GLOBAL LARGE SOLUTIONS TO 3-D INHOMOGENEOUS NAVIER-STOKES SYSTEM WITH ONE SLOW VARIABLE

JEAN-YVES CHEMIN, MARIUS PAICU, AND PING ZHANG

Abstract. In this paper, we are concerned with the global wellposedness of 3-D inhomogeneous incompressible Navier-Stokes equations (1.2) in the critical Besov spaces with the norm of which are invariant by the scaling of the equations and under a nonlinear smallness condition on the isentropic critical Besov norm to the fluctuation of the initial density and the critical anisotropic Besov norm of the horizontal components of the initial velocity which have to be exponentially small compared with the critical anisotropic Besov norm to the third component of the initial velocity. The novelty of this results is that the isentropic space structure to the homogeneity of the initial density function is consistent with the propagation of anisotropic regularity for the velocity field. In the second part, we apply the same idea to prove the global wellposedness of (1.2) with some large data which are slowly varying in one direction.

Keywords: Inhomogeneous Navier-Stokes Equations, Littlewood-Paley Theory, Anisotropic Besov spaces

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1. Introduction

In this paper, we consider the global wellposedness to the following 3-D incompressible inhomogeneous Navier-Stokes equations with initial data in the critical Besov spaces and with the third component of the initial velocity being large:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi &= 0, \\
\text{div } u &= 0, \\
\rho|_{t=0} &= \rho_0, \quad \rho u|_{t=0} = \rho_0 u_0,
\end{align*}
\]

(1.1)

where \(\rho, u = (u_1, u_2, u_3)\) stand for the density and velocity of the fluid respectively, \(\Pi\) is a scalar pressure function. Such system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance.

In [19], O. Ladyženskaja and V. Solonnikov first addressed the question of unique resolvability of (1.1). More precisely, they considered the system (1.1) in bounded domain \(\Omega\) with homogeneous Dirichlet boundary condition for \(u\). Under the assumption that \(u_0\) belongs to \(W^{2-\frac{2}{d}, p}(\Omega)\) with \(p\) greater than \(d\), is divergence free and vanishes on \(\partial \Omega\) and that \(\rho_0\) is \(C^1(\Omega)\), bounded and away from zero, then they proved

- Global well-posedness in dimension \(d = 2\);
- Local well-posedness in dimension \(d = 3\). If in addition \(u_0\) is small in \(W^{2-\frac{2}{d}, p}(\Omega)\), then global well-posedness holds true.

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Similar results were obtained by R. Danchin [14] in $\mathbb{R}^d$ with initial data in the almost critical Sobolev spaces. In general, the global existence of weak solutions with finite energy was established by P.-L. Lions in [20] (see also the reference therein, and the monograph [6]). H. Abidi, G. Gui and the last author established in [3] the large time decay and stability to any given global smooth solutions of (1.1).

When the initial density is away from zero, we denote by $a \overset{\text{def}}{=} \frac{1}{p} - 1$, and then (1.1) can be equivalently formulated as

$$
\begin{align*}
\begin{cases}
\partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t u + u \cdot \nabla u + (1 + a)(\nabla \Pi - \mu \Delta u) = 0, \\
\operatorname{div} u = 0, \\
(a, u)|_{t=0} = (a_0, u_0).
\end{cases}
\end{align*}
$$

(1.2)

Notice that just as the classical Navier-Stokes system (which corresponds to $a = 0$ in (1.2)), the inhomogeneous Navier-Stokes system (1.2) also has a scaling. Indeed if $(a, u)$ solves (1.2) with initial data $(a_0, u_0)$, then for $\forall \ell > 0$,

$$
(a, u)_\ell \overset{\text{def}}{=} (a(\ell^2 - \ell), \ell u(\ell^2 - \ell)) \quad \text{and} \quad (a_0, u_0)_\ell \overset{\text{def}}{=} (a_0(\ell), \ell u_0(\ell))
$$

$(a, u)_\ell$ is also a solution of (1.2) with initial data $(a_0, u_0)_\ell$.

It is easy to check that the norm of $B_{p,1}^{\frac{d}{p}}(\mathbb{R}^d) \times B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ is scaling invariant under the scaling transformation $(a_0, u_0)_\ell$ given by (1.3). In [1], H. Abidi proved in general space dimension $d$ that: if $1 < p < 2d$, $0 < \mu < \mu(p)$, given initial data $(a_0, u_0)$ sufficiently small in $B_{p,1}^{\frac{d}{p}}(\mathbb{R}^d) \times B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$, (1.2) has a global solution. Moreover, this solution is unique if $p$ is in $[1, d]$. This result generalized the wellposedness results of R. Danchin in [13] and [14], which corresponds to the celebrated results by Fujita and Kato [16] devoted to the classical Navier-Stokes system, and was improved by H. Abidi and the second author in [2] with $a_0$ in $B_{q,1}^{\frac{d}{q}}(\mathbb{R}^d)$ and $u_0$ in $B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ for $p, q$ satisfying some technical assumptions. H. Abidi, G. Gui and the last author removed the smallness condition for $a_0$ in [4, 5]. Notice that the main feature of the density space is to be a multiplier on the velocity space and this allows to define the nonlinear terms in the system (1.1). Recently, R. Danchin and P. Mucha proved in [15] a more general wellposedness result of (1.1) by considering very rough densities in some multiplier spaces on the Besov spaces $B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ for $p$ in $[1, 2d]$ which in particular completes the uniqueness result in [1] for $p$ in $[d, 2d]$ in the constant viscosity case.

Motivated by [18, 22, 24] concerning the global wellposedness of 3-D incompressible anisotropic Navier-Stokes system with the third component of the initial velocity field being large, the last two authors relaxed in [23] the smallness condition in [2] so that (1.2) still has a unique global solution (see Theorem 1.1 below for details). We emphasize that the proof in [23] used in a fundamental way the algebraical structure of (1.2). The first step is to obtain energy estimates on the horizontal components of the velocity field on the one hand and then on the vertical component on the other hand. Compared with [18, 22, 24], the additional difficulties with this strategy are that: there appears a hyperbolic type equation in (1.2) and due to the appearance of $a$ in the momentum equation of (1.2), the pressure term is more difficult to be handled. We remark that the equation on the vertical component of the velocity field is a linear equation with coefficients depending on the horizontal components of the velocity field and $a$. Therefore, the equation on the vertical component does not demand any smallness condition. While the equations on the horizontal components of the velocity
field contain bilinear terms in the horizontal components and also terms taking into account the interactions between the horizontal components and the vertical one. In order to solve this equation, we need a smallness condition on $a$ and the horizontal component (amplified by the vertical component) of the initial data. The purpose of this paper is to prove the global wellposedness of (1.2) with initial data, $a_0, u_0 = (u_0^h, u_0^v)$, satisfying some nonlinear smallness condition on the critical isentropic Besov norm to $a_0$ and the critical anisotropic Besov norm to $u_0^h$ which have to be exponentially small in contrast with the critical anisotropic Besov norm to $u_0^3$. Then we apply the same idea to prove the global wellposedness of (1.2) with some large data which are slowly varying in one direction.

Before going further, we recall the functional space framework we are going to use. As in [9], [12] and [21], the definitions of the spaces we are going to work with requires anisotropic dyadic decomposition of the Fourier variables. Let us recall from [7] that

$$
\begin{align*}
\Delta_h a &= F^{-1}(\varphi(2^{-h}|\xi_h|)\hat{a}), \\
\Delta_l a &= F^{-1}(\varphi(2^{-l}|\xi_l|)\hat{a}), \\
S_{h}^b a &= F^{-1}(\chi(2^{-h}|\xi_h|)\hat{a}), \\
S_{l}^b a &= F^{-1}(\chi(2^{-l}|\xi_l|)\hat{a})
\end{align*}
$$

(1.4)

where $\xi_h = (\xi_1, \xi_2)$, $Fa$ and $\hat{a}$ denote the Fourier transform of the distribution $a$, $\chi(x)$ and $\varphi(\tau)$ are smooth functions such that

$$
\begin{align*}
\text{Supp } \varphi &\subset \left\{ \tau \in \mathbb{R} / \ 0 < \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1, \\
\text{Supp } \chi &\subset \left\{ \tau \in \mathbb{R} / \ |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.
\end{align*}
$$

**Definition 1.1.** Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in S'_h(\mathbb{R}^3)$, which means that $u \in S'(\mathbb{R}^3)$ and $\lim_{j \to -\infty} ||S_j u||_{L^\infty} = 0$, we set

$$
||u||_{B^{s}_{p, r}} \overset{\text{def}}{=} \left(2^{qs} ||\Delta_q u||_{L^p} \right)_{q \in \mathbb{N}}.
$$

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $B^s_{p, r}(\mathbb{R}^3) \overset{\text{def}}{=} \{ u \in S'_h(\mathbb{R}^3) \ | \ ||u||_{B^{s}_{p, r}} < \infty \}$.
- If $k \in \mathbb{N}$ and $\frac{3}{p} + k \leq s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $B^s_{p, r}(\mathbb{R}^3)$ is defined as the subset of distributions $u \in S'_h(\mathbb{R}^3)$ such that $\partial^\beta u \in B^{s-k}_{p, r}(\mathbb{R}^3)$ whenever $|\beta| = k$.

**Notations** In all that follows, we shall denote

$$
B^s_{p} \overset{\text{def}}{=} B^s_{p, 1}.
$$

The following theorem was proved by the last two authors in [23]:

**Theorem 1.1.** Let $p$ be in $[1, 6]$. There exist positive constants $c_0$ and $C_0$ such that, for any data $a_0$ in $B^{\frac{3}{p}}_{p}(\mathbb{R}^3)$ and $u_0 = (u_0^h, u_0^v)$ in $B^{-1 + \frac{3}{p}}_{p}(\mathbb{R}^3)$ verifying

$$
\eta \overset{\text{def}}{=} \left(\mu ||a_0||_{B^{\frac{3}{p}}_{p}} + ||u_0^h||_{B^{-1 + \frac{3}{p}}_{p}} \right) \exp \left(\frac{C_0}{\mu} ||u_0^3||^2_{B^{-1 + \frac{3}{p}}_{p}} \right) \leq c_0 \mu,
$$

the system (1.2) has a unique global solution $(a, u)$ in the space

$$
C_b([0, \infty[; B^{\frac{3}{p}}_{p}(\mathbb{R}^3)) \times (C_b([0, \infty[; B^{-1 + \frac{3}{p}}_{p}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+, B^{1 + \frac{3}{p}}_{p}(\mathbb{R}^3))).
$$
We want to prove here an anisotropic version of the above theorem. Let us define the anisotropic Besov space that we are going to use.

**Definition 1.2.** Let $p$ be in $[1, +\infty]$, $s_1 \leq \frac{2}{p}$, $s_2 \leq \frac{1}{p}$ and $u$ in $\mathcal{S}'(\mathbb{R}^3)$, we set

$$\|u\|_{\mathcal{B}^{s_1,s_2}_{p}} \overset{\text{def}}{=} (2^{j+1}2^{ks_2}\|\Delta^j u\|_{L^p})_{l_1}.$$ 

The case when $s_1 > \frac{2}{p}$ or $s_2 > \frac{1}{p}$ can be similarly modified as that in Definition 1.1.

**Notations**

In all that follows, we shall denote

\[ \mathcal{B}_p^0 = \mathcal{B}_p^{-1+\frac{2}{p}+\frac{1}{p}}, \quad \mathcal{B}_p^1 = \mathcal{B}_p^{-1+\frac{2}{p}+\frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}+\frac{1}{p}}, \quad \text{and} \quad \mathcal{B}_p^3 = \mathcal{B}_p^{-1+\frac{2}{p}+\frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}+\frac{1}{p}} \]

Our first result in this paper is as follows:

**Theorem 1.2.** Let $p$ be in $[3, 4]$ and $r$ in $[p, 6]$. Let us consider an initial data $(a_0, u_0)$ in the space $\mathcal{B}_p^\frac{2}{3} \times \mathcal{B}_p^{\frac{2}{3}} \cap \mathcal{B}_r^{-1+\frac{2}{p}}$. Then there exist positive constants $c_0$ and $C_0$ such that if

\[ \eta \overset{\text{def}}{=} (\mu\|a_0\|_{\mathcal{B}_p^0} + \|u_0^h\|_{\mathcal{B}_p^0}) \exp\left(\frac{C_0}{\mu^2}\|u_0^3\|_{\mathcal{B}_p^0}^2\right) \leq c_0 \mu, \]

the system (1.2) has a unique global solution

\[ a \in C_0([0, \infty); \mathcal{B}_p^\frac{2}{3}(\mathbb{R}^3)) \quad \text{and} \quad u \in C_0([0, \infty); \mathcal{B}_r^{-1+\frac{2}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \mathcal{B}_r^{1+\frac{2}{p}}(\mathbb{R}^3)). \]

Moreover, there holds

\[ \|u^h\|_{L^\infty(\mathbb{R}^+; \mathcal{B}_p^0)} + \mu\|a\|_{L^\infty(\mathbb{R}^+; \mathcal{B}_p^0)} + \|u^h\|_{L^1(\mathbb{R}^+; \mathcal{B}_p^0)} \leq C\eta, \]

\[ \|u^3\|_{L^\infty(\mathbb{R}^+; \mathcal{B}_p^0)} + \mu\|u^3\|_{L^1(\mathbb{R}^+; \mathcal{B}_p^0)} \leq 2\|u_0^3\|_{\mathcal{B}_p^0} + c_2 \mu. \]

**Remark 1.1.** (1) We emphasize that for any given function $a, \phi$ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, any $p$ in $[3, 4]$, Theorem 1.2 implies the global wellposedness of (1.2) with initial data of the form

\[ a_0^\varepsilon(x) = (-\ln \varepsilon)^{\delta} \varepsilon^{1+\frac{2}{p}} a(x_1, x_2, \varepsilon x_3) \quad \text{and} \quad u_0^\varepsilon = \varepsilon_0 (-\ln \varepsilon)^{\delta} \varepsilon^{-(1-\frac{2}{p})} \sin\left(\frac{x_1}{\varepsilon}\right)(0, -\varepsilon \partial_3 \phi, \partial_2 \phi)(x_1, x_2, \varepsilon x_3), \]

for $0 < \delta < \frac{1}{2}$, and $\varepsilon, \varepsilon_0$ being sufficiently small. Indeed it is well-known that

\[ \left\| \sin\left(\frac{x_1}{\varepsilon}\right) \nabla \phi(x_1, x_2, x_3) \right\|_{\mathcal{B}_p^0} \leq C_\phi \varepsilon^{1-\frac{2}{p}}, \]

\[ \left\| a(x_1, x_2, \varepsilon x_3) \right\|_{\mathcal{B}_p^0} \leq C \varepsilon^{\frac{1}{p}} \|a\|_{L^p} \varepsilon^{\frac{1}{2}} \|\nabla a\|_{L^p}, \]

which ensures that

\[ (\mu\|a_0^\varepsilon\|_{\mathcal{B}_p^0} + \|u_0^{\varepsilon,h}\|_{\mathcal{B}_p^0}) \exp\left(\frac{C_0}{\mu^2}\|u_0^{\varepsilon,3}\|_{\mathcal{B}_p^0}^2\right) \leq C\varepsilon (-\ln \varepsilon)^{\delta} \exp\left((-\ln \varepsilon)^{2\delta}\right) \to 0, \]

which tends to 0 when $\varepsilon$ tends to 0. Hence Theorem 1.2 implies that (1.2) with initial data $(a_0^\varepsilon, u_0^\varepsilon)$ has a unique global solution $(a^\varepsilon, u^\varepsilon)$. 
(2) In the case when \( \delta = 0 \) in (1.9), the homogeneity of the initial density \( a_0^\varepsilon \) could be much larger. In fact, it follows from the same line as the proof of part (1) that (1.2) with the data
\[
a_0^\varepsilon(x) = \varepsilon^\sigma a(x_1, x_2, \varepsilon x_3) \quad \text{and} \quad u_0^\varepsilon = \varepsilon^{-\epsilon_1} \sin \left( \frac{x_1}{\varepsilon} \right) (0, -\varepsilon \partial_3 \phi, \partial_2 \phi)(x_1, x_2, \varepsilon x_3),
\]
also has a unique global solution for \( \varepsilon_0 \) and \( \varepsilon \) being sufficiently small.

Theorem 1.2 also ensures the global wellposedness of (1.2) with data of the form:
\[
(a_0(x_h, x_3), (\varepsilon u_0^\varepsilon(x_h, \varepsilon x_3), u_0^\varepsilon(x_h, \varepsilon x_3)))
\]
for any smooth divergence free vector field \( u_0 = (u_0^0, u_0^3) \) and with \( \varepsilon, \|a_0\|_{\mathcal{B}^p_q} \), for some \( p \) in \([3, 4]\) being sufficiently small. Notice that the authors [10] proved the global existence of smooth solutions to 3-D classical Navier-Stokes system for some large data which are slowly varying in one direction. The main idea behind the proof in [10] is that the solutions to 3-D Navier-Stokes equations slowly varying in one space variable can be well approximated by solutions of 2-D Navier-Stokes equation. Yet just as the classical 2-D Navier-Stokes system, 2-D inhomogeneous Navier-Stokes equations is also globally wellposed with general initial data (see [14, 19] for instance). This motivates us to study the global wellposedness of (1.2) with large data which are slowly variable in one direction and which do not satisfy the nonlinear smallness condition (1.6).

**Theorem 1.3.** Let \( \sigma \) be a real number greater than \( 1/4 \) and \( a_0 \) a function of \( \mathcal{B}^p_q \cap \mathcal{B}^{-1+\frac{3}{q}}_q \) for some \( p \) in \([3, 4]\) and \( q \in [\frac{3}{2}, 2] \). Let \( v_0^h = (v_0^h, v_0^3) \) be a horizontal, smooth divergence free vector field on \( \mathbb{R}^3 \), belonging, as well as all its derivatives, to \( L^2(\mathbb{R}, \mathcal{H}^1) \). Furthermore, we assume that for any \( \alpha \in \mathbb{N}^3 \), \( \partial^\alpha \partial_3 v_0^h \) belongs to \( \mathcal{B}^{-1+\frac{1}{q}}_q(\mathbb{R}^3) \). Then there exists a positive \( \varepsilon_0 \) such that if \( \varepsilon \leq \varepsilon_0 \), the initial data
\[
(1.10) \quad a_0^\varepsilon(x) = \varepsilon^\sigma a(x_h, \varepsilon x_3), \quad u_0^\varepsilon(x) = (v_0^h(x_h, \varepsilon x_3), 0)
\]
generates a unique global solution \( (a^\varepsilon, u^\varepsilon) \) of (1.2).

**Remark 1.2.**
(1) With \( v_0^h \) being given by Theorem 1.3 and \( u_0 \) a smooth divergence free vector field on \( \mathbb{R}^3 \), I. Gallagher and the first author proved in [10] that there exists a positive \( \varepsilon_0 \) such that if \( 0 < \varepsilon \leq \varepsilon_0 \), the classical Navier-Stokes system (which corresponds to \( a = 0 \) in (1.2)) with the initial data
\[
(1.11) \quad u_0^\varepsilon(x) = (v_0^h + \varepsilon u_0^0, u_0^3)(x_h, \varepsilon x_3)
\]
has a unique global solution.

(2) G. Gui, J. Huang and and the last author proved in [17] similar global wellposedness result for (1.2) with initial data \( a_0^\varepsilon(x) = \varepsilon^\delta_0 a(x_h, \varepsilon x_3) \) and initial velocity given by (1.11) provided that \( a_0 \in W^{1,p} \cap H^2 \) for some \( p \in (1, 2) \) and \( \delta_0 = \frac{1}{p} \). We should point out that one difficulty in [17] is to derive \( L^\infty(\mathbb{R}^3; \mathcal{B}^p_q) \) estimate for the solution \( a \) of the free transport equation in (1.2). Toward this, the authors in [17] assumed more regularities for \( a_0 \) and then use an interpolation argument to get this estimate. The advantage of the argument used in the proof of Theorem 1.3 is that: as observed from the proof of Theorem 1.2, the isentropic regularities of \( a \) is matched with the anisotropic regularities of \( u \), so that we can still work this problem in the scaling invariant spaces, which leads to the improvement of the index \( \sigma > \frac{1}{2} \) in [17] to be \( \sigma > \frac{1}{4} \) here.
(3) It follows from the proof of Theorem 1.3 that we can prove similar wellposedness result for (1.2) with data \((a_0^\delta, u_0^\delta)\) given by (1.11) provided that \(\varepsilon \leq \varepsilon_0\) and \(\|a_0^\delta\|_{B^1_r} + \varepsilon\|a_0^\delta\|_{B^3_1}\) being sufficiently small and for some \(p, q\) satisfying \(p \in [3, 4[\) and \(q \in [\frac{3}{2}, 2[\). Nevertheless, as \(w_0\) part in (1.11) satisfies our nonlinear smallness condition (1.6), we choose to investigate the case (1.10) here.

The organization of this paper is as follows:

In the second section, we prove some lemmas using Littlewood-Paley theory in particular a lemma which explains how to compute the pressure in the case when \(a\) is small in \(B^3_p\) and a lemma of propagation for the transport equation which takes into account some anisotropy.

In the third section, we prove Theorem 1.2.

In the forth section, we prove Theorem 1.3.

Let us complete this section by the notations of the paper:

Let \(A, B\) be two operators, we denote \([A, B] = AB - BA\), the commutator between \(A\) and \(B\). For \(a \lesssim b\), we mean that there is a uniform constant \(C\), which may be different on different lines, such that \(a \leq Cb\). We denote by \((a|b)\) the \(L^2(\mathbb{R}^3)\) inner product of \(a\) and \(b\), \((d_j)_{j \in \mathbb{Z}}\) will be a generic element of \(\ell^1(\mathbb{Z})\) (resp. \(\ell^1(\mathbb{Z}^2)\)) so that \(\sum_{j \in \mathbb{Z}} d_j = 1\).

For \(X\) a Banach space and \(I\) an interval of \(\mathbb{R}\), we denote by \(C(I; X)\) the set of continuous functions on \(I\) with values in \(X\), and by \(C_0(I; X)\) the subset of bounded functions of \(C(I; X)\). For \(q \in [1, +\infty]\), the notation \(L^q(I; X)\) stands for the set of measurable functions on \(I\) with values in \(X\), such that \(t \mapsto \|f(t)\|_X\) belongs to \(L^q(I)\).

2. Some estimates related to Littlewood-Paley analysis

As we shall frequently use the anisotropic Littlewood-Paley theory, and in particular anisotropic Bernstein inequalities. For the convenience of the readers, we first recall the following Bernstein type lemma from \([12, 21]\):

**Lemma 2.1.** Let \(B_h\) (resp. \(B_v\)) a ball of \(\mathbb{R}^2_h\) (resp. \(\mathbb{R}^2_v\)), and \(C_h\) (resp. \(C_v\)) a ring of \(\mathbb{R}^2_h\) (resp. \(\mathbb{R}^2_v\)); let \(1 \leq p_2 \leq p_1 \leq +\infty\) and \(1 \leq q_2 \leq q_1 \leq +\infty\). Then there holds:

If the support of \(a\) is included in \(2^k B_h\), then

\[
\| \partial^\alpha a \|_{L^p_h(L^q_h)} \lesssim 2^{-k(1+|\alpha|+2)\left(\frac{1}{p_2} - \frac{1}{p_1}\right)} \|a\|_{L^p_h(L^q_h)}.
\]

If the support of \(a\) is included in \(2^k B_v\), then

\[
\| \partial^\beta a \|_{L^p_v(L^q_v)} \lesssim 2^{\beta+1\left(\frac{1}{q_1} - \frac{1}{q_2}\right)} \|a\|_{L^p_v(L^q_v)}.
\]

If the support of \(a\) is included in \(2^k C_h\), then

\[
\|a\|_{L^p_h(L^q_h)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \| \partial^\alpha a \|_{L^p_h(L^q_h)}.
\]

If the support of \(a\) is included in \(2^k C_v\), then

\[
\|a\|_{L^p_v(L^q_v)} \lesssim 2^{\beta N} \| \partial^\beta a \|_{L^p_v(L^q_v)}.
\]

To consider the product of a distribution in the isentropic Besov space with a distribution in the anisotropic Besov space, we need the following result which allows to embed isotropic Besov spaces into the anisotropic ones.
Lemma 2.2. Let $s$ and $t$ be positive real numbers. Then for any $p$ in $[1, \infty]$, one has

$$\|f\|_{B_p^{s,t}} \lesssim \|f\|_{B_p^{s+t}}.$$

Proof. Thanks to Definition 1.2, one has

$$\|f\|_{B_p^{s,t}} = \sum_{j, k \in \mathbb{Z}^2} 2^{js} 2^{kt} \|\Delta_j^h \Delta_k^v f\|_{L^p}.$$  

We separate the above sum into two parts, depending on whether $k < j$ or $k \geq j$ and we shall only detail the first case (the second one is identical). We notice that if $k < j$,

$$\|\Delta_j^h \Delta_k^v f\|_{L^p} \lesssim \sum_{|\ell - j| \leq N_0} \|\Delta_{\ell} f\|_{L^p}.$$  

Then we infer from the fact that $t > 0$

$$\sum_{j, k \in \mathbb{Z}^2} 2^{js} 2^{kt} \|\Delta_j^h \Delta_k^v f\|_{L^p} \lesssim \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^p} \sum_{k < j} 2^{kt} \lesssim \sum_{j \in \mathbb{Z}} 2^{j(s+t)} \|\Delta_j f\|_{L^p} \lesssim \|f\|_{B_p^{s+t}}.$$  

And the result follows. $\square$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin-Lerner type spaces $\tilde{L}_{\infty}^s(B_p^r(\mathbb{R}^3))$ (see [7] for instance).

To study product laws between distributions in the anisotropic Besov spaces, we need to modify the isotropic para-differential decomposition of Bony [8] to the setting of anisotropic version. We first recall the isotropic para-differential decomposition from [8]: let $a$ and $b$ be in $\mathcal{S}'(\mathbb{R}^3)$,

$$ab = T(a, b) + R(a, b), \quad \text{or} \quad ab = T(a, b) + \tilde{T}(a, b) + R(a, b),$$

where

$$T(a, b) = \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b, \quad \tilde{T}(a, b) = T(b, a), \quad R(a, b) = \sum_{j \in \mathbb{Z}} \Delta_j a S_{j+2} b,$$

and

$$R(a, b) = \sum_{j \in \mathbb{Z}} \Delta_j a \tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b = \sum_{\ell = j-1}^{j+1} \Delta_\ell a.$$  

(2.1)

In what follows, we shall also use the anisotropic version of Bony’s decomposition for both horizontal and vertical variables.

As an application of the above basic facts on Littlewood-Paley theory, we present the following product laws in the anisotropic Besov spaces.

Lemma 2.3. Let $p \geq q \geq 1$ with $\frac{1}{p} + \frac{1}{q} \leq 1$, and $s_1 \leq \frac{2}{q}$, $s_2 \leq \frac{2}{p}$ with $s_1 + s_2 > 0$. Let $\sigma_1 \leq \frac{1}{q}$, $\sigma_2 \leq \frac{1}{p}$ with $\sigma_1 + \sigma_2 > 0$. Then for $a$ in $\mathcal{B}_q^{s_1, \sigma_1}(\mathbb{R}^3)$ and $b$ in $\mathcal{B}_p^{s_2, \sigma_2}(\mathbb{R}^3)$, the product $ab$ belongs to $\mathcal{B}_p^{s_1+s_2-\frac{2}{q}, \sigma_1+\sigma_2-\frac{1}{q}}(\mathbb{R}^3)$, and

$$\|ab\|_{\mathcal{B}_p^{s_1+s_2-\frac{2}{q}, \sigma_1+\sigma_2-\frac{1}{q}}} \lesssim \|a\|_{\mathcal{B}_q^{s_1, \sigma_1}} \|b\|_{\mathcal{B}_p^{s_2, \sigma_2}}.$$
Proof. We first get by applying Bony’s decomposition (2.1) in both horizontal and vertical variables that
\[ ab = (T^h + T^h + R^h)(T^v + T^v + R^v)(a, b) \]
(2.2)
\[ = T^h T^v(a, b) + T^h T^v(a, b) + T^h R^v(a, b) + T^h T^v(a, b) + T^h T^v(a, b) + R^h T^v(a, b) + R^h T^v(a, b) + R^h R^v(a, b). \]

In what follows, we shall detail the estimates to some typical terms above, the other cases can be followed along the same line. Note that \( \sigma_1 + \sigma_2 > 0 \), we get, by applying Lemma 2.1, that
\[
\| \Delta_j^h \Delta_k^v (T^h R^v(a, b)) \|_{L^p} \lesssim 2 \frac{j}{2} \sum_{|j'-j| \leq 4, k' \geq k} \| S^h_{j' \to j} \Delta_k^v a \|_{L^\infty(L^q_s)} \| \Delta_j^h \Delta_k^v b \|_{L^p} \\
\lesssim 2 \frac{j}{2} \sum_{|j'-j| \leq 4, k' \geq k} d_{j',k'} 2^{-j'(s_1 + s_2 - \frac{2}{q}) 2^{-k' \sigma_1, \sigma_2}} \| a \|_{B_{q_p}^{s_1, \sigma_1}} \| b \|_{B_{q_p}^{s_2, \sigma_2}}. 
\]
The same estimate holds for \( T^h T^v(a, b) \) and \( T^h T^v(a, b) \).

Along the same lines, we obtain
\[
\| \Delta_j^h \Delta_k^v (T^h R^v(a, b)) \|_{L^p} \lesssim 2 \frac{j}{2} \sum_{|j'-j| \leq 4, k' \geq k} \| \Delta_j^h \Delta_k^v a \|_{L^s} \| S^h_{j' \to j} \Delta_k^v b \|_{L^\infty(L^q_s)} \\
\lesssim 2 \frac{j}{2} \sum_{|j'-j| \leq 4, k' \geq k} d_{j',k'} 2^{-j'(s_1 + s_2 - \frac{2}{p}) 2^{-k' \sigma_1, \sigma_2}} \| a \|_{B_{q_p}^{s_1, \sigma_1}} \| b \|_{B_{q_p}^{s_2, \sigma_2}}. 
\]
The same estimate holds for \( T^h T^v(a, b) \) and \( T^h T^v(a, b) \). Finally applying Lemma 2.1 once again and using the fact that \( s_1 + s_2 > 0, \sigma_1 + \sigma_2 > 0 \), gives rise to
\[
\| \Delta_j^h \Delta_k^v (R^h R^v(a, b)) \|_{L^p} \lesssim 2 \frac{j}{2} \sum_{j \geq j' \geq k} \| \Delta_j^h \Delta_k^v a \|_{L^s} \| \Delta_j^h \Delta_k^v b \|_{L^p} \\
\lesssim 2 \frac{j}{2} \sum_{j \geq j' \geq k} d_{j',k'} 2^{-j'(s_1 + s_2) 2^{-k' \sigma_1, \sigma_2}} \| a \|_{B_{q_p}^{s_1, \sigma_1}} \| b \|_{B_{q_p}^{s_2, \sigma_2}}. 
\]
The same estimate holds for \( R^h T^v(a, b) \) and \( R^h T^v(a, b) \). This together with (2.2) completes the proof of Lemma 2.3. \( \square \)

As an application of the laws of product, we state a lemma which will describe the way how to compute the pressure in the case when \( a \) is small.

**Lemma 2.4.** Let \( p \in (1, 4) \), we consider a function \( a \) such that \( \| a \|_{B_{q_p}^{s_1, \sigma_1}} \) is small enough. If \( \Pi \) satisfies
\[
(D) \quad \text{div}((1 + a) \nabla \Pi - f) = 0
\]
with \( f \) in \( \mathfrak{B}_p^0 \), then (D) has a unique solution which satisfies
\[
\|\nabla \Pi\|_{\mathfrak{B}_3^0} \lesssim \|f\|_{\mathfrak{B}_p^0} \quad \text{and thus} \quad (1 + a)\nabla \Pi\|_{\mathfrak{B}_p^0} \lesssim \|f\|_{\mathfrak{B}_p^0}.
\]

Proof. We first write (D) as
\[
\Delta \Pi = -\text{div}(a\nabla \Pi) + \text{div} f.
\]
Applying now the operator \( \nabla \Delta^{-1} \) to this identity implies that
\[
\nabla \Pi = -M_a(\nabla \Pi) + \nabla \Delta^{-1} \text{div} f \quad \text{with} \quad -M_a(g) \overset{\text{def}}{=} \nabla \Delta^{-1} \text{div}(ag).
\]
Laws of product from Lemma 2.3 together with Lemma 2.2 implies that \( \|M_a\|_{\mathcal{L}(\mathfrak{B}_p^0)} \lesssim \|a\|_{\mathfrak{B}_p^0}^{\frac{1}{2}} \) because \( p < 4 \). Thus, if \( \|a\|_{\mathfrak{B}_p^0}^{\frac{1}{2}} \) is small enough, the operator \((\text{Id} - M_a)^{-1}\) is well defined as an element of \( \mathcal{L}(\mathfrak{B}_p^0) \) by the formula
\[
(\text{Id} - M_a)^{-1} = \sum_{k=0}^\infty M_a^k.
\]
As \( \nabla \Delta^{-1} \text{div} \) is a homogenous Fourier multiplier of degree 0, the lemma is proved.

Now, we are going the prove a lemma which is a variation about the classical propagation lemma for regularity of index less than 1.

Lemma 2.5. Let \( a_0 \) be in \( \mathfrak{B}_p^{\frac{3}{2}}(\mathbb{R}^3) \), and \( u = (u^1, u^3) \) be a divergence free vector field such that \( \nabla u \) belongs to \( L^1([0, T], L^\infty(\mathbb{R}^3)) \). Let \( f \) be in \( L^1([0, T]) \) with \( \|\nabla u^3(t)\|_{L^\infty} \leq Cf(t) \) for all \( t \) in \([0, T]\). We denote
\[
a_\lambda \overset{\text{def}}{=} a \exp\left(-\lambda \int_0^t f(t') \, dt'\right).
\]
Then, the unique solution \( a \) of
\[
\tag{2.3}
\partial_t a + u \cdot \nabla a = 0, \quad a|_{t=0} = a_0
\]
satisfies, for any \( t \) in \([0, T]\) and \( \lambda \) large enough,
\[
\tag{2.4}
\|a_\lambda\|_{L^\infty_t(\mathfrak{B}_p^0)} + \frac{\lambda}{2} \int_0^t f(t') \|a_\lambda(t')\|_{\mathfrak{B}_p^0}^{\frac{1}{2}} \, dt' \leq \|a_0\|_{\mathfrak{B}_p^0}^{\frac{1}{2}} + C\|a_\lambda\|_{L^\infty_t(\mathfrak{B}_p^0)} \int_0^t \|\nabla u^3(t')\|_{L^\infty} \, dt'.
\]

Proof. The proof of this lemma basically follows from that of Proposition 3.1 in [23]. The novelty of our observation here is that the \( L^1_T(L^{\infty}(\mathbb{R}^3)) \) estimate of the convection velocity enables us to propagate the \( \mathfrak{B}_p^{\frac{3}{2}} \) regularity for (2.3) when \( p > 3 \).

As both the existence and uniqueness of solutions to (2.3) essentially follows from the estimate (2.4) for some appropriate approximate solutions to (2.3). For simplicity, here we just present the \( a \text{ priori} \) estimate (2.4) for smooth enough solutions of (2.3). In this case, thanks to (2.3), we have
\[
\partial_t a_\lambda + \lambda f(t)a_\lambda + u \cdot \nabla a_\lambda = 0.
\]
Applying \( \Delta_j \) to the above equation and then taking \( L^2 \) inner product of the resulting equation with \( |\Delta_j a_\lambda|^{p-2} \Delta_j a_\lambda \), we obtain
\[
\tag{2.5}
\frac{1}{q} \frac{d}{dt} \|\Delta_j a_\lambda(t)\|_{L^p}^p + \lambda f(t)\|\Delta_j a_\lambda(t)\|_{L^p}^p + (\Delta_j(u \cdot \nabla a_\lambda) \cdot |\Delta_j a_\lambda|^{p-2} \Delta_j a_\lambda) = 0.
\]
While as $\text{div} u = 0$, we get, by using Bony’s decomposition (2.1),

$$u \cdot \nabla a_\lambda = T(u, \nabla a_\lambda) + R(u, \nabla a_\lambda),$$

and a standard commutator’s argument, that

$$\Delta_j(T(u, \nabla a_\lambda)) \cdot |\Delta_j a|^{p-2} \Delta_j a = \sum_{|j'|-j \leq 5} \left( (|\Delta_j; S_{j'-1} u| \Delta_j \nabla a_\lambda | |\Delta_j a_\lambda|^{p-2} \Delta_j a_\lambda) + (S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_j \nabla a_\lambda | |\Delta_j a_\lambda|^{p-2} \Delta_j a_\lambda) \right).$$

Then we deduce from (2.5) that

$$||\Delta_j a_\lambda(t)||_{L^p} + \lambda \int_0^t f(t')||\Delta_j a_\lambda(t')||_{L^p} dt' \leq ||\Delta_j a(0)||_{L^p} + C \left( \sum_{|j'|-j \leq 4} (||\Delta_j; S_{j'-1} u| \Delta_j \nabla a_\lambda||_{L^1_t(L^p)}) + \|R(u, \nabla a_\lambda)||_{L^1_t(L^p)} \right).$$

Applying the classical estimate on commutator (see [7] for instance) leads to

$$\sum_{|j'|-j \leq 4} \|||\Delta_j; S_{j'-1} u| \Delta_j \nabla a_\lambda||_{L^1_t(L^p)} \leq \sum_{|j'|-j \leq 4} \left( ||S_{j'-1} \nabla u^h||_{L^1_t(L^\infty)} ||\Delta_j \nabla a_\lambda||_{L^\infty_t(L^p)} + \int_0^t ||S_{j'-1} \nabla u^3(t')||_{L^\infty} ||\Delta_j a_\lambda(t')||_{L^p} dt' \right) \leq \sum_{|j'|-j \leq 5} \left( d_j 2^{-\frac{3j}{p}} ||\nabla u^h||_{L^1_t(L^\infty)} ||a_\lambda||_{L^\infty_t(L^\frac{2}{3} p)} + \int_0^t f(t') ||a_\lambda(t')||_{L^\frac{2}{3} p} dt' \right).$$

Similarly we get, by applying Lemma 2.1, that

$$\sum_{|j'|-j \leq 4} \|||S_{j'-1} u - S_{j-1} u| \Delta_j \nabla a_\lambda||_{L^1_t(L^p)} \leq \sum_{|j'|-j \leq 4} \left( ||(S_{j'-1} \nabla u^h - S_{j-1} \nabla u^h)||_{L^1_t(L^\infty)} ||\Delta_j \nabla a_\lambda||_{L^\infty_t(L^p)} + \int_0^t ||(S_{j'-1} \nabla u^3 - S_{j-1} \nabla u^3)(t')||_{L^\infty} ||\Delta_j a_\lambda(t')||_{L^p} dt' \right) \leq d_j 2^{-\frac{3j}{p}} ||\nabla u^h||_{L^1_t(L^\infty)} ||a_\lambda||_{L^\infty_t(L^\frac{2}{3} p)} + \sum_{|j'|-j \leq 4} \int_0^t ||\nabla u^3(t')||_{L^\infty} ||\Delta_j a_\lambda(t')||_{L^p} dt' \leq d_j 2^{-\frac{3j}{p}} \left( ||\nabla u^h||_{L^1_t(L^\infty)} ||a_\lambda||_{L^\infty_t(L^\frac{2}{3} p)} + \int_0^t f(t') ||a_\lambda(t')||_{L^\frac{2}{3} p} dt' \right).
On the other hand, as \( p > 3 \) and \( \nabla a \in \tilde{L}^{\infty}_{T} (\mathcal{B}_{p}^{\frac{3}{p} - 1}) \), applying Lemma 2.1 once again gives rise to
\[
\| S_{j'+2} \nabla_{h} a_{\lambda} \|_{L^{\infty}_{T} (L^{p})} \lesssim \sum_{\ell \leq j' - 2} 2^{\ell} \ell a_{\lambda} \|_{L^{\infty}_{T} (L^{p})} \lesssim \sum_{\ell \leq j' - 2} d_{j'} 2^{\ell (1 - \frac{2}{p})} \| a_{\lambda} \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{3})} \lesssim d_{j'} 2^{j' (1 - \frac{2}{p})} \| a_{\lambda} \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{\frac{3}{p}})},
\]
so that
\[
\sum_{j \geq j - N_{0}} \| S_{j'+2} \nabla_{h} a_{\lambda} \|_{L^{\infty}_{T} (L^{p})} \| \Delta_{j'} u^{3} (t') \|_{L^{\infty}_{T}} \lesssim \sum_{j \geq j - N_{0}} d_{j'} 2^{- \frac{3j'}{p}} \| a_{\lambda} \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{3})} \| \nabla u^{3} (t') \|_{L^{1}_{T} (L^{\infty})} \lesssim d_{j'} 2^{- \frac{3j'}{p}} \| \nabla u^{3} \|_{L^{1}_{T} (L^{\infty})} \| a_{\lambda} \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{\frac{3}{p}})}.
\]
It follows from the same lines that
\[
\sum_{j' \geq j - N_{0}} \int_{0}^{t} \| S_{j'+2} \partial_{3} a_{\lambda} (t') \|_{L^{p}} \| \Delta_{j'} u^{3} (t') \|_{L^{\infty}} dt' \lesssim \sum_{j' \geq j - N_{0}} 2^{j' (1 - \frac{2}{p})} \int_{0}^{t} d_{j'} (t') \| a_{\lambda} (t') \|_{\mathcal{B}_{p}^{\frac{3}{p}}} \| \Delta_{j'} u^{3} (t') \|_{L^{\infty}} dt' \lesssim \sum_{j' \geq j - N_{0}} d_{j'} 2^{- \frac{3j'}{p}} \int_{0}^{t} \| a_{\lambda} (t') \|_{\mathcal{B}_{p}^{\frac{3}{p}}} \| \nabla u^{3} (t') \|_{L^{\infty}} dt' \lesssim d_{j'} 2^{- \frac{3j'}{p}} \int_{0}^{t} f (t') \| a_{\lambda} (t') \|_{\mathcal{B}_{p}^{\frac{3}{p}}} dt'.
\]
As a consequence, we obtain
\[
\| \Delta_{j} (\mathcal{R} (u, \nabla a_{\lambda})) \|_{L^{1}_{T} (L^{p})} \lesssim \sum_{j \geq j - N_{0}} \left( \| S_{j'+2} \nabla_{h} a_{\lambda} \|_{L^{\infty}_{T} (L^{p})} \| \Delta_{j'} u^{3} \|_{L^{1}_{T} (L^{\infty})} + \int_{0}^{t} \| S_{j'+2} \partial_{3} a_{\lambda} (t') \|_{L^{p}} \| \Delta_{j'} u^{3} (t') \|_{L^{\infty}} dt' \right) \lesssim d_{j} 2^{- \frac{3j}{p}} \left( \| \nabla u^{3} \|_{L^{1}_{T} (L^{\infty})} \| a_{\lambda} \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{\frac{3}{p}})} + \int_{0}^{t} f (t') \| a_{\lambda} (t') \|_{\mathcal{B}_{p}^{\frac{3}{p}}} dt' \right).
\]
Substituting the above estimates into (2.6) and taking summation for \( j \) in \( \mathbb{Z} \), we arrive at
\[
\| a \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{\frac{3}{p}})} + \lambda \int_{0}^{t} f (t') \| a_{\lambda} (t') \|_{\mathcal{B}_{p}^{\frac{3}{p}}} dt' \leq \| a_{0} \|_{\mathcal{B}_{p}^{\frac{3}{p}}} + C \left( \| \nabla u^{3} \|_{L^{1}_{T} (L^{\infty})} \| a \|_{L^{\infty}_{T} (\mathcal{B}_{p}^{\frac{3}{p}})} + \int_{0}^{t} f (t') \| a_{\lambda} (t') \|_{\mathcal{B}_{p}^{\frac{3}{p}}} dt' \right).
\]
Taking \( \lambda \geq 2C \) in the above inequality, we conclude the proof of (2.4). \( \square \)

Following the same line to the proof of Lemma 2.5, we can also prove the following Lemma, which will be used in the proof of Theorem 1.3 in Section 4.

**Lemma 2.6.** Let \( q \) be in \([1, \infty)\) and \( s \) in \([0, 1]\). Then given an initial data \( a_{0} \in \mathcal{B}_{q}^{s} (\mathbb{R}^{3}) \) and a vector field \( u \) in \( L^{1} ([0, T]; \text{Lip} (\mathbb{R}^{3})) \) with div \( u = 0 \), (2.3) has a unique solution \( a \) in \( C ([0, T]; \mathcal{B}_{q}^{s} (\mathbb{R}^{3})) \cap \tilde{L}^{\infty}_{T} (\mathcal{B}_{q}^{s} (\mathbb{R}^{3})). \) Moreover, there holds
\[
(2.7) \quad \| a \|_{\tilde{L}^{\infty}_{T} (\mathcal{B}_{q}^{s})} \leq \| a_{0} \|_{\mathcal{B}_{q}^{s}} \exp \left( C \int_{0}^{T} \| \nabla u (t') \|_{L^{\infty}} dt' \right).
\]
Proof. Notice once again that the proof of Lemma 2.6 basically follows from (2.7), we shall only detail the proof of (2.7) for smooth enough solutions to (2.3). Indeed it follows from the proof of (2.6) that

\[
\frac{d}{dt} \| \Delta_j a(t) \|_{L^q} \leq \sum_{|j'| - j| \leq 4} \| \Delta_j; S_{j'-1} u \|_\nabla \Delta_j a(t) \|_{L^q} + \sum_{|j'| - j| \leq 4} \| (S_{j'-1} u - S_{j-1} u) \cdot \nabla \Delta_j a(t) \|_{L^q} + \sum_{j' \geq j - N_0} \| \Delta_{j'} u S_{j'+2} \nabla a(t) \|_{L^q}.
\]

We get by using the classical commutator’s estimate (see [7] for instance) that

\[
\sum_{|j'| - j| \leq 4} \| \Delta_j; S_{j'-1} u \|_\nabla \Delta_j a(t) \|_{L^q} \lesssim \sum_{|j'| - j| \leq 4} \| S_{j'-1} (\nabla u(t)) \|_{L^\infty} \| \Delta_j a(t) \|_{L^q} \lesssim d_j(t) 2^{-js} \| \nabla u(t) \|_{L^\infty} \| a(t) \|_{B^s_q}.
\]

The same estimate holds for \( \sum_{|j'| - j| \leq 4} \| (S_{j'-1} u - S_{j-1} u) \cdot \nabla \Delta_j a(t) \|_{L^q} \). Whereas applying Bernstein’s Lemma and using the fact that \( s < 1 \) yields

\[
\sum_{j' \geq j - N_0} \| \Delta_{j'} u S_{j'+2} \nabla a(t) \|_{L^q} \lesssim \sum_{j' \geq j - N_0} \| \Delta_{j'} u(t) \|_{L^\infty} S_{j'+2} \nabla a(t) \|_{L^q} \lesssim d_j(t) 2^{-js} \| \nabla u(t) \|_{L^\infty} \| a(t) \|_{B^s_q}.
\]

As a consequence, we arrive at

\[
\| \Delta_j a \|_{L^\infty_t (L^q)} \leq \| \Delta_j a_0 \|_{L^q} + 2^{-js} \int_0^t d_j(t') \| \nabla u(t') \|_{L^\infty} \| a(t') \|_{B^s_q} dt',
\]

which gives rise to

\[
\| a \|_{L^\infty_t (B^s_q)} \leq \| a_0 \|_{B^s_q} + C \int_0^t \| \nabla u(t') \|_{L^\infty} \| a(t') \|_{B^s_q} dt'.
\]

Applying Gronwall inequality leads to (2.7). \( \square \)

3. THE PROOF OF THEOREM 1.2

We shall only prove that if \((a_0, u_0)\) is a smooth initial data satisfying the smallness condition (1.6) then the associated solution of \((a, u)\) of (1.2) satisfies (1.8), which implies a global control of the \(L^1\) in time with value in \(L^\infty\) for the gradient of \(u\). With this estimate, it is standard to prove the \(L^\infty (\mathbb{R}^+; B^{-1+\frac{2}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; B^{-1+\frac{2}{p}}(\mathbb{R}^3))\) for the velocity field (see [1, 2, 23] for instance). In order to prove the existence part of Theorem 1.2, we regularize the initial data and then pass to the limit. These technical details are omitted. The uniqueness part of Theorem 1.2 follows from Theorem 1 of [15].

Let us denote by \(T^*\) the maximal time of existence of the solution \((a, u)\) of (1.2) associated with the smooth initial data \((a_0, u_0)\). Let us consider \(T^+\) defined by

\[
T^+ \overset{\text{def}}{=} \sup \left\{ T < T^* / \xi_T \overset{\text{def}}{=} \mu \| a \|_{L^\infty_t (B^s_q)} + \| u^h \|_{L^\infty_t (W^s_q)} + \mu \| u^h \|_{L^1_t (W^s_q)} \leq c_0 \mu \right\},
\]

where \(c_0\) will be chosen small enough later on.
We want first to estimate \( \|g\|_{L^1_t(B^0_p)} \) where
\[
g(a, u) \overset{\text{def}}{=} -u \cdot \nabla u + \mu a \Delta u - (1 + a) \nabla \Pi
\]
As in Lemma 2.5, we define
\[
b_\lambda(t) \overset{\text{def}}{=} b(t) \exp \left( -\lambda \int_0^t \|u^3'(t', \cdot)\|_{B^1_p} dt' \right).
\]
This will allow to make the term \( \mu a \Delta u^3 \) integrable thanks to Lemma 2.5. Notice that taking space divergence to the momentum equation of (1.2) gives \( \text{div} \ g(a, u) = 0 \), Lemma 2.4 implies that
\[
\|g(a, u)_\lambda(t)\|_{B^0_p} \lesssim \|\text{div}(u \otimes u) - \mu a \Delta u\|_{B^0_p}.
\]
Let us estimate the right-hand side term. The key point to the estimation is that it does not contain any terms which are quadratic with respect to \( u^3 \).

If \((j, k)\) is in \( \{1, 2\}^2 \), we have, thanks to law of product of Lemma 2.3,
\[
\|\partial_j u^k\|_{B^0_p} \lesssim \|u^h\|_{B^2_p} \|u^h\|_{B^2_p} + \|u^h\|_{B^2_p} \|u^h\|_{B^2_p}.
\]

Because \( p < 4 \), law of product of Lemma 2.3 and \( \text{div} \ u = 0 \) implies that
\[
\|\partial_3 (u^3 u^k)\|_{B^0_p} \lesssim \|u^h\|_{B^2_p} \|u^h\|_{B^2_p} + \|u^3\|_{B^2_p} \|u^h\|_{B^2_p}.
\]
The term \( \partial_3 (u^3)^2 \), which is the only possible quadratic term, is equal to \(-2u^3 \text{div}_h u^h \) thanks to divergence free condition. As above, we have
\[
\|\partial_3 (u^3)^2\|_{B^0_p} \lesssim \|u^3\|_{B^2_p} \|u^h\|_{B^2_p}.
\]

Laws of product of Lemma 2.3 together with Lemma 2.2 gives
\[
\mu \|(a \Delta u^3)\|_{B^0_p} \lesssim \mu \|a\|_{B^2_p} \|u^h\|_{B^2_p} \quad \text{and} \quad \mu \|(a \Delta u^3)\|_{B^0_p} \lesssim \mu \|a\|_{B^2_p} \|u^3\|_{B^2_p}.
\]

Lemma 2.4 and Estimates (3.3)–(3.6) gives, for any positive \( \lambda \),
\[
\|g(a, u)_\lambda(t)\|_{B^0_p} \lesssim \|u^h\|_{B^2_p} \|u^h\|_{B^2_p} + \|u^h\|_{B^2_p} \|u^h\|_{B^2_p} + \|u^3\|_{B^2_p} \|u^h\|_{B^2_p} + \|u^3\|_{B^2_p} \|u^h\|_{B^2_p} + \mu \|a\|_{B^2_p} \|u^h\|_{B^2_p} + \mu \|a\|_{B^2_p} \|u^3\|_{B^2_p}.
\]

Let us first estimate \( u^3 \). As \( u^3 \) satisfies
\[
\partial_t u^3 - \Delta u^3 = (-u \cdot \nabla u + \mu a \Delta u + (1 + a) \nabla \Pi)^3,
\]
we get, by using (3.7) with \( \lambda = 0 \), that
\[
2^j \left( -1 + \frac{2}{p} \right) \frac{1}{2^j} \|\Delta^j \Delta^k u^3\|_{L^2(L^p)} + \mu (2^{2k} + 2^{2j}) \|\Delta^j \Delta^k u\|_{L^1(L^p)}
\]
\[
\lesssim 2^j \left( -1 + \frac{2}{p} \right) \frac{1}{2^j} \|\Delta^j \Delta^k u^3\|_{L^p} + \int_0^T d_{j, k}(t) \left( \|u^h(t)\|_{B^0_p} \|u^h(t)\|_{B^2_p} + \|u^3(t)\|_{B^2_p} \|u^h(t)\|_{B^2_p} + \mu \|a(t)\|_{B^2_p} \left( \|u^h(t)\|_{B^2_p} + \|u^3(t)\|_{B^2_p} \right) dt.
\]
After summation, this gives
\[
\|u^3\|_{L^\infty_T(\mathbb{R}^3_p)} + \mu \| u^3 \|_{L^1_T(\mathbb{R}^3_p)} \lesssim \|u_0^3\|_{\mathbb{R}^3_p} + \int_0^T (\|u^h(t)\|_{\mathbb{R}^3_0} \|u^h(t)\|_{\mathbb{R}^3_p} + \|u^3(t)\|_{\mathbb{R}^3_p} \|u^h(t)\|_{\mathbb{R}^3_p} + \|u^3(t)\|_{\mathbb{R}^3_p}) dt.
\]

By interpolation, we have
\[
\|u^3(t)\|_{\mathbb{R}^3_p} \|u^h(t)\|_{\mathbb{R}^3_p} \leq \|u^3(t)\|_{\mathbb{R}^3_0}^{\frac{1}{2}} \|u^3(t)\|_{\mathbb{R}^3_p}^{\frac{1}{2}} \|u^h(t)\|_{\mathbb{R}^3_p}^{\frac{1}{2}} \|u^h(t)\|_{\mathbb{R}^3_p}^{\frac{1}{2}}.
\]

Using the induction hypothesis (3.1) and Cauchy-Schwarz inequality, we get
\[
\|u^3\|_{L^\infty_T(\mathbb{R}^3_p)} + \mu \| u^3 \|_{L^1_T(\mathbb{R}^3_p)} \lesssim \|u_0^3\|_{\mathbb{R}^3_p} + \frac{\zeta T}{\mu} \|u^3\|_{L^\infty_T(\mathbb{R}^3_p)} \|u^3\|_{L^1_T(\mathbb{R}^3_p)}^{\frac{1}{2}} + \frac{\zeta T}{\mu} (\|u^3\|_{L^\infty_T(\mathbb{R}^3_p)} \|u^3\|_{L^1_T(\mathbb{R}^3_p)})^{\frac{1}{2}}.
\]

Thus, if \(c_0\) is small enough in (3.1), we get
\[
(3.8) \quad \forall T < T^*, \quad \|u^3\|_{L^\infty_T(\mathbb{R}^3_p)} + \mu \| u^3 \|_{L^1_T(\mathbb{R}^3_p)} \lesssim \|u_0^3\|_{\mathbb{R}^3_p} + \zeta T.
\]

The estimate on \(u^h\) is different. Because of the term \(\mu a \Delta u^3\) which has no chance to be small and which appears in the equation of \(u^h\), we need to use conjugating with an exponential weight. Let us point out that \(u_\lambda\) is the solution of
\[
\begin{cases}
\partial_t u_\lambda - \mu \Delta u_\lambda + \lambda \|u^3(t)\|_{\mathbb{R}^3_p} u_\lambda = (-u \cdot \nabla u + \mu a \Delta u - (1 + a) \Pi)\lambda, \\
\text{div } u_\lambda = 0, \quad u_{\lambda t=0} = 0.
\end{cases}
\]

Let us consider any subinterval \(I = [I^-, I^+]\) of \([0, T]\). Then applying (3.7), we infer
\[
\|u^h_\lambda\|_{L^\infty(I; \mathbb{R}^3_p)} + \mu \| u^h_\lambda \|_{L^1(I; \mathbb{R}^3_p)} \lesssim \|u^h_\lambda(I^-)\|_{\mathbb{R}^3_p} + \int_I (\|u^h_\lambda(t)\|_{\mathbb{R}^3_p} \|u^h_\lambda(t)\|_{\mathbb{R}^3_p} + \|u^3(t)\|_{\mathbb{R}^3_p} \|u^h_\lambda(t)\|_{\mathbb{R}^3_p} + \|u^3(t)\|_{\mathbb{R}^3_p} \|u_\lambda(t)\|_{\mathbb{R}^3_p} + \|u^3(t)\|_{\mathbb{R}^3_p} dt).
\]

Using the induction hypothesis (3.1) and Cauchy-Schwarz inequality, this gives
\[
\|u^h_\lambda\|_{L^\infty(I; \mathbb{R}^3_p)} + \mu \| u^h_\lambda \|_{L^1(I; \mathbb{R}^3_p)} \lesssim \|u^h_\lambda(I^-)\|_{\mathbb{R}^3_p} + \frac{\zeta T}{\mu} (\|u^h_\lambda\|_{L^\infty(I; \mathbb{R}^3_p)} + \mu \|u^h_\lambda\|_{L^1(I; \mathbb{R}^3_p)})
\]
\[
+ \|u^3\|_{L^2(I; \mathbb{R}^3_p)} \|u^h_\lambda\|_{L^2(I; \mathbb{R}^3_p)} + \mu \int_I \|a_\lambda(t)\|_{\mathbb{R}^3_p} \|u^3(t)\|_{\mathbb{R}^3_p} dt.
\]

By interpolation, this gives
\[
\|u^h_\lambda\|_{L^\infty(I; \mathbb{R}^3_p)} + \mu \| u^h_\lambda \|_{L^1(I; \mathbb{R}^3_p)} \lesssim \|u^h_\lambda(I^-)\|_{\mathbb{R}^3_p} + \mu \int_I \|a_\lambda(t)\|_{\mathbb{R}^3_p} \|u^3(t)\|_{\mathbb{R}^3_p} dt
\]
\[
+ \left(\frac{\zeta T}{\mu} + \frac{1}{\mu^\frac{1}{2}} \|u^3\|_{L^2(I; \mathbb{R}^3_p)} \right) \|u^h_\lambda\|_{L^\infty(I; \mathbb{R}^3_p)} + \mu \|u^h_\lambda\|_{L^1(I; \mathbb{R}^3_p)}.
\]
The induction hypothesis (3.1) implies that, if $c_0$ is chosen small enough in (3.1), then
\[
\|u_h^k\|_{L^\infty(I;\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1(I;\mathbb{R}_p^{2^2})} \lesssim \|u_h^k(I^-)\|_{\mathbb{R}^3_p} + \mu \int_I \|a_\lambda(t)\|_{B_{p}^2} \|a^3(t)\|_{\mathbb{R}_p^{2^2}} dt \\
+ \frac{1}{\mu^2} \|u^3\|_{L^2(I;\mathbb{R}_p^{2^2})} \left(\|u_h^k\|_{L^\infty(I;\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1(I;\mathbb{R}_p^{2^2})}\right).
\]
Thus, two constant $C_0$ and $C_1$ exist such that, if the interval $I$ satisfies
\[
\int_I \|u^3(t)\|_{\mathbb{R}_p^{2^2}} dt \leq \frac{\mu}{C_1},
\]
then we have
\[
\|u_h^k\|_{L^\infty(I;\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1(I;\mathbb{R}_p^{2^2})} \leq C_0 \left(\|u_h^k(I^-)\|_{\mathbb{R}^3_p} + \mu \int_I \|a_\lambda(t)\|_{B_{p}^2} \|a^3(t)\|_{\mathbb{R}_p^{2^2}} dt\right).
\]
Now let us decompose the interval $[0, T]$ into intervals such that the smallness condition (3.9) is satisfied. Let us define the sequence $(t_j)_{0 \leq j \leq N}$ such that $t_0 = 0, t_N = T$,
\[
\forall j \in \{0, \cdots, N - 2\}, \quad \int_{t_j}^{t_{j+1}} \|u^3(t)\|_{\mathbb{R}_p^{2^2}} dt = \frac{\mu}{C_1} \quad \text{and} \quad \int_{t_{N-1}}^{t_N} \|u^3(t)\|_{\mathbb{R}_p^{2^2}} dt \leq \frac{\mu}{C_1}.
\]
Let us observe that
\[
\int_0^T \|u^3(t)\|_{\mathbb{R}_p^{2^2}} dt \geq \frac{\mu}{C_1} (N - 2)
\]
which implies that the number of intervals $N$ satisfies
\[
N \leq C_1 \frac{\mu}{\mu} \int_0^T \|u^3(t)\|_{\mathbb{R}_p^{2^2}} dt + 2.
\]
Now let us prove by induction that, for any $j \leq N$, we have
\[
(P_j) \quad \|u_h^k\|_{L^\infty([0, t_j];\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1([0, t_j];\mathbb{R}_p^{2^2})} \leq C_0 \left(\|u_0\|_{\mathbb{R}^3_p} + \mu \int_0^{t_j} \|a_\lambda(t)\|_{B_{p}^2} \|a^3(t)\|_{\mathbb{R}_p^{2^2}} dt\right).
\]
For $j = 1$, it is simply (3.10) applied with $I = [0; t_1]$. Now, let us assume $(P_j)$ for $j \leq N - 1$. Applying (3.10) with $I = [t_j, t_{j+1}]$ gives
\[
\|u_h^k\|_{L^\infty([t_j, t_{j+1}];\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1([t_j, t_{j+1}];\mathbb{R}_p^{2^2})} \leq C_0 \left(\|u_h^k(t_j)\|_{\mathbb{R}^3_p} + \mu \int_{t_j}^{t_{j+1}} \|a_\lambda(t)\|_{B_{p}^2} \|a^3(t)\|_{\mathbb{R}_p^{2^2}} dt\right).
\]
The induction hypothesis $(P_j)$ implies that
\[
\|u_h^k\|_{L^\infty([t_j, t_{j+1}];\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1([t_j, t_{j+1}];\mathbb{R}_p^{2^2})} \leq C_0^{j+1} \left(\|u_0\|_{\mathbb{R}^3_p} + \mu \int_0^{t_j} \|a_\lambda(t)\|_{B_{p}^2} \|u^3(t)\|_{\mathbb{R}_p^{2^2}} dt\right) + C_0 \mu \int_{t_j}^{t_{j+1}} \|a_\lambda(t)\|_{B_{p}^2} \|a^3(t)\|_{\mathbb{R}_p^{2^2}} dt
\]
which gives obviously $(P_{j+1})$ and thus $(P_N)$. Because of (3.11), this gives
\[
\|u_h^k\|_{L^\infty(T;\mathbb{R}^3_p)} + \mu \|u_h^k\|_{L^1(T;\mathbb{R}_p^{2^2})} \leq C_0 \mu \int_0^T \|a_\lambda(t)\|_{\mathbb{R}_p^{2^2}} dt.
\]
On the other hand, notice from Lemma 2.1 that \( \| \nabla u^3(t) \|_{L^\infty} \leq C \| u^3(t) \|_{B^{\frac{3}{2}}_p} \), for \( \lambda \) large enough, we get, by applying Lemma 2.5 with \( f(t) = \| u^3(t) \|_{B^{\frac{3}{2}}_p} \), that

\[
\| a_\lambda \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \frac{\lambda}{2} \int_0^T \| u^3(t) \|_{B^{\frac{3}{2}}_p} a_\lambda(t) \, dt \leq \| a_0 \|_{B^{\frac{3}{2}}_p} + C \| u^h \|_{L^1_t(B^{\frac{3}{2}}_p)} a_\lambda \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)}.
\]

Thus as long as \( \zeta_T/\mu \) is chosen sufficiently small, the induction hypothesis (3.1) leads to

\[
\forall T < T^*, \quad \| a_\lambda \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \lambda \int_0^T \| u^3(t) \|_{B^{\frac{3}{2}}_p} a_\lambda(t) \, dt \leq 2 \| a_0 \|_{B^{\frac{3}{2}}_p}.
\]

Substituting (3.13) into (3.12) gives rise to

\[
\| u^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| u^h \|_{L^1_t(B^{\frac{3}{2}}_p)} \exp \left( -\lambda \int_0^T \| u^3(t) \|_{B^{\frac{3}{2}}_p} \, dt \right) \leq \| u_\lambda^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| u_\lambda^h \|_{L^1_t(B^{\frac{3}{2}}_p)}.
\]

As a consequence, we obtain

\[
\| u^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| u^h \|_{L^1_t(B^{\frac{3}{2}}_p)} \exp \left( -\lambda \int_0^T \| u^3(t) \|_{B^{\frac{3}{2}}_p} \, dt \right) \leq \| u_\lambda^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| u_\lambda^h \|_{L^1_t(B^{\frac{3}{2}}_p)}
\]

for some sufficiently large constant \( C_1'' \). This together with (3.8) implies that

\[
\| u^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| a_\lambda \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \| u^h \|_{L^1_t(B^{\frac{3}{2}}_p)} \leq C_2 \left( \| u_0^h \|_{B^{\frac{3}{2}}_p} + \mu \| a_0 \|_{B^{\frac{3}{2}}_p} \right) \exp \left( \frac{C_0}{\mu^2} \| u_0^3 \|_{B^{\frac{3}{2}}_p} \right)
\]

for \( t \leq T^+ \), provided that \( \zeta_T/\mu \leq c_0 \) is small enough.

We now claim that \( T^+ = \infty \) if the initial data \((a_0, u_0)\) satisfies (1.6). Otherwise, we infer from (3.14) that

\[
\| u^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| a_\lambda \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \| u^h \|_{L^1_t(B^{\frac{3}{2}}_p)} \leq C_2 \eta \quad \text{for} \quad t \leq T^+.
\]

In particular if we choose \( \eta \) in (1.6) is so small that \( \eta \leq \frac{c_0}{2C_2} \), one has

\[
\| u^h \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \mu \| a_\lambda \|_{\tilde{L}^\infty_t(B^{\frac{3}{2}}_p)} + \| u^h \|_{L^1_t(B^{\frac{3}{2}}_p)} \leq \frac{c_0}{2} \quad \text{for} \quad t \leq T^+,
\]

which contradicts with the induction hypothesis (3.1), and which in turn shows that \( T^+ = \infty \) under the assumption (1.6). Furthermore, (3.8) and (3.14) ensures (1.8). This completes the proof of Theorem 1.2.
4. The proof of Theorem 1.3

4.1. Outline of proof to Theorem 1.3. The purpose of this section is to present the proof of Theorem 1.3 by following the same line of that to Theorem 1.2. Toward this, we shall first construct the approximate solutions to (1.2) with data (1.10) as a perturbation to the 2-D classical Navier-Stokes system depending on a parameter. Without loss of generality, we may assume that the viscous coefficient $\mu = 1$ in (1.2). The detailed strategy is as follows:

**Step 1.** Construction of the approximate solutions.

As in [10, 17], we denote $(v^h, \Pi_0)$ to be the global smooth solution of the following 2-D Navier-Stokes system depending on a parameter $y_3$:

$$
(NS2D_3) \quad \begin{cases}
\partial_t v^h + v^h \cdot \nabla v^h - \Delta h v^h = -\nabla h \Pi_0, \\
\text{div} v^h = 0, \\
v^h|_{t=0} = v^h_0(\cdot, y_3).
\end{cases}
$$

**Notations** Here and in what follows, we always denote

$$x_h = (x_1, x_2), \quad \nabla h = (\partial_{x_1}, \partial_{x_2}), \quad \Delta h = \partial_{x_1}^2 + \partial_{x_2}^2, \quad \text{and} \quad [b_\varepsilon(x)] = b(x_h, \varepsilon x_3).$$

Then as in [10] and [17], we define the approximate solutions $(\varepsilon_{app}(t, x), \Pi_{app}(t, x))$ as

$$v^\varepsilon_{app}(t, x) \overset{\text{def}}{=} (v^h, 0)(t, x_h, \varepsilon x_3) \quad \text{and} \quad \Pi_{app}^\varepsilon(t, x) \overset{\text{def}}{=} \Pi_0(t, x_h, \varepsilon x_3),$$

which satisfy

$$
\left\{ \begin{array}{l}
\partial_t v^\varepsilon_{app} + v^\varepsilon_{app} \cdot \nabla v^\varepsilon_{app} - \Delta v^\varepsilon_{app} + \nabla \Pi^\varepsilon_{app}(t, x_h, x_3) = F^\varepsilon(t, x_h, x_3), \\
\text{div} v^\varepsilon_{app} = 0, \\
v^\varepsilon_{app}(t, x_h, x_3)|_{t=0} = u_0^\varepsilon(x_h, x_3) \overset{\text{def}}{=} (v^h_0)(x_h, \varepsilon x_3)
\end{array} \right.
$$

with

$$F^\varepsilon(t, x_h, x_3) = \varepsilon F_1(t, x_h, \varepsilon x_3) + F_2^\varepsilon(t, x_h, \varepsilon x_3),$$

where

$$F_1(t, x_h, y_3) \overset{\text{def}}{=} (0, \partial_3 \Pi_0)(t, x_h, y_3) \quad \text{and} \quad F_2^\varepsilon(t, x_h, y_3) \overset{\text{def}}{=} \varepsilon^2(\partial_3^2 v^h_0)(t, x_h, y_3).$$

**Step 2.** The estimate of the error between the true solution and the approximate ones.

Let

$$R^\varepsilon \overset{\text{def}}{=} u^\varepsilon - v^\varepsilon_{app} \quad \text{and} \quad Q^\varepsilon \overset{\text{def}}{=} \Pi^\varepsilon - \Pi^\varepsilon_{app}.$$ 

Then it follows from (1.2) and (4.2) that $(a^\varepsilon, R^\varepsilon, Q^\varepsilon)$ solves

$$
\left\{ \begin{array}{l}
\partial_t a^\varepsilon + (R^\varepsilon + v^\varepsilon_{app}) \cdot \nabla a^\varepsilon = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t R^\varepsilon + R^\varepsilon \cdot \nabla R^\varepsilon + R^\varepsilon \cdot \nabla v^\varepsilon_{app} + v^\varepsilon_{app} \cdot \nabla R^\varepsilon - \Delta R^\varepsilon + \nabla Q^\varepsilon = a^\varepsilon(\Delta R^\varepsilon + \Delta v^\varepsilon_{app} - \nabla \Pi^\varepsilon_{app}) - F^\varepsilon, \\
\text{div} R^\varepsilon = \text{div} v^\varepsilon_{app} = 0, \\
a^\varepsilon(t, x_h, x_3)|_{t=0} = \varepsilon^\delta a_0(x_h, \varepsilon x_3), \quad R^\varepsilon|_{t=0} = 0.
\end{array} \right.
$$

To solve (4.5) globally in the framework of the anisotropic Besov space $\mathfrak{B}_p^0$, in general, one should require $L^1(\mathbb{R}^+; \mathfrak{B}_p^0(\mathbb{R}^3))$ estimate for the source term $F^\varepsilon$ given by (4.3). Nevertheless,
according to Lemma 4.1 below (see also (4.7) of [17]), we do not have the \( L^1(\mathbb{R}^+; \mathfrak{B}_p^0(\mathbb{R}^3)) \) estimate for the term \( F_2^\varepsilon \). To deal with this term, as in [17], we denote
\[
V^h(t, x_h, x_3) \overset{\text{def}}{=} (\partial_3 v^h, 0)(t, x_h, \varepsilon x_3),
\]
then \( F_2^\varepsilon = \varepsilon \partial_3 V^h \). We shall construct \((R_1^\varepsilon, \Pi_\varepsilon^\alpha)\) via
\[
\begin{aligned}
\partial_t R_1^\varepsilon - \Delta R_1^\varepsilon + \nabla \Pi_\varepsilon^\alpha &= -F_2^\varepsilon, \\
\text{div} R_1^\varepsilon &= 0, \\
R_1^\varepsilon|_{t=0} &= 0,
\end{aligned}
\]
(4.6)
then as \( \text{div} V^h = 0 \), we have \( \text{div} F_2^\varepsilon = 0 \) and \( \nabla \Pi_\varepsilon^\alpha = 0 \), so that let
\[
w^\varepsilon \overset{\text{def}}{=} R^\varepsilon - R_1^\varepsilon,
\]
to solve (4.5) for \((a^\varepsilon, R^\varepsilon, Q^\varepsilon)\) is reduced to solve \((a^\varepsilon, w^\varepsilon, Q^\varepsilon)\) through
\[
\begin{aligned}
\partial_t a^\varepsilon + (w^\varepsilon + R_1^\varepsilon + v^\varepsilon_{\text{app}}) \cdot \nabla a^\varepsilon &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t w^\varepsilon + w^\varepsilon \cdot \nabla w^\varepsilon + w^\varepsilon \cdot \nabla (v^\varepsilon_{\text{app}} + R_1^\varepsilon) + (v^\varepsilon_{\text{app}} + R_1^\varepsilon) \cdot \nabla w^\varepsilon - \Delta w^\varepsilon \\
&+ (1 + a^\varepsilon) \nabla Q^\varepsilon = G^\varepsilon,
\end{aligned}
\]
(4.8)
\[
\begin{aligned}
\text{div} w^\varepsilon &= 0, \\
(a^\varepsilon(t, x_h, x_3)|_{t=0} = \varepsilon \delta_0 a_0(x_h, \varepsilon x_3), \quad w^\varepsilon|_{t=0} &= 0, \\
\end{aligned}
\]
with \( F_1^\varepsilon \) given by (4.3) and
\[
G^\varepsilon \overset{\text{def}}{=} a^\varepsilon (\Delta w^\varepsilon + \Delta R_1^\varepsilon + \Delta v^\varepsilon_{\text{app}} - \nabla \Pi_{\text{app}} - R_1^\varepsilon \cdot \nabla (R_1^\varepsilon + v^\varepsilon_{\text{app}}) - v^\varepsilon_{\text{app}} \cdot \nabla R_1^\varepsilon - \varepsilon [F_1]|.\varepsilon.
\]
We shall follow the same line of the proof of Theorem 1.2 to construct the global solution of (4.8). Namely, we shall first estimate \( a^\varepsilon \) in the isentropic Besov spaces
\[
\bar{L}^\infty(\mathbb{R}^+; \mathfrak{B}_p^2(\mathbb{R}^3)) \cap \bar{L}^\infty(\mathbb{R}^+; \mathfrak{B}_q^{1+\frac{2}{q}}(\mathbb{R}^3))
\]
for \( q \in ]\frac{3}{2}, 2[ \) and \( p \) in \([3, 4[\), and then we estimate \( w^\varepsilon \) in the anisotropic Besov spaces
\[
\bar{L}^\infty(\mathbb{R}^+; \mathfrak{B}_p^0(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \mathfrak{B}_p^2(\mathbb{R}^3)).
\]
With these estimates, we repeat the argument at the beginning of Section 3 to construct the unique global solution of (1.2) with data (1.10).

4.2. Technical Lemmas. For simplicity, we shall neglect the subscript \( \varepsilon \) in the rest of this section. Let us first recall Lemma 3.2 and inequality (4.7) of [17].

**Lemma 4.1.** Let \((v^h, \Pi_0)\) be a smooth enough solution of \((NS2D_3)\) and \(R_1\) be determined by (4.6). Then under the assumptions of Theorem 1.3, for any \( \alpha \in \mathbb{N}^3 \), one has
\[
\begin{aligned}
\|\partial^\alpha v^h\|_{\bar{L}^\infty(\mathbb{R}^+; \mathfrak{B}_p^2)} + \|\partial^\alpha v^h\|_{L^1(\mathbb{R}^+; \mathfrak{B}_p^{1+\frac{2}{p}})} + \|\partial^\alpha \Pi_0\|_{L^1(\mathbb{R}^+; \mathfrak{B}_p^{1+\frac{2}{p}})} &\leq C_{v_0}, \\
\|\partial^\alpha \partial_3 v^h\|_{\bar{L}^\infty(\mathbb{R}^+; \mathfrak{B}_p^{1+\frac{2}{p}})} + \|\partial^\alpha \partial_3 v^h\|_{L^1(\mathbb{R}^+; \mathfrak{B}_p^{1+\frac{2}{p}})} &\leq C_{v_0}, \\
\end{aligned}
\]
and
\[
\begin{aligned}
\|R_1\|_{\bar{L}^\infty(\mathbb{R}^+; \mathfrak{B}_p^0)} + \sum_{|\alpha| \leq 1} \|\partial^\alpha \nabla R_1\|_{L^1(\mathbb{R}^+; \mathfrak{B}_p^{1+\frac{2}{p}})} &\leq C_{v_0} \varepsilon.
\end{aligned}
\]
Lemma 4.2. Let \( q \) be in \([1, 2]\), \( p \) in \([3, 4]\), and \( G \) be given by (4.9). Then under the assumptions of Theorem 1.3, one has
\[
\|G\|_{L^1_t(B^p_q)} \lesssim C\|a\|_{L^\infty_t(B^p_q)} \|\Delta w\|_{L^1_t(B^p_q)} + C_{10} (\varepsilon + \|a\|_{L^\infty_t(B^p_q)}) + \varepsilon\|a\|_{L^\infty_t(B^{-1+\frac{4}{p}}_q)}. \tag{4.10}
\]

Proof. Notice that \( p < 4 \), applying Lemma 2.2 and Lemma 2.3 yields
\[
\|a(\Delta w + \Delta_h R_1)\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\Delta w\|_{L^1_t(B^p_q)} + \|\Delta_h R_1\|_{L^1_t(B^p_q)} \right)
\lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\Delta w\|_{L^1_t(B^p_q)} + \|R_1\|_{L^1_t(B^{-1+\frac{4}{p}}_q)} \right).
\]

Similarly as Lemma 4.2. Theorem 1.3, one has
\[
\|a\partial^2_t R_1\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\partial^2_t R_1\|_{L^1_t(B^p_q)} \right)
\lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\partial^2_t R_1\|_{L^1_t(B^p_q)} \right).
\]

It follows the same line that
\[
\|a\Delta v_{app}\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\Delta v\|_{L^1_t(B^p_q)} + \varepsilon^2 \|a\|_{L^\infty_t(B^{-1+\frac{4}{p}}_q)} \right),
\]
and
\[
\|a\nabla \Pi_{app}\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\nabla \Pi\|_{L^1_t(B^p_q)} \right).
\]

Whereas applying Lemma 2.3 twice leads to
\[
\|a R_1 \cdot \nabla (R_1 + v_{app})\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\nabla R_1\|_{L^1_t(B^p_q)} + \|\nabla v\|_{L^1_t(B^p_q)} \right)
\]
and
\[
\|a v_{app} \cdot \nabla R_1\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\nabla v\|_{L^1_t(B^p_q)} \right).
\]

As a consequence, we obtain
\[
\|G\|_{L^1_t(B^p_q)} \lesssim \|a\|_{L^\infty_t(B^p_q)} \left( \|\Delta w\|_{L^1_t(B^p_q)} + \|R_1\|_{L^1_t(B^{-1+\frac{4}{p}}_q)} + \|v\|_{L^1_t(B^{-1+\frac{4}{p}}_q)} + \|\nabla \Pi\|_{L^1_t(B^p_q)} \right)
\]
\[
+ \left( \|R_1\|_{L^1_t(B^p_q)} + \|v\|_{L^1_t(B^p_q)} \left( \|\nabla R_1\|_{L^1_t(B^{-1+\frac{4}{p}}_q)} + \|\nabla v\|_{L^1_t(B^{-1+\frac{4}{p}}_q)} \right) \right)
\]
\[
+ \|a\|_{L^\infty_t(B^{-1+\frac{4}{p}}_q)} \left( \|\partial^2_t R_1\|_{L^1_t(B^p_q)} + \varepsilon^2 \|\partial^2_t v\|_{L^1_t(B^{\frac{1}{2}+rac{1}{p}}_q)} \right) + \varepsilon \|\partial^3 \Pi\|_{L^1_t(B^p_q)},
\]
from which, Lemma 2.1 and Lemma 4.1, we conclude the proof of (4.10).

4.3. The proof of Theorem 1.3. It follows from the argument in Subsection 4.1 that we only need to solve (4.8) globally for \( \varepsilon \) sufficiently small in order to prove Theorem 1.3. Given data (1.10), it is well-known that (1.2) has a unique local solution \((a, u)\) on \((0, T^*)\) for some \( T^* > 0 \). Without loss of generality, we may assume that \( T^* \) is the lifespan of \((a, u)\). Of course, the solution \((a^\varepsilon, w^\varepsilon, Q^\varepsilon)\) of (4.8) entails this lifespan \( T^* \). Similar to the proof of Theorem 1.2 in Section 3, we denote
\[
T^\# \equiv \sup \{ T < T^* / \eta T \mid \|a\|_{L^\infty_t(B^p_q)} + \|w\|_{L^\infty_t(B^p_q)} + \|w\|_{L^1_t(B^p_q)} \leq \delta \},
\]
for some sufficiently small positive constant \( \delta \), which will be chosen later on.
We also define
\[
 f_\lambda(t) \overset{\text{def}}{=} f(t) \exp \left( -\lambda \int_0^t V_h(t') \, dt' \right) \quad \text{with} \quad V_h(t) \overset{\text{def}}{=} \|v^h(t)\|_{B^\frac{3}{2} + \frac{1}{p}}^2 + \|v^h(t)\|_{B^1 + \frac{1}{p} + \frac{1}{q}}^2,
\]
\[
 i(a, w) \overset{\text{def}}{=} w \cdot \nabla w + w \cdot \nabla (v_{\text{app}} + R_1) + (v_{\text{app}} + R_1) \cdot \nabla w - G.
\]

Then thanks to (4.8), \( w_\lambda \) solves
\[
 \partial_t w_\lambda + \lambda V_h(t) w_\lambda - \Delta w_\lambda + (1 + a) \nabla Q_\lambda + i(a, w)_\lambda = 0.
\]

Applying \( \Delta_j^h \Delta_k^v \) to the above equation, then taking the \( L^2 \) inner product of the resulting equation with \( |\Delta_j^h \Delta_k^v w_\lambda|^{p-2} \Delta_j^h \Delta_k^v w_\lambda \) for \( p \in [3, 4] \) and integrating the resulting equation over \([0, t]\), we obtain
\[
 \| \Delta_j^h \Delta_k^v w_\lambda \|_{L^\infty_t(L^p)} + \lambda \int_0^t V_h(t') \| \Delta_j^h \Delta_k^v w_\lambda \|_{L^p} \, dt' + c(2^{2j} + 2^{2k}) \| \Delta_j^h \Delta_k^v w_\lambda \|_{L^1_t(L^p)} \leq \left\| \left( (1 + a) \nabla Q - i(a, w) \right)_\lambda \right\|_{L^1_t(L^p)},
\]
for some \( c > 0 \). After summation, this gives
\[
 \| w_\lambda \|_{\mathfrak{L}^\infty_t(\mathfrak{B}^0_p)} + \lambda \int_0^t V_h(t') \| w_\lambda(t') \|_{\mathfrak{B}^0_p} \, dt' + c \| w_\lambda \|_{L^1_t(\mathfrak{B}^0_p)} \lesssim \left\| \left( (1 + a) \nabla Q - i(a, w) \right)_\lambda \right\|_{L^1_t(\mathfrak{B}^0_p)}.
\]

Lemma 2.4 and (4.8) implies
\[
 \left\| \left( (1 + a) \nabla Q - i(a, w) \right)_\lambda \right\|_{\mathfrak{B}^0_p} \lesssim \left\| i(a, w)_\lambda \right\|_{\mathfrak{B}^0_p}.
\]

And as \( p < 4 \), applying Lemma 2.3 leads to
\[
 \| w \cdot \nabla w_\lambda \|_{\mathfrak{B}^0_p} \lesssim \| w \|_{\mathfrak{B}^0_p} \| \nabla w_\lambda \|_{\mathfrak{B}^1_p + \frac{1}{p}} \lesssim \| w \|_{\mathfrak{B}^0_p} \| w_\lambda \|_{\mathfrak{B}^0_p},
\]
\[
 \| w_\lambda \cdot \nabla R_1 \|_{\mathfrak{B}^0_p} \lesssim \| \nabla R_1 \|_{\mathfrak{B}^1_p + \frac{1}{p}} \| w_\lambda \|_{\mathfrak{B}^0_p},
\]
\[
 \| R_1 \cdot \nabla w_\lambda \|_{\mathfrak{B}^0_p} \lesssim \| R_1 \|_{\mathfrak{B}^2_p} \| w_\lambda \|_{\mathfrak{B}^2_p}.
\]

While notice that for \( b_\varepsilon(x) = b(x_h, \varepsilon x_3) \),
\[
 w_\lambda \cdot \nabla v_{\text{app}} = w_\lambda^h \cdot (\nabla_h v^h)_\varepsilon + \varepsilon w_\lambda^3 [\partial_3 v^h]_\varepsilon,
\]
we get, by applying Lemma 2.3 once again, that
\[
 \| w_\lambda \cdot \nabla v_{\text{app}} \|_{\mathfrak{B}^0_p} \lesssim \| v^h \|_{\mathfrak{B}^1_p + \frac{1}{p}} \| w_\lambda^h \|_{\mathfrak{B}^0_p} + \varepsilon \| \partial_3 v^h \|_{\mathfrak{B}^2_p} \| w_\lambda^3 \|_{\mathfrak{B}^0_p}.
\]

Along the same line, one has
\[
 \| v_{\text{app}} \cdot \nabla w_\lambda \|_{\mathfrak{B}^0_p} \lesssim \| v^h \|_{\mathfrak{B}^1_p + \frac{1}{p}} \| \nabla_h w_\lambda \|_{\mathfrak{B}^0_p}.
\]
Therefore, substituting the above estimates into (4.13), we infer that for any $t \leq T^\bullet$,\
\[
\|w_\lambda\|_{L^\infty_t(B^0_p)} + \lambda \int_0^t V_t(t') \|w_\lambda(t')\|_{B^0_p} dt' + c \|w_\lambda\|_{L^1_t(B^0_p)} \lesssim \|G_\lambda\|_{L^1_t(B^0_p)} + \left( \|\nabla R_1\|_{L^1_t(B^0_p)} + \varepsilon \|\partial_3 v^h\|_{L^1_t(B^0_p)} \right) \|w_\lambda\|_{L^\infty_t(B^0_p)} \\
+ \left( \|w\|_{L^\infty_t(B^0_p)} + \|R_1\|_{L^\infty_t(B^0_p)} \right) \|w_\lambda\|_{L^1_t(B^0_p)} \\
+ \int_0^t \left( \|v^h\|_{B_{p, \frac{2}{p}}^{\frac{1}{p}}(\mathbb{R}^3)} \|w_\lambda\|_{B_{p, \frac{2}{p}}^{\frac{1}{p}}} + \|v^h\|_{B_{p, \frac{1}{p} + \frac{2}{p}}^{\frac{1}{p}}(\mathbb{R}^3)} \|w_\lambda\|_{B^0_p} \right) dt'.
\]
(4.14)

Whereas it follows from the simple interpolation in the anisotropic Besov spaces that\
\[
\int_0^t \|v^h\|_{B_{p, \frac{2}{p}}^{\frac{1}{p}}} \|w_\lambda\|_{B_{p, \frac{2}{p}}^{\frac{1}{p}}} dt' \lesssim \left( \int_0^t \|v^h\|_{B^0_p} dt' \right)^{\frac{1}{2}} \|w_\lambda\|_{L^1_t(B^0_p)},
\]
from which and (4.14), we infer for $t \leq T^\bullet$
\[
\|w_\lambda\|_{L^\infty_t(B^0_p)} + \lambda \int_0^t V_t(t') \|w_\lambda(t')\|_{B^0_p} dt' + c \|w_\lambda\|_{L^1_t(B^0_p)} \lesssim \|G_\lambda\|_{L^1_t(B^0_p)} + \left( \|\nabla R_1\|_{L^1_t(B^0_p)} + \varepsilon \|\partial_3 v^h\|_{L^1_t(B^0_p)} \right) \|w_\lambda\|_{L^\infty_t(B^0_p)} \\
+ \left( \|w\|_{L^\infty_t(B^0_p)} + \|R_1\|_{L^\infty_t(B^0_p)} \right) \|w_\lambda\|_{L^1_t(B^0_p)} + \int_0^t V_t(t') \|w_\lambda(t')\|_{B^0_p} dt',
\]
for $V_t(t)$ defined by (4.12). Taking $\lambda \geq C$ in the above inequality and applying Lemma 4.1 and Lemma 4.2, we obtain for $t \leq T^\bullet$
\[
\|w_\lambda\|_{L^\infty_t(B^0_p)} + \frac{c}{2} \|w_\lambda\|_{L^1_t(B^0_p)} \leq C_{v_0} \left( \varepsilon + \|a\|_{L^\infty_t(B^0_p)} \right) + \varepsilon \|a\|_{L^\infty_t(B^{1+\frac{2}{q}}_q)} \\
+ \varepsilon \|w_\lambda\|_{L^\infty_t(B^0_p)} + C \left( \|a\|_{L^\infty_t(B^0_p)} + \|w\|_{L^\infty_t(B^0_p)} + C_{v_0} \varepsilon \right) \|w_\lambda\|_{L^1_t(B^0_p)}.
\]
(4.15)

Then taking $\delta \leq \frac{c}{8c}$ in (4.11) and $\varepsilon \leq \min \left\{ \frac{c}{8c}, 1 \right\}$, we deduce from (4.15) that
\[
\|w_\lambda\|_{L^\infty_t(B^0_p)} + \frac{c}{4} \|w_\lambda\|_{L^1_t(B^0_p)} \leq 2C_{v_0} \left( \varepsilon + \|a\|_{L^\infty_t(B^0_p)} \right) + \varepsilon \|a\|_{L^\infty_t(B^{1+\frac{2}{q}}_q)}
\]
for $t \leq T^\bullet$.

On the other hand, applying Lemma 2.6 to the free transport equation in (4.8) that for any $s \in [0, 1[$,
\[
\|a\|_{L^\infty_t(B^s_p)} \leq \|a_{0, \varepsilon}\|_{B^s_p} \exp \left( \|\nabla R_1\|_{L^1_t(B^s_p)} + \|\nabla w\|_{L^1_t(B^s_p)} + \|\nabla v^h\|_{L^1_t(B^s_p)} \right) \\
\leq C_{v_0} \varepsilon^{-\frac{1}{s}} \|a_{0, \varepsilon}\|_{B^s_p} \quad \text{for} \quad t \leq T^\bullet.
\]
(4.17)

As $q$ is in $[\frac{3}{2}, 2[$, we can apply his result with $-1 + \frac{2}{q}$. Together with (4.16) this ensures that, for any $t \leq T^\bullet$,
\[
\|w_\lambda\|_{L^\infty_t(B^0_p)} + \frac{c}{4} \|w_\lambda\|_{L^1_t(B^0_p)} \leq 2C_{v_0} \left( \varepsilon + \varepsilon^{-\frac{1}{s}} \|a_{0, \varepsilon}\|_{B^s_p} + \varepsilon^{1+\sigma - \frac{1}{q}} \|a_{0, \varepsilon}\|_{B^{1+\sigma - \frac{1}{q}}_q} \right)
\]
(4.18)
By virtue of (4.12) and (4.18), we obtain
\[
\|w\|_{\tilde{L}^{\infty}_t(B^0_p)} + \frac{c}{4} \|w\|_{L^1_t(B^2_p)} \leq 2C_{\nu_0}(\varepsilon^{\sigma-\frac{1}{p}}\|a_0\|_{B^\frac{1}{p}}^{\frac{1}{2}} + \varepsilon^{1+\sigma-\frac{1}{p}}\|a_0\|_{B^{-1+\frac{3}{p}}}) \\
\times \exp\left(C\left(\|v^h\|_{L^1_t(B^{1+\frac{2}{p}}_p)} + \|v^h\|_{L^2_t(B^{2+\frac{1}{p}}_p)}^2\right)\right) \\
\leq C_{\nu_0}(\varepsilon^{\sigma-\frac{1}{p}}\|a_0\|_{B^\frac{1}{p}}^{\frac{1}{2}} + \varepsilon^{1+\sigma-\frac{1}{p}}\|a_0\|_{B^{-1+\frac{3}{p}}}) \quad \text{for} \quad t \leq T^{\bullet}.
\]
Now as $\sigma > \frac{1}{4}$, we can take $p_0 < 4$ so that $\sigma - \frac{1}{p_0} > 0$. Then for $\varepsilon$ small enough, we conclude that $T^* = T^{\bullet}$, and there holds
\[
(4.19) \quad \|w\|_{\tilde{L}^{\infty}_t(B^0_{p_0})} + \frac{c}{4} \|w\|_{L^1_t(B^2_{p_0})} \leq C_{a_0,\nu_0}\varepsilon^{\sigma-\frac{1}{p_0}} \quad \text{for} \quad t \leq T^*.
\]
With (4.17) and (4.19), it is standard to prove that $T^* = \infty$ and the global solution $(a, u)$ of (1.2) such that
\[
a \in C([0, \infty); B_{p_0}^{\frac{1}{p}}(\mathbb{R}^3)) \cap \tilde{L}^{\infty}(\mathbb{R}^+; B_{p_0}^{\frac{1}{p}}(\mathbb{R}^3))
\]
and
\[
u \in C([0, \infty); B_{p_0}^{-1+\frac{3}{p_0}}(\mathbb{R}^3)) \cap \tilde{L}^{\infty}(\mathbb{R}^+; B_{p_0}^{-1+\frac{3}{p_0}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; B_{p_0}^{1+\frac{3}{p_0}}(\mathbb{R}^3)).
\]
The uniqueness part is guaranteed by Theorem 1 of [15]. We thus complete the proof of Theorem 1.3.

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(J.-Y. Chemin) Laboratoire J.-L. Lions, UMR 7598, Université Pierre et Marie Curie, 75230 Paris Cedex 05, FRANCE

E-mail address: chemin@ann.jussieu.fr

(M. PAICU) Université Bordeaux 1, Institut de Mathématiques de Bordeaux, F-33405 Talence Cedex, France

E-mail address: marius.paicu@math.u-bordeaux1.fr

(P. ZHANG) Academy of Mathematics & Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, The Chinese Academy of Sciences, Beijing 100190, CHINA

E-mail address: zp@amss.ac.cn