Some sharp Hardy inequalities on spherically symmetric domains

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Abstract

We prove some sharp Hardy inequalities for domains with a spherical symmetry. In particular, we prove an inequality for domains of the unit n-dimensional sphere with a point singularity, and an inequality for functions defined on the half-space \( \mathbb{R}^{n+1}_+ \) vanishing on the hyperplane \( \{x_{n+1} = 0\} \), with singularity along the \( x_{n+1} \)-axis. The proofs rely on a one-dimensional Hardy inequality involving a weight function related to the volume element on the sphere, as well as on symmetrization arguments. The one-dimensional inequality is derived in a general form.

Key Words: sharp weighted Hardy inequalities, symmetrization.

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1 Introduction and main results

Sharp Hardy inequalities have attracted a considerable attention in recent years, particularly in view of their applications to differential equations motivated by physics and geometry. Let \( 1 < p < n \). The classical Hardy inequality states that

\[
\int_{\mathbb{R}^n} |Du|^p_n \geq \left( \frac{n - p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p_n}
\]

for all smooth functions \( u \) compactly supported on \( \mathbb{R}^n \), where we set \( |x|^2_n = x_1^2 + \ldots + x_n^2 \) for all \( x \in \mathbb{R}^n \).

A considerable effort has been devoted to extending this inequality to manifolds, to special weight functions as well as to domains exhibiting particular symmetries. See, e.g., [11] for an extensive...
review. Our aim in this note is to derive some sharp Hardy type inequalities specifically tailored for manifolds with a spherical symmetry. It should be mentioned that several recent inequalities have been concerned with the special case of the sphere, see, e.g., [3, 8]. Indeed, certain specific phenomena which do not occur on Euclidean space actually do occur on spheres. For example, in [3] it was shown that the Sobolev inequality admits minimizers on sufficiently large spherical caps.

In order to state our main results we introduce some notation. Let

$$\mathbb{S}^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1\}$$

denote the unit n-sphere, where we set $|x|^2 = |x|^2_{n+1} = x_1^2 + \ldots + x_{n+1}^2$ for all $x \in \mathbb{R}^{n+1}$. For $1 < p < n$ and for $a > 0$ we define the following weight function $\tilde{\eta}_a : (0, a) \rightarrow \mathbb{R}$, which is related to the volume element on $\mathbb{S}^n$, as follows:

$$\tilde{\eta}_a(t) = \frac{\sin t}{\int_0^t \sin s \, ds}.$$ (1.2)

Note that $\lim_{t \rightarrow 0^+} \tilde{\eta}_a(t) = \lim_{t \rightarrow a^-} \tilde{\eta}_a(t) = +\infty$, see (1.2) below. In Lemma 2.3 we will show that there exists $T \in (0, a)$ such that $\tilde{\eta}_a$ decreases in $(0, T)$ and increases in $(T, a)$. Therefore, the following truncated function:

$$\tilde{\eta}_{aT}(t) = \begin{cases} \frac{\sin t}{\int_0^t \sin s \, ds} & \text{if } t \in (0, T) \\ \tilde{\eta}_a(T) & \text{if } x \in [T, a) \end{cases}$$ (1.3)

is decreasing in $(0, a)$. We denote by $\Theta = (\theta_1, \ldots, \theta_{n-1}, \theta_n)$ the angular variables on $\mathbb{S}^n$ and to simplify notation we set $\theta = \theta_n$. The angle $\theta \in [0, \pi]$, satisfying $x_{n+1} = |x| \cos \theta$, will be the only relevant angular variable to our purposes. We denote by $g$ the standard metric on $\mathbb{S}^n$ and by $dV$ the volume element on $\mathbb{S}^n$. For $\alpha \in (0, \pi]$ we denote by $B(\alpha)$ the geodesic ball (spherical cap) on $\mathbb{S}^n$ with radius $\alpha$ centered at the “north pole” $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Namely, we define

$$B(\alpha) = \{x \in \mathbb{S}^n : 0 \leq \theta < \alpha\}.$$ 

Let $N \in \Omega \subset \mathbb{S}^n$ be an open set such that $|\Omega| < |\mathbb{S}^n|$ and let $a^* \in (0, \pi)$ be such that $|B(a^*)| = |\Omega|$. Here, for every measurable set $E \subset \mathbb{S}^n$, $|E|$ denotes the volume of $E$ with respect to the standard Lebesgue measure induced by $g$ on $\mathbb{S}^n$. In turn, we define the following weight function $\rho_a^* : \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}$

$$\rho_a^*(x) = \begin{cases} \frac{n-1}{\alpha-\beta} \tilde{\eta}_{a^*T}(\theta) & \text{if } x \in B(a^*) \setminus \{N\} \\ \frac{\alpha-\beta}{\alpha-\beta} \tilde{\eta}_{a^*T}(T) & \text{if } x \in \mathbb{S}^n \setminus B(a^*) \end{cases}$$ (1.4)

where $\tilde{\eta}_{a^*T}$ is the weight function defined in (1.3) with $a = a^*$. With this notation, our first result is the following.

Theorem 1.1. Let $n \geq 2$ and $1 < p < n$. Let $\Omega \subset \mathbb{S}^n$ be an open set such that $N \in \Omega$ and $|\Omega| < |\mathbb{S}^n|$. Let $a^*$ be such that $|\Omega| = |B(a^*)|$. Then, for every $u \in W^{1,p}_0(\Omega)$, we have

$$\int_{\Omega} |\nabla u|^p \, dV \geq \left( \frac{n-p}{p} \right)^p \int_{\Omega} |u|^p \rho_{a^*}^p \, dV.$$ (1.5)
The constant $[(n-p)/p]^p$ is sharp.

Note that, since $\theta = d_\rho(x,N)$, we have

$$\lim_{x \to N} d_\rho(x,N) \rho_{\ast}(x) = 1,$$  \hspace{1cm} (1.6)

see (2.1) below, so that $\rho_{\ast}$ is a natural extension of the classical singularity $|x|^{-p}$ appearing in (1.1).

Theorem 1.1 together with a Steiner symmetrization with respect to the angular variables, yields an inequality for functions defined on the half-space $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^{n+1} : x_{n+1} > 0 \}$ with singularity along the $x_{n+1}$-axis. More precisely, for every $x \in \mathbb{R}^{n+1}$ let $x' = (x_1, \ldots, x_n, 0)$. We take $a^* = \pi/2$ in (1.2) - (1.4) and we define the singularity $\zeta : \mathbb{R}^n_+ \setminus \{x' = 0\} \rightarrow \mathbb{R}$ as follows:

$$\zeta(x) = \rho_{\pi/2} \left( \frac{x}{|x|} \right).$$

Note that $\zeta$ is singular on the $x_{n+1}$-axis. We have:

**Theorem 1.2.** For every $u \in W^{1,p}_{0}(\mathbb{R}^{n+1}_+)$, the following inequality holds:

$$\int_{\mathbb{R}^n_+} |D\phi u|^p \, dx \geq \left( \frac{n-p}{p} \right)^p \int_{\mathbb{R}^n_+} |u|^p \frac{\zeta^p(x)}{|x|^p} \, dx.$$ \hspace{1cm} (1.7)

Here, $D\phi u(x)$ denotes the projection of the gradient $Du(x)$ on the sphere $\partial B(0, |x|)$. The constant $\left( \frac{n-p}{p} \right)^p$ is sharp.

We observe that for special values of $p$ and $n$ the singularities appearing in (1.3) and (1.7) take particularly simple and explicit forms. More precisely, let $p = (n+1)/2$ and suppose that $\Omega \subset \mathbb{S}^n$ is such that $|\Omega| = |\mathbb{S}^n|/2$. Then, $a^* = \pi/2$, $(p-1)/(n-p) = 1$, $(n-1)/(p-1) = 2$ and therefore $\bar{\eta}_{\pi/2}(t) = \sin^{-2} t (\int_0^{\pi/2} \sin^{-2} \sigma \, d\sigma)^{-1} = (\sin t \cos t)^{-1}$. Consequently, $\bar{T} = \pi/4$ and

$$\rho_{\pi/2}(x) = \begin{cases} (\sin \theta \cos \theta)^{-1} & \text{if } x \in B(\pi/4) \setminus \{N\} \\ 2 & \text{if } x \in \mathbb{S}^n \setminus B(\pi/4) \end{cases}.$$

Note also that $(\sin \theta \cos \theta)^{-1} > \theta^{-1}$ for any $\theta \in (0, \pi/4)$ and $2 > \theta^{-1}$ for any $\theta \in (\pi/4, \pi)$. Therefore, inequality (1.3) implies that

$$\int_{\Omega} |\nabla u|^p \, dV \geq \left( \frac{n-p}{p} \right)^p \int_{\Omega} |u|^p \left[ \frac{1}{d_\rho(x,N)^p} + h \right] \, dV,$$

where $h$ is a positive quantity, thus showing that (1.1) is improved on the sphere in this case. When $p = (n+1)/2$ the same arguments also yield a simple form for (1.7). Indeed, in this case (1.7) may be written in the form

$$\int_{\mathbb{R}^n_+} |D\phi u|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \left( \int_{\mathbb{R}^n_+ \cap \{\theta < \pi/4\}} |u|^p \frac{1}{|x|^p (\cos \theta)^p} \, dx + 2p \int_{\mathbb{R}^n_+ \cap \{\pi/4 < \theta < \pi/2\}} |u|^p \frac{1}{|x|^p} \, dx \right).$$

This special case was also shown to be of interest in [5].
An outline of the proofs may be as follows. Our starting point is the following one-dimensional Hardy inequality:

\[
\int_0^a |u'|^p \sin^{n-1}(t) \, dt \geq \left( \frac{p-1}{p} \right)^p \int_0^a |u|^p \tilde{\eta}_{aT}^p \sin^{n-1}(t) \, dt,
\]

for all \( u \) such that \( u(a) = 0 \), where \( \tilde{\eta}_{aT} \) is the function defined in (1.3). In fact, in Section 2 we shall prove some sharp weighted one-dimensional Hardy inequalities involving a general weight \( \phi \) which reduces to (1.8) when \( \phi(t) = \sin^{n-1}(t) \), see Proposition 2.1 and Proposition 2.2 below. To this end, we extend a method described in [9], see also [13], for the special case \( \phi(t) = 1 \). In fact, one of our efforts is to determine very general conditions on \( \phi \) such that this method is applicable.

This technique was also employed in [7] in the special case \( \phi(t) = (2\pi)^{-1/2} \exp\{-t^2/2\} \) in the context of symmetrization with respect to Gaussian measure. On the other hand, our sharpness considerations as in Proposition 2.2 are new even in these special cases. In Section 3 we employ spherical symmetrization in order to reduce Theorem 1.1 to (1.8). In turn, Theorem 1.1 together with a Steiner symmetrization with respect to the angular variables concludes the proof of Theorem 1.2.

### 2 Some Hardy inequalities on intervals

Our aim in this section is to prove some weighted one-dimensional Hardy inequalities as stated in Proposition 2.1 and Proposition 2.2 below. To this end, as already mentioned in Section 1, we exploit a technique from [9], Theorem 253 p. 175, see also [7, 13]. Let \( a > 0 \), \( p > 1 \) and let \( \phi \in C^1(0, a] \cap C^0([0, a]) \) be such that

\[
\phi(0) = 0, \quad \phi(t) > 0 \text{ in } (0, a], \quad c_1 t^{p-1+\delta} \leq \phi(t) \leq c_2 t^{p-1+\delta},
\]

for some \( c_1, c_2, \delta > 0 \). We denote

\[
W^{1,p}(0, a; \phi) = \left\{ u : [0, a] \to \mathbb{R}; \ u \in L^1_{\text{loc}}[0, a] \text{ and } \int_0^a |u'|^p \phi \, dt < +\infty \right\},
\]

where \( u' \) denotes the distributional derivative of \( u \). We consider the following subspace of \( W^{1,p}(0, a; \phi) \)

\[
\mathcal{E} = \left\{ u \in W^{1,p}(0, a; \phi) : u(a) = 0 \right\},
\]

endowed with the norm \( \|u\| = \left( \int_0^a |u'|^p \phi \right)^{\frac{1}{p}} \). We note that if \( u \in \mathcal{E} \), then \( u \) is absolutely continuous in \([\epsilon, a]\) for all \( \epsilon \in (0, a) \). On the other hand, \( u \) is in general unbounded near the origin. Nevertheless, \( u \) may be approximated in \( \mathcal{E} \) by functions which vanish in 0. More precisely we have the following.

**Lemma 2.1.** \( C_0^1[0, a] \) is dense in \( \mathcal{E} \).

**Proof.** Let \( u \in \mathcal{E} \). By standard properties of Sobolev spaces we may assume that \( u \in C^1[\epsilon, a] \) for all \( \epsilon \in (0, a) \). We first show that \( C_0^1[0, a] \) is dense in \( \mathcal{E} \cap L^\infty(0, a) \). Let \( u \in \mathcal{E} \cap L^\infty(0, a) \). We consider
the sequence:

\[
\begin{array}{ll}
u_k(t) = \\
\quad \\u(t) & \text{if } t \in [k^{-1}, a],
\end{array}
\]

where \( k \in \mathbb{N} \). By the elementary inequality \(|\alpha + \beta|^p \leq 2^{p-1} (|\alpha|^p + |\beta|^p)\), for all \( \alpha, \beta \in \mathbb{R} \), we have

\[
\int_0^a |u' - u_k'|^p \phi \, dt = \int_0^{k^{-1}} |u' - u(k^{-1})|^p \phi \, dt \leq 2^{p-1} \left\{ \int_0^{k^{-1}} |u'|^p \phi \, dt + ||u||_\infty^p k^p \int_0^{k^{-1}} \phi \, dt \right\}. 
\]

(2.2)

Since \( u' \in L^p(0, a; \phi) \), the absolute continuity of the Lebesgue integral implies that

\[
\int_0^{k^{-1}} |u'|^p \phi \, dt = o(1), \quad \text{as } k \to +\infty. 
\]

(2.3)

In view of (2.1), we have

\[
\max_{t \in [0, k^{-1}]} \phi \leq c_2 \max_{t \in [0, k^{-1}]} t^{p-1+\delta} = c_2 k^{-p+1-\delta}. 
\]

Consequently,

\[
k^p \int_0^{k^{-1}} \phi \, dt \leq k^{p-1} \max_{t \in [0, k^{-1}]} \phi \leq c_2 k^{-\delta} = o(1), \quad \text{as } k \to +\infty. 
\]

(2.4)

In view of (2.2)–(2.3)–(2.4) it follows that

\[
\int_0^a |u' - u_k'|^p \phi \, dt = o(1), \quad \text{as } k \to +\infty. 
\]

We conclude by a standard regularization argument. Now suppose that \( u \in \mathcal{E} \). The following sequence of bounded functions

\[
\tilde{u}_k(t) = \begin{cases} 
\quad \\
u(k^{-1}) & \text{if } t \in [0, k^{-1}] \\
u(t) & \text{if } t \in [k^{-1}, a],
\end{cases}
\]

satisfies

\[
\int_0^a |u' - \tilde{u}_k'|^p \phi \, dt = \int_0^{k^{-1}} |u'|^p \phi \, dt = o(1), \quad \text{as } k \to +\infty. 
\]

Hence, we are reduced to the case where \( u \) is bounded, and the claim is established. \(\blacksquare\)

Fix \( a > 0 \). Let

\[
\eta_a(t) = \frac{\phi(t)^{\frac{1}{p-1}}}{\int_t^a \phi(\sigma)^{\frac{1}{p-1}} \, d\sigma}, \quad t \in (0, a). 
\]

(2.5)

We note that \( \eta_a > 0 \) and furthermore the following holds.
Lemma 2.2. The function $\eta_a$ defined by (2.5) satisfies:

$$
\left( \begin{array}{c}
\frac{c_1}{c_2} \\
\frac{\delta}{p-1} \\
\frac{\delta}{t (c_1^p - c_2^p - \phi) - t^p - \phi}
\end{array} \right) \leq \eta_a(t) \leq \left( \begin{array}{c}
\frac{c_2}{c_1} \\
\frac{\delta}{p-1} \\
\frac{\delta}{t (c_1^p - c_2^p - \phi) - t^p - \phi}
\end{array} \right)
$$

for all $t \in (0, a)$, where $c_1, c_2, \delta > 0$ are the constants defined in (2.4).

Proof. We have:

$$
\int_t^a (\sigma^{p-1+\delta})^{p-1} d\sigma = \int_t^a \sigma^{1-\delta/(p-1)} d\sigma = \frac{p-1}{\delta} \left( t^{-\delta/(p-1)} - a^{-\delta/(p-1)} \right).
$$

Consequently,

$$
\frac{(p-1+\delta)^{p-1}}{\int_t^a (\sigma^{p-1+\delta})^{p-1} d\sigma} = \frac{\delta}{p-1} \frac{p}{t (c_1^p - c_2^p - \phi) - t^p - \phi}.
$$

On the other hand, in view of the assumption (2.1) on $\phi$, we have

$$
\left( \begin{array}{c}
\frac{c_1}{c_2} \\
\frac{\delta}{p-1} \\
\frac{\delta}{t (c_1^p - c_2^p - \phi) - t^p - \phi}
\end{array} \right) \leq \eta_a(t) \leq \left( \begin{array}{c}
\frac{c_2}{c_1} \\
\frac{\delta}{p-1} \\
\frac{\delta}{t (c_1^p - c_2^p - \phi) - t^p - \phi}
\end{array} \right)
$$

and the asserted estimate follows.

For later use, we also note that $\eta_a$ satisfies a Riccati equation:

$$
\eta_a \phi' + (p-1)\eta_a = (p-1)\eta_a^2 \quad \text{in } (0, a).
$$

(2.7)

The following Hardy inequality holds.

Proposition 2.1. Let $a > 0$, $p > 1$ and suppose that $\phi$ satisfies (2.1). Let $\eta_a$ be correspondingly defined by (2.5). Then, for every $u \in C\cap (0, a)$, the following inequality holds:

$$
\int_0^a |u|^p \phi dt \geq \left( \frac{p-1}{p} \right)^p \int_0^a |u|^p \eta_a^p \phi dt.
$$

(2.8)

The constant $\left( \frac{p-1}{p} \right)^p$ is sharp.

Proof. In view of Lemma 2.1, we may assume that $u \in C\cap (0, a)$. We recall the elementary convexity inequality $|\alpha|^p \geq |\beta|^p + p|\beta|^{p-2} \beta(\alpha - \beta)$ for all $\alpha, \beta \in \mathbb{R}$. Taking $\alpha = u'$ and $\beta = -\frac{p-1}{p} \eta_a$, we derive:

$$
|u'|^p \geq |\frac{p-1}{p} \eta_a| p - |\frac{p-1}{p} \eta_a|^{p-2} \frac{p-1}{p} \eta_a \left( u' + \frac{p-1}{p} \eta_a \right).
$$

(2.9)

Multiplying by $\phi$ and integrating over $[0, a]$, we obtain:

$$
\int_0^a |u'|^p \phi dt \geq \left( \frac{p-1}{p} \right)^p (1-p) \int_0^a |u|^p \eta_a^p \phi dt - p \left( \frac{p-1}{p} \right)^{p-1} \int_0^a |u|^{p-2} uu' \eta_a^{p-1} \phi dt.
$$
Integration by parts yields
\[
\int_0^a |u'|^p \phi \, dt \geq \left( \frac{p-1}{p} \right)^p (1-p) \int_0^a |u|^p \eta_p^p \phi \, dt + \left( \frac{p-1}{p} \right)^{p-1} \int_0^a |u|^p \left( \eta_p^{p-1} \phi \right)' \, dt \\
- \left( \frac{p-1}{p} \right)^{p-1} \left[ |u|^p \eta_p^{p-1} \phi \right]_0^a.
\]

Now we observe that by (2.6) and the fact that \( u \in C_0^1 ([0,a]) \), we have \( u \eta_p \in L^\infty (0,a) \). Therefore, the boundary terms vanish and we obtain
\[
\int_0^a |u'|^p \phi \, dt \geq (1-p) \left( \frac{p-1}{p} \right)^p \int_0^a |u|^p \eta_p^p \phi \, dt + \left( \frac{p-1}{p} \right)^{p-1} \int_0^a |u|^p \left( \eta_p^{p-1} \phi' + (p-1) \eta_p^{p-2} \phi \right) \, dt \\
= (1-p) \left( \frac{p-1}{p} \right)^p \int_0^a |u|^p \eta_p^p \phi \, dt + \left( \frac{p-1}{p} \right)^{p-1} \int_0^a |u|^p \eta_p^{p-2} \left( \eta_p \phi' + (p-1) \eta_p \phi \right) \, dt.
\]

In view of (2.6), we have
\[
\int_0^a |u|^p \eta_p^{p-2} \left( \eta_p \phi' + (p-1) \eta_p \phi \right) \, dt = (p-1) \int_0^a |u|^p \eta_p^p \phi \, dt.
\]

It follows that
\[
\int_0^a |u'|^p \phi \, dt \geq \left[ \left( \frac{p-1}{p} \right)^p - \frac{(p-1)^p}{p^{p-1}} \right] \int_0^a |u|^p \eta_p^p \phi \, dt + \frac{(p-1)^p}{p^{p-1}} \int_0^a |u|^p \eta_p^p \phi \, dt = \left( \frac{p-1}{p} \right)^p \int_0^a |u|^p \eta_p^p \phi \, dt.
\]

Hence, (2.8) is satisfied.

Now we verify sharpness. To this end, we consider the sequence of functions \( \{ U_k \}_{k \in \mathbb{N}} \subset E \) defined by
\[
U_k(t) = \begin{cases} 
\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^{\frac{p-1}{p}} & \text{if } t \in \left[ 0, \frac{1}{k} \right) \\
\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^{\frac{p-1}{p}} & \text{if } t \in \left[ \frac{1}{k}, a \right].
\end{cases}
\]

Then,
\[
U_k(t) = \begin{cases} 
0 & \text{if } t \in \left[ 0, \frac{1}{k} \right) \\
- \frac{p-1}{p} \left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^{- \frac{1}{p}} \phi^{- \frac{1}{p-1}} (t) & \text{if } t \in \left[ \frac{1}{k}, a \right].
\end{cases}
\]

We claim that
\[
\lim_{k \to +\infty} \int_0^a |U_k'|^p \phi \, dt = \left( \frac{p-1}{p} \right)^p \int_0^a U_k^p \eta_p \phi \, dt.
\]

Indeed, we note that
\[
(n_p \phi)(t) = \left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^{\frac{p-1}{p}}.
\]
for all $t \in (0, a)$. Therefore, we may write
\[
\int_0^a U_k^p \eta_0^p \phi \, dt = A_k + B_k,
\]
where
\[
A_k \equiv \int_0^1 U_k^p \eta_0^p \phi \, dt = \left( \int_0^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^{p-1} \int_0^1 \frac{\phi^{- \frac{1}{p-1}}(t)}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt \quad (2.11)
\]
and
\[
B_k \equiv \int_1^a U_k^p \eta_0^p \phi \, dt = \int_1^a \left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^{p-1} \frac{\phi^{- \frac{1}{p-1}}(t)}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt = \int_1^a \frac{\phi^{- \frac{1}{p-1}}(t)}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt. \quad (2.12)
\]
We claim that
\[
\lim_{k \to +\infty} A_k = \frac{1}{p-1}. \quad (2.13)
\]
Indeed, we first observe that in view of (2.11) we have
\[
\int_1^a \frac{\phi^{- \frac{1}{p-1}}(t)}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt \to +\infty \quad (2.14)
\]
as $k \to \infty$. Hence, by L'Hospital's rule,
\[
\lim_{k \to +\infty} A_k = \lim_{k \to +\infty} \frac{\int_0^1 \frac{\phi^{- \frac{1}{p-1}}}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt}{\int_0^a \frac{\phi^{- \frac{1}{p-1}}}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt} = \frac{1}{p-1},
\]
and (2.13) follows. We conclude that
\[
\int_0^a U_k^p \eta_0^p \phi \, dt = \int_0^a \frac{\phi^{- \frac{1}{p-1}}(t)}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt + \frac{1}{p-1} + o(1).
\]
On the other hand, we have
\[
\int_0^a |U_k'|^p \phi \, dt = \left( \frac{p-1}{p} \right)^p \int_0^a \frac{\phi^{- \frac{1}{p-1}}(t)}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt.
\]
Hence, recalling (2.14), we obtain
\[
\lim_{k \to +\infty} \frac{\int_0^a |U_k'|^p \phi \, dt}{\int_0^a U_k^p \eta_0^p \phi \, dt} = \left( \frac{p-1}{p} \right)^p \lim_{k \to +\infty} \frac{\int_0^a \frac{\phi^{- \frac{1}{p-1}}}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt}{\int_0^a \frac{\phi^{- \frac{1}{p-1}}}{\left( \int_t^a \phi^{- \frac{1}{p-1}} \, d\sigma \right)^p} \, dt} = \left( \frac{p-1}{p} \right)^p.
\]
Hence, the sharpness is also established.
Now we show that under an extra simple assumption for \( \phi \), the corresponding function \( \eta_a \) defined by (2.5) has exactly one critical point, corresponding to the absolute minimum of \( \eta_a \) in \((0,a)\).

**Lemma 2.3.** Suppose that \( \phi : [0,a] \to \mathbb{R} \) satisfies (2.1). Furthermore, suppose that \( \phi \) is twice differentiable in \((0,a)\) and that

\[
(\log \phi)''(t) = \left( \frac{\phi'}{\phi} \right)'(t) < 0 \quad \text{for all } t \in (0,a).
\] (2.15)

Then, there exists a unique \( T \in (0,a) \) such that \( \eta''_a(t) < 0 \) in \((0,T)\) and \( \eta''_a(t) > 0 \) in \((T,a)\).

**Proof.** Differentiating the Riccati equation (2.7) we obtain

\[
\eta'_a \phi' + \eta_a \left( \frac{\phi'}{\phi} \right)' + (p-1)\eta''_a = 2(p-1)\eta_a \eta_a'.
\] (2.16)

Suppose that \( \eta'_a(\hat{t}) = 0 \). Then, (2.16) implies that

\[
\eta''_a(\hat{t}) = -\frac{1}{p-1} \eta_a(\hat{t}) \left( \frac{\phi'}{\phi} \right)'(\hat{t}) > 0.
\]

It follows that any critical point for \( \eta_a \) is necessarily a strict minimum point. In view of Lemma 2.2, it follows that \( \eta_a \) admits a unique minimum point and the existence of \( T \) is established.

Let \( \phi \) be twice differentiable and suppose that \( \phi \) satisfies (2.1) and (2.15). Then, the following function obtained by truncating \( \eta_a \) at the point \( T \), is non-increasing:

\[
\eta_{aT}(t) = \begin{cases} 
\eta_a(t) & \text{for } t \in (0,T) \\
\eta_a(T) & \text{for } t \in (T,a).
\end{cases}
\] (2.17)

Since \( \eta_{aT} \leq \eta_a \) pointwise, it is clear that Proposition 2.1 still holds with \( \eta_a \) replaced by \( \eta_{aT} \). On the other hand, it is not a priori clear whether or not, with such a replacement, the constant \( |(p-1)/p|^p \) is still sharp. In the next proposition we show that this is indeed the case.

**Proposition 2.2.** Suppose that \( \phi \) is twice differentiable and satisfies (2.1) and (2.15). Let \( \eta_{aT} \) be defined by (2.17). Then,

\[
\int_0^a |u|^p \phi \, dt \geq \left( \frac{p-1}{p} \right)^p \int_0^a |u|^p \eta_{aT} \phi \, dt, \quad \forall u \in \mathcal{E}.
\]

Furthermore, the constant \( \left( \frac{p-1}{p} \right)^p \) is sharp.

**Proof.** We need only check sharpness. For \( k \in \mathbb{N}, 1/k < T \), we consider the sequence \( \{V_k\}_{k \in \mathbb{N}} \subset \mathcal{E} \) defined by

\[
V_k(t) = \begin{cases} 
U_k(t) & \text{if } t \in [0,T) \\
\left( \int_T^a \phi^\frac{p-1}{p} \, d\sigma \right)^\frac{p}{p-1} \frac{2a-T}{T-a} & \text{if } t \in [T,(a+T)/2) \\
0 & \text{if } t \in [(a+T)/2,a]
\end{cases}
\] (2.18)
where \( \{U_k\}_{k \in \mathbb{N}} \) is the sequence defined in (2.10). Then,

\[
V'_k(t) = \begin{cases} 
U'_k(t) & \text{if } t \in [0,T) \\
\int_T^t \phi^{-\frac{1}{p-1}} d\sigma & \text{if } t \in [T,(a+T)/2) \\
0 & \text{if } t \in [(a+T)/2,a] 
\end{cases}
\]

We claim that

\[
\lim_{k \to +\infty} \frac{\int_0^a |V'_k|^p \phi dt}{\int_0^a |V_k|^p \eta_a \phi dt} = \left( \frac{p-1}{p} \right)^p.
\]

Let \( C_1, C_2 > 0 \) be defined by

\[
C_1 = \left( \frac{p}{p-1} \right)^p \int_T^{(T+a)/2} |V'_k|^p \phi dt, \quad C_2 = \int_T^{(T+a)/2} |V_k|^p \eta_a \phi dt.
\]

Note that \( C_1, C_2 \) are independent of \( k \). Then, we have:

\[
\frac{\int_0^a |V'_k|^p \phi dt}{\int_0^a |V_k|^p \eta_a \phi dt} = \left( \frac{p-1}{p} \right)^p \frac{\int_T^{(T+a)/2} \phi^{-\frac{1}{p-1}} \phi^{-\frac{1}{p-1}} d\sigma + C_1}{A_k + \int_T^{(T+a)/2} \phi^{-\frac{1}{p-1}} \phi^{-\frac{1}{p-1}} d\sigma + C_2} = \left( \frac{p-1}{p} \right)^p \frac{\int_T^{(T+a)/2} \phi^{-\frac{1}{p-1}} \phi^{-\frac{1}{p-1}} d\sigma + C_1}{1 + o(1) + \int_T^{(T+a)/2} \phi^{-\frac{1}{p-1}} \phi^{-\frac{1}{p-1}} d\sigma + C_2},
\]

where \( A_k \) is defined in (2.11). This establishes the claim.

\[ \square \]

3 Proofs of Theorem 1.1 and Theorem 1.2

In this section we apply Proposition 2.2 in order to prove Theorem 1.1 and Theorem 1.2. In what follows we assume that \( 1 < p < n \). We let \( a \in (0, \pi) \) and we take \( \phi = \tilde{\phi} \), where

\[
\tilde{\phi}(t) = \sin^{n-1}(t).
\]

We note that \( \tilde{\phi} \) satisfies assumptions (2.1) with \( \delta = n - p \). The weight function corresponding to \( \tilde{\phi} \) defined according to (2.5) is given by (1.2), namely

\[
\tilde{\eta}_a(t) = \frac{\sin^{-\frac{n-1}{p-1}}(t)}{\int_0^a (\sin \sigma)^{-\frac{n-1}{p-1}} d\sigma}.
\]

Furthermore, \( \tilde{\phi} \) is twice differentiable and we have

\[
\left( \log \tilde{\phi} \right)''(t) = \frac{-n-1}{\sin^2 t}
\]
for all $t \in (0, \pi)$. In particular, $\tilde{\phi}$ satisfies assumption (2.15). Using L’Hospital’s rule we have:

$$
\frac{p - 1}{n - p} \lim_{t \to 0^+} t \eta_n(t) = \frac{p - 1}{n - p} \lim_{t \to 0^+} t (\sin t) \frac{\frac{-n-1}{p-1}}{\sin \sigma \frac{-n-1}{p-1}} = \frac{p - 1}{n - p} \lim_{t \to 0^+} (1 + o(1)) \frac{t \frac{-n-1}{p-1}}{\sin \sigma \frac{-n-1}{p-1}} = 1,
$$

and therefore (1.6) follows. The following elementary facts will be used in the sequel. Recall that for $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ we set $x_{n+1} = |x| \cos \theta$.

**Lemma 3.1.** Let $\Omega \subset S^n$ and suppose that $u : \Omega \to \mathbb{R}$ depends on $\theta$ only. Then:

$$
|\nabla u|^2 = \left(\frac{\partial u}{\partial \theta}\right)^2,
$$

$$
\int_{B(a)} u(\theta) dV = \omega_{n-1} \int_0^a u(\theta) \tilde{\phi}(\theta) d\theta
$$

where $\omega_{n-1} = (2\pi)^{n/2}/\Gamma(n/2)$ denotes the volume of $S^{n-1}$.

We shall also need the following basic facts concerning spherical rearrangements, see, e.g., [4, 12].

For every $a \in [0, \pi]$, let

$$
A(a) = |B(a)| = \omega_{n-1} \int_0^a \tilde{\phi}(\theta) d\theta.
$$

Let $\Omega \subset S^n$ be an open set and let $u : \Omega \to \mathbb{R}$ be a measurable function. For every $t > 0$, let

$$
\mu(t) = |\{x \in \Omega : |u(x)| > t\}|
$$

denote the distribution function of $u$. Then the decreasing rearrangement $u^*$ of $u$ is defined by

$$
u^*(s) = \inf \{t \geq 0 : \mu(t) \leq s\}
$$

for every $s \in [0, |\Omega|]$. Let $\Omega^* = B(a^*)$, where $a^* = A^{-1}(|\Omega|)$. Then, the spherical rearrangement $u^*$ of $u$ is defined by

$$
u^*(x) = u^*(A(\theta)), \quad x \in \Omega^*.
$$

It follows that $u^*$ is a decreasing function of $\theta$, and that its level sets are geodesic balls (spherical caps) centered at $N = (0, 0, \ldots, 1) \in S^n$. Since $|u|$ and $u^*$ have the same distribution function, we have

$$
\int_{\Omega} |u|^q dV = \int_{\Omega^*} (u^*)^q dV,
$$

for all $q \geq 1$. We shall use two standard inequalities involving rearrangements. The following lemma is a special case of the well-known Hardy-Littlewood inequality and may be found, e.g., in [6], Theorem 2.2 p. 44.

**Lemma 3.2 (Hardy-Littlewood inequality).** Let $\Omega \subset S^n$ be an open set and suppose that $u, v : \Omega \to \mathbb{R}$ are measurable and finite a.e. Then,

$$
\int_{\Omega} uv dV \leq \int_{\Omega^*} u^* v^* dV.
$$

(3.5)
The following inequality is a special case of the Pólya-Szego principle, and may be found in [3], Proposition 2.17, p. 41, see also [12], Theorem p. 325.

**Lemma 3.3** (Pólya-Szego principle). Let \( q \geq 1 \) and let \( u \in W^{1,q}(\mathbb{S}^n) \). Then,

\[
\int_{\mathbb{S}^n} |\nabla u|^q \, dV \geq \int_{\mathbb{S}^n} |\nabla u^*|^q \, dV.
\]

(3.6)

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** For every fixed \( r > 0 \) we consider the function obtained by restricting \( u \) to \( \mathbb{S}^n \cap \mathbb{R}^{n+1}_+ \). Therefore, it suffices to show that

\[
\int_\Omega |\nabla u|^p \, dV \geq \int_\Omega |\nabla u^*|^p \, dV = \omega_{n-1} \int_0^{a^*} \left| \frac{\partial u^*}{\partial \theta} \right|^p \tilde{\phi}(\theta) d\theta.
\]

On the other hand, in view of Lemma 3.2 we have:

\[
\int_\Omega |u|^p \rho^{p^*}_a \, dV \leq \int_\Omega |u^*|^p \rho^{p^*}_a \, dV = \omega_{n-1} \int_0^{a^*} |u^*|^p \rho^{p^*}_a \tilde{\phi}(\theta) d\theta.
\]

Therefore, it suffices to show that

\[
\int_0^{a^*} \left| \frac{\partial u^*}{\partial \theta} \right|^p \tilde{\phi}(\theta) d\theta - \left( \frac{n-p}{p} \right) \int_0^{a^*} |u^*|^p \rho^{p^*}_a \tilde{\phi}(\theta) d\theta \geq 0.
\]

The above inequality holds by definition of \( \rho^{p^*}_a \), as in (1.4), and by Proposition 2.2.

In order to show that the constant \( \left( \frac{n-p}{p} \right) \) is sharp it suffices to use, as test functions, the sequence \( \{T_k\}_{k \in \mathbb{N}} \) obtained by setting \( \phi = \phi, a = a^* \) and \( T = T \) in (2.18). Namely,

\[
\tilde{T}_k(\theta) = \begin{cases} 
\left( \int_{\frac{1}{k}}^{a^*} \tilde{\phi} \, d\sigma \right)^{\frac{p-1}{p}} & \text{if } \theta \in \left[0, \frac{1}{k}\right) \\
\left( \int_{\frac{1}{k}}^{T} \tilde{\phi} \, d\sigma \right)^{\frac{p-1}{p}} & \text{if } \theta \in \left[\frac{1}{k}, T\right) \\
\left( \int_{\frac{1}{k}}^{T} \tilde{\phi} \, d\sigma \right)^{\frac{p-1}{p}} \frac{2(a^* - T)}{T - a^*} & \text{if } \theta \in \left[T, (a^* + T)/2\right) \\
0 & \text{if } \theta \in \left[(a^* + T)/2, a^*\right]
\end{cases}
\]

(3.7)

Now the proof of Theorem 1.1 is complete.

In order to prove Theorem 1.2 we use a Steiner-type symmetrization on \( \mathbb{R}^{n+1}_+ \) with respect to the angular variables. See, e.g., [1] [10] for the main results on Steiner symmetrization. Let \( u \in C^1_0(\mathbb{R}^{n+1}_+) \). For every fixed \( r > 0 \) we consider the function obtained by restricting \( u \) to \( \mathbb{S}^n \cap \mathbb{R}^{n+1}_+ \). Namely, we consider the function

\[
\Theta \in \mathbb{S}^n \cap \mathbb{R}^{n+1}_+ \rightarrow u(r, \Theta),
\]

(3.8)

where \( \Theta = (\theta_1, ..., \theta_{n-1}, \theta) \) is the set of all angular variables. We denote by \( u^*(r, \cdot) \) the decreasing rearrangement of the function in (3.8), according to the definition given in (3.4). Finally we introduce the Steiner rearrangement \( u^\sharp \) of \( u \) as follows:

\[
u^\sharp(r, \theta) = u^*(r, A(\theta)), \quad \theta \in [0, \pi/2],
\]

(3.9)
where \( A(\theta) \) is defined in (3.3). We denote by \( g_r \) the standard metric on \( S^n_r \) and by \( dV_r \) the volume element on \( S^n_r \). Then, we have \( D_\Theta u = \nabla_{g_r} u \) and therefore, in view of Lemma 3.1 and a rescaling argument,

\[
|D_\Theta u|^p = \frac{1}{r^p} \left| \frac{\partial u^*}{\partial \theta} \right|^p.
\]  

(3.10)

We claim that:

\[
\zeta(x) \frac{|x|}{|x'|} \leq C \frac{\theta r}{r \sin \theta} \leq C \frac{C}{|x'|}.
\]  

(3.11)

Consequently, for any \( R > 0 \) we have:

\[
\int_{|x'|,|x_{n+1}|<R} \zeta(x) \frac{|x|}{|x'|} \, dx = \int_0^R \int_{|x'|<R} \zeta(x) \frac{|x|}{|x'|} \, dx'.
\]

Now (3.11) follows in view of the assumption \( p < n \).

**Proof of Theorem 1.2.** By density, it suffices to consider \( u \in C^1_0(\mathbb{R}^{n+1}_+) \). By rescaling, if \( \Omega \subset S^n_r \) and \( u : \Omega \to \mathbb{R} \) depends on \( \theta \) only, then rescaling (3.2) we obtain

\[
|\nabla_{g_r} u|^2 = 1 \frac{\partial u}{\partial \theta}^2 \int_{B_r(\alpha)} u(\theta) \, dV_r = \omega_{n-1} r^n \int_0^\alpha u(\theta) \tilde{\phi}(\theta) \, d\theta.
\]  

(3.12)

By Fubini’s Theorem and in view of Lemma 3.3 we have:

\[
\int_{\mathbb{R}^{n+1}} |D_\Theta u|^p \, dx \geq \int_{\mathbb{R}^{n+1}} |\nabla_{g_r} u|^p \, d\sigma_r \geq \int_{\mathbb{R}^{n+1}} |\nabla_{g_r} u^*|^p \, d\sigma_r
\]  

(3.13)

where \( u^* = u^*(r, \theta) \) is defined in (3.9). Consequently, from (3.10), (3.13) and in view of Theorem 1.1 with \( a^* = \pi/2 \), we derive

\[
\int_{\mathbb{R}^{n+1}} |D_\Theta u|^p \, dx \geq \omega_{n-1} \int_0^{+\infty} \left( \int_0^{\frac{\pi}{2}} \left( \frac{1}{r^p} \left| \frac{\partial u^*}{\partial \theta} \right|^p \sin^{n-1} \theta \, d\theta \right) \right) r^n \, dr
\]

\[
\geq \omega_{n-1} \left( \frac{n-p}{p} \right)^p \int_0^{+\infty} \left( \frac{1}{r^p} \int_0^{\frac{\pi}{2}} |u^*|^p \zeta^p \sin^{n-1} \theta \, d\theta \right) r^n \, dr
\]

\[
\geq \left( \frac{n-p}{p} \right)^p \int_{\mathbb{R}^{n+1}} \frac{|u|^p}{r^p} \zeta^p \, dx.
\]

We are left to prove sharpness. To this end, we consider the sequence \( u_k(\theta, r) = \Theta_k(\theta) R_k(r) \), \( k \in \mathbb{N} \), where \( R_k \in C_0(0, +\infty) \) satisfies \( R_k > 0 \) and \( R_k^p(r) \rightharpoonup \delta_1(r) \), weakly in the sense of measures. Here \( \delta_1 \)
denotes the Dirac mass on \((0, +\infty)\) centered at \(r = 1\), and \(\Theta_k(\theta) = \tilde{V}_k(\theta)\), where \(\tilde{V}_k\) is the sequence defined in \((3.7)\), with \(a^* = \pi/2\). We have

\[
\lim_{k \to +\infty} \int_0^{+\infty} R_k^n(r)^n \, dr = \lim_{k \to +\infty} \int_0^{+\infty} R_k^n(r)^{n-p} \, dr = 1.
\]

Now, the claim follows since

\[
\int_{S^{n+1}_+} |\Theta_k(\theta)|^p \, d\sigma = \int_0^{\pi/2} |\Theta_k'(\theta)|^p \, \theta^p \, d\theta + o(1) = \left( \frac{n-p}{p} \right)^p + o(1),
\]

where \(o(1)\) vanishes as \(k \to \infty\). \(\square\)

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