Differentially Private Clustering: Tight Approximation Ratios

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Abstract

We study the task of differentially private clustering. For several basic clustering problems, including Euclidean DensestBall, 1-Cluster, k-means, and k-median, we give efficient differentially private algorithms that achieve essentially the same approximation ratios as those that can be obtained by any non-private algorithm, while incurring only small additive errors. This improves upon existing efficient algorithms that only achieve some large constant approximation factors. Our results also imply an improved algorithm for the Sample and Aggregate privacy framework. Furthermore, we show that one of the tools used in our 1-Cluster algorithm can be employed to get a faster quantum algorithm for ClosestPair in a moderate number of dimensions.

1 Introduction

With the significant increase in data collection, serious concerns about user privacy have emerged. This has stimulated research on formulating and guaranteeing strong privacy protections for user-sensitive information. Differential Privacy (DP) [DMNS06, DKM+06] is a rigorous mathematical concept for studying user privacy and has been widely adopted in practice [EPK14, Sha14, Gre16, App17, DKY17, Abo18]. Informally, the notion of privacy is that the algorithm’s output (or output distribution) should be mostly unchanged when any one of its inputs is changed. DP is quantified by two parameters $\epsilon$ and $\delta$; the resulting notion is referred to as pure-DP when $\delta = 0$, and approximate-DP when $\delta > 0$. See Section 2 for formal definitions of DP and [DR14, Vad17] for an overview. Clustering is a central primitive in unsupervised machine learning [XW08, ACT13]. An algorithm for clustering in the DP model informally means that the cluster centers (or the distribution on cluster centers) output by the algorithm should be mostly unchanged when any one of the input points is changed. Many real-world applications involve clustering sensitive data. Motivated by these, a long line of work has studied clustering algorithms in the DP model [BDMN05, NRS07, FFKN09, GLM+10, MTS+12, WWST15, NSV16, NCBN16, SCL+16, FXZR17, BDL+17, NS18, HL18, NCBN16, NS18, SK18, Ste20]. In this work we focus on several basic clustering problems in the DP model and obtain efficient algorithms with tight approximation ratios.

Clustering Formulations. The input to all our problems is a set $X$ of $n$ points, each contained in the $d$-dimensional unit ball. There are many different formulations of clustering. In the popular $k$-means problem [Llo82], the goal is to find $k$ centers minimizing the clustering cost, which is the sum of squared distances from each point to its closest center. The $k$-median problem is similar to $k$-means except that the distances are not squared in the definition of the clustering cost. Both problems are NP-hard, and there is a large body of work dedicated to determining the best possible

1For the formal definitions of $k$-means and $k$-median, see Definition 3 and the paragraph following it.
approximation ratios achievable in polynomial time (e.g. [Bar96, CCGG98, CCGTS02, V01, LMS02, AGK^04, KMN^04, AV07, LSW17, ACKS15, BPR17, LSW17, ANSW17, CK19]), although the answers remain elusive. We consider approximation algorithms for both these problems in the DP model, where a \((w, t)\)-approximation algorithm outputs a cluster whose cost is at most the sum of \(t\) and \(w\) times the optimum; we refer to \(w\) as the approximation ratio and \(t\) as the additive error. It is important that \(t\) is small since without this constraint, the problem could become trivial. (Note also that without privacy constraints, approximation algorithms typically work with \(t = 0\).)

We also study two even more basic clustering primitives, \textsc{DensestBall} and 1-Cluster, in the DP model. These underlie several of our results.

\textbf{Definition 1 (DensestBall).} Given \(r > 0\), a \((w, t)\)-approximation for the \textsc{DensestBall} problem is a ball \(B\) of radius \(w \cdot r\) such that whenever there is a ball of radius \(r\) that contains at least \(T\) input points, \(B\) contains at least \(T - t\) input points.

This problem is NP-hard for \(w = 1\) [BS00, BES02, She15]. Moreover, approximating the largest number of points within any ball of radius \(r\) and up some constant factor is also NP-hard [BES02]. On the other hand, several polynomial-time approximation algorithms achieving \((1 + \alpha, 0)\)-approximation for any \(\alpha > 0\) are known [AHPV05, She13, BES02].

\textsc{DensestBall} is a useful primitive since a DP algorithm for it allows one to “peel off” one important cluster at a time. This approach has played a pivotal role in a recent fruitful line of research that obtains DP approximation algorithms for \textsc{\(k\)-means} and \textsc{\(k\)-median} [SK18, Ste20].

The 1-Cluster problem studied, e.g., in [NSV16, NS18] is the “inverse” of \textsc{DensestBall}, where instead of the radius \(r\), the target number \(T\) of points inside the ball is given. Without DP constraints, the computational complexities of these two problems are essentially the same (up to logarithmic factors in the number of points and the input universe size), as we may use binary search on \(r\) to convert a \textsc{DensestBall} algorithm into one for 1-Cluster, and vice versa. These two problems are generalizations of the MinimumEnclosingBall (aka MinimumBoundingSphere) problem, which is well-studied in statistics, operations research, and computational geometry.

As we elaborate below, \textsc{DensestBall} and 1-Cluster are also related to other well-studied problems, such as learning halfspaces with a margin and the Sample and Aggregate framework [NRS07].

\textbf{Main Results.} A common highlight of most of our results is that for the problems we study, our algorithms run in polynomial time (in \(n\) and \(d\)) and obtain tight approximation ratios. Previous work sacrificed one of these, i.e., either ran in polynomial time but produced sub-optimal approximation ratios or took time exponential in \(d\) to guarantee tight approximation ratios.

(i) For \textsc{DensestBall}, we obtain for any \(\alpha > 0\), a pure-DP \((1 + \alpha, \tilde{O}_\epsilon(\frac{1}{\alpha}))\)-approximation algorithm and an approximate-DP \((1 + \alpha, \tilde{O}_\epsilon(\frac{\sqrt{d}}{\epsilon}))\)-approximation algorithm\footnote{To reduce from 1-Cluster to \textsc{DensestBall}, one can binary-search on the target radius. In this case, the number of iterations needed for the binary search depends logarithmically on the ratio between the maximum possible distance between two input points and the minimum possible distance between two (distinct) input points. In the other direction (i.e., reducing from \textsc{DensestBall} to 1-Cluster), one can binary-search on the number of points inside the optimal ball, and here the number of iterations will be logarithmic in the number of input points.}. The runtime of our algorithms is \(\text{poly}(nd)\). Table 1 shows our results compared to previous work. To solve \textsc{DensestBall} with DP, \(\epsilon\)-DP algorithms for \(\text{Cluster}\) and \(\text{MinimumEnclosingBall}\) (aka \text{MinimumBoundingSphere}) are also related to other well-studied problems, such as learning halfspaces with a margin and the Sample and Aggregate framework [NRS07].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Reference & \(w\) & \(t\) & Running time \\
\hline
[NSV16], \(\delta > 0\) & \(O(\sqrt[3]{n})\) & \(O(\sqrt[3]{n} \cdot \text{poly log } \frac{1}{\delta})\) & \(\text{poly}(n, d, \log \frac{1}{\delta})\) \\
[NS18], \(\delta > 0\) & \(O(1)\) & \(\tilde{O}_\epsilon(\sqrt[3]{n} \cdot n^{0.1} \cdot \text{poly log } \frac{1}{\delta})\) & \(\text{poly}(n, d, \log \frac{1}{\delta})\) \\
Exp. Mech. [AM10], \(\delta = 0\) & \(1 + \alpha\) & \(O_{\alpha}(\frac{d}{\epsilon} \cdot \text{log } \frac{1}{\epsilon})\) & \(O\left(\left(\frac{1}{\alpha}\right)^d\right)\) \\
\hline
\end{tabular}
\caption{Comparison of \((\epsilon, \delta)\)-DP algorithms for \((w, t)\)-approximations for \textsc{DensestBall} given \(r\).}
\end{table}

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we introduce and solve two problems: efficient list-decodable covers and private sparse selection. These could be of independent interest.

(ii) For 1-Cluster, informally, we obtain for any $\alpha > 0$, a pure-DP $(1 + \alpha, \tilde{O}(\frac{d}{\epsilon}))$-approximation algorithm running in time $(nd)^{O(\alpha)}$. We also obtain an approximate-DP $(1 + \alpha, \tilde{O}(\frac{\sqrt{d}}{\epsilon}))$-approximation algorithm running in time $(nd)^{O(\alpha)}$. The latter is an improvement over the previous work of [NST18] who obtain an $(\tilde{O}(1 + \frac{1}{\epsilon}), \tilde{O}(\epsilon n^{1/2}d))$-approximation. In particular, they do not get an approximation ratio $w$ arbitrarily close to 1. Even worse, the exponent $\phi$ in the additive error $t$ can be made close to $0$ only at the expense of blowing up $w$. Our algorithm for 1-Cluster follows by applying our DP algorithm for DensestBall, along with “DP binary search” similarly to [NSV16].

(iii) For $k$-means and $k$-median, we prove that we can take any (not necessarily private) approximation algorithm and convert it into a DP clustering algorithm with essentially the same approximation ratio, and with small additive error and small increase in runtime. More precisely, given any $w^*$-approximation algorithm for $k$-means (resp., $k$-median), we obtain a pure-DP $(w^*(1 + \alpha), \tilde{O}(\frac{k\epsilon d}{\alpha}O(\epsilon)))$-approximation algorithm and an approximate-DP $(w^*(1 + \alpha), \tilde{O}(\frac{k\sqrt{d}+kO(\epsilon)}{\alpha}O(\epsilon)))$-approximation algorithm for $k$-means (resp., $k$-median). The current best known non-private approximation algorithms achieve $w^* = 6.358$ for $k$-means and $w^* = 2.633$ for $k$-median [ANSW17]. Our algorithms run in time polynomial in $n$, $d$ and $k$, and improve on those of [SK18] who only obtained some large constant factor approximation ratio independent of $w^*$.

It is known that $w^*$ can be made arbitrarily close to 1 for (non-private) $k$-means and $k$-median if we allow fixed parameter tractable algorithms [BTIP02, DLVKK03, KSS04, KSS05, Che06, FMS07, FL11]. Using this, we get a pure-DP $(1 + \alpha, \tilde{O}(\frac{k\epsilon d}{\alpha}O(\epsilon)))$-approximation, and an approximate-DP $(1 + \alpha, \tilde{O}(\frac{k\sqrt{d}+kO(\epsilon)}{\alpha}O(\epsilon)))$-approximation. The algorithms run in time $2^{O(k \log k)}\poly(nd)$.

Overview of the Framework. All of our DP clustering algorithms follow this three-step recipe:

(i) Dimensionality reduction: we randomly project the input points to a lower dimension.

(ii) Cluster(s) identification in low dimension: we devise a DP clustering algorithm in the low-dimensional space for the problem of interest, which results in cluster(s) of input points.

(iii) Cluster center finding in original dimension: for each cluster found in step (ii), we privately compute a center in the original high-dimensional space minimizing the desired cost.

Applications. Our DP algorithms for 1-Cluster imply better algorithms for the Sample and Aggregate framework of [NRSS07]. Using a reduction from 1-Cluster due to [NSV16], we get an algorithm that privately outputs a stable point with a radius not larger than the optimal radius by a $1 + \alpha$ factor, where $\alpha$ is an arbitrary positive constant. For more context, please see Section 5.2.

Moreover, by combining our DP algorithm for DensestBall with a reduction of [BS00, BES02], we obtain an efficient DP algorithm for agnostic learning of halfspaces with a constant margin. Note that this result was already known from the work of Nguyen et al. [NUZ20]; we simply give an alternative proof that employs our DensestBall algorithm as a blackbox. For more on this and related work, please see Section 5.3.

Finally, we provide an application of one of our observations outside of DP. In particular, we give a faster (randomized) history-independent data structure for dynamically maintaining ClosestPair in a moderate number of dimensions. This in turn implies a faster quantum algorithm for ClosestPair in a similar setting of parameters.

Organization. Section 2 contains background on DP and clustering. Our algorithms for DensestBall are presented in Section 3 and those for $k$-means and $k$-median are given in Section 4. Applications to 1-Cluster, Sample and Aggregate, agnostic learning of halfspaces with a margin, and ClosestPair are described in Section 5. We conclude with some open questions in Section 6. All missing proofs are deferred to the Appendix.\footnote{Recall that an algorithm is said to be fixed parameter tractable in $k$ if its running time is of the form $f(k) \cdot \poly(n)$ for some function $f$, and where $n$ is the input size [DF13].}
2 Preliminaries

Notation. For a finite universe \( \mathcal{U} \) and \( \ell \in \mathbb{N} \), we let \( \binom{\mathcal{U}}{\leq \ell} \) be the set of all subsets of \( \mathcal{U} \) of size at most \( \ell \). Let \( [n] = \{1, \ldots, n\} \). For \( v \in \mathbb{R}^d \) and \( r \in \mathbb{R}_{\geq 0} \), let \( B(v, r) \) be the ball of radius \( r \) centered at \( v \). For \( k \in \mathbb{R}_{\geq 0} \), denote by \( \mathbb{B}^d_k \) the quantized \( d \)-dimensional unit ball with discretization step \( k \). We throughout consider closed balls.

Differential Privacy (DP). We next recall the definition and basic properties of DP. Datasets \( X \) and \( X' \) are said to be neighbors if \( X' \) results from removing or adding a single data point from \( X \).

Definition 2 (Differential Privacy (DP)). Let \( \epsilon, \delta \in \mathbb{R}_{\geq 0} \) and \( n \in \mathbb{N} \). A randomized algorithm \( \mathcal{A} \) taking as input a dataset is said to be \((\epsilon, \delta)\)-differentially private if for any two neighboring datasets \( X \) and \( X' \), and for any subset \( S \) of outputs of \( \mathcal{A} \), it holds that \( \Pr[\mathcal{A}(X) \in S] \leq e^{\epsilon} \cdot \Pr[\mathcal{A}(X') \in S] + \delta \). If \( \delta = 0 \), then \( \mathcal{A} \) is said to be \( \epsilon \)-differentially private.

We assume throughout that \( 0 < \epsilon \leq O(1) \), \( 0 < \alpha < 1 \), and when used, \( \delta > 0 \).

Clustering. Since many of the proof components are common to the analyses of \( k \)-means and \( k \)-median, we will use the following notion, which generalizes both problems.

Definition 3 ((\( k, p \))-Clustering). Given \( k \in \mathbb{N} \) and a multiset \( X = \{x_1, \ldots, x_n\} \) of points in the unit ball, we wish to find \( k \) centers \( c_1, \ldots, c_k \in \mathbb{R}^d \) minimizing \( \text{cost}_X(c_1, \ldots, c_k) := \sum_{i \in [n]} \left( \min_{j \in [k]} \|x_i - c_j\| \right)^p \). Let \( \text{OPT}^{p,k}_X \) denote \( \min_{c_1, \ldots, c_k \in \mathbb{R}^d} \text{cost}_X^p(c_1, \ldots, c_k) \). A \((w, t)\)-approximation algorithm for \((k, p)\)-Clustering outputs \( c_1, \ldots, c_k \) such that \( \text{cost}_X^p(c_1, \ldots, c_k) \leq w \cdot \text{OPT}^{p,k}_X + t \). When \( X \), \( p \), and \( k \) are unambiguous, we drop the subscripts and superscripts.

Note that \((k, 1)\)-Clustering and \((k, 2)\)-Clustering correspond to \( k \)-median and \( k \)-means respectively. It will also be useful to consider the Discrete \((k, p)\)-Clustering problem, which is the same as in Definition 3 except that we are given a set \( C \) of “candidate centers” and we can only choose the centers from \( C \). We use \( \text{OPT}^{p,k}_X(C) \) to denote \( \min_{c_1, \ldots, c_k \in C} \text{cost}_X^p(c_1, \ldots, c_k) \).

Centroid Sets and Coresets. A centroid set is a set of candidate centers such that the optimum does not increase by much even when we restrict the centers to belong to this set.

Definition 4 (Centroid Set [Mat00]). For \( w, t > 0 \), \( p \geq 1 \), \( k, d \in \mathbb{N} \), a set \( C \subseteq \mathbb{R}^d \) is a \((p, k, w, t)\)-centroid set of \( X \subseteq \mathbb{R}^d \) if \( \text{OPT}^{p,k}_X(C) \leq w \cdot \text{OPT}^{p,k}_X + t \). When \( k \) and \( p \) are unambiguous, we simply say that \( C \) is a \((w, t)\)-centroid set of \( X \).

A coreset is a (multi)set of points such that, for any possible \( k \) centers, the cost of \((k, p)\)-Clustering of the original set is roughly the same as that of the coreset (e.g., [HM04]).

Definition 5 (Coreset). For \( \gamma, t > 0 \), \( p \geq 1 \), \( k, d \in \mathbb{N} \), a set \( X' \) is a \((p, k, \gamma, t)\)-coreset of \( X \subseteq \mathbb{R}^d \) if for every \( C = \{c_1, \ldots, c_k\} \subseteq \mathbb{R}^d \), we have \( (1 - \gamma) \cdot \text{cost}_X^p(C) - t \leq \text{cost}_{X'}^p(C) \leq (1 + \gamma) \cdot \text{cost}_X^p(C) + t \). When \( k \) and \( p \) are unambiguous, we simply say that \( X' \) is a \((\gamma, t)\)-coreset of \( X \).

3 Private DensestBall

In this section, we obtain pure-DP and approximate-DP algorithms for DensestBall.

\(^3\)Whenever we assume that the inputs lie in \( \mathbb{R}^d_+ \), our results will hold for any discretization as long as the minimum distance between two points at least \( k \).

\(^4\)This definition of DP is sometimes referred to as removal DP. Some works in the field consider the alternative notion of replacement DP where two datasets are considered neighbors if one results from modifying (instead of removing) a single data point of the other. We remark that \((\epsilon, \delta)\)-removal DP implies \((2\epsilon, 2\delta)\)-replacement DP. Thus, our results also hold (with the same asymptotic bounds) for the replacement DP notion.

\(^5\)The cost is sometimes defined as the \((1/p)\)th power.
Theorem 6. There is an \(\epsilon\)-DP (resp., \((\epsilon, \delta)\)-DP) algorithm that runs in time \(\left(\omega d\right)^{O(1)} \cdot \text{poly} \log(1/r)\) and, w.p. \(0.99\), returns a \(\left(1 + \alpha, \Omega\left(\frac{\epsilon}{\alpha} \cdot \log \left(\frac{2}{\delta}\right)\right)\right)\)-approximation (resp., \(\epsilon\)) for \(\text{DensestBall}\).

To prove this, we follow the three-step recipe from Section 1. Using the Johnson–Lindenstrauss (JL) lemma \([\text{JL84}]\) together with the Kirszbraun Theorem \([\text{Kir34}]\), we project the input to \((\log n)/\alpha^2\) dimensions in step (i). It turns out that step (iii) is similar to (ii), as we can repeatedly apply a low-dimensional \(\text{DensestBall}\) algorithm to find a center in the high-dimensional space. Therefore, the bulk of our technical work is in carrying out step (ii), i.e., finding an efficient, DP algorithm for \(\text{DensestBall}\) in \(O((\log n)/\alpha^2)\) dimensions. We focus on this part in the rest of this section; the full proof with the rest of the arguments can be found in Appendix D.2.

3.1 A Private Algorithm in Low Dimensions

Having reduced the dimension to \(d' = \omega((\log n)/\alpha^2)\) in step (i), we can afford an algorithm that runs in time \(\exp(O_n(d')) = n^{O(1)}\). With this in mind, our algorithms in dimension \(d'\) have the following guarantees:

Theorem 7. There is an \(\epsilon\)-DP (resp., \((\epsilon, \delta)\)-DP) algorithm that runs in time \(\left(1 + 1/\alpha\right)^{O(d')} \cdot \text{poly} \log(1/r)\) and, w.p. \(0.99\), returns a \(\left(1 + \alpha, \Omega\left(\frac{\epsilon}{\alpha} \cdot \log \left(\frac{2}{\delta}\right)\right)\right)\)-approximation (resp., \(\epsilon\)) for \(\text{DensestBall}\).

As the algorithms are allowed to run in time exponential in \(d'\), Theorem 7 might seem easy to devise at first glance. Unfortunately, even the Exponential Mechanism \([\text{MT07}]\), which is the only known algorithm achieving approximation ratio arbitrarily close to 1, still takes \(\Theta((1/r)d')\) time, which is \(\exp(\omega(d'))\) for \(r = \omega(1)\). (In fact, in applications to \(k\)-means and \(k\)-median, we set \(r\) to be as small as \(1/n\), which would result in a running time of \(n^\Theta(\log n)\). To understand, and eventually overcome this barrier, we recall the implementation of the Exponential Mechanism for \(\text{DensestBall}\):

- Consider any \((\alpha r)\)-cover \(C\) of the unit ball \(B(0, 1)\).
- For every \(c \in C\), let \(\text{score}[c]\) be the number of input points lying inside \(B(c, (1 + \alpha)r)\).
- Output a point \(c^* \in C\) with probability \(\frac{e^{(\alpha/2) \cdot \text{score}[c^*]}}{\sum_{c \in C} e^{(\alpha/2) \cdot \text{score}[c]}}\).

By the generic analysis of the Exponential Mechanism \([\text{MT07}]\), this algorithm is \(\epsilon\)-DP and achieves a \(\left(1 + \alpha, \Omega\left(\frac{\epsilon}{\alpha} \cdot \log \left(\frac{2}{\delta}\right)\right)\right)\)-approximation as in Theorem 7. The existence of an \((\alpha r)\)-cover of size \(\Theta((1/r)d')\) is well-known and directly implies the \(\Theta((1/r)d')\) running time stated above.

Our main technical contribution is to implement the Exponential Mechanism in \(\Theta((1/r)d') \cdot \text{poly} \log \frac{1}{\epsilon}\) time instead of \(\Theta((1/r)d')\). To elaborate on our approach, for each input point \(x_i\), we define \(S_i\) to be \(C \cap B(x_i, (1 + \alpha)r)\), i.e., the set of all points in the cover \(C\) within distance \((1 + \alpha)r\) of \(x_i\). Note that the score assigned by the Exponential Mechanism is \(\text{score}[c] = \{i \in [n] \mid c \in S_i\}\), and our goal is to privately select \(c^* \in C\) with as large a score as possible. Two main questions remain: (1) How do we find the \(S_i\)’s efficiently? (2) Given the \(S_i\)’s, how do we sample \(c^*\)? We address these in the following two subsections, respectively.

3.1.1 Efficiently List-Decodable Covers

In this section, we discuss how to find \(S_i\) in time \((1 + 1/\alpha)^{O(d')}\). Motivated by works on error-correcting codes (see, e.g., \([\text{Gur06}]\)), we introduce the notion of list-decodability for covers:

Definition 8 (List-Decodable Cover). A \(\Delta\)-cover is \(\ell\)-list-decodable at distance \(\Delta' \geq \Delta\) with list size \(\ell\) if for any \(x \in B(0, 1)\), we have that \(\{c \in C \mid ||c - x|| \leq \Delta'\}\) \(\leq \ell\). Moreover, the cover is efficiently list-decodable if there is an algorithm that returns such a list in time \(\ell \cdot (d', \log(1/\Delta'))\).

In the main body of the paper, we state error bounds that hold with probability 0.99. In the appendix, we extend all our bounds to hold with probability \(1 - \beta\) for any \(\beta > 0\), with a mild dependency on \(\beta\) in the error.

A \(\zeta\)-cover \(C\) of \(B(0, 1)\) is a set of points such that for any \(y \in B(0, 1)\), there is \(c \in C\) with \(||c - y|| \leq \zeta\).
We prove the existence of efficiently list-decodable covers with the following parameters:

**Lemma 9.** For every $0 < \Delta < 1$, there exists a $\Delta$-cover $C_\Delta$ that is efficiently list-decodable at any distance $\Delta' \geq \Delta$ with list size $(1 + \Delta'/\Delta)^{O(d)}$.

In this terminology, $S_i$ is exactly the decoded list at distance $\Delta' = (1 + \alpha)r$, where $\Delta = \alpha r$ in our cover $C$. As a result, we obtain the $(1 + 1/\alpha)^{O(r)}$ bound on the time for computing $S_i$, as desired.

The proof of Lemma 9 includes two tasks: (i) bounding the size of the list and (ii) coming up with an efficient decoding algorithm. It turns out that (i) is not too hard: if our cover is also an $\Omega(\Delta)$-packing then a standard volume argument implies the bound in Lemma 9. However, carrying out (ii) is more challenging. To do so, we turn to lattice-based covers. A lattice is a set of points that can be written as an integer combination of some given basis vectors. Rogers [Rog59] (see also [Mic14]) constructed a family of lattices that are both $\Delta$-covers and $\Omega(\Delta)$-packings. Furthermore, known lattice algorithms for the so-called Closest Vector Problem [MV13] allow us to find a point $c \in C_{\Delta}$ that is closest to a given point $x$ in time $2^{O(d')}$. With some more work, we can “expand” from $c$ to get the entire list in time polynomial in $\ell$. This concludes the outline of our proof of Lemma 9.

### 3.1.2 SparseSelection

We now move to (2): given $S_i$’s, how to privately select $c^*$ with large $\text{score}[c^*] = |\{i \mid c^* \in S_i\}|$?

We formalize the problem as follows:

**Definition 10 (SparseSelection).** For $\ell \in \mathbb{N}$, the input to the $\ell$-SparseSelection problem is a list $S_1, \ldots, S_n$ of subsets, where $S_1, \ldots, S_n \in \binom{C}{\leq \ell}$ for some finite universe $C$. An algorithm solves $\ell$-SparseSelection with additive error $t$ if it outputs a universe element $c^* \in C$ such that $|\{i \mid c^* \in S_i\}| \geq \max_{c \in C} |\{i \mid c \in S_i\}| + t$.

The crux of our SparseSelection algorithm is the following. Since $\text{score}[c^*] = 0$ for all $c^* \notin S_1 \cup \cdots \cup S_n$, to implement the Exponential Mechanism it suffices to first randomly select (with appropriate probability) whether we should sample from $S_1 \cup \cdots \cup S_n$ or uniformly from $C$. For the former, the sampling is efficient since $S_1 \cup \cdots \cup S_n$ is small. This gives the following for pure-DP:

**Lemma 11.** Suppose there is a poly log $|C|$-time algorithm $O$ that samples a random element of $C$ where each element of $C$ is output with probability at least $0.1/|C|$. Then, there is a poly$(n, \ell, \log |C|)$-time $\ell$-DP algorithm that, with probability 0.99, solves $\ell$-SparseSelection with additive error $O\left(\frac{1}{\ell} \cdot \log |C|\right)$.

We remark that, in Lemma 11, we only require $O$ to sample approximately uniformly from $C$. This is due to a technical reason that we only have such a sampler for the lattice covers we use. Nonetheless, the outline of the algorithm is still exactly the same as before.

For approximate-DP, it turns out that we can get rid of the dependency of $|C|$ in the additive error entirely, by adjusting the probability assigned to each of the two cases. In fact, for the second case, it even suffices to just output some symbol $\downarrow$ instead of sampling (approximately) uniformly from $C$. Hence, there is no need for a sampler for $C$ at all, and this gives us the following guarantees:

**Lemma 12.** There is a poly$(n, \ell, \log |C|)$-time $(\epsilon, \delta)$-DP algorithm that, with probability 0.99, solves $\ell$-SparseSelection with additive error $O\left(\frac{1}{\ell} \cdot \log \left(\frac{\ln n}{\epsilon \delta}\right)\right)$.

### 3.1.3 Putting Things Together

With the ingredients ready, the DensestBall algorithm is given in Algorithm 1. The pure- and approximate-DP algorithms for SparseSelection in Lemmas 11 and 12 lead to Theorem 7.

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10 A $\zeta$-packing is a set of points such that each pairwise distance is at least $\zeta$. 
4 Private $k$-means and $k$-median

We next describe how we use our DensestBall algorithm along with additional ingredients adapted from previous studies of coresets to obtain DP approximation algorithms for $k$-means and $k$-median with nearly tight approximation ratios and small additive errors as stated next:

**Theorem 13.** Assume there is a polynomial-time (not necessarily DP) algorithm for $k$-means (resp., $k$-median) in $\mathbb{R}^d$ with approximation ratio $w$. Then, there is an $\epsilon$-DP algorithm that runs in time $kO_{\alpha}(1)\poly(nd)$ and, with probability 0.99, produces a $(w(1+\alpha), O_{w,\alpha}(\frac{k^{d+O_{\alpha}(1)}}{\epsilon} \poly n))$-approximation for $k$-means (resp., $k$-median). Moreover, there is an $(\epsilon,\delta)$-DP algorithm with the same runtime and approximation ratio but with additive error $O_{w,\alpha}(\left(\frac{k\sqrt{d}}{\epsilon} \cdot \poly (\frac{k}{\delta})\right) + \left(\frac{kO_{\alpha}(1)}{\epsilon} \cdot \poly n\right))$.

To prove Theorem 13 as for DensestBall, we first reduce the dimension of the clustering instance from $d$ to $d' = O_{\alpha}(\log k)$, which can be done using the recent result of Makarychev et al. [MMR19].

**Theorem 14.** Under the same assumption as in Theorem 13 there is an $\epsilon$-DP algorithm that runs in time $2^O_{\alpha}(d')\poly(n)$ and, with probability 0.99, produces a $(w(1+\alpha), O_{\alpha,w}(\frac{k^22^O_{\alpha}(d')}{{\epsilon}} \poly n))$-approximation for $k$-means (resp., $k$-median).

We point out that it is crucial for us that the reduced dimension $d'$ is $O_{\alpha}(\log k)$ as opposed to $O_{\alpha}(\log n)$ (which is the bound from a generic application of the JL lemma), as otherwise the additive error in Theorem 14 would be poly$(n)$, which is vacuous, instead of poly$(k)$. We next proceed by (i) finding a “coarse” centroid set (satisfying Definition 5 with $w = O(1)$), (ii) turning the centroid set into a DP coreset (satisfying Definition 5 with $w = 1 + \alpha$), and (iii) running the non-private approximation algorithm as a black box. We describe these steps in more detail below.

4.1 Finding a Coarse Centroid Set via DensestBall

We consider geometrically increasing radii $r = 1/n, 2/n, 4/n, \ldots$. For each such $r$, we iteratively run our DensestBall algorithm $2k$ times, and for each returned center, remove all points within a distance of $8r$ from it. This yields $2k\log n$ candidate centers. We prove that they form a centroid set with a constant approximation ratio and a small additive error:

**Lemma 15.** There is a polynomial time $\epsilon$-DP algorithm that, with probability 0.99, outputs an $(O(1), O\left(\frac{k^{d'}}{\epsilon} \poly n\right))$-centroid set of size $2k\log n$ for $k$-means (resp., $k$-median).

We point out that the solution to this step is not unique. For example, it is possible to run the DP $k$-means algorithm from [SK18] instead of Lemma 15. However, we choose to use our algorithm since its analysis works almost verbatim for both $k$-median and $k$-means, and it is simple.

4.2 Turning a Coarse Centroid Set into a Coreset

Once we have a coarse centroid set from the previous step, we follow the approach of Feldman et al. [FFKN09], which can turn the coarse centroid and eventually produce a DP coreset:

**Lemma 16.** There is a $2^O_{\alpha}(d')\poly(n)$-time $\epsilon$-DP algorithm that, with probability 0.99, produces an $(\alpha, O_{\alpha}\left(\frac{k^22^O_{\alpha}(d')}{{\epsilon}} \poly n\right))$-coreset for $k$-means (and $k$-median).

Roughly speaking, the idea is to first “refine” the coarse centroid by constructing an exponential cover around each center $c$ from Lemma 15. Specifically, for each radius $r = 1/n, 2/n, 4/n, \ldots$, we consider all points in the $(ar)$-cover of the ball of radius $r$ around $c$. Notice that the number of points in such a cover can be bounded by $2^O_{\alpha}(d')$. Taking the union over all such $c, r$, this result in a new fine centroid set of size $2^O_{\alpha}(d') \cdot \poly(k, \log n)$. Each input point is then snapped to the closest point in this set; these snapped points form a good coreset [HM04]. To make this coreset private, we add an appropriately calibrated noise to the number of input points snapped to each point in the fine centroid set. The additive error resulting from this step scales linearly with the size of the fine centroid set, which is $2^O_{\alpha}(d') \cdot \poly(k, \log n)$ as desired.
We note that, although our approach in this step is essentially the same as Feldman et al. [FKN09], they only fully analyzed the algorithm for $k$-median and $d \leq 2$. Thus, we cannot use their result as a black box and hence, we provide a full proof that also works for $k$-means and for any $d > 0$ in Appendix C.

4.3 Finishing Steps

Finally, we can simply run the (not necessarily DP) approximation algorithm on the DP coreset from Lemma 16, which immediately yields Theorem 14.

5 Applications

Our DensestBall algorithms imply new results for other well-studied tasks, which we now describe.

5.1 1-Cluster

Recall the 1-Cluster problem from Section 1. As shown by [NSV16], a discretization of the inputs is necessary to guarantee a finite error with DP, so we assume that they lie in $\mathbb{B}_2^d$. For this problem, they obtained an $O(\sqrt{\log n})$ approximation ratio, which was subsequently improved to some large constant by [NS18] albeit with an additive error that grows polynomially in $n$. Using our DensestBall algorithms we get a $1 + \alpha$ approximation ratio with additive error polylogarithmic in $n$.

Theorem 17. For $0 < \kappa < 1$, there is an $\epsilon$-DP algorithm that runs in $(nd)^{O_\kappa(1)} \log^{(1)}(\frac{1}{\epsilon})$ time and with probability 0.99, outputs a $(1 + \alpha, O_\alpha \left( \frac{\sqrt{d}}{\epsilon} \log \left( \frac{nd}{\alpha} \right) \right))$-approximation for 1-Cluster. For any $\delta > 0$, there is an $(\epsilon, \delta)$-DP algorithm with the same runtime and approximation ratio but with additive error $O_\alpha \left( \frac{\sqrt{d}}{\epsilon^2} \log \left( \frac{nd}{\alpha} \right) \right) + O \left( \frac{1}{\epsilon} \cdot \log \left( \frac{1}{\delta} \right) \cdot g^{\log^+(d/\kappa)} \right)$.

5.2 Sample and Aggregate

Consider functions $f : U^* \to \mathbb{B}^d$ mapping databases to the discretized unit ball. A basic technique in DP is Sample and Aggregate [NRS07], whose premise is that for large databases $S \subseteq U^*$, evaluating $f$ on a random subsample of $S$ can give a good approximation to $f(S)$. This method enables bypassing worst-case sensitivity bounds in DP (see, e.g., [DK14]) and it captures basic machine learning primitives such as bagging [Yyd19]. Concretely, a point $c \in \mathbb{B}_2^d$ is an $(m, r, \zeta, \alpha)$-stable point of $f$ on $S$ if $\text{Pr}[|f(S') - c|_2 \leq r] \geq \zeta$ for $S'$ a database of $m$ i.i.d. samples from $S$. If such a point exists, $f$ is $(m, r, \zeta)$-stable on $S$, and $r$ is a radius of $c$. Via a reduction to 1-Cluster, [NSV16] find a stable point of radius within an $O(\sqrt{\log n})$ factor from the smallest possible while [NRS07] got an $O(\sqrt{d})$ approximation, and a constant factor is subsequently implied by [NS18]. Our 1-Cluster algorithm yields a $1 + \alpha$ approximation:

Theorem 18. Let $d, m, n \in \mathbb{N}$ and $0 < \epsilon, \zeta, \alpha, \delta, \kappa < 1$ with $m \leq n$, $\epsilon \leq \frac{\zeta}{2}$ and $\delta \leq \frac{\zeta}{300}$. There is an $(\epsilon, \delta)$-DP algorithm that takes $f : U^* \to \mathbb{B}^d$ and parameters $m, \epsilon, \delta, \zeta$, runs in time $(\frac{nd}{m} \ln) O_\alpha(1) \log(\frac{1}{\epsilon})$ plus the time for $O(\frac{m}{n})$ evaluations of $f$ on a dataset of size $m$, and whenever $f$ is $(m, r, \zeta)$-stable on $S$, with probability 0.99, the algorithm outputs an $(m, (1 + \alpha)r, \frac{\zeta}{2})$-stable point of $f$ on $S$, provided that $m \geq 2 \cdot O_\alpha \left( \frac{\sqrt{d}}{\epsilon^2} \log \left( \frac{nd}{\alpha} \right) + \frac{1}{\epsilon} \cdot \log \left( \frac{1}{\delta} \right) \cdot g^{\log^+(d/\kappa)} \right)$.

5.3 Agnostic Learning of Halfspaces with a Margin

We next apply our algorithms to the well-studied problem of agnostic learning of halfspaces with a margin (see, e.g., [BS09, BM02, McA03, SSS09, BST12, DCM19, DCM20]). Denote the error rate of a hypothesis $h$ on a distribution $D$ on labeled samples by $\text{err}^D(h)$, and the $\mu$-margin error rate of halfspace $h_\mu(x) = \text{sgn}(u \cdot x)$ on $D$ by $\text{err}^D_\mu(u)$. (See Appendix D for precise definitions.) Furthermore, let $\text{OPT}'_\mu := \min_{u \in \mathbb{B}^d} \text{err}^D_\mu(u)$. The problem of learning halfspaces with a margin in the agnostic PAC model [Hau92, KSS94] can be defined as follows.

Definition 19. Let $d \in \mathbb{N}$ and $\mu, t \in \mathbb{R}^+$. An algorithm properly agnostically PAC learns halfspaces with margin $\mu$, error $t$ and sample complexity $m$, if given as input a training set
which is an easier problem than only data structure that can quickly answer whether provoked. To make the algorithm work when this assumption does not hold, we simply keep a history-independent extend our results to other metric spaces.

is to obtain practical implementations of DP clustering algorithms that could scale to large datasets preserving the tight non-private approximation ratios that we achieve. Another important direction interesting research direction is to study the smallest possible additive error for DP clustering while

In this work, we obtained tight approximation ratios for several fundamental DP clustering tasks. An

Corollary 23. There exists a quantum algorithm that solves (offline) ClosestPair with probability 0.99 in time $2^{O(d)} n^{2/3} \log^2(n, L)$.}

5.4 ClosestPair

Finally, we depart from the notion of DP and instead give an application of efficiently list-decodable covers to the ClosestPair problem:

Definition 21 (ClosestPair). Given points $x_1, \ldots, x_n \in \mathbb{Z}^d$, where each coordinate of $x_i$ is represented as an $L$-bit integer, and an integer $\xi \in \mathbb{Z}$, determine whether there exists $1 \leq i < j \leq n$ such that $\|x_i - x_j\|^2 \leq \xi$.

In the dynamic setting of ClosestPair, we start with an empty set $S$ of points. At each step, a point maybe added to and removed$^{11}$ from $S$, and we have to answer whether there are two distinct points in $S$ whose squared Euclidean distance is at most $\xi$. Our main contribution is a faster history-independent data structure for dynamic ClosestPair. Recall that a deterministic data structure is said to be history-independent if, for any two sequences of updates that result in the same set of points, the states of the data structure must be the same in both cases. For a randomized data structure, we say that it is history-independent if, for any two sequences of updates that result in the same set of points, the distribution of the state of the data structure must be the same.

Theorem 22. There is a history-independent randomized data structure for dynamic ClosestPair that supports up to $n$ updates, with each update takes $2^{O(d)} \log^2(n, L)$ time, and uses $O(nd \cdot \log(n, L))$ memory.

We remark that the data structure is only randomized in terms of the layout of the memory (i.e., state), and that the correctness always holds. Our data structure improves that of Aaronson et al. [ACL+20], in which the running time per update operation is $d^{O(d)} \log(n, L)$.

Aaronson et al. [ACL+20] show how to use their data structure together with quantum random walks from [MNRS11] (see also [Amb07, Sz2011]) to provide a fast quantum algorithm for ClosestPair in low dimensions which runs in time $d^{O(d)} n^{2/3} \log^2(n, L)$. With our improvement above, we immediately obtain a speed up in terms of the dependency on $d$ under the same model$^{12}$.

Corollary 23. There exists a quantum algorithm that solves (offline) ClosestPair with probability 0.99 in time $2^{O(d)} n^{2/3} \log^2(n, L)$.}

6 Conclusion and Open Questions

In this work, we obtained tight approximation ratios for several fundamental DP clustering tasks. An interesting research direction is to study the smallest possible additive error for DP clustering while preserving the tight non-private approximation ratios that we achieve. Another important direction is to obtain practical implementations of DP clustering algorithms that could scale to large datasets with many clusters. We focused in this work on the Euclidean metric; it would also be interesting to extend our results to other metric spaces.

$^{11}$Throughout, we assume without loss of generality that $x$ must belong to $S$ before “remove $x$” can be invoked. To make the algorithm work when this assumption does not hold, we simply keep a history-independent data structure that can quickly answer whether $x$ belongs to $S$ [Amb07, BJLM13].

$^{12}$The model assumes the presence of gates for random access to an $m$-qubit quantum memory that takes time only $\log(m)$. As discussed in [Amb07], such an assumption is necessary even for element distinctness, which is an easier problem than ClosestPair.
Broader Impact

Our work lies in the active area of privacy and its broader impact should be interpreted in light of ongoing debates in academia and industry. The primary goal of our work is to develop efficient differentially private algorithms for clustering data, with quality approaching that of clustering algorithms that are indifferent to privacy.

Being able to cluster data without compromising privacy but with quality almost as good as without privacy considerations, we believe, has a few societal benefits. Firstly, it could compel applications that deal with sensitive data and that already use off-the-shelf clustering algorithms to switch to using private clustering since the quality losses of our algorithm are guaranteed to be minimal and our algorithms are only modestly more expensive to run. Secondly, since clustering is a fundamental primitive in machine learning and data analysis, our work can enable privacy in more intricate applications that depend on clustering. Thirdly, we believe our work can spur further research into making other private machine learning algorithms attain quality comparable to non-private ones. In other words, it can lead to the following state: preserving privacy does not entail a compromise in quality. This will have far-reaching effects on how researchers develop new methods.

On the other hand, there are possible negative consequences of our work. Since our work has not been tested in practice, it is conceivable that practitioners might be dissuaded from using it on their own. Further, there might be unintended or malicious applications of private clustering, where privacy might be used in a negative way; our work might become a latent enablers of such activity.

Overall we believe that protecting privacy is a net positive for the society and our work contribute towards this larger goal in a positive way.

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References

[Abo18] John M Abowd. The US Census Bureau adopts differential privacy. In KDD, pages 2867–2867, 2018.

[AC13] Charu C. Aggarwal and K. R. Chandan. Data Clustering: Algorithms and Applications. Chapman and Hall/CRC Boca Raton, 2013.

[ACKS15] Pranjal Awasthi, Moses Charikar, Ravishankar Krishnaswamy, and Ali Kemal Sinop. The hardness of approximation of Euclidean k-means. In SoCG, pages 754–767, 2015.

[ACL+20] Scott Aaronson, Nai-Hui Chia, Han-Hsuan Lin, Chunhao Wang, and Ruizhe Zhang. On the Quantum Complexity of Closest Pair and Related Problems. In CCC, pages 16:1–16:43, 2020.

[ADS15] Divesh Aggarwal, Daniel Dadush, and Noah Stephens-Davidowitz. Solving the closest vector problem in \(2^n\) time - the discrete Gaussian strikes again! In FOCS, pages 563–582, 2015.

[AGK+04] Vijay Arya, Naveen Garg, Rohit Khandekar, Adam Meyerson, Kamesh Munagala, and Vinayaka Pandit. Local search heuristics for k-median and facility location problems. SIAM J. Comput., 33(3):544–562, 2004.

[AHPV05] Pankaj K Agarwal, Sariel Har-Peled, and Kasturi R Varadarajan. Geometric approximation via coresets. Combinatorial and Computational Geometry, 52:1–30, 2005.

[Amb07] Andris Ambainis. Quantum walk algorithm for element distinctness. SIAM J. Comput., 37(1):210–239, 2007.
[ANSW17] Sara Ahmadian, Ashkan Norouzi-Fard, Ola Svensson, and Justin Ward. Better guarantees for $k$-means and Euclidean $k$-median by primal-dual algorithms. In FOCS, pages 61–72, 2017.

[App17] Apple Differential Privacy Team. Learning with privacy at scale. Apple Machine Learning Journal, 2017.

[AS18] Divesh Aggarwal and Noah Stephens-Davidowitz. Just take the average! an embarrassingly simple $2^n$-time algorithm for SVP (and CVP). In SODA, pages 12:1–12:19, 2018.

[AV07] David Arthur and Sergei Vassilvitskii. $k$-means++: the advantages of careful seeding. In SODA, pages 1027–1035, 2007.

[Bar96] Yair Bartal. Probabilistic approximations of metric spaces and its algorithmic applications. In FOCS, pages 184–193, 1996.

[BDL+17] Maria-Florina Balcan, Travis Dick, Yingyu Liang, Wenlong Mou, and Hongyang Zhang. Differentially private clustering in high-dimensional Euclidean spaces. In ICML, pages 322–331, 2017.

[BDMN05] Avrim Blum, Cynthia Dwork, Frank McSherry, and Kobbi Nissim. Practical privacy: the sulq framework. In PODS, pages 128–138, 2005.

[BEL03] Shai Ben-David, Nadav Eiron, and Philip M. Long. On the difficulty of approximately maximizing agreements. JCSS, 66(3):496–514, 2003.

[Bes98] Sergei Bespamyatnikh. An optimal algorithm for closest-pair maintenance. Discret. Comput. Geom., 19(2):175–195, 1998.

[BES02] Shai Ben-David, Nadav Eiron, and Hans Ulrich Simon. The computational complexity of densest region detection. JCSS, 64(1):22–47, 2002.

[BF13] Karl Bringmann and Tobias Friedrich. Exact and efficient generation of geometric random variates and random graphs. In ICALP, pages 267–278, 2013.

[BHIP02] Mihai Bădaiou, Sariel Har-Peled, and Piotr Indyk. Approximate clustering via coresets. In STOC, pages 250–257, 2002.

[BJLM13] Daniel J. Bernstein, Stacey Jeffery, Tanja Lange, and Alexander Meurer. Quantum algorithms for the subset-sum problem. In PQCrypto, pages 16–33, 2013.

[BM02] Peter L. Bartlett and Shahar Mendelson. Rademacher and Gaussian complexities: Risk bounds and structural results. JMLR, 3:463–482, 2002.

[BPR+17] Jaroslaw Byrka, Thomas W. Pensyl, Bartosz Rybicki, Aravind Srinivasan, and Khoa Trinh. An improved approximation for $k$-median and positive correlation in budgeted optimization. ACM Trans. Algorithms, 13(2):23:1–23:31, 2017.

[BS76] Jon Louis Bentley and Michael Ian Shamos. Divide-and-conquer in multidimensional space. In STOC, pages 220–230, 1976.

[BS00] Shai Ben-David and Hans Ulrich Simon. Efficient learning of linear perceptrons. In NIPS, pages 189–195, 2000.

[BS12] Aharon Birnbaum and Shai Shalev-Shwartz. Learning halfspaces with the zero-one loss: Time-accuracy tradeoffs. In NIPS, pages 935–943, 2012.

[BST14] Raef Bassily, Adam D. Smith, and Abhradeep Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In FOCS, pages 464–473, 2014.

[BU17] Mitali Bafna and Jonathan Ullman. The price of selection in differential privacy. In COLT, pages 151–168, 2017.
[CCGG98] Moses Charikar, Chandra Chekuri, Ashish Goel, and Sudipto Guha. Rounding via
trees: Deterministic approximation algorithms for group Steiner trees and $k$-median.
In STOC, pages 114–123, 1998.

[CGTS02] Moses Charikar, Sudipto Guha, Éva Tardos, and David B. Shmoys. A constant-factor
approximation algorithm for the $k$-median problem. JCSS, 65(1):129–149, 2002.

[Che06] Ke Chen. On $k$-median clustering in high dimensions. In SODA, pages 1177–1185,
2006.

[CK19] Vincent Cohen-Addad and Karthik C. S. Inapproximability of clustering in $l_p$
metrics. In FOCS, pages 519–539, 2019.

[CMS11] Kamalika Chaudhuri, Claire Monteleoni, and Anand D. Sarwate. Differentially pri-
vate empirical risk minimization. JMLR, 12:1069–1109, 2011.

[DF13] Rodney G. Downey and Michael R. Fellows. Fundamentals of Parameterized Com-
plexity. Texts in Computer Science. Springer, 2013.

[DG03] Sanjoy Dasgupta and Anupam Gupta. An elementary proof of a theorem of Johnson
and Lindenstrauss. Random Struct. Algorithms, 22(1):60–65, 2003.

[DJW13] John C. Duchi, Michael I. Jordan, and Martin J. Wainwright. Local privacy and
statistical minimax rates. In FOCS, pages 429–438, 2013.

[DKM′06] Cynthia Dwork, Krishnaram Kenthapadi, Frank McSherry, Ilya Mironov, and Moni
Naor. Our data, ourselves: Privacy via distributed noise generation. In EURO-
CRYPT, pages 486–503, 2006.

[DKM19] Ilias Diakonikolas, Daniel Kane, and Pasin Manurangsi. Nearly tight bounds for
robust proper learning of halfspaces with a margin. In NeurIPS, pages 10473–10484,
2019.

[DKM20] Ilias Diakonikolas, Daniel M. Kane, and Pasin Manurangsi. The complexity of
adversarially robust proper learning of halfspaces with agnostic noise. CoRR,
abs/2007.15220, 2020.

[DKY17] Bolin Ding, Janardhan Kulkarni, and Sergey Yekhanin. Collecting telemetry data
privately. In NIPS, pages 3571–3580, 2017.

[DLVKKR03] W Fernandez De La Vega, Marek Karpinski, Claire Kenyon, and Yuval Rabani.
Approximation schemes for clustering problems. In STOC, pages 50–58, 2003.

[DMNS06] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise
to sensitivity in private data analysis. In TCC, pages 265–284, 2006.

[DNR+09] Cynthia Dwork, Moni Naor, Omer Reingold, Guy N. Rothblum, and Salil P. Vadhan.
On the complexity of differentially private data release: efficient algorithms and
hardness results. In STOC, pages 381–390, 2009.

[DR14] Cynthia Dwork and Aaron Roth. The Algorithmic Foundations of Differential Pri-
vacy. Foundations and Trends in Theoretical Computer Science, 9(3-4):211–407,
2014.

[DRV10] Cynthia Dwork, Guy N. Rothblum, and Salil P. Vadhan. Boosting and differential
privacy. In FOCS, pages 51–60, 2010.

[EPK14] Úlfar Erlingsson, Vasyl Pihur, and Aleksandra Korolova. RAPPOR: Randomized
aggregatable privacy-preserving ordinal response. In CCS, pages 1054–1067, 2014.

[FFKN09] Dan Feldman, Amos Fiat, Haim Kaplan, and Kobbi Nissim. Private coresets. In
STOC, pages 361–370, 2009.

[FL11] Dan Feldman and Michael Langberg. A unified framework for approximating and
clustering data. In STOC, pages 569–578, 2011.
[FMS07] Dan Feldman, Morteza Monemizadeh, and Christian Sohler. A PTAS for $k$-means clustering based on weak coresets. In SoCG, pages 11–18, 2007.

[FXZR17] Dan Feldman, Chongyuan Xiang, Ruihao Zhu, and Daniela Rus. Coresets for differentially private $k$-means clustering and applications to privacy in mobile sensor networks. In IPSN, pages 3–16, 2017.

[GLM⁺10] Anupam Gupta, Katrina Ligett, Frank McSherry, Aaron Roth, and Kunal Talwar. Differentially private combinatorial optimization. In SODA, pages 1106–1125, 2010.

[Gre16] Andy Greenberg. Apple’s “differential privacy” is about collecting your data – but not your data. Wired, June, 13, 2016.

[Gur06] Venkatesan Guruswami. Algorithmic Results in List Decoding. Foundations and Trends in Theoretical Computer Science, 2(2), 2006.

[Hau92] David Haussler. Decision theoretic generalizations of the PAC model for neural net and other learning applications. Information and Computation, 100(1):78–150, 1992.

[HL18] Zhiyi Huang and Jinyan Liu. Optimal differentially private algorithms for $k$-means clustering. In PODS, pages 395–408, 2018.

[HM04] Sariel Har-Peled and Soham Mazumdar. On coresets for $k$-means and $k$-median clustering. In STOC, pages 291–300, 2004.

[Jef14] Jeffery, Stacey. Frameworks for Quantum Algorithms. PhD thesis, University of Waterloo, 2014.

[JKT12] Prateek Jain, Pravesh Kothari, and Abhradeep Thakurta. Differentially private online learning. In COLT, pages 24.1–24.34, 2012.

[JL84] William B. Johnson and Joram Lindenstrauss. Extensions of Lipschitz mappings into Hilbert space. Contemporary mathematics, 26:189–206, 1984.

[JMS02] Kamal Jain, Mohammad Mahdian, and Amin Saberi. A new greedy approach for facility location problems. In STOC, pages 731–740, 2002.

[JV01] Kamal Jain and Vijay V. Vazirani. Approximation algorithms for metric facility location and $k$-median problems using the primal-dual schema and lagrangian relaxation. J. ACM, 48(2):274–296, 2001.

[JYvdS19] James Jordon, Jinsung Yoon, and Mihaela van der Schaar. Differentially private bagging: Improved utility and cheaper privacy than subsample-and-aggregate. In NeurIPS, pages 4325–4334, 2019.

[Kir34] Mojzesz Kirszbraun. Über die zusammenziehende und Lipschitzsche transformationen. Fundamenta Mathematicae, 22(1):77–108, 1934.

[KM19] Karthik C. S. and Pasin Manurangsi. On closest pair in Euclidean metric: Monochromatic is as hard as bichromatic. In ITCS, pages 17:1–17:16, 2019.

[KMN⁺04] Tapas Kanungo, David M. Mount, Nathan S. Netanyahu, Christine D. Piatko, Ruth Silverman, and Angela Y. Wu. A local search approximation algorithm for $k$-means clustering. Comput. Geom., 28(2-3):89–112, 2004.

[KS96] Sanjiv Kapoor and Michiel H. M. Smid. New techniques for exact and approximate dynamic closest-point problems. SIAM J. Comput., 25(4):775–796, 1996.

[KSS94] Michael J Kearns, Robert E Schapire, and Linda M Sellie. Toward efficient agnostic learning. Machine Learning, 17(2-3):115–141, 1994.

[KSS04] A Kumar, Y Sabharwal, and S Sen. A simple linear time $(1 + \epsilon)$-approximation algorithm for $k$-means clustering in any dimensions. In FOCS, pages 454–462, 2004.
[KSS05] Amit Kumar, Yogish Sabharwal, and Sandeep Sen. Linear time algorithms for clustering problems in any dimensions. In ICALP, pages 1374–1385, 2005.

[KST12] Daniel Kifer, Adam D. Smith, and Abhradeep Thakurta. Private convex optimization for empirical risk minimization with applications to high-dimensional regression. In COLT, pages 25.1–25.40, 2012.

[Llo82] Stuart Lloyd. Least squares quantization in PCM. IEEE TOIT, 28(2):129–137, 1982.

[LS92] Hans-Peter Lenhof and Michiel H. M. Smid. Enumerating the k closest pairs optimally. In FOCS, pages 380–386, 1992.

[LS16] Shi Li and Ola Svensson. Approximating k-median via pseudo-approximation. SIAM J. Comput., 45(2):530–547, 2016.

[LSW17] Euiwoong Lee, Melanie Schmidt, and John Wright. Improved and simplified inapproximability for k-means. Inf. Process. Lett., 120:40–43, 2017.

[Mat00] Jivri Matouvsek. On approximate geometric k-clustering. Discret. Comput. Geom., 24(1):61–84, 2000.

[McA03] David McAllester. Simplified PAC-Bayesian margin bounds. In Learning theory and Kernel machines, pages 203–215. Springer, 2003.

[MG12] Daniele Micciancio and Shafi Goldwasser. Complexity of Lattice Problems: A Cryptographic Perspective, volume 671. Springer Science & Business Media, 2012.

[Mic04] Daniele Micciancio. Almost perfect lattices, the covering radius problem, and applications to Ajtai’s connection factor. SIAM J. Comput., 34(1):118–169, 2004.

[MMR19] Konstantin Makarychev, Yury Makarychev, and Ilya P. Razenshteyn. Performance of Johnson–Lindenstrauss transform for k-means and k-medians clustering. In STOC, pages 1027–1038, 2019.

[MNRS11] Frédéric Magniez, Ashwin Nayak, Jérémie Roland, and Miklos Santha. Search via quantum walk. SIAM J. Comput., 40(1):142–164, 2011.

[MT07] Frank McSherry and Kunal Talwar. Mechanism design via differential privacy. In FOCS, pages 94–103, 2007.

[MTS12] Prashanth Mohan, Abhradeep Thakurta, Elaine Shi, Dawn Song, and David Culler. GUPT: privacy preserving data analysis made easy. In SIGMOD, pages 349–360, 2012.

[MV13] Daniele Micciancio and Panagiotis Voulgaris. A deterministic single exponential time algorithm for most lattice problems based on Voronoi cell computations. SIAM J. Comput., 42(3):1364–1391, 2013.

[NCBN16] Richard Nock, Raphaël Canyasse, Roksana Boreli, and Frank Nielsen. k-variates++: more pluses in the k-means++. In ICML, pages 145–154, 2016.

[Nov62] Albert B.J. Novikoff. On convergence proofs on perceptrons. In Proceedings of the Symposium on the Mathematical Theory of Automata, volume 12, pages 615–622, 1962.

[NRS07] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In STOC, pages 75–84, 2007.

[NS18] Kobbi Nissim and Uri Stemmer. Clustering algorithms for the centralized and local models. In ALT, pages 619–653, 2018.

[NSV16] Kobbi Nissim, Uri Stemmer, and Salil P. Vadhan. Locating a small cluster privately. In PODS, pages 413–427, 2016.
[NUZ20] Huy Lê Nguyên, Jonathan Ullman, and Lydia Zakynthinou. Efficient private algorithms for learning large-margin halfspaces. In ALT, pages 704–724, 2020.

[Rab76] Michael O. Rabin. Probabilistic algorithms. In Proceedings of a Symposium on New Directions and Recent Results in Algorithms and Complexity, Computer Science Department, Carnegie-Mellon University, April 7-9, 1976, pages 21–39, 1976.

[Rog59] Claude A Rogers. Lattice coverings of space. Mathematika, 6(1):33–39, 1959.

[Ros58] Frank Rosenblatt. The Perceptron: a probabilistic model for information storage and organization in the brain. Psychological Review, 65:386–407, 1958.

[Sal91] Jeffrey S. Salowe. Shallow interdistnace selection and interdistance enumeration. In WADS, pages 117–128, 1991.

[SCL+16] Dong Su, Jianpeng Cao, Ninghui Li, Elisa Bertino, and Hongxia Jin. Differentially private $k$-means clustering. In CODASPY, pages 26–37, 2016.

[SH75] Michael Ian Shamos and Dan Hoey. Closest-point problems. In FOCS, pages 151–162, 1975.

[Sha14] Stephen Shankland. How Google tricks itself to protect Chrome user privacy. CNET, October, 2014.

[She13] Vladimir Shenmaier. The problem of a minimal ball enclosing $k$ points. Journal of Applied and Industrial Mathematics, 7(3):444–448, 2013.

[She15] Vladimir Shenmaier. Complexity and approximation of the smallest $k$-enclosing ball problem. Eur. J. Comb., 48:81–87, 2015.

[SK18] Uri Stemmer and Haim Kaplan. Differentially private $k$-means with constant multiplicative error. In NeurIPS, pages 5436–5446, 2018.

[Smi92] Michiel Smid. Maintaining the minimal distance of a point set in polylogarithmic time. Discrete & Computational Geometry, 7(4):415–431, 1992.

[SSS09] S. Shalev Shwartz, O. Shamir, and K. Sridharan. Agnostically learning halfspaces with margin errors. TTI Technical Report, 2009.

[Ste20] Uri Stemmer. Locally private $k$-means clustering. In SODA, pages 548–559, 2020.

[SU17] Thomas Steinke and Jonathan Ullman. Tight lower bounds for differentially private selection. In FOCS, pages 552–563, 2017.

[Sze04] Mario Szegedy. Quantum speed-up of Markov chain based algorithms. In FOCS, pages 32–41, 2004.

[Ull18] Jonathan Ullman. Tight lower bounds for locally differentially private selection. CoRR, abs/1802.02638, 2018.

[Vad17] Salil Vadhan. The complexity of differential privacy. In Tutorials on the Foundations of Cryptography, pages 347–450. Springer, 2017.

[WWS15] Yining Wang, Yu-Xiang Wang, and Aarti Singh. Differentially private subspace clustering. In NIPS, pages 1000–1008, 2015.

[WYX17] Di Wang, Minwei Ye, and Jinhui Xu. Differentially private empirical risk minimization revisited: Faster and more general. In NIPS, pages 2722–2731, 2017.

[XW08] Rui Xu and Don Wunsch. Clustering, volume 10. John Wiley & Sons, 2008.