Continuity of the Jones’ set function $\mathcal{T}$

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Abstract

Given a continuum $X$, for each $A \subseteq X$, the Jones’ set function $\mathcal{T}$ is defined by
\[
\mathcal{T}(A) = \{x \in X : \text{for each subcontinuum } K \text{ such that } x \in \text{Int}(K), \text{ then } K \cap A \neq \emptyset\}.
\]
We show that $D = \{\mathcal{T}\{x\} : x \in X\}$ is decomposition of $X$ when $\mathcal{T}$ is continuous.
We present a characterization of the continuity of $\mathcal{T}$ and answer several open questions posed by D. Bellamy.

1 Introduction

Given a continuum $X$, for each $A \subseteq X$, the Jones’ set function $\mathcal{T}$ is defined by
\[
\mathcal{T}(A) = \{x \in X : \text{for each subcontinuum } K \text{ such that } x \in \text{Int}(K), \text{ then } K \cap A \neq \emptyset\}.
\]
The set function $\mathcal{T}$ was defined by F. Burton Jones \cite{Jones} in order to study some properties related with aposyndesis of continua. Since then it has been studied extensively as an aid to classify continua (see for instance \cite{Bellamy}, \cite{Bromberg}, \cite{Bromberg2}, \cite{Brown}, \cite{Brown2}, \cite{Brown3}, \cite{Brown4}, \cite{Brown5}, \cite{Brown6}, \cite{Brown7} and \cite{Brown8}).

A continuum is a compact connected and nonempty metric space. Given a continuum $X$, $2^X$ denotes the collection of all closed and nonempty subsets of $X$; $2^X$ is itself a continuum if it is topologized by the Hausdorff metric. It is known that $\mathcal{T}(A) \in 2^X$, for each $A \in 2^X$ and thus is natural to look for conditions under which $\mathcal{T}: 2^X \to 2^X$ is continuous. This problem has been addressed in several places (see \cite{Bellamy}, \cite{Bromberg}, \cite{Bromberg2}, \cite{Bromberg3}, \cite{Bromberg4}, \cite{Bromberg5} and \cite{Bromberg6}) and is the objective of this article. There are two trivial cases where $\mathcal{T}$ is continuous: (a) If $X$ is locally connected, then $\mathcal{T}(A) = A$ for all $A \in 2^X$ and (b) if $X$ is indecomposable, then $\mathcal{T}(A) = X$ for all $A \in 2^X$. Bellamy \cite{Bellamy} asked if any nonlocally connected continuum for which $\mathcal{T}$ is continuous had to be indecomposable, later he proved this is not the case by showing that $\mathcal{T}$ is continuous for the circle of pseudo-arcs \cite{Bromberg2}. Macías \cite{Macias} generalized Bellamy’s example showing that if $X$ is one dimensional, has a terminal decomposition in pseudo-arcs and its decomposition space is locally connected, then $\mathcal{T}$ is continuous.

From the very beginning it was recognized that $D = \{\mathcal{T}\{x\} : x \in X\}$ plays a crucial role in the study topological properties of continua using the function $\mathcal{T}$. Bellamy \cite{Bromberg2} Theorem 5, p. 9] shows that if there exist a locally connected continuum $Y$ and a monotone and open map $f: X \to Y$ such that $\mathcal{T}(A) = f^{-1}(f(A))$, for each $A \in 2^X$, then $\mathcal{T}$ is continuous and $D$ is clearly a decomposition of $X$. Later Macías \cite{Macias} proved the converse, if $\mathcal{T}$ is continuous and $D$ is a decomposition of $X$, then the conditions in the hypothesis of Bellamy’s theorem
hold. In this paper, we prove that the continuity of $\mathcal{T}$ suffices, that is to say, it implies that $\mathcal{D}$ is a decomposition (see Theorem 3.11). This was the missing piece to answer several questions posed by Bellamy [5, p. 390] and by Macías in [14, Chapter 7] (see Corollary 3.8, Theorem 3.11, Theorem 3.12, Corollary 3.13).

Macías [12] proved that if $X$ is a continuously irreducible continuum, the set function $\mathcal{T}$ is continuous. We will show that in the class of decomposable irreducible continua, the continuously irreducible continua are the only one for which the set function $\mathcal{T}$ is continuous (see Theorem 3.14).

2 Preliminaries

If $X$ is a metric space, then given $A \subseteq X$ and $\epsilon > 0$, the open ball about $A$ of radius $\epsilon$ is denoted by $B(A, \epsilon)$, the closure of $A$ is denoted by $\overline{A}$ and its interior is denoted by $A^\circ$. Given a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ and $x \in X$, $x_n \to x$ means that $(x_n)_{n \in \mathbb{N}}$ converges to $x$. In this paper every map will be a continuous function. If $f : X \to Y$ is a map and $A \subseteq X$, $f|_A$ denotes the restriction of $f$ to $A$.

Given a topological space $X$, a decomposition of $X$ is a family $\mathcal{G}$ of nonempty and mutually disjoint subsets of $X$ such that $\bigcup \mathcal{G} = X$. A decomposition $\mathcal{G}$ of a topological space $X$ is said to be continuous if the quotient map $q : X \to X/\mathcal{G}$ is both closed and open.

A compactum is a compact metric space. A continuum is a compact connected and nonempty metric space. Given a continuum $X$, we define

$$2^X = \{A \subseteq X : A \text{ is closed and nonempty}\}.$$ 

For each $A, B \in 2^X$, we define

$$H(A, B) = \inf \{r > 0 : A \subseteq N(B, r) \text{ and } B \subseteq N(A, r)\},$$

where $N(D, s) = \{x \in X : d(x, z) < s, \text{ for some } z \in D\}$, $D \in 2^X$ and $s > 0$. $H$ is the Hausdorff metric on $2^X$. It is known that $2^X$ is a continuum.

Let $C_1, ..., C_n$ be subsets of $X$. We define

$$\langle C_1, ..., C_n \rangle = \{A \in 2^X : A \subseteq \bigcup_{i=1}^n C_i \text{ and } A \cap C_i \neq \emptyset, \text{ for each } i \in \{1, ..., n\}\}.$$ 

Then $\{\langle U_1, ..., U_n \rangle : U_i \text{ is open of } X \text{ for each } i\}$ is a base of the topology generated by the Hausdorff metric in $2^X$.

**Definition 2.1.** Let $X$ be a compactum. Define $\mathcal{T} : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\mathcal{T}(A) = X \setminus \{x \in X : \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in W^\circ \subseteq W \subseteq X \setminus A\},$$

for each $A \in \mathcal{P}(X)$. The function $\mathcal{T}$ is called Jones’ set function $\mathcal{T}$. 

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It is not difficult to show that $\mathcal{T}(A)$ is closed, for each $A \in \mathcal{P}(X)$. Thus, the restriction $\mathcal{T}|_{2^X} : 2^X \to 2^X$ is well defined. When we say that $\mathcal{T}$ is continuous, we refer to $\mathcal{T}|_{2^X}$. It is known that $\mathcal{T}$ is upper semicontinuous [14, Theorem 3.2.1]. We say that $\mathcal{T}$ is idempotent provided that $\mathcal{T}^2(A) = \mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$, for each $A \in \mathcal{P}(X)$. We say that $X$ is idempotent on closed sets provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$, for each $A \in 2^X$. Clearly, if $\mathcal{T}$ is idempotent, then it is idempotent on closed sets.

Example 2.2. There exists a continuum $X$ such that $X$ is idempotent on closed sets, but it is not idempotent.

We define $[a, b] = \{x \in \mathbb{R}^2 : x = at + b(1 - t), \ t \in [0, 1]\}$, for each $a, b \in \mathbb{R}^2$. Let $X = \bigcup_{n=0}^{\infty} [v, u_n]$, where $v = (0, 1)$, $u_0 = (0, 0)$ and $u_n = \left(\frac{1}{n}, 0\right)$, for each $n \in \mathbb{N}$. $X$ is known as the harmonic fan. Let $U = [v, u_1] \setminus \{v\} \subseteq X$. It is not difficult to see that $\mathcal{T}(U) = [v, u_1]$ and $\mathcal{T}^2(U) = [v, u_1] \cup [v, u_0]$. Hence, $\mathcal{T}^2(U) \neq \mathcal{T}(U)$ and $\mathcal{T}$ is not idempotent. Let $A \in 2^X$. It is easy to show that

$$\mathcal{T}(A) = \begin{cases} A & \text{if } A \cap [v, u_0] = \emptyset; \\ A \cup [w, u_0] & \text{if } w \in A \cap [v, u_0] \text{ and } [v, w] \cap A = \{w\}. \end{cases}$$

Thus, we may verify that $\mathcal{T}^2(A) = \mathcal{T}(A)$, for each $A \in 2^X$; i. e., $\mathcal{T}$ is idempotent on closed sets.

A continuum $X$ is $\mathcal{T}$–additive provided that for each pair, $A$ and $B$, of closed subsets of $X$, $\mathcal{T}(A \cup B) = \mathcal{T}(A) \cup \mathcal{T}(B)$. Furthermore, $X$ is $\mathcal{T}$–symmetric if for each closed sets $A$ and $B$, we have that $A \cap \mathcal{T}(B) = \emptyset$ if and only if $\mathcal{T}(A) \cap B = \emptyset$. Finally, a continuum $X$ is point $\mathcal{T}$–symmetric provided that for each $p, q \in X$, $p \notin \mathcal{T}(\{q\})$ if and only if $q \notin \mathcal{T}(\{p\})$.

3 Main results

In this section, we prove in Theorem 3.7 that if $\mathcal{T}$ is continuous, then $\mathcal{D} = \{\mathcal{T}(\{x\}) : x \in X\}$ is a continuous decomposition of $X$ such that the quotient space $X/\mathcal{D}$ is locally connected. We show some interesting consequences of this result.

Proposition 3.1. Let $X$ be a continuum and let $A_n, A, L \in 2^X$, for each $n \in \mathbb{N}$. If $A_n \to A$ and $\mathcal{T}(A_n) \to L$, then $L \subseteq \mathcal{T}(A) \subseteq \mathcal{T}(L)$.

Proof. We show first that $L \subseteq \mathcal{T}(A)$. Let $x \in L$ and $W$ be a continuum such that $x \in W^\circ$. Since $\mathcal{T}(A_n) \to L$, there is $n_0$ such that $\mathcal{T}(A_n) \cap W^\circ \neq \emptyset$ for all $n \geq n_0$. Then, by the definition of $\mathcal{T}$, $W \cap A_n \neq \emptyset$ for all $n \geq n_0$. By compactness, we can find an increasing sequence of integer $(n_k)_{k \in \mathbb{N}}$ and $z_k \in W \cap A_{n_k}$ such that $z_k \to z$ for some $z \in X$. Thus $z \in A \cap W$. Since $W$ was arbitrary, then $x \in \mathcal{T}(A)$. We have shown that $L \subseteq \mathcal{T}(A)$.

Now we show that $\mathcal{T}(A) \subseteq \mathcal{T}(L)$. Since $A_n \subseteq \mathcal{T}(A_n)$, for each $n \in \mathbb{N}$, $A_n \to A$ and $\mathcal{T}(A_n) \to L$, we have that $A \subseteq L$. Therefore, $\mathcal{T}(A) \subseteq \mathcal{T}(L)$. 

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In [14, Theorem 3.2.8], it is proved that if \( T \) is continuous then \( T \) is idempotent. The following result characterizes the continuity of \( T \) in terms of idempotence.

**Theorem 3.2.** Let \( X \) be a continuum. Then, \( T \) is continuous if and only if \( T \) is idempotent on closed sets and \( T(2^X) \) is closed in \( 2^X \).

**Proof.** Notice that if \( T \) is continuous, then \( T \) is idempotent, by [14, Theorem 3.2.8], and clearly \( T(2^X) \) is closed in \( 2^X \). Conversely, suppose \( T \) is idempotent on closed sets and \( T(2^X) \) is closed in \( 2^X \). We will show that \( T \) is continuous. Suppose \( A_n, A \in 2^X, A_n \to A \). By passing to a subsequence we assume that \( T(A_n) \to L \) for some \( L \in 2^X \). By Proposition 3.1 \( L \subseteq T(A) \subseteq T(L) \). Since \( T(2^X) \) is closed in \( 2^X \), there is \( B \in 2^X \) such that \( L = T(B) \). But \( T(L) = T(T(B)) = T(B) = L \) as \( T \) is idempotent on closed sets. Hence \( L = T(A) \).

We will need the following result of Macías.

**Lemma 3.3.** ([13, Theorem 3.7]) Let \( X \) be a continuum such that \( T \) is idempotent on closed sets. If \( x \in X \), then there exists \( x_0 \in X \) such that \( T(\{x_0\}) \subseteq T(\{x\}) \) and \( T(\{z\}) = T(\{x_0\}) \) for each \( z \in T(\{x_0\}) \).

**Theorem 3.4.** Let \( X \) be a continuum. If \( T|_{F_1(X)} : F_1(X) \to 2^X \) is continuous and \( T \) is idempotent on closed sets, then \( D = \{T(\{x\}) : x \in X\} \) is a decomposition of \( X \).

**Proof.** Let \( M = \{x \in X : \text{for each } z \in T(\{x\}), T(\{z\}) = T(\{x\})\} \).

Observe that \( T(\{x\}) \subseteq M \), for each \( x \in M \). Also, by Lemma 3.3 \( M \neq \emptyset \) and \( T(\{x\}) \cap M \neq \emptyset \), for each \( x \in X \).

**Claim 3.5.** \( M \) is closed in \( X \).

To show the claim, let \( (x_n)_{n=1}^\infty \subseteq M \) and \( x \in X \) be such that \( x_n \to x \). We will prove that \( x \in M \). Let \( z \in T(\{x\}) \). Since \( T|_{F_1(X)} \) is continuous, \( T(\{x_n\}) \to T(\{x\}) \). Hence, there exists \( z_n \in T(\{x_n\}) \), for each \( n \in \mathbb{N} \), such that \( z_n \to z \). Thus, \( T(\{z_n\}) \to T(\{z\}) \), by the continuity of \( T|_{F_1(X)} \). Observe that \( T(\{z_n\}) = T(\{x_n\}) \), for each \( n \in \mathbb{N} \), as \( x_n \in M \). Therefore, \( T(\{z\}) = T(\{x\}) \) and \( x \in M \). This proves Claim 3.5

**Claim 3.6.** \( M = X \)

To prove the claim we assume that \( M \neq X \) to get a contradiction. We consider two cases:

1. Suppose \( M \) is disconnected. Let \( M = M_1 \cup M_2 \), where \( M_1 \) and \( M_2 \) are disjoint nonempty compact sets. Let \( U_1 \) and \( U_2 \) be open sets such that \( M_1 \subseteq U_1 \), \( M_2 \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \). For each \( x \in M_1 \), as \( T(\{x\}) \) is connected and \( x \in T(\{x\}) \), then \( T(\{x\}) \subseteq M_1 \subseteq U_1 \). Since \( T \) is upper semicontinuous, for each \( x \in M_1 \), there is an open set \( V_x \subseteq U_1 \) such that \( x \in V_x \) and \( T(\{z\}) \subseteq U_1 \), for each \( z \in V_x \). Let \( V_1 = \bigcup\{V_x : x \in M_1\} \). It is clear that \( V_1 \subseteq U_1 \), \( M_1 \subseteq V_1 \) and \( T(\{x\}) \subseteq U_1 \) for each \( x \in V_1 \). Analogously, we can find an open
Therefore, for each decomposition of $X$ sets:

$$ R = \{ x \in X : \mathcal{T}(\{x\}) \cap M_1 \neq \emptyset \text{ and } \mathcal{T}(\{x\}) \cap M_2 \neq \emptyset \} $$

$$ R_i = \{ x \in X : \mathcal{T}(\{x\}) \subseteq X \setminus M_i \} \cong \mathcal{T}^{-1}|_{F_i(X)}((X \setminus M_i)), \text{ for } i \in \{1, 2\}. $$

Notice that $R$ is a closed set in $X$ and $R_i$ is an open set in $X$ for $i \in \{1, 2\}$.

Also, $V_2 \subseteq R_1, V_1 \subseteq R_2$ and $(V_1 \cup V_2) \cap R = \emptyset$. Then $R_i \neq \emptyset$, $i \in \{1, 2\}$. By Lemma 3.3, if $x \in X$, then $\mathcal{T}(\{x\}) \cap M \neq \emptyset$. Hence, $X \subseteq R \cup R_1 \cup R_2$ and thus $X = R \cup R_1 \cup R_2$.

Notice that if $x \in R_1 \cap R_2$, then $\mathcal{T}(\{x\}) \subseteq X \setminus (M_1 \cup M_2)$, which contradicts Lemma 3.3. Therefore, $R_1 \cap R_2 = \emptyset$. Since $X$ is connected and $R_1$ and $R_2$ are open, then $R \neq \emptyset$.

There is $x \in R$ such that $x \in R_1 \cup R_2$, without lost of generality, we assume that there is $(x_n)_{n=1}^\infty \subseteq R_2$ such that $x_n \rightarrow x$. Since $\mathcal{T}(\{x\}) \cap M_2 \neq \emptyset$ and $M_2 \subseteq V_2$, we have that $\mathcal{T}(\{x\}) \in (X, V_2)$. By the continuity of $\mathcal{T}$, there is $k \in \mathbb{N}$ such that $\mathcal{T}(\{x_k\}) \in (X, V_2)$. Hence there is $z \in \mathcal{T}(\{x\}) \cap V_2$. By the choice of $V_2$, $\mathcal{T}(\{z\}) \subseteq U_2$. By [13, Theorem 3.7], $\mathcal{T}^\prime(\{z\}) \cap M \neq \emptyset$. As $M_1 \cup U_2 = \emptyset$, then $\mathcal{T}^\prime(\{z\}) \cap M_2 \neq \emptyset$. Since $\mathcal{T}(\{z\}) \subseteq \mathcal{T}(\mathcal{T}(\{x_k\})) = \mathcal{T}(\{x_k\}) \cap M_2 \neq \emptyset$ which contradicts that $x_k \in R_2$.

(2) Suppose $M$ is connected. Let $y \in X \setminus M$. By Lemma 3.3, there is $x_0 \in M$ such that $\mathcal{T}(\{x_0\}) \subseteq \mathcal{T}(\{y\})$. Let $U$ and $V$ be open sets of $X$ such that $M \subseteq U$, $y \in V$ and $U \cap V = \emptyset$. Since $\mathcal{T}$ is upper semicontinuous, there is an open set $W$ of $X$ such that $x_0 \in W$ and $\mathcal{T}(\{z\}) \subseteq U$, for each $z \in W$. Let $K = \bigcup \{\mathcal{T}(\{z\}) : z \in W\} \cup M$. Since $\mathcal{T}(\{z\}) \cap M \neq \emptyset$, for each $z \in W$, and $M$ is connected, we have that $\bigcup \{\mathcal{T}(\{z\}) : z \in W\} \cup M$ is connected. Thus, $K$ is a subcontinuum of $X$. It is clear that $K \subseteq U \subseteq X \setminus V$ and $x_0 \in W \subseteq K$. Therefore, $K \cap \{y\} = \emptyset$ and $x_0 \notin \mathcal{T}(\{y\})$, which contradicts that $\mathcal{T}(\{x_0\}) \subseteq \mathcal{T}(\{y\})$.

This completes the proof that $X = M$, which clearly implies that $\mathcal{T}(\{x\}) : x \in X$ is a decomposition of $X$.

**Theorem 3.7.** Let $X$ be a continuum. If $\mathcal{T} : 2^X \rightarrow 2^X$ is continuous, then $\mathcal{D} = \{\mathcal{T}(\{x\}) : x \in X\}$ is a continuous decomposition such that $X/\mathcal{D}$ is locally connected.

**Proof.** Since $\mathcal{T}$ is continuous, $\mathcal{T}$ is idempotent, by [14, Theorem 3.2.8]. Thus, $\mathcal{D}$ is a decomposition, by Theorem 3.3. It is clear that since $\mathcal{T}$ is continuous, $\mathcal{T}|_{F_1(X)}$ is continuous and $\mathcal{D}$ is a continuous decomposition. Finally, $X/\mathcal{D}$ is locally connected, by [13, Theorem 3.4].

If $\mathcal{D} = \{\mathcal{T}(\{x\}) : x \in X\}$ is a decomposition of $X$, then it is easy to see that $p \notin \mathcal{T}(\{q\})$ if and only if $q \notin \mathcal{T}(\{p\})$, for $p, q \in X$, $p \neq q$; i.e., $X$ is point $\mathcal{T}$–symmetric. Thus, by Theorem 3.7 and [14, Corollary 3.2.15], we have the following result which give us a positive answer to Question 7.2.1. of [14].

**Corollary 3.8.** Let $X$ be a continuum. If $\mathcal{T} : 2^X \rightarrow 2^X$ is continuous, then $X$ is both $\mathcal{T}$–symmetric and $\mathcal{T}$–additive.
Theorem 3.9. Let $X$ be a continuum. Then, $X$ is locally connected if and only if $T$ is onto and idempotent on closed sets.

Proof. If $X$ is locally connected then $T(K) = K$ for all $K \in 2^X$ [14, Corollary 3.1.25]. Thus, $T$ is onto and idempotent on closed sets. Suppose $T$ is onto and idempotent. By Theorem 3.2, $T$ is continuous. Since $T$ is onto, then necessarily $T(\{x\}) = \{x\}$ for all $x \in X$. Therefore by Theorem 3.7, $X$ is locally connected as the decomposition $D$ is trivial.

Question 3.10. Suppose $T$ is onto, is $X$ locally connected? or equivalently, is $T$ idempotent on closed sets?

The following result gives a positive answer to Problem 162 of the Houston Problem Book [5, p. 390] and characterizes all continua for which $T$ is continuous.

Theorem 3.11. Let $X$ be a continuum. Then, $T$ is continuous if and only if there exist a locally connected continuum $Y$ and a monotone and open map $f : X \to Y$ such that $T(A) = f^{-1}(f(A))$, for each $A \in 2^X$.

Proof. Suppose that $T$ is continuous. By Theorem 3.7, $Y = X/D$ is locally connected and $q : X \to Y$, the quotient map, is monotone and open. It is clear that $T(\{x\}) = q^{-1}(q(x))$, for each $x \in X$. Note that $X$ is $T-$additive, by Corollary 3.8. By [14, Corollary 3.1.46], $T(A) = \bigcup_{a \in A} T(\{a\})$, for each $A \in 2^X$. Therefore, $T(A) = \bigcup_{a \in A} T(\{a\}) = \bigcup_{a \in A} q^{-1}(q(a)) = q^{-1}(q(A))$.

Conversely, suppose $f : X \to Y$ is a map as in the hypothesis. Since $T(A) = f^{-1}(f(A))$, for each $A \in 2^X$, and $f$ is open and surjective, then $T$ is continuous by Theorems 1.8.22 and 1.8.24 of [14].

With the next two results, we answer Questions 7.2.3 and 7.25 of [14], in positive.

Theorem 3.12. Let $X$ be a continuum such that $T : 2^X \to 2^X$ is continuous. If $T(\{p\})$ has nonempty interior for some $p \in X$, then $X$ is indecomposable.

Proof. Since $T$ is continuous, $D = \{T(\{x\}) : x \in X\}$ is a decomposition of $X$, by Theorem 3.4. Notice that if $T(\{p\}) = X$, then $T(\{x\}) = X$, for each $x \in X$, and $X$ is indecomposable [14, Theorem 3.1.34]. Thus, suppose that $T(\{p\}) \neq X$. Since $T(\{p\})$ is closed, there exists $(x_n)_{n=1}^\infty \subseteq X \setminus T(\{p\})$ such that $x_n \to x_0$, $x_0 \in T(\{p\})$. Since $D$ is a decomposition (Theorem 3.4), $T(\{x_0\}) = T(\{p\})$ and $T(\{x_n\}) \cap T(\{x_0\}) = \emptyset$, for each $n \in \mathbb{N}$. Let $U$ be a nonempty open subset of $X$ such that $U \subseteq T(\{x_0\})$. It is clear that $T(\{x_0\}) \in \langle X, U \rangle$ and, since $T(\{x_n\}) \cap T(\{x_0\}) = \emptyset$, $T(\{x_n\}) \cap U = \emptyset$, for each $n \in \mathbb{N}$. Thus, $T(\{x_n\}) \notin \langle X, U \rangle$ contrary to our assumption that $T$ is continuous and $x_n \to x_0$. Therefore, $T(\{p\}) = X$ and $X$ is indecomposable.

Corollary 3.13. Let $X$ be a continuum such that $T : 2^X \to 2^X$ is continuous. Then, $X$ is decomposable if and only if $T(\{p\})$ has empty interior, for each $p \in X$. 

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Proof. If $X$ is decomposable, then $\mathcal{T}(\{p\})$ has empty interior, for each $p \in X$, by Theorem 3.12. Conversely, suppose that $\mathcal{T}(\{p\})$ has empty interior, for each $p \in X$. Thus, $X/D$ is a nondegenerate locally connected, where $D = \{\mathcal{T}(\{x\}) : x \in X\}$, by Theorem 3.7. Therefore, $X/D$ is decomposable [9, Theorem 2, p. 207] and, since the quotient map $q: X \to X/D$ is monotone, we have that $X$ is decomposable.

A continuum $X$ is irreducible if there are two points $p$ and $q$ of $X$ such that no proper subcontinuum of $X$ contains both $p$ and $q$. A continuum $X$ is of type $\lambda$ provided that $X$ is irreducible and each indecomposable subcontinuum of $X$ has empty interior. It is known that a continuum $X$ is of type $\lambda$ if and only if admits a finest monotone upper semicontinuous decomposition $\mathcal{G}$ such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc [15, Theorem 10, p. 15]. We say that a continuum $X$ is continuously irreducible provided that $X$ is of type $\lambda$ and its decomposition $\mathcal{G}$ is continuous. We characterize continuously irreducible continua, compare the following result with [12, Theorem 3.2].

**Theorem 3.14.** Let $X$ be an irreducible continuum. Then, $X$ is either indecomposable or continuously irreducible continuum if and only if the set function $\mathcal{T}$ is continuous.

**Proof.** If $X$ is continuously irreducible, then $\mathcal{T}$ is continuous, by [12, Theorem 3.2]. It is clear that $\mathcal{T}$ is continuous, if $X$ is indecomposable [14, Theorem 3.1.34].

Conversely, suppose that $\mathcal{T}: 2^X \to 2^X$ is continuous and $X$ is not indecomposable. By Theorem 3.11 there exist a locally connected continuum $Y$ and an open and monotone map $f: X \to Y$ such that $\mathcal{T}(A) = f^{-1}(f(A))$ for each $A \in 2^X$. Since $X$ is irreducible, $Y$ is irreducible, by [9, Theorem 3, p.192]. Hence, $Y$ is homeomorphic to $[0,1]$. Furthermore, $\mathcal{T}(\{x\})$ has empty interior for each $x \in X$, by Theorem 3.12. Thus, each indecomposable subcontinuum of $X$ has empty interior. Therefore, $X$ is a continuum of type $\lambda$ and $X$ is continuously irreducible, by [12, Theorem 3.2].

**References**

[1] D. P. Bellamy, Continua for which the set function $\mathcal{T}$ is continuous, Trans. Amer. Math. Soc., 151 (1970), 581-587.

[2] D. P. Bellamy, Some Topics in Modern Continua Theory, in Continua Decompositions Manifolds, (R. H. Bing, W. T. Eaton and M. P. Starbird, eds.), University of Texas Press, (1983), 1-26.

[3] D. P. Bellamy and J. J. Charatonik, The set function $\mathcal{T}$ and contractibility of continua, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 25 (1977), 47-49.

[4] D. P. Bellamy, L. Fernandez and S. Macías, On $\mathcal{T}$—closed sets, Topology Appl., 195 (2015), (Special issue honoring the memory of Mary Ellen Rudin), 209-225.
[5] H. Cook, W. T. Ingram, and A. Lelek, A list of problems known as Houston Problem Book, in: Continua with the Houston Problem Book, H. Cook, W. T. Ingram, K. T. Kuperberg, A. Lelek, and P. Minc (eds.), Lecture Notes in Pure and Appl. Math. 170, Dekker, New York, (1995), 365-398.

[6] A. Illanes and S. B. Nadler Jr., Hyperspaces, Fundamentals and recent advances, Pure and Applied Mathematics, Vol. 216, Marcel Dekker, New York, 1999.

[7] F. B. Jones, Concerning nonaposyndetic continua, Amer. J. Math., 70 (1948), 403-413.

[8] L. Fernández, On strictly point $\mathcal{T}$–symmetric continua. Topol. Proc., 35 (2010), 91-96.

[9] K. Kuratowski, Topology, Vol II, Academic Press, New York, N. Y., 1968.

[10] S. Macías, A class of one-dimensional, nonlocally connected continua for which the set function $\mathcal{T}$ is continuous, Houston J. Math. 32 (2006) 161-165.

[11] S. Macías, Homogeneous continua for which the set function $\mathcal{T}$ is continuous, Topology Appl., 153 (2006), 3397–3401.

[12] S. Macías, On continuously irreducible continua, Topology Appl., 156 (2009), 2357-2363.

[13] S. Macías, A decomposition theorem for a class of continua for which the set function $\mathcal{T}$ is continuous, Colloq. Math., 109 (2007), 163-170.

[14] S. Macías, Topics on Continua, Pure and Applied Mathematics Series, Vol. 275, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, London, New York, Singapore, 2005.

[15] E. S. Thomas Jr., Monotone decompositions of irreducible continua, Dissertationes Math. (Rozprawy Mat.) 50 (1966) 1-74.

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