here are permutations, and their product is a permutation, all indistinguishable from \(I\). A simplifying assumption, e.g., the even intersections \((u_i \otimes v_j) \cong 0\), yields the same type:

\[
PQ \cong I + \sum_i u_i \otimes u_i + \sum_i v_i \otimes v_i = I + \sum_i u_i' \otimes u_i'. \tag{16}
\]

So, \(\mathcal{K}^2\) is distinguishable from \(\mathcal{K}\), e.g., for odd \(n\) and odd blocks, we may obtain an asymmetric matrix:

\[
\begin{pmatrix}
\top & \top & O \\
\top & \top & O \\
O & O & I
\end{pmatrix} \cdot 
\begin{pmatrix}
I & O & O \\
O & T & I \\
O & I & T
\end{pmatrix} = 
\begin{pmatrix}
\top & O & H \\
H & T & O \\
O & O & T
\end{pmatrix} \cong I + 
\begin{pmatrix}
H & O & H \\
H & O & H \\
O & O & T
\end{pmatrix} \notin \mathcal{K}
\]

When all blocks are even, as expected in \([16]\), the product equals

\[
\begin{pmatrix}
\top & \top & O \\
\top & \top & O \\
O & O & T
\end{pmatrix} \cong I + 
\begin{pmatrix}
H & O & H \\
H & O & H \\
O & O & T
\end{pmatrix} \in \mathcal{K}.
\]

Although \(\mathcal{O}\) is large, it is finite, so the sequence \(\mathcal{K}^k\) must eventually become constant. Two questions emerge, the first one is whether the entire \(\mathcal{O}\) can be reached by this procedure. If the answer is affirmative, the second question is how fast this task can be achieved. However, we observe that the number of simple matrices \([15]\) is just a humble big number, compared to the total count.

**Proposition 3.8** Denote by \(p_0(n)\) the number of distinguishable \(P = I + A\) of type \([15]\) with disjoint vectors. Then the range of \(p_0(n)\) is approximately equal to \(c c^d \sqrt{n}\) as \(n \to \infty\), for suitable constants \(c, d > 0\).

**Proof.** The number of canonical \(P\)s is related to the partition function \(p(m)\), the number of ways \(m\) can be written as a sum of positive integers with order irrelevant (cf. \([15], (23)\) for the definition and properties). When \(n\) is odd, \(n = 2m + 1\), we must select an odd number \(2j + 1, 0 \leq j \leq m\), to make the dimension of \(I\). When \(n\) is even, \(n = 2m\), we may either select \(I\) of dimension \(2j, j \leq 2m\), or skip it. Therefore, the number \(p_0(n)\) equals

\[
\text{if } n = 2m + 1: \quad p_0(n) = \sum_{j=0}^{m} j p(m - j), \\
\text{if } n = 2m: \quad p_0(n) = \sum_{j=0}^{m} j p(m - j) + p(m).
\]

13
The Hardy and Ramanujan (1918) approximation \[15, (23)\] of the partition function is given by
\[ p(m) \approx a \frac{e^{b\sqrt{m}}}{m}, \quad \text{where} \quad a = \frac{1}{4\sqrt{3}}, \quad b = \frac{\pi\sqrt{2}}{\sqrt{3}}. \]

Put \( f(t) = e^{b\sqrt{t+1}}/(t+1) \) and \( F(t) = \int_0^t f(x) \, dx \), yielding the convolution
\[ \int_0^t (t-x)f(x) \, dx = \int_0^t F(x) \, dx \approx \frac{2}{b} \int_0^t \frac{e^{b\sqrt{x+1}}}{\sqrt{x+1}} \, du \approx \frac{4}{b^2} e^{b\sqrt{t}}. \]

With \( t \approx n/2 \) we adjust the constants, omitting the lower order term \( p(m) \) when \( n \) is even.

The relatively low count warns us about possible inefficiency of the product procedure. However, this count applies only to canonical \( P \) but they do not commute with permutations, and these would bring the factorial multiplier. Therefore, in spite of the initial simplicity the products of such basic matrices quickly become rich and complicated, which suggests that there might be a multiplication algorithm leading to an arbitrary D-orthogonal matrix.

**Theorem 3.9** Every D-orthogonal matrix of dimension \( n \) is indistinguishable from a product of \( n \) D-orthogonal matrices of the form \( I + u \otimes u \).

**Proof.** It suffices to show that a D-orthogonal matrix \( P \) admits a sequence \( K_i = I + u_i \otimes u_i \) and a permutation matrix \( \pi \) such that \( K_n \cdots K_1 P = \pi \). To this end, note the implication
\[ \langle pc \rangle \equiv 0 \quad \text{and} \quad \langle p \rangle \equiv 1 \quad \Rightarrow \quad (I + (p + c) \otimes (p + c)) \, p \equiv c. \]

Whence \( (I + u \otimes u) \, p \equiv 1_k \) for an odd \( p \neq 1 \) and \( p_k = 0 \), denoting \( u = p + 1_k \).

Let us focus on the first column \( p_1 \neq 1 \) of \( P \) and create an even vector \( u_1 \) by the above recipe. Then the first column of \( (I + u_1 \otimes u_1)P \) contains a solitary 1. Since the product is D-orthogonal hence this 1 is solitary in the corresponding row. The new columns are still odd and have even intersections with other columns, and there is no 1. Then, the procedure, repeated for the second (new) column, then for the third one, etc., without affecting the previously obtained single 1s, entails a permutation matrix \( \pi \) at the end.

### 4 Types of signatures

We will classify finite or infinite AC generators with respect to the distribution of signatures. The presence of one or more additional negative elements that commute with all others provides new means of controlling signatures. We will discuss this case in Section 5.3.

Recall that \( s_+ \) or \( s_- \) denote the counts of powers \( e^p \) that are negative depending upon the generator \( e \) being positive or negative, respectively. We will evaluate the counts.

**Proposition 4.1** Consider a pure anticommutative generator \( e \) of length \( n \).

1. We have \( s_- = s_+ \) if and only if \( n \) is divisible by 4. In this case positive and negative generators are replaceable.

2. The counts are as follows:
\[ s_+ = 2^{n-2} - 2^{n/2-1} \left( \cos \frac{n\pi}{4} + \sin \frac{n\pi}{4} \right) \]
\[ s_- = 2^{n-2} - 2^{n/2-1} \left( \cos \frac{n\pi}{4} - \sin \frac{n\pi}{4} \right) \]
Proof. We will use $b_q$ from (5). The 1834 Ramus identity (cf. [16, 17]) involved a period $p$ and the $p^{th}$ root $\omega = e^{2\pi i/p}$ of 1:

$$b_q(p) = \sum_j \left( \frac{n}{q + pj} \right) = \frac{1}{p} \sum_{k=1}^{p} \omega^{-qk}(1 + \omega^k)^n.$$  

It follows by the binomial formula used in the right hand side and the fact that for $z = \omega^r$

$$\sum_{k=1}^{p} z^k \neq 0 \iff p|r,$$

in which case the sum equals $k$. However, we just need its very elementary version, because the period $p = 4$ entails $\omega = i$:

$$b_q = \frac{1}{4} \sum_{k=1}^{4} i^{-qk}(1 + i^k)^n = 2^{n-2} + \frac{c_q(n)}{i^q}, \quad \text{where} \quad c_q(n) = \frac{(1+i)^n}{i^q} + \frac{(1-i)^n}{i^{3q}}. \quad \text{(17)}$$

Hence

$$c_1 = \frac{(1+i)^n - (1-i)^n}{i} = -c_3, \quad c_2 = -(1+i)^n - (1-i)^n. \quad \text{(18)}$$

Therefore, $s_+ = s_-$ if and only if $c_1 = 0$, i.e., $(1+i)^n = (1-i)^n$, or, in other words

$$t^n = \left( \frac{1+i}{1-i} \right)^n = 1 \iff 4|n.$$  

So, consider $n = 4m$. For $m = 1$ let us choose $A = \{ 1110, 1101, 1011, 0111 \}$. Then the four triple products $e^p$, $p \in A$, anticommute, since $pq - \langle pq \rangle = 9 - 2 = 7$ for $p, q \in A$. If $e$ is negative then $e^p$ are positive since their signatures are $(-1)^{p(p+1)/2} = (-1)^6$. If $e$ is positive then $e^p$ are negative since their signatures are $(-1)^{p(p-1)/2} = (-1)^3$.

If $m > 1$, we partition $e$ into $m$ disjoint quadruples and apply the above signature changing process within each quadruple to bring all elements to the same signature. The new elements from disjoint quadruples anticommute because their binary marks $p$ and $q$ produce the form $pq - \langle pq \rangle = 9 - 0$. This completes the proof of the first statement.

Now, we combine the sums in (5) with (17) and (18):

$$c_1 = 2^{n/2-1} \sin \frac{n\pi}{4} = -c_3, \quad c_2 = -2^{n/2-1} \cos \frac{n\pi}{4}.$$  

In the periodic trigonometric sequences $t_\pm = \cos \frac{n\pi}{4} \pm \sin \frac{n\pi}{4}$ of period 8 the radicals (or 0) appear only at some odd indices, so they simplify due to the factor $2^{n/2-1}$:

$$n = ... 0, 1, 2, 3, 4, 5, 6, 7, ...
\begin{align*}
t_+ & : ... 1, \sqrt{2}, 1, 0, -1, -\sqrt{2}, -1, 0, \ldots \\
t_- & : ... 1, 0, -1, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \ldots 
\end{align*}$$

This concludes the proof.

The sign matrix $C$ of a finite group with an anticommutative generator is unique. However, the signatures may be arbitrary and may change under AC replacements. Let us denote by $N_-(e) = N_-(e)$ and $N_+(e) = N_+(e)$ the number of negative and positive elements, respectively, in an AC generator $e$. If its length is $n$, so $N_- + N_+ = n$. Clearly, if $N_-(e) = N_-(e')$ then the generated groups are isomorphic. Yet, $N_-(e)$ is not an isomorphism invariant and it is not immediately clear how the quantity behaves under admissible replacement, $e \mapsto e^p$, where $P$ is DI and AC-preserving.
Example 4.2 Let us write \((m, k) = \left( N_+, N_+ \right) \).

1. The replacement \( F \mapsto F(f_0) \) from Example 3.5.1, where an element \( f_0 \) is positive, changes the signatures as follows

\[
(m, k) \leftrightarrow (k + 1, m - 1).
\]

2. Let \( n = 4 \) and \( e = (e_1, e_2, e_3, e_4) \) be pure. Then \( P = T_4 \) defines the admissible replacement \( e' = e^P = (e_2e_3e_4, e_1e_3e_4, e_1e_2e_4, e_1e_2e_3) \) with the signatures swapped:

\[
(4, 0) \leftrightarrow (0, 4).
\]

For the other configurations: \((3, 1) \leftrightarrow (2, 2)\), which is also covered by the previous replacement, \((1, 3) \leftrightarrow (4, 0)\). Thus we obtain two non-isomorphic signed groups, with equivalent distribution of \((+, -)\) elements:

\[
\{(4, 0), (0, 4), (1, 3)\} \quad \text{and} \quad \{(3, 1), (2, 2)\}.
\]

We will prove it in the general case.

Theorem 4.3 Let \( n \geq 2 \) be the length of an AC generator of a signed group of order \( 2^{n+1} \).

Up to an isomorphism,

1. if \( n \) is even then there are two non-isomorphic groups;

2. if \( n \) is odd then there are three non-isomorphic groups;

3. if \( n = \infty \) then there is only one signed group.

Proof. There are \( n + 1 \) possible replacements. We record the replacement within same group based on transformations in the above Example, either \( m = N_+ \) belongs to an arithmetic sequence of step 4, or being subject to the transformation \((m, k) \leftrightarrow (k + 1, m - 1)\).

If \( n = 2 \), two groups \((2, 0) \leftrightarrow (1, 1)\) and \((0, 2)\) are not isomorphic because pure generators yield different counts of negative elements by Proposition 4.1. So let \( n \geq 3 \).

For \( n = 4k + 2 \) we obtain exactly two non-isomorphic groups. The first group has a pure negative generator while the second one has a pure positive generator, hence the total counts \( N(G) \) differ by Proposition 4.1. The negative count is isomorphism invariant.

\[
\begin{align*}
(4i, 4(k - i) + 2) & \quad i = 0, \ldots, k \\
\downarrow & \quad (2k + 1 \text{ replacements}) \\
(4(k - i) + 3, 4i - 1) & \quad i = 1, \ldots, k
\end{align*}
\]

\[
\begin{align*}
(4i + 2, 4(k - i)) & \quad i = 0, \ldots, k \\
\uparrow & \quad (2k + 2 \text{ replacements}) \\
(4(k - i) + 1, 4i + 1) & \quad i = 0, \ldots, k
\end{align*}
\]

In other words, the two types in the case of \( n = 2 \) (mod 4) follow again the remainder of \( N_+ / 4 \):

- type \( R_{n, 2}(0, 3) \): if \( N_+ = 0 \) (mod 4) or \( N_+ = 3 \) (mod 4);
- type \( R_{n, 2}(1, 2) \): if \( N_+ = 1 \) (mod 4) or \( N_+ = 2 \) (mod 4).

In particular, the split \((N_+, N_-) = (0, n)\) characterizes \( R_{n, 2}(0, 3) \).
For \( n = 4k + 3 \) we obtain at most three non-isomorphic groups:

\[
\begin{align*}
(4i, 4(k - i) + 3) & \quad i = 0, \ldots, k \\
\downarrow & \quad \text{the same} \\
\end{align*}
\]

\[
\begin{align*}
(4i + 1, 4(k - i) + 2) & \quad i = 0, \ldots, k \\
\downarrow & \quad \text{the same} \\
(4(k - i) + 3, 4i) & \quad i = 0, \ldots, k \\
\end{align*}
\]

\[
\begin{align*}
(4i + 2, 4(k - i) + 1) & \quad i = 0, \ldots, k \\
\downarrow & \quad \text{the same,} \\
\end{align*}
\]

\[
\begin{align*}
\{ & (k + 1 \text{ replacements}) \\
\{ & (2k + 2 \text{ replacements}) \\
\{ & (k + 1 \text{ replacements}) \\
\end{align*}
\]

In other words, for \( n = 3 \pmod{4} \) three types emerge:

- type \( \mathcal{R}_{n;3}(0) \): if \( N_+ = 0 \pmod{4} \);
- type \( \mathcal{R}_{n;3}(1,3) \): if \( N_+ = 1 \pmod{4} \) or \( N_+ = 3 \pmod{4} \);
- type \( \mathcal{R}_{n;3}(2) \): if \( N_+ = 2 \pmod{4} \);

They are not isomorphic. Indeed, by Proposition 4.1 groups of the first and of the second type are not isomorphic because they have pure generators of opposite signatures. Also, each of them contains the subgroup of type \( \mathcal{R}_{n-1;2}(0,3) \), i.e., one with generator \((N_+, N_-) = (0, n - 1)\) of length \( n - 1 \), which does not show among subgroups of a group of type \( \mathcal{R}_{n;3}(2) \), because the maximum \( N_- = 4k + 1 = n - 2 \).

For \( n = 4k \) two groups emerge:

\[
\begin{align*}
(4i, 4(k - i)) & \quad i = 0, \ldots, k \\
\downarrow & \quad \text{the same} \\
(4(k - i) + 1, 4i - 1) & \quad i = 1, \ldots, k \\
\end{align*}
\]

\[
\begin{align*}
(4i + 2, 4(k - i) - 2) & \quad i = 0, \ldots, k - 1 \\
\downarrow & \quad \text{the same} \\
(4(k - i - 1) + 3, 4i + 1) & \quad i = 0, \ldots, k - 1 \\
\end{align*}
\]

\[
\begin{align*}
\{ & (2k + 1 \text{ replacements}) \\
\{ & (2k \text{ replacements}) \\
\end{align*}
\]

In other words, when \( n = 0 \pmod{4} \):

- type \( \mathcal{R}_{n;0}(0,1) \): if \( N_+ = 0 \pmod{4} \) or \( N_+ = 1 \pmod{4} \);
- type \( \mathcal{R}_{n;0}(2,3) \): if \( N_+ = 2 \pmod{4} \) or \( N_+ = 3 \pmod{4} \).

The lack of isomorphism follows again by a subgroup argument. That is, a group of type \( \mathcal{R}_{n;0}(0,1) \) contains a subgroup \( \mathcal{R}_{n-1;3}(0) \), represented by \((N_+, N_-) = (0, n - 1)\), which is absent among subgroups of \( \mathcal{R}_{n-1;3}(2,3) \) for which the maximum \( N_- = 4k - 2 = n - 2 \).
The last case \( n = 4k + 1 \) follows similarly with three groups at hand.

\[
\begin{align*}
(4i, 4(k - i) + 1) & \quad i = 0, \ldots, k \\
& \uparrow \\
(4(k - i) + 2, 4i - 1) & \quad i = 1, \ldots, k \\
(4i + 1, 4(k - i)) & \quad i = 0, \ldots, k \\
& \uparrow \\
& \quad \text{the same} \\
(4i + 3, 4(k - i - 1) + 2) & \quad i = 0, \ldots, k - 1 \\
& \uparrow \\
& \quad \text{the same,}
\end{align*}
\]

\( (2k + 1 \text{ replacements}) \)

\( (k + 1 \text{ replacements}) \)

\( (k \text{ replacements}) \)

Like before, \( n = 1 \pmod{4} \) yields three types of groups:

- type \( \mathcal{R}_{n:1}(0, 2) \): if \( N_+ = 0 \pmod{4} \) or \( N_+ = 2 \pmod{4} \);
- type \( \mathcal{R}_{n:1}(1) \): if \( N_+ = 1 \pmod{4} \);
- type \( \mathcal{R}_{n:1}(3) \): if \( N_+ = 3 \pmod{4} \);

\( \mathcal{R}_{n:1}(0, 2) \) with a positive generator and \( \mathcal{R}_{n:1}(1) \) with a negative generator are not isomorphic. Also, each contains \( \mathcal{R}_{n-1:0}(0, 1) \), displaying a generator with \( N_- = n - 1 \), as a subgroup in contrast to \( \mathcal{R}_{n:1}(3) \) where the maximum \( N_- = 4k - 2 = n - 3 \).

Let us now turn to the proof of the third statement. Using again the classification \( (N_+, N_-) \), we first reduce all possible groups to four cases:

1. \( (0, \infty), (4, \infty), \ldots \leftrightarrow (\infty, 3), (\infty, 7), \ldots \);
2. \( (1, \infty), (5, \infty), \ldots \leftrightarrow (\infty, 0), (\infty, 4), \ldots \);
3. \( (2, \infty), (6, \infty), \ldots \leftrightarrow (\infty, 1), (\infty, 5), \ldots \);
4. \( (3, \infty), (7, \infty), \ldots \leftrightarrow (\infty, 2), (\infty, 6), \ldots \),

each represented by a sequence that begins with 0, 1, 2, or 3 positive elements. Then, focusing on the first four elements, we use the type \( \mathcal{R}_{n:0}(0, 1) \) to merge the cases 1 and 2 into one case, represented by \((0, \infty)\). Then we use \( \mathcal{R}_{n:0}(2, 3) \) to merge the cases 3 and 4 to one case, represented by \((2, \infty)\).

Then we cut off the first five elements and consider \((0, 5)\) versus \((2, 3)\) to see that \( \mathcal{R}_{5:1}(0, 2) \) makes one group, while the rest of the sequence, which is negative, remains unaffected.

\section{Classification of finite signed systems}

We consider a signed group \( G \) together with its basic subsets \( E \), possibly ordered \( E \leftrightarrow \emptyset \). It is convenient to refer to the 0-1 equivalent \( D \) of the AC-matrix \( C \), cf. \( \cite{2} \). For a time being we disregard the signatures of elements because they do not affect the AC-matrix. By convention, we consider a singleton \( \{g\} \) and \( \emptyset \) as both commutative and anticommutative. We denote by \( c^{-} = c^{-}(E) \) (also called the ‘AC-count’) the count of the negative signs in the AC-matrix \( C \) (or 1s in \( D \)) of the group generated by the generator \( E \). The quantity is a group invariant, thus different counts yield non-isomorphic groups. We will see that the inverse implication is conditionally true, i.e., equal counts imply an isomorphism provided that the signature patterns agree.

A commutative signed generator happens if and only if the generated group is commutative. A more interesting situation occurs when some elements anticommute. Although arbitrary or even random signs may be assigned to a generator, possibly entailing chaos of signs in the AC-matrix of the generated group, yet the opposite happens. We will see that
quite ‘orderly’ replacements exist that enjoy clear sign patterns. In particular, a subset of elements commuting with all others can be put aside.

**Example 5.1** With $D$ of size $n \times n$, if 1s are placed at random in the upper triangle $D^{u}$ then the probability of obtaining at least one element commuting with all others is

$$1 - \prod_{k=1}^{n-1} (1 - 2^{-k}) \to 1 - \phi(1/2) \approx 0.7112 \text{ as } n \to \infty.$$ 

Indeed, we may view $D$ as the adjacency matrix of a graph with $n$ vertices. If $p_n$ denotes the probability that the graph is connected, then $p_2 = 2^{-1} = 1 - 2^{-1}$ and, conditioning on the first vertex, $p_n = (1 - 2^{n-1}) p_{n-1}$ for $n \geq 3$.

### 5.1 Equivalents of AC generators

Below we will give meaning to two actions: **integration** of smaller (desirably simple) matrices of the given type to form a larger matrix of the same type, and the inverse action of **disintegration**.

**Proposition 5.2**

1. When $n = 2$ or $n = 3$ then at least one pair of anticommuting elements yields an anticommutative generator.

The case $n = 2$ is trivial. Let $n = 3$ and let $D^{u}$ have either a single 0 or a single 1, e.g., neglecting equivalent permutations:

$$
\begin{pmatrix}
0 & 1 \\
\cdot & 0 \\
\cdot & \cdot
\end{pmatrix} \mapsto (e_1, e_2 e_3), \quad \begin{pmatrix}
1 & 0 \\
\cdot & 0 \\
\cdot & \cdot
\end{pmatrix} \mapsto (e_1, e_2, e_1 e_2 e_3).
$$

2. A chain-like sequence $u$ such that $u_k \circ u_{k+1} = -1$ and $u_i \circ u_j = 1$ when $|i - j| \geq 2$ can be replaced by the AC-generator $e_k = u_1 \cdots u_k$. Conversely, $u_k = e_{k-1} e_k$ (here $e_0 = 1$).

Indeed, the needed properties follow by inspection.

3. An AC generator $E$ of even or infinite size and a chain (i.e., the union) of commuting AC doubletons $D_k = \{d_{2k-1}, d_{2k}\}$ (i.e., $D_i \circ D_j = 1$ for $i \neq j$ and $d_{2k-1} \circ d_{2k} = -1$) are mutually replaceable.

Indeed, the repetitive replacement $E' = D \cup DE$, where $D = \{e_1, e_2\}$, yields a chain. Explicitly, the products $f_k = e_1 \cdots e_k$ entail $E' = \bigcup_n D_n$ with $D_n = \{d_{2n-1}, d_{2n}\}$, where

$$d_1 = e_1, \quad d_2 = e_2, \quad \text{and for } n \geq 2, \quad d_{2n-1} = f_{2n-2} e_{2n-1}, \quad d_{2n} = f_{2n-2} e_{2n}.$$

The replacement is self-invertible:

$$e_{2n-1} = d_1 \cdots d_{2n-2} d_{2n-1}, \quad e_{2n} = d_1 \cdots d_{2n-2} d_{2n}.$$ 

4. In particular, a finite even or infinite AC generator $E$ can be replaced by a generator $E'$ that admits a partition $E' = F_1 \cup F_2 \cup \cdots$ with mutually commuting AC components of finite even or infinite size. For an infinite $E$ the number of components can be either finite or infinite.
Indeed, the aforementioned integrations of doubletons or disintegrations into doubletons can be combined at will to obtain an arbitrary described replacement.

5. Let $E = K \cup M$, where $K \circ M = 1$, $K$ is finite and AC, and $M$ is commutative. If the size of $K$ is even and $M \neq \emptyset$, then $K$ can be increased by one, while if $K$ is of odd size it may be decreased by one.

Indeed, for $K = \{k_1, \ldots, k_{2j}\}$ and $m_0 \in M$, we define $k_0 = k_1 \cdots k_{2j} m_0$. Since $k_0 \circ K = -1$ then we can replace $K' = K \cup \{k_0\}$ and $M' = M \setminus \{m_0\}$. If $K = \{k_0, k_1, \ldots, k_{2j}\}$, then we augment $M$ by the total product $k_0 k_1 \cdots k_{2j}$, which commutes with all of $K$, and remove $k_0$ from $K$.

An AC generator is shown in the diagram below. Possible additional elements, independent of and commuting with the generator, are not displayed.

An AC generator is replaceable as described in Points 2 through 4:

In Point 5 we toggle between odd and even anticommutative portions of generators:

Remark 5.3

1. The aforementioned replacements can be expressed in terms of matrix transformations $e \mapsto P e$ (see the paragraph above (3)). For example, the replacement in (9) corresponds to $P = C_1$ from Example 3.5.1.

2. Commuting positive doubletons appeared before, isomorphically defined by (3). Negative doubletons required ‘complexification’ as shown below (3).

5.2 Orderly partitions

We assume that all sequences or corresponding sets appearing below are basic with at least two anticommuting elements.

Proposition 5.4 Let an element $g$ be independent of a finite anticommutative basic set $F$. Then there are three distinct possibilities:

1. $g$ commutes with $F$,

2. $g$ anticommutates with at least one element of $F$, in which case $F \cup \{g\}$ can be replaced
(a) either by an anticommutative generator,
(b) or by \( F' \cup \{ g' \} \), where \( F' \) is anticommutative and \( g' \) commutes with \( F' \).

Proof. The element \( g \) entails a partition of \( F = F_a \cup F_c \), where \( g \circ F_c = -1 \) and \( g \circ F_c = 1 \). Recall (Example 5.3.1) that for \( f_0 \in F \) the replacement \( F(f_0) = \{ f_0 \} \cup \{ f_0 f : f \in F, f \neq f_0 \} \) preserves anticommutativity. In addition we observe that

If \( F_a = \{ f_0 \} \) then \( g \) anticommutes with the anticommutative generator \( F(f_0) \). (19)

Suppose that \( g \) anticommutes with at least one element of \( F \), i.e., \( F_a \neq \emptyset \). If \( F_a \) is a singleton, then (19) yields case (a). Suppose there are at least two distinct elements \( f_0, f_1 \in F_a \).

Replace \( F \) by \( F' = F(f_0) \) and \( g \) by \( g' = f_0 f_1 g \). Then we check that \( |F'_a| = |F_a| - 2 \), preserving parity:

\[
g \circ \left\{ f_0, f_1, \ldots, f_a \right\} \mapsto f_0 f_1 g \circ \left\{ f_0, f_0 f_1, \ldots, f_a f_1 \right\},
\]

with values of the commutativity function marked beneath the elements. Therefore, by recursion we can reduce \( F_a \) either to the empty set, yielding (b), or to a singleton and then we use (19) to arrive in case (a).

\[\square\]

Remark 5.5 A partial extension is valid for infinite \( F \) when \( F_a \) or \( F_c \) is finite:

1. If \( F_a \) is finite then \( F \cup \{ g \} \) can be replaced

   (a) either by \( F' \cup \{ g' \} \) with an AC set \( F' \) and \( g' \circ F' = 1 \), for even \( |F_a| \),

   (b) or by an AC set \( F' \), for odd \( |F_a| \).

2. If \( F_c \) is finite and \( F_a \neq \emptyset \), then w.l.o.g. we may assume that \( F_a \) is finite.

The proof for a finite \( F_a \) mimics the corresponding proof above. The case of finite \( F_c \) follows by switching to \( F(f) \), where \( f \in F_a \).

\[\square\]

Recall the simple yet useful property (19). The equivalence relation \( G(e) = G(e') \) (i.e., sequences are mutually replaceable) between basic sequences extends to the relation of partial order \( G(e) \subset G(e') \) between (equivalence classes of) basic sequences. The first ‘orderly pattern’ will appear as follows:

\[
E \leftrightarrow E' \leftrightarrow D' = \begin{pmatrix}
T & O & O & \cdots & O \\
O & T & O & \cdots & O \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O & \cdots & O & T & O \\
O & \cdots & O & O & O
\end{pmatrix}
\]

(21)

with \( T \)'s of some varying sizes at least 2 along the diagonal. Suppose that all sizes are even. Then some or all of \( T \)'s may be turned to be odd if the commuting part appearing in the lower right corner is large enough, which can be always assumed by augmenting the system. See Remark 5.4 for more details.

Theorem 5.6 Let \( E \) be a signed basic set of length \( n \geq 2 \) with at least one pair of anticommuting elements. Then for some \( k = 1, \ldots, n \) there exists a replacement \( E' \) of \( E \) and its disjoint partition \( E' = F_0 \cup F_1 \cup \cdots \cup F_k \) such that for each \( j \geq 1 \) the set \( F_j \) is anticommutative of at least length 2, \( F_0 \) is commutative, and \( F_j \circ F_j = 1 \) for \( j \neq j', 0 \leq j, j' \leq k \).
Proof. If a finite signed basic set $E$ has at least one anticommutative pair, then there exists a nonempty anticommutative basic $F$ of maximal size. If $G(F) = G(E)$, we finish with $F_1 = F$ and $F_0 = \emptyset$. If $G(F) \neq G(E)$, we put $G = E \setminus F$ and then for any $g \notin G(F)$ we arrive in Case 1 of Proposition 5.4 since $F$ is maximal. That is, $F \cup \{g\}$ is replaced by $F' \cup \{g'\}$ with an anticommutative $F'$ and $g' \circ F' = 1$. If $G$ is commutative and commutes with $F$, then we are done with $F_1 = F$ and $F_0 = G$.

Otherwise, consider an anticommutative basic set $G_2 \subset G(E) \setminus G(F)$, maximal in size. If $G_2 \circ F = 1$, we put $F_1 = F$ and $F_2 = G_2$. If $G_2$ partially anticommutates with $F$, so each its element anticommutates with an even number of elements of $F$, since Proposition 5.4 enforces Case 1 in view of the maximality of $F$. Then, browsing $G$, we skip elements $g$ that commute with $F$ (i.e., that ‘even number’ is 0), and attend its subset $G'$ of elements $g$ admitting anticommutants in $F$. Then, for each of these elements $g'$ we apply (20) as many times as necessary to replace $F \cup \{g'\}$ by $F' \cup \{g'\}$. Note that the former elements $g$ still satisfy $g \circ F' = 1$ and $g \circ g' = -1$. This part of the algorithm ends when we reach the last element in $G$, finishing with $F_1 = F'$ and $F_2 = G \cup G'$.

Suppose that we arrived in a partition $E = E_k \cup G$, where $E_k = F_1 \cup \cdots \cup F_k$ with anticommutative maximal sets $F_i$ that commute among themselves. Suppose that the partition is not final. We need to show how to find a subset of $G$ to create $F_{k+1}$. So, let $G_{k+1}$ be a maximal in size anticommutative set independent of $E_k$ that partially anticommutates with $E_k$. We skip its subset that fully commutes with $E_k$, and attend elements, one by one, that have anticommutants in $E_k$. For each of these elements we apply procedure (20) with respect to consecutive $F_k$’s bearing anticommutants. We observe that the replacements $F \mapsto F(f)$ preserves the commutativity among $F_k$’s, and at the same time preserves the anticommutativity among $g$’s. This leads to $F_{k+1}$.

The algorithm must end after finitely many steps with the sought-for partition. ■

Let us summarize our findings, allowing some redundancy, to exhibit two extremal patterns within the framework of (21): one extreme a single $T$ and on the other doubletons yielding $2 \times 2$ tiny matrices $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**Theorem 5.7** Let $E$ be a basic signed set of size $n$ with at least two anticommuting elements. Then the following replacements exist.

1. There exists a replacement and its partition $E' = K \cup M$ such that $K$ is anticommutative, $M$ is commutative, and may be empty, and $K$ and $M$ commute.

2. Let $k$ denote the size of $K$ and $m$ denote the size of $M$. Then we may replace

   $$(k, m) \mapsto (k - 1, m + 1) \quad \text{if } k \text{ is odd}$$

   $$(k, m) \mapsto (k + 1, m - 1) \quad \text{if } k \text{ is even and } m \geq 1$$

3. There is a $K$ with the maximum even size $k = 2j$. In this case there is a replacement and partition $K' = K_1 \cup \cdots \cup K_j$ into the union of anticommutative doubletons that commute with each other and with $M$. Also, the $AC$-count is $c^-(E) = 2^{2n-1}(2^j - 1)$.

**Proof.** Consider any $F_i = \{f_1, f_2, \ldots\}$ in the partition stated in Theorem 5.6. The replacement of $F_i$:

$$(f_1, f_2, \ldots, f_n) \mapsto (f_1, f_2, \ldots, f_1, f_2, f_1, \ldots) = (f_1, f_2) (f_1, f_2, f_1, f_2, f_1, \ldots)$$

yields the partition of $F_i = F_{i1} \cup F_{i2}$ into two commuting anticommutative sets that still commute with all other sets $F_j$, $j \neq i$. We repeat the procedure while the size of $F_2$ is at least 3, ending with a partition of $F_i$ into doubletons with desired properties and perhaps a single leftover that commutes with everything. In the latter case we augment $F_0$ by that singleton.
The repetition of the algorithm for all components of the original partition entails Case 3, with a sequence of commuting doubletons. Now, given an anticommuting doubleton \{f_1, f_2\} and an anticommutating \(F' = \{f'\}\), and both commute, we replace their union by the anticommutating \(\{f_1, f_2, f_1 f_2 f' : f' \in F'\}\). The replacement does not change the commutativity with the remaining elements. Thus we end up with a \(K = \{f_1, \ldots, f_k\}\) of an even size \(k\), which proves the first statement.

However, we may still modify \(K\), as described in the second statement. If \(k < n\), i.e., there is at least one element \(f_0\) that commutes with all \(f_k\), then \(f_0 f_1 \cdots f_k\) anticommutates with \(K\), which increases the size of \(K\) by 1. The inverse replacement reduces the size of \(K\) by 1.

**Theorem 5.8** Let \(e = (e_1, \ldots, e_k)\) be a basic sequence of length \(n\), consisting of subsequences \(e_i\) of length \(\ell_i\) (so \(\ell_1 + \cdots + \ell_k = n\)), such that \(e_i \circ e_j = 1\) for \(i \neq j\). Let a commutative basic sequence \(d\) of length \(m\) commute with \(e\). Denote by \(c_i = c^-(e_i)\), the AC-count of the group generated by \(e_i\). Then

\[
c^-(e) = \frac{1}{2} \left(2^{2n} - \prod_{i=1}^{k} (2^{2\ell_i} - 2c_i)\right), \quad c^-(e, d) = 2^m c^-(e).
\]

In particular,

1. for an even \(n\) and the entire AC \(e\), or equivalently, for mutually commutative pairs of AC anticommutating elements \(e_i, i = 1, \ldots, k; c^-(e) = 2^{n-1}(2^n - 1)\)

2. for an odd \(n\) and the entire AC \(e\), the count equals \(c^-(e) = 2^{n-2}(2^{n-1} - 1)\).

**Proof.** Consider two arrays \(P = [p_1, \ldots, p_k], Q = [q_1, \ldots, q_k]\) and the sign

\[
s(P, Q) = e^p_1 \cdots e^p_k \circ e^q_1 \cdots e^q_k = (-1)^{C(p, q_1) + \cdots + C(p, q_k)}.
\]

Then AC-counts equal

\[
c^-(e_i) = \frac{1}{2} \sum_{p_i, q_i \in D_{c_i}} (1 - s_i(p_i, q_i)), \quad \frac{1}{2} \sum_{P, Q} (1 - s(P, Q)) = \frac{1}{2} \sum_{P, Q} (1 - s_1 \cdots s_k),
\]

which yields the sought-for formula. Denote for the sake of brevity \(N = c^-(e)\). Then, for the augmented sequences \((e, d)\) the AC-count follows from the diagram

|       | \(e^p\)       | \(d^q\)       | \(e^p d^q, q \neq 0\) |
|-------|---------------|---------------|------------------------|
| \(e^p\) | \(N\)         | \(2^m - 1\)   | \(N\)                  |
| \(d^q\) | \(\cdots\)    | \(\cdots\)    | \(\cdots\)             |
| \(e^p d^q, q \neq 0\) | \(2^m - 1\) \(N\) | \(2^m - 1\) \(N\) |

For the pairs, the counts \(s_i = 6\) entail the special case when \(n\) is even and \(e_i\) are commuting AC doubletons.

Finally, we invoke Theorem 5.7 that allows us to pool doubletons together or make a replacement with a chain of doubletons, and an odd AC generator may be reduced by one, leaving a commuting element outside.

**Remark 5.9** Recall partition (21). While even commuting AC basic sequences can be integrated into one AC sequence, whose size is the sum of the sizes of parts, this does not occur when at least one of the sizes is odd. Each odd sequence must be first reduced by one to make it even, and only then the resulting even sequences can be integrated into one.
For example, consider the sizes 9:7:2:(0), splitting a sequence of length 18, where the size of the commutative sub-generator commuting with all others appears in parentheses. Then we reduce two first sequences to arrive at the ratio 8:6:2:(2), setting aside two extra elements commuting with all. Next, we integrate the three even sequences into one AC sequence of length 16, i.e., 16:(2), which could be disintegrated at will, e.g., to 4:4:4:4:(2) or 12:4:(2), or else. Then the two extra elements can be added to some sequences, yielding, e.g., 5:5:4:4:(0) in the first case, or 13:5:(0) in the second case.

Remark 5.10 If $E$ is infinite, we encounter two ascending mutually commuting groups, with AC generators $K_n$ and commutative $M_n$. Unfortunately, our finite algorithms, developed thus far, even augmented by a handful of infinite procedures, are not suitable for infinite (countable) signed groups. In an infinite signed group $G$ with infinitely many anticommuting elements we can find an ascending sequence of proper subgroups $G_n$. The question is whether or not the union $\bigcup_n G_n$ is proper. Should the answer be affirmative, its sub-generator $K$ would entail other patterns listed above or in Proposition 5.2. The lack of a quick answer is tied to our method of consecutive enlargements that are based on replacements, i.e., while groups grow their generators constantly change with no stabilization detected.

5.3 Partitions with signatures

An AC-doubleton always has a pure generator, either negative (of quaternion type), or positive (of Pauli type). Indeed, a mixed generator $(e_1, e_2)$ with $e_1^2 = 1, e_2^2 = -1$ has the positive replacement $(e_1, e_1 e_2)$. Invoke Theorem 5.4 A commuting generator $M$ of size $m$ is either positive, or it has a replacement with any number $m' \geq 1$ of negative elements. Choose $m' = 1$ and name that single positive or negative element $g$ (one stands for all) a ‘leftover’.

Consider a generator $E$ with an AC-pair and invoke the chain of doubletons described in Theorem 5.3. Let $p$ and $q$ denote the number of positive and negative doubletons, respectively. That is, doubletons are either of Pauli type or of quaternion type. Assume that $M = \emptyset$ (so $E$ is even) or $M = \{g\}$ (so $E$ is odd). A larger positive commuting $M$ that commutes with everything else is irrelevant in our context, so we disregard it.

Therefore, we can assign the following temporary characteristic augmented by the signature $s = g^2$ of a leftover $g$ which appears only when the size $n$ of $E$ is odd:

\[
\langle p, q \rangle, \quad \text{for even } n = 2j, p + q = j, \\
\langle p, q; s \rangle, \quad \text{for odd } n = 2j + 1, p + q = j.
\]

We will see that there are factually either two or three characteristics, depending on parity.

We write $\langle p, q \rangle = \langle p', q' \rangle$ if the corresponding generators are mutually replaceable. We will see below that only the parity of the number of negative (or positive) doubletons matters. Thus, we arrive at the following taxa

\[
\langle 0, j \rangle, \langle 1, j - 1 \rangle, \quad \text{or} \quad \langle 0, j; + \rangle, \langle 1, j - 1; + \rangle, \langle 0, j; - \rangle,
\]

containing two groups for even $n = 2j$, or three groups for odd $n = 2j + 1$, respectively.
Theorem 5.11 If \( n = 2j \) is even, then \( \langle p, q \rangle = \langle p', q' \rangle \) iff \( p \cong p' \). In other words, we obtain two types of groups:

\[
\langle 0, j \rangle = \langle 2, j - 2 \rangle = \cdots \quad \text{or} \quad \langle 1, j - 1 \rangle = \langle 3, j - 3 \rangle = \cdots
\]

i.e., the first group has an even number while the second group has an odd number of positive doubletons.

If \( n = 2j + 1 \) is odd, then for \( s = - \) all decompositions \( p + q = j \) are equivalent, and for \( s = + \) two above groups appear.

Proof. Let \( n \geq 4 \) be even. We examine the last step in the integration and disintegration \((e_1, e_2, \ldots, e_n) \leftrightarrow (e_1 e_2) (e_3 e_4) \ldots \), described in the proof of Theorem 5.7 when a generator has length 4. As shown in Theorem 4.3 there are two types \((4, 0) \leftrightarrow (1, 3) \leftrightarrow (0, 4)\) and \((2, 2) \leftrightarrow (3, 1)\). We verify that for \( n = 4 \)

\[
\langle 2, 0 \rangle \leftrightarrow \langle 2, 2 \rangle \leftrightarrow \langle 0, 2 \rangle.
\]

(22)

The relation is just a symbolic expression of the explicit integration:

\[
(++)(++) \leftrightarrow (+ + +) \quad \text{and} \quad (- -)(-) \leftrightarrow (- - +).
\]

Also, \( \langle 1, 1 \rangle \leftrightarrow \langle 4, 0 \rangle \) since \((++)(-) \leftrightarrow (+ + +).\) This means that \( \langle 2, 0 \rangle = \langle 0, 2 \rangle\) and proves the statement for \( n = 4 \). Let \( n \geq 6 \). Then relation (22) says that every pair of doubletons of the same parity can be replaced by a pair of doubletons of the opposite parity. This completes the proof in the case of an even \( n \).

Let \( n \) be odd. If \( s = 1 \) (i.e. the leftover is positive), then the case is reducible to the above case in virtue of Theorem 5.7.2. So, let \( s = -1 \) (a negative leftover). Let us consider the chain of doubletons and integrate one of them with the leftover into a generator of length 3. Among three types of groups, \((3, 0) \leftrightarrow (1, 2)\) is of interest. By inspection, the integration/disintegration proceeds as follows:

\[
(++)(-) \leftrightarrow (++++) \quad \text{and} \quad (- -)(-) \leftrightarrow (- - +)
\]

Equivalently, in our symbolic notation:

\[
\langle 1, 0; - \rangle \leftrightarrow \langle 3, 0 \rangle \leftrightarrow \langle 1, 2 \rangle \leftrightarrow \langle 0, 1; - \rangle.
\]

In other words, with \( s = -1 \) at hand we may switch the parity of any doubleton, still within the same group. This completes the proof in the odd case.

5.4 Dual decomposition

It is natural to wonder about a dual decomposition, i.e. about the possibility of replacing \( E \) by \( E' = F_0 \cup F \), where \( F = F_1 \cup \cdots \cup F_k \) such that each \( F_k \) is commutative for \( k \geq 1 \), \( F_0 \) is anticommutative, and \( F_j \circ F_j' = -1 \) for \( j \neq j', 1 \leq j, j' \leq k \). \( F_0 \) would commute with other components.

Conversely, when we face such pattern we may try to find an ‘explanation’, i.e., a simple generator that ‘causes’ it. We will show how simple counts help to fulfill this objective.

Example 5.12 Let \( (f, g) \) have length \( N \), where \( f \circ g = -1 \) and both sequences are nonempty and commutative. Then the count \( c^- = 3 \cdot 2^{2N-3} \) and the pattern is ‘caused’ by just two anticommuting elements.
Proof. Consider $f^p g^q \circ f^s g^r = (-1)^{ps + qr}$. Let $f$ have length $i$ and denote the length of $g$ by $j = N - i$. Then
\[
e^{-} = \frac{1}{2} \sum_{p,r,q,s} \left(1 - (-1)^{ps + qr}\right) = \frac{2^{2N} - a^2}{2},
\]
where the number $a$ is easily computable, yielding the claimed count:
\[
a = \sum_{q,r} (-1)^{qr} = \sum_{q \geq 0, r} 1 + \sum_{q \geq 1, r \geq 0} 1 - \sum_{q \geq 1, r \geq 1} 1
\]
\[
= 2^{i - 1} \cdot 2j + 2^{j - 1} \cdot 2^{i - 1} - 2^{j - 1} \cdot 2^{i - 1} = 2^{N - 1}.
\]

Now we look for the even $n$, the maximal size of an AC basic sequence $e$ that together with $m = N - n$ commuting elements, also commuting with $e$, forms a generator. In virtue of Theorem 3.8, the AC count equals $2^{2m+n-1}(2^n - 1) = 2^{2N-n-1}(2^n - 1)$. Since the quantity is a group invariant, it also equals $3 \cdot 2^{2N-3}$. For an even $n$, we easily check that $2^n - 1$ is an odd multiple of 3. Hence, necessarily, $2N - n - 1 = 2N - 3$, so $n = 2$. In other words, just a single pair of AC elements 'causes' the given pattern, although not uniquely.

Several 'recipes' follow. Let $e_1 \circ e_2 = -1$, $d_p \circ d_q = 1$ for $p, q \in \{1, \ldots, N - 2\}$, and $e \circ d = 1$.

Recipe 1. In the first example we put
\[
f_1 = e_1, f_p = e_1 d_p, p = 2, \ldots, i - 1, \quad \text{and} \quad g_q = e_1 e_2 d_q, q = N - j - 1, \ldots, N - 2.
\]
Observe that $N - j - 1 = i - 1$, i.e., the last member in the first group is $e_1 d_{i-1}$ while the first member in the second group is $e_1 e_2 d_{i-1}$, so both generate $e_2$ and thus, together with $e_1$, all $d_p$'s. E.g., for $i = 5$ and $j = 3$:
\[
F = \{e_1, e_1 d_1, e_1 d_2, e_1 d_3, e_1 d_4\}, \quad G = \{e_1 e_2 d_4, e_1 e_2 d_5, e_1 e_2 d_6\}.
\]

Here both sets are pure, $\sigma(F) = \sigma(e_1)$ while $\sigma(G) = -\sigma(e_1) \sigma(e_2)$.

Recipe 2. In another example the sets may have arbitrary signatures $\pm 1$. First, let us 'pile' commuting elements as follows and then assign each product to one of two disjoint sets, $D_1$ of size $i$ or $D_2$ of size $j$, $i + j = N - 2$:
\[
d_1, d_4 d_2, \ldots, d_1 d_2 \cdots d_{N-2}.
\]

Then the commuting sets $F = \{e_1\} \cup e_1 D_1$ and $G = \{e_2\} \cup e_2 D_2$ anticommute and are pure of signatures of $e_1$ or $e_2$.

Recipe 3. Alternatively, let us use $e_1$ and $e_2$ of opposite signatures to build an arbitrarily mixed $F$, and let $e_1 e_2$ yield $G$, which must be pure.

General Recipe. The latter recipe immediately generalizes to an arbitrary number of anticommuting commutative generators. The signatures can be controlled by signatures of commuting elements $d_l$. Since the issue turns out to be rather elementary, we omit further details.

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