A-D-E Singularity and the Seiberg-Witten Theory

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Abstract

We study the low-energy effective theory of $N = 2$ supersymmetric Yang-Mills theory with $ADE$ gauge groups in view of the spectral curves of the periodic Toda lattice and the $A$-$D$-$E$ singularity theory. We examine the exact solutions by using the Picard-Fuchs equations for the period integrals of the Seiberg-Witten differential. In particular, we find that the superconformal fixed point in the strong coupling region of the Coulomb branch is characterized by the $ADE$ superpotential. We compute the scaling exponents, which agree with the previous results.

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1 Introduction

Seiberg and Witten showed that the low-energy effective theory of $N = 2$ supersymmetric gauge theory in four dimensions is determined by the prepotential, a holomorphic function of the period integrals of the meromorphic one-form (the Seiberg-Witten differential) on a Riemann surface\[1\]. For a simple Lie group $G$, it has been proposed in \[2, 3\] that the spectral curve of the periodic Toda lattice associated with the dual affine Lie algebra $\hat{G}$ provides the Riemann surface which describes the Coulomb branch of $N = 2$ supersymmetric Yang-Mills theory with the gauge group $G$. In the case of gauge theories with some matter hypermultiplets, the spectral curves and related integrable systems are discussed in \[4\]. Other systematic approaches based on the heterotic/type II duality\[5\] or the M5 branes \[6\] are also studied extensively.

In the present work we will study the exact solution of the low-energy effective theory from the viewpoint of the spectral curve of the periodic Toda lattice. For $ADE$ type gauge groups, the spectral curves are shown to be the sum of the superpotential of two-dimensional topological Landau-Ginzburg models of $ADE$ type and that of the topological $CP^1$ model. In a series of papers\[7, 8, 9, 10\], we have studied various aspects of the exact solution of the Seiberg-Witten theories with $ADE$ gauge groups by using two-dimensional topological field theories.

This paper is organized as follows: In sect. 2, we introduce the spectral curve of the periodic Toda lattice associated with the dual of the affine Lie algebra $\hat{G}$ for the gauge theory with gauge group $G$. In sect. 3, we consider the $ADE$ gauge groups and express the spectral curve as the sum of the superpotential of the topological $CP^1$ model and the $ADE$ minimal model. Using the flat coordinates in the $A$-$D$-$E$ singularity theory, we derive the Picard-Fuchs differential equations obeyed by the period integral of the Seiberg-Witten differential. We then show that these equations are equivalent to the Gauss-Manin system for the $ADE$ minimal model and the $CP^1$ model and the scaling relation for the Seiberg-Witten differential. In sect. 4, we study an exact solution in the strong coupling region. Argyres and Douglas\[11\] showed that there exists a non-trivial RG fixed point in the Coulomb branch such that the massless solitons with mutually non-local charges coexist and the theory corresponds to $N = 2$ superconformal field theory,
where the gauge invariant order parameters have fractional dimensions with respect to
the BPS mass. We investigate this Argyres-Douglas point in the Coulomb branch of the
$N = 2$ supersymmetric Yang-Mills theory for $ADE$ gauge groups.

2 Spectral Curves and $A-D-E$ Singularity

The low-energy properties of the Coulomb branch of $N = 2$ supersymmetric gauge theories
with gauge group $G$ are exactly described by holomorphic data associated with certain
algebraic curves. In particular, the BPS mass formula is expressed in terms of the period
integrals of the so-called Seiberg-Witten (SW) differential $\lambda_{SW}$:

$$m_{BPS} = |n^I a_I + m^I a_{DI}|$$  \hspace{1cm} (1)

where $n^I$ and $m^I$ are integers and

$$a_I = \oint_{A_I} \lambda_{SW}, \hspace{1cm} a_{DI} = \oint_{B_I} \lambda_{SW}, \hspace{1cm} I = 1, \ldots, r$$  \hspace{1cm} (2)

along one-cycles $A_I$, $B_I$ with appropriate intersections on the curve. Here $r$ is the rank
of the gauge group $G$. The low-energy effective theory effective action is described by the
prepotential $\mathcal{F}(a)$. The dual period $a_{DI}$ is then given by $\partial \mathcal{F}(a)/\partial a_I$.

We define the spectral curve for the periodic Toda lattice for the (twisted) affine Lie
algebra $(\hat{G})^\vee$. Let $G$ be a simple Lie algebra with rank $r$. Let $\alpha_1, \ldots, \alpha_r$ be simple
roots of the Lie algebra and $\alpha_0 = -\theta$, where $\theta$ denotes the highest root. We consider
a representation $\mathcal{R}$ with $d$ dimensions. Let $E_\alpha$ be generators associated with the roots
$\alpha$ and $H^I$ those of the Cartan subalgebra, which are realized by $d \times d$ matrices in the
representation $\mathcal{R}$. Introduce matrices $A$ and $B$ by

$$A(z) = \sum_{i=1}^r b_i H_i + a_i(E_{\alpha_i} + E_{-\alpha_i}) + a_0(z E_{\alpha_0} + z^{-1} E_{-\alpha_0})$$

$$B(z) = \sum_{i=1}^r b_i H_i + a_i(E_{\alpha_i} - E_{-\alpha_i}) + a_0(z E_{\alpha_0} - z^{-1} E_{-\alpha_0}),$$  \hspace{1cm} (3)

where $z$ is called as the spectral parameter. The equation of motion of the periodic Toda
lattice is defined in the Lax form

$$\frac{dA}{dt} = [A, B].$$  \hspace{1cm} (4)
The spectral curve is defined by the characteristic polynomial of the matrix $A(z)$:

$$P_G^R(x; u_1, \ldots, u_r, z) \equiv \det_R(x1_d - A(z)) = 0. \quad (5)$$

Here $u_i$ ($i = 1, \ldots, r$) are the $i$-th order Casimirs of $G$ with degree $q_i$ where $q_i$ is order of the Casimirs. The exponents $e_i$ of $G$ is related to $q_i$ by $q_i = e_i + 1$. The second order Casimir $u_2$ has degree 2 and the top Casimir $u_r$ has degree $q_r = h$, where $h$ is the Coxeter number. The curve depends on the scale parameter $\mu^2 = \prod_{i=0}^r a_i^{n_i}$, where non-negative integers $n_i$’s are the Dynkin labels of the affine roots. The spectral parameter $z$ and $\mu$ have degree $h^\vee$, the dual Coxeter number of $G$. The exponents and the (dual) Coxeter number are given in the table 1. The spectral curve is invariant under the transformation $z \rightarrow \mu^2/z$. For simply laced Lie algebra $G$, we have $h = h^\vee$. Therefore the top Casimir $u_r$ and the spectral parameter $z$ or its dual $\mu^2/z$ have the same degree. It is found that for the representation $\mathcal{R}$ of $G = ADE$, the spectral curve is given by

$$P_G^\mathcal{R}(x; u_1, \ldots, u_r + z + \frac{\mu^2}{z}) = 0, \quad (6)$$

where $P_G^\mathcal{R}(x; u_1, \ldots, u_r)$ is the characteristic polynomial of the representation $\mathcal{R}$ of $G$ and expressed in the form of

$$P_G^\mathcal{R}(x; u_1, \ldots, u_r) = \prod_{i=1}^d (x - \lambda_i \cdot a). \quad (7)$$
Here \( \lambda \) denote the weight vector of the representation \( \mathcal{R} \). We list the explicit form of the spectral curve for \( A_r, D_r \) and \( E_6 \) Lie algebras for the \( d \)-dimensional representation \( d \):

- **\( A_r \) (\( A_r^{(1)}, r + 1 \))**
  \[
  x^{r+1} - u_1 x^{r-1} - \cdots - u_r - \left( z + \frac{\mu^2}{z} \right) = 0 \tag{8}
  \]

- **\( D_r \) (\( D_r^{(1)}, 2r \))**
  \[
  x^{2r} - u_1 x^{2r-2} - \cdots - u_{r-2} x^4 - u_r x^2 - u_{r-1}^2 - x^2 \left( z + \frac{\mu^2}{z} \right) = 0 \tag{9}
  \]

- **\( E_6 \) (\( E_6^{(1)}, 27 \))** \([12]\)
  \[
  \frac{1}{2} x^3 \left( z + \frac{\mu^2}{z} + u_6 \right)^2 - q_1(x) \left( z + \frac{\mu^2}{z} + u_6 \right) + q_2(x) = 0, \tag{10}
  \]

where

\[
q_1 = 270 x^{15} + 342 u_1 x^{13} + 162 u_2^2 x^{11} - 252 u_2 x^{10} + (26 u_1^3 + 18 u_3) x^9 - 162 u_1 u_2^2 x^8 + (6 u_1 u_3 - 27 u_4) x^7 - (30 u_2^2 u_2 - 36 u_5) x^6 + (27 u_2^2 - 9 u_1 u_4) x^5 - (3 u_2 u_3 - 6 u_1 u_5) x^4 - 3 u_1 u_2 x^3 - 3 u_2 u_5 x - u_2^3,
\]

\[
q_2 = \frac{1}{2 x^3 (q_1^2 - p_1^2 p_2)},
\]

\[
p_1 = 78 x^{10} + 60 u_1 x^8 + 14 u_1^2 x^6 - 33 u_2 x^5 + 2 u_2 x^4 - 5 u_1 u_2 x^3 - u_4 x^2 - u_5 x - u_2^2,
\]

\[
p_2 = 12 x^{10} + 12 u_1 x^8 + 4 u_1^2 x^6 - 12 u_2 x^5 + u_3 x^4 - 4 u_1 u_2 x^3 - 2 u_4 x^2 + 4 u_5 x + u_2^2.
\]

For simply-laced Lie algebra \( G \), \( \hat{G} \) is self-dual, i.e. \( (G^{(1)})^\vee = G^{(1)} \). For non-simply laced Lie algebras, we have \( \hat{B}_r^\vee = A_{2r-1}^{(2)}, \hat{C}_r^\vee = D_{r+1}^{(2)}, \hat{F}_4^\vee = E_6^{(2)} \) and \( \hat{G}_2^\vee = D_4^{(3)} \). Thus we need the twisted affine Lie algebra to construct the spectral curve. The characteristic polynomial can be obtained by folding procedure of the corresponding Dynkin diagram. Due to \( h \neq h^\vee \), the spectral parameter \( z \) or its dual \( \mu^2/z \) appears in the spectral curve in a nontrivial way. Now we will write the explicit form of the spectral curves for \( d \)-dimensional representation \( d \) of non-simply laced Lie algebra \( G \).
• For $B_r \ (A_{2r-1}^{(2)}, 2r)$ case, the spectral curve of the representation $2r$ is obtained from the characteristic polynomial $P_{A_{2r-1}}^{2r}(x; u_1, \ldots, u_{2r})$ by the restriction $u_2 = \cdots = u_{2r-4} = 0$ and $u_{2r-2} = z + \mu^2/z$:

$$x^{2r} - u_1 x^{2r-2} - \cdots - u_{2r-1} - x\left(z + \frac{\mu^2}{z}\right) = 0. \quad (12)$$

• For $C_r \ (D_{r+1}^{(2)}, 2r + 2)$ case, the curve is obtained from the characteristic polynomial $P_{D_{r+1}}^{2r+2}(x; u_1, \ldots, u_r, u_{r+1})$ by restricting $u_r = x - \mu^2/z$:

$$x^{2r+2} - u_1 x^{2r} - \cdots - u_{r-1} x^4 - u_{r+1} x^2 - \left(z - \frac{\mu^2}{z}\right)^2 = 0. \quad (13)$$

• For $F_4 \ (E_6^{(2)}, 27)$ case, the curve is obtained from the characteristic polynomial $P_{E_6}^{27}(x; u_1, u_2, u_3, u_4, u_5, u_6)$ by the restriction $u_2 = 0$ and $u_5 = -6(z + \mu^2/z);

$$-8 \left(z + \frac{\mu^2}{z}\right)^3 + a_1(x) \left(z + \frac{\mu^2}{z}\right)^2 + a_2(x) \left(z + \frac{\mu^2}{z}\right) + a_3(x) = 0, \quad a_1(x) = -636x^9 - 300u_1 x^7 - 48u_1^2 x^5 - 5u_3 x^3 + 2u_4 x,$$

$$a_2(x) = -168x^{18} - 348u_1 x^{16} - 276u_1^2 x^{14} + (-116u_1^3 + 14u_3) x^{12} + (-92u_4 - 20u_1^4 - 8u_1 u_3) x^{10} + (-42u_1 u_4 - 6u_1^2 u_3) x^8 + (-4u_6 - 10u_1^2 u_4 - \frac{2}{3} u_2^2) x^6 + (\frac{1}{3} u_3 u_4 - \frac{2}{3} u_6 u_1) x^4,$$

$$a_3(x) = x^{27} + 6u_1 x^{25} + 15u_1^2 x^{23} + (20u_1^3 + u_3) x^{21} + (5u_4 + 4u_1 u_3 + 15u_1^4) x^{19} + (6u_1^2 u_3 + 12u_1 u_4 + 6u_1^5) x^{17} + (\frac{1}{3} u_3^2 + 5u_6 + 4u_1^3 u_3 + \frac{26}{3} u_1^2 u_4 + u_1^6) x^{15} + (\frac{4}{3} u_1^3 u_4 + \frac{19}{3} u_6 u_1 + u_1^4 u_3 + \frac{4}{3} u_3 u_4 + \frac{2}{3} u_3^2 u_1) x^{13} + (\frac{1}{3} u_1^2 u_3^2 - \frac{1}{3} u_1^4 u_4 - \frac{15}{4} u_4^2 + 3u_6 u_1^2) x^{11} + (\frac{1}{3} u_6 u_3 + \frac{4}{9} u_1^2 u_4 + \frac{15}{4} u_4^2 + 3u_6 u_1^2) x^9 + (-\frac{9}{9} u_3^2 u_4 - \frac{1}{2} u_6 u_4 + \frac{1}{9} u_6 u_1 u_3 - \frac{7}{36} u_1^2 u_4^2) x^7 + (\frac{1}{12} u_4^2 u_3 - \frac{1}{6} u_6 u_1 u_4) x^5 + (-\frac{1}{54} u_4^3 - \frac{1}{108} u_6^2) x^3. \quad (14)$$

• For $G_2 \ (D_4^{(3)}, 8)$ case, the spectral curve is obtained from $P_{D_4}^{8}(x; u_1, u_2, u_3, u_4)$ by the restriction $u_1 = 2u$, $u_2 = -u^2 - z + \mu^2/z$, $u_3 = \sqrt{3}(z - \mu^2/z)$ and $u_4 = \ldots$
\[ v + 2u(z + \mu^2/z); \]
\[ 3 \left( z - \frac{\mu^2}{z} \right)^2 - x^8 + 2ux^6 - \left[ u^2 + \left( z + \frac{\mu^2}{z} \right) \right] x^4 + \left[ v + 2u \left( z + \frac{\mu^2}{z} \right) \right] x^2 = 0. \quad (15) \]

We may write the spectral curve in the form of
\[ z + \frac{\mu^2}{z} = W^R_G(x, u_1, \cdots, u_r), \quad (16) \]

namely we regard the curve as an fibration over \( \mathbb{C}P^1 \) with the fiber characterized by the function \( W^R_G(x) \). For example, in the case of \( A_r, D_r \) and \( E_6 \) gauge groups, we obtain

- **\( A_r \)**
  \[ W^r_{A^r} = x^{r+1} - u_1x^r - \cdots - u_{r-1}x - u_r, \quad (17) \]

- **\( D_r \)**
  \[ W^{2r}_{D^r} = x^{2r-2} - u_1x^{2r-4} - \cdots - u_{r-2}x^2 - \frac{u_{r-1}^2}{x^2} - u_r, \quad (18) \]

- **\( E_6 \)**
  \[ W^{27}_{E_6} = \frac{1}{x^3} (q_1 \pm p_1 \sqrt{p_2}) - u_0, \quad (19) \]

where \( p_1, p_2 \) and \( q_1 \) are given in \((11)\).

It is important to notice that \( W_{ADE}(x) \) is nothing but the superpotential of the two-dimensional topological Landau-Ginzburg (LG) model of type \( ADE \). \( W^{r+1}_{A^r}(x) \) and \( W^{2r}_{D^r}(x) \) are familiar superpotentials for the \( A_r \) and \( D_r \) type minimal models. For \( E_6 \) case, the function \( W^{27}_{E_6} \) looks like very different from the usual deformation of the \( E_6 \) singularity written in terms of three variables

\[ W_{E_6}(x_1, x_2, x_3) = x_1^4 + x_2^3 + x_3^2. \quad (20) \]

It is, however, found in \((13)\) that \( W^{27}_{E_6} \) is a single-variable version of the LG superpotential for the \( E_6 \) minimal model. On the other hand, the singularity of the form of \((20)\) is obtained by considering the fibration of ALE spaces \((5)\). The relation of these two description of the Seiberg-Witten curves are discussed in \((12)\).

For non-simply laced cases, we have rather nontrivial \( W_G(x) \) from the spectral curves although in the two dimensional case, the LG superpotentials are obtained from the simply laced one by the folding procedure \((14)\). The functions \( W^R_G(x) \) are given as follows:
• $B_r$

$$W_{B_r}^{2r}(x; u_1, \cdots, u_r) = \frac{W_{BC}(x; u_1, \cdots, u_r)}{x},$$

(21)

where the LG potential of $BC$ type

$$W_{BC}(x; u_1, \cdots, u_r) = x^{2r} - \sum_{i=1}^{r} u_i x^{2r-2i},$$

(22)

is obtained from the $A_{2r-1}$ superpotential $W_{A_{2r-1}}(x; \tilde{u}_1, \cdots, \tilde{u}_{2r-1})$ by the restriction $\tilde{u}_{2k} = 0$ ($k = 1, \cdots, r-1$) and setting $u_k = \tilde{u}_{2k-1}$ ($k = 1, \cdots, r$).

• $C_r$

$$W_{C_r}^{2r+2}(x; u_1, \cdots, u_r) = (x^2 W_{BC}(x; u_1, \cdots, u_r)^2 + 4\mu^2)^{1/2}.$$  

(23)

• $F_4$

$$W_{F_4}^{27} = \frac{a_1(x)}{24} - \frac{1}{2} \left\{ \left( -q + \sqrt{q^2 + 4p^3} \right)^{1/3} + \left( -q - \sqrt{q^2 + 4p^3} \right)^{1/3} \right\},$$

(24)

where

$$p(x) = -\frac{a_2}{6} - \frac{a_3^2}{144},$$

$$q(x) = \frac{1}{27} \left( \frac{a_1^3}{32} + \frac{9}{8} a_1 a_2 + 27a_3 \right),$$

(25)

and $a_1, a_2$ and $a_3$ are defined in (14).

• $G_2$

$$W_{G_2}^{8} = \frac{1}{6}(p_1 + \sqrt{p_1^2 + 12p_2}),$$

(26)

where

$$p_1 = 6x^4 - 2ux^2, \quad p_2 = x^8 - 2ux^6 + u^2x^4 - ux^2 + 12\mu^4.$$  

(27)

Note that for $C_r$ and $G_2$ cases, the superpotentials $W_{G}^{R}(x)$ depend on the scale parameter $\mu$ explicitly. So far we have seen the SW spectral curves for general gauge groups. The SW differential defined on these spectral curves take the simple form

$$\lambda_{SW} = \frac{1}{2\pi i} \frac{dz}{z}.$$  

(28)

In the next section we will study the period integrals of the SW differential using the two-dimensional topological LG theory.
3 Picard-Fuchs Equations and 2D Topological Landau-Ginzburg Models

In this section we consider the ADE gauge groups. To describe the moduli space of the Coulomb branch we adopt the flat coordinate system \((t_1, t_2, \cdots, t_r)\) developed in the A-D-E singularity theory instead of the conventional Casimir coordinates \((u_1, u_2, \cdots, u_r)\). The coordinate transformation is read off from the residue integral

\[
t_i = c_i \oint dx W_R^G(x, u) \frac{dt_i}{t_i}, \quad i = 1, \cdots, r
\]

with a suitable constant \(c_i\) \cite{13, 15}. Notice that the overall degree of \(W_R^G\) is equal to \(h\).

The flat coordinates \(t_i\) are expressed as polynomials in \(u_i\).

Firstly we discuss the role of flat coordinates in the two-dimensional topological Landau-Ginzburg models. We define the primary fields \(\phi_i(x,t)\) as the derivatives of the superpotential \(W_G(x,t)\):

\[
\phi_i(x,t) = \frac{\partial W_G(x,t)}{\partial t_i}.
\]

We choose the normalization factor \(c_r\) such that \(\phi_r(x,t) = 1\). We now consider two-dimensional topological gravity coupled to topological LG model \cite{16}. In this case, the primary field \(\phi_r\) is regarded as the puncture operator \(P\). In the flat coordinate system, the topological metric \(\eta_{ij} = \langle \phi_i \phi_j P \rangle\) is independent of \(t_i\) and takes the form:

\[
\eta_{ij} = \delta_{e_i + e_j, h}.
\]

Furthermore, the primary fields obey the operator product expansions

\[
\phi_i(x,t) \phi_j(x,t) = \sum_{k=1}^{r} C_{ij}^k(t) \phi_k(x,t) + Q_{ij}(x,t) \partial_x W_G(x,t).
\]

The flatness condition implies that the function \(Q_{ij}(x,t)\) in (32) satisfies

\[
\frac{\partial^2 W_G(x,t)}{\partial t_i \partial t_j} = \partial_x Q_{ij}(x,t).
\]

The structure constants \(C_{ij}^k(t)\) are related to the three-point function \(F_{ijk}(t) = \langle \phi_i \phi_j \phi_k \rangle\) by the relation \(F_{ijk}(t) = \eta_{kl} C_{ij}^l(t)\). In two-dimensional topological theory, all the topological correlation functions are determined by the free energy \(F(t)\). The three point function \(F_{ijk}(t)\) is given by \(\partial^3 F(t)/\partial t_i \partial t_j \partial t_k\).
The associativity of the chiral ring \((\phi_i \phi_j) \phi_k = \phi_i (\phi_j \phi_k)\) implies the relation \(C^t_{ij} C^n_{kl} = C^t_{jk} C^n_{il}\). Let us introduce \(r \times r\) matrices \(C_i, F_i\) and \(\eta\) by \((C_i)^k = C^k_{ij}\), \((F_i)_{jk} = F_{ijk}\) and \(\eta = (\eta_{ij})\). The associativity condition leads to the commutativity of the matrices \(C_i\);

\[ C_i C_j = C_j C_i. \tag{34} \]

Using \(F_i = \eta C_i\), we obtain the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation:

\[ F_i \eta^{-1} F_j = F_j \eta^{-1} F_i, \tag{35} \]

which is one of the important relations in two-dimensional topological theory.

Now we go back to the four-dimensional gauge theory. We consider the period integral of the SW differential \(\lambda_{SW}\) (28):

\[ \Pi = \int_\gamma \lambda_{SW} \tag{36} \]

along the certain one-cycle \(\gamma\) on the spectral curve (14). In terms of the superpotential \(W_G(x, t)\), the SW differential takes the form:

\[ \lambda_{SW} = \frac{1}{2\pi i} \frac{x \partial_x W_G(x, t)}{\sqrt{W_G(x, t)^2 - 4\mu^2}} dx. \tag{37} \]

It is shown in [7] that \(\Pi\) obeys the set of differential equations:

\[ \partial_i \partial_j \Pi = \sum_{k=1}^{r} C^k_{ij}(t) \partial_k \partial_t \Pi, \tag{38} \]

which is called as the Gauss-Manin system in the singularity theory.

In addition to the Gauss-Manin system, the SW differential satisfies another two differential equations. From the scaling relation to the superpotential

\[ \left( \sum_{i=1}^{r} q_i t_i \partial_{t_i} + x \partial_x \right) W_G(x, t) = h W_G(x, t), \tag{39} \]

we obtain the scaling relation for the period \(\Pi\):

\[ \left( \sum_{i=1}^{r} q_i t_i \partial_{t_i} + h \gamma \mu \partial_\mu - 1 \right) \Pi = 0. \tag{40} \]

The final differential equation is obtained by regarding the l.h.s. of the spectral curve (10) as the superpotential of the topological \(CP^1\) model [18]:

\[ W_{CP^1}(z) = z + \frac{\mu^2}{z} - t_r. \tag{41} \]
Since \( \log \mu^2 \) and \( t_r \) are flat coordinates of the \( CP^1 \) model, we obtain the \( CP^1 \) relation:

\[
\left( (\mu \partial_\mu)^2 - 4 \mu^2 \partial_\mu^2 \right) \Pi = 0.
\]  

(42)

Combining the scaling relation \((10)\) and the \( CP^1 \) relation \((42)\), we obtain the differential equation

\[
\left\{ \left( \sum_{i=1}^r q_i t_i \frac{\partial}{\partial t_i} - 1 \right)^2 - 4 \mu^2 h^2 \frac{\partial^2}{\partial t_r^2} \right\} \Pi = 0
\]

(43)

By solving the Gauss-Manin system \((38)\) and the scaling equation \((43)\), we may analyze the exact solutions in the Coulomb branch. For classical gauge groups, the present Picard-Fuchs equations are shown to be the same as those appeared in the previous works \([19]\), in which various gauge theories with or without hypermultiplets are discussed.

In the weak coupling region where the scale parameter \( \Lambda^2 = 4 \mu^2 \) are small, the solutions of these Picard-Fuchs equations are studied extensively using various methods \([19, 20]\), which are shown to agree with the microscopic instanton calculation \([21]\). In the next section, we study the solutions in the strong coupling region. But before going to the next section, we discuss an important consequence of the Gauss-Manin system. Since the dual period \( a_{DI} \) also satisfies the Gauss-Manin system \((38)\), this Gauss-Manin system provides the third-order differential equation for the prepotential \( F(a) \):

\[
\tilde{F}_{ijk} = \sum_{l=1}^r C_{ijl} \tilde{F}_{lrk},
\]

(44)

where

\[
\tilde{F}_{ijk} = \partial_{t_i} a_l \partial_{t_j} a_j \partial_{t_k} a_K F_{LK},
\]

\[
F_{LK} = \partial_{a_l} \partial_{a_j} \partial_{a_K} F(a)
\]

(45)

The equations \((14)\) is very similar to \( F_{ijk}(t) = C_{ij}^{kl} \eta_{kl} \) in two-dimensional topological theory. Let us introduce matrices \( \tilde{F}_i \), \( \mathcal{G} \) and \( \mathcal{F}_I \) defined by \( (\tilde{F}_i)_{jk} = \tilde{F}_{ijk} \), \( \mathcal{G} = \tilde{F}_r \), and \( (\mathcal{F}_I)_{JK} = F_{IJK} \), respectively. Then we find the WDVV equations in the Seiberg-Witten theory \([22, 4]\):

\[
\tilde{F}_i \mathcal{G}^{-1} \tilde{F}_j = \tilde{F}_j \mathcal{G}^{-1} \tilde{F}_i,
\]

(46)

which may be written in the form

\[
\mathcal{F}_j \mathcal{F}_K^{-1} \mathcal{F}_J = \mathcal{F}_J \mathcal{F}_K^{-1} \mathcal{F}_I.
\]

(47)
In addition to the WDVV equation, the prepotential satisfies the scaling equation\[23]\):
\[
\left( \sum_{I=1}^{r} a_I \partial a_I + h^\mu \mu \partial_\mu \right) F(a) = 2 F(a).
\] (48)
This scaling equation is important to calculate the instanton correction to the prepotential in the weak coupling region\[8\]. Recently it is shown that the prepotential satisfies further non-trivial equations \[24\] obtained from the Whitham hierarchy. Instanton correction to the prepotential has been calculated in this framework\[25\] for some gauge theories. It would be interesting to study the relation between the WDVV equation approach and this formulation.

\section{Superconformal point}

One of interesting phenomena in the strong coupling physics of $N = 2$ supersymmetric gauge theory is the existence of non-trivial $N = 2$ superconformal fixed point in the Coulomb branch\[11, 26, 27\]. At this point, massless solitons of mutually nonlocal charges coexist. The superconformal field theory is characterized by the scaling operators with non-trivial (fractional) scaling dimensions. In particular, the calculations of the scaling exponents based on the exact solution suggest that the superconformal fixed points are characterized by the $A-D-E$ classification\[27\]. We will study this RG fixed point in view of the Picard-Fuchs equations obtained in the previous section.

For $G = ADE$, the superconformal fixed points exist at
\[
t_1 = \cdots = t_{r-1} = 0, \quad t_r = \pm 2\mu.
\]
We take the plus sign without loss of generality. Let us introduce new flat coordinates $\tilde{t}_i$ by shifting $t_r$ by $2\mu$:
\[
t_i = \tilde{t}_i, \quad (i = 1, \cdots, r - 1), \quad t_r = \tilde{t}_r + 2\mu
\] (49)
Since the OPE coefficients $C_{ij}^k(t)$ are independent of $t_r$\[8\], the Gauss-Manin system does not change its form under this change of coordinates:
\[
\left( \partial_{\tilde{t}_i} \partial_{\tilde{t}_j} - \sum_{k=1}^{r} C_{ij}^k(\tilde{t}) \partial_{\tilde{t}_k} \partial_{\tilde{t}_i} \right) \Pi = 0,
\] (50)
The scaling equation (43), on the other hand, becomes

\[
\left\{ \left( \sum_{i=1}^{r} q_i \tilde{t}_i \partial_{\tilde{t}_i} - 1 \right) \right\}^2 + 4 \mu h \left[ \left( \sum_{i=1}^{r} q_i \tilde{t}_i \partial_{\tilde{t}_i} - 1 \right) \partial_{\tilde{t}_i} \frac{h}{2} \partial_{\tilde{t}_i} \right] \Pi = 0. \tag{51}
\]

Introduce the scaling parameter \( \epsilon \) by

\[
\tilde{t}_i = \epsilon^n \rho_i, \quad (i = 1, \cdots, r - 1), \quad \tilde{t}_r = \epsilon^h
\]

and consider the limit \( \epsilon \to 0 \). We are interested in the solution of the SW periods which behave like

\[
\Pi = \epsilon^\alpha f(\rho) + \cdots,
\]

as \( \epsilon \to 0 \). Since

\[
\partial_{\tilde{t}_i} = \epsilon^{-q_i} \partial_{\rho_i}, \quad \partial_{\tilde{t}_r} = \frac{1}{h} \epsilon^{-h} \left( \epsilon \partial_{\epsilon} - \sum_{i=1}^{r-1} q_i \rho_i \partial_{\rho_i} \right),
\]

the Gauss-Manin system (50) for \( i, j < r \) becomes the differential equations for \( f(\rho) \) with respect to \( \rho \):

\[
\left( \partial_{\rho_i} \partial_{\rho_j} - \frac{1}{h} \sum_{k=1}^{r-1} \tilde{C}^k_{ij}(\rho) \partial_{\rho_k} \left[ \alpha - \sum_{l=1}^{r-1} q_l \rho_l \partial_{\rho_l} \right] \right) f(\rho) = 0
\]

where \( \tilde{C}^k_{ij}(\rho) = C^k_{ij}(\tilde{t}) e^{q_i + q_j - q_k - h} \). As for the scaling equation (51), the second term is dominant in the superconformal limit. We thus find that \( f(\rho) \) should satisfy the equation

\[
\left( \alpha - \frac{h + 2}{2} \right) \left[ \alpha - \sum_{l=1}^{r-1} q_l \rho_l \partial_{\rho_l} \right] f(\rho) = 0 \tag{56}
\]

This equation determines the exponent \( \alpha \) in (53) such as

\[
\alpha = \frac{h + 2}{2} \tag{57}
\]

The superconformal field theory is characterized by the scaling operator \( \text{tr} \phi^q \), whose conformal dimension is measured with respect to the BPS mass (1). From (53) and (57), we have

\[
\langle \text{tr} \phi^q \rangle \sim \left( m_{BPS} \right)^{\frac{2q}{\pi+2}} \tag{58}
\]
Thus the scaling dimension of $\langle \text{tr} \phi^q \rangle$ is $2 q_i / (h + 2)$, which agrees with the result of [27].

We could examine the above arguments from the viewpoint of the SW curve. In terms of the parameters $\tilde{t}$, the SW curve becomes

$$z + \frac{\mu^2}{z} = W_G(x; t) = W_G(x; \tilde{t}) - 2 \mu. \quad (59)$$

Let us introduce $\xi$ by

$$\xi = \sqrt{z} + \frac{\mu}{\sqrt{z}}. \quad (60)$$

Then the curve is expressed in the form of the ADE superpotential with the Gaussian part $\xi^2$:

$$\xi^2 = W_G(x; \tilde{t}). \quad (61)$$

The SW differential then becomes

$$\lambda_{SW} = \frac{1}{2 \pi i} x \frac{dz}{z} = 2 \pi i \frac{1}{\sqrt{-\mu}} \frac{d\xi}{\sqrt{\xi^2 - 4 \mu}}. \quad (62)$$

In the superconformal limit, expanding the above formula around $\epsilon = 0$ we obtain

$$\lambda_{SW} = - \frac{1}{2 \pi i \sqrt{-\mu}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{W_G(x, \tilde{t}) 2n+1}{(2n + 1)(4\mu)^n}, \quad (63)$$

up to the total derivative term. After rescaling $x = \epsilon \tilde{x}$, the leading term in (63) is

$$\lambda_{SW} = - \frac{1}{2 \pi i \sqrt{-\mu}} \epsilon^{h+2} \sqrt{W_G(\tilde{x}; \rho_1, \cdots, \rho_{r-1}, 1)} d\tilde{x} + \cdots, \quad (64)$$

which also leads to the exponent (57). Note that the derivative of the SW period integrals:

$$\frac{\partial \Pi}{\partial \rho_i} \sim \epsilon^{h+2} \int \frac{\partial \rho_i W_G(\tilde{x}; \rho)}{\sqrt{W_G(\tilde{x}; \rho)}} d\tilde{x}$$

is the period of the curve

$$y^2 = W_G(\tilde{x}; \rho_1, \cdots, \rho_{r-1}, 1). \quad (65)$$

Thus we find that the superconformal fixed point is simply characterized by the ADE superpotential $W_G(x, \tilde{t})$. The SW curve reduce to the hyperelliptic type curve (65). For example, let us consider the $A_2$ case. The SW curve (65) is given by

$$y^2 = x^3 - \rho x - 1,$$
which is nothing but the curve of $SU(2)$ gauge theory with $N_f = 1$ at the superconformal point (the small torus in [11]). The Gauss-Manin system (50) becomes

$$\left[(4\rho^3 - 27)\partial^2 - \frac{5}{4}\rho\right]f(\rho) = 0. \quad (66)$$

One may solve this equation around $\rho = 0$ found that the results agree with those obtained in [11].

5 Discussion

In this paper, we have seen the close relationship between the four-dimensional gauge theory with $ADE$ gauge group and two-dimensional topological LG models coupled to topological gravity. We have examined the exact solutions around the superconformal points using the Picard-Fuchs equations and showed that the superconformal points are simply characterized by the $A$-$D$-$E$ singularity. For other gauge theories with or without matter hypermultiplets, on the other hand, it is difficult to extend the present results because of the complexity of the superpotential. But at the superconformal point we would expect that the curves become simple and are classified by the $A$-$D$-$E$ singularity[27]. The two-dimensional topological field theory would provide a useful tool for analyzing the physics in both weak and strong coupling region.

It seems interesting to compare the expansion (63) of the SW differential around the superconformal point with that around the origin ($t_i = 0$) of the Coulomb branch:

$$\lambda_{SW} = -\frac{1}{2\pi i} \frac{1}{\sqrt{-4\mu^2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{(2n + 1)(4\mu^2)^n} W_G(x, t)^{2n+1} dx. \quad (67)$$

In [10], it has been shown that the period integrals of (67) are expressed directly in terms of the one-point function $\langle \sigma_n(\phi_i) \rangle$ of the $n$-th gravitational descendant $\sigma_n(\phi_i)$ of the primary field $\phi_i$ in two-dimensional topological LG models coupled to topological gravity. The one-point function satisfies the same Gauss-Manin system, which is evaluated by the residue integrals[13]:

$$\langle \sigma_n(\phi_i) \rangle = b_{n,i} \sum_{j=1}^{r} \eta_{ij} \oint W_G(x, t)^{\epsilon_j + n+1}, \quad (68)$$
where $b_{n,i}$ is certain constant. In this formulation, the Gauss-Manin system [38] is also derived from the topological recursion relation [17]:

$$
\langle \sigma_n(\phi_i)XY \rangle = \sum_j \langle \sigma_{n-1}(\phi_i)\phi_j \rangle \langle \phi_j XY \rangle
$$

(69)

and the puncture equation (in the small phase space):

$$
\langle P \prod_{i=1}^s \sigma_{n_i}(\phi_{\alpha_i}) \rangle = \sum_{i=1}^s \langle \prod_{j=1}^s \sigma_{n_j-\delta_{ji}}(\phi_{\alpha_j}) \rangle.
$$

(70)

One might expect that the similar relation would be hold in the superconformal case, which would be important to understand the relation between the SW theory and $d < 1$ topological strings.

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