ON THE AC SPECTRUM OF ONE-DIMENSIONAL RANDOM SCHRÖDINGER OPERATORS WITH
MATRIX-VALUED POTENTIALS

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ABSTRACT. We consider discrete one-dimensional random Schrödinger operators with decaying matrix-valued, inde-
pendent potentials. We show that if the $\ell^2$-norm of this potential has finite expectation value with respect to the product
measure then almost surely the Schrödinger operator has an interval of purely absolutely continuous (ac) spectrum. We
apply this result to Schrödinger operators on a strip. This work provides a new proof and generalizes a result obtained by
Delyon, Simon, and Souillard [8].

1. MODEL AND STATEMENT OF RESULTS

In this paper we are interested in the absolutely continuous (ac) spectrum of quasi one-dimensional random
Schrödinger operators with decaying potentials. To this end, it is convenient to formulate the problem in terms of
matrix-valued potentials on the one-dimensional lattice, $\mathbb{Z}$.

Let us first introduce some standard notation that is used throughout this paper. If $H$ is an operator on some Hilbert
space $\mathcal{H}$, then we denote by $\rho(H)$, $\sigma(H)$, $\sigma_{ac}(H)$, $\sigma_{ess}(H)$ its resolvent set, spectrum, ac spectrum, respectively its
essential spectrum. By $\|H\|$ we denote the operator norm of $H$.

For some $m \in \mathbb{N}$, let $\text{Sym}(m)$ denote the set of real symmetric $m \times m$ matrices. Let $D \in \text{Sym}(m)$ be some fixed
matrix and let $q = (q_n)_{n \in \mathbb{Z}}$ be a family of independent $\text{Sym}(m)$-valued random variables. We assume here that (i) the
mean of each random variable $q_n$ is zero and (ii) there is a compact set $K \subset \text{Sym}(m)$ so that the support of each $q_n$
is contained in $K$. By $\nu_n$ we denote the probability measure of $q_n$. The probability measure for $q$ is then the product
measure $\nu = \otimes_{n \in \mathbb{Z}} \nu_n$. We use the notation $E$ to denote the expectation value with respect to this product measure, $\nu$.

On the Hilbert space $\ell^2(\mathbb{Z}; \mathbb{C}^m)$ (of $\mathbb{C}^m$-valued functions on $\mathbb{Z}$ equipped with the usual Euclidean norm) we con-
sider the operator

\begin{equation}
H := \Delta + D + q, 
\end{equation}

which is defined as

\begin{equation}
(H \varphi)(n) := -\varphi(n-1) - \varphi(n+1) + D \varphi(n) + q_n \varphi(n), \quad \varphi \in \ell^2(\mathbb{Z}; \mathbb{C}^m), \ n \in \mathbb{Z}.
\end{equation}

To state the first result of this paper we introduce the following set which depends on the (eigenvalues of the) constant
“potential” $D$,

\begin{equation}
I_D := \bigcap_{\lambda \in \sigma(D)} [\lambda - 2, \lambda + 2].
\end{equation}

**Theorem 1.** Let $\mathbb{E}[\sum_{n \in \mathbb{Z}} \|q_n\|^2] < \infty$. Then almost surely $\sigma_{ac}(H) \supseteq I_D$ and the spectrum of $H$ is purely absolutely
continuous in the interior of $I_D$.

**Date:** October 7, 2009.
Theorem 1 will be used to prove the second result of this paper.

**Theorem 2.** Let \( \mathbb{E}[\sum_{n \in \mathbb{Z}} \|q_n\|^2] < \infty \). Then almost surely \( \sigma_{ac}(H) \supseteq \sigma(\Delta + D) \).

**Remarks:** The case of a random potential with \( m = 1 \) has been analyzed in great detail by Delyon, Simon, and Souillard [8]. For \( m = 1 \) they not only prove Theorem 1 (even under weaker conditions on the measures \( \nu_n \)) but also that the rate of decay of \( q_n \) is necessary in order to have absolutely continuous spectrum. A deterministic version of a result in the direction of Theorem 2 has been announced by Molchanov and Vainberg [20] but, to the best of our knowledge, has not yet been published. Other previous work by Kirsch, Krishna, Obermeit and Sinha on decaying potentials can be found in [17], [14] and [18]. We would also like to mention also the work of Kotani and Simon [16] and Schulz-Baldes [22] on random Schrödinger operators on the strip.

**Example.** The most important application of Theorem 2 is to Schrödinger operators on a strip. More generally, let \( C := \{1, 2, \ldots, L\}_d \) denote the discrete \( d \)-dimensional cube with side length \( L \). Then \( \ell^2(C) \cong \mathbb{C}^m \) with \( m = L^d \) and \( \ell^2(\mathbb{Z}; \ell^2(C)) \cong \ell^2(\mathbb{Z} \times C) \). We introduce the multi-index \( n := (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d \) with \( |n| := |n_1| + |n_2| + \cdots + |n_d| \). Let \( D \) be the Dirichlet Laplacian on \( C \), i.e., for all \( \psi \in C \),

\[
(D \psi)(n) := -\sum_{m \in C : |m-n|=1} \psi(m), \quad \psi \in \ell^2(C).
\]

Note that \( \Delta + D \) is equivalent to the (nearest neighbor) Dirichlet Laplace operator on \( \ell^2(\mathbb{Z} \times C) \). The eigenvalues of \( D \) are indexed by \( \lambda \in C \) and are given by \(-2 \sum_{i=1}^{d} \cos(\pi n_i/(L+1))\). Observe that

\[
I_D = \{ \lambda \in \mathbb{R} : |\lambda| \leq 2 - d \cos(\pi/(L+1)) \}
\]

If \( d \geq 2 \) this set is empty unless \( L = 1 \). If \( d = 1 \), then \( I_D \) is non-empty but its length converges to 0 as \( L \) tends to infinity. By Theorem 2 \( \sigma_{ac}(H) \supseteq \sigma(\Delta + D) = [-2 - 2d \cos(\pi/(L+1)), 2 + 2d \cos(\pi/(L+1))] \). By formally setting \( L \) to infinity the last interval becomes \([-2(d + 1), 2(d + 1)]\).

**Remarks:** On the full two-dimensional lattice \( \mathbb{Z}^2 \), Bourgain [3] proved \( \sigma_{ac}(\Delta + q) \supseteq \sigma(\Delta) \) for Bernoulli and Gaussian distributed, independent random potentials whose variances decay faster than \( |n|^{-1/2} \). In [4], Bourgain improves this result to the weaker \( |n|^{-1/3} \) decay rate. For a deterministic potential, \( q \), on \( \mathbb{Z}^d \), Simon [23] conjectured that if \( (q_n/\sqrt{1+|n|^{d-1}})_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d) \), then \( \sigma_{ac}(\Delta + q) = \sigma(\Delta) \). In dimension one this was proved by Deift and Killip [7]. A recent improvement of this result has been obtained by Denisov [11]. In the analogous continuous setting, progress has been made towards this \( L^2 \)-conjecture e.g. by Denisov [9] and Laptev, Naboko, and Safronov [19]. For additional references see [5], [24].

2. **Proofs of Theorem 1 and 2**

In order to prove the two main theorems in this paper we will study the Green’s functions defined by

\[
G_n := P_n(H-\lambda)^{-1}P_n, \quad n \in \mathbb{Z}.
\]
Here, $P_n$ denotes the orthogonal projections of $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^m)$ onto the subspace $\ell^2(\{n\}; \mathbb{C}^m) \cong \mathbb{C}^m$, and $\lambda$ denotes the spectral parameter. Let $P_n^+ := \sum_{k\geq n} P_k$ and $P_n^- := \sum_{k\leq n} P_k$ be the orthogonal projections of $\mathcal{H}$ onto the subspaces $\ell^2(\{n, n+1, \ldots\}; \mathbb{C}^m)$ and $\ell^2(\{\ldots, n-1, n\}; \mathbb{C}^m)$, respectively. Let

$$G_n := P_n (P_n^+(H_n - \lambda)P_n^+)^{-1} P_n, \quad n \in \mathbb{Z} \tag{5}$$

be the so-called forward and backward Green’s functions. Then we have the recursion relation

$$G_n = -(G_{n+1}^+ + G_{n-1}^- + \lambda - D - q_n)^{-1}, \quad n \in \mathbb{Z}, \tag{6}$$

which follows by using the decomposition $\mathcal{H} = \text{Ran} P_n^+ \oplus \text{Ran} P_n^-$ and a resolvent identity.

If $\text{Im} \lambda > 0$, it is elementary to see that for each $n \in \mathbb{Z}$

$$G_n, G_n^+ \in \mathbb{SH}_m := \{Z = X + iY : X, Y \in \text{Sym}(m), Y > 0\}. \tag{7}$$

Henceforth we will assume that $\text{Im} \lambda > 0$. We equip the vector space $\mathbb{SH}_m$ with the metric

$$d(Z, W) := \cosh^{-1} \left(1 + \frac{1}{2} \text{cd}(Z, W)\right), \quad Z, W \in \mathbb{SH}_m,$$

where we have introduced

$$\text{cd}(Z, W) := \text{tr} \left[(\text{Im}Z)^{-1}(Z-W)^* (\text{Im}W)^{-1}(Z-W)\right].$$

The space $(\mathbb{SH}_m, d)$ is called Siegel half space and is a generalization of the usual Poincaré upper half plane.

By symmetry it will suffice to study $G_0^+$. Using the decomposition $\text{Ran} P_0^+ = \text{Ran} P_0^+ \oplus \text{Ran} P_1^+$, we see that

$$G_0^+ = \Phi_{q_0}(G_1^+), \tag{8}$$

with the following mapping on $\mathbb{SH}_m$

$$\Phi_\delta(Z) := -(Z + \lambda - D - \delta)^{-1}, \quad \delta \in \text{Sym}(m), Z \in \mathbb{SH}_m.$$

Iterating Eq. (8) we arrive at

$$G_0^+ = \Phi_{q_0} \circ \Phi_{q_1} \circ \cdots \circ \Phi_{q_n}(G_{n+1}^+), \quad n \in \mathbb{N}_0. \tag{9}$$

We will use the following theorem, which is a special case of a theorem obtained in [12].

**Theorem 3.** Let $\text{Im} \lambda > 0$ and $(\Lambda_n)_{n \in \mathbb{N}_0} \subset \mathbb{SH}_m$ be any sequence. Then

$$G_0^+ = \lim_{n \to \infty} \Phi_{q_0} \circ \cdots \circ \Phi_{q_n}(\Lambda_n).$$

We present a direct proof of this theorem in Appendix A, which in this case is simpler than the proof given in [12] for more general graphs.

The next theorem measures the distance of $G_0^+$ from the free forward Green’s function, which is determined by the following fixed point relation in $\mathbb{SH}_m$:

$$Z_k = \Phi_0(Z_k). \tag{10}$$
Solving for $Z_\lambda$ yields

$$Z_\lambda = \frac{D - \lambda}{2} + i \sqrt{1 - \left(\frac{D - \lambda}{2}\right)^2}.$$  

Note that for real $\lambda$, we have Im$Z_\lambda > 0$ if and only if $\lambda$ is in the interior of $I_D$. To formulate the next theorem we define

$$\text{cd}_\lambda(Z) := \text{cd}(Z_\lambda, Z), \quad Z \in \mathbb{H}_m.$$  

**Theorem 4.** Suppose $\mathbb{E}[\sum_{n \in \mathbb{Z}} \|q_n\|^2] < \infty$. Let $J$ be a closed subset of the interior of $I_D$. Then

$$\sup_{\lambda \in J + i(0,1]} \mathbb{E}\left[\text{cd}_\lambda^2(G_0^+)\right] < \infty.$$  

**Proof.** By symmetry it suffices to consider without loss of generality $G_0^+$. We assume $\lambda \in J + i(0,1]$ is fixed. By Theorem 7 we know that $G_0^+ = \lim_{n \to \infty} Z_{0,n}$, where $Z_{0,n} = \Phi_{q_0} \circ \Phi_{q_1} \circ \cdots \circ \Phi_{q_n}(Z_\lambda)$. Moreover, by Lemma 7 from Appendix B, we know that there exists a hyperbolic ball $B \subset \mathbb{H}_m$ such that $Z_{0,n} \in B$ for all $n \geq 2$ and potentials $q$ with $q_k \in K$. By continuity of the function $Z \mapsto \text{cd}_\lambda^2(Z)$, we have

$$\lim_{n \to \infty} \text{cd}_\lambda^2(Z_{0,n}) = \text{cd}_\lambda^2(G_0^+).$$  

Since $\text{cd}_\lambda^2(Z)$ is bounded on the ball $B$, it follows from dominated convergence that

$$\mathbb{E}[\text{cd}_\lambda^2(G_0^+)] = \lim_{n \to \infty} \mathbb{E}[\text{cd}_\lambda^2(Z_{0,n})].$$  

It remains to show that the right-hand side is bounded uniformly in $\lambda \in J + i(0,1]$. To this end we set $Z_{0,n} := \Phi_{q_0} \circ \Phi_{q_1} \circ \cdots \circ \Phi_{q_n}(Z_\lambda)$. Note that $Z_{0,n} = \Phi_{q_0}(Z_{0,n+1})$. Using the inequality of Lemma 5 below, we find

$$\mathbb{E}[\text{cd}_\lambda^2(Z_{0,n})] + 1 = \int_{K^{n+1}} (\text{cd}_\lambda^2(Z_{0,n}) + 1) d\nu_0(q_0) \cdots d\nu_n(q_n)$$

$$= \int_{K^{n+1}} \frac{\text{cd}_\lambda^2(\Phi_{q_0}(Z_{1,n})) + 1}{\text{cd}_\lambda^2(Z_{1,n}) + 1} (\text{cd}_\lambda^2(Z_{1,n}) + 1) d\nu_0(q_0) \cdots d\nu_n(q_n)$$

$$\leq \int_K (1 + A(Z_{1,n}, q_0) + C_0 \|q_0\|^2) d\nu_0(q_0) \int_K (\text{cd}_\lambda^2(Z_{1,n}) + 1) d\nu_1(q_1) \cdots d\nu_n(q_n)$$

$$= (1 + C_0 \mathbb{E}[\|q_0\|^2]) \int_K (\text{cd}_\lambda^2(Z_{1,n}) + 1) d\nu_1(q_2) \cdots d\nu_n(q_n)$$

$$\vdots$$

$$\leq \prod_{i=0}^n (1 + C_0 \mathbb{E}[\|q_i\|^2])$$

$$\leq \exp(C_0 \sum_{i=0}^\infty \mathbb{E}[\|q_i\|^2]),$$

where we have used $\int A(z, q) d\nu_i(q) = 0$, which follows from the assumption that $q_i$ is a random variable with mean zero. \hfill \Box
Lemma 5. Suppose $K$ is a compact subset of $\mathbb{SH}_m$. Let $J$ be a closed interval contained in the interior of $I_0$. Then there exists a constant $C_0$ and a linear functional $A(Z, \cdot) : \text{Sym}(m) \to \mathbb{R}$, depending continuously on $Z \in \mathbb{SH}_m$, such that for all $\lambda \in J + i(0, 1)$ and $Z \in \mathbb{SH}_m$,

\[
\frac{\cd_{\lambda}^2(\Phi_\delta(Z)) + 1}{\cd_{\lambda}^2(Z) + 1} \leq 1 + A(Z, \delta) + C_0\|\delta\|^2, \quad \forall \delta \in K.
\]

Proof. Using that $\Phi_0$ is a hyperbolic contraction, we see $\cd_{\lambda}(\Phi_\delta(Z)) = \cd(\Phi_\delta(Z), Z_\lambda) \leq \cd(Z - \delta, Z_\lambda) = \cd_{\lambda}(Z - \delta)$.

By the definition of the distance function we have

\[
\cd(Z - \delta, Z_\lambda) = \cd(Z, Z_\lambda) + a(Z, \delta) + b(Z, \delta),
\]

where (with $Z_\lambda = X_\lambda + iY_\lambda$ and $Z = X + iY$),

\[
a(Z, \delta) := -\text{tr}[Y_\lambda^{-1/2}\delta Y^{-1}(Z - Z_\lambda)Y_\lambda^{-1/2}] - \text{tr}[Y_\lambda^{-1/2}(Z - Z_\lambda)^*Y^{-1}\delta Y_\lambda^{-1/2}],
\]

\[
b(Z, \delta) := \text{tr}(Y_\lambda^{-1/2}\delta Y^{-1}\delta Y_\lambda^{-1/2}).
\]

Using the Cauchy-Schwarz inequality it follows that

\[
\text{L. H. S. of } (14) \leq 1 + A(Z, \delta) + C(Z, \delta),
\]

with

\[
A(Z, \delta) := \frac{2\cd_{\lambda}(Z)a(Z, \delta)}{\cd_{\lambda}^2(Z) + 1},
\]

\[
C(Z, \delta) := \frac{2a(Z, \delta)^2 + 2\cd_{\lambda}(Z)b(Z, \delta) + 2b(Z, \delta)^2}{\cd_{\lambda}^2(Z) + 1}.
\]

It remains to show that $C(Z, \delta) \leq C_0\|\delta\|^2$ for some $C_0$. Let us use the bounds,

\[
a(Z, \delta)^2 \leq 4\cd_{\lambda}(Z)b(Z, \delta),
\]

\[
b(Z, \delta) \leq \|Y_\lambda^{-1}\|^2\|\delta\|^2\|\text{tr}(Y_\lambda^{-1/2}Y^{-1}Y_\lambda^{-1/2})\|.
\]

(15) follows from the Cauchy-Schwarz inequality. The trace in the function $b$ can be written as $\text{tr}(EFE)$ with $E := Y_\lambda^{-1/2}\delta Y_\lambda^{-1/2}$ and $F := Y_\lambda^{-1/2}Y^{-1}Y_\lambda^{-1/2}$. This trace is estimated from above by $\|E^2\|\text{tr}F$. Then use $\|E^2\| \leq \|Y_\lambda^{-1/2}\|^2\|Y_\lambda^{-1}\|\|\delta\|^2$. Since $Y_\lambda$ is self-adjoint $\|Y_\lambda^{-1/2}\|^2 = \|Y_\lambda^{-1}\|$, and (16) follows.

The next estimate allows us to bound the right-hand side of (16) in terms of $\cd_{\lambda}(Z)$.

\[
\text{tr}(Y_\lambda^{-1/2}Y^{-1}Y_\lambda^{-1/2})
\]

\[
\leq \text{tr}(Y_\lambda^{-1/2}Y^{-1}Y_\lambda^{-1/2}) + \text{tr}(Y_\lambda^{-1/2}Y^{-1}Y_\lambda^{-1/2})
\]

\[
= \text{tr}[Y_\lambda^{-1/2}(Y - Y_\lambda)Y^{-1}(Y - Y_\lambda)Y_\lambda^{-1/2}] + 2m
\]

\[
\leq \text{tr}[Y_\lambda^{-1/2}(Y - Y_\lambda)Y^{-1}(Y - Y_\lambda)Y_\lambda^{-1/2}] + \text{tr}[Y_\lambda^{-1/2}(X - X_\lambda)Y^{-1}(X - X_\lambda)Y_\lambda^{-1/2}] + 2m
\]

\[
= \cd_{\lambda}(Z) + 2m.
\]

The claim now follows by inserting the above estimates and using that $\|Y_\lambda\|$ and $\|Y_\lambda^{-1}\|$ are uniformly bounded for $\lambda \in J + i(0, 1)$ and that $\delta$ is contained in a bounded set.\[\square\]
Proof of Theorem

Step 1: Almost surely $\sigma(H) \supseteq \sigma(\Delta + D)$.

The condition $\mathbb{E}[\sum_{n \in \mathbb{Z}} \|q_n\|^2]$ implies that almost all potentials are in $\ell^2$ and thus decay at infinity. $H$ is thus a compact perturbation of $\Delta + D$ and hence $\sigma(\Delta + D) = \sigma_{\text{ess}}(\Delta + D) = \sigma_{\text{ess}}(H) \subseteq \sigma(H)$ by Weyl’s Theorem.

Step 2: Let $J$ be any closed interval contained in the interior of $I_D$. Let $W_\lambda := -(2Z_\lambda + \lambda - D)^{-1}$. Then

$$\sup_{\lambda \in J \cap [0,1]} \mathbb{E} \left[ \text{cd}^2(G_n, W_\lambda) \right] < \infty.$$

If we use the recursion relation (6), the fact that $Z \mapsto -Z^{-1}$ is a hyperbolic isometry, and the inequalities of Lemma 8 (given in the Appendix B) we find that

$$\text{cd}(G_n, W_\lambda) \leq \text{cd}(G_{n+1}^+ + G_{n-1}^- + q_n, 2Z_\lambda)$$
$$\leq \text{cd}(G_{n+1}^+ + q_n/2, Z_\lambda) + \text{cd}(G_{n-1}^- + q_n/2, Z_\lambda)$$
$$\leq C \left[ 1 + \text{cd}_\lambda(G_{n+1}^+, Z_\lambda) + \text{cd}_\lambda(G_{n-1}^-, Z_\lambda) \right] \left( 1 + \|q_n\|^2 \right).$$

Then,

$$\mathbb{E} \left[ \text{cd}^2(G_n, W_\lambda) \right] \leq C \left[ 1 + \text{cd}_\lambda(G_{n+1}^+) + \text{cd}_\lambda(G_{n+1}^-) \right],$$

and Step 2 follows from Theorem 4.

Step 3: Almost surely $H$ has purely ac spectrum in the interior of $I_D$.

For $x \in \mathbb{Z} \times \{1, \ldots, m\}$ let $\mu_x$ denote the spectral measure of $H$ for the indicator function at $x$, $1_x \in \mathcal{H} \cong \ell^2(\mathbb{Z} \times \{1, \ldots, m\})$. Step 2 implies that almost surely $\mu_x$ is absolutely continuous on any closed subset of the interior of $I_D$. This can be seen for example by applying Lemma 1 in [13] and noting that for any closed subset $J$ contained in the interior of $I_D$ there exists a constant $C$ such that (see Lemma 8) $\text{tr}(\text{Im}Z) \leq C \left( \text{cd}_\lambda(Z) + 1 \right)$ for all $\lambda \in J$ and $Z \in \mathbb{S}_{m}$; see also [15, Theorem 4.1]. Now choosing a sequence of closed subsets $(J_n)_{n \in \mathbb{N}}$ of the interior of $I_D$, such that $J_n \subset J_{n+1}$ and $\bigcup_{n=1}^{\infty} J_n = I_D$, and using that countable unions of sets of measure zero have again measure zero, we find almost surely that for all $x \in \mathbb{Z} \times \{1, \ldots, m\}$ the spectral measure $\mu_x$ is absolutely continuous on the interior of $I_D$.

The theorem now follows by combining Steps 1 and 3.

To prove Theorem 3 we will use Theorem 1 in combination with Theorem 6 below. Theorem 6 is an extension of a theorem by Denisov [10, Theorem 1.2]. A proof can also be found in Albeverio and Konstantinov [11]. We give a proof in Appendix C following arguments given in [2, 10].
Theorem 6 (Denisov). Let $H_1$ and $H_2$ be two bounded self-adjoint operators on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that for $a < b$, $[a, b] \subseteq \sigma_{ac}(H_1)$ and $\sigma_{\text{ess}}(H_2) \subseteq (-\infty, a] \cup [b, \infty)$. Let $V : \mathcal{H}_2 \to \mathcal{H}_1$ be a Hilbert-Schmidt operator (i.e., $V^*V$ and $VV^*$ are trace class operators on $\mathcal{H}_2$ respectively $\mathcal{H}_1$ and let $H_V := \begin{bmatrix} H_1 & V \\ V^* & H_2 \end{bmatrix}$.

Then, $[a, b] \subseteq \sigma_{ac}(H_V)$.

Proof of Theorem 2 Let $\{\mu_1, \ldots, \mu_m\}$ be the eigenvalues of $D$ and let $\lambda \in \mathbb{R}$. Then the eigenvalues of $Z_{\lambda}$ are given by $z_{\lambda,k} = (\mu_k - \lambda)/2 + i\sqrt{1 - ((\mu_k - \lambda)/2)^2}$. If $\lambda \in [\mu_k - 2, \mu_k + 2]$ then $z_{\lambda,k}$ lies on the unit semicircle above the real axis. Otherwise $z_{\lambda,k}$ lies on the real axis outside the unit circle (see diagram)

Since $Z_{\lambda}$ is related to the Green’s function for $\Delta + D$, a point $\lambda$ lies in $\sigma(\Delta + D)$ if and only if at least one of the $z_{\lambda,k}$ lies on the semicircle, and thus has positive imaginary part. Let $m(\lambda)$ denote the number of $z_{\lambda,k}$ on the semicircle. As we vary $\lambda$, the function $m(\lambda)$ is locally constant, with jumps when one of the $z_{\lambda,k}$ moves in or out of the semicircle. Pick a $\lambda_0$ and let $I$ be the largest interval containing $\lambda_0$ on which $m(\lambda)$ is constant. Notice that $\sigma(\Delta + D)$ is a finite disjoint union of such intervals. The collection of $z_{\lambda,k}$ that remain in the semicircle for $\lambda \in I$ corresponds to a subset of eigenvalues of $D$, and thus to a spectral projection $P_I$ on $\mathbb{C}^m$. We identify the range of $P_I$ with $\mathbb{C}^{m(\lambda)}$. We use the same notation for the projection on $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^m)$ where $P_I$ acts as a (constant) multiplication operator. Introducing $\overline{P}_I := 1 - P_I$ we have the decomposition

$$H = \Delta + D + q = \begin{bmatrix} P_I(\Delta + D + q)P_I & P_Iq\overline{P}_I \\ \overline{P}_IqP_I & \overline{P}_I(\Delta + D + q)\overline{P}_I \end{bmatrix}.$$  

Note that $\Delta_I := P_I\Delta P_I$ is just the Laplace operator (2) on $\ell^2(\mathbb{Z}; \mathbb{C}^{m(\lambda)})$. Furthermore, let $D_I$ be the restriction of $D$ onto $\mathbb{C}^{m(\lambda)}$. By Theorem 1 $P_I(\Delta + D + q)P_I = \Delta_I + D_I + P_IqP_I$ has almost surely $ac$ spectrum on $I$, since $I \subseteq I_{D_I}$, see (3). Since almost surely $q$ is in $\ell^2$ and thus decays at infinity the essential spectrum of $\overline{P}_I(\Delta + D + q)\overline{P}_I$ is contained in the complement of the interior of $I$. Since $P_Iq\overline{P}_I$ is Hilbert-Schmidt almost surely, we can apply Theorem 5 and hence conclude that almost surely $I \subseteq \sigma_{ac}(H + D + q)$. Repeating the above arguments for the remaining intervals of non-zero length in the decomposition of the spectrum of $\Delta + D$ yields the claim. 

ACKNOWLEDGEMENT

W.S. wants to thank the University of British Columbia for hospitality and financial support. D. H. wants to acknowledge the summer research grant awarded by the College of William & Mary.

APPENDIX A: PROOF OF THEOREM 5

Let us start with the following lemma.

Lemma 7. Suppose that $|\lambda|, \|\delta_1\|, \|\delta_2\| \leq C$ and $\text{Im}\lambda \geq 1/C$. Then there exists a compact set $B \subset \mathbb{SH}_m$ (depending on $C$) such that $\Phi_{\delta_1} \circ \Phi_{\delta_2}(\mathbb{SH}_m) \subseteq B$. 

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Proof. Applying $\Phi_{\delta}$ once yields an upper bound on the norm, which can be seen from the basic inequality

$$\|(Z + \lambda - D - \delta)^{-1}\| \leq (\text{Im}\lambda)^{-1}, \quad Z \in \mathbb{SH}_m.$$  

Applying $\Phi_{\delta}$ a second time yields a lower bound on the imaginary part, which can be seen by the following estimate

$$\text{Im}\left[-(Z + \lambda - D - \delta)^{-1}\right] = (Z^* + \lambda^* - D - \delta)^{-1} \text{Im}(Z + \lambda) (Z + \lambda - D - \delta)^{-1} \geq \frac{\text{Im}\lambda}{\|Z + \lambda - D - \delta\|^2}.$$  

□

Proof of Theorem 3

Step 1: Let $B$ be a compact subset of $\mathbb{SH}_m$ as in the previous Lemma. Then there exists a $\gamma < 1$, such that for all $Z, W \in B$,  

$$d(\Phi_{\delta}(Z), \Phi_{\delta}(W)) \leq \gamma d(Z, W).$$

Using that the maps $Z \mapsto -Z^{-1}$ and $Z \mapsto Z - D - \delta$ are hyperbolic isometries on $\mathbb{SH}_m$, we find that  

$$d(\Phi_{\delta}(Z), \Phi_{\delta}(W)) = d(Z + \lambda, W + \lambda).$$

In order to estimate the last expression we use that  

$$[\text{Im}(W + \lambda)]^{-1} = [\text{Im}(W)]^{-1/2} [\text{Im}(W + \lambda)]^{-1/2} [\text{Im}(W)]^{-1/2} \leq \sqrt{\gamma} [\text{Im}(W)]^{-1}$$

for some real number $\gamma < 1$ since $W$ is in a bounded set. If we apply this estimate also to $Z$ we obtain that  

$$d(\Phi_{\delta}(Z), \Phi_{\delta}(W)) = d(Z + \lambda, W + \lambda) \leq \gamma d(Z, W).$$

Step 2: The sequence $(\Phi_{q_0} \circ \cdots \circ \Phi_{q_n}(\Lambda_n))_{n \in \mathbb{N}_0}$ converges to a limit independent of the choice of $(\Lambda_n)_{n \in \mathbb{N}_0}$.

Suppose $(\widetilde{\Lambda}_n)_{n \in \mathbb{N}_0}$ is a different sequence. Then

$$(18) \quad d(\Phi_{q_0} \circ \cdots \circ \Phi_{q_n}(\Lambda_n), \Phi_{q_0} \circ \cdots \circ \Phi_{q_n}(\widetilde{\Lambda}_n)) \leq \gamma^{n-2} C \to 0, \quad (n \to \infty),$$

with $C := \sup_{(Z, W) \in B^2} d(Z, W)$. We conclude that if the limit exists it must be independent of the sequence $(\Lambda_n)_{n \in \mathbb{N}_0}$. On the other hand the sequence $(\Phi_{q_0} \circ \cdots \circ \Phi_{q_n}(\Lambda_n))_{n \in \mathbb{N}_0}$ is a Cauchy sequence, which can be seen by inserting $\Lambda_{n + m} := \Phi_{q_{n+1}} \circ \cdots \circ \Phi_{q_{n+m}}(\Lambda_{n+m})$ for $m \in \mathbb{N}$ into (18).

The theorem now follows from Step 2 and Eq. (17). □
Appendix B: Some Inequalities

**Lemma 8.** Let $Z_i \in \mathcal{SH}_m, i \in \{0, 1, 2\}$ and $\delta \in \text{Sym}(m)$. Then

(a) $\text{cd}(2Z_0, Z_1 + Z_2) \leq \frac{1}{2} [\text{cd}(Z_0, Z_1) + \text{cd}(Z_0, Z_2)]$.

(b) $\text{cd}(Z_0, \delta + Z_1) \leq C (1 + \|\delta\|^2) [\text{cd}(Z_0, Z_1) + 1]$ for some constant $C$ that depends on the (norm of the) imaginary part of $Z_0$.

(c) For $\lambda \in \mathcal{D}$ (see definition (3)) there is a constant $C$ (depending on $\lambda$ and $m$) so that $\text{tr}(\text{Im}Z_0) \leq C (\text{cd}_\lambda(Z_0) + 1)$.

**Proof.** (a). Let us define

$$A := \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad B := \begin{bmatrix} Y_1^{1/2}(Y_1 + Y_2)^{-1/2} \\ Y_2^{1/2}(Y_1 + Y_2)^{-1/2} \end{bmatrix}$$

with $Y_i := \text{Im}(Z_i)$ and $U_i = Y_i^{-1/2}(Z_0 - Z_i)Y_0^{-1/2}$. Then

$$\text{cd}(2Z_0, Z_1 + Z_2) = \frac{1}{2} \text{tr} \left[ (U_1 Y_1^{1/2} + U_2 Y_2^{1/2})(Y_1 + Y_2)^{-1}(Y_1^{1/2} U_1 + Y_2^{1/2} U_2) \right] = \frac{1}{2} \text{tr} [A^* B B^* A].$$

Since $B^* B = 1$, $BB^*$ is a projection and hence $BB^* \leq 1$. Therefore, $\text{tr} [A^* BB^* A] \leq \text{tr} [A^* A] = \text{tr} [U_1^* U_1] + \text{tr} [U_2^* U_2] = \text{cd}(Z_0, Z_1) + \text{cd}(Z_0, Z_2)$.

(b). By expanding the product in the trace and using $\text{tr}[A^* B] \leq (\text{tr}|A|^2)^{1/2}(\text{tr}|B|^2)^{1/2} \leq \frac{1}{2} \text{tr}|A|^2 + \frac{1}{2} \text{tr}|B|^2$ we obtain

$$\text{cd}(Z_0, \delta + Z_1) \leq 2 \text{cd}(Z_0, Z_1) + 2 \text{tr}(Y_0^{-1/2}\delta Y_1^{-1}\delta Y_0^{-1/2})$$

$$\leq 2 \text{cd}(Z_0, Z_1) + 2 \|Y_0^{-1/2}\delta Y_0^{-1/2}\|^2 \text{tr}(Y_0^{1/2}Y_1^{-1}Y_0^{1/2}).$$

Now we use $\|Y_0^{-1/2}\delta Y_0^{-1/2}\| \leq \|\delta\| \|Y_0^{-1/2}\| \|Y_0^{-1/2}\|^2$ and inequality (17), i.e., $\text{tr}(Y_0^{1/2}Y^{-1}Y_0^{-1/2}) \leq \text{cd}(Z_0, Z_1) + 2m$, which, all put together, proves the claimed inequality.

(c). We have $\text{cd}_\lambda(Z_0) \geq \text{tr}[Y_\lambda^{1/2}(Y_0 - Y_\lambda)Y_0^{-1}(Y_0 - Y_\lambda)Y_\lambda^{-1/2}] \geq \text{tr}[Y_\lambda^{-1/2}Y_0 Y_\lambda^{-1/2}] - 2m \geq C^\prime \text{tr}(Y_0) - 2m$, where the constant $C'$ depends on $\lambda$ through the estimate on $Y_\lambda^{-1}$. Finally, we choose the constant $C := 2m/C'$ and the stated inequality follows.

$\Box$

Appendix C: Proof of Theorem 6

To prove Theorem 6 we will use the following theorem by Albeverio, Makarov, and Motovilov [2]. For the convenience of the reader we also present a proof that follows closely the one given in [2] but uses analytic perturbation theory to obtain the graph subspaces.

**Theorem 9** (Albeverio-Makarov-Motovilov). Let $H_1, H_2$ be two bounded self-adjoint operators on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\sigma(H_1) \subset (a, b) \subset \rho(H_2)$ and $V : \mathcal{H}_2 \to \mathcal{H}_1$ is a Hilbert-Schmidt operator. Let

$$H_0 := \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad W := \begin{bmatrix} 0 & V \\ V^* & 0 \end{bmatrix}, \quad H_V := H_0 + W.$$

Then $\sigma_{ac}(H_V) = \sigma_{ac}(H_0)$. 


Proof. Since finite rank perturbations $F$ do not change the ac spectrum and such operators are norm-dense in the space of Hilbert-Schmidt operators we can replace $V$ by $V + F$ and achieve that its norm is small. Henceforth we assume that $\|V\|$ is small. Let $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$, and for $i = 1, 2$ we denote by $p_i$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_i$.

Step 1: For $\|V\|$ sufficiently small, there exist orthogonal projections $P_1$ and $P_2$ such that $P_1 + P_2 = 1$, $P_i H = H P_i$, and $P_i P_j : \mathcal{H}_i \to \mathcal{H}_j$ is bijective, for $i \in \{1, 2\}$. Furthermore, $P_i - p_i$ is Hilbert-Schmidt and its norm can be made arbitrarily small if we choose $\|V\|$ small enough.

Let $\Gamma_1 [\Gamma_2]$ be a counter-clockwise contour in $\mathbb{C}$ around the spectrum of $H_1 [H_2]$ and contained in the resolvent set of $H_2 [H_1]$. Then we define the spectral projections

$$P_i := \frac{1}{2\pi i} \int_{\Gamma_i} \frac{dz}{z - H_V}, \quad i \in \{1, 2\}. \quad (20)$$

The first two properties follow from this representation. In order to verify the other statements we use a similar representation for the projections $p_i$ and the resolvent identity so that

$$P_i - p_i = \frac{1}{2\pi i} \int_{\Gamma_i} (z - H_0)^{-1} W(z - H_V)^{-1} dz.$$

Hence, $P_i - p_i$ is Hilbert-Schmidt. If $\|V\|$ is small then $\|p_i P_1 P_i - p_i\|$ is small, too, and therefore $p_i P_1 P_i$ can be inverted on $\mathcal{H}_i$.

Step 2: For $\|V\|$ sufficiently small, there exist operators $Q_1 : \mathcal{H}_1 \to \mathcal{H}_2$ and $Q_2 : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\text{Ran} P_1 = \{(x, Q_1 x) \mid x \in \mathcal{H}_1\}$ and $\text{Ran} P_2 = \{(Q_2 x, x) \mid x \in \mathcal{H}_2\}$. Moreover $Q_2 = -Q_1^*$ and $Q_i$, for $i \in \{1, 2\}$, is Hilbert-Schmidt and its norm can be chosen arbitrarily small for $\|V\|$ sufficiently small.

Let $i \neq j$. First observe that the operator $p_j P_1 = p_j (P_1 - p_i)$ as well as its adjoint are Hilbert-Schmidt and can be made arbitrarily small by Step 1. Using the identity $P_1 p_1 + P_2 p_2 = 1 - P_1 p_2 - P_2 p_1$ and noting that the r.h.s. can be made arbitrarily close to one, we see that $\mathcal{H} = \text{Ran} P_1 p_1 + \text{Ran} P_2 p_2$. This implies that $\text{Ran} P_i = \text{Ran} P_i p_i$. Define $Q_i := p_j P_1 (p_i P_i) - 1$. If we set $x := p_i (P_j x)$ for $z \in \mathcal{H}_i$, then $p_j (P_{jz}) = Q_i x$. Hence, the range of $P_j P_i$ equals the graph of $Q_i$. The statement $Q_2 = -Q_1^*$ follows by orthogonality.

Step 3: For $\|V\|$ sufficiently small $H_V$ is unitary equivalent to

$$\begin{bmatrix}
H_1 + T_1 & 0 \\
0 & H_2 + T_2
\end{bmatrix},$$

where $T_i$ are trace class operators.

Since by Step 2, $H_V$ leaves the graphs of the operators $Q_1$ and $Q_2$ invariant, there exist operators $A_i \in \mathcal{H}_i$ such that

$$H_V (1 + Q) = (1 + Q) A, \quad (21)$$
By Step 2, let \( z = (x, Q_1x) \in \text{Ran} P_1 \). By Step 1, \( H_V z \in \text{Ran} P_1 \), and again by Step 2, \( H_V z = (y, Q_1 y) \) for some uniquely determined \( y \in \mathcal{H}_1 \). This defines the operator \( A_1 : \mathcal{H}_1 \to \mathcal{H}_1 \) by setting \( A_1 x := y \). A similar construction gives the operator \( A_2 \). As a result we obtain (21). Expanding the product in (21) we see more concretely that \( A_1 = H_1 + VQ_1 \) and \( A_2 = H_2 + V^* Q_2 \). Since \( Q \) has purely imaginary spectrum the operator \( 1 + Q \) is bijective. Using the polar decomposition \( 1 + Q = U |1 + Q| \), with \( U \) unitary, we find

\[
(22) \quad U^* H_V U = |1 + Q| A |1 + Q|^{-1}.
\]

Note that

\[
|1 + Q| = \begin{bmatrix} (1 + Q_1^* Q_1)^{1/2} & 0 \\ 0 & (1 + Q_2^* Q_2)^{1/2} \end{bmatrix}.
\]

Using \( 0 \leq (1 + Q_1^* Q_1)^{1/2} - 1 \leq Q_1^* Q_1 \), the operator \( (1 + Q_1^* Q_1)^{1/2} (H_1 + VQ_1) (1 + Q_1^* Q_1)^{-1/2} - H_1 \) is trace class. A similar statement holds for the second diagonal operator on the right-hand side of (22).

Step 4: \( \sigma_{ac}(H) = \sigma_{ac}(H_0) \).

This follows from the result of Step 3 and the fact the trace class perturbations preserve \( ac \) spectrum.

Our proof of Theorem 6 follows closely the one given by Denisov [10], but uses almost analytic functional calculus (cf. [6]) to control the function of an operator.

**Proof of Theorem 6** Fix \( \epsilon > 0 \). We will show that \( [a + \epsilon, b - \epsilon] \subset \sigma_{ac}(H_V) \). Since finite-rank perturbations do not change the \( ac \) and the essential spectrum and \( \sigma_{ess}(H_2) \subset (-\infty, a] \cup [b, \infty) \), we can assume w.l.o.g. that \( \sigma(H_2) \subset (-\infty, a + \epsilon/2] \cup [b - \epsilon/2, \infty) \).

**Step 1:** Define \( \tilde{H}_1 := H_1\chi_{[a+\epsilon, b-\epsilon]}(H_1) \). Then \([a + \epsilon, b - \epsilon] \subset \sigma_{ac}(\tilde{H}_1) \), where \( \tilde{H}_V := \begin{bmatrix} \tilde{H}_1 & V \\ V^* & H_2 \end{bmatrix} \).

This follows directly from Theorem 5 by noting that \( \sigma(\tilde{H}_1) = [a + \epsilon, b - \epsilon] \subset \rho(H_2) \).

**Step 2:** Let \( f \in C^\infty_0([a + \epsilon, b - \epsilon]) \). Then \( f(H_V) - f(\tilde{H}_V) \) is trace class.

To show Step 2 we use almost analytic functional calculus. Let \( \tilde{f} \in C^\infty_0(\mathbb{C}) \) be an almost analytic extension of \( f \), satisfying \( \tilde{f}|_\mathbb{R} = f \), \( \tilde{f}(x + iy) = 0 \) if \( x \notin \text{supp} f \), and that for some constant \( C \) we have \( |\partial_z \tilde{f}(z)| \leq C |\text{Im} z|^\alpha \) for all \( z \in \mathbb{C} \). [Of course, \( \partial_z := \frac{1}{2i}(\partial_x + i\partial_y) \) for \( z = x + iy \in \mathbb{C} \) and \( \overline{z} := x - iy \).] Setting \( R(A, z) := (z - A)^{-1} \) for a bounded...
self-adjoint operator $A$ we recall the Helffer–Sjöstrand formula (see [6] (5))

$$f(A) := -\frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) R(A, z) \, dx \, dy.$$  

(23)

Setting $W := \begin{bmatrix} 0 & V \\ V^* & 0 \end{bmatrix}$ and $\hat{H}_0 := \begin{bmatrix} \hat{H}_1 & 0 \\ 0 & H_2 \end{bmatrix}$, and applying the resolvent identity twice, we find

$$f(H_V) - f(\hat{H}_0) = -\frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) R(H_0, z) \, dx \, dy$$

$$- \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) [R(H_0, z) WR(H_0, z) - R(\hat{H}_0, z) WR(\hat{H}_0, z)] \, dx \, dy$$

and thus vanishes. The third term on the right-hand side is a trace class operator. The second term also vanishes, as we now show. It is a combination of terms of the following form,

$$-\frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) R(H_1, z) VR(H_2, z) \, dx \, dy = -\frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) R(H_1, z) V \int_{\sigma(H_2)} \frac{1}{z - t} \, dP_{H_2}(t) \, dx \, dy$$

$$= \int_{\sigma(H_2)} f(H_1) \frac{1}{H_1 - t} V \, dP_{H_2}(t).$$

In the first line we have applied the Spectral Theorem for $R(H_2, z)$ with the spectral projections $P_{H_2}$ of $H_2$. In the last equality we have used that $\tilde{f}(z)/(z - t)^{-1}$ is an almost analytic extension of $f(x)/(x - t)^{-1}$ since $\partial_{\bar{z}}(z - t)^{-1} = 0$ for $z \neq t$, and the Helffer–Sjöstrand formula (23). By inspection, the right-hand side of the last displayed formula does not change if we replace $H_1$ by $\hat{H}_1$.

Step 3: $[a + \varepsilon, b - \varepsilon] \in \sigma_{\text{ac}}(H_V)$.

By the statement of Step 2 and the Kato-Rosenblum Theorem [21] Theorem XI.8] we know that $\sigma_{\text{ac}}(f(H_V)) = \sigma_{\text{ac}}(\tilde{f}(H_V))$ for all $f \in C_0^\infty([a + \varepsilon, b - \varepsilon])$. By the Spectral Theorem we conclude that $\sigma_{\text{ac}}(H_V) \cap [a + \varepsilon, b - \varepsilon] = \sigma_{\text{ac}}(\tilde{H}_V) \cap [a + \varepsilon, b - \varepsilon] = [a + \varepsilon, b - \varepsilon]$, where the second equality follows from the result of Step 1. \hfill \Box

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