On Fractional Volterra integrodifferential equations with fractional integrable impulses

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Abstract

We consider a class of nonlinear fractional Volterra integrodifferential equation with fractional integrable impulses and investigate the existence and uniqueness results in the Bielecki’s normed Banach spaces. Further, Bielecki–Ulam type stabilities have been demonstrated on a compact interval. A concrete example is provided to illustrate the outcomes we acquired.

Key words: Fractional Volterra integrodifferential equation, Integral Impulses, Banach contraction principle, existence of solutions, Bielecki norm, Bielecki-Ulam type Stability

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1 Introduction

Famous “Ulam stability problem” of functional differential equation raised by Ulam [1] have been extended to different kinds of equations. Wang et al.[2] are the first mathematicians who investigated the Ulam stability and data dependence for fractional differential equations. Thereafter, several interesting works on different Ulam type stabilities of fractional differential and integral equations have been reported (see for instance [3],[4],[5],[6],[7]).

An overview pertaining to impulsive differential equations with instantaneous impulses and its applicability in the practical dynamical systems have provided in the monograph [8],[9],[10]. The impulsive differential equations with time variable impulses dealt in an interesting papers [11],[12],[13]. Wang et al. [14],[15] studied existence and uniqueness of solutions and established generalized \( \beta \)-Ulam-Hyers-Rassias stability to differential equations with not instantaneous impulses in a \( P\beta \)-normed Banach space.

Recently, Wang and Zang [16] introduced a new class of impulsive differential equations of the form

\[
\begin{align*}
x'(t) &= f(t, x(t)), \ t \in (s_i, t_{i+1}], \ i = 0,1,\cdots, m, \\
x(t) &= I_{t_i,t}^\beta h_i(t, x(t)), \ t \in (t_i, s_i], \ i = 1,2,\cdots, m, \ \beta \in (0,1). 
\end{align*}
\]

and examined the existence and uniqueness of solutions in Bielecki’s normed Banach spaces. Further, demonstrated that the corresponding equations are Bielecki-Ulam’s-Hyer stable. It is seen that such sort of formulations are adequate to depict the memory procedures of the drugs in the circulation system and the subsequent absorption for the body.

Motivated by the work of Wang and Zang [16], we consider the following class of nonlinear fractional Volterra integrodifferential equation with fractional order integral impulses
of the form
\[
\begin{cases}
\frac{\varepsilon_i}{\sigma_i^\alpha} D_\tau^\alpha x(\tau) = f \left( \tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma)) d\sigma \right), & \tau \in (\sigma_i, \tau_{i+1}], \ i = 0, 1, \cdots, m, \ \alpha \in (0, 1), \\
x(\tau) = I^\beta_{\tau_i, \tau} h_i(\tau, x(\tau)), & \tau \in (\tau_i, \sigma_i], \ i = 1, 2, \cdots, m, \ \beta \in (0, 1),
\end{cases}
\]
and research the existence and uniqueness of solutions and examine the outcomes relating to Bielecki–Ulam type stabilities viz. Bielecki–Ulam–Hyers and Bielecki–Ulam–Hyers–Rassias stabilities on a compact interval. Here \( \tau_i \) and \( \sigma_i \) are pre-fixed numbers satisfying \( 0 = \tau_0 = \sigma_0 = \tau_1 \leq \sigma_1 \leq \tau_2 < \cdots < \tau_m \leq \sigma_m = \tau_{m+1} = T \), \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( h : [0, T] \times \mathbb{R} \to \mathbb{R} \) and for each \( i = 1, 2 \cdots m \), \( h_i : (\tau_i, \sigma_i] \times \mathbb{R} \to \mathbb{R} \) are continuous functions, \( \frac{\varepsilon_i}{\sigma_i^\alpha} D_\tau^\alpha \) is the Caputo fractional derivative of order \( \alpha \) with lower terminal at \( \sigma_i \) and \( I^\beta_{\tau_i, \tau} \) is the Riemann-Liouville fractional integral of order \( \beta \) with lower terminal \( \tau_i \).

We comment that within this scope, the class of equations considered in the present paper is more broad and the outcomes acquired are the generalization of the fundamental results obtained by Wang and Zang [16]. We support our main results with the examples.

In section 2, we introduce some preliminaries and auxiliary lemmas related to fractional calculus. In Section 3, we prove existence and uniqueness results for (1.1) by using Banach contraction principle via Bielecki norm. In Section 4, adopting the idea of Wang and Zang [16] we examine different Bielecki-Ulams type stability for the problem (1.1). Finally, an example has been provided to illustrate the results we obtained.

2 Preliminaries

Let \( J = [0, T] \). Let \( C(J, \mathbb{R}) = \{ x : J \to \mathbb{R} ; x \text{ is continuous function} \} \) be the Banach space endowed with a Bielecki norm
\[
\| x \|_B = \sup_{\tau \in J} \frac{|x(\tau)|}{e^{\theta \tau}},
\]
where \( \theta > 0 \) is a fixed real number. Let
\[
PC(J, \mathbb{R}) = \left\{ x : J \to \mathbb{R} : x \in C((\tau_i, \tau_{i+1}], \mathbb{R}), \text{ both } x(\tau_i^+) \text{ and } x(\tau_i^-) \text{ exists and } x(\tau_i) = x(\tau_i^+) = x(\tau_i^-), \ i = 0, 1, \cdots, m \right\}.
\]
If \( PC(J, \mathbb{R}) \) is endowed with the norm
\[
\| x \|_{PB} = \sup_{\tau \in J} \frac{|x(\tau)|}{e^{\theta \tau}}, \ \theta > 0
\]
then \( (PC(J, \mathbb{R}), \| \cdot \|_{PB}) \) is a Banach space. The space
\[
PC^1(J, \mathbb{R}) = \left\{ x \in PC(J, \mathbb{R}) : x' \in PC(J, \mathbb{R}) \right\},
\]
with the norm \( \| x \|_{PB'} = \max \{ \| x \|_{PB}, \| x' \|_{PB} \} \) then \( (PC^1(J, \mathbb{R}), \| \cdot \|_{PB'}) \) is a Banach space.

Next, we use definitions and the results listed below from fractional calculus. For more details, we refer the readers to the monographs [17].
Lemma 2.2 Let $0 < \alpha \leq 1$ then the Caputo fractional derivative $^{c}\mathcal{D}_{a}^{\alpha}$ of order $\alpha$ with lower terminal $a$ of a function $g \in C^{1}[a,T]$ is defined as

$$^{c}\mathcal{D}_{a}^{\alpha}g(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{\tau} (\tau - \sigma)^{-\alpha - 1} g'(\sigma) d\sigma, \quad \tau > a \geq 0.$$ 

Lemma 2.1 Let $m-1 < \alpha \leq m \in \mathbb{N}$ and $g \in C^{m}[a,T]$. Then

$$I_{a,\tau}^{\alpha}[^{c}\mathcal{D}_{a}^{\alpha}g(\tau)] = g(\tau) - \sum_{k=0}^{m-1} \frac{g^{(k)}(a)}{\Gamma(k+1)} \tau^{k}, \quad \tau > a \geq 0.$$ 

Following lemma plays an important role to obtain our results.

Lemma 2.2 [18] Let $\alpha, \beta, \gamma$ and $p$ be constants such that $\alpha > 0$, $p(\gamma - 1) + 1 > 0$ and $p(\beta - 1) + 1 > 0$. Then

$$\int_{0}^{\tau} (\tau^{\alpha} - \sigma^{\alpha})^{p(\beta - 1)} \sigma^{p(\gamma - 1)} d\sigma = \frac{\tau^{\theta}}{\alpha} \mathbb{B} \left( \frac{p(\gamma - 1) + 1}{\alpha}, p(\beta - 1) + 1 \right), \text{ where } \tau \in \mathbb{R}_{+}$$

where

$$\mathbb{B}(\xi, \sigma) = \int_{0}^{1} \sigma^{\xi-1}(1 - \sigma)^{\sigma-1} d\sigma, \quad Re(\xi) > 0, \quad Re(\sigma) > 0$$

is a Beta function and $\theta = p[\alpha(\beta - 1) + \gamma - 1] + 1$

3 Existence and Uniqueness Results

Lemma 3.1 A function $x \in PC^{1}(J,R)$ is a classical solution of the problem

$$\begin{cases}
^{c}\mathcal{D}_{\tau}^{\alpha}x(\tau) = f \left( \tau, x(\tau), \int_{\sigma_{i}}^{\tau} h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (\sigma_{i}, \tau_{i+1}], \quad i = 0, 1, \ldots, m, \quad \alpha \in (0, 1) \\
x(\tau) = I_{\sigma_{i},\tau}^{\beta}h_{i}(\tau, x(\tau)), \quad \tau \in (\tau_{i}, \sigma_{i}], \quad i = 1, 2, \ldots, m, \quad \beta \in (0, 1) \\
x(0) = x_{0}
\end{cases} \tag{3.1}$$

if $x$ satisfies the fractional Volterra integral equation

$$x(\tau) = \begin{cases}
x_{0}, \quad \tau = 0, \\
I_{\sigma_{i},\tau}^{\beta}h_{i}(\tau, x(\tau)), \quad \tau \in (\tau_{i}, \sigma_{i}], \quad i = 1, \ldots, m, \\
x_{0} + I_{0,\tau}^{\alpha}f \left( \tau, x(\tau), \int_{0}^{\tau} h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (0, \tau_{i}], \\
I_{\sigma_{i},\tau}^{\beta}h_{i}(\sigma_{i}, x(\sigma_{i})) + I_{\sigma_{i},\tau}^{\gamma}f \left( \tau, x(\tau), \int_{\sigma_{i}}^{\tau} h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (\sigma_{i}, \tau_{i+1}], \quad i = 1, \ldots, m.
\end{cases}$$
Proof: For \( i = 0 \), on operating Riemann-Liouville fractional integral operator \( \mathcal{I}_{\sigma_i, \tau}^\alpha \) on both sides of fractional differential equation (3.1), we get
\[
\mathcal{I}_{0, \tau}^\alpha \left[ \mathcal{D}_\tau^\alpha x(\tau) \right] = \mathcal{I}_{0, \tau}^\alpha f \left( \tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (0, \tau_1)
\]
As \( 0 < \alpha < 1 \), in view of lemma (2.1) we obtain
\[
x(\tau) - x(0) = \mathcal{I}_{0, \tau}^\alpha f \left( \tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (0, \tau_1)
\]
Therefore
\[
x(\tau) = x_0 + \mathcal{I}_{0, \tau}^\alpha f \left( \tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (0, \tau_1)
\]
Similarly, for each \( i (i = 1, 2, \cdots, m) \), operating \( \mathcal{I}_{\sigma_i, \tau}^\alpha \) on both sides of (3.1), we get
\[
x(\tau) = x(\sigma_i) + \mathcal{I}_{\sigma_i, \tau}^\alpha f \left( \tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (\sigma_i, \tau_{i+1})
\]
But from equation (3.1), we have
\[
x(\sigma_i) = \left[ \mathcal{I}_{0, \tau}^\beta h_i(\tau, x(\tau)) \right]_{\tau = \sigma_i} = \mathcal{I}_{0, \tau}^\beta h_i(\sigma_i, x(\sigma_i)), \quad i = 1, 2, \cdots, m
\]
Therefore
\[
x(\tau) = \mathcal{I}_{\sigma_i, \tau}^\beta h_i(\sigma_i, x(\sigma_i)) + \mathcal{I}_{\sigma_i, \tau}^\alpha f \left( \tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma)) d\sigma \right), \quad \tau \in (\sigma_i, \tau_{i+1}).
\]
\[\square\]
We list the following hypotheses in order to establish our main results.

(H1) The function \( f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) satisfies the Lipschitz condition
\[
|f(\tau, x, y) - f(\tau, \bar{x}, \bar{y})| \leq M_f|x - \bar{x}| + N_f|y - \bar{y}|, \quad \tau \in J; \ x, \bar{x}, y, \bar{y} \in \mathbb{R},
\]
where \( M_f > 0, N_f > 0 \).

(H2) The function \( h \in C(J \times \mathbb{R}, \mathbb{R}) \) satisfies the Lipschitz condition
\[
|h(\tau, x) - h(\tau, \bar{x})| \leq K_h|x - \bar{x}|, \quad \tau \in J; \ x, \bar{x} \in \mathbb{R},
\]
where \( K_h > 0 \).

(H3) For each \( i = 1, 2, \cdots, m; h_i \in C((\sigma_i, \tau_i] \times \mathbb{R}, \mathbb{R}) \) and satisfies Lipschitz condition
\[
|h_i(\tau, x) - h_i(\tau, \bar{x})| \leq L_{h_i}|x - \bar{x}|, \quad \text{for each } \tau \in (\sigma_i, \tau_i]; \ x, \bar{x} \in \mathbb{R},
\]
where \( L_{h_i} > 0 \).

Theorem 3.2 Assume that hypotheses (H1)-(H3) holds. Then the problem (3.1) has a unique solution, provided that \( \alpha, \beta \in \left( \frac{1}{2}, 1 \right) \).
Proof: Define an operator $T: PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by
\[
(Tx)(\tau) = \begin{cases}
 x_0, & \text{if } \tau = 0, \\
 \alpha_i \int_0^\tau h_i(\tau, x(\tau))d\tau, & \text{if } \tau \in (\tau_i, \sigma_i], \ i = 1, \cdots, m, \\
 x_0 + \alpha_i \int_0^\tau f(\tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma))d\sigma, & \text{if } \tau \in (0, \tau_i],
\end{cases}
\]
We shall show that the operator $T$ is contraction with respect to Bielecki norm. Let $x, y \in PC(J, \mathbb{R})$ and $\tau \in (\tau_i, \sigma_i), \ i = 1, 2, \cdots, m$, then using hypothesis (H3), we have
\[
|\alpha_i \int_0^\tau h_i(\tau, x(\tau))d\tau - \alpha_i \int_0^\tau h_i(\tau, y(\tau))d\tau|
\]
\[
= \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - \sigma)^{\beta-1} h_i(\sigma, x(\sigma))d\sigma - \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - \sigma)^{\beta-1} h_i(\sigma, y(\sigma))d\sigma
\]
\[
\leq \frac{L}{\Gamma(\beta)} \int_0^\tau (\tau - \sigma)^{\beta-1} |x(\sigma) - y(\sigma)|d\sigma
\]
\[
\leq \frac{L}{\Gamma(\beta)} \|x - y\|_{PB} \int_0^\tau (\tau - \sigma)^{\beta-1} \|e^{\theta\sigma}x(\sigma) - e^{\theta\sigma}y(\sigma)\|d\sigma
\]
By Holder’s inequality, for $\beta \in (\frac{1}{2}, 1)$ and $\tau \in (\tau_i, \sigma_i)$, we have
\[
\int_0^\tau (\tau - \sigma)^{\beta-1} e^{\theta\sigma}d\sigma \leq \left(\int_0^\tau (\tau - \sigma)^{2(\beta-1)}d\sigma\right)^{\frac{1}{2}} \left(\int_0^\tau e^{2\theta\sigma}d\sigma\right)^{\frac{1}{2}}
\]
\[
= \left(\frac{(\tau - \tau_i)^{2\beta-1}}{2\beta - 1}\right)^{\frac{1}{2}} \left(\frac{e^{2\theta\tau} - e^{2\theta\tau_i}}{2\theta}\right)^{\frac{1}{2}}
\]
\[
\leq \left(\frac{2\beta - 1}{2\beta - 1}\right)^{\frac{1}{2}} \left(\frac{e^{2\theta\tau} - e^{2\theta\tau_i}}{2\theta}\right)^{\frac{1}{2}}
\]
\[
\leq \left(\frac{\sigma_i - \tau_i}{2\beta - 1}\right)^{\frac{1}{2}} \left(e^{\theta\tau} - e^{\theta\tau_i}\right)^{\frac{1}{2}}
\]
Thus for any $x, y \in PC(J, \mathbb{R})$ and $\tau \in (\tau_i, \sigma_i)$, $i = 1, 2, \cdots, m$, we have
\[
\left|\alpha_i \int_0^\tau h_i(\tau, x(\tau))d\tau - \alpha_i \int_0^\tau h_i(\tau, y(\tau))d\tau\right| \leq \frac{L}{\Gamma(\beta)} \left(\frac{\sigma_i - \tau_i}{\sigma_i - \tau_i}\right)^{\frac{1}{2}} \|x - y\|_{PB}. \tag{3.3}
\]
Now, for any $x, y \in PC(J, \mathbb{R})$ and $\tau \in (\sigma_i, \tau_{i+1})$, $i = 1, 2, \cdots, m$, by using hypotheses (H1) – (H2), we obtain
\[
\left|\alpha_i \int_0^\tau h(\sigma, x(\sigma))d\sigma - \alpha_i \int_0^\tau h(\sigma, y(\sigma))d\sigma\right|
\]
\[
\leq \alpha_i \int_0^\tau \left|f(\tau, x(\tau)) - f(\tau, y(\tau))\right|d\sigma
\]
\[
\leq \alpha_i \int_0^\tau \left|f(\tau, x(\tau)) - f(\tau, y(\tau))\right|d\sigma
\]
\[
\leq T_{\sigma,\tau}^\alpha (M_f|x(\tau) - y(\tau)| + N_f |T_{\sigma,\tau}^{1} h(\tau, y(\tau)) - T_{\sigma,\tau}^{1} h(\tau, y(\tau))) \\
\leq T_{\sigma,\tau}^\alpha (M_f|x(\tau) - y(\tau)| + N_f T_{\sigma,\tau}^{1} h(\tau, x(\tau)) - h(\tau, y(\tau))) \\
\leq T_{\sigma,\tau}^\alpha (M_f|x(\tau) - y(\tau)| + N_f T_{\sigma,\tau}^{1} K_h|x(\tau) - y(\tau)|) \\
= M_f T_{\sigma,\tau}^\alpha|x(\tau) - y(\tau)| + N_f K_h T_{\sigma,\tau}^{+1} (T_{\sigma,\tau}^{1} |x(\tau) - y(\tau)|) \\
= M_f T_{\sigma,\tau}^\alpha|x(\tau) - y(\tau)| + N_f K_h T_{\sigma,\tau}^{+1} |x(\tau) - y(\tau)| \\
= \frac{M_f}{\Gamma(\alpha)} \int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha-1} |x(\sigma) - y(\sigma)| d\sigma + \frac{N_f K_h}{\Gamma(\alpha + 1)} \int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha} |x(\sigma) - y(\sigma)| d\sigma \\
= \frac{M_f}{\Gamma(\alpha)} \int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha-1} e^{\theta \sigma} \left( \sup_{\sigma \in (\sigma_i, \sigma_{i+1}]} e^{-\theta \sigma} |x(\sigma) - y(\sigma)| \right) d\sigma \\
+ \frac{N_f K_h}{\Gamma(\alpha + 1)} \int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha} e^{\theta \sigma} \left( \sup_{\sigma \in (\sigma_i, \sigma_{i+1}]} e^{-\theta \sigma} |x(\sigma) - y(\sigma)| \right) d\sigma \\
\leq \frac{M_f}{\Gamma(\alpha)} \|x - y\|_{PB} \int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha-1} e^{\theta t} d\sigma + \frac{N_f K_h}{\Gamma(\alpha + 1)} \|x - y\|_{PB} \int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha} e^{\theta t} d\sigma. \quad (3.4)
\]

Since \( \alpha \in (\frac{1}{2}, 1) \), from the inequality (3.2), for any \( \tau \in (\sigma_i, \sigma_{i+1}], i = 1, 2, \cdots, m \), we have
\[
\int_{\sigma_i}^{\tau} (\tau - \sigma)^{\alpha-1} e^{\theta \sigma} d\sigma \leq \frac{(\tau - \sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha - 1)^{\frac{3}{2}} (2\theta)^{\frac{3}{2}}} e^{\theta \tau} \leq \frac{(\tau_{i+1} - \sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha - 1)^{\frac{3}{2}} (2\theta)^{\frac{3}{2}}} e^{\theta \tau} \quad (3.5)
\]
By replacing \( \alpha \) by \( \alpha + 1 \) in the above inequality we get
\[
\int_{\sigma_i}^{t} (t - \sigma)^{\alpha} e^{\theta \sigma} d\sigma \leq \frac{(t_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha + 1)^{\frac{3}{2}} (2\theta)^{\frac{3}{2}}} e^{\theta t}, \; t \in (\sigma_i, t_{i+1}], \; i = 1, 2, \cdots, m \quad (3.6)
\]
Using the inequalities (3.5) and (3.6), the inequality (3.4) takes the form
\[
\left| T_{\sigma,\tau}^\alpha \left( \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma)) d\sigma \right) - T_{\sigma,\tau}^\alpha \left( \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)) d\sigma \right) \right| \leq \frac{M_f}{\Gamma(\alpha)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha - 1)^{\frac{3}{2}} (2\theta)^{\frac{3}{2}}} e^{\theta \tau} + \frac{N_f K_h}{\Gamma(\alpha + 1)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha + 1)^{\frac{3}{2}} (2\theta)^{\frac{3}{2}}} \|x - y\|_{PB} \quad (3.7)
\]
From the definition of an operator \( T \) and hypothesis (H1)-(H3), for any \( x, y \in PC(J, \mathbb{R}) \), we obtain that
Case 1: For \( \tau \in (\tau_i, \sigma_i], \; i = 1, 2, \cdots, m \), from the inequality (3.3), we have
\[
|Tx(\tau) - Ty(\tau)| e^{-\theta \tau} \leq \left| T_{\tau_i,\tau}^\beta h_i(\tau, x(\tau)) - T_{\tau_i,\tau}^\beta h_i(\tau, y(\tau)) \right| e^{-\theta \tau} \leq \frac{L_{h_i}}{\Gamma(\beta)} \frac{(\sigma_i - \tau_i)^{\beta-\frac{1}{2}}}{(2\beta - 1)^{\frac{3}{2}} (2\theta)^{\frac{3}{2}}} \|x - y\|_{PB}.
\]
Case 2: For \( \tau \in (0, \tau_1] \), on similar line of (3.7), we have
\[
|Tx(\tau) - Ty(\tau)| e^{-\theta \tau} = \left| T_{0,\tau}^{\alpha} \left( \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma)) d\sigma \right) - T_{0,\tau}^{\alpha} \left( \int_{0}^{\tau} h(\sigma, y(\sigma)) d\sigma \right) \right| e^{-\theta \tau}
\]
From the cases 1-3, we can conclude that

\[ T \in \text{e}^{-\theta \tau} \text{ and satisfies Lipschitz condition} \]

Thus, \( T \) becomes a contraction mapping and has a unique fixed point due to Banach contraction principle. This fixed point of \( T \) act as unique solution of the problem (3.1).

Choose sufficiently large value of \( \theta \) so that \( L < 1 \). Thus, \( T \) becomes a contraction mapping and has a unique fixed point due to Banach contraction principle. This fixed point of \( T \) act as unique solution of the problem (3.1).

Next, our aim is to extend the restriction of \( \alpha, \beta \in (1, 1) \) to \( \alpha, \beta \in (0, 1) \). For this purpose we use the following variants of the hypotheses (H2) and (H3):

\[ \hat{H}1 \] The function \( f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) satisfies the Lipschitz condition

\[ |f(\tau, x, y) - f(\tau, \bar{x}, \bar{y})| \leq M_f \tau^{\gamma_f} |x - \bar{x}| + N_f |y - \bar{y}|, \ \tau \in [0, T]; \ x, \bar{x}, y, \bar{y} \in \mathbb{R}, \]

where \( M_f, N_f > 0 \) and \( \gamma_f > -\alpha \).

\[ \hat{H}3 \] For each \( i = 1, 2, \cdots m \); \( h_i \in C([\tau_i, \tau_{i+1}] \times \mathbb{R}, \mathbb{R}) \) and satisfies Lipschitz condition

\[ |h_i(\tau, x) - h_i(\tau, \bar{x})| \leq \gamma N_{h_i} |x - \bar{x}|, \ \tau \in [\tau_i, \tau_{i+1}); \ x, \bar{x} \in \mathbb{R}, \]

where \( N_{h_i} > 0 \) and \( \gamma > -\beta \).
Theorem 3.3 Assume that hypotheses (H1), (H2) and (H3) holds. Then the problem (3.1) has a unique solution.

Proof: Consider the operator $T$ defined in the Theorem 5.2. By (H1), $\gamma_f > -\alpha = (1-\alpha)-1$ where $\alpha \in (0, 1)$. Choose $\sigma > 1$ such that $\sigma \gamma_f > \sigma(1-\alpha)-1$ and $\sigma(\alpha-1)+1 > 0$. Define $\sigma^* = \frac{\sigma}{\sigma-1}$. Then $\sigma + \sigma^* = 1$. By (H3), for any $\tau \in (\sigma_i, \tau_i+1]$, we have

\[
\left| I_{\sigma_i}^\alpha f \left( \tau, x(\tau) \right) - I_{\sigma_i}^\alpha f \left( \tau, y(\tau) \right) \right| \leq M \left( \int_{\sigma_i}^\tau \left| x(\tau) - y(\tau) \right| d\sigma \right)^{\sigma^*} \leq I \leq I
\]

Thus, the problem (3.1) has a unique solution.

Using Holders inequality, we get

\[
\int_{\sigma_i}^\tau (\tau - \sigma)^{\sigma^*} e^{\theta \sigma} d\sigma \leq \left( \int_{\sigma_i}^\tau (\tau - \sigma)^{\sigma(\alpha-1)} \sigma^{\gamma_f} d\sigma \right)^{\frac{1}{\sigma}} \left( \int_{\sigma_i}^\tau e^{\theta \sigma} d\sigma \right)^{\frac{1}{\theta}}
\]

But

\[
\left( \int_{\sigma_i}^\tau e^{\theta \sigma} d\sigma \right)^{\frac{1}{\theta}} = \left( \frac{e^{\theta \sigma^*} - e^{\theta \sigma^*}}{\theta \sigma^*} \right)^{\frac{1}{\theta}} \leq \left( \frac{e^{\theta \sigma^*}}{\theta \sigma^*} \right)^{\frac{1}{\theta}} = \frac{e^{\theta \tau}}{(\theta \sigma^*)^{\frac{1}{\theta}}}
\]

By lemma 2.2

\[
\left( \int_{\sigma_i}^\tau (\tau - \sigma)^{\sigma(\alpha-1)} \sigma^{\gamma_f} d\sigma \right)^{\frac{1}{\sigma}} \leq \left( \int_0^\tau (\tau - \sigma)^{\sigma(\alpha-1)} \sigma^{\gamma_f} d\sigma \right)^{\frac{1}{\sigma}} = \left( \sigma^{\sigma(\alpha-1)+\sigma \gamma_f+1} B(\gamma_f+1, \sigma(\alpha-1)+1) \right)^{\frac{1}{\sigma}} \leq \omega_1,
\]

where $\omega_1 = \left( T^{\sigma(\alpha-1)+\sigma \gamma_f+1} B(\gamma_f+1, \sigma(\alpha-1)+1) \right)^{\frac{1}{\sigma}}$. Therefore

\[
\int_{\sigma_i}^\tau (\tau - \sigma)^{\sigma^*} e^{\theta \sigma} d\sigma \leq \omega_1 \frac{e^{\theta \tau}}{(\theta \sigma^*)^{\frac{1}{\theta}}}
\]

(3.9)
Using inequalities (3.9) and (3.10), from (3.8) we get

\[
\left| T_{\alpha,i}^\beta f\left(\tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma)) d\sigma - T_{\alpha,i}^\beta f\left(\tau, y(\tau), \int_{\sigma_i}^\tau h(\sigma, y(\sigma)) d\sigma\right)\right| e^{-\theta \tau}
\leq \left( \omega_1 \frac{M_f}{\Gamma(\alpha)(\theta \sigma^\ast)^{2\alpha}} + \frac{N_f K_h}{\Gamma(\alpha + 1) (2\alpha + 1)^2 (2\theta)^2} \right) \|x - y\|_{PB} \tag{3.10}
\]

On the similar line by (H3), as discussed above, we can chose \(\sigma_1 > 1\) such that \(\sigma_1 \gamma > \sigma_1 (1 - \beta) - 1\) and \(\sigma_1 (\beta - 1) + 1 > 0\). Then for \(\sigma_1^* = \frac{\sigma_1}{\sigma_1^*}\) we have \(\sigma_1 + \sigma_1^* = 1\). By (H3) and the Holders inequality, for any \(\tau \in (\tau_i, \sigma_i)\),

\[
\left| T_{\alpha,i}^\beta f\left(\tau, x(\tau)\right) - T_{\alpha,i}^\beta f\left(\tau, y(\tau)\right)\right| e^{-\theta \tau}
\leq \frac{L_h}{\Gamma(\beta)} \int_{\tau_i}^\tau (\sigma - \sigma_1)^{\beta - 1} \sigma^\gamma |x(\sigma) - y(\sigma)| d\sigma
\leq \frac{L_h}{\Gamma(\beta)} \|x - y\|_{PB} \int_{\tau_i}^\tau (\sigma - \sigma_1)^{\beta - 1} e^{\theta \sigma} d\sigma
\leq \frac{L_h}{\Gamma(\beta)} \|x - y\|_{PB} \left( \int_{\tau_i}^\tau (\sigma - \sigma_1)^{\beta - 1} \sigma^\gamma d\sigma \right)^{\frac{1}{\sigma_1}} \left( \int_{\tau_i}^\tau e^{\theta \sigma^\ast \sigma} d\sigma \right)^{\frac{1}{\sigma_1}}
\leq \frac{L_h}{\Gamma(\beta)} \|x - y\|_{PB} \left( \int_{\tau_i}^\tau (\sigma - \sigma_1)^{\beta - 1} \sigma^\gamma d\sigma \right)^{\frac{1}{\sigma_1}} \frac{e^{\theta \tau}}{(\theta \sigma^\ast)^{\frac{1}{\sigma_1}}}
\leq \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma^\ast)^{\frac{1}{\sigma_1}}} e^{\theta \tau} \|x - y\|_{PB},
\]

where \(\omega_2 = (T_{\sigma_1(\beta - 1) + \sigma_1 \gamma + 1}B(\sigma_1 \gamma + 1, \sigma_1 (\beta - 1) + 1))^{\frac{1}{\sigma_1}}\). Thus

\[
\left| T_{\alpha,i}^\beta f\left(\tau, x(\tau)\right) - T_{\alpha,i}^\beta f\left(\tau, y(\tau)\right)\right| e^{-\theta \tau} \leq \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma^\ast)^{\frac{1}{\sigma_1}}} \|x - y\|_{PB}, \quad \tau \in (\tau_i, \sigma_i) \tag{3.11}
\]

By definition of an operator \(T\) and using inequations (3.10) and (3.11), for any \(x, y \in PC(J, \mathbb{R})\), we have:

Case (i): For \(\tau \in (\sigma_i, \tau_{i+1}), i = 1, 2, \cdots m,\)

\[
|Tx(\tau) - Ty(\tau)| e^{-\theta \tau}
\leq T_{\alpha,i} f\left(\tau, x(\sigma_i), x(\sigma_i)\right) - h_i(\sigma_i, y(\sigma_i))|e^{-\theta \tau}
+ T_{\alpha,i} \left| f\left(\tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma)) d\sigma\right) - f\left(\tau, y(\tau), \int_{\sigma_i}^\tau h(\sigma, y(\sigma)) d\sigma\right)\right| e^{-\theta \tau}
\leq \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma^\ast)^{\frac{1}{\sigma_1}}} + \omega_1 \frac{M_f}{\Gamma(\alpha)(\theta \sigma^\ast)^{2\alpha}} + \frac{N_f K_h}{\Gamma(\alpha + 1) (2\alpha + 1)^2 (2\theta)^2} \|x - y\|_{PB}
\]

Case (ii): For \(\tau \in (0, \tau_1),\)

\[
|Tx(\tau) - Ty(\tau)| e^{-\theta \tau}
\]
We adopt the idea of Wang and Zang [16] to investigate the concepts of Bielecki-Ulam type stability for the class of nonlinear fractional order Volterra integrodifferential equation (1.1).

For any $\theta > 0$, $\epsilon > 0$, $\psi \geq 0$, $\varphi \in PC(J, \mathbb{R}^+)$ is nondecreasing and $\alpha, \beta \in (0, 1)$, consider the following inequalities.

\[
\begin{align*}
\left| \frac{\alpha}{\alpha - 1} \mathcal{D}^\alpha_{\sigma_i} y(\tau) - f(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)) d\sigma) \right| & \leq \epsilon, \quad \tau \in (\sigma_i, \tau_{i+1}], \quad i = 0, 1, \cdots, m, \\
\left| y(\tau) - \mathcal{I}^\beta_{\tau_{i+1}} h_i(\tau, y(\tau)) \right| & \leq \epsilon, \quad \tau \in (\sigma_i, \tau_i], \quad i = 1, 2, \cdots, m.
\end{align*}
\]

and

\[
\begin{align*}
\left| \frac{\alpha}{\alpha - 1} \mathcal{D}^\alpha_{\tau} y(\tau) - f(\tau, y(\tau), \int_{\tau}^{\tau} h(\sigma, y(\sigma)) d\sigma) \right| & \leq \varphi(\tau), \quad \tau \in (\sigma_i, \tau_{i+1}], \quad i = 0, 1, \cdots, m, \\
\left| y(\tau) - \mathcal{I}^\beta_{\tau_{i+1}} h_i(\tau, y(\tau)) \right| & \leq \psi, \quad \tau \in (\tau_i, \sigma_i], \quad i = 1, 2, \cdots, m.
\end{align*}
\]

and

\[
\begin{align*}
\left| \frac{\alpha}{\alpha - 1} \mathcal{D}^\alpha_{\tau} y(\tau) - f(\tau, y(\tau), \int_{\tau}^{\tau} h(\sigma, y(\sigma)) d\sigma) \right| & \leq \epsilon \varphi(\tau), \quad \tau \in (\sigma_i, \tau_{i+1}], \quad i = 0, 1, \cdots, m, \\
\left| y(\tau) - \mathcal{I}^\beta_{\tau_{i+1}} h_i(\tau, y(\tau)) \right| & \leq \epsilon \psi, \quad \tau \in (\tau_i, \sigma_i], \quad i = 1, 2, \cdots, m.
\end{align*}
\]

By choosing sufficient large value of $\theta$, we get $L_1 < 1$ and in this case $T$ is a contraction and hence $T$ has a unique fixed point, which is the unique solution of (3.1). 

4 Bielecki-Ulam-Hyers Stability

We adopt the idea of Wang and Zang [16] to investigate the concepts of Bielecki-Ulam-Hyers type stability for the class of nonlinear fractional order Volterra integrodifferential equation (1.1).
Definition 4.1 The equation (1.1) is Bielecki-Ulam-Hyers stable if there exists a real number $C_{f,\theta,h,\alpha,\beta,h_i}>0$ such that for each $\epsilon>0$ for each solution $y\in PC^1(J,\mathbb{R})$ of inequality (1.1) there exists a solution $x\in PC^1(J,\mathbb{R})$ of equation (1.1) with
\[
|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i}\epsilon, \quad \tau \in J.
\]

Definition 4.2 The equation (1.1) is generalized Bielecki-Ulam-Hyers stable if there exists $\theta_{f,\theta,h,\alpha,\beta,h_i}\in C(\mathbb{R}^+,\mathbb{R}^+)$, $\theta_{f,\theta,h,\alpha,\beta,h_i}(0)=0$ such that for each $\epsilon>0$ for each solution $y\in PC'(J,\mathbb{R})$ of inequality (1.1) there exists a solution $x\in PC^1(J,\mathbb{R})$ of equation (1.1) with
\[
|y(\tau) - x(\tau)|e^{-\theta\tau} \leq \theta_{f,\theta,h,\alpha,\beta,h_i}(\epsilon), \quad \tau \in J.
\]

Definition 4.3 The equation (1.1) is Bielecki-Ulam-Hyers-Rassias stable with respect to $(\phi,\psi)$ if there exists $C_{f,\theta,h,\alpha,\beta,h_i,\phi,\psi}>0$ such that for each $\epsilon>0$ for each solution $y\in PC^1(J,\mathbb{R})$ of inequality (1.1) there exists a solution $x\in PC^1(J,\mathbb{R})$ of equation (1.1) with
\[
|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i,\phi,\psi}(\epsilon), \quad \tau \in J.
\]

Definition 4.4 The equation (1.1) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to $(\phi,\psi)$ if there exists $C_{f,\theta,h,\alpha,\beta,h_i,\phi,\psi}>0$ such that for each $\epsilon>0$ for each solution $y\in PC^1(J,\mathbb{R})$ of inequality (1.1) there exists a solution $x\in PC^1(J,\mathbb{R})$ of equation (1.1) with
\[
|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i,\phi,\psi}(\epsilon), \quad \tau \in J.
\]

Lemma 4.1 If $y\in PC^1(J,\mathbb{R})$ is a solution of inequality (4.3) then $y$ satisfies the following integral inequalities
\[
\begin{align*}
|y(\tau) - I_{\tau_i,\tau}^\beta h_i(\tau,y(\tau))| &\leq \epsilon \psi, \quad \tau \in (\tau_i,\sigma_i], \ i = 1,2,\ldots,m, \\
|y(\tau) - y(0) - I_{\tau,\tau}^\alpha f(\tau,y(\tau),\int_{\tau_i}^{\tau} h(\sigma,y(\sigma))d\sigma)| &\leq \epsilon \int_{\tau_i}^{\tau} I_{\sigma,\tau}^\alpha \phi(\sigma)d\sigma, \ \text{if} \ \tau \in (0,\tau_i], \\
|y(\tau) - I_{\tau_i,\tau}^\beta h_i(\sigma_i,y(\sigma_i)) - I_{\sigma_i,\tau}^\alpha f(\tau,y(\tau),\int_{\sigma_i}^{\tau} h(\sigma,y(\sigma))d\sigma)| &\leq \epsilon \left(\psi + \int_{\sigma_i}^{\tau} I_{\sigma,\tau}^\alpha \phi(\xi)d\xi\right) \\
&\tau \in (\sigma_i,\tau_{i+1}], \ i = 1,2,\ldots,m.
\end{align*}
\]

Proof: If $y\in PC^1(J,\mathbb{R})$ is a solution of the inequality (4.3) then there exists $H\in PC(J,\mathbb{R})$ and constants $H_i$, $i = 1,2,\ldots,m$ (which depend on $y$) such that
\[
\begin{align*}
(i) \ |H(\tau)| &\leq \epsilon \varphi(\tau), \ \tau \in J \ \text{and} \ |H_i| \leq \epsilon \psi \ \text{for} \ i = 1,2,\ldots,m. \\
(ii) \ &\left\{\begin{array}{l}
\sigma_i \mathcal{D}_\tau^\alpha y(\tau) = f(\tau,y(\tau),\int_{\sigma_i}^{\tau} h(\sigma,y(\sigma))d\sigma) + H(\tau), \ \tau \in (\sigma_i,\tau_{i+1}], \ i = 0,1,\ldots,m, \\
y(\tau) = I_{\sigma_i,\tau}^\beta h_i(\tau,y(\tau)) + H_i, \ \tau \in (\sigma_i,\tau_i], \ i = 1,2,\ldots,m,
\end{array}\right.
\end{align*}
\]
In the view of Lemma 4.1 above equation is equivalent to the integral equations
\[
y(\tau) = \begin{cases}
I_{\tau_i,\tau}^\beta h_i(\tau,y(\tau)) + \epsilon H_i, \ \tau \in (\tau_i,\sigma_i], \ i = 1,\ldots,m, \\
y(0) + I_{\tau,\tau}^\alpha [f(\tau,y(\tau),\int_{\tau_i}^{\tau} h(\sigma,y(\sigma))d\sigma) + \epsilon H(\tau)], \ \tau \in [0,\tau_i], \\
I_{\sigma_i,\tau}^\beta h_i(\sigma_i,y(\sigma_i)) + \epsilon H_i + I_{\sigma_i,\tau}^\alpha \left[f(\tau,y(\tau),\int_{\sigma_i}^{\tau} h(\sigma,y(\sigma))d\sigma) + \epsilon H(\tau)\right], \\
\tau \in (\sigma_i,\tau_{i+1}], i = 1,\ldots,m.
\end{cases}
\]
For any $\tau \in (\sigma_i, \tau_{i+1}]$, $i = 1, 2, \cdots m$,
\[
\left| y(\tau) - J^\beta_{\sigma_i, \tau} h_i(\sigma, y(\sigma)) - J^\alpha_{\sigma_i, \tau} f \left( \tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)) \, d\sigma \right) \right| = |H_i + J^\alpha_{\sigma_i, \tau} H(\tau)| \\
\leq |H_i| + J^\alpha_{\sigma_i, \tau} |H(\tau)| \\
\leq \epsilon \left( \psi + J^\alpha_{\sigma_i, \tau} \varphi(\tau) \right).
\]

For $\tau \in (0, \tau_1]$,\[
\left| y(\tau) - y(0) - J^\alpha_{0, \tau} f \left( \tau, y(\tau), \int_{0}^{\tau} h(\sigma, y(\sigma)) \, d\sigma \right) \right| = J^\alpha_{0, \tau} |H(\tau)| \leq \epsilon J^\alpha_{0, \tau} \varphi(\tau).
\]
For $\tau \in (\sigma_i, \sigma_{i+1}]$, $i = 1, 2, \cdots m$,
\[
\left| y(\tau) - J^\beta_{\sigma_i, \tau} h_i(\tau, y(\tau)) \right| \leq |H_i| \leq \epsilon \psi, \quad \tau \in (\sigma_i, \sigma_{i+1}], \quad i = 1, 2, \cdots m.
\]

The last three inequalities are the required equivalent integral inequalities in (4.4). \quad \Box

Following additional assumption is needed to prove the Bielecki-Ulam-Hyers-Rassias stability of equation (1.1).

(H4) Let $\varphi \in C(J, \mathbb{R}_+)$ is nondecreasing and there exists $c_{\varphi} > 0$ such that
\[
J^\alpha_{0, \tau} \varphi \leq c_{\varphi} \varphi(\tau), \quad \tau \in J.
\]

**Theorem 4.2** Assume that hypotheses (H1), (H2), (H3) and (H4) holds. Then, the equation (1.1) is Bielecki-Ulam-Hyers-Rassias stable with respect to $(\varphi, \psi)$, where $\alpha, \beta \in (0, 1)$.

**Proof:** Let $y \in PC^1(J, \mathbb{R})$ be a solution of inequality (4.3). Then by lemma 4.1, $y$ satisfies the integral inequalities
\[
\begin{cases}
|y(\tau) - J^\beta_{\sigma_i, \tau} h_i(\tau, y(\tau))| \leq \epsilon \psi, \quad \tau \in (\sigma_i, \sigma_{i+1}], \quad i = 1, 2, \cdots , m, \\
|y(\tau) - y(0) - J^\alpha_{0, \tau} f \left( \tau, y(\tau), \int_{0}^{\tau} h(\sigma, y(\sigma)) \, d\sigma \right) | \leq c_{\varphi} \varphi(\tau), \quad \tau \in (0, \tau_1], \\
|y(\tau) - J^\beta_{\sigma_i, \tau} h_i(\sigma_i, y(\sigma)) - J^\alpha_{\sigma_i, \tau} f \left( \tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)) \, d\sigma \right) | \leq \epsilon \left( \psi + c_{\varphi} \varphi(\tau) \right), \\
\tau \in (\sigma_i, \tau_{i+1}], \quad i = 1, 2, \cdots m.
\end{cases}
\]

Denote by $x$ the classical solution of fractional Volterra integrodifferential equations
\[
\begin{cases}
D^\alpha_{\sigma_i} x(\tau) = f \left( \tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma)) \, d\sigma \right), \quad \tau \in (\sigma_i, \tau_{i+1}], \quad i = 0, 1, \cdots , m, \\
x(\tau) = J^\beta_{\sigma_i, \tau} h_i(\tau, x(\tau)), \quad \tau \in (\sigma_i, \sigma_{i+1}], \quad i = 1, 2, \cdots , m, \\
x(0) = y(0)
\end{cases}
\]

Then, in the view of lemma 3.1, $x$ satisfies the fractional integral equations
\[
\begin{cases}
x(0), \quad \tau = 0, \\
J^\beta_{\sigma_i, \tau} h_i(\tau, x(\tau)), \quad \tau \in (\sigma_i, \sigma_{i+1}], \quad i = 1, \cdots , m, \\
y(0) + J^\alpha_{0, \tau} f \left( \tau, x(\tau), \int_{0}^{\tau} h(\sigma, x(\sigma)) \, d\sigma \right), \quad \tau \in (0, \tau_1], \\
J^\beta_{\sigma_i, \tau} h_i(\sigma_i, x(\sigma)) + J^\alpha_{\sigma_i, \tau} f \left( \tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma)) \, d\sigma \right), \quad \tau \in (\sigma_i, \tau_{i+1}], \quad i = 1, \cdots m
\end{cases}
\]
Therefore, for any $\tau \in (\sigma_i, \tau_{i+1})$, we obtain

$$|y(\tau) - x(\tau)| e^{-\theta \tau} \leq \left| y(\tau) - \left( I_{\tau_i, \sigma_i}^h h_i(\sigma_i, x(\sigma_i)) + I_{\sigma_i, \tau}^\alpha f \left( \tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma)) d\sigma \right) \right) \right| e^{-\theta \tau}$$

This gives

$$\sup_{\tau \in (\sigma_i, \tau_{i+1})} e^{-\theta \tau} |x(\tau) - y(\tau)| \leq \varepsilon (1 + c_\varphi) (\psi + \varphi(\tau)) e^{-\theta \sigma_i} + \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma_i^*)\frac{1}{\sigma_i}} + \frac{\omega_1 M_f}{\Gamma(\alpha)(\theta \sigma^*)\frac{1}{\sigma_i}} + \frac{N f K_h (\tau_{i+1} - \sigma_i)^{\alpha + \frac{1}{2}}}{\Gamma(\alpha + 1)(2\alpha + 1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} \sup_{\tau \in (\sigma_i, \tau_{i+1})} e^{-\theta \tau} |x(\tau) - y(\tau)|$$

Therefore,

$$\sup_{\tau \in (\tau_i, \tau_{i+1})} e^{-\theta \tau} |x(\tau) - y(\tau)| \leq \varepsilon (1 + c_\varphi) (\psi + \varphi(\tau)) e^{-\theta \sigma_i} + \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma_i^*)\frac{1}{\sigma_i}} + \frac{\omega_1 M_f}{\Gamma(\alpha)(\theta \sigma^*)\frac{1}{\sigma_i}} + \frac{N f K_h (\tau_{i+1} - \sigma_i)^{\alpha + \frac{1}{2}}}{\Gamma(\alpha + 1)(2\alpha + 1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} \sup_{\tau \in (\sigma_i, \sigma_i)} e^{-\theta \tau} \left| x(\tau) - y(\tau) \right|$$

Now, for any $\tau \in (\tau_i, \sigma_i)$, $i = 1, 2, \cdots, m$,

$$|y(\tau) - x(\tau)| e^{-\theta \tau} \leq |y(\tau) - I_{\tau_i, \tau}^\beta h_i(\tau, x(\tau))| e^{-\theta \tau}$$

$$\leq |y(\tau) - I_{\tau_i, \tau}^\beta h_i(\tau, y(\tau))| e^{-\theta \tau} + |I_{\tau_i, \tau}^\beta h_i(\tau, y(\tau)) - I_{\tau_i, \tau}^\beta h_i(\tau, x(\tau))| e^{-\theta \tau}$$

Therefore,

$$\sup_{\tau \in (\tau_i, \sigma_i)} e^{-\theta \tau} \left| x(\tau) - y(\tau) \right| \leq \varepsilon (1 + c_\varphi) (\psi + \varphi(\tau)) e^{-\theta \sigma_i} + \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma_i^*)\frac{1}{\sigma_i}} \sup_{\tau \in (\tau_i, \sigma_i)} e^{-\theta \tau} \left| x(\tau) - y(\tau) \right|$$
Thus we have
\[
\sup_{\tau \in [\tau_i, \sigma_i]} e^{-\theta \tau} |x(\tau) - y(\tau)| \leq \frac{\epsilon \psi}{\left(1 - \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma^*_1)^{\frac{1}{\sigma_1}}}\right)^{\epsilon \psi}} e^{-\theta \tau_i}
\] (4.9)

Next, for any \(\tau \in (0, \tau_1]\),
\[
|y(\tau) - x(\tau)| e^{-\theta \tau}
\leq |y(\tau) - y(0) - I_{0, \tau} f (\tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma)) d\sigma)| e^{-\theta \tau}
\leq |y(\tau) - y(0) - I_{0, \tau} f (\tau, y(\tau), \int_0^\tau h(\sigma, y(\sigma)) d\sigma)| e^{-\theta \tau}
+ I_{0, \tau} |f(\tau, y(\tau), \int_0^\tau h(\sigma, y(\sigma)) d\sigma) - f(\tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma)) d\sigma)| e^{-\theta \tau}
\leq \epsilon c \varphi(t) + \left(\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta \sigma^*)^{\frac{1}{\sigma}}} + \frac{N_f K_h}{\Gamma(\alpha + 1)(2\alpha + 1) \frac{1}{2} (2\theta)^{\frac{1}{2}}} \right) \sup_{\tau \in (0, \tau_1]} e^{-\theta \tau} |x(\tau) - y(\tau)|
\]

Just computed as above, we get
\[
\sup_{\tau \in [0, \tau_1]} |y(\tau) - x(\tau)| e^{-\theta \tau} \leq \frac{\epsilon c \varphi(\tau)}{1 - \left(\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta \sigma^*)^{\frac{1}{\sigma}}} + \frac{N_f K_h}{\Gamma(\alpha + 1)(2\alpha + 1) \frac{1}{2} (2\theta)^{\frac{1}{2}}} \right)}
\] (4.10)

Since
\[J = (0, T) = \bigcup_{i=0}^{m} (\tau_i, \tau_{i+1}] \bigcup_{i=1}^{m} (\tau_i, \sigma_i] \],
from the inequalities (4.8), (4.9) and (4.10), we obtain
\[
\sup_{\tau \in (0, T]} |y(\tau) - x(\tau)| e^{-\theta \tau} \leq \sup_{\tau \in (0, \tau_1]} |y(\tau) - x(\tau)| e^{-\theta \tau} + \sum_{i=1}^{m} \sup_{\tau \in (\sigma_i, \tau_{i+1}]} |y(\tau) - x(\tau)| e^{-\theta \tau}
\]
\[
+ \sum_{i=1}^{m} \sup_{\tau \in (\tau_i, \sigma_i]} |y(\tau) - x(\tau)| e^{-\theta \tau}
\leq \epsilon c \psi \varphi(t) + \varphi(\tau)
\] (4.11)

where
\[
\epsilon_{f, \theta, h, \alpha, \beta, h, \varphi} = \frac{c \varphi}{1 - \left(\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta \sigma^*)^{\frac{1}{\sigma}}} + \frac{N_f K_h}{\Gamma(\alpha + 1)(2\alpha + 1) \frac{1}{2} (2\theta)^{\frac{1}{2}}} \right)} + \sum_{i=1}^{m} \frac{\psi}{1 - \left(1 - \frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma^*_1)^{\frac{1}{\sigma_1}}}\right)} + \sum_{i=1}^{m} \frac{1 + c \varphi}{1 - \left(\frac{L_h \omega_2}{\Gamma(\beta)(\theta \sigma^*_1)^{\frac{1}{\sigma_1}}} + \frac{\omega_1 M_f}{\Gamma(\alpha)(\theta \sigma^*)^{\frac{1}{\sigma}}} + \frac{N_f K_h(\tau_{i+1} - \sigma_i)^{\alpha + \frac{1}{2}}}{\Gamma(\alpha + 1)(2\alpha + 1) \frac{1}{2} (2\theta)^{\frac{1}{2}}} \right)}
\]
Finally, from inequality (4.11), we have

$$|y(\tau) - x(\tau)|e^{-\theta \tau} \leq \epsilon c_{f,\theta,h,\alpha,\beta,h_1,\psi}(\psi + \varphi(\tau)), \tau \in J \quad (4.12)$$

This shows that equation (1.1) Bielecki-Ulam-Hyers-Rassias stable with respect to $$(\varphi, \psi)$$. □

**Corollary 4.3** Assume that hypotheses $$(\tilde{H}1), (H2), (\tilde{H}3)$$ and $$(H4)$$ holds. Then, the equation (1.1) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to $$(\varphi, \psi)$$, where $$\alpha, \beta \in (0,1)$$.

**Proof:** Set $$\epsilon = 1$$ in the proof of Theorem 5.3, we obtain

$$|y(\tau) - x(\tau)|e^{-\theta \tau} \leq \epsilon c_{f,\theta,h,\alpha,\beta,h_1,\psi}(\psi + \varphi(\tau)), \tau \in J. \quad (4.13)$$

This proves the equation (1.1) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to $$(\varphi, \psi)$$. □

**Corollary 4.4** Assume that hypotheses $$(\tilde{H}1), (H2), (\tilde{H}3)$$ and $$(H4)$$ holds. Then, the equation (1.1) is Bielecki-Ulam-Hyers stable, where $$\alpha, \beta \in (0,1)$$.

**Proof:** Set $$\psi = 1$$ and $$\varphi(\tau) = 1, \tau \in J$$ in theorem proof of Theorem 4.2. Then $$\varphi \in C(J, \mathbb{R}_+)$$ and $$T_{0,\tau}^\alpha \varphi \leq c_\varphi \varphi(\tau), \tau \in J$$, where $$c_\varphi = \frac{T_\alpha}{\Gamma(1+\alpha)}$$. Thus the hypothesis $$(H4)$$ is satisfied. Further, we have

$$|y(\tau) - x(\tau)|e^{-\theta \tau} \leq \epsilon c_{f,\theta,h,\alpha,\beta,h_1}, \tau \in J. \quad (4.14)$$

Therefore the equation (1.1) is Bielecki-Ulam-Hyers stable. □

**Corollary 4.5** Assume that hypotheses $$(\tilde{H}1), (H2), (\tilde{H}3)$$ and $$(H4)$$ holds. Then, the equation (1.1) is generalized Bielecki-Ulam-Hyers stable, where $$\alpha, \beta \in (0,1)$$.

**Proof:** Define $$\theta_{f,\theta,h,\alpha,\beta,h_1} : \mathbb{R}_+ \to \mathbb{R}_+$$ by $$\theta_{f,\theta,h,\alpha,\beta,h_1}(\epsilon) = \epsilon c_{f,\theta,h,\alpha,\beta,h_1}$$. Then $$\theta_{f,\theta,h,\alpha,\beta,h_1} \in C(\mathbb{R}_+, \mathbb{R}_+)$$ and $$\theta_{f,\theta,h,\alpha,\beta,h_1}(0) = 0$$. Further from the inequation (4.14), we have

$$|y(\tau) - x(\tau)|e^{-\theta t} \leq \theta(\epsilon), \tau \in J,$$

which shows that (1.1) is generalized Bielecki-Ulam-Hyers stable. □

**Remark 4.6** Under the assumptions of Theorem 3.2 one can obtain the Bielecki-Ulam-Hyers stability and Bielecki-Ulam-Hyers-Rassias stability of equation when $$\alpha, \beta \in \left(\frac{1}{2}, 1\right)$$.
5 Example

Example 5.1: Consider Caputo fractional differential equation with fractional integrable impulse

\[ c_{\alpha_i}D_{\alpha_i}^2 x(\tau) = f \left( \tau, x(\tau), \int_{\sigma_i}^\tau h(\sigma, x(\sigma))d\sigma \right), \tau \in (\sigma_i, \tau_{i+1}), i = 0, 1 \]  \tag{5.1}

\[ x(\tau) = \mathcal{I}_{\sigma_1}^\tau h_1(\tau, x(\tau)), \tau \in (\tau_1, \sigma_1], \]  \tag{5.2}

\[ x(0) = 0 \]  \tag{5.3}

where \( 0 = \sigma_0 = \tau_0 < \tau_1 = 1 < \sigma_1 = 2 < \tau_2 = 3 \) and the functions \( f : [0,3] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}; h : [0,3] \times \mathbb{R} \to \mathbb{R}, h_1 : (1,2) \times \mathbb{R} \to \mathbb{R} \) are defined as follows:

\[ f \left( \tau, x(\tau), \int_{\sigma_1}^\tau h(\sigma, x(\sigma))d\sigma \right) = g(\tau) + \frac{e^{-\tau^2}}{4} (\sin x(\tau) + \cos x(\tau)) + \int_{\sigma_1}^\tau \frac{\sigma}{e^{\sigma^2}} \sin x(\sigma)d\sigma \]

\[ h(\tau, x(\tau)) = \frac{\tau}{e^{\tau^2}} \sin(x(\tau)) \]

\[ h_1(\tau, x(\tau)) = \frac{1}{\Gamma\left(\frac{3}{4}\right)} \left(\frac{\tau-1}{\tau-4} \right) \left(\frac{|x(\tau)| - 3}{|x(\tau)| + 1}\right) \]

where

\[ g(\tau) = \begin{cases} \frac{9}{2\Gamma\left(\frac{7}{4}\right)} \tau^\frac{3}{2} - \frac{1}{4}, & \tau \in [0,1] \\ 0, & \tau \in (1,2] \\ \frac{9}{2\Gamma\left(\frac{7}{4}\right)} (\tau - 2)^\frac{3}{2} - \frac{1}{4} \cos 4 + \sin 4, & \tau \in (2,3], \end{cases} \]

For any \( x, y, \bar{x}, \bar{y} \in PC(J, \mathbb{R}) \) and \( \tau \in [0,3], \)

\[ |f(\tau, x, y) - f(\tau, \bar{x}, \bar{y})| \]

\[ = \left| \left( g(\tau) + \frac{e^{-\tau^2}}{4} (\sin x + \cos x) + y \right) - \left( g(\tau) + \frac{e^{-\tau^2}}{4} (\sin \bar{x} + \cos \bar{x}) + \bar{y} \right) \right| \]

\[ \leq \frac{e^{-\tau^2}}{4} \left( |\sin x - \sin \bar{x}| + |\cos x - \cos \bar{x}| \right) + |y - \bar{y}| \]

\[ = \frac{e^{-\tau^2}}{4} \left( 2 \cos \left(\frac{x + \bar{x}}{2}\right) \sin \left(\frac{x - \bar{x}}{2}\right) \right) + |y - \bar{y}| \]

\[ \leq \frac{e^{-\tau^2}}{2} |x - \bar{x}| + |y - \bar{y}| \leq \frac{1}{2} |x - \bar{x}| + |y - \bar{y}| \]

Further, for \( \tau \in [1,2] \)

\[ |h_1(\tau, x) - h_1(\tau, \bar{x})| = \left| \frac{1}{\Gamma\left(\frac{3}{4}\right)} \left(\frac{\tau-1}{\tau-4} \right) \left(\frac{|x| - 3}{|x| + 1}\right) - \frac{1}{\Gamma\left(\frac{3}{4}\right)} \left(\frac{\tau-1}{\tau-4} \right) \left(\frac{|y| - 3}{|y| + 1}\right) \right| \]

\[ \leq \frac{1}{\Gamma\left(\frac{3}{4}\right)} \left|\frac{|x| - 3}{|x| + 1} - \frac{|y| - 3}{|y| + 1}\right| \leq \frac{4}{\Gamma\left(\frac{3}{4}\right)} \frac{|x| - |y|}{(|x| + 1)(|y| + 1)} \]

\[ \leq \frac{4}{\Gamma\left(\frac{3}{4}\right)} |x - y| \]
and for $\tau \in [0, 3]$, we have

\[
|h(\tau, x(\tau)) - h(\tau, \bar{x})| = \left| \frac{\tau}{e^{\tau^2}} \sin x - \frac{\tau}{e^{\tau^2}} \sin \bar{x} \right| = \frac{\tau}{e^{\tau^2}} |\sin x - \sin \bar{x}|
\]

\[
= \frac{\tau}{e^{\tau^2}} |\sin x - \sin \bar{x}| = \frac{\tau}{e^{\tau^2}} \left| 2 \cos\left(\frac{x + \bar{x}}{2}\right) \sin\left(\frac{x - \bar{x}}{2}\right) \right|
\]

\[
\leq 6 \left| \frac{x - \bar{x}}{2} \right| = 3|x - \bar{x}|
\]

Thus $f, h$ and $h_1$ satisfies Lipschitz condition with Lipschitz constants

\[
M_f = \frac{1}{2}, \quad N_f = 1, \quad K_h = 3, \quad L_{h_1} = \frac{4}{\Gamma\left(\frac{4}{3}\right)}.
\]

Thus by Theorem 3.2, the problem (5.1)-(5.3) has a unique solution on $[0, 3]$.

Let $\varphi(\tau) = 2.8361\mathbb{E}_\frac{2}{3}\left(\tau^\frac{2}{3}\right)$, $\tau \in J$ and $\psi = 0$. Then $\varphi \in C(J, \mathbb{R}_+)$ is nondecreasing and satisfy the condition

\[
\mathcal{I}_{0, \tau}^\frac{2}{3}\varphi(\tau) = 2.8361\mathcal{I}_{0, \tau}^\frac{2}{3}\left(\mathbb{E}_\frac{2}{3}\left(\tau^\frac{2}{3}\right)\right) \leq 2.8361\mathbb{E}_\frac{2}{3}\left(\tau^\frac{2}{3}\right) = c_\varphi \varphi(\tau), \quad \tau \in J,
\]

where $c_\varphi = 1$. Note that all the assumptions of the Corollary 4.3 holds. therefore problem (5.1)-(5.2) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to $(\varphi, \psi)$.

Next, we shall discuss the generalized Bielecki-Ulam-Hyers-Rassias stability of the equation (5.1)-(5.2) by showing that there exist an exact solution $x(\tau)$ of the problem (5.1)-(5.3) corresponding to $(\varphi, \psi)$ and the given solution $y(\tau)$ of the inequalities

\[
\left| \frac{1}{\sigma} \mathcal{D}_\tau^\frac{4}{3} y(\tau) - f(\tau, y, \tau), \int_{\sigma_i}^\tau h(\sigma, y(\sigma))d\sigma \right| \leq \varphi(\tau), \quad \tau \in (\sigma_i, \tau_{i+1}], \quad i = 0, 1 \quad (5.4)
\]

\[
\left| y(\tau) - \mathcal{I}_{\sigma, \tau}^\frac{4}{3} h_1(y(\tau)) \right| \leq 0, \quad \tau \in (\tau_1, \sigma_1]. \quad (5.5)
\]

where $0 = \sigma_0 = \tau_0 < \tau_1 = 1 < \sigma_1 = 2 < \tau_2 = 3$.

Let

\[
y(\tau) = \begin{cases} \tau, & \tau \in [0, 1] \cup (2, 3] \\ (\tau - 1), & \tau \in (1, 2] \end{cases}
\]

Then for $\tau \in (0, 1]$,

\[
\left| \frac{1}{\sigma} \mathcal{D}_\tau^\frac{4}{3} y(t) - g(\tau) - \frac{e^{-\tau^2}}{4} (\sin y(\tau) + \cos y(\tau)) - \int_0^\tau \frac{\sigma}{e^{\sigma^2}} \sin y(\sigma)d\sigma \right|
\]

\[
= \left| \frac{1}{\sigma} \mathcal{D}_\tau^\frac{4}{3} (\tau^\frac{2}{3}) - \frac{9}{2\Gamma\left(\frac{4}{3}\right)} \tau^\frac{4}{3} + \frac{1}{4} - \frac{e^{-\tau^2}}{4} (\sin \tau + \cos \tau) - \int_0^\tau \frac{\sigma}{e^{\sigma^2}} \sin \sigma d\sigma \right|
\]

\[
\leq \frac{1}{\Gamma\left(\frac{4}{3}\right)} \tau^\frac{4}{3} + \frac{9}{2\Gamma\left(\frac{4}{3}\right)} \tau^\frac{4}{3} + \frac{1}{4} + \frac{e^{-\tau^2}}{4} (|\sin \tau| + |\cos \tau|) + \int_0^\tau \frac{\sigma}{e^{\sigma^2}} |\sin \sigma| d\sigma
\]

\[
\leq 4.5495
\]
and for $\tau \in (2, 3]$

$$\left| \frac{c}{2} D_{\tau}^{\frac{3}{4}} y(\tau) - g(\tau) - \frac{e^{-\tau^2}}{4} (\sin y(\tau) + \cos y(\tau)) - \int_{2}^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin(y(\sigma)) d\sigma \right|$$

$$= \left| \frac{c}{2} D_{\tau}^{\frac{3}{4}} [(\tau - 2) + 2] - \frac{9}{2 \Gamma(\frac{3}{4})} (\tau - 2)^{\frac{3}{4}} + \frac{1}{4e^4} (\cos 4 + \sin 4)$$

$$- \frac{e^{-\tau^2}}{4} (\sin \tau + \cos \tau) - \int_{2}^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin \sigma d\sigma \right|$$

$$\leq \frac{1}{\Gamma(\frac{3}{4})} (\tau - 2)^{\frac{3}{4}} + 0 + \frac{9}{2 \Gamma(\frac{3}{4})} (\tau - 2)^{\frac{3}{4}}$$

$$+ \frac{1}{4e^4} (|\cos 4| + |\sin 4|) + \frac{1}{4e^4} (|\sin \tau| + |\cos \tau|) + \int_{2}^{\tau} \frac{\sigma}{e^{\sigma^2}} |\sin \sigma| d\sigma$$

$$\leq 2.8361$$

Therefore for $\tau \in (\sigma_i, \tau_i]$, $i = 0, 1$, we have

$$\left| \frac{c}{2} D_{\tau}^{\frac{3}{4}} y(\tau) - g(\tau) - \frac{e^{-\tau^2}}{4} (\sin y(\tau) + \cos y(\tau)) - \int_{\sigma_i}^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin(y(\sigma)) d\sigma \right|$$

$$\leq \min \{4.5495, 2.8361\} = 2.8361 \leq 2.8361 E_{\frac{3}{4}}(\tau^{\frac{3}{4}}) = \varphi(\tau)$$

Also, for $\tau \in (1, 2]$ we get

$$|y(\tau) - \mathcal{L}_{y, \tau}^{\frac{3}{4}} (h_1(\tau, y(\tau)))| = |(\tau - 1) - (\tau - 1)| = 0$$

Hence $y(\tau)$ is a solution of an inequality (5.4)-(5.5).

Next, one can easily verify that

$$x(\tau) = \begin{cases} 
\tau^2, & \tau \in (0, 1] \cup (2, 3] \\
(\tau - 1), & \tau \in (1, 2],
\end{cases}$$

is the unique solution of the problem (5.1)-(5.3).

As discussed in the proof of the Theorem 3.2, we have

$$L = \max \left\{ \frac{L_{h_i}}{\Gamma(\beta)} (\sigma_i - \tau_i)^{\beta - \frac{3}{4}} \frac{1}{\sqrt{2\beta - 1} \sqrt{2\theta}} + \frac{M_{k_i}}{\Gamma(\alpha)} (\tau_i - \sigma_i)^{\alpha - \frac{1}{2}} \frac{1}{\sqrt{2\alpha - 1} \sqrt{2\theta}} + \frac{N_{f L_{h_i}}}{\Gamma(\alpha + 1)} (\tau_i - \sigma_i)^{\alpha + \frac{1}{2}} \frac{1}{\sqrt{2\alpha + 1} \sqrt{2\theta}}; i = 0, 1 \right\}$$

$$= \max \left\{ \frac{M_{f}}{\Gamma(\alpha)} \frac{1}{\sqrt{2\alpha - 1} \sqrt{2\theta}} + \frac{N_{f K_{h_i}}}{\Gamma(\alpha + 1)} \frac{1}{\sqrt{2\alpha \sqrt{2\theta}}}, \right.\left. \frac{L_{h_i}}{\Gamma(\beta)} \frac{1}{\sqrt{2\beta - 1} \sqrt{2\theta}} + \frac{M_{f}}{\Gamma(\alpha)} \frac{1}{\sqrt{2\alpha - 1} \sqrt{2\theta}} + \frac{N_{f L_{h_i}}}{\Gamma(\alpha + 1)} \frac{1}{\sqrt{2\alpha + 1} \sqrt{2\theta}} \right\}$$

$$= \frac{7.5188 \sqrt{3}}{\Gamma(\frac{3}{4}) \sqrt{2\theta}}$$
Choose \( \theta > \left( \frac{7.5188\sqrt{3}}{\sqrt{2}\Gamma\left(\frac{2}{3}\right)} \right)^2 = 46.2473 \) so that \( L < 1 \). For this choice of \( \theta \), we have: for \( \tau \in (0, 1) \),

\[
|y(\tau) - x(\tau)|e^{-\theta\tau} = |\tau - \tau^2|e^{-\theta\tau} \leq (\tau + \tau^2) \leq 2,
\]

for \( \tau \in (1, 2) \),

\[
|y(\tau) - x(\tau)|e^{-\theta\tau} = 0,
\]

and for \( \tau \in (2, 3) \),

\[
|y(\tau) - x(\tau)|e^{-\theta\tau} = |\tau - \tau^2|e^{-\theta\tau} \leq (\tau + \tau^2)e^{-2\theta} = 11e^{-2\theta}.
\]

Thus,

\[
|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i,\varphi}(\psi + \varphi(\tau)), \quad \tau \in J = [0, 3],
\]

where \( C_{f,\theta,h,\alpha,\beta,h_i,\varphi} = 1, \psi = 0 \) and \( \varphi(\tau) = 2.8361E_2^3(\tau^\frac{2}{3}) \).

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