Cohomology with causally restricted supports

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Abstract
De Rham cohomology with spacelike compact and timelike compact supports has recently been noticed to be of importance for understanding the structure of classical and quantum field theories on curved spacetimes. We compute these cohomology groups for globally hyperbolic spacetimes in terms of their standard de Rham cohomologies. The calculation exploits the fact that the de Rham-d’Alambert wave operator can be extended to a chain map that is homotopic to zero and that its causal Green function fits into a convenient exact sequence. This method extends also to the Calabi (or Killing-Riemann-Bianchi) complex and possibly other differential complexes. We also discuss generalized causal structures and functoriality.

1 Introduction
Recently, a number of works on the structure of classical and quantum field theory on curved spacetimes [15, 46, 6, 19, 5, 31, 35, 36] have made use of de Rham cohomology with spacelike compact supports. It appears in the characterizations of the center of Poisson (or quantum) algebra of observables of the Maxwell field and also of the degeneracy of the bilinear pairing between spacelike compactly supported solutions and compactly supported smearing functions. Similar considerations appear in more general field theories [36, 35], though involving cohomologies of complexes that are different from the de Rham one.

It was noticed long ago [1] that non-trivial spacetime topology can influence in a non-trivial way the construction of the classical and quantum field theories. However, these effects had not been systematically investigated until recently. This may explain why neither the standard literature on differential geometry and topology, nor the literature on relativity seem to have considered cohomologies with supports restricted by causal relations (like spacelike or timelike compactness). So, given their growing importance, they deserve independent investigation, which is the subject of this work.

In Section 2 we briefly outline some well known geometric properties of the de Rham complex on a Lorentzian spacetime, as well as some basic facts of homological algebra. These properties are then used in Section 3 to express
the various cohomologies with causally restricted supports in terms of the stand-
dard de Rham cohomologies with unrestricted and compact supports. Then,
Section 4 makes a few remarks and lists some generalizations of the method of
Section 3. Sections 4.1 and 4.2 deal with the behavior of the causally restricted
cohomology groups under changes of causal structure and under embeddings.
Section 4.3 presents the Calabi differential complex and repeats the calculations
of Section 3 for it. The Calabi complex plays a role in linearized gravity on a
constant curvature background analogous to that of the de Rham complex for
Maxwell theory. Then, Section 4.4 briefly describes how the method is applied
to the de Rham and Calabi examples could be generalized to other differential
complexes that arise in the study of general field theories with constrains and
gauge invariance [36, 35]. Finally, Section 5 concludes with a discussion of our
results.

It should be mentioned that results very similar to those in Section 3 have
been obtained independently in the recent work [5], though by a different
method. On the other hand, the content of Section 4 goes beyond [5] in several
directions.

2 Preliminaries

Fix an $n$-dimensional smooth manifold $M$ with a Lorentzian metric $g$ such that
$(M, g)$ is an oriented, time-oriented space, globally hyperbolic spacetime [51
32 42 4]. Recall that, according to the Geroch splitting theorem, there exists
a diffeomorphism $M \cong \mathbb{R} \times \Sigma$ (non-unique, of course) where the corresponding
projection $t: M \to \mathbb{R}$ is a Cauchy temporal function [25 9 8].

Let $\Omega^p(M)$ denote the linear space of differential $p$-forms on $M$ and $d: \Omega^p(M) \to
\Omega^{p+1}(M)$ the de Rham differential, which together form the de Rham complex

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \rightarrow 0. \quad (1)$$

Its cohomology is denoted by $H^p(M)$. It is well known that this de Rham co-
homology is isomorphic, $H^p(M) \cong H^p(M, \mathbb{R})$, to the singular cohomology of $M$ with coefficients in $\mathbb{R}$, to the Čech cohomology of $M$ with coefficients in $\mathbb{R}$, and
to the sheaf cohomology of $M$ with coefficients in the sheaf of locally constant
$\mathbb{R}$-valued functions, all of which being isomorphic are denoted by $H^p(M, \mathbb{R})$. If we replace $\Omega^p(M)$ in (1) with $\Omega^p_0(M)$, the linear space of differential $p$-forms with compact support, the corresponding de Rham cohomology of $M$ with compact supports is isomorphic to the singular homology of $M$ with coefficients in $\mathbb{R}$, $H^p_0(M) \cong H_p(M, \mathbb{R})$, under the same hypotheses as before [11]. There also exists a non-degenerate bilinear pairing between $\Omega^p(M)$ and $\Omega^{n-p}_0(M)$,

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta, \quad (2)$$

which descends to a non-degenerate bilinear pairing between $H^p(M)$ and $H^{n-p}_0(M)$.

This result is known as Poincaré duality.

Using the Hodge star operator $*: \Omega^p(M) \to \Omega^{n-p}(M)$ associated to the met-
ric $g$, we can define the de Rham co-differential $\delta = *d*: \Omega^p(M) \to \Omega^{p-1}(M)$. Next,
we define the so-called de Rham-d’Alambertian or wave operator $\Box_\gamma: \Omega^p(M) \to
\Omega^{p+2}(M)$.
This operator differs from the simple tensor d’Alambertian \( \nabla_a \nabla^a \) by terms of lower differential order. From its very definition, we see that the d’Alambertian is a cochain map from the de Rham complex to itself, \( d^2 = 0 \), which is moreover cochain homotopic to zero, with the co-differential \( \delta \) the corresponding cochain homotopy. That is, it induces the zero map from \( H^p(M) \) to itself. The following diagram illustrates the discussion:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(M) & \rightarrow & 0 \\
0 & \xrightarrow{\Box} & \Omega^0(M) & \xrightarrow{\delta} & \Omega^1(M) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & \Omega^n(M) & \xrightarrow{\delta} & 0
\end{array}
\]

where the rows constitute (de Rham) complexes, the solid arrows commute, and the dashed arrows illustrate the cochain homotopy. This is an important observation that will be used in an essential way in Section 3. Note that the formula (3) is analogous to the formula for the Hodge-de Rham Laplacian in Riemannian geometry. There, the observation that this Laplacian is homotopic to zero lies at the foundation of Hodge theory [27, 33].

The causal structure on \( M \) defined by the Lorentzian metric \( g \) allows us to restrict the supports of differential forms in other ways as well. A closed set \( S \subseteq M \) is said to be retarded if \( S \subseteq J^-(K) \) for some compact \( K \), advanced if \( S \subseteq J^+(K) \) for some compact \( K \), spacelike compact if it \( S \subseteq J(K) \) for some compact \( K \), past compact if \( S \cap J^-(K) \) is compact for every compact \( K \), future compact if \( S \cap J^+(K) \) is compact for every compact \( K \), and timelike compact if \( S \) is both past and future compact [13, 2]. Timelike compactness is also equivalent to the property of having compact intersection with every spacelike compact set. Let \( \Omega^p_X(M) \), with \( X = +, -, sc, pc, fc \) or \( tc \), denote the linear space of differential \( p \)-forms with, respectively, retarded, advanced, spacelike compact, past compact, future compact or timelike compact supports. For brevity, we refer to these spaces as space of forms with causally restricted supports.

Of course, since differential operators preserve supports, it also restricts to \( \Box: \Omega^p_\alpha(M) \rightarrow \Omega^p_\beta(M) \). By the same reasoning, the spaces of forms with causally restricted supports are also preserved by both \( d \) and \( \Box \). We define de Rham cohomology with causally restricted supports in the obvious way and denote it by \( H^p_{\alpha}(M) \), with \( X = +, -, sc, pc, fc \) or \( tc \). Let \( \Omega^p_\alpha(M) \) and \( \Omega^p_{\alpha,X}(M) \) denote the kernel of the wave operator \( \Box \), also known as its solution space, in the spaces of forms with corresponding supports. Finally, by the cochain map property, the de Rham differential restricts to the kernel of the wave operator, hence defining the de Rham cohomology groups \( H^p_{\alpha}(M) \) and \( H^p_{\alpha,X}(M) \) of solutions.

The wave operator on a globally hyperbolic Lorentzian manifold is well known to be Green hyperbolic. That is, it has advanced and retarded Green functions denoted respectively \( G_+ \) and \( G_- \), \( G_{\pm}: \Omega^p_\alpha(M) \rightarrow \Omega^p_\beta(M) \). Since \( \Box \) commutes with \( d \), then so do \( G_+ \) and \( G_- \). The form \( \beta = G_\pm[\alpha] \) is the unique solution of \( \Box \beta = \alpha \) with, respectively, retarded or advanced support. The domain of definition of the Green functions can be extended, in a unique way, to \( \Omega^p_\alpha(M) \) for \( X = +, -, pc \) or \( fc \). Then, the maps

\[
\Box: \Omega^p_\alpha(M) \rightarrow \Omega^p_\beta(M), \quad G_X: \Omega^p_\alpha(M) \rightarrow \Omega^p_\beta(M)
\]
are mutually inverse bijections, whenever $X = +$ and $Y = +$ or $pc$, or $X = -$ and $Y = -$ or $fc$. The combination $G = G_+ - G_-$ is known as the causal Green function and fits into the following, in our terminology Green-hyperbolic, exact sequences $[3, 26, 36, 35, 2]$.

$$0 \to \Omega^p_0(M) \xrightarrow{\Box} \Omega^p_0(M) \xrightarrow{G} \Omega^p_{sc}(M) \xrightarrow{\Box} \Omega^p_{sc}(M) \to 0, \quad (6)$$

$$0 \to \Omega^p_c(M) \xrightarrow{\Box} \Omega^p_c(M) \xrightarrow{G} \Omega^p(M) \xrightarrow{\Box} \Omega^p(M) \to 0. \quad (7)$$

Note that, according to the above formulas, we can represent the space of solutions with spacelike compact or unrestricted support either as

$$\Omega^p_{\Box, X}(M) = \ker \Box \subset \Omega^p_X(M) \quad (8)$$

or

$$\Omega^p_{C, X}(M) = G[\Omega^p_Y(M)] = \Omega^p_Y(M)/\Box \Omega^p_Y(M), \quad (9)$$

with $X = sc$ and $Y = 0$, or $X$ empty and $Y = tc$, respectively. On the other hand, we have trivial solutions spaces $\Omega^p_{\Box, X}(M) = \{0\}$ when $X = +, -, pc$ or $fc$.

The existence of the Green-hyperbolic exact sequences will allow us to later make use of the following elementary result of homological algebra. Let $A^\bullet = (A^p, d)$ be a cochain complex, and similarly for $B^\bullet$ and $C^\bullet$. It is well known that a short exact sequence of cochain maps,

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0, \quad (10)$$

induces a long exact sequence in cohomology,

$$0 \to H^0(A^\bullet, d) \to H^0(B^\bullet, d) \to H^0(C^\bullet, d) \to H^1(A^\bullet, d) \to H^1(B^\bullet, d) \to H^1(C^\bullet, d) \to \ldots \quad (11)$$

3 Computation of cohomology groups

In this section, we state and prove our main results on de Rham cohomology with causally restricted supports. We rely essentially on the properties of the wave operator and its Green functions, as summarized in Section 2. The important properties are that the wave operator $\Box$ is cochain homotopic to zero, and the way its range and kernel characterized using the causal Green function $G$.

**Theorem 1.** De Rham cohomology $H^p_X(M)$, with $X = +, -, pc$ or $fc$, is trivial.

**Proof.** Let $X = +, -, pc$ or $fc$. Then, as was noted in Section 2 the wave operator is a cochain map of the corresponding de Rham complex into itself, is invertible [Equation (5)] and cochain homotopic to zero [Equation (3)]. Therefore, it is an elementary consequence of homological algebra [11] that this complex is homotopy equivalent to the trivial complex and hence has vanishing cohomology. \qed
Theorem 2. We have the isomorphisms

\[ H^p_{\text{sc}}(M) \cong H^{p+1}_0(M), \quad H^p_{\Box,\text{sc}} \cong H^p_0(M) \oplus H^{p+1}_0(M), \quad (12) \]
\[ H^p_{\text{tc}}(M) \cong H^{p-1}_0(M), \quad \text{and } H^p_{\Box}(M) \cong H^p(M) \oplus H^{p-1}_0(M), \quad (13) \]

with the convention that all cohomologies vanish in degree \( p \) for \( p < 0 \) or \( p > n \).

Proof. Recall again from Section 2 that both the wave operator \( \Box \) and its causal Green function \( G \) commute with \( d \) and hence constitute cochain maps between the de Rham complexes with appropriate supports, inducing maps in cohomology. Moreover, since \( \Box \) is cochain homotopic to zero [Equation (3)], it induces the zero map in cohomology.

Let us start with spacelike compact supports. We can break the exact sequence in (6) into two short exact sequences of complexes:

\[
0 \rightarrow \Omega^p_0(M) \xrightarrow{\Box} \Omega^p_{\Box,\text{sc}}(M) \xrightarrow{G} \Omega^p_{\Box,\text{sc}}(M) \rightarrow 0, \quad (14)
\]
\[
0 \rightarrow \Omega^p_{\Box,\text{sc}}(M) \xrightarrow{C} \Omega^p_{\text{sc}}(M) \xrightarrow{\Box} \Omega^p_{\text{sc}}(M) \rightarrow 0. \quad (15)
\]

Because \( \Box \) always induces the zero map, the corresponding long exact sequences in cohomology [cf. Equation (11)] break up into the following short exact sequences:

\[
0 \rightarrow H^p_0(M) \xrightarrow{G} H^p_{\Box,\text{sc}}(M) \xrightarrow{d} H^{p+1}_0(M) \rightarrow 0, \quad (16)
\]
\[
0 \rightarrow H^{p-1}_0(M) \xrightarrow{d} H^p_{\Box,\text{sc}}(M) \xrightarrow{C} H^p_{\text{sc}}(M) \rightarrow 0, \quad (17)
\]

again with the convention that any \( H^p_0(M) \) vanishes for \( p < 0 \) or \( p > n \). By inspection, it is not hard to see that the above short exact sequences induces the following isomorphisms:

\[ H^p_{\text{sc}}(M) \cong H^{p+1}_0(M), \quad (18) \]
\[ H^p_{\Box,\text{sc}}(M) \cong H^p_{\text{sc}}(M) \oplus H^{p-1}_0(M) \cong H^p_0(M) \oplus H^{p+1}_0(M). \quad (19) \]

Applying the same argument to the exact sequence (7), we obtain the isomorphisms

\[ H^p_{\text{tc}}(M) \cong H^{p-1}_0(M), \quad (20) \]
\[ H^p_{\Box}(M) \cong H^p_{\text{tc}}(M) \oplus H^{p+1}_0(M) \cong H^p(M) \oplus H^{p-1}(M). \quad (21) \]

This completes the proof. \( \square \)

Let \( \Sigma \subseteq M \) be a Cauchy surface. Recall that, by the smooth Geroch splitting theorem, we can always smoothly factor \( M \cong \mathbb{R} \times \Sigma \). This observation results in

Corollary 3. We have the isomorphisms

\[ H^p_{\text{sc}}(M) \cong H^p_0(\Sigma), \quad H^p_{\Box,\text{sc}}(M) \cong H^p_0(\Sigma) \oplus H^{p+1}_0(\Sigma) \quad (22) \]
\[ H^p_{\text{tc}}(M) \cong H^{p-1}_0(\Sigma), \quad \text{and } H^p_{\Box}(M) \cong H^p(\Sigma) \oplus H^{p-1}(\Sigma), \quad (23) \]

with the convention that all cohomologies vanish in degree \( p \) for \( p < 0 \) or \( p > n \).
Proof. The splitting $M \cong \mathbb{R} \times \Sigma$ shows that $M$ is homotopic to $\Sigma$. Hence, by the homotopy invariance of de Rham cohomologies with unrestricted supports, we have the isomorphism $H^p(M) \cong H^p(\Sigma)$. On the other hand, Poincaré duality induces the isomorphism $H^p_0(M) \cong H^{n-p}_0(\Sigma)$. Therefore, the desired conclusion follows directly from these identities in combination with Theorem 2.

Finally, knowing the respective de Rham cohomologies with spacelike and timelike compact supports, we have the following generalization of the Poincaré lemma.

**Corollary 4.** The non-degenerate bilinear pairing between $\Omega^p_\text{sc}(M)$ and $\Omega^{n-p}_\text{sc}(M)$ descends to a non-degenerate bilinear pairing between $H^p_\text{sc}(M)$ and $H^{n-p}_\text{sc}(M)$. There exists also a non-degenerate bilinear pairing between $H^p_{\Box,\text{sc}}(M)$ and $H^{n-p}_{\Box,\text{sc}}(M)$.

Proof. A consequence of Theorem 2 is that $H^p_\text{sc}(M) \cong H^0_0(M) \oplus H^{p+1}_0(M)$ and $H^{n-p}_\text{sc}(M) = H^{n-p-1}(M)$. So, the usual Poincaré duality establishes that $H^p_\text{sc}(M)^* \cong H^{n-p}_\text{sc}(M)$. The isomorphism can be exhibited by bilinear pairing, which descends from the standard bilinear pairing between $\Omega^p_\text{sc}(M)$ and $\Omega^{n-p}_\text{sc}(M)$, tracing its effect throughout the proof of Theorem 2. Its non-degeneracy is also a consequence of the Poincaré lemma applied to $H^p_\text{sc}(M)$ and $H^{n-p}_\text{sc}(M)$. It also follows from Theorem 2 that $H^p_{\Box,\text{sc}}(M) \cong H^0_\text{sc}(M) \oplus H^{p+1}_0(M)$ and $H^{n-p}_{\Box,\text{sc}}(M) \cong H^{n-p}(M) \oplus H^{n-p-1}(M)$. Again, the usual Poincaré duality establishes the isomorphism $H^p_{\Box,\text{sc}}(M) \cong H^{n-p}_\text{sc}(M)$. The isomorphism can be exhibited by a bilinear pairing between $\Omega^p_{\Box,\text{sc}}(M)$ and $\Omega^{p+1}_\text{sc}(M) \oplus \Omega^{n-p}_\text{sc}(M)$, defined by the latter identity and the self-adjointness of $\Box$ with respect to our pairing between forms. Again, tracing this pairing through the proof of Theorem 2 and appealing to the standard Poincaré duality establishes its non-degeneracy.

As already discussed in the introduction, the importance of knowing the above cohomology groups is important for understanding the (pre)symplectic and Poisson structure of classical field theories, as emphasized in [36] [35] [46] [6] [5]. The same result as Corollary 3 was obtained independently in [5]. As a matter of fact, the method of [5] can be seen as a special case of our homological algebraic calculation, as discussed more explicitly at the end of Section 4.1.2.

### 4 Notes and generalizations

#### 4.1 Generalized causal structures

The notion of a causal structure on a manifold or even a topological space (in the sense of a partial order on events) can be generalized quite far beyond the context of Lorentzian geometry [37] [22]. We will stick with the context of differential geometry, where a natural generalization consists of introducing at every point of a manifold an arbitrary convex cone in the tangent bundle. Such a manifold could be called a *conal manifold* [11] [38] [49] [35]. Various notions generated by the causal structure on Lorentzian manifolds survive almost without modification on conal manifolds, including spacelike and timelike compactness.

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1 One could equally do so in the cotangent bundle, and produce a tangent cone by convex (or polar) duality.
The main question we will try to answer in this section is the following: Is it possible to use the methods of Section 3 to compute causally restricted cohomologies on a conal manifold? We shall see that the answer is yes, even if the conal manifold is not Lorentzian.

4.1.1 Conal manifolds

Before dealing with spacelike and timelike compactly supported forms, let us introduce the basics of conal manifolds and causal relations on them. Let $M$ be a smooth manifold and $C \subset TM$ be an open subset, such that $C_x = C \cap T_x M$ is an open, convex cone in $T_x M$ that does not contain any affine line. It can be shown that the interior $C^\circ_x$ of the polar dual (or convex dual) cone $T^*_x \supset C^*_x = \left\{ p \in T^*_x M \mid \forall v \in C_x : p \cdot v \geq 0 \right\}$ satisfies the same conditions, with $C^\circ = \cup_{x \in M} C^\circ_x$. The pair $(M, C)$ or $(M, C^\circ)$ is called a conal manifold, with $C$ (or $C^\circ$) called the tangent (or cotangent) cone distribution or cone bundle. For example, the subset of non-vanishing, future-pointing, timelike vectors on a Lorentzian manifold with a time orientation satisfies the above conditions. In general, the cones $C_x$ need not even have elliptic cross sections, thus not be associated to any Lorentzian metric. The cones of future pointing timelike vectors of linear symmetric hyperbolic PDE systems also satisfy the same properties [35, Sec. 4.1]. Sometimes, it is also convenient to admit degenerate cases where the cones are not open or contain some affine lines, but some special care must be taken in those situations.

Given a conal manifold $(M, C)$ we can define a chronological order relation on the points of $M$. Namely, $x \ll y$ if there exists a smooth curve $\gamma : [0, 1] \to M$, such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot{\gamma}(t) \in C$ for all $t \in [0, 1]$. It can be shown that the chronological order relation $I^+ \subset M \times M$ is open and transitive. We can also define the reverse chronological order, $I^-$, and chronological influence, $I = I^+ \cup I^-$, relations in the obvious way. We avoid defining the analog of the causal order relation usually denoted by $J^+$, simply because we have not made any hypotheses about the regularity of the set of causal vectors $(C_x \subset T_x M)$. Given any set $K \subseteq M$, we denote by $I^\pm(K)$ the set of all points of $M$ that respectively chronologically precede or are preceded by the points of $K$. In general, $I^\pm(K)$ is not closed, even if $K$ is. So, for convenience we define $\overline{I^\pm(K)} = I^\pm(K)$. We also use the notation $I(K) = I^+(K) \cup I^-(K)$ and $\overline{I(K)} = \overline{I^+(K)} \cup \overline{I^-(K)}$. Note that $\overline{I^\pm} \subseteq M \times M$ need not be transitive as relations.

The definition of a Cauchy surface $\Sigma \subset M$ is the usual one, every inextensible smooth curve with timelike tangents must intersect $\Sigma$ exactly once. It has recently been shown that the smooth version of the Geroch splitting theorem [25, 0, 8] generalizes to conal manifolds [17]. So, globally hyperbolicity can be simply characterized by the existence of a Cauchy surface. Also, the results of [35] should also directly carry over to conal manifolds. Finally, we define the notions of advanced, retarded, spacelike compact, timelike compact, future compact and past compact exactly in the same way as in Section 3, with the exception that we use the relations $I^\pm$ and $\overline{I}$ instead of the relations $J^\pm$ and $\overline{J}$.

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2We are not concerned with possible minor inconsistencies this substitution introduces in the case of Lorentzian manifolds with ill-behaved causal structures. In any case, we shall only apply these notions for globally hyperbolic spacetimes, where these differences do not appear.
4.1.2 Cohomology with causally restricted supports

Let \( M \) be a globally hyperbolic conal manifold and \( g \) an auxiliary globally hyperbolic Lorentzian metric that induces another conal structure on \( M \) that is “slower” than the original one. That is, \( \Omega^p_{f,sc}(M) \subseteq \Omega^p_{f,c}(M) \), which also implies that \( \Omega^p_{sc}(M) \subseteq \Omega^p_{c}(M) \), while \( \Omega^p_{f,pc}(M) \supseteq \Omega^p_{f,pc}(M) \), and hence \( \Omega^p_{sc}(M) \supseteq \Omega^p_{tc}(M) \). Any conal manifold admits a nowhere vanishing vector field (contract each cone to a ray and select a vector from it), which is moreover everywhere future directed. So, that such an auxiliary Lorentzian metric always exists follows from general arguments showing the existence of Lorentzian metrics on manifolds with vanishing Euler characteristic (i.e., admitting a nowhere vanishing vector field) [4, 42], which the “slowness” requirement implemented by making sure that the Lorentzian timelike cones closely hug the direction of the vanishing vector field) [4, 42], which the “slowness” requirement implemented by making sure that the Lorentzian timelike cones closely hug the direction of the vanishing vector field.

Let \( G_\pm \) denote once again the advanced and retarded Green functions of the wave operator \( \Box_g \) defined with respect to \( g \). Then it is easy to see that the Green functions are still well defined and injective as maps \( G_\pm : \Omega^p_0(M) \to \Omega^p_0(M) \). Appealing to the same logic as in the standard proofs [3, 26, 35, 2], we can extend the Green functions to bijective maps \( G_\pm : \Omega^p_0(M) \to \Omega^p_0(M) \) and \( G_\pm : \Omega^p_{f,pc}(M) \to \Omega^p_{f,pc}(M) \), from which it is straightforward to establish exactness of the following sequences

\[
0 \longrightarrow \Omega^p_0(M) \xrightarrow{\Box} \Omega^p_0(M) \xrightarrow{G} \Omega^p_{sc}(M) \xrightarrow{\Box} \Omega^p_{tc}(M) \xrightarrow{G} \Omega^p(M) \longrightarrow 0, \tag{24}
\]

\[
0 \longrightarrow \Omega^p_0(M) \xrightarrow{\Box} \Omega^p_0(M) \xrightarrow{G} \Omega^p(M) \longrightarrow 0, \tag{25}
\]

where the supports are restricted by the given conal structure on \( M \) and not by that induced by the auxiliary Lorentzian metric \( g \). Note that the proofs would make use of the hypothesis that the given conal structure is globally hyperbolic, specifically in the construction of explicit splitting maps that demonstrate exactness [36, Lem.2.1]. Thus, repeating the arguments Section 3 we establish the following generalization of Theorems 1 and 2.

**Theorem 5.** Consider a globally hyperbolic conal manifold \( M \). Its de Rham cohomology \( H^p_X(M) \) with causally restricted supports \( X = +, -, pc \) or \( fc \) is trivial. Moreover, we have the isomorphisms

\[
H^p_{fc}(M) \cong H^{p+1}(M), \quad H^p_{f,sc}(M) \cong H^p_0(M) \oplus H^{p+1}_0(M), \tag{26}
\]

\[
H^p_{tc}(M) \cong H^{p-1}(M), \quad H^p_0(M) \cong H^p(M) \oplus H^{p-1}(M), \tag{27}
\]

with the convention that all cohomologies vanish in degree \( p \) for \( p < 0 \) or \( p > n \).

It should be clear from the preceding discussion that there is nothing inherently special in our use of the d’Alambertian \( \Box_g \), when it comes to the calculation of de Rham cohomologies with causally restricted supports on a globally hyperbolic conal manifold \( M \). It is merely one of multiple possible auxiliary hyperbolic differential operators that can serve the same purpose. Here are the key required properties for such an operator \( h \): \( h \) must be a cochain map that is homotopic to zero with respect to the de Rham complex, it must possess retarded and advanced Green functions, these Green functions must be...
causal with respect to the given conal structure on $M$. In fact, the conclusion of our Theorem 2 was reached independently in the recent paper [5] by following an argument structurally similar to ours, with the d’Alambertian replaced by the Lie derivative $\mathcal{L}_v$ with respect to a complete timelike vector field $v$. It is clearly (Green) hyperbolic [3, 2, 35, 36] with Green functions simply given by integration (into the future or past) along the flow lines of $v$. Moreover, it is cochain homotopic to zero because of the well known magic formula of Cartan: $\mathcal{L}_v = \iota_v d + d \iota_v$.

4.2 Functoriality

Recall that ordinary de Rham cohomology is defined on any finite dimensional manifold and the pullback of differential forms along a smooth map between manifolds induces a map between their cohomologies (in the direction opposite the original smooth map). This observation has the following well-known formalisation: de Rham cohomology in degree $p$, $H^p(-)$, is a contravariant functor from the category of smooth manifolds to the category of real vector spaces. The same cannot be said for de Rham cohomology with compact supports, $H^p_0(-)$, because the pullback of a compactly supported form need not be compactly supported itself. This pullback problem is fixed by considering only proper smooth maps between manifolds. So, given a proper smooth map $f: M \to N$, pullback along it induces a contravariant map between de Rham cohomologies in degree $p$ with compact support, $f^*: H^p_0(N) \to H^p_0(M)$. If the map $f$ satisfies an additional restrictive condition, namely that it is an open embedding, it is possible to define a covariant pushforward map $f_*: H^p_0(M) \to H^p_0(N)$: we can identify $M$ with its image $f(M)$, an open subset of $N$, and extend by zero any compactly supported form defined on $M$ to all of $N$. In short, de Rham cohomology with compact supports, $H^p_0(-)$, defines a contravariant functor on the category of smooth manifolds with proper maps as morphisms, when paired with the pullback, while it defines a covariant functor on the category of smooth manifolds with open embeddings as morphisms, when paired with the pushforward.

A natural question is the following: do similar properties hold, and under what precise conditions, for de Rham cohomologies with causally restricted supports? For instance, this question was briefly raised, but without any definite answer, in [5]. In fact, it is straightforward to present causally restricted cohomologies as functors, provided we modify the domain category by adding generalized causal structures to manifolds (as in Section 4.1) and by modifying the notion of a proper map with respect to the causal structure.

Consider two conal manifolds $M$ and $N$, with a smooth map $f: M \to N$ between them. We call the map $f$ reflectively spacelike-proper if the preimage of any spacelike compact set is also spacelike compact, while we call it reflectively timelike-proper if the preimage of any timelike compact set is also timelike compact. When the map $f$ is an open embedding, we also introduce the terminology monotonically spacelike-proper for the case when the image of any spacelike compact set is itself spacelike-compact and monotonically timelike-proper.

\footnote{We shall not delve here into the details of category theory. It suffices to say that any statement that we shall make involving functors and categories will be simply a very terse transcription of some other property that will be spelled out in more elementary terms. More details about the functorial properties of de Rham cohomology can be found in [11].}

\footnote{A continuous map is proper if the preimage of any compact set is compact.}
for the case when the image of any timelike compact set is timelike compact. 
We should note that the above terminology is partly inspired by some general 
notions from the theory of partially ordered sets. A map \( f : M \to N \) between 
two partially ordered sets \((M, \leq)\) and \((N, \leq)\) is said to be monotonic if \( x \leq y \) 
implies \( f(x) \leq f(y) \) and, on the other hand, it is said to be order-reflecting if 
\( f(x) \leq f(y) \) implies \( x \leq y \). The following theorem is a straight forward gen-
eralization of the previous arguments for the simpler case of compact supports.

**Theorem 6.** Let \( \mathcal{CMan}_{sc} \) and \( \mathcal{CMan}_{tc} \) be the categories of conal manifolds 
with, respectively, reflectively spacelike-proper and reflectively timelike-proper, 
smooth maps as morphisms, while the \( \mathcal{CMan}_{sc}^{e} \) and \( \mathcal{CMan}_{tc}^{e} \) categories have, 
respectively, monotonically spacelike-proper and monotonically timelike-proper 
open embeddings as morphisms. The following theorem is a straight forward gener-
eralization of the previous arguments for the simpler case of compact supports.

**Proof.** The proof is a direct parallel of the above arguments for the case with 
compact supports, since the definitions have been specifically adapted to that 
argument.

To show that the definitions of spacelike- and timelike-proper maps are in 
some sense natural, we give a couple of examples.

**Lemma 7.** Let \( M \) be a manifold and two conal structures on it, \( C \subseteq C' \subseteq TM \) (\( C \) is “slower” than \( C' \)) (Section 4.1). The identity map is a reflectively 
spacelike-proper from \((M, C')\) to \((M, C)\) and reflectively timelike-proper from 
\((M, C')\) to \((M, C)\).

**Proof.** Let \( K \subseteq M \) be any compact subset. Then, by hypothesis, the \( C \)-influence 
set is smaller than the \( C' \)-influence set, \( T_{C}(K) \subseteq T_{C'}(K) \). Therefore, any \( C \)- 
spacelike compact set is also \( C' \)-spacelike and hence the identity from \((M, C')\) 
to \((M, C)\) is reflectively spacelike-proper. On the other hand, if \( U \subseteq M \) is \( C' \)- 
timelike compact, then we have the inclusion \( T_{C}(K) \cap U \subseteq T_{C'}(K) \cap U \), the 
latter being compact. Therefore, \( U \) is also \( C \)-timelike compact and the identity 
from \((M, C)\) to \((M, C')\) is reflectively timelike-proper.

**Lemma 8.** Let \((M, g)\) and \((N, h)\) be two globally hyperbolic Lorentzian man-
ifolds and \( f : M \to N \) an open isometric embedding, such that the image of 
a Cauchy surface of \( M \) is a Cauchy surface of \( N \). Then, \( f \) is monotonically 
timelike-proper.

**Proof.** Let \( U \subseteq M \) be timelike compact. According to \([35]\), this is equivalent 
to \( U \) being contained between two Cauchy surfaces in \((M, g)\), say \( \Sigma_{1}, \Sigma_{2} \subset M \). 
This means that the image, \( f(U) \) is contained between \( f(\Sigma_{1}) \) and \( f(\Sigma_{2}) \), with 
the latter, by hypothesis, being Cauchy surfaces in \((N, h)\). Thus, \( f(U) \) is also 
timelike compact and the map \( f \) is monotonically timelike-proper.

### 4.3 Calabi or Killing-Riemann-Bianchi complex

In \([35] [35]\), it was pointed out that the construction of the symplectic and Pois-
son structures on the phase space of field theories with constraints and/or gauge
invariance can be done using a general framework, provided a given field theory satisfies certain geometric conditions. These conditions include the existence of certain differential complexes that extend the operators that constitute the constraints and that generate the gauge transformations. For Maxwell (and similar) theories, all of these complexes are invariably part of the de Rham complex \[36\, \text{Secs.4.2–3}\]. On the other hand, for linearized gravity, one has to use something different. Unfortunately, the explicit form of these differential complexes is not currently known for linearized gravity on an arbitrary background \[36, \text{Sec.4.4}\]. However, in the special case of constant curvature backgrounds, the answer is known and it is the so-called Calabi complex \[13\]. It is likely that, once an explicit understanding of the corresponding differential complexes for more general backgrounds is achieved, the general framework of \[36, \text{35}\] would supersede recent covariant treatments of the quantization of linearized gravity like \[18, \text{30}\].

The Calabi complex provides a fine resolution \[12\] of the sheaf of Killing vectors, similarly to how the de Rham complex provides a fine resolution of the sheaf of locally constant functions. As such, it has been studied in some literature on the deformation of constant curvature geometric structures \[13, 7, 29, 23, 24, 43, 16\]. Because its structure is substantially different from the de Rham complex, we summarize some of its relevant properties before concentrating on its causally restricted cohomologies. Many of these properties are scattered throughout or are simply not available in the existing literature. We defer a fuller discussion of the Calabi complex, which collects these properties and their proofs, to \[34\].

### 4.3.1 Tensor bundles

We will present later a differential complex whose nodes are sections of tensor bundles that are not so easy to express in conventional notation. So, let us introduce the following short-hands. We denote the cotangent bundle by \(V_M = T^*M\) and the bundle of metrics (symmetric, covariant 2-tensors) by \(S^2M = S^2T^*M\). Let \(RM \subset (T^*)^4M\) denote the sub-bundle of covariant 4-tensors that satisfy the algebraic symmetries of the Riemann tensor \((R_{(abcd)} = R_{(ab)(cd)} = R_{abcd} - R_{[a[b]c]d} = 0)\). Next, we let \(BM \subset (T^*)^5M\) denote target bundle of the Bianchi operator \(\nabla_{[a} R_{bc]de}\). At this point it is convenient to notice that the fiber of each of these bundles carries \[20\] an irreducible representation of \(\text{GL}(n)\), with \(n = \text{dim} M\). In fact, it is easiest to describe the remaining tensor bundles in terms of the irreducible \(\text{GL}(n)\) representation carried by their fibers. So let \(C_l M \subset (T^*)^{l+2}M\) (with \(C\) standing for Calabi) denote the sub-bundles of covariant \((l + 2)\)-tensors with the corresponding irreducible representations listed in Table 1, which also lists their fiber ranks. It is consistent for us to assign \(C_0M \cong VM\), \(C_1M \cong S^2M\) and \(C_2M \cong RM\) and \(C_3M \cong BM\). Recall that, on an \(n\)-dimensional manifold, the largest rank of a fully antisymmetric tensor is \(n\). So the bundles \(C_l M\) become trivial (zero fiber rank) for \(l > n\).

Given two \(S^2M\) tensors, we can construct an \(RM\) tensor out of them using the formula

\[
(g \otimes h)_{abcd} = g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}. \tag{28}
\]

In fact, the above formula represents a \(\text{GL}(n)\)-equivariant map between \(S^2 \otimes S^2\) and \(R\) (where we use the bundle prefixes to stand in for the corresponding
Table 1: It is conventional to label irreducible GL($n$) representations by Young diagrams \cite{21}. Recall that a Young diagram with $k$ cells of type $(r_1, r_2, \ldots)$ consists of a number of rows of non-increasing lengths $r_i$, $r_{i+1} \leq r_i$, such that $\sum_i r_i = k$. Given a Young diagram with $k$ cells, an instance of the corresponding irreducible GL($n$) representation class can be realized as the image of the space of covariant $k$-tensors after two projections: assign an independent tensor in each cell of the diagram, symmetrize over each row, anti-symmetrize over each column.

The table below lists the tensor bundles of the Calabi complex, the corresponding irreducible GL($n$) representations (labeled by Young diagrams), and their fiber ranks, for $\dim M = n$. The rank is given by the famous hook formula, which is the following fraction. The numerator is the product of the following numbers: place $n$ in the top left cell, increase by 1 to the right and decrease by 1 down, until all cells are filled. The denominator is the product of the following numbers: fill a given cell with the number of cells constituting a hook with vertex at the given location, extending to the right and down \cite{21}.

| bundle | Young diagram | fiber rank |
|--------|---------------|------------|
| $VM \cong C_0 M$ | | $n$ |
| $S^2 M \cong C_1 M$ | | $\frac{n(n+1)}{2}$ |
| $RM \cong C_2 M$ | | $\frac{n^2(n^2-1)}{12}$ |
| $BM \cong C_3 M$ | | $\frac{n^2(n^2-1)(n-2)}{24}$ |
| $C_4 M$ | | $\frac{n^2(n^2-1)(n-2)(n-l+1)}{2(l+1)l(l-2)!}$ |

\[\vdots\]
| $C_l M$ | | $\vdots$ |

\[
\frac{n^2(n^2-1)(n-2)(n-l+1)}{2(l+1)l(l-2)!}
\]
irreducible representations). The decomposition of the $S^2 \otimes S^2$ tensor product has only one copy of $R$, so by Schur’s lemma such a map is unique, up to an overall rescaling. The same argument can be repeated for the tensor product $S^2 \otimes Y$, where $Y$ corresponds to any other Young diagram. This tensor product decomposes into irreducible subrepresentations without multiplicities. Then the projection onto any of the subrepresentations $Y'$ is well defined up to a rescaling. If we fix a sections $g$ of $S^2 M$ and $h$ of $Y M$, these projections define a bilinear operation between $g$ and $h$ with the result a section of $Y' M$. We use the following explicit formulas:

\[(g \odot t)_{abc;de} = +g_{ad}t_{bc:e} + g_{bd}t_{ca:e} + g_{cd}t_{ab:e} - g_{ae}t_{bc:d} - g_{be}t_{ca:d} - g_{ce}t_{ab:d},\]  

\[(g \odot t)_{abcd;ef} = +g_{ac}t_{bd:e} - g_{bc}t_{ad:e} + g_{cd}t_{ab:e} - g_{af}t_{bd:e} + g_{bf}t_{ca:e} - g_{cf}t_{ab:e} - g_{df}t_{abc:e}.\]  

Note that a tensor with indices written as in $t_{abc;de}$ has the symmetry type $(2, 2, 1)$, while $t_{abcd;ef}$ corresponds to the symmetry type $(2, 1, 1)$, and so on.

The metric $g_{ab}$ itself, an $S^2 M$ tensor, can now be used to produce an $R M$ tensor,

\[(g \odot g)_{abcd} = 2(g_{ac}g_{bd} - g_{bc}g_{ad}),\]  

which is obviously covariantly constant. In fact, a constant curvature spacetime must have (covariant) Riemann tensor, Ricci tensor and Ricci scalar of the following form

\[
\bar{R}_{abcd} = \frac{k}{n(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad}), \quad \bar{R}_{ac} = \frac{k}{n}g_{ac}, \quad \bar{R} = k.
\]  

We have decorated these quantities with a bar to indicate the fact that we shall fix a constant curvature background metric $g$ and consider perturbations on it. For our purposes, we also require that the Lorentzian manifold $(M, g)$ is globally hyperbolic.

We should note that solutions of Einstein equations (including a possible cosmological constant term) with constant curvature includes Minkowski space ($k = 0$), de Sitter space ($k > 0$) and anti-de Sitter space ($k < 0$). The latter is not globally hyperbolic, so it is excluded from part of our discussion. All three examples are simply connected. Other examples may be obtained by taking quotients thereof with respect to a discrete subgroup, thus changing the topology. The list of possibilities is thus exhausted.

### 4.3.2 Differential operators

Now, we introduce a number of differential operators between the tensor bundles that we have defined. These operators fit into the following (almost) commutative diagram, where the tensor bundles also stand in for their spaces of sections:

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & C_0 M & \longrightarrow & C_1 M & \longrightarrow & C_2 M & \longrightarrow & \cdots & \longrightarrow & C_n M & \longrightarrow & 0 \\
& & p_1 \downarrow & & & l_2 \downarrow & & \ddots & & & l_n & \downarrow & \\
0 & \longrightarrow & C_0 M & \longrightarrow & C_1 M & \longrightarrow & C_2 M & \longrightarrow & \cdots & \longrightarrow & C_n M & \longrightarrow & 0 \\
& & p_1 \downarrow & & & l_2 \downarrow & & \ddots & & & l_n & \downarrow & \\
\end{array}
\]  

\[(33)\]
where all the solid arrows commute and the rows constitute (cochain) complexes. The vertical maps are then necessarily cochain maps. They happen to satisfy the identities $P_l = E_{l+1} \circ B_{l+1} + B_l \circ E_l$, which means that they are null-homotopic, with the $E_l$ supplying the corresponding cochain homotopy.

Below, we give explicit formulas for all these differential operators in dimension $n = 4$. More details can be found in [34], which draws from the earlier works [13, 7, 29, 23, 24, 43, 16]. As we shall see, for low indices they are well known in the relativity literature. However, the relations between them in terms of fitting into the above commutative diagram do not seem to have been fully noted.

The **Calabi differential complex** is given by

\[
B_1[v]_{ab} = \nabla_a v_b + \nabla_b v_a, \tag{34}
\]

\[
B_2[h]_{ab;cd} = \frac{1}{2} \left( \nabla_a \nabla_c h_{bd} - \nabla_b \nabla_c h_{ad} - \nabla_a \nabla_d h_{bc} + \nabla_b \nabla_d h_{ac} \right) + k \frac{1}{2n(n-1)} (g \circ h)_{ab;cd}, \tag{35}
\]

\[
B_3[r]_{ab;ced} = 3 \nabla_a r_{bc;de} = \nabla_a r_{bc} \delta_{de} + \nabla_b r_{ca;de} + \nabla_c r_{ab;de}, \tag{36}
\]

\[
B_4[b]_{abcd;ef} = 4 \nabla_a b_{bcd;ef} - \nabla_b b_{cde;af} - \nabla_c b_{def;ab} - \nabla_d b_{abcdef}. \tag{37}
\]

The details showing that these operators have the desired symmetry properties and indeed define a complex, $B_{l+1} \circ B_l = 0$, which is moreover elliptic\(^6\), can be found in [34].

It is interesting to note the following relations with well known differential operators in relativity. The **Killing operator** is $K[h] = B_1[h]$. The **linearized Riemann tensor** is $\hat{R}[h] = -\frac{1}{2} B_2[h] + k \frac{2}{n(n-1)} (g \circ h)$, where the all covariant non-linear Riemann tensor is expanded as $R[g + \lambda h]_{ab;cd} = \hat{R}_{ab;cd} + \lambda \hat{R}[h]_{ab;cd}$ (convention of [51]). The background **Bianchi operator** is $\hat{B}[r] = B_3[r]$, with $\hat{B}[\hat{R}] = 0$. Finally, though the name is not standard, it is meaningful to call $B_4[b]$ a **higher Bianchi operator**. Thus, it would also make sense to refer to the Calabi complex as the **Killing-Riemann-Bianchi complex**. This complex also happens to be locally exact\(^5\). Thus, according to the general machinery of sheaf theory [12], the Calabi complex provides a fine resolution of the sheaf of Killing vectors (or Killing sheaf) $K_M$ on the Lorentzian manifold $(M, g)$. This observation immediately gains us the following

**Proposition 9** (Calabi [13, 34]). The (unrestricted) cohomology $H^i(M, g) = \ker B_{i+1} / \im B_i$ of the Calabi complex is isomorphic to the sheaf cohomology $H^*(M, K_M)$ of the sheaf of Killing vectors on any spacetime $(M, g)$ of constant curvature.

---

\(^5\)A complex of differential operators is **elliptic** if the corresponding complex of symbol maps is exact for every non-zero covector.

\(^6\)A differential complex on a manifold $M$ is **locally exact** if every $x \in M$ has a neighborhood such that the complex restricted to it becomes exact. For example, this condition is fulfilled for the de Rham complex thanks to the Poincaré lemma.
The homotopy differential operators are given by

\[ E_1[h]_a = \nabla^b h_{ab} - \frac{1}{2} \nabla_a h, \]

\[ E_2[r]_a^b = r_{a c b}^c, \]

\[ E_3[b]_{a b c d} = \frac{1}{2} (\nabla^e b_{a c d e} + \nabla^e b_{b c d a}) - \frac{1}{2} (\nabla_b b_{c d e a}^e + \nabla_c b_{a b e d}^e - \nabla_d b_{a b c e}^e), \]

\[ E_4[b]_{a b c d e} = \frac{1}{3} \left( 2 \nabla^f b_{a b c d e} + \nabla^f b_{d a c b e} + \nabla^f b_{b d e c a} + \nabla^f b_{b d e c a} \right) + \frac{1}{6} \left( 2 \nabla^f b_{a b c d e} - 2 \nabla^f b_{a b c d e} \right) - \nabla_a b_{d e c f}^c \]

\[ + \nabla_a b_{d e c f}^d + \nabla_b b_{d c e f}^c + \nabla_c b_{d d e f}^c + \nabla_d b_{d e c f}^e. \]

Their desired symmetry properties are demonstrated in [34]. Again, we find the following relations with classical differential operators from relativity. The de Donder operator is \( D[h] = E_1[h] \). The trace from the Riemann to the Ricci tensors is given by \( R_{a b} = R_{a c b d} = E_2[R]_{a b} \). The higher homotopy operators \( E_i \) do not seem to be part of the classical literature. However, they are essentially modified divergence operators and are thus reminiscent of the de Rham co-differentials.

Finally, the cochain maps \( P_i = E_{i+1} \circ B_{i+1} + B_i \circ E_i \) (with the edge cases \( P_0 = E_1 \circ B_1 \) and \( P_n = B_n \circ E_n \)) are given by

\[ P_0[v]_a = \Box v_a + \frac{1}{n} v_a, \]

\[ P_1[h]_{a b} = \Box h_{a b} - \frac{1}{n(n-1)} h_{a b} + \frac{2}{n(n-1)} g_{a b} \text{tr}[h], \]

\[ P_2[r]_{a b c d} = \Box r_{a b c d} - \frac{2}{n} r_{a b c d} + \frac{2}{n(n-1)} (g \circ \text{tr}[r])_{a b c d}, \]

\[ P_3[b]_{a b c d e} = \Box b_{a b c d e} - \frac{1}{n(n-1)} b_{a b c d e} - \frac{2}{n(n-1)} (g \circ \text{tr}[b])_{a b c d e}, \]

\[ P_4[b]_{a b c d e f} = \Box b_{a b c d e f} - \frac{2}{n(n-1)} b_{a b c d e f} + \frac{2}{n(n-1)} (g \circ \text{tr}[b])_{a b c d e f}, \]

where we have defined the traces as \( \text{tr}[h] = h_{a}^a \), \( \text{tr}[r]_{a b} = r_{a c b}^c \), \( \text{tr}[b]_{a b c d} = b_{a b c d}^e \), \( \text{tr}[b]_{a b c d e} = b_{a b c d e}^f \). The required null-homotopy identities \( P_i = E_{i+1} \circ B_{i+1} + B_i \circ E_i \) (including the edge cases \( P_0 = E_1 \circ B_1 \) and \( P_n = B_n \circ E_n \)) are demonstrated in [34]. These identities for \( P_0[v] \) and \( P_1[v] \) are well known and are tightly linked with the de Donder gauge fixing condition in linearized gravity [31][18]. The higher cochain maps and the corresponding identities appear to be new. Though, the identity for \( P_2[r] \) is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the Lichnerowicz Laplacian [39] Sec.1.3] (see also [14 Sec.7.1], [10 Exr.15.2], [10 Eq.35]).
4.3.3 Cohomology with unrestricted and compact supports

Let us denote the cohomology of the Calabi complex by $H^C_X(M, g)$, where $X = 0, +, −, fc, pc, sc, tc$ or empty, according to the conventions of Section 2. As in the case of the de Rham complex in Section 3 we will later relate the cohomology with causally restricted supports to that with unrestricted or compact supports. It remains still to find a means to calculate these cohomology groups. We will state some results in that direction below, referring to [34] for a fuller discussion.

An important observation is that each of the $P_l$ operators is wave-like, that is, it has the same principal symbol as the wave operator $\Box g$ with respect to the background Lorentzian metric $g$. This observation has a dual role. First, this means that each of the $P_l$ operators is Green hyperbolic [3, 2], while being cochain homotopic to zero, opening the door to using the methods of Section 3 to compute the cohomology with causally restricted supports.

The second role is more subtle. Note that the principal symbols of the $B_l$ maps in the Calabi complex are actually $GL(n)$-equivariant and so do not actually involve the background metric $g$. On the other hand, the principal symbols of the cochain maps $P_l$ do depend on $g$. This dependence comes purely from the cochain homotopy operators $E_l = E_l^g$ and the identity $P_l = P_l^g = E_{l+1}^g \circ B_{l+1} + B_l \circ E_l^g$, where we have used the subscript $g$ to indicate that the background metric was used for covariant differentiation and index raising. On the other hand, we are completely free to define a different set of cochain maps $P_l^{gn} = E_{l+1}^{gn} \circ B_{l+1} + B_l \circ E_l^{gn}$, which now depend on a different metric $g_R$ with Riemannian signature. It is crucial to note that the principal symbol of $P_l^{gn}$ depends only on the principal symbols of the $E_l^{gn}$ and $B_l$. So, in fact, it is equal to the principal symbol of $P_l^g$, but with the Lorentzian metric $g$ replaced by the Riemannian metric $g_R$. In other words, each of the $P_l^{gn}$ operators is elliptic, since its principal symbol coincides with the Laplace operator $\Delta_{gn}$. Of course, $P_l^{gn}$ would differ much more radically from the formulas we have given for $P_l^g$ in the terms of subleading differential orders.

The existence of such an “elliptic” cochain homotopy results in the following

**Proposition 10** (Calabi [13, 34]). (a) The cohomology $H^C(M, g)$ of the Calabi complex $(\Gamma(C(M), B_l)$ is isomorphic to the cohomology $H^* M, K_g)$ of the sheaf $K_g$ of Killing vectors on $(M, g)$. (b) If $(M, g)$ is a simply connected, constant curvature Lorentzian manifold, then $H^* M, K_g)$ isomorphic to $H^*(M) \otimes V_g$, where $V_g$ is the vector space of all Killing vectors $\times g)$ is the de Rham cohomology group.

Killing vectors (or rather covectors in our notation) are solutions $v \in \Gamma(T^* M)$ of the Killing equation $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a = 0$. On simply connected, constant curvature $n$-dimensional spacetimes, $\dim V_g = (n+1)$. Note also that the simple connectedness condition implies that $H^1(M) = 0$. The precise definition of a sheaf and its cohomology is not of particular importance for the moment. For present purposes, it suffices that the above result, at the very least, this answers the question of what $H^C(M, g)$ is for Minkowski ($\mathbb{R}^n$), de Sitter ($\mathbb{R} \times S^{n-1}$) and anti-de Sitter ($\mathbb{R}^n$) spacetimes. The proof, together with a partial discussion of the non-simply connected case, can be found in [34].

It remains to discuss the cohomologies with compact supports $H^C^0(M, g)$. First, we note that the chain complex $(\Gamma(C^*_M, B_l^*)$ formally adjoint to the Calabi complex has the interesting property that equation $B^*_l[0] = 0$ is equiv-
alent to the (rank-2) Killing-Yano equation \( Y[w]_{abc} = \nabla_a w_{bc} + \nabla_b w_{ac} \), where a solution with \( w_{ab} = w_{ba} \) is called a (rank-2) Killing-Yano tensor on \((M, g)\). Since taking formal adjoints preserves the homotopy identities and ellipticity, appealing to the same arguments as above we also have

**Proposition 11** \((\text{[34]}\)) . (a) The homology \( H\Gamma_1(M, g) = \ker B_1^* / \text{im} B_{l-1}^* \) of the adjoint Calabi complex \( \Gamma(C^*_l M) \), \( B_1^* \) is isomorphic to the cohomology \( H^*(M, \KY_g) \) of the sheaf \( \KY_g \) of Killing-Yano tensors on \((M, g)\). (b) If \((M, g)\) is a simply connected, constant curvature Lorentzian manifold, then \( H^*(M, \KY_g) \cong H^*(M) \otimes W_g \), where \( W_g \) is the vector space of all Killing-Yano tensors and \( H^*(M) \) is the de Rham cohomology group.

On simply connected, constant curvature \( n \)-dimensional spacetimes, \( \dim W_g = (n + 1)^2 / 12 \). Furthermore, using some general results from the theory of elliptic differential complexes (see Example 5.1.11 of \([50]\), which relies on the results of \([47]\)), we have the following duality isomorphisms.

**Proposition 12**. When finite dimensional, the cohomology with compact supports of the Calabi complex is the linear dual of the homology of the formally adjoint Calabi complex, \( H\Gamma_1(M, g) = H\Gamma_1(M, g)^* \), while the homology with compact supports of the adjoint Calabi complex is the linear dual of the cohomology of the Calabi complex, \( H\Gamma_{1,0}(M, g) = H\Gamma_0^l(M, g)^* \). In both cases, the duality can be exhibited via the non-degeneracy of the pairing descended from the natural pairing between the chains and cochains of corresponding complexes.

### 4.3.4 Cohomology with causally restricted supports

With the above discussion in mind, we can see immediately that we are in a situation very similar to that of Section 3 with the de Rham complex replaced by the Calabi complex and the wave operators \( \Box \) replaced by the operators \( P_1 \), which have wave-like principal symbols and are Green hyperbolic. So, repeating the arguments of Section 3 we immediately have the following

**Theorem 13**. Consider a globally hyperbolic, constant curvature Lorentzian manifold \((M, g)\). Its Calabi cohomology \( HC^X_l(M, g) \) with the causally restricted supports \( X = +, -, pc \) or \( fc \) is trivial. Moreover, for the cases \( X = sc, tc \), we have the isomorphisms

\[
HC_{sc}^l(M, g) \cong HC_0^{l+1}(M, g), \quad HC_{P, sc}^l(M, g) \cong HC_0^l(M, g) \oplus HC_{l+1}^0(M, g), \quad (48)
\]

\[
HC_{tc}^l(M, g) \cong HC_{l-1}^{l-1}(M, g), \quad HC_{P}^l(M, g) \cong HC_{l}^l(M, g) \oplus HC_{l-1}^{l-1}(M, g), \quad (49)
\]

with the convention that all cohomologies vanish in degree \( l \) for \( l < 0 \) or \( l > n \).

The Calabi cohomology with spacelike compact support in degree \( l = 1 \) is important in understanding the symplectic and Poisson structure of the classical field theory (and of course the quantization) of linearized gravitons on a background of constant curvature. This was pointed out explicitly in \([36\text{, Sec.4.4]}\) as a special case of a more general phenomenon (also discussed in \([35]\)).

**Remark 1.** Using the above theorem and the results of Section 4.3.3 we can assert that for \( n \)-dimensional Minkowski space \( HC_{sc}^l \) vanishes in all degrees except \( l = n - 1 \), while \( HC_{P, sc}^l \) vanishes in all degrees except \( l = n, n - 1 \). For \( n \)-dimensional de Sitter space \( HC_{sc}^l \) vanishes in all degrees except \( l = n - 1 \), while \( HC_{P, sc}^l \) vanish in all degrees except \( l = 0, n - 1, n \).
4.4 Other differential complexes

Our interest in computing the de Rham and Calabi cohomologies with causally restricted supports has been motivated by their importance in understanding the geometric structure of classical and quantum field theories [15, 46, 6, 19, 5, 31, 35, 36]. Namely, for a general class of linear field theories, one can formulate sufficient conditions for the non-degeneracy of the theory's Poisson structure and the completeness of compactly supported smeared fields as physical observables in terms of the cohomologies of corresponding differential complexes. Non-linear field theories can be studied in terms of their linearizations about arbitrary background solutions. To Maxwell electrodynamics corresponds the de Rham complex [36, Sec.4.2]. To linearized gravity on constant curvature backgrounds, corresponds the Calabi complex [36, Sec.4.4]. Similarly, to Yang-Mills linearized about a flat connection corresponds a twisted de Rham complex.

Each of these examples can be treated using the methods presented in this paper. Few other explicit examples of differential complexes corresponding to other field theories of physical interest seem to be known. In particular, they do not seem to be known for linearized gravity on non-constant curvature backgrounds and, perhaps, not even for Yang-Mills linearized about non-flat connections. On the other hand, there are strong abstract reasons to believe that such differential complexes do indeed exist [14, 29, 43].

If such a differential complex also shares the apparently crucial property of admitting cochain homotopies that generate hyperbolic and elliptic cochain maps (cf. the $E_g, P_g, E_R, P_R$ maps of Sections 4.3.2 and 4.3.3), then its causally restricted cohomologies can be related to those with unrestricted and compactly supported ones, as in Theorems 2 and 13.

If, in addition, such a differential complex could also be seen as resolving a locally constant sheaf, its unrestricted cohomologies could be computed by algebraic means, without actually solving complicated systems of differential equations, as in Section 4.3.3. The latter requirement is closely related to the initial differential operator in the complex having only a finite dimensional space of solutions (being of finite type), as is the case for the locally constant (de Rham) and Killing (Calabi) conditions.

The compactly supported cohomologies could also be obtained if the corresponding formally adjoint complex satisfied similar requirements, as illustrated in Section 4.3.3 by the appearance of the locally constant sheaf of Killing-Yano tensors.

5 Discussion

We have shown how to compute the de Rham cohomology with causally restricted supports (retarded, advanced, past compact, future compact, spacelike compact and timelike compact) on a globally hyperbolic Lorentzian spacetime. The result (Theorems 1, 2 and Corollary 4) expresses these causally restricted cohomologies in terms of the standard de Rham cohomologies of the spacetime manifold, with either unrestricted or compact supports. These results, confirm and generalize the independent similar results of the recent work [5], with the generalizations described below.

Further, we showed how the special geometric features of the de Rham com-
plex, which we used in the cohomology calculation, can be interpreted in terms of homological algebra and applied to other complexes of differential operators that could arise in the investigation of the geometry of classical and quantum field theories. These applications are illustrated on the specific example of the Calabi (or Killing-Riemann-Bianchi) complex, which plays for linearized gravity on constant curvature backgrounds a role analogous to the de Rham complex for Maxwell-like field theories. We have also made comments about the covariance of causally restricted cohomologies under specific types of morphisms between spacetimes, adapted to their causal structure, and under changes of the causal structure itself.

A fuller discussion of the Calabi complex, including its relevant geometric properties that are difficult to locate in or are absent from the current literature, is deferred to future work [34]. In the future, it will also be interesting to find the analogs of the Calabi complex on more general Lorentzian backgrounds, which would consist of differential complexes resolving the locally constant sheaf of Killing vectors on a given background. It is also a separate challenge to compute independently and effectively compute the cohomology of the Killing sheaf based on the topological and geometric information on a given background.

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