THE RELATIVE MODULAR OBJECT AND FROBENIUS EXTENSIONS OF FINITE HOPF ALGEBRAS

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Abstract. For a certain kind of tensor functor $F : C \to D$ between tensor categories, we introduce the notion of the relative modular object $\chi_F \in D$ as the ‘difference’ between a left adjoint and a right adjoint of $F$. Our main result claims that, if $C$ and $D$ are finite, then $\chi_F$ can be written in terms of a categorical analogue of the modular function on a Hopf algebra. Applying this result to the restriction functor associated to an extension $A/B$ of finite-dimensional Hopf algebras, we recover the result of Fischman, Montgomery and Schneider on the Frobenius type property of $A/B$. As an application, we give an analogous result for an extension of Hopf algebras in a braided finite tensor category.

1. Introduction

For a finite-dimensional Hopf algebra $H$ over a field $k$, the (right) modular function $\alpha_H : H \to k$ (also called the distinguished grouplike element) is defined by

$$h \cdot \Lambda = \alpha_H(h)\Lambda$$

for $h \in H$, where $\Lambda \in H$ is a non-zero right integral. It is known that the modular functions control the Frobenius properties of an extension of finite-dimensional Hopf algebras: For such an extension $A/B$, the relative modular function $\chi = \chi_{A/B}$ and the relative Nakayama automorphism $\beta = \beta_{A/B}$ are given respectively by

$$\chi = \alpha_A(b(1))\alpha_B(S(b(2)))$$

and

$$\beta(b) = \chi(b(1))b(2)$$

for $b \in B$, where $S$ is the antipode and $\Delta(b) = b(1) \otimes b(2)$ is the comultiplication of $b$ in the Sweedler notation [FMS97, Definition 1.6]. The map $\beta$ is in fact an algebra automorphism of $B$ and thus, for a left $B$-module $M$, the $B$-module $\beta M$ is defined by twisting the action of $B$ on $M$ by $\beta$. Note that $A$ is a free $B$-module by the Nichols-Zoeller theorem. Fischman, Montgomery and Schneider showed that the extension $A/B$ is $\beta$-Frobenius, i.e., there exists an isomorphism

$$B A_A \cong \beta \text{Hom}_B(A, B_B)$$

(1.1)

of $B$-$A$-bimodules [FMS97, Theorem 1.7]. The starting point of this paper is to understand this result in the setting of finite tensor categories [EO04], a class of tensor categories including the representation category of a finite-dimensional Hopf algebra.

To formulate [FMS97, Theorem 1.7] in a categorical setting, we recall that the Frobenius property of the extension $A/B$ can be described in terms of adjoint functors of the restriction functor $\text{Res}_A^B : \text{mod-}A \to \text{mod-}B$ between the categories of right modules: By the basic theory of algebras, the functors

$$L := (-) \otimes_B A$$

and

$$R := \text{Hom}_B(A, -) \cong (-) \otimes_B \text{Hom}_B(A, B)$$

(1.2)
are a left adjoint and a right adjoint of \( \text{Res}^A_B \), respectively (where the isomorphism for \( R \) follows from the Nichols-Zoeller theorem). Hence (1.1) can be read as a relation between the functors \( L \) and \( R \). Based on this observation, we define the relative modular object \( \chi_F \in D \) for a certain kind of tensor functor \( F : \mathcal{C} \to \mathcal{D} \). Our main result is that \( \chi_F \) can be expressed in terms of a categorical analogue of the modular function if \( \mathcal{C} \) and \( \mathcal{D} \) are finite tensor categories. As an application, we give a generalization of \([\text{FMS97, Theorem 1.7}]\) to an extension of Hopf algebras in a braided finite tensor categories.

Now we explain the organization of this paper: In Section 2, we collect basic results on finite tensor categories and their module categories.

In Section 3, we introduce the modular object \( \alpha_C \in \mathcal{C} \) of a finite tensor category \( \mathcal{C} \) as a categorical analogue of the modular function. After a brief discussion on the Deligne tensor product, we introduce an algebra \( A \) in \( \mathcal{C} \text{env} \colonequals \mathcal{C} \boxtimes \mathcal{C} \text{rev} \). If \( M \) and \( N \) are finite left \( \mathcal{C} \)-module categories (in the sense of Definition 2.7), then \( \mathcal{C} \text{env} \) acts on the category \( \text{REX}(M,N) \) of \( k \)-linear right exact functors from \( M \) to \( N \). A key observation is that an \( A \)-module in \( \text{REX}(M,N) \) is precisely a \( \mathcal{C} \)-module functor. Based on this observation, we define the modular object in a quite abstract way. It turns out that \( \alpha_C \) is isomorphic to the dual of the distinguished invertible object \( D \in \mathcal{C} \) introduced in \([\text{ENO04}]\) whenever \( D \) is defined (Lemma 3.12). Our definition is useful in later sections, however, we do not know whether \( \alpha_C \) is invertible in the case where \( \mathcal{C} \text{env} \) is not rigid.

In Section 4, we consider a tensor functor \( F : \mathcal{C} \to \mathcal{D} \) between tensor categories (in the sense of §4.1) having a left adjoint \( L \) and a right adjoint \( R \). The results of the first-half part of this section are summarized as follows:

**Theorem (Lemma 4.3 and Theorem 4.6).** With the above notation, the following assertions are equivalent:

1. \( L \) has a left adjoint.
2. \( R \) has a right adjoint.
3. There exists an object \( \chi_F \in \mathcal{D} \) such that \( R \cong L(\chi_F \otimes -) \).

Such an object \( \chi_F \) is unique up to isomorphism if it exists, is an invertible object, and satisfies the following relations:

\[
L(\chi_F \otimes -) \cong R \cong L(- \otimes \chi_F) \quad \text{and} \quad R(\chi_F^* \otimes -) \cong L \otimes R(- \otimes \chi_F^*).
\]

If, moreover, \( \mathcal{C} \) and \( \mathcal{D} \) are finite tensor categories, then the above three conditions are equivalent to each of the following four conditions:

4. \( L \) is exact.
5. \( R \) is exact.
6. \( F(P) \) is projective for every projective object \( P \in \mathcal{C} \).
7. \( F(P) \) is projective for a projective generator \( P \in \mathcal{C} \).

We call \( \chi_F \) the relative modular object of \( F \). We note that this theorem may be an instance of a general principle in the category theory. Indeed, similar results are obtained in different settings in \([\text{Bal14, BDS15}]\). In any case, this theorem is not sufficient as a generalization of \([\text{FMS97, Theorem 1.7}]\); their result describes the relation between \( L \) and \( R \) by the relative modular function \( \chi_{A/B} \), while the above theorem does not give any information about \( \chi_F \). Our main result is the following formula of the relative modular object:

**Theorem (Theorem 4.7).** \( \chi_F \cong F(\alpha_C) \otimes \alpha_D^* \) if either \( F(\alpha_C) \) or \( \alpha_D \) is invertible.
If $F = \text{Res}^A_B$ is the restriction functor associated to an extension $A/B$ of finite-dimensional Hopf algebras, then the object $\chi_F$ is the right $H$-module corresponding to the the relative modular function $\chi_{A/B}$. One can derive [FMS97, Theorem 1.7] by combining this result with (1.2); see §4.3.

To obtain a meaningful consequence from our result, we need an expression of the modular object of a given finite tensor category. In Section 5, we determine the modular object of the category $B_H$ of right modules over a Hopf algebra $H$ in a braided finite tensor category $B$. The modular function $\alpha_H : H \to 1$ is defined in a similar way as the ordinary case, however, the right $H$-module corresponding to $\alpha_H$ is not the modular object of $B_H$ in general. We express the modular object of $B_H$ by the modular function $\alpha_H$, the modular object $\alpha_B$ and the object $\text{Int}(H)$ of integrals of $H$ (Theorem 5.2). As an application, we obtain the following ‘braided version’ of [FMS97, Theorem 1.7]:

**Theorem** (Theorem 5.4). Let $B$ be a braided finite tensor category whose modular object is invertible, and let $A/B$ be an extension of Hopf algebras in $B$. Then the following assertions are equivalent:

1. The restriction functor $\text{Res}^A_B : B_A \to B_B$ is a Frobenius functor, i.e., a left adjoint of $F$ and a right adjoint of $F$ are isomorphic.
2. $\text{Int}(A) \cong \text{Int}(B)$ and $\alpha_A \circ i = \alpha_B$, where $i : B \to A$ is the inclusion.

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2. Preliminaries

2.1. Monoidal categories. Recall that a monoidal category [ML98, VII.1] is a category $\mathcal{C}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the tensor product), an object $1 \in \mathcal{C}$ (called the unit object) and natural isomorphisms

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \quad \text{and} \quad 1 \otimes X \cong X \cong X \otimes 1$$

obeying the pentagon and the triangle axiom. If these natural isomorphisms are the identity, then $\mathcal{C}$ is said to be strict. By the Mac Lane coherence theorem, we may assume that all monoidal categories are strict. Given a monoidal category $\mathcal{C}$, we denote by $\mathcal{C}^{\text{rev}}$ the same category but with the reversed tensor product given by $X \otimes^{\text{rev}} Y = Y \otimes X$.

Let $L$ and $R$ be objects of $\mathcal{C}$, and let $\varepsilon : L \otimes R \to 1$ and $\delta : 1 \to R \otimes L$ be morphisms in $\mathcal{C}$. If $((\varepsilon \otimes \text{id}_L) \circ (\text{id}_L \otimes \delta))$ and $(\text{id}_R \otimes \varepsilon) \circ (\delta \otimes \text{id}_R)$ are identities, then we say that the triple $(L, \varepsilon, \eta)$ is a left dual object of $R$ and the triple $(R, \varepsilon, \eta)$ is a right dual object of $L$.

We say that $\mathcal{C}$ is rigid if every object of $\mathcal{C}$ has a left dual object and a right dual object. If this is the case, we denote by $(V^*, \text{ev}, \text{coev})$ the (fixed) left dual object of $V \in \mathcal{C}$. The assignment $V \mapsto V^*$ extends to an equivalence $(\cdot)^* : \mathcal{C} \to \mathcal{C}^{\text{op, rev}}$ of monoidal categories, which we call the left duality functor. A quasi-inverse of $(\cdot)^*$, denoted by $^\ast(\cdot)$ and called the right duality functor, is given by taking a right dual object. For simplicity, we assume that $(\cdot)^*$ and $^\ast(\cdot)$ are strict monoidal and mutually inverse.
2.2. Modules over a monoidal category. Let $C$ be a monoidal category. A $left C$-module category is a category $M$ endowed with a functor $\otimes : C \times M \to M$ (called the action) and natural isomorphisms

$$a_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M) \quad \text{and} \quad \ell_M : 1 \otimes M \to M$$

obeying certain axioms similar to those for a monoidal category. A $right C$-module category and a $C$-bimodule category are defined analogously. Now let $M$ and $N$ be left $C$-module categories. A (left) lax $C$-module functor from $M$ to $N$ is a functor $F : M \to N$ endowed with a natural transformation

$$\xi_{X,M} : X \otimes F(M) \to F(X \otimes M) \quad (X \in C, M \in M)$$

compatible with the natural isomorphisms $a$ and $\ell$; see, e.g., [EGNO09] for the precise definitions of these notions. If the natural transformation $\xi_{X,M}$ is invertible, then $F$ is said to be strong. The following lemma is remarked in [Ost03]; see [DSS14] for the detailed proof.

**Lemma 2.1.** If $C$ is rigid, then every lax $C$-module functor is strong.

Thus, in the case where $C$ is rigid, lax $C$-module functors and strong $C$-module functors are simple called $C$-module functors.

Note that the opposite category $M^{op}$ of a left $C$-module category $M$ is naturally a left $C^{op}$-module category. Formally, we can define a colax $C$-module functor from $M$ to $N$ to be a lax $C^{op}$-module functor from $M^{op}$ to $N^{op}$. The reader can find a proof of the following lemma in [DSS14]:

**Lemma 2.2.** Let $F : M \to N$ be a functor between left $C$-module categories, and suppose that $F$ has a right adjoint $G : N \to M$ with unit $\eta$ and counit $\varepsilon$. If $G$ is a lax $C$-module functor, then $F$ has a natural structure of a colax $C$-module functor. Similarly, if $F$ is a colax $C$-module functor, then $G$ is a lax $C$-module functor.

We omit the definition of morphisms of lax $C$-module functors; see [Ost03]. Left $C$-module categories, lax $C$-module functors and their morphisms form a 2-category. An equivalence of left $C$-module categories is defined to be an equivalence in this 2-category. By Lemma 2.2, a strong $C$-module functor $F : M \to N$ is an equivalence of left $C$-module categories if and only if it is an equivalence between the underlying categories [DSS14].

2.3. Modules over an algebra. Let $A$ be an algebra (= a monoid [ML98 VII.3]) in a monoidal category $C$. Recall that a left $A$-module is an object $M \in C$ endowed with a morphism $\triangleright_M : A \otimes M \to M$ (called the action) such that

$$\triangleright_M \circ (m_A \otimes \text{id}_M) = \triangleright_M \circ (\text{id}_A \otimes \triangleright_M) \quad \text{and} \quad \triangleright_M \circ (u_A \otimes \text{id}_M) = \text{id}_M$$

where $m_A : A \otimes A \to A$ and $u_A : 1 \to A$ are the multiplication and the unit of the algebra $A$, respectively. Left $A$-modules form a category, which we denote by $AC$. If $B$ is another algebra in $C$, then the category $CB$ of right $B$-modules and the category $ACB$ of $A$-$B$-bimodules are defined analogously. The following lemma is well-known:

**Lemma 2.3.** Suppose that $C$ is rigid. If $M \in AC$, then $M^* \in CA$ by

$$\llangle_{M^*} : M^* \otimes A \xrightarrow{(\triangleright_M)^* \circ \text{id}_A} M^* \otimes A^* \otimes A \xrightarrow{id_{M^*} \otimes \triangleright} M^*.$$
Similarly, $N \in {_A\mathcal{C}}$ if $N \in \mathcal{C}_A$. These constructions give anti-equivalences

$(-)^* : _A\mathcal{C} \to \mathcal{C}_A$ and $^*(-) : \mathcal{C}_A \to _A\mathcal{C}$.

It is convenient to extend the notion of modules over an algebra in the following way: Note that a left $\mathcal{C}$-module category $\mathcal{M}$ is the same thing as a category $\mathcal{M}$ endowed with a strong monoidal functor from $\mathcal{C}$ to the category $\mathcal{M}^\mathcal{M}$ of endofunctors on $\mathcal{M}$. Hence, if $\mathcal{M}$ is a left $\mathcal{C}$-module category, an algebra $A$ in $\mathcal{C}$ defines an algebra $A \otimes (-)$ in $\mathcal{M}^\mathcal{M}$, i.e., a monad on $\mathcal{M}$.

**Definition 2.4.** Given an algebra $A$ in $\mathcal{C}$, we denote by $A\mathcal{M}$ the Eilenberg-Moore category of the monad $A \otimes (-)$ on $\mathcal{M}$. An object of the category $A\mathcal{M}$ will be referred to as a left $A$-module in $\mathcal{M}$. A right $A$-module in a right $\mathcal{C}$-module category and an $A$-$B$-bimodule in a $\mathcal{C}$-bimodule category are also defined analogously.

Note that $\mathcal{C}$ is a $\mathcal{C}$-bimodule category by the tensor product. The notation and the terminology given in Definition 2.4 are consistent with those introduced at the beginning of this subsection.

2.4. Closed module categories. Let $\mathcal{C}$ be a monoidal category, and let $\mathcal{M}$ be a left $\mathcal{C}$-module category. We say that $\mathcal{M}$ is closed if, for every object $M \in \mathcal{M}$, the functor $\mathcal{C} \to \mathcal{M}$ defined by $X \mapsto X \otimes M$ has a right adjoint (cf. the definition of closed monoidal categories).

Suppose that $\mathcal{M}$ is closed. For an object $M \in \mathcal{M}$, we denote by $\text{Hom}_\mathcal{M}(M, -)$ a right adjoint of the functor $(-) \otimes M$. By the parameter theorem for adjunctions, the assignment $(M, N) \mapsto \text{Hom}_\mathcal{M}(M, N)$ extends to a functor from $\mathcal{M}^{\text{op}} \times \mathcal{M}$ to $\mathcal{C}$ such that there is a natural isomorphism

$$\text{Hom}_\mathcal{M}(X \otimes M, M') \cong \text{Hom}_\mathcal{C}(X, \text{Hom}_\mathcal{M}(M, M'))$$

for $M, M' \in \mathcal{M}$ and $X \in \mathcal{C}$. The functor $\text{Hom}_\mathcal{M}$ is called the internal Hom functor for $\mathcal{M}$ and makes $\mathcal{M}$ a $\mathcal{C}$-enriched category. For simplicity, we often write $\text{Hom}_\mathcal{M}$ as $\text{Hom}$ if $\mathcal{M}$ is obvious from the context.

2.5. Finite tensor categories. Given an algebra $A$ over a field $k$ (= an associative unital algebra over $k$), we denote by $\text{mod-}A$ the category of finitely generated right $A$-modules. The following variant of the Eilenberg-Watts theorem is well-known and will be used extensively in this paper:

**Lemma 2.5.** Let $A$ and $B$ be finite-dimensional algebras over $k$. For a $k$-linear functor $F : \text{mod-}A \to \text{mod-}B$, the following assertions are equivalent:

1. $F$ is right exact.
2. $F$ has a right adjoint.
3. $F \cong (-) \otimes_A M$ for some finite-dimensional $A$-$B$-bimodule $M$.

A $k$-linear abelian category is said to be finite if it is $k$-linearly equivalent to $\text{mod-}A$ as a $k$-linear category for some finite-dimensional algebra $A$ over $k$. By the above lemma, a $k$-linear functor between finite abelian categories has a right adjoint if and only if it is right exact.

**Definition 2.6.** A finite tensor category over $k$ is a rigid monoidal category $\mathcal{C}$ such that $\mathcal{C}$ is a finite abelian category over $k$ and the tensor product of $\mathcal{C}$ is $k$-linear in each variable.
Unlike [EO04] (and like [DSS13, DSS14]), we do not assume that the unit object of a finite tensor category is a simple object (thus our finite tensor category is in fact a finite multi-tensor category in the sense of [EO04]).

2.6. Finite module categories. Let \( C \) be a finite tensor category. We mainly consider the following class of left \( C \)-module categories:

**Definition 2.7.** A finite left \( C \)-module category is a left \( C \)-module category \( M \) such that \( M \) is a finite abelian category and the action \( \otimes : C \times M \to M \) is \( k \)-linear in each variable and right exact in the first variable.

For \( k \)-linear abelian categories \( M \) and \( N \), we denote by \( \text{Rex}(M, N) \) the category of \( k \)-linear right exact functors from \( M \) to \( N \). A finite left \( C \)-module category is the same thing as a finite abelian category \( M \) endowed with a \( k \)-linear strong monoidal functor \( C \to \text{Rex}(M, M) \).

**Example 2.8.** Every finite abelian category \( M \) over \( k \) is naturally a finite module category over \( \text{mod-}k \) by the action \( \bullet \) determined by \( \text{Hom}_M(V \bullet M, M') \sim \text{Hom}_k(V, \text{Hom}_M(M, M')) \) \( (V \in \text{mod-}k, M, M' \in M) \). Namely, the action \( \bullet \) is defined so that the internal Hom functor coincides with the usual Hom functor. We note that every \( k \)-linear functor between finite abelian categories is a \( \text{mod-}k \)-module functor.

**Example 2.9.** Let \( A \) be an algebra in a finite tensor category \( C \). Then the category \( C_A \) of right \( A \)-modules in \( C \) is a finite left \( C \)-module category. The internal Hom functor for \( C_A \), denoted by \( \text{Hom}_A \), is given by
\[
\text{Hom}_A(M, N) = (M \otimes_A N)^* \quad (M, N \in C_A),
\]
where \( \otimes_A \) is the tensor product over \( A \); see [Ost03].

Let \( M \) be a finite \( C \)-module category. Then \( M \) is closed by Lemma 2.5. We also note that the action \( \otimes : C \times M \to M \) is exact in the second variable. Indeed, for each \( X \in C \), the functors \( X^* \otimes (-) \) and \( *X \otimes (-) \) are a left adjoint and a right adjoint of \( X \otimes (-) \), respectively.

We fix an object \( M \in M \) and consider the functor \( Y_M := \text{Hom}(M, -) \) from \( M \) to \( C \). We say that \( M \) is \( C \)-projective if \( Y_M \) is exact, and call \( M \) a \( C \)-generator if \( Y_M \) is faithful. The object \( A := \text{Hom}(M, M) \) is an algebra in \( C \) by the composition, and the functor \( Y_M \) induces a functor
\[
K_M : M \to C_A, \quad M' \mapsto \text{Hom}(M, M') \quad (M \in C),
\]
where the action of \( A \) on \( \text{Hom}(M, M') \) is given by the composition. The functor \( K_M \) is in fact the comparison functor of (2.1). Applying the Barr-Beck monadicity theorem [ML98, VI.7], we obtain:

**Theorem 2.10 ([EGNO09], [DSS14]).** The functor \( K_M \) is an equivalence of left \( C \)-module categories if and only if \( M \) is a \( C \)-projective \( C \)-generator.

In view of this theorem, it is important to study the properties of \( C \)-projective objects and \( C \)-generators. The following two lemmas are due to [DSS14]:

**Lemma 2.11.** For an object \( M \in M \), the following assertions are equivalent:

1. \( M \) is a \( C \)-generator.
(2) The component $\varepsilon_M : \text{Hom}(M, M') \otimes M \to M'$ of the counit of the adjunction is an epimorphism for all $M' \in \mathcal{M}$.

(3) Every $M' \in \mathcal{M}$ is a quotient of an object of the form $X \otimes M$, $X \in \mathcal{C}$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is well-known; see e.g., [ML98 IV.3]. It is obvious that (2) implies (3). To show that (3) implies (1), we suppose that $f : M' \to M''$ is a morphism in $\mathcal{M}$ such that $\text{Hom}(M, f) = 0$. By the definition of the internal Hom functor, there is a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_M(X, \text{Hom}(M, M')) & \xrightarrow{2.1} & \text{Hom}_M(X \otimes M, M') \\
\text{Hom}_C(X, \text{Hom}(M, f)) \downarrow & & \downarrow \text{Hom}_M(X \otimes M, f) \\
\text{Hom}_M(X, \text{Hom}(M, M'')) & \xrightarrow{2.1} & \text{Hom}_M(X \otimes M, M'')
\end{array}
\]

for all $X \in \mathcal{C}$. By the assumption, there exists an object $X \in \mathcal{C}$ and an epimorphism $p : X \otimes M \to M'$. Chasing $p$ around the above diagram, we have $f \circ p = 0$. Since $p$ is epic, we have $f = 0$. Hence (1) follows. \qed

Lemma 2.12. For an object $M \in \mathcal{M}$, the following assertions are equivalent:

1. $M$ is $\mathcal{C}$-projective.
2. $P \otimes M$ is projective for every projective object $P \in \mathcal{C}$.
3. $P \otimes M$ is projective for a projective generator $P \in \mathcal{C}$.

Proof. If $M$ is $\mathcal{C}$-projective and $P \in \mathcal{C}$ is projective, then

\[
\text{Hom}(P \otimes M, -) \cong \text{Hom}_C(P, -) \circ \text{Hom}(M, -)
\]

is an exact functor as the composition of exact functors. Hence (1) implies (2). The implication (2) $\Rightarrow$ (3) is obvious. Now we show that (3) implies (1). Let $P \in \mathcal{C}$ be a projective generator. Then, since $\text{Hom}_C(P, -)$ reflects exact sequences, the exactness of $\text{Hom}(M, -)$ follows from (2.2). Hence $M$ is $\mathcal{C}$-projective. \qed

Hence, an exact left $\mathcal{C}$-module category [EO04 Definition 3.1] is a finite left $\mathcal{C}$-module category whose every object is $\mathcal{C}$-projective.

Lemma 2.13. A projective object of $\mathcal{M}$ is $\mathcal{C}$-projective.

Proof. Let $Q \in \mathcal{M}$ be a projective object. Then $X \otimes Q$ is also projective for all object $X \in \mathcal{C}$, since $\text{Hom}_M(X \otimes Q, -) \cong \text{Hom}_M(Q, -) \circ (\text{Hom}(X, -))$ is exact as the composition of exact functors. Hence, by Lemma 2.12, $Q$ is $\mathcal{C}$-projective. \qed

By the above three lemmas, $\mathcal{M}$ has a $\mathcal{C}$-projective $\mathcal{C}$-generator and therefore it is equivalent to $\mathcal{C}_A$ for some algebra $A$ in $\mathcal{C}$ as a $\mathcal{C}$-module category [DSS14]. As a consequence, the action $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is exact in each variable (although we only assume that the action is right exact in the second variable).

2.7. Eilenberg-Watts type theorem. Let $A$ and $B$ be algebras in a finite tensor category $\mathcal{C}$ over a field $k$. Since the tensor product of $\mathcal{C}$ preserves coequalizers, every $A$-$B$-bimodule $M \in \mathcal{C}$ defines a left $\mathcal{C}$-module functor

\[
(2.3) \quad (-) \otimes_A M : \mathcal{C}_A \to \mathcal{C}_B.
\]

A right adjoint of this functor is given as follows: By the definition of the internal Hom functor, there is a canonical isomorphism

\[
\text{Hom}_B(A \otimes M, M) \cong \text{Hom}_C(A, \text{Hom}_B(M, M)).
\]
Let $\rho : A \to \text{Hom}_B(M, M)$ be the morphism corresponding to the action $\triangleright_M$ under the above isomorphism. Since $\rho$ is a morphism of algebras, the algebra $A$ acts on any object of the form $\text{Hom}_B(M, M')$ through $\rho$. Hence we get a functor

$$\text{Hom}_B(M, -) : C_B \to C_A$$

and this functor is right adjoint to (2.3). See also Pareigis [Par77a, Par77b], where the same claim is proved in a more general setting.

For finite left $C$-module categories $M$ and $N$, we denote by $\text{REX}_C(M, N)$ the category of $k$-linear right exact $C$-module functors from $M$ to $N$. The following generalization of Lemma 2.5 is also found in [Par77b]:

**Theorem 2.14.** The following functor is an equivalence of categories:

$$A C_B \xrightarrow{\cong} \text{REX}_C(C_A, C_B), \ M \mapsto (-) \otimes_A M. $$

### 3. Modular object

#### 3.1. Ends and coends.

The aim of this section is to introduce a categorical analogue of the modular function, which we call the modular object. We first recall from [ML98] the notion of ends and coends. Let $C$ and $V$ be categories, and let $S, T : C^{op} \times C \to V$ be functors. A dinatural transformation $\xi : P \Rightarrow Q$ is a family

$$\xi = \{\xi_X : S(X, X) \to T(X, X)\}_{X \in C}$$

of morphisms in $V$ such that

$$T(id_X, f) \circ \xi_X \circ S(f, id_X) = T(f, id_Y) \circ \xi_Y \circ S(id_Y, f)$$

for all morphism $f : X \to Y$ in $C$. We now regard an object $X \in V$ as the constant functor from $C^{op} \times C$ to $V$ sending all objects to $X$. Then an end of $S$ is a pair $(E, p)$ consisting of an object $E \in V$ and a dinatural transformation $p : S \Rightarrow E$ satisfying a certain universal property. Dually, a coend of $T$ is a pair $(C, i)$ consisting of an object $C \in V$ and a ‘universal’ dinatural transformation $i : T \Rightarrow C$. The end of $S$ and the coend of $T$ are expressed as

$$E = \int_{X \in C} S(X, X) \quad \text{and} \quad C = \int_{X \in C} T(X, X),$$

respectively; see [ML98] for more details.

We now suppose that a coend $(C, i)$ of the functor $T$ exists. If $C$ has an equivalence $(-)^* : C \to C^{op}$ of categories (e.g., $C$ is a rigid monoidal category), then the pair $(T, i^*)$, where $i^*_X = p_{X^*}$, is a coend of the functor $(X, Y) \mapsto T(Y^*, X^*)$. This result can be expressed symbolically as follows:

$$\int_{X \in C} S(X, X)^* = \int_{X \in C} S(X^*, X^*).$$

If $V$ has an equivalence $(-)^* : V \to V^{op}$, then the pair $(C^*, p)$, where $p_X = i_X^*$, is an end of the functor $(X, Y) \mapsto T(Y, X)^*$. Symbolically, we have

$$\left(\int_{X \in C} S(X, X)\right)^* = \int_{X \in C} S(X, X)^*.$$
3.2. The Deligne tensor product of abelian categories. In what follows, we will consider functors between categories whose objects are functors. Let, for example, $E$ be the category of endofunctors on a monoidal category $C$. If $\Psi : E \rightarrow E$ is a functor, then the expression “$\Psi((-) \otimes X)(V)$” for $V, X \in C$ makes sense. However, such a notation is open to misunderstanding. To avoid confusion, we adopt the following notation:

**Notation 3.1.** For a functor $\Psi$ whose source is a category consisting of functors, we usually write $\Psi[F]$ instead of $\Psi(F)$.

The **Deligne tensor product** of $k$-linear abelian categories $\mathcal{M}$ and $\mathcal{N}$, denoted by $\mathcal{M} \boxtimes \mathcal{N}$, is a $k$-linear abelian category endowed with a functor $\boxtimes : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \boxtimes \mathcal{N}$ such that $\boxtimes$ is $k$-linear and right exact in each variable and the functor

$$\text{REX}(\mathcal{M} \boxtimes \mathcal{N}, A) \rightarrow \text{REX}_2(\mathcal{M}, \mathcal{N}; A), \quad T \mapsto T \circ \boxtimes$$

is an equivalence for any $k$-linear abelian category $A$, where $\text{REX}_2(\mathcal{M}, \mathcal{N}; A)$ is the category of functors from $\mathcal{M} \times \mathcal{N}$ to $A$ being $k$-linear and right exact in each variable. If $A$ and $B$ are finite-dimensional algebras over $k$, then

$$(\text{mod-}A) \boxtimes (\text{mod-}B) = \text{mod-}(A \otimes_k B)$$

with $M \boxtimes N = M \otimes_k N$ [Del90]. The following lemma can be proved by using this realization of the Deligne tensor product:

**Lemma 3.2.** Let $\mathcal{M}$ and $\mathcal{N}$ be finite abelian categories over $k$. Then $\mathcal{M} \boxtimes \mathcal{N}$ is a finite abelian category over $k$, and the functor $\boxtimes : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \boxtimes \mathcal{N}$ is exact in each variable. For $M, M' \in \mathcal{M}$ and $N, N' \in \mathcal{N}$, there is an isomorphism

$$\text{Hom}_{\mathcal{M} \boxtimes \mathcal{N}}(M \boxtimes M', N \boxtimes N') \cong \text{Hom}_{\mathcal{M}}(M, M') \otimes_k \text{Hom}_{\mathcal{N}}(N, N').$$

We also note the following lemma (cf. [Shi14]):

**Lemma 3.3.** For finite abelian categories $\mathcal{M}$ and $\mathcal{N}$ over $k$, the following functor $\Phi$ is an equivalence of $k$-linear categories:

$$\Phi : \mathcal{M}^\text{op} \boxtimes \mathcal{N} \rightarrow \text{REX}(\mathcal{M}, \mathcal{N}), \quad M \boxtimes N \mapsto \text{Hom}_{\mathcal{M}}(-, M^* \bullet N),$$

where $\bullet$ is given in Example 2.8. A quasi-inverse of $\Phi$ is given by

$$(3.3) \quad \overline{\Phi} : \text{REX}(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{M}^\text{op} \boxtimes \mathcal{N}, \quad F \mapsto \int_{X \in \mathcal{M}} X \boxtimes F(X).$$

**Proof.** We may assume that $\mathcal{M} = \text{mod-}A$ and $\mathcal{N} = \text{mod-}B$ for some finite-dimensional algebras $A$ and $B$ over $k$. Then the following functor is an equivalence:

$$\mathcal{M}^\text{op} \boxtimes \mathcal{N} \rightarrow \text{A-mod-B}, \quad M \boxtimes N \mapsto M^* \otimes_k N,$$

where $\text{A-mod-B}$ is the category of finite-dimensional $A$-$B$-bimodules. We also have an equivalence

$$\text{A-mod-B} \rightarrow \text{REX}(\mathcal{M}, \mathcal{N}), \quad X \mapsto (-) \otimes_A X$$

by Lemma 2.5. The functor $\Phi$ is an equivalence, since it is obtained by composing the above two equivalences. Now let $\overline{\Phi}$ be a quasi-inverse of $\Phi$. Then we have

$$\text{Hom}_{\mathcal{M}^\text{op} \boxtimes \mathcal{N}}(M \boxtimes N, \overline{\Phi}[F]) \cong \text{Nat}(\Phi(M \boxtimes N), F)$$

$$\cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{N}}(\text{Hom}_{\mathcal{M}}(X, M)^* \bullet N, F(X))$$

$$\cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(X, M) \otimes \text{Hom}_{\mathcal{N}}(N, F(X))$$

$$\cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}^\text{op} \boxtimes \mathcal{N}}(M \boxtimes N, X \boxtimes F(X))$$
for $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $F \in \text{REX}(\mathcal{M}, \mathcal{N})$ in the category $\mathcal{V}$ of all vector spaces over $k$. Since every object of $\mathcal{M}^{\text{op}} \boxtimes \mathcal{N}$ is a colimit of objects of the form $M \boxtimes N$, the above computation implies that $\Phi(F)$ represents the functor

$$\mathcal{M}^{\text{op}} \boxtimes \mathcal{N} \to \mathcal{V}^{\text{op}}, \quad L \mapsto \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}^{\text{op}} \boxtimes \mathcal{N}}(L, X \boxtimes F(X)).$$

Hence the end in (3.3) indeed exists and is isomorphic to $\Phi[F]$. $\square$

### 3.3. Tensor product of module categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be finite tensor categories over a field $k$. We consider the functor

$$(\mathcal{C} \boxtimes \mathcal{D} \times \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}, \quad (X, Y, X', Y') \mapsto (X \otimes X') \boxtimes (Y \otimes Y')).$$

This functor is $k$-linear and right exact in each variable. Hence, by the universality of the Deligne tensor product, it induces a $k$-linear right exact functor

$$(3.4) \quad \mathcal{C} \boxtimes \mathcal{D} \boxtimes \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}, \quad X \boxtimes Y \boxtimes X' \boxtimes Y' \mapsto (X \otimes X') \boxtimes (Y \otimes Y').$$

Now we define $\otimes : (\mathcal{C} \boxtimes \mathcal{D}) \times (\mathcal{C} \boxtimes \mathcal{D}) \to \mathcal{C} \boxtimes \mathcal{D}$ to be the composition:

$$(3.5) \quad \otimes : (\mathcal{C} \boxtimes \mathcal{D}) \times (\mathcal{C} \boxtimes \mathcal{D}) \to \mathcal{C} \boxtimes \mathcal{D} \boxtimes \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}.$$

The category $\mathcal{C} \boxtimes \mathcal{D}$ is a monoidal category with this tensor product and with unit object $1 \boxtimes 1$. Moreover, since (3.5) is right exact in each variable, $\mathcal{C} \boxtimes \mathcal{D}$ is a closed monoidal category. However, it is not rigid in general. We note that $\mathcal{C} \boxtimes \mathcal{D}$ is rigid (and therefore a finite tensor category) if the base field $k$ is perfect [Del90].

Now let $\mathcal{M}$ and $\mathcal{N}$ be finite module categories over $\mathcal{D}$ and $\mathcal{C}$, respectively. In a similar way as above, we can define an action of $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$ on $\mathcal{M}^{\text{op}} \boxtimes \mathcal{N}$ by

$$((X \boxtimes Y) \otimes (M \boxtimes N)) = (\ast_Y \otimes M) \boxtimes (X \otimes N)$$

for $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $M \in \mathcal{M}$ and $N \in \mathcal{N}$. By definition, this action is right exact in each variable. We also define an action of $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$ on $\text{REX}(\mathcal{M}, \mathcal{N})$ by

$$((X \boxtimes Y) \otimes F)(M) = X \otimes F(Y \otimes M)$$

for $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $M \in \mathcal{M}$ and $F \in \text{REX}(\mathcal{M}, \mathcal{N})$.

### Lemma 3.4.

The equivalence $\Phi : \mathcal{M}^{\text{op}} \boxtimes \mathcal{N} \to \text{REX}(\mathcal{M}, \mathcal{N})$ given in Lemma 3.3 is an equivalence of $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$-module categories.

**Proof.** For $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $M, M' \in \mathcal{M}^{\text{op}}$ and $N \in \mathcal{N}$, we have

$$\Phi((X \boxtimes Y) \otimes (M \boxtimes N))(M') = \text{Hom}_{\mathcal{M}}(M', \ast_Y \otimes M) \ast (X \otimes N)$$

$$\cong X \otimes (\text{Hom}_{\mathcal{M}}(Y \otimes M', M) \ast N)$$

$$= ((X \boxtimes Y) \otimes \Phi(M \boxtimes N))(M').$$

Since both sides are $k$-linear and right exact in $X$, $Y$, $M$ and $N$, we get

$$\Phi(Z \otimes L)(M') \cong (Z \otimes \Phi(L))(M')$$

for all $Z \in \mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$, $L \in \mathcal{M}^{\text{op}} \boxtimes \mathcal{N}$, and $M' \in \mathcal{M}$. Hence $\Phi$ is a strong $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$-module functor. Since $\Phi$ is an equivalence of categories, the result follows. $\square$

The following technical lemma will be used in Section 5.
Lemma 3.5. Let \(G : \mathcal{M}_2 \to \mathcal{M}_1\) and \(E : \mathcal{N}_1 \to \mathcal{N}_2\) be \(k\)-linear right exact module functors between finite module categories over \(\mathcal{D}\) and \(\mathcal{C}\), respectively. Then

\[
\text{Rex}(G, E) : \text{Rex}((\mathcal{M}_1, \mathcal{N}_1)) \to \text{Rex}((\mathcal{M}_2, \mathcal{N}_2)), \quad F \mapsto E \circ F \circ G
\]

is a \(k\)-linear right exact strong \(\mathcal{C} \otimes \mathcal{D}^{\text{env}}\)-module functor.

Proof. Let \(R : \mathcal{M}_1 \to \mathcal{M}_2\) be a right adjoint functor of \(G\). Since \(R\) is \(k\)-linear and left exact, the functor \(R^\text{op} : \mathcal{M}_1^{\text{op}} \to \mathcal{M}_2^{\text{op}}\) induced by \(R\) is \(k\)-linear and right exact. Hence we have a \(k\)-linear right exact functor

\[
H := R^\text{op} \otimes E : \mathcal{M}_1^{\text{op}} \otimes \mathcal{N}_1 \to \mathcal{M}_2^{\text{op}} \otimes \mathcal{N}_2.
\]

The functor \(H\) is obviously a strong \(\mathcal{C} \otimes \mathcal{D}^{\text{env}}\)-module functor. Now we compute

\[
\left(\text{Rex}(G, E) \left[\Phi(M_1 \otimes N_1)\right]\right)(M_2) \cong \text{Hom}_{\mathcal{M}_1}(G(M_2), M_1)^* \ast E(N_1)
\]

\[
\cong \text{Hom}_{\mathcal{M}_2}(M_2, R(M_1))^* \ast E(N_1)
\]

\[
= \Phi(H(M_1 \otimes N_1))(M_2)
\]

for \(M_1 \in \mathcal{M}_1^{\text{op}}, M_2 \in \mathcal{M}_2\) and \(N_1 \in \mathcal{N}_1\), where \(\Phi\)'s are the equivalences given in Lemma 3.3. Thus, in a sense, \(\text{Rex}(G, E)\) is 'conjugate' of \(H\) by \(\Phi\). The functor \(\text{Rex}(G, E)\) has the desired properties since \(H\) does. \(\square\)

3.4. Monadic description of module functors. Let \(\mathcal{C}\) be a finite tensor category over \(k\). We define \(A \in \mathcal{C}^{\text{env}}\) to be the coend of \((X, Y) \mapsto X^* \otimes Y\). Namely,

\[
A = \int^{\mathcal{X} \in \mathcal{C}} X^* \otimes X \in \mathcal{C}^{\text{env}}
\]

(see [KL01] §5 or [Shi14] for the existence of such a coend). Let \(i_X : X^* \otimes X \to A\) be the universal dinatural transformation for the coend. By the Fubini theorem for coends, \(A \otimes A\) is a coend of \((X_1, X_2, Y_1, Y_2) \mapsto (X_1^* \otimes Y_1) \otimes (X_2^* \otimes Y_2)\). Thus there exists a unique morphism \(m\) such that the diagram

\[
\begin{array}{ccc}
(X^* \otimes X) \otimes (Y^* \otimes Y) & \overset{i_X \otimes i_Y}{\longrightarrow} & A \otimes A \\
\parallel & & \parallel \\
(Y \otimes X)^* \otimes (Y \otimes X) & \overset{i_Y \otimes i_X}{\longrightarrow} & A
\end{array}
\]

commutes for all \(X, Y \in \mathcal{C}\). We also define \(u : \mathbb{1} \otimes \mathbb{1} \to A\) by \(u = i_1\). The proof of the following lemma is straightforward:

Lemma 3.6. The triple \((A, m, u)\) is an algebra in \(\mathcal{C}^{\text{env}}\).

Let \(\mathcal{M}\) and \(\mathcal{N}\) be finite left \(\mathcal{C}\)-module categories. As we have seen, \(\text{Rex}(\mathcal{M}, \mathcal{N})\) is a left \(\mathcal{C}^{\text{env}}\)-module category by the action given by (3.6). We now consider the category of left \(A\)-modules in \(\text{Rex}(\mathcal{M}, \mathcal{N})\) in the sense of Definition 2.4.

Lemma 3.7. \(A_{\text{Rex}}(\mathcal{M}, \mathcal{N}) \cong \text{Rex}_A(\mathcal{M}, \mathcal{N})\).

Proof. Day and Street [DS07] showed that the functor \(Z(V) = \int_{\mathcal{X} \in \mathcal{C}} X^* \otimes V \otimes X\) \((V \in \mathcal{C})\) has a structure of a monad such that the category of \(Z\)-modules can be identified with the monoidal center \(Z(\mathcal{C})\) of \(\mathcal{C}\). The proof is essentially same as their proof of this fact: For \(F \in \text{Rex}(\mathcal{M}, \mathcal{N})\) and \(M \in \mathcal{M}\), we have

\[
(A \otimes F)(M) = \int_{\mathcal{X} \in \mathcal{C}} X^* \otimes F(X \otimes M).
\]
Hence a morphism $A \otimes F \to F$ in $\text{REX}(M, N)$ is the same thing as a family
\[
\rho = \{ \rho_{M,X} : X^* \otimes F(X \otimes M) \to F(M) \}_{M \in M, X \in \mathcal{C}}
\]
of morphisms in $N$ which is natural in $M$ and dinatural in $X$. Since $X^* \otimes (-)$ is
left adjoint to $X \otimes (-)$, such a family corresponds to a family
\[
\xi = \{ \xi_{M,X} : F(X \otimes M) \to X \otimes F(M) \}_{M \in M, X \in \mathcal{C}}
\]
of morphisms natural in $M$ and $X$. We see that the family $\rho$ makes $F$ an $A$-module
if and only if the corresponding natural transformation $\xi$ makes $F$ a colax $\mathcal{C}$-module
functor. Hence, by Lemma 2.1, we obtain a bijection between the objects of the
two categories. This gives rise to an isomorphism of categories. \hfill $\square$

The above lemma implies:

**Corollary 3.8.** The forgetful functor $\text{REX}(\mathcal{M}, N) \to \text{REX}(\mathcal{M}, N)$ is monadic.

Now we set $\text{REX}(\mathcal{C}) := \text{REX}(\mathcal{C}, \mathcal{C})$. We consider the functor
\[
(3.7) \quad \Phi_\mathcal{C} : \mathcal{C}^{\text{env}} \to \text{REX}(\mathcal{C}), \quad V \boxtimes W \mapsto \text{Hom}_\mathcal{C}(-, *W)^* \bullet V \quad (V, W \in \mathcal{C})
\]

obtained by composing the duality and the equivalence given in Lemma 3.3. The
equivalence $\Phi_\mathcal{C}$ is in fact an equivalence of $\mathcal{C}^{\text{env}}$-module categories.

**Corollary 3.9.** The following functor is an equivalence of categories:
\[
\Psi_\mathcal{C} : A(\mathcal{C}^{\text{env}}) \to \mathcal{C}, \quad M \mapsto \Phi_\mathcal{C}(M)(1).
\]

**Proof.** Since $\Phi_\mathcal{C}$ is an equivalence of left $\mathcal{C}^{\text{env}}$-module categories, it induces an
equivalence between the categories of $A$-modules. The functor $\Psi_\mathcal{C}$ is an equivalence, since
it is obtained by the following composition:
\[
A(\mathcal{C}^{\text{env}}) \xrightarrow{\cong} A \text{REX}(\mathcal{C}) \xrightarrow{\cong} \text{REX}_\mathcal{C}(\mathcal{C}) \xrightarrow{\cong} \mathcal{C}. \quad \square
\]

**Remark 3.10.** Suppose that $\mathcal{C}^{\text{env}}$ is rigid. Then $\mathcal{C}$ is a finite $\mathcal{C}^{\text{env}}$-module category
by the action given by $(X \boxtimes Y) \otimes V = X \otimes V \otimes Y$ $(V, X, Y \in \mathcal{C})$. Let $\text{Hom}$
denote the internal Hom functor for the $\mathcal{C}^{\text{env}}$-module category $\mathcal{C}$. Then $\text{Hom}(1, 1)$
is isomorphic to the algebra $A$ of Lemma 3.6. Hence, applying Theorem 2.14 to
$M = 1$, we obtain an equivalence of left $\mathcal{C}^{\text{env}}$-module categories
\[
(3.8) \quad K : \mathcal{C} \to (\mathcal{C}^{\text{env}})_A, \quad V \mapsto \text{Hom}(1, V) \cong (V \boxtimes 1) \otimes A
\]
(Etingof-Nikshych-Ostrik [ENO04 Proposition 2.3]). Their result can be thought of as a categorical analogue of the fundamental theorem of Hopf bimodules. There is
the following relation between $K$ and $\Psi_\mathcal{C}$ of Corollary 3.9
\[
(3.9) \quad \Psi_\mathcal{C}(^*K(V^*)) \cong V \quad (V \in \mathcal{C}).
\]
To see this, we note that a quasi-inverse of $\Phi_\mathcal{C} : \mathcal{C}^{\text{env}} \to \text{REX}(\mathcal{C})$ is given by
\[
(3.10) \quad \Phi_\mathcal{C} : \text{REX}(\mathcal{C}) \to \mathcal{C}^{\text{env}}, \quad F \mapsto \int_{X \in \mathcal{C}} F(X) \boxtimes X^*.
\]
(cf. Lemma 3.3). By (3.2) and (3.10), we have isomorphisms
\[
^*A \otimes (V \boxtimes 1) \cong \int_X (^*X^* \boxtimes X) \otimes (V \boxtimes 1) \cong \int_X (X \otimes V) \boxtimes X^* \cong \Phi_\mathcal{C}(-) \otimes V
\]
in $\mathcal{C}^{\text{env}}$ (we do not mention their $A$-module structures, since $\Psi_\mathcal{C}$ factors through the
functor forgetting the $A$-module structure). Hence,
\[
\Psi_\mathcal{C}(^*K(V^*)) \cong \Psi_\mathcal{C}( ^*A \otimes (V \boxtimes 1)) \cong 1 \otimes V = V.
3.5. Modular object. Let $C$ be a finite tensor category over a field $k$. We consider the (right) Cayley functor defined by

$$\Upsilon_C : C \to \text{REX}(C), \quad X \mapsto (-) \otimes X \quad (X \in C).$$

If we identify $C$ with $\text{REX}_C(C)$ by Theorem 2.14, the Cayley functor $\Upsilon_C$ corresponds to the forgetful functor from $\text{REX}_C(C)$. Thus, by Corollary 3.8 it has a left adjoint functor, say $\Upsilon_C^\ast$.

Definition 3.11. The modular object $\alpha_C \in C$ is defined to be the image of

$$J_C := \text{Hom}_C(-(\bullet \otimes \cdot)) \in \text{REX}(C)$$

under a left adjoint of the Cayley functor. Namely,

$$\alpha_C := \Upsilon_C^\ast[J_C].$$

We will show that the modular object defined here is isomorphic to the dual of the distinguished invertible object $D \in C$ introduced by Etingof, Nikshych and Ostrik in [ENO04] provided that $C$ is rigid; see Lemma 3.12 below. Our definition is useful in later sections and makes sense even in the case where $C$ is rigid, however, we do not know whether $\alpha_C$ is invertible in general.

To study the modular object, we describe a left adjoint of $\Upsilon_C$ in detail. Let $\Phi_C$ be the equivalence given by (3.7). Then the following diagram commutes:

$$\begin{array}{ccccc}
A(C) & \xrightarrow{\Phi_C} & A \text{REX}(C) & \cong & \text{REX}_C(C) \\
\downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{\Upsilon_C} & \text{REX}(C) & \cong & C
\end{array}$$

where the unlabeled vertical arrows are the forgetful functors. By considering left adjoints of functors in the above diagram, we have

$$\Upsilon_C^\ast[F] = \int_X X^* \otimes F(X) \cong \int_X X \otimes F(X^*)$$

for $F \in \text{REX}(C)$. Since the composition along the top row is the equivalence $\Psi_C$ given in Corollary 3.9 we also have

$$\Upsilon_C^\ast[\Phi_C(M)] \cong \Psi_C(A \otimes M)$$

for $M \in C$. For a while, we suppose that $C$ is rigid. Let $K : C \to (C)_{\text{env}}$ be the equivalence given by (3.8). Then the distinguished invertible object $[\text{ENO04}]$ is defined to be the unique (up to isomorphism) object $D \in C$ such that $K(D) \cong A^\ast$. We note that the object $D$ is invertible (see [ENO04] for the case where $k$ is algebraically closed and $1 \in C$ is simple and [Shi14] for the general case). We compute

$$D^\ast \cong \Psi_C(\text{env}(D)) \cong \Psi_C(A) \cong \Upsilon_C^\ast[\Phi_C(1 \otimes 1)] = \Upsilon_C^\ast[J_C] = \alpha_C$$

by (3.10), (3.12), (3.13) and (3.15). Summarizing the above argument, we conclude:

Lemma 3.12. If the monoidal category $C$ is rigid (e.g., the case where the base field $k$ is perfect), then $\alpha_C$ is isomorphic to the dual of the distinguished invertible object of $[\text{ENO04}]$. In particular, if this is the case, $\alpha_C$ is invertible.
We go back to the general situation. By (3.12), (3.13) and (3.14), we have the following formula of the modular object:

**Theorem 3.13.** For a finite tensor category $\mathcal{C}$ over $k$, we have

$$\alpha_{\mathcal{C}} \cong \int_{X \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(\mathbb{1}, X)^* \bullet X.$$

Since the above formula does not involve the tensor product, we have:

**Corollary 3.14.** Let $F : \mathcal{C} \to \mathcal{D}$ be a $k$-linear equivalence between finite tensor categories $\mathcal{C}$ and $\mathcal{D}$ over $k$ such that $F(\mathbb{1}) \cong \mathbb{1}$. Then we have $F(\alpha_{\mathcal{C}}) \cong \alpha_{\mathcal{D}}$.

This corollary may be useful to find the modular object. For example:

**Corollary 3.15.** For a finite tensor category $\mathcal{C}$ over $k$, we have $\alpha_{\mathcal{C}^{\text{rev}}} \cong \alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{C}^{\text{op}}} \cong \alpha_{\mathcal{C}^{\text{op}}}$.

**Proof.** Apply Corollary 3.14 to $\text{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\text{rev}}$ and $(-)^* : \mathcal{C} \to \mathcal{C}^{\text{op}}$. □

Further properties of the modular object can be found in [Shi14].

### 3.6. Radford $S^4$-formula

Let $\mathcal{C}$ be a finite tensor category over $k$. The category $\text{REX}(\mathcal{C})$ is a right $\text{C}^{\text{env}}$-module category with the action determined by

$$F \otimes (X \boxtimes Y) = F(- \otimes **Y) \otimes X \quad (F \in \text{REX}(\mathcal{C}), X, Y \in \mathcal{C}).$$

The equivalence $\Phi_{\mathcal{C}}$ given by (3.7) is also an equivalence of right $\text{C}^{\text{env}}$-module categories. Thus we can consider the category $\text{REX}(\mathcal{C})_A$ of right $A$-modules in $\text{REX}(\mathcal{C})$ in the sense of Definition 2.4. To describe this category, we introduce the following notation: Given a right $\mathcal{C}$-module category $\mathcal{M}$ and a strong monoidal functor $T : \mathcal{C} \to \mathcal{C}$, we denote by $\mathcal{M} \langle T \rangle$ the category $\mathcal{M}$ with the action twisted by $T$.

**Lemma 3.16.** $\text{REX}(\mathcal{C})_A$ is isomorphic to the category of $k$-linear right $\mathcal{C}$-module functors from $\mathcal{C}_{(S^2)}$ to $\mathcal{C}_{(S^2)}$, where $S = (-)^*$ is the left duality functor on $\mathcal{C}$.

**Proof.** We first note that a $k$-linear right $\mathcal{C}$-module functor $\mathcal{C}_{(S^2)} \to \mathcal{C}_{(S^2)}$ is automatically exact. Indeed, if $F$ is such a functor, then

$$F(X) = F(\mathbb{1} \otimes X**) \cong F(\mathbb{1}) \otimes X** = F(\mathbb{1}) \otimes X^{****} \quad (X \in \mathcal{C}).$$

The proof is almost the same as Lemma 3.7 (and thus it is essentially same as the argument due to Day and Street). For $F \in \text{REX}(\mathcal{C})$, a morphism $F \otimes A \to F$ in $\text{REX}(\mathcal{C})$ is the same thing as a family

$$\rho = \{ \rho_{V,X} : F(V \otimes **X) \otimes X^* \to F(V) \}_{V,X \in \mathcal{C}}$$

of morphisms in $\mathcal{C}$ which is natural in $V$ and dinatural in $X$, and hence corresponds to a natural transformation

$$\xi = \{ \xi_{V,X} : F(V \otimes **X) \to F(V) \otimes X** \}_{V,X \in \mathcal{C}}.$$

The morphism $\rho$ makes $F$ a right $A$-module if and only if $\xi$ makes $F$ a colax $\mathcal{C}$-module functor from $\mathcal{C}_{(S^2)}$ to $\mathcal{C}_{(S^2)}$. This correspondence gives rise to an isomorphism of the two categories. □

Now we prove the following categorical analogue of the Radford $S^4$-formula:
Theorem 3.17. Set $\alpha = \alpha_C$ for simplicity. There is a natural isomorphism

$$\gamma_X : X \otimes \alpha \to \alpha \otimes X^{****} \quad (X \in \mathcal{C})$$

such that $\gamma_1 = \text{id}_\alpha$ and $\gamma_{X \otimes Y} = (\gamma_X \otimes \text{id}_Y^{****}) \circ (\text{id}_X \otimes \gamma_Y)$ for all $X, Y \in \mathcal{C}$.

Proof. Let $\Psi : \mathcal{C} \to \mathcal{D}$ be the equivalence given in Corollary 3.9. By (3.15), we have $\alpha \cong \Psi_C(A)$. Applying the equivalence $\mathcal{C} \approx \mathcal{Rex}(\mathcal{C})$ of Theorems 2.13 to both sides, we obtain isomorphisms

$$(-) \otimes \alpha \cong (-) \otimes \Psi_C(A) \cong \Phi_C(A)$$

in $\mathcal{Rex}(\mathcal{C})$. Now we set $G = (-) \otimes \alpha$. Since $A \in \mathcal{C}^{\text{env}}$ is a right $A$-module in $\mathcal{C}^{\text{env}}$, the functor $G$ is a right $A$-module in $\mathcal{Rex}(\mathcal{C})$. Hence, by the previous lemma, there exists a natural isomorphism

$$\xi_{V,X} : G(V \otimes **X) \to G(V) \otimes X^{**} \quad (V, X \in \mathcal{C})$$

making $G$ a (colax) $\mathcal{C}$-module functor from $\mathcal{C}^{(S^{-2})}$ to $\mathcal{C}^{(S^2)}$. The natural isomorphism defined by $\gamma_X = \xi_{1,X^{**}}$ ($X \in \mathcal{C}$) has the desired property. \hfill \Box

By the above theorem, we obtain a monoidal natural transformation

$$\alpha^* \otimes X \otimes \alpha \xrightarrow{\text{id}_{\alpha^*} \otimes \gamma_X} \alpha^* \otimes \alpha \otimes X^{****} \xrightarrow{\text{ev}_\alpha \otimes \text{id}^{****}_X} X^{****} \quad (X \in \mathcal{C}),$$

which is invertible if and only if the modular object $\alpha \in \mathcal{C}$ is. Etingof, Nikshych and Ostrik [ENO04] proved this result under the assumption that $k$ is algebraically closed and $1 \in \mathcal{C}$ is simple. Douglas, Schommer-Pries and Snyder [DSS13] proved this result from the viewpoint of local topological field theory under the assumption that $k$ is perfect. Theorem 3.17 holds for arbitrary $k$, however, the invertibility of $\alpha$ is not proved in the general case. Our contribution in this section is, rather, a new framework to deal with the modular object.

4. The Relative Modular Object

4.1. Tensor functors. Let $k$ be a field. By a tensor category, we mean an abelian rigid monoidal category over $k$ (thus a finite tensor category is precisely a tensor category whose underlying category is a finite abelian category). For a $k$-linear functor $T : \mathcal{C} \to \mathcal{D}$ between tensor categories $\mathcal{C}$ and $\mathcal{D}$, we define

$$T^!(X) = T^*(X^*) \quad \text{and} \quad T(X) = T^!(X)^* \quad (X \in \mathcal{C}).$$

Now let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be $k$-linear functors. The following easy lemma is important:

Lemma 4.1. $F \dashv G$ implies $G^! \dashv F^!$ and $!G \dashv !F$.

Proof. Suppose that $F \dashv G$. Then, for $V \in \mathcal{D}$ and $X \in \mathcal{C}$, we have

$$\text{Hom}_\mathcal{C}(G^!(V), X) \cong \text{Hom}_\mathcal{C}(X^*, G(V^*)) \quad (by \ \text{the duality})$$

$$\cong \text{Hom}_\mathcal{D}(F(X^*), V^*) \quad (by \ \ F \dashv G)$$

$$\cong \text{Hom}_\mathcal{D}(V, F^!(X)) \quad (by \ \text{the duality}).$$

Hence $G^! \dashv F^!$. One can prove $!G \dashv !F$ in a similar way. \hfill \Box
By a tensor functor, we mean a $k$-linear strong monoidal functor between tensor categories. We note that a tensor functor $F : C \to D$ preserves the duality. Thus, we have $F^! \cong F \cong \overline{1} F$. If $F$ has a right adjoint $R$, then $R^! \dashv F$ and $\overline{1} R \dashv F$ by the above lemma. Similarly, if $L \dashv F$, then $F \dashv L^!$ and $F \dashv \overline{1} L$. Summarizing, we have the following result \cite[Lemma 3.5]{BV12}:

**Lemma 4.2.** A tensor functor $F$ has a left adjoint if and only if it has a right adjoint. If $L \dashv F \dashv R$, then $L \cong L \cong R^!$ and $L^! \cong R \cong L^!$.

4.2. **The relative modular object.** Now we introduce the notion of the relative modular object for a tensor functor with nice properties, which is a categorical analogue of the relative modular function. We first prove the following lemma:

**Lemma 4.3.** Let $F : C \to D$ be a tensor functor between tensor categories $C$ and $D$, and suppose that it has a left adjoint $L$ and a right adjoint $R$. Then the following assertions are equivalent:

1. $L$ has a left adjoint.
2. $R$ has a right adjoint.
3. There exists an object $\chi_F \in D$ such that $R \cong L(- \otimes \chi_F)$.

Such an object $\chi_F \in D$ is unique up to isomorphism if it exists. More precisely, if the above conditions hold, then $\chi_F$ is determined up to isomorphism by

$$\chi_F \cong {}^*G(1), \quad \text{where } F \dashv R \dashv G.$$  

If, moreover, $C$ and $D$ are finite tensor categories, then the above three conditions are equivalent to each of the following four conditions:

4. $L$ is exact.
5. $R$ is exact.
6. $F(P)$ is projective for every projective object $P \in C$.
7. $F(P)$ is projective for a projective generator $P \in C$.

**Proof.** The equivalence (1) $\iff$ (2) follows from Lemmas \ref{lem:4.1} and \ref{lem:4.2}. More precisely, if $L$ has a left adjoint $E$, then $E'$ is right adjoint to $R \cong L^!$. Similarly, if $R$ has a right adjoint $G$, then $G'$ is left adjoint to $L \cong R^!$.

To show (2) $\iff$ (3), we note that $D$ is a left $C$-module category by the action given by $X \otimes V = F(X) \otimes V$ ($X \in C, V \in D$). Suppose that $R$ has a right adjoint, say $G$, and set $\chi = {}^*G(1)$. Since $F : C \to D$ is a $C$-module functor, so is $R$ by Lemmas \ref{lem:2.1} and \ref{lem:2.2} and thus so is $G$. Hence, for $X \in C$, we have

$$G(X) = G(X \otimes 1) \cong X \otimes G(1) \cong F(X) \otimes \chi^*.$$  

By definition, $R$ is left adjoint to $G$. On the other hand,

$$\text{Hom}_C(V, G(X)) \cong \text{Hom}_D(V \otimes \chi, F(X)) \cong \text{Hom}_D(L(V \otimes \chi), X)$$

for $V \in D$ and $X \in C$. Thus $R \cong L(- \otimes \chi)$. Conversely, if such an object $\chi$ exists, then we see that $F(-) \otimes \chi^*$ is right adjoint to $R$ by a computation similar to the above. Hence we have proved (2) $\iff$ (3). Since $F$ preserves the unit, the last argument also shows \ref{lem:4.1}.

Now suppose that $C$ and $D$ are finite. Then (1) $\iff$ (4) and (2) $\iff$ (5) follow from Lemma \ref{lem:2.5}. To show that (5), (6) and (7) are equivalent, we make the category $D$ a left $C$-module category by $F$ as above. For $V, W \in D$ and $X \in C$, there is a natural isomorphism

$$\text{Hom}_D(X \otimes V, W) \cong \text{Hom}_C(X, R(W \otimes V^*)).$$  

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Namely, $\text{Hom}(V, W) := R(W \otimes V^*)$ is the internal Hom functor for the $C$-module category $D$. By Lemma 2.12 we see that each of the conditions (4), (5) and (6) is equivalent to that $D$ is an exact $C$-module category. \hfill \square

**Definition 4.4.** We call $\chi_F$ of Lemma 4.3 the **relative modular object** of $F$.

A functor is said to be **Frobenius** if it has a left adjoint $L$ and a right adjoint $R$ such that $L \cong R$. It is obvious from the definition that a tensor functor $F$ satisfying the equivalent conditions of Lemma 4.3 is a Frobenius functor if and only if $\chi_F \cong 1$.

**Example 4.5.** Let $C$ be a finite tensor category such that $C \cong C^{rev}$ is rigid, and let $\mathcal{Z}(C)$ denote its monoidal center. The main result of [Shi14] can be rephrased as follows: The relative modular object for the forgetful functor $U : \mathcal{Z}(C) \to C$ is given by $\chi_U = \alpha_C^*$ (where $\alpha_C$ is the modular object), and thus $U$ is a Frobenius functor if and only if $C$ is unimodular (see also Remark 3.10). This result is one of the motivations of this research.

The relative modular object has the following properties:

**Theorem 4.6.** Let $F : C \to D$ be a tensor functor with left adjoint $L$ and right adjoint $R$. Suppose that $F$ satisfies the equivalent conditions of Lemma 4.3. Then the relative modular object $\chi_F \in D$ of $F$ is an invertible object, and there are natural isomorphisms

\[(4.3) \quad L(V \otimes \chi) \cong R(V) \cong L(\chi \otimes V) \quad \text{and} \quad R(\chi^* \otimes V) \cong L(V) \cong R(V \otimes \chi^*)\]

for $\chi = \chi_F$ and $V \in D$. Each one of these natural isomorphisms characterizes the relative modular object: If $\chi \in D$ is an object such that one of the isomorphisms in (4.3) exists, then $\chi \cong \chi_F$.

**Proof.** Set $\chi = \chi_F$. We first show that $\chi^{**} \cong \chi$. For this purpose, let $E$ be a left adjoint of $L$ (which exists by the assumption). By Lemma 4.1 both $G := E^\dagger$ and $G' := E'$ are right adjoint to $R$. Hence, by (4.1), we have

\[(4.4) \quad \chi^{**} \cong *(G(1))^{**} \cong E(1) \cong *(G'(1)) \cong \chi.\]

For later use, we also note the following result:

\[(4.5) \quad \chi_F \cong E(1), \quad \text{where} \quad E \dashv L \dashv F.\]

Now we establish the isomorphisms in (4.3). The first one is trivial. The second one is obtained by applying Lemma 4.3 to the tensor functor $F^{rev} : C^{rev} \to D^{rev}$ induced by $F$ (note that (4.3) implies $\chi_{F^{rev}} = \chi_F$). The third one follows from Lemma 4.2 and the first one as follows:

\[L(V) \cong R(V^*) \cong *(V \otimes \chi) \cong *(\chi^* \otimes V) \cong R(\chi^* \otimes V) \quad (V \in D).\]

The fourth one is obtained from the second one in a similar way.

Next, we prove that $\chi = \chi_F$ is invertible. By (4.3), we have

\[L(\chi^* \otimes V \otimes \chi) \cong R(\chi^* \otimes V) \cong L(V) \cong R(V \otimes \chi^*) \cong L(\chi \otimes V \otimes \chi^*)\]

for $V \in D$. We consider right adjoint functors of them. By (4.4), we get

\[\chi \otimes F(X) \otimes \chi^* \cong F(X) \cong \chi^* \otimes F(X) \otimes \chi\]

for $X \in C$. The invertibility of $\chi$ follows by letting $X = 1$.

To see that each of the isomorphisms in (4.3) characterizes the relative modular object, consider adjoints of them and then use (4.1) or (4.5). For example, if $\mu \in D$
is an object such that $L \cong R(- \otimes \mu^*)$, then we have $E \cong F(-) \otimes \mu^*$ by taking left adjoints of both sides. Hence $\mu \cong \chi_F$ by (4.4). The other cases are proved analogously. □

We note that Balan [Bal14] proved similar results from the viewpoint of Hopf monads. Moreover, Balmer, Dell’Ambrogio and Sanders [BDS15] showed similar results in the quite general (but symmetric) setting of tensor-triangulated categories. Mentioning these results, we could say that the results of this subsection are only an instance of a very general theorem in the category theory.

In any case, our results are not sufficient as a generalization of the theorem of Fischman, Montgomery and Schneider mentioned in Introduction: Their result can be thought of as an explicit formula of the relative modular object for $\text{Res}^A_0$ in terms of the modular function (see Remark 4.8 below), while our results do not give any information about the relative modular object. In the next subsection, under certain assumptions, we express the relative modular object in an explicit way in terms of the modular object.

4.3. A formula for the relative modular object. The second main result of this section is the following relation between the relative modular object and the modular objects:

**Theorem 4.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be an exact tensor functor between finite tensor categories satisfying the equivalent conditions Lemma 4.2. Then

$$\chi_F \otimes \alpha_D \cong F(\alpha_C) \cong \alpha_D \otimes \chi_F.$$

We have shown that $\chi_F$ is an invertible object. Thus, by this theorem, $F(\alpha_C)$ is invertible if and only if $\alpha_D$ is. Provided that either $F(\alpha_C)$ or $\alpha_D$ is invertible (e.g., the base field $k$ is perfect), we have the following isomorphisms:

$$\alpha_D^* \otimes F(\alpha_C) \cong \chi_F \cong F(\alpha_C) \otimes \alpha_D^*.$$

**Proof.** Let $L$ and $R$ be a left and a right adjoint of $F$, respectively. As before, we make $\mathcal{D}$ a left $\mathcal{C}$-module category by $F$. By Lemmas 2.1 and 2.2 the functor $R$ is a $\mathcal{C}$-module functor. This means that there is a natural isomorphism $R(F(X) \otimes V) \cong X \otimes R(V)$ for $X \in \mathcal{C}$ and $V \in \mathcal{D}$. In other words, there is an isomorphism

$$\Upsilon_C \circ R \cong \text{REX}(F, R) \circ \Upsilon_D,$$

(4.6)

where $\Upsilon_C$ and $\Upsilon_D$ are the Cayley functors introduced in §3.5.

As in §3.5, we denote by $\Upsilon_{\square}$ a left adjoint of $\Upsilon_{\square}$ ($\square = \mathcal{C}, \mathcal{D}$). It is trivial that $F$ is left adjoint to $R$. To get a left adjoint of $\text{REX}(F, R)$, we note that there are natural isomorphisms

$$\text{Nat}(F \circ T_1, T_2) \cong \text{Nat}(T_1, R \circ T_2) \quad \text{and} \quad \text{Nat}(T_3 \circ R, T_4) \cong \text{Nat}(T_3, T_4 \circ F)$$

for $T_1 : \mathcal{C} \to \mathcal{C}$, $T_2 : \mathcal{D} \to \mathcal{C}$, $T_3 : \mathcal{C} \to \mathcal{D}$ and $T_4 : \mathcal{D} \to \mathcal{D}$ (see [ML98] X.5 and X.7 for these natural isomorphisms). Hence we have

$$\text{Nat}(T, R \circ T' \circ F) \cong \text{Nat}(F \circ T, T' \circ F) \cong \text{Nat}(F \circ T \circ R, F)$$

for $T : \mathcal{C} \to \mathcal{C}$ and $T' : \mathcal{D} \to \mathcal{D}$. This means $\text{REX}(R, F) \dashv \text{REX}(F, R)$. Thus, taking left adjoints of both sides of (4.6), we get

$$F \circ \Upsilon_C^* \cong \Upsilon_D^* \circ \text{REX}(R, F).$$

(4.7)
We compute the image of $J_C \in \text{REX}(C)$ under \eqref{eq:nichols-zoeller}. By the definition of the modular object, we have $F(T_C^*(J_C)) = F(\alpha_C)$. On the other hand,

$$
\left( \text{REX}(R, F)[J_C] \right)(V) \cong F(\text{Hom}_C(R(V), 1)^* \cdot 1)
\cong \text{Hom}_C(L(V \otimes \chi_F), 1)^* \cdot F(1)
\cong \text{Hom}_D(V \otimes \chi_F, F(1))^* \cdot 1
\cong J_D(V \otimes \chi_F)
$$

for $V \in C$. If we define an action of $D$ on $\text{REX}(D)$ by $V \otimes T = T(- \otimes V)$ for $V \in D$ and $T \in \text{REX}(D)$, then the above result reads:

$$
\text{REX}(R, F)[J_C] \cong \chi_F \otimes J_D.
$$

Since $\Upsilon_D$ is a left $D$-module functor, so is $\Upsilon_D^*$. Thus,

$$
\Upsilon_D^* \left[ \text{REX}(R, F)[J_C] \right] \cong \Upsilon_D^*[\chi_F \otimes J_D] \cong \chi_F \otimes \Upsilon_D^*[J_D] = \chi_F \otimes \alpha_D.
$$

Hence we obtain the first isomorphism $\chi_F \otimes \alpha_D \cong F(\alpha_C)$. The second isomorphism is obtained from the first one and the Radford $S^4$-formula, as follows:

$$
\alpha_D \otimes \chi_F \cong \alpha_D \otimes \chi_F^{***} \cong \chi_F \otimes \alpha_D \cong F(\alpha_C).
$$

\makebox[\textwidth]{\hfill $\square$}

Remark 4.8. We shall explain how this theorem implies a result of Fischman, Montgomery and Schneider \cite{FMS97}. Let $A/B$ be an extension of finite-dimensional Hopf algebras over $k$, and let $F := \text{Res}_B^A : \text{mod}-A \to \text{mod}-B$ be the restriction functor. As we have mentioned in Introduction, the functors

$$
L := (-) \otimes_B A \quad \text{and} \quad R := (-) \otimes_B \text{Hom}_B(A_B, B_B)
$$

are a left adjoint and a right adjoint of $F$, respectively. Recall that we have used the Nichols-Zoeller theorem to obtain the above expression of $R$. The theorem also allows us to apply Theorems 4.6 and 4.7 to the functor $F$. As a consequence,

$$
(4.8) \quad L(\chi_F \otimes_k X) \cong R(X) \quad (X \in \text{mod}-B),
$$

where $\chi_F$ is the right $B$-module corresponding to the algebra map

$$
\chi_{A/B}(b) = \alpha_A(b_{(1)})\alpha_B(S(b_{(2)}) \quad (b \in B),
$$

i.e., the relative modular function introduced in \cite{FMS97}. The relative Nakayama automorphism $\beta = \beta_{A/B}$ corresponds to the functor $\chi_F \otimes (-)$. Since

$$
L(\chi_F \otimes_k X) \cong L(X_\beta) \cong (X_\beta) \otimes_B A \cong X \otimes_B (\beta^{-1}A),
$$

we get an isomorphism $\beta^{-1}A_A \cong \text{Hom}_A(A_B, B)$ of $B$-$A$-bimodules from \eqref{eq:nichols-zoeller}. In other words, the extension $A/B$ is $\beta$-Frobenius \cite{FMS97}, Theorem 1.7.\footnote{Theorem 1.7}

Remark 4.9. We have used the Nichols-Zoeller theorem to apply Theorem 4.7 in the above. Like this, some non-trivial results will be needed to apply our results. Here we note the following criteria: An exact tensor functor $F : C \to D$ between finite tensor categories satisfies the equivalent conditions of Lemma 4.8 if it is faithful and surjective in the sense that every object of $D$ is a quotient of $F(X)$ for some $X \in C$ \cite[Theorem 2.5 and Section 3]{EOO04} (notice that our terminology slight differs from theirs).
4.4. Quasi-tensor functors. A tensor functor is, by definition, a functor $F$ between tensor categories endowed with an isomorphism $1 \cong F(1)$ and a natural isomorphism $F(X) \otimes F(Y) \cong F(X \otimes Y)$ satisfying certain coherence conditions. The suspicious reader may notice that we did not use the coherence condition essentially to prove the results in this section. Here we observe what happens if we consider ‘incoherent’ tensor functors in our theory.

Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories. By a quasi-tensor functor from $\mathcal{C}$ to $\mathcal{D}$, we mean a $k$-linear functor $F : \mathcal{C} \to \mathcal{D}$ endowed with (natural) isomorphisms

$$1 \to F(1), \quad F(X) \otimes F(Y) \to F(X \otimes Y) \quad \text{and} \quad F(X^*) \to F(X^*)$$

for $X, Y \in \mathcal{C}$ (note that this term has been used in slight different meaning in, e.g., [EO04]). Thanks to $F(X^*) \cong F(X)^*$, we have $F^\sharp \cong F \cong F^\prime$. Hence Lemma 4.11 holds if we replace ‘tensor functor’ with ‘quasi-tensor functor’.

Our results are based on the theory of module categories. To deal with quasi-tensor functors, we need to introduce a ‘quasi’ version of module categories: A (left $\mathcal{C}$-)quasi-module category is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, called the quasi-action, and natural isomorphisms

$$1 \otimes M \cong M, \quad (X \otimes Y) \otimes M \cong X \otimes (Y \otimes M),$$

$$\text{Hom}_\mathcal{M}(M, X \otimes N) \cong \text{Hom}_\mathcal{M}(X^* \otimes M, N)$$

for $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$. A (left $\mathcal{C}$-)quasi-module functor is a functor $F : \mathcal{M} \to \mathcal{N}$ between quasi-module categories endowed with a natural isomorphism

$$F(X \otimes M) \cong X \otimes F(M)$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$. If a quasi-module functor $F : \mathcal{M} \to \mathcal{N}$ has a right adjoint $R$, then $R$ is also a quasi-module functor. Indeed,

$$\text{Hom}_\mathcal{M}(M, R(X \otimes N)) \cong \text{Hom}_\mathcal{N}(F(M), X \otimes N) \cong \text{Hom}_\mathcal{N}(X^* \otimes F(M), N)$$

$$\cong \text{Hom}_\mathcal{M}(F(X^* \otimes M), N) \cong \text{Hom}_\mathcal{M}(X^* \otimes M, R(N)) \cong \text{Hom}_\mathcal{M}(M, X \otimes R(N))$$

for $X \in \mathcal{C}$, $M \in \mathcal{M}$ and $N \in \mathcal{N}$. In a similar way, we see that a left adjoint of $F$ is a quasi-module functor.

We can also define the notion of a finite left $\mathcal{C}$-quasi-module category and their internal Hom functors in an obvious way. Lemma 2.10 does not make sense in our ‘quasi’ setting, since $\text{Hom}(M, M)$ is not an algebra in general. On the other hand, Lemmas 2.11 and 2.12 still hold if we replace ‘module’ with ‘quasi-module’. Now the following theorem is proved along the completely same line as before:

**Theorem 4.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor between tensor categories, and suppose that it has adjoints. If $L \dashv F \dashv R$, then the following assertions are equivalent:

1. $L$ has a left adjoint.
2. $R$ has a right adjoint.
3. There exists an object $\chi_F \in \mathcal{D}$ such that $R \cong L(\otimes \chi_F)$.

Such an object $\chi_F$ is unique up to isomorphism if it exists, is an invertible object, and satisfies the following relations:

$$L(\chi_F \otimes -) \cong R \cong L(- \otimes \chi_F) \quad \text{and} \quad R(\chi_F^* \otimes -) \cong L \otimes R(- \otimes \chi_F^*).$$

If, moreover, $\mathcal{C}$ and $\mathcal{D}$ are finite tensor categories, then the above three conditions are equivalent to each of the following four conditions:
(4) $L$ is exact.
(5) $R$ is exact.
(6) $F(P)$ is projective for every projective object $P \in \mathcal{C}$.
(7) $F(P)$ is projective for a projective generator $P \in \mathcal{C}$.

Furthermore, there are isomorphisms $\chi_F \otimes \alpha_D \cong F(\alpha_C) \cong \alpha_D \otimes \chi_F$.

5. Braided Hopf algebras

5.1. Main result of this section. In this section, we give a description of the modular object of the category of right modules over a Hopf algebra in a braided finite tensor category (often called a braided Hopf algebra). To state our result, we first fix some notations related to Hopf algebras in a braided monoidal category.

Let $\mathcal{B}$ be a braided monoidal category with braiding $\sigma$, and let $H$ be a Hopf algebra with multiplication $m$, unit $u$, comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$. For $M, N \in \mathcal{B}_H$, their tensor product $M \otimes N \in \mathcal{B}_H$ is a right $H$-module by

$$\triangleright M \otimes N = (\triangleright M \otimes \triangleright N) \circ (\text{id}_M \otimes \sigma_{N,H} \otimes \text{id}_H) \circ (\text{id}_M \otimes \text{id}_N \otimes \Delta),$$

where $\triangleright_M$ and $\triangleright_N$ are the actions of $H$ on $M$ and $N$, respectively. The category $\mathcal{B}_H$ is a monoidal category with this operation. In a similar way, the category $\mathcal{H}_B$ is also a monoidal category. We note that $\mathcal{B}_H$ and $\mathcal{H}_B$ are rigid if $\mathcal{B}$ is rigid and $S$ is invertible.

Now suppose that $\mathcal{B}$ is rigid and admits equalizers. Then the antipode of $H$ is invertible. An (X-based) right integral in $H$ is a morphism $t : X \rightarrow H$ in $\mathcal{B}$ such that $m \circ (t \otimes H) = t \otimes \varepsilon$. The category of right integrals in $H$ (defined as the full subcategory of the category $(\mathcal{B} \downarrow H)$ of objects over $H$ [ML98 II.6]) has a terminal object. We write it as $\Lambda : \text{Int}(H) \rightarrow H$ and call $\text{Int}(H) \in \mathcal{B}$ the object of integrals in $H$. It is known that $\text{Int}(H)$ is an invertible object (see [Tak99] and [BKLT00] for these results). Thus the following definition makes sense:

**Definition 5.1.** The (right) modular function on $H$ is a morphism $\alpha_H : H \rightarrow \mathcal{B}$ of algebras in $\mathcal{B}$ determined by the following equation:

$$\alpha_H \otimes \text{id}_{\text{Int}(H)} = m \circ (\text{id}_H \otimes \Lambda).$$

We regard an object $V \in \mathcal{B}$ as a right $H$-module by defining the action of $H$ by the counit. We also identify a morphism $\alpha : H \rightarrow \mathcal{B}$ of algebras with the right $H$-module whose underlying object is $\mathcal{B}$ and whose action is given by $\alpha$. Note that the modular function $\alpha_H$ is a morphism of algebras in $\mathcal{B}$. With the above notation, the main result of this section is stated as follows:

**Theorem 5.2.** Let $\mathcal{B}$ be a braided finite tensor category over a field $k$, and let $H$ be a Hopf algebra in $\mathcal{B}$. The modular object of $\mathcal{C} = \mathcal{B}_H$ is given by

$$\alpha_C = \text{Int}(H)^* \otimes \alpha_H \otimes \alpha_B.$$ 

The left modular function $\overline{\alpha}_H$ is given by $\overline{\alpha}_H = \alpha_H \circ S$. Replacing $\mathcal{B}$ with $\mathcal{B}^{\text{ev}}$ in the above theorem, we obtain the following description of the modular object of the category of left $H$-modules:

**Corollary 5.3.** The modular object of $\mathcal{C} = \mathcal{H}_B$ is given by

$$\alpha_C = \text{Int}(H)^* \otimes \overline{\alpha}_H \otimes \alpha_B.$$
By an extension $A/B$ of Hopf algebras in a braided monoidal category $B$, we mean a pair of Hopf algebras $A$ and $B$ endowed with a morphism $i_{A/B} : B \to A$ of Hopf algebras in $B$ being monic as a morphism in $B$. As in the ordinary case, the functor $B_A \to B_B$ induced by $i_{A/B}$ is called the restriction functor. As an application of Theorems 4.7 and 5.2, we will show the following theorem:

**Theorem 5.4.** Let $B$ be a braided finite tensor category over $k$ whose modular object is invertible. For an extension $A/B$ of Hopf algebras in $B$, the following assertions are equivalent:

1. The restriction functor $B_A \to B_B$ is a Frobenius functor.
2. $\text{Int}(A) \cong \text{Int}(B)$ and $\alpha_A \circ i_{A/B} = \alpha_B$.

The rest of this paper is devoted to the proofs of Theorems 5.2 and 5.4.

5.2. **Nakayama automorphism.** Let $B$ be a braided rigid monoidal category with braiding $\sigma$. We use the graphical techniques to express morphisms in $B$. Our convention is that a morphism goes from the top of the diagram to the bottom (cf. [Tak99]). Following, the braiding $\sigma$, its inverse, the evaluation $\text{ev} : X^* \otimes X \to \text{1}$ and the coevaluation $\text{coev} : \text{1} \to X \otimes X^*$ are expressed as follows:

\[
\sigma_{X,Y} = \begin{array}{c} X \psi Y \\ Y \psi X \end{array}, \quad \sigma^{-1}_{X,Y} = \begin{array}{c} Y \psi X \\ X \psi Y \end{array}, \quad \text{ev} = \bigcup_{X \otimes X^*} X, \quad \text{coev} = \bigcup_{X \otimes X^*} X^*
\]

The axioms for Hopf algebras in $B$ are expressed as in Figure 1. Here, for a Hopf algebra $H$ in $B$, we depict its structure morphisms as follows:

\[
m = \begin{array}{c} H \psi H \\ H \psi H \end{array}, \quad u = \begin{array}{c} \text{1} \psi H \\ H \psi \text{1} \end{array}, \quad \Delta = \begin{array}{c} H \psi H \\ H \psi H \end{array}, \quad \varepsilon = \begin{array}{c} H \psi \text{1} \\ \text{1} \psi H \end{array}, \quad S = \begin{array}{c} H \psi H \\ H \psi H \end{array}, \quad S^{-1} = \begin{array}{c} \text{1} \psi H \\ H \psi \text{1} \end{array}
\]

Suppose that $B$ admits equalizers. Let $H$ be a Hopf algebra in $B$. As in the previous subsection, we fix a terminal object $\Lambda : \text{Int}(H) \to H$ of the category of right integrals in $H$. By definition, we have

\[
m \circ (\Lambda \otimes \text{id}_H) = \Lambda \otimes \varepsilon.
\]
It is known that there exists a unique morphism \( \lambda : H \to \text{Int}(H) \) such that

\[
(5.3) \quad (\lambda \otimes \text{id}_H) \circ \Delta = \lambda \otimes u \quad \text{and} \quad \lambda \circ \Lambda = \text{id}_{\text{Int}(H)}.
\]

The paring \( \phi_H : H \to \text{Int}(H) \) is non-degenerate in the sense that

\[
(5.4) \quad \phi_H : H \xrightarrow{H \otimes \text{coev}} H \otimes H \otimes H^* \xrightarrow{(\lambda \circ \text{coev}) \otimes H^*} \text{Int}(H) \otimes H^*
\]

is invertible (see [Tak99] and [BKLT00] for these results). Hence the following definition makes sense:

**Definition 5.5** (Doi-Takeuchi [DT00]). The **Nakayama automorphism** of \( H \) is the unique morphism \( N : H \to H \) in \( B \) such that

\[
(5.5) \quad \lambda \circ m \circ \sigma_{H,H} = \lambda \circ m \circ (\text{id}_H \otimes N).
\]

Let \( K \in B \) be an invertible object. The **monodromy around** \( K \) is the natural transformation \( \Omega(K) : \text{id}_B \to \text{id}_B \) defined by

\[
(5.6) \quad \text{id}_K \otimes \Omega(K)V = \sigma_{V,K} \circ \sigma_{K,V}
\]

for \( V \in B \). The definition of a braiding implies

\[
(5.7) \quad \Omega(\text{Id}) = \text{id}, \quad \Omega(K \otimes K') = \Omega(K) \circ \Omega(K') \quad \text{and} \quad \Omega(K^*) = \Omega(K)^{-1}
\]

for invertible objects \( K \) and \( K' \).

Lemma 5.6 below is proved by Doi and Takeuchi under the assumption that \( B \) is built on the category of vector spaces \([DT00, \text{Proposition 13.1}]\). Since their proof cannot be applied to our general setting, we give a proof.

**Lemma 5.6.** The **Nakayama automorphism** of \( H \) is given by

\[
(5.8) \quad N = S^{-2} \circ (\alpha_H \otimes \text{id}_H) \circ \Delta \circ \Omega(\text{Int}(H))_H
\]

**Proof.** Set \( I = \text{Int}(H) \), \( \alpha = \alpha_H \) and \( \omega = \Omega(I)_H \). Then (5.8) is equivalent to

\[
(5.9) \quad S^2 \circ N = (\alpha \otimes \text{id}_H) \circ \Delta \circ \omega.
\]

To prove this equation, we first prove

\[
(5.10) \quad (m \otimes \text{id}_H) \circ (\text{id}_H \otimes \sigma_{H,H}) \circ (\Delta \otimes \text{id}_H) = (\text{id}_H \otimes m) \circ (\Delta m \otimes S) \circ (\text{id}_H \otimes \Delta)
\]

as in Figure 2 (this result reads “\( a_{(1)}b \otimes a_{(2)} = (ab_{(1)})_{(1)} \otimes (ab_{(1)})_{(2)}S(b_{(2)}) \)” in an ordinary Hopf algebra with the Sweedler notation). Using (5.2), (5.3) and (5.10), we obtain the three formulas depicted in Figure 3. Now (5.9) is proved as in Figure 4. \( \square \)

**5.3. The fundamental theorem for Hopf bimodules.** Let \( B \) and \( H \) be as in the previous subsection. For an \( H \)-bimodule \( M \in B_H \), we express the left action \( \triangleright_M \) and the right action \( \triangleleft_M \) of \( H \) on \( M \) respectively as

\[
\triangleright_M = \begin{array}{c}
H \\
M
\end{array} \quad \text{and} \quad \triangleleft_M = \begin{array}{c}
M \\
H
\end{array}
\]
Given $M, N \in {}_H\mathcal{B}_H$, we define $M \tilde{\otimes} N$ to be the $H$-bimodule with the underlying object $M \otimes N$ and with actions

\[(5.11) \quad \triangledown_{M \tilde{\otimes} N} = \quad \text{and} \quad \triangledown_{M \tilde{\otimes} N} = \]

The category ${}_H\mathcal{B}_H$ is a rigid monoidal category with respect to $\tilde{\otimes}$. Note that the left dual object of $M \in {}_H\mathcal{B}_H$ (with respect to $\tilde{\otimes}$), denoted by $M^\vee$, is the $H$-bimodule
with underlying object $M^*$ and with actions

\begin{equation}
\triangleright_{M^*} = \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$H$};
\node (b) at (1,0) {$M^*$};
\path (a) edge (b);
\end{tikzpicture}
\end{array}
\quad \text{and} \quad
\triangleleft_{M^*} = \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$M^*$};
\node (b) at (1,0) {$H$};
\path (a) edge (b);
\end{tikzpicture}
\end{array}
\end{equation}

The object $H \in \mathcal{B}$ is an $H$-bimodule by $m$. Moreover, the coalgebra $(H, \Delta, \varepsilon)$ in $\mathcal{B}$ is in fact a coalgebra in $(H,B_H,\otimes,1)$. We denote by $H^H_B$ the category of left $H$-comodules in $H_BH$ and refer to an object of this category as a Hopf $H$-bimodule. By definition, an object of $H^H_B$ is an $H$-bimodule $M$ in $\mathcal{B}$ endowed with a left $H$-comodule structure $\delta_M : M \to H \otimes M$ in $\mathcal{B}$ such that

\begin{equation}
\delta_M = \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$H$};
\node (b) at (1,0) {$M$};
\node (c) at (2,0) {$M$};
\node (d) at (3,0) {$H$};
\path (a) edge (b) edge (d);
\path (b) edge (c) edge (d);
\end{tikzpicture}
\end{array}
\quad \text{and} \quad
\delta_M = \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$M$};
\node (b) at (1,0) {$H$};
\node (c) at (2,0) {$M$};
\node (d) at (3,0) {$H$};
\path (a) edge (b) edge (d);
\path (b) edge (c) edge (d);
\end{tikzpicture}
\end{array}
\end{equation}

(here, the coaction $\delta_M$ is expressed by the up-side down of the action $\triangleright_M$). Thus a Hopf $H$-bimodule is the same thing as a two-fold Hopf bimodule in the sense of Bespalov and Drabant [BD98, Definition 3.6.1].

For $M \in H^H_B$, we set $\pi_M := \triangleright_M \circ (S \otimes \lambda_M) \circ \delta_M : M \to M$. The coinvariant of $M$, denoted by $M^{\text{co}H}$, is defined to be the equalizer of $\pi_M$ and $\lambda_M$. It is also an equalizer of $\delta_M$ and $u \otimes \lambda_M$. Thus, symbolically, we have

\begin{equation}
M^{\text{co}H} = \text{Eq}(\pi_M, \lambda_M) = \text{Eq}(\delta_M, u \otimes \lambda_M).
\end{equation}

Let $M_{\text{ad}}$ be the right $H$-module with underlying object $M$ and with action

\begin{equation}
\triangleleft_{M_{\text{ad}}} = \triangleright_M \circ (S \otimes \sigma_M) \circ (\sigma_{M,H} \otimes \lambda_H) \circ (\lambda_M \otimes \Delta).
\end{equation}

We call $\triangleleft_{M_{\text{ad}}}$ the adjoint action and express it by the same diagram as a right action but with labeled ‘ad’ (as in the first diagram in Figure 5). The morphism $\pi_M$ is in fact an $H$-linear idempotent on the right $H$-module $M_{\text{ad}}$. Hence $M^{\text{co}H}$ is a right $H$-module as a submodule of $M_{\text{ad}}$.

An object of the form $H \otimes M$, $M \in H_BH$, is a Hopf $H$-bimodule as a free left $H$-comodule. We regard a right $H$-module as an $H$-bimodule by defining the left action by $\varepsilon$. The fundamental theorem for Hopf bimodules (Bespalov and Drabant [BD98 Proposition 3.6.3]) states that the functor

\begin{equation}
H \otimes (-) : \mathcal{B}_H \to H_BH, \quad V \mapsto H \otimes V \quad (V \in \mathcal{B}_H)
\end{equation}

is an equivalence of categories with quasi-inverse $(-)^{\text{co}H}$.

The left dual object of a right $H$-comodule is a left $H$-comodule in a natural way (cf. Lemma 2.3). We are interested in the Hopf bimodule $H^\vee \in H_BH$ (where $H$ is regarded as a right $H$-comodule in $H_BH$ by the comultiplication). In view of the fundamental theorem, it is essential to determine its coinvariant.

**Lemma 5.7.** $(H^\vee)^{\text{co}H} \cong \text{Int}(H)^* \otimes \alpha_H$ as right $H$-modules.

**Proof.** As remarked in [BKL10 §3.1], $\lambda$ is a coequalizer of

\begin{equation}
f := \lambda_H \otimes ^* u \quad \text{and} \quad g := (\lambda_H \otimes \text{ev}) \circ (\Delta \otimes \lambda_H).
\end{equation}
Note that $g^*$ is the coaction of $H$ on $H^\vee \in H^\vee B_H$ (cf. Lemma 2.3). By (5.14), $\lambda^*$ is the coinvariant of $H^\vee$. Hence the claim of this lemma is equivalent to

\begin{equation}
\langle^a_{H^\vee} \circ (\lambda^* \otimes \text{id}_H) = \lambda^* \otimes \alpha_H \rangle
\end{equation}

For simplicity, we set $I = \text{Int}(H)$, $\alpha = \alpha_H$ and $\Xi = \Omega(I)^{-1}$. The adjoint action is computed as in Figure 3. The equation equivalent to (5.17) via

$$\text{Hom}_B(I^* \otimes H, H^*) \cong \text{Hom}_B(H \otimes H, I)$$

is proved as in Figure 6. \hfill \Box

5.4. Proof of Theorem 5.2. We now prove Theorem 5.2. We recall the assumptions: $B$ is a braided finite tensor category over $k$ with braiding $\sigma$, and $H$ is a Hopf algebra in $B$. We regard $B$ as a full subcategory of $C := B_H$ by regarding an object of $B$ as a right $H$-module by the coinvert of $H$. There are forgetful functors

$$\text{Rex}_C(C) \xrightarrow{\Theta_C} \text{Rex}(C) \quad \text{and} \quad \text{Rex}_C(C) \xrightarrow{\Theta_{C/H}} \text{Rex}_B(C) \xrightarrow{\Theta_B} \text{Rex}(C)$$

such that $\Theta_C = \Theta_{C/B} \circ \Theta_B$. Since the Cayley functor $\Upsilon_C$ corresponds to $\Theta_C$ under the identification $\text{Rex}_C(C) \approx C$, the composition

$$\Theta_C^* : \text{Rex}(C) \xrightarrow{\Upsilon_C^*} C \xrightarrow{\approx \text{Theorem 2.14}} \text{Rex}_C(C)$$

is left adjoint to $\Theta_C$. Now we set $J_C = \text{Hom}_C(\cdot, \mathbb{1})^* \mathbb{1} \in \text{Rex}(C)$. By the definition of the modular object, we have:

**Lemma 5.8.** $\Theta_C^*[J_C] \cong (-) \otimes \alpha_C$ in $\text{Rex}_C(C)$.

Our main idea of the proof of Theorem 5.2 is to compute $\Theta_C^*[J_C]$ in terms of left adjoints of $\Theta_B$ and $\Theta_{C/B}$. We first prove:

**Lemma 5.9.** The functor $\Theta_B$ has a left adjoint, say $\Theta_B^*$. We have

$$\Theta_B^*[J_C] \cong (-) \otimes_H \alpha_B,$$

where $\alpha_B \in B$ is regarded as an $H$-bimodule by the coinvert of $H$.

**Proof.** That $\Theta_B$ has a left adjoint follows from Corollary 5.8. To prove the claim, we note that the inclusion functor $i : B \to C$ has a left adjoint

$$T : C \to B, \quad X \mapsto X \otimes_H \mathbb{1}$$

by the argument of 2.7. Since $i$ and $T$ are $k$-linear right exact $B$-module functors, they induce a $k$-linear right exact strong $B^\text{env}$-module functor

$$\Omega := \text{Rex}(T, i) : \text{Rex}(B) \to \text{Rex}(C), \quad F \mapsto i \circ F \circ T$$

by Lemma 3.6. Now let $A = \int_{X \in B} X \boxtimes X^*$ be the algebra in $B^\text{env}$ considered in Lemma 3.6. Since $\Omega$ is a strong $B^\text{env}$-module functor, it induces a functor (also denoted by $\Omega$) between the categories of $A$-modules in such a way that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc}
\text{Rex}(B) & \xrightarrow{A \otimes (-)} & A\text{Rex}(B) \\
\Omega & \downarrow & \Omega \\
\text{Rex}(C) & \xrightarrow{A \otimes (-)} & A\text{Rex}(C)
\end{array} \xrightarrow{\text{Lemma 3.6}} \begin{array}{ccc}
\text{Rex}_B(B) & \cong & \text{Rex}_B(B) \\
\Omega & \downarrow & \Omega \\
\text{Rex}_C(C) & \cong & \text{Rex}_C(C)
\end{array}$$

\hfill \Box
Figure 5. The computation of the adjoint action

Figure 6. The computation of the action of $H$ on $(H^\vee)^{coH}$
Now we chase $J_B := \text{Hom}_B(1, -) \bullet 1 \in \text{Rex}(\mathcal{B})$ around this diagram. Since the composition along the bottom row is $\Theta^*_B$, we have

$$\Theta^*_B \Omega[J_B] \cong \Omega[A \otimes J_B]$$

in $\text{Rex}_B(\mathcal{C})$. Since $T$ is left adjoint to $i$, we have

$$\Theta^*_B[J_B] \cong \Omega\left[A \otimes J_B \right] \cong T(-) \otimes \alpha_B \cong (-) \otimes_H \alpha_B$$

in $\text{Rex}_B(\mathcal{C})$. Hence we have $\Theta^*_B[J_C] \cong [-] \otimes_H \alpha_B$.

□

We now consider the forgetful functor $\Theta_{\mathcal{C}/\mathcal{B}} : \text{Rex}_\mathcal{C}(\mathcal{C}) \to \text{Rex}_\mathcal{B}(\mathcal{C})$.

**Lemma 5.10.** $\Theta_{\mathcal{C}/\mathcal{B}}$ has a left adjoint, say $\Theta^*_\mathcal{C}/\mathcal{B}$. For $M \in H_B$, we have

$$\Theta^*_\mathcal{C}/\mathcal{B}\left((-) \otimes_H M\right) \cong (-) \otimes (H^\vee \hat{\otimes} M)^{\circ H}.$$ 

**Proof.** We consider the following diagram:

$$\begin{array}{ccc}
\text{Rex}_\mathcal{C}(\mathcal{C}) & \xrightarrow{\Theta_{\mathcal{C}/\mathcal{B}}} & \text{Rex}_\mathcal{B}(\mathcal{C}) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{B}_H & \xrightarrow{H \hat{\otimes} (-)} & H^H_B, \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{B}_H & \xrightarrow{H \hat{\otimes} (-)} & H^H_B, \\
\end{array}$$

where the vertical arrows are the equivalences given by Theorem 2.14 and $F$ is the functor forgetting the comodule structure. Since there is an isomorphism

$$X \otimes_H (H \hat{\otimes} V) \cong X \otimes V \quad (X, V \in \mathcal{C}),$$

the diagram commutes up to isomorphisms. The functors in the diagram are equivalences except $F$ and $\Theta_{\mathcal{C}/\mathcal{B}}$, and the functor tensoring $H^\vee$ is left adjoint to $F$. Thus $\Theta_{\mathcal{C}/\mathcal{B}}$ has a left adjoint as the composition of functors having left adjoints. Hence we get the following diagram commuting up to isomorphism:

$$\begin{array}{ccc}
\text{Rex}_\mathcal{C}(\mathcal{C}) & \xrightarrow{\Theta^*_\mathcal{C}/\mathcal{B}} & \text{Rex}_\mathcal{B}(\mathcal{C}) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{B}_H & \xrightarrow{(-)^{\circ H}} & H^H_B, \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{B}_H & \xrightarrow{H \hat{\otimes} (-)} & H^H_B, \\
\end{array}$$

Now the claim is obtained by chasing $M \in H_B$ around this diagram. □

**Proof of Theorem 5.2.** Since $\Theta_\mathcal{C} = \Theta_\mathcal{B} \circ \Theta_{\mathcal{C}/\mathcal{B}}$, we have

$$\Theta^*_\mathcal{C}[J_C] = (\Theta^*_\mathcal{C}/\mathcal{B} \circ \Theta^*_\mathcal{B})[J_C].$$

(5.18)
The left-hand side is \((-\otimes\alpha_C\) by Lemma 5.8. Set \(I = \text{Int}(H)\). The right-hand side is computed as follows:

\[
(\Theta^*_C\otimes\Theta^*_B)[J_C] \cong \Theta^*_C\otimes\Theta^*_B(-\otimes H\alpha_B) \\
\cong (-\otimes(H^\vee\otimes\alpha_B)^{coh}) \\
\cong (-\otimes(H\otimes(I^*\otimes H\otimes\alpha_B)^{coh}) \\
\cong (-\otimes\alpha_H\otimes\alpha_B).
\]

Now the result is obtained by evaluating the both sides of (5.18) at \(\delta_B\).

5.5. **Proof of Theorem 5.4.** We now prove Theorem 5.2. We recall the assumptions: \(B\) is a braided finite tensor category over \(k\) with invertible modular object, and \(i_{A/B} : B \to A\) is an extension of Hopf algebras in \(B\).

**Proof of Theorem 5.4.** Let \(F : B_A \to B_B\) be the restriction functor. It is sufficient to show that Theorem 4.7 is applicable to \(F\). For \(M \in B_B\), we denote its underlying object by \(M_0\) for clarity. As before, we regard \(B \subset B_B\) (and thus \(M_0 \in B_B\)). For \(X \in B_B\), we consider the morphism

\[
X_0 \otimes I_0 \otimes (B^A)^* \xrightarrow{id_X \otimes 1 \otimes i_{A/B}} X_0 \otimes I_0 \otimes (B^B)^* \xrightarrow{id_X \otimes \phi^{-1}_B} X_0 \otimes B \xrightarrow{\delta_B} X,
\]

where \(I = \text{Int}(B)\) and \(\phi_B\) is the isomorphism given by (5.4) with \(H = B\). This is an epimorphism of right \(B\)-modules. Obviously, \(X_0 \otimes I_0 \otimes (B^A)^*\) is a restriction of an \(A\)-module. Thus, by Remark 4.9 we can apply Theorem 4.7 to \(F\). □

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