SUBMANIFOLDS OF SYMPLECTIC MANIFOLDS WITH CONTACT BORDER

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Abstract

We construct symplectic submanifolds of symplectic manifolds with contact border. The boundary of such submanifolds is shown to be a contact submanifold of the contact border. We also give a topological characterization of the constructed submanifolds by means of a “relative Lefschetz hyperplane Theorem”. We sketch some of the applications of the results.
1. Introduction.

A number of works have been developed from the foundational paper [Do96] that exploit the idea of ampleness in the symplectic and contact category. The key idea has been to adapt the concept of “linear system” to these cases. The techniques have provided a new insight in symplectic topology, giving as byproduct new symplectic invariants [Au99b, Do99]. In the contact case, it has been possible to mimic the symplectic counterpart [IMP99]. The results contained in [Pr00] open the way for constructing contact invariants although it will be less direct because, contrary to the symplectic situation, we do not have canonical constructions (up to symplectic isotopy).

The aim of this paper is to put together the two constructions, in the symplectic and contact category to push-forward the progress in the study of symplectic submanifolds with contact border. This concept appears naturally when defining “cobordisms” in the symplectic category.

We call sym-con category the category defined by symplectic manifolds with contact border. An object in this category is a set \((M, \omega, C, \theta)\) such that \((M, \omega)\) is an open symplectic manifold with compactification \(\overline{M} = M \cup C\) and \((C, \theta)\) is a cooriented contact manifold whose structure is compatible with the symplectic one in the usual sense (see Subsection 2.1 for details). Along the paper the dimension on the symplectic manifold \(M\) will be \(2n+2\), except when an explicit mention is made. The hermitian line bundle \(L\) whose curvature is \(-i\omega\) will be called prequantizable line bundle.

In this article we study the possibility of constructing submanifolds in this sym-con category with topological properties similar to divisors in complex projective geometry. We provide a complete topological characterization of these submanifolds which are approximately holomorphic in the sense of [Do95, Au97, IMP99]. The main result of this paper is

**Theorem 1.1.** Let \((M, \omega)\) be a symplectic manifold of integer class with prequantizable bundle \(L\) and with contact border \((C, \theta)\). Fix a rank \(r\) complex vector bundle \(E\) over \(M\). For \(k\) large enough, there exists a symplectic submanifold \(W\) of \(M\) which is Poincaré dual of \(c_r(L^k \otimes E)\), satisfying that \(W \cap C\) is a contact submanifold of \(C\). Moreover the inclusion \(i : W \rightarrow M\), induces an isomorphism in relative homology groups through the natural morphism \(H_p(W, W \cap C) \rightarrow H_p(M, C)\) for \(p < n - r\) and an epimorphism for \(p = n - r\).

First we do notice that Theorem 1.1 could follow by a more or less straightforward combination of ideas in [Au97] and [IMP99]. For this one might only to extend the contact submanifolds constructed in [IMP99] to a small neighborhood of \(C\) in \(M\). However, this kind of approach presents problems of difficult solution. We detail a little more this question, using the approximately holomorphic tools, in Subsection 2.3 (cfr. Remark 2.13).

So we have chosen an alternative way to attack the problem. We will define directly global sections which solve the problem near the border. For this we need to revise the local theory developed in [IMP99] to [Pr00]. The solution goes through the use of approximation theory and a refined Jackson’s theorem. Jackson’s theorems [Ch66] provide bounding for the error made when we approximate a differentiable function by a polynomial of a given degree in terms of the derivatives of the function. This kind of results
were used by S. Donaldson implicitly in the foundational work [Do96], when he approximated asymptotically holomorphic functions by polynomials. But now, we need a similar result for any function such that we only control the norm of the derivatives.

In Section 2 we will state the approximately holomorphic theory in the sym-con category, following the notations of [Do99, Pr00]. We reduce the proof of Theorem 1.1 to a transversality result in the border. In Section 4 we prove the relative Lefschetz hyperplane theorem as stated in Theorem 1.1 and, also we characterize the Chern classes of the constructed submanifolds.

We must stress that the topological results are new even in the integrable complex case and offer a new insight in the topological structure of the sym-con manifolds, showing again that the idea of Eliashberg of studying contact manifolds as the more natural border definition in the symplectic and kähler category is powerful and rich in consequences.

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2. Definitions and results.

We state along this Section the basic notions in symplectic and contact topology needed in what follows. Also we state the main result in terms of the tools introduced and sketch the idea of the proof of Theorem 1.1.

2.1. Sym-con manifolds. The symplectization $S_D(C)$ of a contact manifold $(C, D)$ is defined as

$$S_D(C) = \{ \theta \in T^*C : \text{Ker } \theta = D(\pi(\theta)) \},$$

where $\pi : T^*C \to C$ is the standard projection. It is easy to check that this manifold has a canonical exact symplectic structure provided by the exterior differential of the Liouville 1-form

$$\alpha(v_\theta) = \theta(\pi_*(v)), \forall v_\theta \in TS_D(C).$$

The symplectization $S_D(C)$ has structure of a $\mathbb{R}^*$-principal bundle over $C$. We are particularly interested on the cooriented case (called exact case as well). The contact manifold is said to be cooriented if there exists a global 1-form $\theta$ in $C$ satisfying that $\text{Ker } \theta = D$. In that case the form $\theta$ provides a section of $S_D(C)$ and this becomes a trivial bundle. Fixing a 1-form $\theta$ we can identify canonically $S_D(C) = C \times (-\infty, 0) \cup (0, \infty)$. In this paper we will only use the cooriented case and from now on we call symplectization to the connected component $C \times (0, \infty)$ instead of the total set. We can give an explicit formula for the symplectic form in that manifold, we set up the following (canonical) isomorphism:

$$S_D(C) \to C \times (0, \infty)$$

$$\lambda \cdot \theta(x) \to (\pi(\theta(x)), \lambda).$$

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1 Geometría simpléctica con técnicas algebraicas, CSIC-UC3M, 2000.
Then we obtain
\[ d\alpha = d\lambda \wedge \pi^*\theta + \lambda \cdot \pi^*d\theta. \]
The contact manifold \( C \) can be embedded in the symplectization through the graph of \( \theta \), namely as the contact hypersurface \( \tilde{C}_\epsilon = \{(p, \lambda) \in C \times \mathbb{R}^+ : \lambda = \epsilon\} \). This embedding will be called the \( \epsilon \)-embedding of \( C \) in \( S_D(C) \), denoted as \( i_\epsilon \). If there is not risk of confusion, we usually denote \( C_1 \) by \( \tilde{C} \). Recall that through this family of embeddings the distribution \( D \) defines a distribution \( \tilde{D} \) of 2-codimensional spaces in \( S_D(C) \), it is obvious that \( \tilde{D} \) is symplectic with respect to the canonical symplectic structure in the symplectization.

Recall that a symplectic manifold \((M, \omega)\) has a contact border \( C \) when we can identify through a symplectomorphism a neighborhood \( V \) of the border \( C \) with one of the two standard models: \( C \times [a, b) \) or \( C \times (a, b] \), for some \( a, b \in \mathbb{R}^+ \), with \( a < b \). In the first case we will say that the manifold has concave border and in the second one convex border. Moreover we can generalize the definition to include mixed cases. So in general a symplectic manifold has concave border and in the second one convex border. Morever we denote as \( \phi \) the symplectization.

An important observation is that convex and concave models are not equivalent and provide very different problems in the sym-con category (see i.e. the pseudo-holomorphic curves construction in [El98]).

**Remark 2.1.** In the literature, definitions above are called sometimes “strictly convex (or concave) contact borders” to distinguish them from a weaker notion defined by Eliashberg as follows. We will say that a symplectic manifold \((M, \omega)\) has “weak” contact borders \((C_1^1, C_2^1, \ldots, C_1^\infty, C_2^1, \ldots, C_2^\infty)\), where each \( C_i^j \) is a connected contact cooriented manifold, if we can decompose a neighborhood \( V \) of the border into \( cc + cv \) connected components \( V_1^i, V_2^j \) \((i = 1, \ldots, cc \text{ and } j = 1, \ldots, cv)\) such that each of the \( V_1^i \) is symplectomorphic to the local concave model defined by \( C_i^1 \) and respectively with \( V_2^j \) and the convex model.

As in the closed manifold case we can construct a complex line bundle, \( L \), over a symplectic manifold \((M, \omega)\) whose curvature form is \(-i\omega\), provided an integrality condition is satisfied, namely \([\omega/2\pi]\) has to be the lifting of an integer class. This bundle is usually called a prequantizable bundle, because of the geometric quantization setting. Moreover the precedent considerations assure that the bundle extends to the border defining a line bundle whose curvature form is \(-i\alpha\) (under the standard models identifications) in each connected component. We will denote by \( L \) the prequantizable bundle in \( M \) and also its extension to \( C \).
**Definition 2.2.** A sym-con manifold \((M, \omega, C, \theta)\) is a symplectic manifold \((M, \omega)\) with compactification \(\bar{M}\) satisfying that \(\bar{M} - M = C\) admits a contact structure \(\theta\) which defines a contact border for \(M\).

The following trivial result is the symplectic analogue of the connected sum theorem in topology.

**Lemma 2.3.** Given two sym-con manifolds \((M_1, \omega_1, C, \theta)\) and \((M_2, \omega_2, C, \theta)\) with connected convex and concave borders respectively. Then, for a suitable nonzero constant \(\lambda\), there exists a closed symplectic manifold \((\tilde{M}, \omega)\) and two symplectic embeddings \(\varphi_1 : (M_1, \omega_1) \rightarrow (M, \omega)\) and \(\varphi_2 : (M_2, \lambda \omega_2) \rightarrow (M, \omega)\) satisfying that \(\varphi_1(M_1) \cup \varphi_2(M_2) = M\).

The manifold \(M\) is usually denoted as \(M_1 \cup_C M_2\) and topologically is a connected sum along \(C\).

**Proof:** We have only to use the standard models of the borders to glue the manifolds symplectically. Say that near the border the local model for \(M_1\) is \(C \times (a_1, b_1]\) and for \(M_2\) is \(C \times [a_2, b_2)\). If we find that \((a_1, b_1] \cap [a_2, b_2) \neq \emptyset\) then we are finished. If not we substitute the symplectic form \(\omega_2\) by \(\lambda \omega_2\). This produces a change in the local model of \(M_2\) which is now \(C \times [\lambda a_2, \lambda b_2)\). Obviously a suitable choice of \(\lambda\) reduces the problem to the precedent one.

Recall from the proof that it is not very important that the contact forms chosen in the two borders coincide, if the distribution is the same. In fact, the symplectic connected sum along a contact border does not depend on this choice, because the symplectic structure of the symplectization does not depend on the choice of contact form.

We can always add a symplectic collar in the border of a sym-con manifold. This is the content of the following

**Corollary 2.4.** Let \((M, \omega, C, \theta)\) a sym-con manifold, then we can find a manifold \((M', \omega', C, \theta)\) such that there exists a symplectic embedding of \((M, \omega)\) in \((M', \omega')\) satisfying that the compactification of \(M\) does not intersect the border \(C\) of \(M'\).

**Proof:** It is a direct application of the precedent Proposition choosing \(M_1 = M\) and \(M_2 = C \times [1/2, 3/2)\) if the border of \(M\) is convex (resp. \(M_2 = C \times (1/2, 3/2]\) if the border is concave).

**Definition 2.5.** A contact hypersurface \((C, \theta)\) in a symplectic manifold \((M, \omega)\) is a hypersurface in \(M\) supporting a 1-form \(\theta\) such that \(D = \text{Ker} \theta\) is a contact distribution and \(d\theta|_D = \omega|_D\).

If, using Corollary 2.4, we add a symplectic collar to a sym-con manifold \((M, \omega, C, \theta)\) then, the submanifold \(C\) is a contact hypersurface in the enlarged manifold \(M'\). We will use this idea afterwards.

We define a compatible almost-complex structure \(J\) in a sym-con manifold \((M, \omega, C, \theta)\) as a compatible almost-complex structure in \((M, \omega)\) such that the restriction of \(J\) to the contact border \(C\) leaves invariant the distribution \(D\). By using the local model it is obvious that in this case the restriction of \(J\) to the distribution \(D\) provides a compatible almost-complex structure in the symplectic bundle \(D\). It is easy to check that the moduli
space of such structures is contractible. For this we use the same arguments that in the symplectic and in the contact case.

As always, when we fix a compatible almost-complex structure, we automatically obtain a metric \( g \) on the manifold \((M, \omega)\) as \( g(v, w) = \omega(v, Jw) \). We refer to this metric as the symplectic metric. We define also the \( k \)-rescaled symplectic metric as \( g_k = kg \).

2.2. Contact manifolds. Now, we recall some basic ideas about contact geometry. We assume that \((C, D)\) is a cooriented contact manifold where we have fixed a contact form \( \theta \). This contact form determines a vector field \( R \) by the conditions:

\[
i_R \theta = 1, \quad i_R d\theta = 0,
\]

which is called the Reeb vector field. As in the symplectic case when we fix a compatible almost-complex structure \( J \) we obtain a metric in the contact manifold as \( g(v, w) = \theta(v)\theta(w) + d\theta(v, Jw) \), which is called the contact metric. The \( k \)-rescaled contact metric is defined as \( g_k = kg \). We are abusing notation by using the same letter to denote the symplectic and contact metrics, but it is easy to check that in a sym-con manifold the restriction of the \( k \)-rescaled symplectic metric to the contact border coincides with the precedent definition, justifying our notation. However, an important change of behaviour appears in the contact case. Namely the \( k \)-rescaled symplectic metric is the symplectic metric associated to the form \( k\omega \), but in the contact case the \( k \)-rescaled contact metric is not the contact metric associated to \( k\theta \). This difference is fundamental to develop the theory and will reflect, in the contact case, the localization process which appears in Donaldson’s theory. We formalize this idea with the following definitions.

Definition 2.6. The maximum angle between two subspaces \( U, V \in \text{Gr}_R(r, n) \) is defined as

\[
\angle_M(U, V) = \max_{u \in U} \max_{v \in V} \angle(u, v).
\]

This angle defines a distance in the topological space \( \text{Gr}_R(r, n) \) (for details see [MPS99]).

Definition 2.7. Let \( D_k \) be a sequence of contact distributions in \( \mathbb{R}^{2n+1} \). The sequence is called \( c \)-asymptotically flat in the set \( U \in \mathbb{R}^{2n+1} \) if

\[
\angle_M(D_k(0), D_k(x)) \leq ck^{-1/2}, \text{ for all } x \in U.
\]

The sequence is called asymptotically flat if there exist some \( c \) for which it is \( c \)-asymptotically flat.

The standard contact structure in \( \mathbb{R}^{2n+1} \) is defined as \( \theta_0 = ds + \sum_{j=1}^n x_j dy_j \), where \((x_j, y_j, s) \in \mathbb{R}^{2n+1} \). The \( k \)-rescaled contact metric is the contact metric associated to \( \theta_{k^{1/2}} = k^{1/2}ds + \sum_{j=1}^n x_j dy_j \), which is obtained from \( \theta_0 \) scaling the coordinates by a factor \( k^{1/2} \). So, it is clear that, at any small neighborhood of a given point, when we apply the set of metrics \( g_k \) we obtain as a result, passing to a fixed Darboux trivialization, a sequence of contact forms \( \theta_{k^{1/2}} \), by scaling with a factor \( k^{1/2} \) in \( \mathbb{R}^{2n+1} \), which is obviously asymptotically flat in any bounded set in \( \mathbb{R}^{2n+1} \).
Given any asymptotically flat sequence of distributions $D_k$ in $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$, satisfying that $D_k(0) = \mathbb{C}^n \times \{0\}$, there exists a canonical almost-complex structure in a neighborhood of the origin, for $k$ large enough. We only have to lift the complex structure defined in $\mathbb{C}^n$ to the distribution $D_k$ using the pull-back of the vertical projection (which is an isomorphism near the origin for $k$ large enough).

Finally if we have a contact hypersurface $(C, \theta)$ in a symplectic manifold $(M, \omega)$ we can choose a compatible almost-complex structure which makes the distribution $D J$-invariant. In fact, in this case we can identify symplectically a neighborhood of $C$ with a neighborhood of the 1-embedding of $C$ in the symplectization $SD(C)$ and the almost-complex structure can be chosen to make the distribution $D J$-invariant in this neighborhood (through the identification). This kind of almost complex structures will be called compatible with the hypersurface. Suppose that we have added a symplectic collar to a sym-con manifold. A compatible almost-complex structure $J$ in the sym-con manifold admits an extension to an almost complex structure $\tilde{J}$ in the enlarged manifold which is compatible with the contact hypersurface $C$.

2.3. Sequences of bounded sections. We recall from [Do99, Pr00] the approximately holomorphic setting. We adapt it to our present work and ideas. A uniform constant, polynomial, etc. is a constant, polynomial, etc. which does not depend on the chosen point of the sym-con manifold nor the integer $k$ appearing in the context.

Now, we introduce the notion of asymptotically holomorphic sections which is one of the key points. All the norms in the definitions to follow are defined with respect to the sequence of metrics $g_k$.

**Definition 2.8 ([Do99]).** A sequence of sections $s_k$ of the hermitian bundles $E_k$ over the symplectic manifold $(M, \omega)$ has $C^r$-bounding $c$ at the point $x$ if it satisfies

$$|s_k(x)| < c, \\
|\nabla^j s_k(x)| < c, \forall j = 1, \ldots, r, \\
|\nabla^{j-1}\bar{\partial}s_k(x)| < ck^{1/2}, \forall j = 1, \ldots, r.$$ 

The sequence has uniform $C^r$-bounding $c$ if it satisfies these boundings at every point.

**Definition 2.9 ([Pr00]).** A sequence of sections $s_k$ of the hermitian bundles $E_k$ over the contact manifold $(C, \theta)$ has mixed $C^r$-boundings $(c_D, c_R)$ at the point $x$ if it satisfies

$$|s_k(x)| < c_D, \\
|\nabla_D s_k(x)| < c_D, \forall j = 1, \ldots, r, \\
|\nabla_R s_k(x)| < c_R, \forall j = 1, \ldots, r, \\
|\nabla^{j-1}\bar{\partial}s_k(x)| < c_Rk^{-1/2}, \forall j = 1, \ldots, r.$$ 

The sequence has uniform mixed $C^r$-boundings $(c_D, c_R)$ if it satisfies these boundings at every point.
Definition 2.10. A sequence of sections \( s_k \) of the hermitian bundles \( E_k \) over the sym-con manifold \((M, \omega, C, \theta)\) has global \( C^r \)-boundings \( (c, c_D, c_R) \) at a point \( x \in C \) if \( s_k \) restricted to \( M \) has \( C^r \)-bounding \( c \) and restricted to \( C \) has uniform mixed \( C^r \)-boundings \((c_D, c_R)\).

As usual the \( C^r \)-openness is important in this kind of definitions. Namely, if we have sections \( s_k^1 \) and \( s_k^2 \) with global \( C^r \)-boundings \((c^1, c_D^1, c_R^1)\) and \((c^2, c_D^2, c_R^2)\) then \( s_k^1 + s_k^2 \) has global \( C^r \)-bounds \((c^1 + c^2, c_D^1 + c_D^2, c_R^1 + c_R^2)\). An analogous property is satisfied by the other types of boundings.

Definition 2.11 also applies to sequences of sections defined over a symplectic manifold \( M \) which contains a contact hypersurface \( C \), being the definition in this case the natural one.

The other key ingredient is the notion of transversality with estimates. We set it up in a general way following [MP99].

Definition 2.11. Let \( s \) be a section of a complex vector bundle \( E \) over the Riemannian manifold \( X \) with distribution \( D \), and \( \eta > 0 \). The section \( s \) is said to be \( \eta \)-transverse to \( 0 \) along \( D \) at a point \( x \in X \) if it is satisfied at least one the following conditions:

1. \( |s(x)| < \eta \),
2. the covariant derivative restricted to \( D \), \( \nabla_D s : D_x \subset T_x X \to E_x \), is surjective and has a right inverse of norm less than \( \eta^{-1} \).

The section is \( \eta \)-transverse to \( 0 \) along \( D \) in a set \( U \), if it is \( \eta \)-transverse at all the points of \( U \).

In the symplectic case \( D = TM \) and in the contact case \( D \) is the contact distribution. This definition is \( C^1 \)-open in the sense that there exists a constant \( c_0 \), only depending in the dimensions, such that if \( s \) is \( \epsilon \)-transverse to \( 0 \) along \( D \) and \( |\sigma - s| < \alpha \) then \( \sigma \) is \( (\epsilon - c_0 \alpha) \)-transverse to \( 0 \) along \( D \). It is possible to precise a little more in the contact case. Namely, again there exists a constant \( c'_0 \) such that if \( s \) is \( \epsilon \)-transverse to \( 0 \) along \( D \) and \( |\sigma - s| \) has mixed \( C^1 \)-boundings \((\alpha, c_R)\) then \( \sigma \) is \( (\epsilon - c'_0 \alpha) \)-transverse to \( 0 \) along \( D \).

With these definitions at hand we reduce the proof of Theorem 1.1 to the following:

Proposition 2.12. Let \((M, \omega)\) be a symplectic manifold of integer class with contact border \((C, D)\). Let \( U \) a compact set in \( M \) which does not intersect \( C \). Fix a rank \( r \) complex vector bundle \( E \) over \( M \). Fix a constant \( \epsilon > 0 \) and a compatible almost-complex structure in the sym-con manifold. Let \( s_k \) be a sequence of sections with global \( C^3 \)-boundings of the bundles \( E \otimes L^\otimes k \). Then there exists a sequence of sections \( \sigma_k \) with global \( C^3 \)-boundings such that \( |s_k - \sigma_k|_{C^1, U} < \epsilon \) and satisfying that \( \sigma_k \) is \( \eta \)-transverse to \( 0 \) in \( M \) and \( \eta \)-transverse to \( 0 \) along \( D \) in \( C \).

Observe that near the border we cannot control the \( C^0 \)-norm of the perturbation.

Proof of the existence part of Theorem 1.1: Take a sequence of sections \( \sigma_k \) given by Proposition 2.12. We only have to apply Lemma 5 of [MP99] to the manifold \( M \) and to the border \( C \) respectively to obtain that the zero sets are symplectic and contact. Only notice that the asymptotically holomorphic sequences of that article correspond to our \( C^r \) and mixed \( C^r \)-boundings. \( \square \)
Remark 2.13. A direct approach for proving Proposition 2.12 should be to define a mixed $C^r$-bounded sequence of sections in the border $C$ which be $η$-transverse to $0$ provided by [IMP99, Pr00] and try to extend the sequence to the symplectic manifold. In fact, the boundings in the derivatives work to produce this extension and the holomorphicity condition gives us the derivatives of a given section in the normal direction. But making the computations in detail we find that we are able to extend the construction in an asymptotically holomorphic way only to a strip of $g_1$-radius $O(k^{-1/2})$ from the border. This is not enough to multiply by a cut-off function and so to define the section all over $M$, because the global boundings are destroyed. We would need a strip of $g_1$-radius $k^{-1/3}$, but the arrangement to get the boundings in this strip is not clear. In the next paragraphs we explain the method of proof that we use to overcome this difficulty.

2.4. Proof of Proposition 2.12. We state from the results of [Au97] the following

Theorem 2.14 (Adaptation of Theorem 2 in [Au97]). Let $E$ be a complex vector bundle of rank $r$ over a symplectic manifold $(M, ω)$ of integer class (not necessarily compact). Let $J$ be a compatible almost-complex structure. Fix a constant $ε > 0$ and a compact set $U$ in $M$, and let $s_k$ a sequences of sections with $C^r$-bounding $c$ of the bundles $E \otimes L^\otimes k$.

Then there exists a uniform constant $η > 0$ (depending only on $ε$ and $c$) and a sequence $σ_k$ of sections with $C^r$-bounding $ε$ such that $s_k + σ_k$ is $η$-transverse to $0$ over $U$.

Proof: The only difference with respect to Auroux’ result is that we do not impose the closedness of the manifold $M$. But Auroux techniques are purely local. So there is no reason to impose the closedness of the manifold $M$. The only important point is to guarantee the compactness condition and this is assured by restricting ourselves to a compact set $U \subset M$. □

The existence of the border makes impossible to set up the 1-parametric discussion of [Au97] as is shown in [IMP99, Pr00]. We want to reduce the proof of Proposition 2.12 to the following result

Theorem 2.15. Let $ε > 0$, $α > 0$. Given a cooriented contact manifold $(C, θ)$ and given the 1-embedding of $C$ in the symplectization $S_D(C) = C \times \mathbb{R}$, fix a complex vector bundle $E$ over the symplectization. Then given a global $(c, c_D, c_R)$-bounded in $C^r$-norm sequence of sections $s_k$ of the bundles $E \otimes L^\otimes k$, there exists another sequence of sections $τ_k$ with global $(c', ε, c'_R)$ $C^r$-boundings satisfying, for $k$ large enough, that

1. $τ_k$ is supported in $C \times (1 − α, 1 + α)$.
2. The restriction to $\hat{C}$ of $s_k + τ_k$ is $η$-transverse to $0$ along the distribution $D$ in $\hat{C}$, for some uniform constant $η > 0$.

We assume this result, which will be proved in Section 3 and then we prove:

Proof of Proposition 2.12: Fix a compatible almost-complex structure in the sym-con manifold. Enlarge, adding a symplectic collar, the sym-con manifold $(M, ω, C, θ)$ to obtain a new symplectic manifold $M'$ where $C$ is a closed contact hypersurface. For $α > 0$ small enough we can identify
symplectically a neighborhood $V$ of $C$ with the neighborhood $C \times (1 - \alpha, 1 + \alpha)$. We can extend the almost-complex structure with one compatible with the hypersurface. Fix a sequence of sections $s_k$ with uniform $C^r$-bounding $c$ in $(M', \omega')$, obviously $s_k$ has global $C^r$-bounds $(c, c, c)$ in the initial manifold $M$.

Then we apply Theorem 2.13 to $V \simeq C \times (1 - \alpha, 1 + \alpha)$ perturbing the sequence $s_k$ to obtain a new sequence $\sigma_k$ which is $\eta$-transverse along the distribution $D$ on $C$. To finish we perform a perturbation $\tau_k$ of $C^1$-norm less than $\frac{\eta}{2c_0}$, where $c_0$ is the constant of $C^1$-openness, satisfying that $\sigma_k + \tau_k$ is $\eta'$-transverse in the compact set $M \cup C \times [1 - 1/2\alpha, 1 + 1/2\alpha]$. Use the $C^1$-openness of the transversality of sections along the distribution $D$ in $C$ to assure that $\sigma_k + \tau_k$ is still $\eta/2$-transverse to $0$ along $D$ in $C$. This finishes the proof.

**Remark 2.16.** Observe that the process followed in the proof is not symmetrical, i.e. we cannot perturb first the sequence to obtain symplecticity and later on to obtain contactness, because the perturbations needed to get contactness are not $C^1$ small and so they destroy the achieved simplicity.

One of the most surprising points of the result is that we cannot assure $C^0$-closedness between the initial and the perturbed sections. But, curiously, Donaldson’s techniques which are based in this phenomenon continue applying. For this we will need to control the behaviour of the sections in a certain sense which will be apparent along the proofs.

## 3. Achieving transversality in local neighborhoods.

Along this Section we are going to prove Theorem 2.13. We will assume all the local transversality results developed in [Do96] and [Pr01], but we need a further refinement to prove the result.

### 3.1. Approximately holomorphic models

We will use the following Lemma to trivialize the sym-con manifold in the border. (As before we will enlarge a little the manifold to change the border into a contact hypersurface). We denote by $C_0$ the subspace of $\mathbb{R}^{2n+2}$ defined as

$$\{(0, y_0, x_1, y_1, \ldots, x_n, y_n) : x_i \in \mathbb{R}, y_j \in \mathbb{R}\}.$$ 

Moreover we will identify $C_0$ with $\mathbb{C}^n \times \mathbb{R}$ in the natural way.

**Lemma 3.1.** Given a closed contact manifold $(C, \theta)$ and a compatible almost-complex structure $J$, construct the 1-embedding of $C$ into the symplectization $S_D(C) \simeq C \times \mathbb{R}^+$, and denote it by $\hat{C}$. Fix a point $x \in \hat{C}$. There exists a uniform constant $c > 0$ and a symplectic Darboux chart $\varphi : (B_g(x, c), \omega) \to (\mathbb{R}^{2n+2}, \omega_0)$ satisfying that: $\varphi(x) = 0$, $\varphi^*J_0(0) = J(x)$, $\varphi^*D(x)$ is a complex subspace, $\varphi^{-1}(C_0) = \hat{C}$ and also

$$\frac{1}{2}g(v, w) \leq \langle (\varphi)_y v, (\varphi)_y w \rangle \leq 2g(v, w), \forall y \in B_g(x, c), \quad v, w \in T_yS_D(C).$$

This implies that $|\nabla^r \varphi| = O(1)$ and $|\nabla^r \varphi^{-1}| = O(1)$, for $r=1,2,3$. Also $|\partial \varphi(y)| \leq c'd(x, y)$, for a uniform constant $c'$. Denote by $\hat{\varphi}$ the restriction of $\varphi$ to $\hat{C}$. The distribution $\hat{\varphi}_*D$ of $C_0 \simeq \mathbb{C}^n \times \mathbb{R}$ can be equipped with the canonical almost-complex structure $\hat{J}_0$ (obtained by vertical
lifting) and then \( |\partial \hat{\varphi}(y)| \leq c'd(x,y) \), for all \( y \in B_y(x,c) \cap \hat{C} \), where the operator \( \hat{\partial} \) is computed respect to \( \hat{J} \) and \( \hat{J}_0 \).

**Proof:** We choose a symplectic Darboux chart at \( x \), \( \varphi : B_{g_k}(x, c) \to V \subset \mathbb{R}^{2n+2} \). The constant \( c \) can be chosen in a uniform way because of the compactness of \( C \). We need to assure also that the standard complex structure \( J_0 \) in \( \varphi_* D \subset \mathbb{R}^{2n+1} \) and \( \varphi_* J \) coincide at \( \hat{\varphi}(x) = 0 \). We only have to modify the Arnold’s proof of the contact case. We first use the symplectic Darboux Theorem to obtain Darboux coordinates \( \varphi(y) = (p_0, \ldots, p_n, q_0, \ldots, q_n) \).

Following [Ar80] we can assure that the embedding of the contact manifold is locally given by the equation \( p_0 = 0 \). Notice that in general \( D_x = \hat{\varphi}_* D(x) \neq \{ p_0 = q_0 = 0 \} \). But we can choose a standard symplectic basis \( (e_1, \ldots, e_n, f_1, \ldots, f_n) \) in \( D_x \). Also we can choose a standard symplectic basis \( (e_0, f_0) \) in \( D_x^\perp \), assuring that \( p_0(e_0) = 0 \). The orthogonal operation is made with respect to the symplectic form in the symplectization. Now, we define the transformation:

\[
\eta : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2} \\
\frac{\partial}{\partial p_i} \to e_i \\
\frac{\partial}{\partial q_i} \to f_i.
\]

The map \( \eta \) is symplectic and if we compose \( \eta \circ \hat{\varphi} \) we obtain that, in these new Darboux coordinates, denoted again by \( (p_0, q_0, \ldots, p_n, q_n) \), \( C \) is locally defined by the equation \( p_0 = 0 \) and also \( D_x \) is complex, in fact \( D_x = \{ p_0 = q_0 = 0 \} \). Finally performing a symplectic transformation in \( D_x \) we can assure that \( J|_D = (J_0)|_D \). Observe that \( D_x^\omega \) is also complex and then a \( Sp(2) \) transformation there makes that \( J(x) = J_0 \).

So we have checked that \( \varphi^*(J_0)(x) = J(x) \) and \( \hat{\varphi}^*(J_0)|_D(x) = J|_D(x) \) at the point \( x \). We cannot assure more because the two complex structures are related through a, in general non-vanishing, Nijenhuis type tensor at the origin. The last inequalities in the statement of the Lemma are assured by the fact that \( \varphi \) is a isometry at \( x \) and by the compactness of \( C \). Now following the discussion in Section 2 of [Do96] it is easy to verify that the boundings in the antiholomorphic parts are as given. \( \square \)

After scaling, the precedent result appears as

**Lemma 3.2.** Given a closed contact manifold \( (C, \theta) \) and a compatible almost-complex structure \( J \). Construct the 1-embedding of \( C \) into the symplectization \( S_D(C) \cong C \times \mathbb{R}^+ \), and denote it by \( \hat{C} \). Fix a point \( x \in \hat{C} \). Then there exists a uniform constant \( c > 0 \) and a symplectic Darboux chart \( \varphi_k : (B_{g_k}(x,c), k\omega) \to (\mathbb{R}^{2n+2}, \omega_0) \) satisfying that \( \varphi_k(x) = 0 \), \( \varphi_k^*(J_0) = J(x) \), \( (\varphi_k)_* D(x) \) is a complex subspace, \( \varphi_k^{-1}(C_0) = \hat{C} \) and also

\[
\frac{1}{2} g_k(v, w) = ((\varphi_k)_y v, ((\varphi_k)_y w) \leq 2 g_k(v,w), \forall y \in B_y(x,c), v, w \in T_y S_D(C).
\]

This implies that \( |\nabla^r \varphi_k| = O(1) \) and \( |\nabla^r \varphi_k^{-1}| = O(1) \), for \( r=1,2,3 \). Also \( |\partial \varphi_k(y)| = O(1) \), for a uniform constant \( c' \). Denote by \( \hat{\varphi}_k \) the restriction of \( \varphi_k \) to \( \hat{C} \). Then the distribution \( (\varphi_k)_* D \) is a sequence of asymptotically flat contact distributions in \( \mathbb{R}^{2n+1} \) that are
equipped with the canonical complex structure $\hat{J}_0$ (obtained by vertical lifting) and then $|\nabla^r \tilde{\varphi}_k(y)|_{g_k} = O(k^{-1/2})$, for all $y \in B_{g_k}(x, c) \cap \hat{C}$ and $r = 0, 1, 2$, where the operator is computed respect to $J$ and $\hat{J}_0$.

**Proof:** It follows by composing the map $\varphi$ obtained in Lemma [3.1] with the scaling map $\lambda_k : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ defined as $\lambda_k(z) = k^{1/2}z$. Then all the boundings are automatic. The only point is to assure that $|\nabla^r \tilde{\varphi}_k(y)|_{g_k} = O(k^{-1/2})$. For $r = 0$ it is a trivial consequence of Lemma [3.1]. For $r \geq 1$ follows from $|\nabla^r \varphi_k| = O(k^{-(r-1)/2})$. The same occurs with $\partial \tilde{\varphi}_k$ and its derivatives. \hfill $\Box$

**Definition 3.3.** A sequence of sections $s_k$ of hermitian bundles $E_k$ with connections has Gaussian decay in $C^r$-norm away from the point $x \in M$ if there exists a uniform polynomial $P$ and a uniform constant $\lambda > 0$ such that for all $y \in M$, $|s(y)|$, $|\nabla s(y)|_{g_k}$, ..., $|\nabla^r s(y)|_{g_k}$ are bounded by $P(d_k(x, y)) \exp(-\lambda d_k(x, y))$. Here $d_k$ is the distance associated to the metric $g_k$.

The following result is used to trivialize bundles in an approximately holomorphic way.

**Lemma 3.4 ([Do96, An97]).** Given any point $x \in M$, for $k$ large enough, there exist $(c, c, c)$-bounded sections in $C^r$-norm $s_{k,x}^{ref}$ of $L^{\otimes k}$ over $M$ satisfying the following bounds: $|s_{k,x}^{ref}| > c_s$ at every point of a ball of $g_k$-radius 1 centered at $x$, for some uniform constant $c_s > 0$; the sections $s_{k,x}^{ref}$ have Gaussian decay away from $x$ in $C^r$-norm.

### 3.2. Some results of approximation theory

We give by completeness some basic ideas about the behaviour of the Tchebycheff polynomials for interpolating differentiable functions. Finally we prove an easy, but not standard, result.

In what follows we will study functions $f : [-1, 1] \rightarrow \mathbb{C}$ and our objective will be to approximate them by polynomials. We introduce the following

**Definition 3.5.** The Tchebycheff polynomials $T_n(x)$ are defined inductively as follows

1. $T_0(x) = 1$,
2. $T_1(x) = x$,
3. $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

We define the Tchebycheff inner product of two functions $f, g : [-1, 1] \rightarrow \mathbb{C}$ as

$$\langle f, g \rangle = \frac{2}{\pi} \int_{-1}^{1} f(x)g(x) \frac{dx}{\sqrt{1 - x^2}}$$

Tchebycheff polynomials satisfy the following simple properties

**Lemma 3.6.**

1. The system of polynomials $\frac{T_n}{\sqrt{2}}, T_1, T_2, \ldots$ is an orthonormal system in the space of differentiable functions with respect to the Tchebycheff inner product.
2. $T_n(x) = \cos(n \arccos x)$. 
**Proof:** It is a simple computation. □

Using the precedent result we can compute the orthogonal projection of any given function to the orthonormal basis $T_0/\sqrt{2}, T_1$, etc. So the order $n$ Fourier expansion of a given function $f$ is

$$T_n f = \sum_{j=0}^{n} A_j T_j,$$

where

$$A_j = \frac{2}{\pi} \int_{-1}^{1} f(x) \bar{T}_j(x) \frac{dx}{\sqrt{1-x^2}}.$$

The result we will use is the following technical Lemma, it is nothing but a slight, and not very precise, adaptation of a classical Jackson’s theorem.

**Lemma 3.7.** Given $-1 < a < b < 1$ and given a $C^2$ function $f : [-1, 1] \rightarrow \mathbb{C}$ which satisfies that $|f'(x)| < \epsilon$ and $|f''(x)| < \epsilon$ for all $x \in [a, b]$ and $f'(x) = 0$, $f''(x) = 0$ otherwise. Then we have

1. $|A_j| \leq \frac{4}{\pi j^2} |\arccos b - \arccos a|$
2. $|f - T_n f|_{C^0} \leq \frac{4\epsilon}{\pi n} |\arccos b - \arccos a|$. Therefore $T_n f$ converges to $f$ in $C^0$-norm.

**Proof:** The second property follows from the first one by a simple computation summing the error (and checking that the Fourier expansion converges, which is direct from the Weirstrass $M$-test).

To check the first property, we compute it directly. We perform the change of variable $x = \cos \theta$ and denote $g(\theta) = \bar{f}(\cos \theta)$, then we can write

$$A_j = \frac{2}{\pi} \int_{0}^{\pi} \cos(j \theta) g(\theta) d\theta. \quad (1)$$

Integrating by parts,

$$A_j = \frac{2}{\pi} \int_{0}^{\pi} \frac{1}{j} \sin(j \theta) g'(\theta) d\theta.$$

A new integration by parts leads us to

$$A_j = \frac{-2}{\pi j^2} \int_{0}^{\pi} \cos(j \theta) g''(\theta) d\theta.$$

Now checking that $g''(\theta) = \bar{f}''(\cos \theta) \sin^2 \theta - \bar{f}'(\cos \theta) \cos \theta$ we obtain that $|g''(\theta)| \leq 2\epsilon$ and so

$$|A_j| \leq \frac{2}{\pi j^2} 2\epsilon \int_{\arccos a}^{\arccos b} \cos(j \theta) d\theta \leq \frac{4\epsilon}{\pi j^2} |\arccos b - \arccos a|.$$

So we obtain the required expression. □

### 3.3. Local result.

The key point is as usual the local study. We prove in this Subsection the following

**Proposition 3.8.** Let $f_k : B \times [-1, 1] \rightarrow \mathbb{C}^m$ be a sequence of functions where $B$ is the ball of radius 1 in $\mathbb{C}^n$ and $B \times [0, 1]$ is equipped with a sequence of contact forms $\theta(k)$ whose distributions are asymptotically flat. Let $0 < \delta < 1/2$ be a constant, $\sigma = \delta(\log(\delta^{-1}))^{-p}$, where $p$ is an integer...
depending only on the dimensions. Assume that \( f_k \) satisfies over \( B \times [-1,1] \)
the following bounds
\[
|f_k| \leq 1, \quad |\bar{\partial}_0 f_k| \leq \sigma, \quad |\nabla \bar{\partial}_0 f_k| \leq \sigma,
\]
\[
|\partial f_k/\partial s| < \sigma, \quad |\partial \nabla f_k/\partial s| < \sigma,
\]
for \( k \) large enough, where \( \bar{\partial}_0 \) is the \((0,1)\) operator defined in \( D(k) = \ker \theta(k) \)
by the complex structure \( J_0 \) and \( s \) is the real coordinate. Then for \( k \) large
enough there exists a holomorphic polynomial \( t_k : \mathbb{C} \to \mathbb{C}^m \) such that
\[
|t_k| < \delta \quad \text{on the set} \quad [-2k^{1/6}, 2k^{1/6}] \times \{0\} \subset \mathbb{C}
\]
and such that the function \( s_k(z_1, \ldots, z_{n+1}) = f_k(z_1, \ldots, z_{n+1}) - t_k(z_{n+1}) \)
is \( \sigma \)-transverse along the distribution \( D(k) \) to zero on \( B(0,1/2) \times [-1,1] \subset \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1} \) for \( k \) large
enough. Moreover, the modulus of \( t_k \) and of its first and second derivatives
can be bounded above by a fixed real polynomial \( b_\delta \) depending only on \( \delta \).

This Proposition is a consequence of the local transversality results of the contact category which are stated in all generality in \([\text{Pr00}]\) as

**Proposition 3.9** (Proposition 4.4 in \([\text{Pr00}]\)). Let \( f_k : B \times [0,1] \to \mathbb{C}^m \) be
a sequence of functions where \( B \) is the ball of radius 1 in \( \mathbb{C}^n \) and \( B \times [0,1] \)
is equipped with a sequence of contact forms \( \theta(k) \) whose distributions are
asympotically flat. Let \( 0 < \delta_0 < 1/2 \) be a constant and let \( \sigma = \delta_0 \log(\delta_0^{-1})^{-p} \), where \( p \) is an integer depending only on the dimensions. Assume that \( f_k \) satisfies over \( B \times [0,1] \) the following bounds
\[
|f_k| \leq 1, \quad |\bar{\partial}_0 f_k| \leq \sigma, \quad |\nabla \bar{\partial}_0 f_k| \leq \sigma,
\]
for \( k \) large enough, where \( \bar{\partial}_0 \) is the \((0,1)\) operator defined in \( D(k) = \ker \theta(k) \)
by vertical projection of the standard complex structure \( J_0 \). Then for \( k \) large
enough there exists a smooth curve \( w_k : [0,1] \to \mathbb{C}^m \) such that \( |w_k| < \delta_0 \)
and the function \( f_k - w_k \) is \( \sigma \)-transverse to zero on \( B(0,1/2) \times [0,1] \). Moreover,
if \( |\partial f_k/\partial s| < \sigma \) and \( |\partial \nabla f_k/\partial s| < \sigma \), we can choose \( w_k \) such that \( |d^j w_k/\partial s^j| < \Phi(\delta_0) \), \( (i = 1,2) \); \( d^j w_k/\partial s^j(0) = 0 \) and \( d^j w_k/\partial s^j(1) = 0 \), for all \( j \in \mathbb{N} \), where \( c \) is a uniform constant and \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function depending only on the dimensions.

**Proof of Proposition 3.9**. Our hypothesis coincide with the ones in
Proposition 3.9. We choose \( \delta_0 = \delta/2 \). So we obtain a function \( w_k : [-1,1] \to \mathbb{C}^m \), such that \( f_k - w_k \) is \( \sigma \)-transverse and satisfying also that \( |w_k| \leq \delta/2 \).
The idea is to approximate \( w_k \) by a complex polynomial. First we extend
\( w_k \) to the whole real line as
\[
\hat{w}_k(t) = \begin{cases} 
  w_k(-1) & \text{if } x \leq -1, \\
  w_k(t) & \text{if } -1 \leq x \leq 1, \\
  w_k(1) & \text{if } x \geq 1.
\end{cases}
\]
Now we scale the real coordinate constructing a new function \( h_k(x) = \hat{w}_k(2k^{1/6}x) \). Obviously we have the following boundings \( |h_k(x)| \leq \delta \), \( |\partial h_k/\partial s| \leq 2k^{1/6}\Phi(\delta) \) for \( j = 1,2 \). Moreover \( |\partial^j h_k/\partial s^j| = 0 \) if \( x \in [-1/2k^{1/6}, 1/2k^{1/6}] \).

Decompose \( h_k = (h_k^1, \ldots, h_k^m) \). Then each of the components \( h_k^i \) is in
the hypothesis of Lemma 3.7, when restricted to the segment \([-1,1] \). So we have that the associated Tchebycheff polynomial of degree \( d \) satisfies
\[
|T_d h_k^i| \leq \frac{8\Phi(\delta)k^{1/6}}{\pi d} \arccos\left(\frac{1}{2k^{1/6}}\right) - \arccos\left(-\frac{1}{2k^{1/6}}\right).
\]
We substitute $\pi/2 - |x| \leq |\arccos(x)| \leq \pi/2 + |x|$. Summing up all the components we find

$$|h_k - T_dh_k|_{C^0} \leq \frac{8m\Phi(\delta)}{\pi d},$$

Then increasing enough $d$ we can assure that

$$(2) \quad |h_k - T_dh_k| \leq \frac{\sigma}{2c_u},$$

$$(3) \quad |h_k - T_dh_k| \leq \frac{\delta}{2}.$$

In fact, we need $d = O(\max\{\sigma^{-1}\Phi(\delta), \delta^{-1}\})$, where $c_u$ is the uniform constant of $C^1$-openness for the property of being transverse to $0$ along the distribution $D(k)$. Define $t_k(z) = T_dh_k\left(\frac{z}{2k^1/6}\right)$. So, we claim that, imposing (2) and (3), $f_k - t_k$ is $\sigma/2$-transverse to $0$ along $D$ in $B \times [-1, 1]$ as we wanted, and also that $|t_k(z)| \leq \delta$ for all $x \in [-2k^{1/6}, 2k^{1/6}] \times \{0\}$. To prove it, we extend to $\mathbb{C}^n \times \mathbb{R}$ the functions $w_k$ and $t_k$ as $w_k(z_1, \ldots, z_n, s) = w_k(s)$ and $t_k(z_1, \ldots, z_n) = t_k(s)$. Now it is easy to check that $f_k - w_k$ is $\sigma$-transverse along $D$, we obtain that $f_k - t_k$ is $\sigma/2$-transverse to $D$ in $B \times [-1, 1]$.

To finish we need to bound above the modulus of $t_k$, or equivalently $T_dh_k$, by a fixed polynomial. For this we need only to recall the first property of Lemma 3.7 which translates in our case

$$|\hat{A}_j^l| \leq \frac{4\Phi(\delta)}{\pi j^2},$$

where $\hat{A}_j^l$ is the $A_j$ component of the polynomial $T_dh_k^l$ once the rescaling $2k^{1/6}$ is introduced. It implies that the coefficient $\hat{A}_j$ is bounded above by a function of $\delta$. So, for a fixed $\delta$ the degree $d$ is constant and the coefficients of the Tchebycheff aproximation are bounded above by a constant. Then it is obvious that there exists a fixed real polynomial bounding above the modulus of $t_k$ and of its derivatives. This finishes the proof.

The following result has a more geometrical appearance.

**Proposition 3.10.** Let $C$ be a cooriented contact manifold and let $s_k$ be a sequence of sections with global $C^3$-boundings $(c_2, c_D, c_R)$ of the bundles $E \otimes L^{\otimes k}$ over the symplectization $S_D(C)$. Then given a point $x$ in the 1-embedding $\hat{C}$ and $\delta > 0$ there exists a sequence of sections $\tau_{k,x}$ of $E \otimes L^{\otimes k}$ and $\sigma = \delta (\log(\delta^{-1}))^{-p}$ (for some integer $p > 0$) satisfying that:

1. $\tau_{k,x}$ has global $C^3$-boundings for $k$ large (depending on $\delta$)

   $$(c_u c_R P_\delta(d_k(x,y)) \exp(-\lambda d_k(x,y)^2), c_u c_D \delta Q(d_k(x,y)) \exp(-\lambda d_k(x,y)^2), c_u c_R P_\delta(d_k(x,y)) \exp(-\lambda d_k(x,y)^2))$$

   at any point $y$,

2. $(s_k + \tau_{k,x})_C$ is $\sigma$-transverse to 0 along $D$ in $B_{g_k}(x, \hat{c}) \cap \hat{C}$

for $k$ large enough, where $\lambda$ and $p$ are constants depending only on the dimensions, $P^{\delta}$ is a uniform polynomial (depending on $\delta$), $Q$ is a uniform polynomial (not depending on $\delta$), $\hat{c}$ and $c_u$ are uniform constants.
Proof: We choose the trivializations $\varphi$ and $\varphi_k$ defined by Lemmas 3.1 and 3.2. Also we fix a section $s_{k,x}^{ref}$ as defined in Lemma 3.4. Fix a unitary basis $\{e_1(x), \ldots, e_r(x)\}$ in $E_x$ and extend it by parallel transport along radial directions to a frame $\{e_1, \ldots, e_r\}$ in a neighborhood of $x$. It is easy to check that $|\nabla^r e_i|_{g_k} = O(k^{-r/2})$ and so the sequence of sections $s_k^r = e_j$ has $c$ bounding in $C^r$-norm, for a uniform $c > 0$. Now we define the frame:

$$\sigma_j = e_j \otimes s_{k,x}^{ref},$$

which is bounded in $C^r$-norm by construction. Moreover $|\sigma_j| > c_s$ for any $y \in B_{g_k}(x, 1)$. Finally choosing a sufficiently small uniform $c$, we have that $\sigma_1, \ldots, \sigma_r$ is approximately unitary for any $y \in B_{g_k}(x, c)$. 

Now we construct an application $\tilde{f}_k : B_{g_k}(x, c) \to \mathbb{C}^r$ imposing the condition

$$s_k(y) = \tilde{f}_k(y) \cdot \sigma_1(y) + \ldots + \tilde{f}_k(y) \cdot \sigma_r(y).$$

Using that $\sigma = (\sigma_1, \ldots, \sigma_r)$ is approximately unitary, namely, interpreted in each fiber as a linear application $\sigma : \mathbb{C}^r \to E_y$, $\sigma$ has an inverse with uniformly bounded norm, we find

$$|\tilde{f}_k| \leq c_u, \quad |\nabla^r \tilde{f}_k| \leq c_u, \quad |\nabla^r \tilde{f}_k| \leq c_u k^{-1/2},$$

for $r = 1, 2, 3$. Finally we use the chart $\varphi_k$ to define an application $f_k = \tilde{f}_k \circ \varphi_k^{-1}$. Scaling the chart by an appropriate uniform constant we can assure that $\varphi_k(B_{g_k}(x, \hat{c}/8)) \subset B(0, 1/2) \subset B(0, 2) \subset \varphi_k(B_{g_k}(x, \hat{c}))$. This is possible, perhaps after shrinking uniformly $\hat{c}$, because of the approximately isometry property of Lemma 3.2. From (4) and the boundings of Lemma 3.2 we obtain

$$|f_k| \leq c_u, \quad |\nabla^r f_k| \leq c_u, \quad |\nabla^r \tilde{f}_k| \leq c_u k^{-1/2},$$

Without loss of generality we suppose that $f_k$ satisfies the boundings required in Proposition 3.8 (in fact, we only have to multiply it by a non-zero uniform constant to assure this). Then the precedent Proposition applies, once $k$ is large enough, and we obtain a polynomial $t_k$ satisfying the conditions of Proposition 3.8. We extend the definition of $t_k$ to $\mathbb{C}^{n+1}$ as

$$t_k(z_1, \ldots, z_{n+1}) = t_k(z_{n+1}).$$

Now we define $\tilde{t}_k = t_k \circ \varphi_k$. Recall that this is defined in a ball of $g_k$-radius $O(k^{1/2})$, which is the domain of $\varphi_k$ (obviously, it has the same domain than $\varphi$). Then, taking into account that $s_{k,x}^{ref}$ has support in a ball of $g_k$-radius $O(k^{1/6})$, define

$$\tau_{k,x} = \tilde{t}_k^{-1} \sigma_1 + \ldots + \tilde{t}_k^r \sigma_r.$$

Recall that $s_k + \tau_{k,x}$ and $f_k + t_k$ are related through uniform scaling constants and through the approximately unitary basis $\sigma$. So, by construction, the property 2 of the statement is satisfied, except by a uniform multiplying factor which can be eliminated by increasing uniformly the integer $p$.

Recall that we can bound $t_k$ by a fixed polynomial $b_\delta$. Define $\hat{P}_\delta(r) = \max|z|=\{b_\delta(r)\}$. We can find a fixed polynomial $P_\delta(r)$ satisfying that $P_\delta(r) \geq \hat{P}_\delta(r)$. Then we easily conclude that $\tau_k$ has global boundings

$$(c_u c_3 P_\delta(d_k(x, y)) \exp(-\lambda d_k(x, y)^2), c_u c_4 P_\delta(d_k(x, y)) \exp(-\lambda d_k(x, y)^2),$$

$$c_u c_8 P_\delta(d_k(x, y)) \exp(-\lambda d_k(x, y)^2),$$
for any $y \in \hat{C}$. The first and third boundings are trivial. For the second one we proceed as follows. The bounding of $|\tau_{k,x}(y)|$ follows form the condition that $|\dot{t}_k(y)| < \delta$ for all the points of the set $\mathbb{C}^n \times [-2k^{1/6}, 2k^{1/6}] \times \{0\}$, which implies that $|\dot{t}_k(y)| < c_u \delta$ at any point $y \in \hat{C} \cap B_{g_k}(x, k^{1/6})$. For bounding the derivatives, we denote $D_0$ the pull-back through the map $\hat{\varphi}_k$ of the distribution $\mathbb{C}^n \times \{0\}$. We easily bound

$$\angle_M(D(y), D_0(y)) \leq ck^{-1/2}d_k(x,y),$$

where $c > 0$ is certain uniform constant. By construction,

$$\nabla D_0 \dot{t}_k = 0.$$ 

And so using (3), (7) and the bounding polynomial $P_3$ we find

$$|\nabla D_\dot{t}_k(y)| = ck^{-1/2}d_k(x,y)P_3(d_k(x,y)).$$

We change the polynomial $P_3(t)$ by $t \cdot P_3(t)$ and so

$$\nabla D_\dot{t}_k(y) = ck^{-1/2}P_3(d_k(x,y)).$$

Therefore

$$|\nabla D(\dot{t}_k \cdot \sigma)| = |\nabla D_\dot{t}_k \cdot \sigma + \dot{t}_k \cdot \nabla D \sigma| =$$

$$\leq ck^{-1/2}P_3(d_k(x,y)) \exp(-\lambda d_k(x,y)^2) +$$

$$+ \delta Q(d_k(x,y)) \exp(-\lambda d_k(x,y)^2),$$

which, for $k$ large enough, satisfies the required bounding because the first term is arbitrarily small and the polynomial $Q$ does not depend on $\delta$ as required. The boundings on $|\nabla^r \tau_{k,x}|$ are obtained in the same way. □

3.4. Globalization process. As in [Do96, MP99], the final point will be to construct a global perturbation of the sequence of sections from a sequence of localized perturbations added in a suitable way. Along this Subsection we adapt Donaldson’s framework to our case. This development is given by

**Proof of Theorem 2.15**: Donaldson’s globalization argument works with some slight variations. Choose a finite set of points $S$ satisfying the following conditions:

1. $\cup_{x \in S} B_{g_k}(x, \epsilon) \supset \hat{C}$.
2. There exist a partition $S = \cup_{j \in J} S_j$ verifying that $d_{g_k}(x,y) > N$ if $x, y \in S_j$, $N$ will be fixed along the proof.
3. The value of $J$ is $O(N^{2n+1})$.

Recall that the starting sequence of sections has global $C^3$-boundings $(c, c_D, c_R)$. We proceed by steps perturbing at each $S_j$ at a time. Let us find a perturbation centred on each of the points of $S_1$ to achieve transversality at a neighborhood of $S_1$. Fix $x \in S_1$, use Proposition 3.10 with certain $\delta = \delta_1 > 0$ to be chosen. We find out a sequence of perturbations $\tau_{k,x}$ with global boundings

$$(c_u c_R P_\delta(d_k(x,y)) \exp(-\lambda d_k(x,y)^2), c_u c_D \delta Q(d_k(x,y)) \exp(-\lambda d_k(x,y)^2),$$

$$c_u c_R P_\delta(d_k(x,y)) \exp(-\lambda d_k(x,y)^2)).$$

We take $\delta_1$ to assure that the second bounding is uniformly less that $\epsilon/2$ (Recall that $\epsilon$ is the maximum first mixed $C^3$-bounding admitted). In fact, we can choose $\delta_1 \leq c_p \epsilon$, for a certain uniform $c_p > 0$. This is possible since $Q$ does not depend on $\delta$! We have now a perturbation centred on each of
the points of $S_1$ which solves the problem in the balls $B_{g_k}(x, \hat{c})$. But the perturbations are not independent. This is the moment when the integer $N$ comes into play. Analyze a fixed $x \in S_1$. We can compute the maximum first mixed $C^3$-bounding of the perturbations (the boundings in the distribution directions) of the rest of the points of $S_1$ in the ball $B_{g_k}(x, c)$. This “bad” perturbation is bounded by $c_u \delta \exp(-\lambda N^2)$. Again, it is very important to assure that $Q$ does not depend on $\delta$ to find $c_u$ independent of $\delta$, otherwise the globalization process does not hold. To avoid the destruction of the achieved transversality a sufficient condition is so

$$c_u \delta_1 \exp(-\lambda N^2) \leq \delta_1 (\log(\delta_1^{-1}))^{-p},$$

for a uniform constant $c_u$ not depending on $\delta$. In this first stage we may choose $N$ to satisfy (8). So adding all the perturbations we find a sequence of sections $\tau_k^1$ which added to $s_k$ achieve $\sigma_1$-transversality in $\bigcup_{x \in S_1} B_{g_k}(x, \hat{c}) \cap \hat{C}$. Moreover we find that the sequence $\tau_k^1$ has global $C^3$-boundings $(\epsilon'_1, \epsilon/2, \epsilon_2')$. We only know that $\epsilon'_1$ and $\epsilon_2'$ depend on $\delta$, but the important point is that “they exist”. Now in the second stage we choose $\delta_2$ to assure that the final sequence of perturbations $\tau_k^2$ has boundings $(\epsilon_2', \min\{\epsilon/4, \sigma_1/2c_u\}, \epsilon_2')$. The second bounding is imposed to guarantee that the sequence has controled boundings in the distribution $D$ directions and also that do not destroy the achieved transversality in the $\hat{c}$-neighborhood of $S_1$ ($c_u$ is the constant of $C^1$-openness of the transversality to 0 along $D$).

Repeating the process we find $\tau_k = \sum_{j=1}^{q} \tau_k^j$ that has global $C^3$-boundings $(\epsilon', \epsilon/4, \epsilon_2')$, which are independent of $k$ because $q$ is independent. Moreover $s_k + \tau_k$ is $\sigma$-transverse to 0 along $D$ all over $\hat{C}$. Again, the constant $\sigma > 0$ is uniform because the number of steps is independent of $k$.

Only one important question has to be checked. The expression (8) must hold in all the steps of the process. Namely we must assure

$$c_u \delta_j \exp(-\lambda N^2) \leq \delta_j (\log(\delta_j^{-1}))^{-p}.$$  

But, the asymptotic analysis of the expression $(\log(\delta_j^{-1}))^{-p}$ provides this condition if we choose $N$ large enough (for a proof of this fact see Section 2 in [Do96]).

4. Topological considerations.

In this Section we characterize the topological properties of the constructed submanifolds.

4.1. Relative Lefschetz hyperplane theorem. We prove now the second part of Theorem [11]. The started point is a sym-con manifold $(M, \omega, C, \theta)$ where we have found a sequence of sym-con submanifolds $(W_k, \omega, C_k, \theta)$ obtained as zero sets of a sequence of sections $s_k$ of the bundles $E \otimes L^\otimes k$ with global boundings which are transverse to 0 in the symplectic manifold and in the contact border.

We take as a tool the functions $f_k(p) = \log |s_k(p)|^2$. Then we follow the Proof of Proposition 2 in Section 5.1 of [Au97] to conclude that the critical points of $f_k$ in $M$ have at least index $n - r + 2$, for $k$ large enough. In the same way following [IMP99], we conclude that the critical points of $f_k$ in the border $C$ have at least index $n - r + 1$ (again, for $k$ sufficiently large). So,
being the border a closed manifold, this proves that the inclusion \( i : C_k \to C \)
duces isomorphism in homology groups \( H_j \) (resp. homotopy groups) for \( j \leq n - r - 1 \) and surjection for \( j = n - r \). This is the content of the Lefschetz
theorem in the contact case.

We are going to define the double copy \( M^d \) of the manifold \( M \) as the topological connected sum \( M \cup_C M \). We can arrange this topological operation, for each \( k \), to assure the smoothness of the submanifold \( W^d_k = W_k \cup_{C_k} W_k \).

Now we perturb \( f_k \) into a new function \( \hat{f}_k \) to assure that the natural extension to the double copy is smooth. For this we only need to assure that \( \frac{d}{n}(c) = 0 \) for any \( c \in C \) where \( n \) is the normal direction to \( C \) respect to the metric \( h_k \). Use that in a small neighborhood \( V \) of \( C \) we can trivialize \( M \) as \( C \times [0, \epsilon) \) assuring also that \( n = \frac{\partial}{\partial s} \) being \( s \) the real coordinate. Therefore we perturb \( f_k \) in this small neighborhood as

\[
\hat{f}_k(c, s) = f_k(c, \beta(s)),
\]

where \( \beta : [0, \epsilon] \to [0, \epsilon] \) is a smooth function satisfying

1. \( \beta(0) = 0 \) and \( \beta(\epsilon) = \epsilon \).
2. \( \beta'(x) > 0 \) for all \( x \in (0, \epsilon) \).
3. \( \frac{d^r\beta(0)}{ds^r} = 0 \), for all \( r \in \mathbb{N}^* \).
4. \( \beta'(\epsilon) = 1 \) and \( \frac{d^r\beta(\epsilon)}{ds^r} = 0 \) for \( r = 2, 3, \ldots \)

It is easy to check that \( \hat{f}_k \) extends to a smooth function, again denoted, \( \hat{f}_k \) in \( M^d \). Moreover the critical points of \( f_k \) and \( \hat{f}_k \) coincide in \( M \) because \( \beta \) only performs a diffeomorphism outside the border. The indices do not change. In \( C \) (interpreted as a submanifold in \( M^d \)) we obtain that the critical points of \( f_k \) are now critical points of \( \hat{f}_k \) and the index of these critical points is at least \( n - r + 1 \).

Summarizing, the manifold \( W^d_k \) is a smooth submanifold of \( M^d \). The function \( \hat{f}_k \) has critical points of index at least \( n - r + 1 \) for \( k \) large enough.

This implies, by standard Morse theory, that the inclusion

\[
i_d : W^d_k \to M^d
\]

induces isomorphisms in homology and homotopy groups for dimension less than or equal to \( n - r \) and surjection for \( n - r + 1 \).

We denote the first and second copies of \( M \) in \( M^d \) by \( M^1 \) and \( M^2 \) respectively. The same for \( W_k \) with copies \( W^1_k \) and \( W^2_k \). The natural diffeomorphism defined in \( M^d \) interchanging the copies is denoted as \( e : M^d \to M^d \), namely \( e(M^1) = M^2, e(M^2) = M^1 \) and \( e(C) = C \).

Our objective is to prove that the natural morphism

\[
i_j : H_j(W^d_k, C_k) \to H_j(M, C)
\]

is actually an isomorphism when \( j \leq n - r \) (and an epimorphism for \( j = n - r + 1 \)). There are several ways to prove this result we choose a constructive one which is a little longer than others because it clarifies a bit the topological ideas involved in the proof.

First let us prove that \( i_j \) is epimorphism in the required cases. We choose \( \alpha^1 \in H_j(M, C) \). Identify \( \bar{M} \simeq M^1 \), then we define \( \alpha^2 = -e_*\alpha_1 \). Construct \( \alpha^d = \alpha_1 + \alpha_2 \) which is an element of \( H_j(M^d) \). If \( j \leq n - r + 1 \), then there exists \( \gamma^d \in H_j(W^d_k) \) such that \( \alpha^d - \gamma^d = \partial \epsilon \), for some \( \epsilon \in H_{j+1}(M^d) \).
After a small isotopic perturbation, we can suppose that all the elemental chains defining \( \gamma^d \) and \( \epsilon \) are transverse to \( C \). We claim that we can find an element homologous to \( \gamma^d \) of the form
\[
\gamma^1 + f + \gamma^2 - f,
\]
where \( \gamma^i \in H_j(W^i_k, C_k) \), \( i = 1, 2 \) and \( f \) is a chain in \( C \). For this recall that given an elemental 1-chain \( c : [0, 1] \to M^d \) transverse to \( C \) we can define a 1-chain \( c^1 \) in \( M^d \) as follows. The chain intersects \( c \) in points \( a_1, b_1, a_2, \ldots \) (suppose that \( c(0) \in M^1 \)). Then we define \( c^1 = c[0, a_1] + [c[b_1, a_2] + \cdots \). The same hold for any \( j \)-chain transverse to \( C \) using an adequate triangulation. In fact, the morphism \( c \to c^1 \) is the explicit way of the composition
\[
H_j(W^d_k) \to H_j(W^d_k, \bar{W}^2_k) \to H_j(\bar{W}^1_k, C_k),
\]
where the first morphism is the restriction and the second one is generated by excision of \( W^2_k \). Then by the construction we find that \( (\alpha^d) = \alpha^1 \) and \( \alpha^1 - \gamma^1 - f = \partial e_f^1 \), for certain chain \( f \) in \( C \). This implies (9) and \( \alpha_1 \) and \( \gamma^1 \) are homologous relative to the border \( C \). So the morphism \( i_j \) is surjective in the expected cases.

Now we study the injectivity. Choose \( \gamma^1 \in H_j(W^1_k, C_k) \) and suppose that there exists a \( (j + 1) \)-chain \( \epsilon^1 \) in \( \bar{M} \) such that \( \partial \epsilon^1 = \gamma^1 - c \), for some chain \( c \) of \( C_k \). The question is whether we are able to find a chain \( \epsilon^1 \) in \( W^1_k \). The argument is analogous to the precedent one. We construct \( \gamma^2 = -\epsilon, \gamma^1 \) and therefore \( \gamma^d = \gamma^1 + \gamma^2 \) is an element of \( H_j(W^d_k) \). In the same way we construct \( \epsilon^d \) satisfying \( \partial \epsilon^d = \gamma^d \). If we assume that \( j \leq n - r \) then the Lefschetz hyperplane theorem in \( M^d \) assures that there exists a \( (j + 1) \)-chain \( \rho \) in \( W^d_k \) satisfying \( \partial \rho = \gamma^d \). Now we construct \( \rho^1 \), which is in \( W^1_k \), as in the precedent case. Finally, we obtain \( \partial \rho^1 = \gamma^1 - c \), for some chain \( c \) in \( C_k \).

4.2. Homology and Chern numbers of the submanifolds. To finish we state the following straightforward result.

**Proposition 4.1.** Given any sequence of sections \( s_k \) with global \( C^3 \)-boundings of bundles \( E \otimes L^{\otimes k} \) which are transverse to zero, then the Chern classes of the symplectic zero sets \( Z(s_k) \) are given by
\[
c_t(TZ(s_k)) = (-1)^l \left( r + l - 1 \right) \left( k \left[ \frac{\omega}{2\pi} \right] l + O(k^{l-1}) \right).
\]

**Proof:** Denote \( Z(s_k) = W_k \). The formula follows directly from the relation
\[
i^* c(TX) = i^* c(E \otimes L^{\otimes k}) \cdot c(TW_k).
\]

REFERENCES

[Ar80] V. Arnold. Mathematical Methods of Classical Mechanics. Springer-Verlag (1980).

[Au97] D. Auroux. Asymptotically holomorphic families of symplectic submanifolds. Geom. Funct. Anal., 7, 971-995 (1997).

[Au99] D. Auroux. Théorèmes de structure des variétés symplectiques compactes via des techniques presque complexes. Ph. D. Thesis. (1999).

[Au99b] D. Auroux, L. Katzarkov. Branched coverings of \( \mathbb{C}P^2 \) and associated invariants of symplectic 4-manifolds. Preprint. École Polytechnique. Paris. (1999).

[Ch66] E. W. Cheney. Introduction to Approximation Theory. McGraw-Hill Book Company. New York. (1966).
[Do96] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. J. Diff. Geom., 44, 666-705 (1996).
[Do99] S. K. Donaldson. Lefschetz pencils on symplectic manifolds. Preprint (1999).
[Eli98] Y. Eliashberg. ICM 98, Berlin (1998).
[IMP99] A. Ibort, D. Martínez, F. Presas. On the construction of contact submanifolds with prescribed topology. Preprint. Universidad Carlos III de Madrid. (1999).
[MPS99] V. Muñoz, F. Presas, I.Sols. Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds. Preprint. Universidad Complutense de Madrid. (1999).
[Pr00] F. Presas. Lefschetz type pencils on contact manifolds. Preprint. Universidad Complutense de Madrid. (2000).