The critical behavior and phase transition in the 2+1 dimensional Bañados, Teitelboim, and Zanelli (BTZ) black holes are discussed. By calculating the equilibrium thermodynamic fluctuations in the microcanonical ensemble, canonical ensemble, and grand canonical ensemble, respectively, we find that the extremal spinning BTZ black hole is a critical point, some critical exponents satisfy the scaling laws of the “first kind”, and the scaling laws related to the correlation length suggest that the effective spatial dimension of extremal black holes is one, which is in agreement with the argument that the extremal black holes are the Bogomol’nyi saturated string states. In addition, we find that the massless BTZ black hole is a critical point of spinless BTZ black holes.

I. INTRODUCTION

In 1973, Bardeen, Carter, and Hawking [1] established a remarkable mathematical analogy existed between the laws of thermodynamics and the laws of black hole mechanics derived from general relativity. If the formal replacements $E \rightarrow M$, $T \rightarrow c\kappa$, and $S \rightarrow A/8\pi c$ (c is a constant) are made in the laws of thermodynamics, one can obtain immediately the laws governing the mechanics of black holes, where $E$, $T$, and $S$ are the internal energy, temperature, and entropy of an ordinary thermodynamic system, while $M$, $\kappa$, and $A$ denote the mass, surface gravity, and horizon area of a black hole, respectively. The physical analogy, however, seems to have problems due to the fact that in classical general relativity the thermodynamic temperature of a black hole should be absolute zero. When quantum effects are taken into account, Hawking [2] found that a black hole absorbs and emits particles as a black body at the temperature $T = \kappa/2\pi$. The discovery of Hawking resolved the puzzle of this physical analogy and provides a basis of the black hole entropy suggested firstly by Bekenstein [3]. Since then, most people have believed that a black hole is a thermodynamic system, and the thermodynamics of black holes has been developed [4,5].

The phase transition is an important phenomenon in the ordinary thermodynamics. Therefore, it is natural to ask whether the phase transitions exist in the black hole thermodynamics. According to the infinite discontinuity of the heat capacity of Kerr-Newman black holes, Davies [4] claimed that a second-order phase transition takes place when the black holes leap over the discontinuity point of heat capacity. Although some researches on this singularity point have been carried out [6,7,8,9,10], and Lousto [9] even claimed that the phase transition of Davies satisfies the scaling laws of critical points, the reasons that the Davies’ point is regarded as a phase transition point are insufficient, because the event horizon does not lose its regularity and the internal state of the black hole is unaffected at those points. Furthermore, by investigating the stability of black holes in the different environments, Kaburaki and his collaborators [11,12] found that the Davies’ points are in fact related to the presence of the changes of stability, and further claimed that the divergence of heat capacity at those points does not mean the occurrence of phase transitions.

On the other hand, more than a decade ago, based on the analogy between the Kerr black hole and a laser system, Curir [13] claimed that a critical point exists at the extremal limit of the hole and a phase transition takes place from the extremal to nonextremal Kerr black hole. By making use of Landau-Lifshitz theory of nonequilibrium thermodynamic fluctuations, Pavón and Rubí [14] calculated, respectively, some second moments of relevant quantities for Schwarzschild black holes, Kerr black holes, and Reissner-Nordström (RN) black holes, and found that some second moments diverge for the extremal Kerr black holes and extremal RN black holes, but are always finite for Schwarzschild black holes. The most important thing is that nothing special happens at the Davies’ points. The divergence of second moments leads to the argument of Pavón and Rubí that a second-order phase transition occurs from the extremal to nonextremal black holes. This conclusion is in agreement with the one of Curir although the two methods they employed are different.

To find the essence and universality of divergence of second moments, Cai, Su and Yu discussed the nonequilibrium and equilibrium fluctuations for various charged dilaton black holes [15], and found the difference of the outer and inner horizons plays a crucial role in the divergence of second moments. Thus, they suggested that the difference plays the role of an order parameter in the second-order phase transition of black holes [16]. More recently, Kaburaki [17] has shown that the critical exponents in the extremal Kerr-Newman black holes also obeys the scaling laws. As is well known, the extremal black hole is very different from the nonextremal one, for example, in the geometric structure, thermodynamics [18,19], the essence of radiations, etc.. Wilczek et al [20] argued that the thermodynamic description is inadequate for some extremal black holes and the behavior of these holes resembles the normal elementary particles’ or
strings’ behavior.

To investigate the properties of black holes, one of the powerful methods is to study the interaction between quantum fields and black holes. Traschen [21] has recently finished such a issue. By studying the behavior of a massive charged scalar field on the background of RN black holes, Traschen has found that the spacetime geometry near the horizon of the extremal RN black holes has a scaling symmetry, which is absent for the nonextremal holes, a scale being introduced by the surface gravity. The scaling symmetry results in that an external source has a long range influence on the extremal background, compared to a correlation length scale which falls off exponentially fast in the case of nonextremal holes. The long range correlation is just the characteristic of phase transitions. Although these evidences are in favor of the existence of phase transitions in the extremal black holes, further study is needed in order to better understand the behavior of the critical point because the Hawking temperature of an extremal black hole vanishes and the ordinary thermodynamics is invalid in this limit.

In recent years, one of the important progresses in general relativity is the discovery of the 2+1 dimensional black holes by Bañados, Teitelboim, and Zanelli (BTZ) [22]. A review paper on the 2+1 dimensional black holes can be seen in Ref. [23]. Although the BTZ black holes are the counterparts of Kerr black holes in 2+1 dimensions, their thermodynamic properties have some differences. An important difference is that the heat capacity of BTZ black holes is always positive and has no discontinuous jump, while the heat capacity of Kerr holes is negative for the small angular momentum, and positive for the large angular momentum, that is, it has an infinite discontinuity at the Davies’ point. This property results in that not only the microcanonical ensemble, but also the canonical ensemble can describe the gas of BTZ black holes, and no limited temperature exists [24], unlike the gas of the 1+1 dimensional dilaton black holes, which has a limited temperature [25].

The purpose of this paper is to investigate the equilibrium thermodynamic fluctuations of BTZ black holes in the different environments and the behavior of critical points. We find that similar to the Kerr black holes, a critical point exists in the extremal BTZ black holes. Furthermore, the massless BTZ black hole is found to be a critical point as well, which is different from the case of Schwarzschild black holes. In particular, we find that the effective spatial dimension of extremal black holes is one, which is surprisingly in agreement with the arguments that the extremal black holes are the Bogomol’nyi saturated string states and the extremal black holes can be regarded as elementary particles like states in string theory [26].

The paper is organized as follows. In the next section we will briefly review the BTZ black holes and introduce the fluctuation theory of equilibrium thermodynamics. In Sec. III we will discuss the equilibrium fluctuations of thermodynamic quantities for the spinning BTZ black holes in the microcanonical ensemble, canonical ensemble, and grand canonical ensemble, respectively, study the critical behavior of extremal BTZ black holes, and calculate some critical exponents of relevant quantities. We will discuss the case of the spinless BTZ black holes, in Sec. IV, and argue that the massless BTZ black hole is a critical point of the spinless BTZ black holes, and a second-order phase transition takes place from the massless BTZ black hole to a generic spinless BTZ black hole. The conclusion and discussion are included in Sec. V.

II. BTZ BLACK HOLES AND THE THEORY OF EQUILIBRIUM FLUCTUATIONS

In 2+1 dimensions, the Einstein gravity theory with a negative cosmological constant has the BTZ black hole solution, which can be described by the line element [22]

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2$$ (1)

where

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4\pi^2}, \quad N^\phi(r) = -\frac{J}{2\pi l^2}.$$ (2)

$M$ and $J$ are two integration constants and represent the mass and angular momentum of the BTZ black holes (here the unit $8G = 1$ is used), respectively, and $l^{-2}$ denotes the cosmological constant.

The horizons are given by the condition that the lapse function $N^2(r) = 0$, and read

$$r_{\pm} = \left[\frac{M}{2} l^2 (1 \pm \triangle)\right]^{1/2}, \quad \triangle = [1 - (J/M l)^2]^{1/2}.$$ (3)

In order that solution (1) has the structure of black holes, evidently, we must impose the condition

$$M \geq 0 \quad \text{and} \quad J \leq M l.$$ (4)

The BTZ black hole solution (1) is neither asymptotically flat nor asymptotically de Sitter, but asymptotically anti-de Sitter. If $M = -1$ and $J = 0$, the 2+1 dimensional anti-de Sitter space is restored

$$ds^2_{\text{ADS}} = -(1 + \frac{r^2}{l^2}) dt^2 + (1 + \frac{r^2}{l^2})^{-1} dr^2 + r^2 d\phi^2.$$ (5)

The BTZ black hole solution can be obtained by identifying points in this anti-de Sitter space (5) and using the orbits of a spacelike Killing vector field. In the BTZ black hole solution (1), a special case exists. That is, if $M = 0$ and $J = 0$, the solution (1) then reduces to

$$ds^2_{\text{VAC}} = -\frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2,$$ (6)

which is the ground state of the BTZ solutions, and describes a massless black hole. The massless BTZ solution
(6) is of some interesting properties. First, the solution (6) has an infinitely long throat for small $r > 0$, which is reminiscent of the extremal RN black hole solution in 3+1 dimensions. This leads the solution (6) to have similar properties as those of the extremal RN black holes. Second, the solution (6) has two exact supersymmetries, while a generic extremal BTZ black hole has only one supersymmetry [27].

Define
\[ \Omega_H \equiv - \frac{g_{t\phi}}{g_{\phi\phi}} \bigg|_{r_+} = \frac{J}{2r_+^2}, \] (7)

analogous to its counterpart in the Kerr black holes, $\Omega_H$ can be regarded as the angular velocity of the BTZ black holes. In addition, the surface $r^s$ is the surface of infinite redshift where $g_{tt}(r^s)$ vanishes,
\[ r^s = \sqrt{Ml}. \] (8)

Obviously, $r^s \geq r_+$. The region
\[ r_+ \leq r \leq r^s \] (9)
is called as the ergosphere of the BTZ black hole. Because of the appearance of the ergosphere, the Penrose process and Misner superradiation can take place in the BTZ black holes, even under the extremal condition ($J = Ml$). When $J = 0$, the ergosphere disappears and the superradiation thereby is killed.

By analyzing the geometry of solution (1), one can find that it has two Killing vectors, timelike $k_{(t)}$ and spacelike $m_{(\phi)}$. Hence,
\[ p^\mu = k_{(t)}^\mu + \Omega_H m_{(\phi)}^\mu, \] (10)
is also a Killing vector. According to the definition of surface gravity, $k_{(r)}^{\mu} p_{(r)}^\nu = \kappa l^\mu$, we obtain the surface gravity of BTZ black holes at the horizon $r_+$,
\[ \kappa = \frac{r_+^2 - r_-^2}{r_+ l^2} = \frac{M \Delta}{r_+}. \] (11)

Furthermore, making use of the definition of mass,
\[ M = \frac{1}{4\pi} \int_{r_+} k_{(t)}^{\mu\nu} d\Sigma_{\mu\nu}, \] (12)
we can get the Smarr-type mass formula of BTZ black holes,
\[ M = \frac{\kappa}{2\pi} L + \Omega_H J, \] (13)
where $L = 2\pi r_+$ is the perimeter of the horizon $r_+$. The Hawking temperature of the black holes is
\[ T = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi r_+ l^2}. \] (14)

With the aid of Euclidean action method, one can easily obtain the black hole entropy [22]
\[ S = 4\pi r_+ = 4\pi \left[ \frac{M}{2} l^2 (1 + \Delta) \right]^{1/2}, \] (15)
which is the twice perimeter of the horizon. Varying the mass formula (13) yields the first law of thermodynamics for the BTZ black holes
\[ dM = TdS + \Omega_H dJ. \] (16)
In terms of the formula $C_J = \left( \frac{\partial M}{\partial T} \right)_J$, we get the heat capacity of the hole,
\[ C_J = \frac{4\pi r_+ \Delta}{2 - \Delta}. \] (17)

Because of $0 \leq \Delta \leq 1$, $C_J \geq 0$ is always satisfied. It is very different from that of Kerr black holes [4]. When $J = 0$, i.e., $\Delta = 1$, one has $C_J \equiv C = 4\pi l\sqrt{M}$, which means the temperature to increase with the mass, and to $T = C = 0$ as $M = 0$. When $J = Ml$, i.e., $\Delta = 0$, one has $C_J = 0$, which corresponds to the extremal BTZ black holes, and the vanishing Hawking temperature.

We now turn to the fluctuation theory of equilibrium thermodynamics. In general, the fluctuations of thermodynamic quantities of a thermodynamic system equilibrating with different environments are different. In some textbooks on thermodynamics [28-30], however, this point is not distinguished clearly. In order that the Einstein’s theory of thermodynamic fluctuations is applicable in the various thermodynamic environments, Kaburaki [12] developed the Einstein’s theory and made it be convenient to discuss the fluctuations in various ensembles.

The equilibrium states of a thermodynamic system in a given environment are characterized by the Massieu function, $\Phi$, and the infinitesimal variation of the function $\Phi$ can be written as [12]
\[ d\Phi = \sum_{i=1}^{n} X_i dx_i, \] (18)
where the variables $\{x_i\}$ are called the intrinsic variables and are specified directly by the given thermodynamic environment. The $\{X_i\}$ forms a set of variables conjugate to $\{x_i\}$, and can be expressed as functions of their intrinsic variables: $X_i = (\partial \Phi / \partial x_i)_{\bar{x}_i}$, where $\bar{x}_i$ denotes the set of variables $\{x_i\}$ eliminating the $x_i$. Thus the probability of the system deviating the equilibrium state can be given by the macroscopic distribution function
\[ P(\xi_1 . . . \xi_n) d\xi_1 . . . d\xi_n \sim \exp[k_B^{-1}(\bar{\Phi} - \Phi)]d\xi_1 . . . d\xi_n, \] (19)
where $k_B$ is the Boltzmann’s constant, $\xi_i \equiv \delta X_i$ stands for the deviation of the $i$th conjugate variable from its
equilibrium value and $\hat{\Phi}$ represents the Massieu function. Analytically continued to the nonequilibrium sequence near the equilibrium sequence in the phase space. By expanding the Massieu function with respect to conjugate variables up to the second order, one has

$$P(\{\xi_i\}) \sim \exp[-(2k_B)^{-1} \sum_{i=1}^{n} \lambda_i \xi_i^2],$$  \hspace{2cm} (20)

where $\lambda_i = (\partial x_i / \partial X_i)_{\bar{x}_i} = (\partial^2 \hat{\Phi} / \partial x_i^2)_{\bar{x}_i}^{-1}$ are the eigenvalues of the fluctuation modes $\delta X_i$. From (20), as is expected the average of each fluctuation mode always vanishes, $< \delta X_i > = 0$, and the second moments of equilibrium fluctuations are

$$< \delta X_i \delta X_j > = \frac{k_B}{\lambda_i} \delta_{ij} = k_B \left( \frac{\partial^2 \hat{\Phi}}{\partial x_i^2} \right)_{\bar{x}_i} \delta_{ij}. \hspace{2cm} (21)$$

From (21) we can see that the second moments diverge only if $\lambda_i = 0$, i.e., $(\partial X_i / \partial x_i)_{\bar{x}_i} = \pm \infty$. According to the turning point method of stability analysis developed by Katz [31], the change of stability occurs only at the turning point where the tangent of the curve changes its sign through a infinity (a vertical tangent) in the “conjugate diagrams”, in which the conjugate variables $\{X_i\}$ are plotted against a intrinsic variable (other intrinsic variables are fixed). The Davies’ points are just such turning points. The divergence of some second moments at the Davies’ points only means the change of stability for Kerr-Newman black holes, and does not have close relation to the second-order phase transitions [11,12]. Instead the divergence of some second moments at the extremal limit might imply the presence of phase transitions because the extremal limit is not the turning point.

**III. FLUCTUATIONS OF SPINNING BTZ BLACK HOLES AND SCALING LAWS**

In this section, by using the fluctuation theory developed in the previous section, we will discuss the equilibrium fluctuations of spinning BTZ black holes in the microcanonical ensemble, canonical ensemble, and grand canonical ensemble, respectively, and find that all the second moments are finite in these ensembles except for the extremal BTZ black holes. The extremal limit is a critical point and critical exponents obey the scaling laws. The effective spatial dimension of extremal black holes is found to be one. The spinless BTZ black holes will be discussed separately in the next section.

**A. Equilibrium fluctuations of spinning BTZ black holes**

The proper Massieu functions are different for different environments. Therefore, the corresponding fluctuations and second moments are different in the various ensembles. Let us discuss the case of microcanonical ensemble first.

(1) Microcanonical ensemble. In this ensemble, nothing of the thermodynamic system can be exchanged with its environment. The proper Massieu function is the entropy of system. For the BTZ black holes, rewriting the first law of BTZ black holes (16), we have

$$d\Phi_1 = dS = \beta dM - \mu dJ,$$ \hspace{2cm} (22)

where $\beta = 1/T$ and $\mu = \Omega_H$. Thus, in this ensemble, the intrinsic variable are $\{M,J\}$ and the corresponding conjugate variables are $\{\beta,-\mu\}$. The eigenvalues for fluctuation modes are

$$\lambda_{1m} = \left( \frac{\partial M}{\partial \beta} \right)_J = -T^2 C_J,$$ \hspace{2cm} (23)

$$\lambda_{1j} = - \left( \frac{\partial J}{\partial \mu} \right)_M = -T I_M,$$ \hspace{2cm} (24)

where $C_J$ is the heat capacity given by Eq. (17), and

$$I_M \equiv \beta \left( \frac{\partial J}{\partial \mu} \right)_M = \frac{1}{2} + \frac{J^2}{8M^2 r_+^4} \Delta + \frac{J^2}{2M^2 r_+^2 \Delta},$$ \hspace{2cm} (25)

is the “moment of inertia” of BTZ black holes. Some second moments can be easily given,

$$< \delta \beta \delta \beta > = -k_B \frac{\beta^2}{C_J},$$

$$< \delta \mu \delta \mu > = -k_B \frac{\beta}{I_M},$$

$$< \delta \Omega_H \delta \Omega_H > = -k_B \left[ \frac{T}{I_M} + \frac{\Omega_H^2}{C_J} \right],$$

$$< \delta \Omega_H \delta \beta > = k_B \frac{\beta \Omega_H}{C_J}. \hspace{2cm} (26)$$

Obviously, these second moments are finite for the nonextremal BTZ black holes. When the extremal limit is approached, i.e., $\Delta \to 0$, however, the eigenvalues $\lambda_{1m}$ and $\lambda_{1j}$ approach zero, and all of these second moments (26) diverge. The divergence means that the extremal BTZ black hole is a critical point.

(2) Canonical ensemble. In this ensemble, the BTZ black hole can only exchange the heat with surroundings. The variation of the proper Massieu function is

$$d\Phi_2 = dS - d(\beta M),$$

$$= -Md\beta - \mu dJ,$$ \hspace{2cm} (27)

which gives the intrinsic variables $x_i = \{\beta,J\}$ and the conjugate variables $X_i = \{-M,-\mu\}$. The corresponding eigenvalues are
\[ \lambda_{2\beta} = -\left( \frac{\partial \beta}{\partial M} \right)_J = \frac{\beta^2}{C_J}, \quad (28) \]
\[ \lambda_{2j} = -\left( \frac{\partial J}{\partial \mu} \right)_\beta = -\frac{I_\beta}{\beta}, \quad (29) \]

where
\[ I_\beta \equiv \beta \left( \frac{\partial J}{\partial \mu} \right)_\beta, \]
\[ = \left[ \frac{1}{2r_+} + \frac{J}{r_+^2} \left( \frac{J}{M^2l^2\Delta} \right) \right]^{-1}, \quad (30) \]
and
\[ G = \frac{J}{M^2l^2\Delta} \left( \frac{Ml^2}{4r_+} - \frac{r_+}{\Delta} \right) \left( \frac{(1 + \Delta)^2}{4r_+} \right) \]
\[ - \frac{r_+}{m} + \frac{J^2}{M^2l^2\Delta} \left( \frac{Ml^2}{4r_+} - \frac{r_+}{\Delta} \right) \right]^{-1}. \]

With the help of Eqs. (20) and (21), in this ensemble, some nonvanishing second moments are
\[ < \delta M \delta M > = k_B T^2 C_J, \]
\[ < \delta S \delta M > = k_B T C_J, \]
\[ < \delta \mu \delta \mu > = -k_B \frac{\beta}{I_\beta}, \]
\[ < \delta \Omega_H \delta \Omega_H > = -k_B \frac{T}{I_\beta}, \]
\[ < \delta S \delta S > = k_B C_J. \quad (31) \]

These second moments are also finite for nonextremal BTZ black holes. In the extremal limit, \( \lambda_{2\beta} \) diverges, but \( \lambda_{2j} \) does not. Thus, \( < \delta M \delta M >, < \delta S \delta M >, \) and \( < \delta S \delta S > \) are finite, but \( < \delta \mu \delta \mu > > < \delta \Omega_H \delta \Omega_H > \) diverge.

(3) Grand canonical ensemble. This ensemble implies that the BTZ black holes not only can exchange the heat with surroundings, but also do work to the surroundings. The variation of proper Massieu function in this case is
\[ d\Phi_3 = d\Phi_2 + d(\mu J) \]
\[ = -Md\beta + Jd\mu. \quad (32) \]

The intrinsic variables become \( x_i = {\beta, \mu} \), and the corresponding conjugate variables \( X_i = {-M, J} \). The eigenvalues of fluctuation modes are
\[ \lambda_{3\beta} = -\left( \frac{\partial \beta}{\partial M} \right)_J = \frac{\beta^2}{C_\mu}, \quad (33) \]
\[ \lambda_{3\mu} = \left( \frac{\partial \mu}{\partial J} \right)_\beta = \frac{\beta}{I_\beta}, \quad (34) \]
where
\[ C_\mu = \frac{\partial M}{\partial T} \mu, \]
\[ = 2\pi \left( \frac{\Delta}{2r_+} + \left( \frac{J}{M r_+ + r_+^2 \Delta} \right)^{-1} - \left( \frac{M}{r_+ + r_+^2 \Delta} \right) \right), \quad (35) \]

and
\[ \mathcal{H} = \left[ \frac{J}{M^2 r_+} + \frac{I^2 (1 + \Delta) J}{4 M r_+^4 \Delta} \right] \]
\[ + \frac{J^2}{M^2 l^2} \left( \frac{J}{M r_+ + r_+^2 \Delta} + \frac{J M l^2}{4 M r_+^4 \Delta} \right) \]
\[ \left[ \left( \frac{1}{M r_+} + \frac{J}{M^2 l^2} \left( \frac{J}{M r_+ + r_+^2 \Delta} + \frac{J M l^2}{4 M r_+^4 \Delta} \right) \right]^{-1}. \]

The nonvanishing second moments are easily given and read
\[ < \delta M \delta M > = k_B T^2 C_\mu, \]
\[ < \delta J \delta J > = k_B T I_\beta, \]
\[ < \delta S \delta S > = k_B (C_\mu + \mu^2 T I_\beta), \]
\[ < \delta \Omega_H \delta \Omega_H > = k_B T C_\mu, \]
\[ < \delta \mu \delta \mu > = -k_B \mu^2 T I_\beta. \quad (36) \]

In this case, the eigenvalues \( \lambda_{3\beta} \) and \( \lambda_{3\mu} \) diverge under the extremal condition, but, all of these second moments are finite and vanish for the extremal BTZ black holes.

**B. Scaling laws at the critical point**

To discuss the critical behavior of an isolated black holes, it is appropriate to choose the microcanonical ensemble, in which the proper Massieu function is the entropy of black holes. From the previous subsection we see that in the microcanonical ensemble, the eigenvalues of fluctuation modes are zero and nonvanishing second moments diverge for extremal BTZ black holes. Therefore, the extremal limit is a critical point of BTZ black holes. The extremal and nonextremal BTZ black holes are two different phases. In the ordinary thermodynamic system, the order parameters are defined usually as the differences of conjugate variables between the two phases. Due to the differences between black holes and ordinary thermodynamic system, however, the corresponding order parameters in black hole thermodynamics are defined as the differences of conjugate variables between the two phases. This is because that the intensive quantities are different in the two phases while extensive ones are common. Correspondingly, Kaburaki [17] defined various critical exponents associated with some thermodynamic quantities of Kerr-Newman black holes. For the
BTZ black holes, in this case, the intrinsic variables \( x_i = \{M, J\} \), and the conjugate variables \( X_i = \{\beta, -\mu\} \). Thus, \( \eta_\beta = \beta_+ - \beta_- \) and \( \eta_J = \mu_+ - \mu_- \) can be regarded as the order parameters of BTZ black holes, where the suffixes “+” and “−” mean that the quantity is taken at the outer horizon and inner horizon, respectively. The second-order derivatives of entropy with respect to the intrinsic variables are the inverse eigenvalues,

\[
\tilde{\zeta}_1 \equiv \left( \frac{\partial^2 S}{\partial M^2} \right)_J = \lambda_1^{-1} = -\frac{\beta^2}{C_J}, \\
\tilde{\zeta}_2 \equiv \left( \frac{\partial^2 S}{\partial J^2} \right)_M = \lambda_2^{-1} = -\frac{\beta}{I_M}.
\]

According to the definition of Kaburaki [17], some critical exponents can be given as

\[
\tilde{\zeta}_1 \sim \varepsilon_M^{\alpha} \quad \text{(for } J \text{ fixed)}, \\
\sim \varepsilon_J^{\varphi} \quad \text{(for } M \text{ fixed)}, \\
\tilde{\zeta}_2 \sim \varepsilon_M^{\gamma} \quad \text{(for } J \text{ fixed)}, \\
\sim \varepsilon_J^{\sigma} \quad \text{(for } M \text{ fixed)}, \\
\eta_J \sim \varepsilon_M^{\beta} \quad \text{(for } J \text{ fixed)}, \\
\sim \varepsilon_J^{\mu} \quad \text{(for } M \text{ fixed)},
\]

where \( \varepsilon_M \) and \( \varepsilon_J \) represent the infinitesimal deviations of \( M \) and \( J \) from their limit values. From Eqs. (39)-(41), these critical exponents are easily obtained,

\[
\alpha = \varphi = \gamma = \sigma = 3/2, \quad \beta = \delta^{-1} = -1/2.
\]

These values are exactly the same as those of Kerr-Newman black holes [17]. It shows the universality of critical behavior for extremal black holes. Naturally, those critical exponents satisfy the scaling laws of the “first kind”,

\[
\alpha + 2\beta + \gamma = 2, \\
\beta(\delta - 1) = \gamma, \\
\varphi(\beta + \gamma) = \alpha.
\]

The scaling laws are related to the fact that the entropy (15) of BTZ black holes is a homogeneous function,

\[
S(\lambda M, \lambda J) = \lambda^{1/2} S(M, J),
\]

where \( \lambda \) is a positive constant. Because the extremal black holes separate the nonextremal black holes from the naked singularities and the thermodynamic knowledge for the naked singularity is still lacking, there exist some shortcomings in these calculations. For example, these critical exponents can be computed only in one side, and the ordinary thermodynamics is invalid in the extremal limit. Despite these drawbacks, the differences between extremal and nonextremal black holes seem to be enough to show that the extremal limit is a critical point and a second-order phase transition occurs when the extremal black holes become the nonextremal black holes.

To the best of our knowledge, there exist at least three aspects of differences between extremal and nonextremal black holes: (1) Symmetry and geometric structure. As is well known, the extremal RN black holes are the solutions of the \( N = 2 \) supergravity, and so are of the supersymmetry [32,33]. In fact, many extremal black holes are also of the supersymmetry [34]. More recently, Coussaert and Henneaux [27] have proved that the supersymmetry also exists in the extremal BTZ black holes. But these supersymmetric properties are absent for nonextremal black holes. According to the general arguments to second-order phase transitions, we know that the presence of phase transition is always accompanied by the change of symmetry of a thermodynamic system. Therefore, The occurrence of a phase transition at the extremal black holes is in agreement with the change of symmetry of black holes. The extremal black holes are in the disordered phases, while the nonextremal black holes are in the ordered phases. In addition, it is known that the differences of topological structures exist in the extremal and nonextremal black holes [18,19]. For example, in the Euclidean manifolds of black hole solutions, a conical singularity exists at the event horizon for a nonextremal black hole and results in the periodicity of the Euclidean time in order to remove the singularity, which gives the Hawking temperature of the black hole. The periodicity is absent for the extremal black holes. This gives rise to the argument that an extremal black hole can be in equilibrium with a heat bath at an arbitrary temperature [18]. (2) The difference in thermodynamics. A nonextremal black hole can be described very well by thermodynamics, but the thermodynamic description is inadequate for some extremal black holes [20]. (3) The kinds of radiations. As is well known, the nonextremal black holes have both the thermal Hawking evaporation and the nonthermal superradiation. However, the Hawking evaporation disappears and only the superradiation exists in extremal black holes because the Hawking temperature vanishes for extremal black holes.

To further explore the properties of extremal black holes, it is of interest to compute other critical exponents. The two-point correlation function is a powerful tool to study the critical behavior. Near the critical points, the correlation function has usually the form at a large distance [28-30],

\[
G(r) \sim \frac{\exp(-r/\xi)}{r^{d-2+\eta}},
\]

where \( \eta \) is the Fisher’s exponent, \( d \) is the effective spatial dimension of the system under consideration, and \( \xi \) is the correlation length which is also divergent at critical points. The critical exponents related to the correlation length are defined as

\[
\xi \sim \varepsilon_M^{\nu} \quad \text{(for } J \text{ fixed)}, \\
\sim \varepsilon_J^{\mu} \quad \text{(for } M \text{ fixed)},
\]

for the BTZ black holes. Combining these critical exponents \( \nu, \mu, \) and \( \eta \) with the effective spatial dimension \( d \)
and critical exponents (42) yields the scaling laws of the “second kind”,
\[ \nu(2 - \eta) = \gamma, \quad \nu d = 2 - \alpha, \quad \mu(\beta + \gamma) = \nu. \] (49)

Unfortunately, we have not as yet knowledge about the two-point correlation function of quantum black holes. To proceed to our discussion, here we use the correlation function of a scalar field in the black hole background to mimic the one of black holes. The work of Traschen [21] tells us that for the extremal RN black holes, the scalar field has the scaling symmetry and long range correlations, i.e., the effect of the source falls off \( y^{-1} \); while for the nonextremal black holes, there is no scaling symmetry, and the influence of the source falls off exponentially fast, like \( e^{2\kappa y} \), where \( \kappa \) is the surface gravity of black holes and \( y \) is the usual tortoise coordinate. Therefore, the inverse surface gravity of black holes plays the role of correlation length. In the BTZ black holes, this statement is also valid [35,36]. With the help of (48) and (11), we get
\[ \nu = \mu = 1/2. \] (50)

Substituting (50) into (49), we have
\[ \eta = -1, \quad d = 1. \] (51)

Due to the universality of scaling laws, we surprisingly find that the effective dimension of extremal black holes is one, at least for the BTZ black holes and Kerr-Newman black holes because their corresponding critical exponents are same [17]. The result is in agreement with the point of view that the extremal black holes are the Bogomol’nyi saturated string states, or more generally, the extremal black hole is a special kind of strings [26]. Of course, in order to confirm completely the assumption, further investigation for extremal black holes is necessary. As to the BTZ black hole, an important fact is that the BTZ black hole is also an exact solution of string theory in three dimensions [37]. Horowitz and Strominger [38] have shown that some extremal black extended solutions in string theories are precisely the supersymmetric solutions describing the macroscopic fundamental strings or \( p \)-branes. Therefore, our conclusion is consistent with these evidences. In the meantime, it seems to imply the critical behavior of extremal black holes could be confirmed in the string theories. In addition to the BTZ black hole solution, the three dimensional string theory has the black string solution, dual solution of the BTZ black hole. The black string has the similar quantum properties as those of the BTZ black hole [39]. So we argue that the extremal black string is also a critical point and the similar critical behavior will appear in the extremal black strings.

In addition, in the case of extremal black holes, the correlation function (47) becomes
\[ G(r) \sim r^2 \exp(-r/\xi), \] (52)

which shows that the radiation of extremal black holes is nonthermal. It is again in agreement with the fact that the Hawking evaporation is killed in the extremal black holes and only the nonthermal superradiation is left.

All these discussions manifest that the extremal black hole is a critical point, and a phase transition takes place from the extremal to nonextremal black holes. In this phase transition point, the scaling laws of the “first kind” is satisfied automatically, and the “second kind” suggests that the effective spatial dimension of extremal black holes is one. This is the common feature of extremal black holes. In the BTZ black holes, we find that the massless BTZ black hole is also a critical point. This is a special example in the known black holes. In the next section, we will discuss the case.

IV. THE CRITICAL POINT IN SPINLESS BTZ BLACK HOLES

In the spinning BTZ black hole solution (1), if \( J = 0 \), Eq. (1) then describes a spinless BTZ black hole,
\[ ds^2 = -dt^2 + (t^2 + r^2)^{-1} dr^2 + r^2 d\phi^2. \] (53)
The spinless BTZ black holes have only one physical parameter, mass \( M \), and a horizon \( r_+ = \sqrt{M} \). So it has no extremal limit in the usual sense. The Hawking temperature of the hole is \( \beta^{-1} = \sqrt{M}/(2\pi) \), the heat capacity is \( C = 4\pi \sqrt{M} \), and the entropy \( S = 4\pi \sqrt{M} \). When \( M \rightarrow 0 \), all these quantities approach zero. In the microcanonical ensemble, from Eqs. (23) and (26) we see that the eigenvalue of temperature fluctuation mode is zero and \( \delta \beta \delta \beta > 0 \) diverges as \( M \) approaches zero. Therefore, the massless BTZ black hole may be also a critical point. In fact, it indeed corresponds to the extremal black holes in the usual sense. First, the massless BTZ black hole has zero Hawking temperature, zero entropy, and vanishing heat capacity. These are the same as the corresponding quantities of usual extremal black holes. Although the usual extremal black holes have the nonzero area of horizon, their entropies vanish identically [18,19]. So no difference of thermodynamics exists between the massless BTZ black holes and usual extremal black holes. The only difference is that \( M = 0 \) for the spinless BTZ black hole in the extremal limit. Second, Coussaert and Henneaux [27] have shown that the massless BTZ black hole has two exact supersymmetries, and the supersymmetry is absent for a generic spinless BTZ black hole. This manifests that the massless hole and generic spinless BTZ black hole is in the disordered phase and the generic spinless BTZ black hole in the ordered phase. Moreover, the supersymmetry shows the massless BTZ black hole corresponds...
to the usual extremal black hole. Third, the geometric structure of the massless BTZ black hole looks like the one of extremal RN black holes. Both of them have an infinite long throat near their horizons and scaling symmetry. Although the massless BTZ black hole has zero-length horizon, the singularity which coincides with the horizon, is invisible to an observer at infinity \[27\]. So, in this paper the massless BTZ solution is referred to as the massless BTZ black hole. Finally, unlike the flat Minkowski spacetime, the background of massless BTZ black holes is not flat, and the Casimir energy exists, which results in the quantum instability of the massless BTZ black hole \[35\]. When the reaction effect of quantum fields is taken into account, a non-zero horizon will be developed in the massless BTZ black hole. According to this point, Lifschytz and Ortiz \[35\] argued that the massless hole is not the end of Hawking evaporation and the end might be the 2+1 dimensional anti-de Sitter spacetime \(5\). Thus, the massless hole separates the generic spinless BTZ black hole from the anti-de Sitter space.

Combining the above points, we argue that the massless BTZ black hole is equivalent to the usual extremal black hole in certain senses, and is a critical point of spinless BTZ black holes. A second-order phase transition will take place from \(M = 0\) to \(M \neq 0\) black holes because the entropy is continuous in this process.

V. CONCLUSION AND DISCUSSION

In this work we have investigated the critical behavior and phase transition in the 2+1 dimensional BTZ black holes. By calculating the equilibrium fluctuations of BTZ black holes, we have found that some second moments are divergent for extremal BTZ black holes in the microcanonical ensemble and canonical ensemble, but all second moments are finite in the grand canonical ensemble even under the extremal condition. The divergence of second moments in the microcanonical ensemble shows the extremal BTZ black hole is the critical point of the equilibrium sequence of BTZ black holes. Some critical exponents have been calculated and they satisfy the scaling laws of the “first kind”. By using the correlation function of scalar fields in the black hole background to replace the one of quantum black holes, the effective spatial dimension of extremal black holes is found to be one. This is surprisingly in agreement with the assumption that the extremal black holes are the Bogomol’nyi saturated strings states and the extremal black hole can be regarded as the elementary particles like states in string theories, although we do not as yet know whether there exist the essential relations between our result and the equivalent assumption of extremal black holes and saturated string states. In addition, we have argued that the massless BTZ black hole is a critical point of the spinless BTZ black hole, and a second-order phase transition will take place from the extremal to nonextremal BTZ black holes. Because of the similarity of thermodynamics between BTZ black holes and three dimensional black strings \[39\], we believe that the similar critical behavior also appears in the black strings.

To better understand the critical behavior of extremal black holes, further investigation is needed. For example, are the critical exponents universal? Is the effective spatial dimension of other extremal black holes also one? Could we obtain a reduced quantum model of black holes, in which the critical behavior can be confirmed? These issues are currently under investigation.

Finally, we make a comment on the phase transition of black holes. First, according to the second and third laws of black hole thermodynamics, an extremal black hole cannot be reached by a physical process. The extremal black hole, however, can be produced by the pair creation \[18\]. Therefore, the second and third laws cannot exclude the phase transition from the extremal black holes to nonextremal black holes. Second, if the entropy of extremal black holes still satisfies the Bekenstein-Hawking area formula, the phase transition is second-order because the entropy is continuous from the extremal to nonextremal black holes. Instead if one accepts the view of point that the entropy of extremal black holes vanishes identically, the phase transition is higher than second-order, because in this case the entropy is discontinuous. But, the phase transition in the spinless BTZ black holes is still a second-order one because of the continuity of entropy.

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