DYNAMICS OF A DEPLETION-TYPE GIERER-MEINHARDT MODEL WITH LANGMUIR-HINSHELWOOD REACTION SCHEME

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Abstract. A depletion-type reaction-diffusion Gierer-Meinhardt model with Langmuir-Hinshelwood reaction scheme and the homogeneous Neumann boundary conditions is introduced and investigated in this paper. Firstly, the boundedness of positive solution of the parabolic system is given, and the constant steady state solutions of the model are exhibited by the Shengjin formulas. Through rigorous theoretical analysis, the stability of the corresponding positive constant steady state solution is explored. Next, a priori estimates, the properties of the nonconstant steady states, non-existence and existence of the nonconstant steady state solution for the corresponding elliptic system are investigated by some estimates and the Leray-Schauder degree theory, respectively. Then, some existence conditions are established and some properties of the Hopf bifurcation and the steady state bifurcation are presented, respectively. It is showed that the temporal and spatial bifurcation structures will appear in the reaction-diffusion model. Theoretical results are confirmed and complemented by numerical simulations.

1. Introduction. The self-organized concentration patterns will often occur when the steady states keep stable to the spatially homogeneous perturbation, while becoming unstable to the spatially inhomogeneous one due to the spatial diffusion. This is the basic ideological motivation of the Turing instability [27]. In view of the diffusive effects of the reactants and products or activators and inhibitors, nonlinear reaction-diffusion systems are frequently constructed to explore the dynamical behaviors induced by diffusion. To this end, many chemical models have been proposed and their dynamical behaviors have been extensively investigated. Such as, the Brusselator model [24, 9], the Lengyel-Epstein model [1, 17], the Seelig model [31, 20], the Degn-Harrison model [30, 11], Gierer-Meinhardt model [8, 14] and other well-known chemical reaction-diffusion models, see [33, 2].

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In 1972, Gierer and Meinhardt [7] proposed a prototypical depletion-type chemical model, which is described as follows

\[
\begin{align*}
\frac{\partial a}{\partial t} &= \rho_0 \rho' + c \rho' a^k f(h) - \mu_0 a + D_a \frac{\partial^2 a}{\partial x^2}, \\
\frac{\partial h}{\partial t} &= c_0 - c' \rho' a^k f(h) - \nu_0 h + D_h \frac{\partial^2 h}{\partial x^2},
\end{align*}
\]

(1)

where \(a = a(x,t)\) and \(h = h(x,t)\) are concentrations of two chemical substances at time \(t\) and location \(x\), \(\rho'\) is the source density for activator, the distribution of which is activated by the activator concentration \(a(x,t)\). The other substance \(h(x,t)\) is assumed to be consumed by activation or some indirect effect of activation and may be derived from a large area, being produced everywhere at a constant rate \(c_0\). \(D_a\) and \(D_h\) are diffusion coefficients of \(a(x,t)\) and \(h(x,t)\), respectively; \(k\) is an integer with \(k > 1\); \(f(h)\) is a function increasing with respect to \(h\); all parameters in (1) are positive constants.

Gierer and Meinhardt proposed a simplest form when taking \(k = 2\) and \(f(h) = h\), namely

\[
\begin{align*}
\frac{\partial a}{\partial t} &= \rho_0 \rho' + c \rho' a^2 h - \mu_0 a + D_a \frac{\partial^2 a}{\partial x^2}, \\
\frac{\partial h}{\partial t} &= c_0 - c' \rho' a^2 h - \nu_0 h + D_h \frac{\partial^2 h}{\partial x^2}.
\end{align*}
\]

It is noticed that this model was used to describe pigmentation patterns in sea shells and the ontogeny of ribbing on ammonoid shells [16, 3]. In [16], this model was extended to describe the formation of complex patterns, such as shell patterns and fish-bone like patterns. The switching-induced Turing instability of the above model was reported in [3], where it was showed that the appearance of the induced spatiotemporal structures depends on the ratio of two characteristic times. The stability, the Hopf bifurcation and the Turing instability by the technique of stability theory, normal form theory and center manifold reduction were carried out in [29].

Now we assume that

\[f(h) = \frac{h}{1 + mh},\]

it is often known as the Langmuir-Hinshelwood law and generally refers to a chemical reaction process. In such process, each reactant is provided to adsorb on a surface site and there is no energetic interaction among substances adsorbed on the surface. After surface reactions occur among the adsorbed reactants, then surface products are generated and the products are desorbed from the surface site. This function is similar to the Michelis-Menten functional response, describing the kinetics of enzyme-catalyzed reactions. Also, we note that \(\lim_{h \to \infty} \frac{h}{1 + mh} = \frac{1}{m}\), this implies that when substance \(h\) is large enough, the term \(f(h) = \frac{h}{1 + mh}\) will reach saturation during the reaction, so the activator \(a(x,t)\) will possesses saturated production. Accordingly, system (1) with the Langmuir-Hinshelwood law or saturated product is more controllable than that with \(f(h) = h\) from a chemical reaction point of view, see Figure 1 for some details. The Gierer-Meinhardt model with saturation effect has been reported in some literatures. Detailed analysis of the existence, stability and dynamics of ring and near-ring solutions for a saturated Gierer-Meinhardt model in the semi-strong diffusion regime were reported by Moyles and Ward in [19]. Chen et al. [5] investigated the stability of the equilibria, the Hopf bifurcation and the steady state bifurcation of a Gierer-Meinhardt model with saturation in the activator production. Kurata et al. [10] explored the multi-peak stationary solution of a saturated Gierer-Meinhardt model with a bounded smooth domain in \(\mathbb{R}^N\).
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stripe solution, spotted solution, mixed solution and labyrinthine-like solution in a two-dimensional space of a saturated Gierer-Meinhardt model were discussed by Song et al. [26]. For more results on the Gierer-Meinhardt system with saturation effect one could refer to [18, 12] and references cited therein. With this idea in mind, a new reaction-diffusion system takes the form:

\[
\begin{align*}
\frac{\partial a}{\partial t} &= \rho_0 \rho' + \frac{c' \rho' a^2 h}{1+m h} - \mu_0 a + D_a \frac{\partial^2 a}{\partial x^2}, \\
\frac{\partial h}{\partial t} &= c_0 - \frac{c' \rho' a^2 h}{1+m h} - \nu_0 h + D_h \frac{\partial^2 h}{\partial x^2}.
\end{align*}
\]  

(2)

Figure 1. Comparison of \( f(h) = h \) and \( f(h) = \frac{h}{1 + mh} \) in the plane of \( h - f(h) \).

Introduce the non-dimensional transformations

\[
t = \frac{\tau}{\nu_0}, \quad a = \frac{\rho_0 \rho' A}{\nu_0}, \quad h = \frac{\nu_0^2 H}{c_0 \rho_0 \rho'^2}, \quad m = \frac{c_0 \rho_0 \rho'^2 m}{\nu_0^2}, \quad \mu_0 = \nu_0 \tilde{\mu},
\]

\[
\beta = \frac{c_0 \rho_0 \rho'^2}{\nu_0^2}, \quad \alpha = \frac{c' \rho_0 \rho'^3}{\nu_0^2}, \quad d_1 = \frac{D_a}{\nu_0}, \quad d_2 = \frac{D_h}{\nu_0},
\]

and let

\[
\tau \to t, \quad A \to u, \quad H \to v, \quad \tilde{m} \to m, \quad \tilde{\mu} \to r,
\]

then system (2) becomes

\[
\begin{align*}
\frac{\partial u}{\partial t} &= 1 - ru + \frac{u^2 v}{1 + mv} + d_1 \frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial v}{\partial t} &= \beta - v - \frac{\alpha u^2 v}{1 + mv} + d_2 \frac{\partial^2 v}{\partial x^2}.
\end{align*}
\]  

(3)

Consider the homogeneous Neumann boundary conditions and nonnegative initial values, a final version of the system has the form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + \frac{u^2 v}{1 + mv} - ru + 1, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + \beta - v - \frac{\alpha u^2 v}{1 + mv}, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega,
\end{align*}
\]  

(4)

where \( \Delta u \) and \( \Delta v \) are unbiased movement for \( u \) and \( v \) respectively, and \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a bounded domain with the boundary \( \partial \Omega \), \( \mathbf{n} \) is the outward unit normal vector on \( \partial \Omega \), and \( \partial \mathbf{n} \) denotes the operator of directional derivative along the direction \( \mathbf{n} \). All parameters in system (4) are positive for chemical reaction.

In this paper, we shall study abundant dynamics of system (4). Firstly, we study the boundedness of system (4) by the comparison principle of parabolic equations.
It is showed that any non-negative solution of the system is bounded and the system may have the persistence property for any non-negative initial concentration conditions. Then the existence and the stability of positive constant steady state solutions of the system will be considered. However, we find that it is difficult to solve the constant steady state solutions directly due to the cubic polynomial, and the exact expression of positive constant steady state solutions can not be exhibited via the Descarte’s rule of signs. Thereby, the Shengjin formulas [6] is adopted to overcome this difficulty. Moreover, the local and global stability of the positive constant steady state solution are investigated by the technique of linear analysis and construction of a Lyapunov function, respectively. Next, the non-existence and existence of non-constant steady state solutions of the fully elliptic system of (4) are considered. A priori estimates of arbitrary positive solution of the elliptic system are given by means of maximum principle [15]. After that, some estimates about the positive non-constant steady state solution \((u(x), v(x))\) are presented via the Cauchy-Schwarz inequality and the Poincaré’s inequality. Especially, an auxiliary function \(\omega(x) = \alpha d_1 u(x) + d_2 v(x)\) is verified to be useful to obtain these estimates. By employing Cauchy-Schwarz inequality, the Poincaré’s inequality and Young’s inequality, the non-existence of the non-constant steady state solution of the elliptic system is established. The main results show that the diffusion coefficients \(d_1\) and \(d_2\) have important influence on the non-existence of non-constant steady state solutions. Furthermore, the existence of positive non-constant steady state solution is investigated via the homotopy invariance of the Leray-Schauder degree [21, 4]. Finally, the steady state bifurcation and the Hopf bifurcation are investigated, respectively. We find that it is very complicated to derive the existence of these bifurcations directly, instead, we first study the properties of the steady state bifurcation and the Hopf bifurcation curves respectively, then the existence results of the Hopf bifurcation with spatial homogeneity and inhomogeneity, and the steady state bifurcation could be obtained in view of these bifurcation curve properties. Also, the validity of the theoretical results are well verified by numerical simulations.

The layout of this paper is designed as follows. In Section 2, the boundedness of the positive solution, the existence of the positive constant steady state solution and its stability are presented. Some estimates about the solutions, nonexistence and existence of the non-constant steady state of the corresponding elliptic system are investigated in Section 3. In Section 4, the steady state bifurcation and the Hopf bifurcation are analyzed respectively. Some discussions and conclusions are made in Section 5.

2. Boundedness, constant steady state and its stability.

2.1. Boundedness. In this subsection, the boundedness of solution to the reaction-diffusion system (4) will be considered.

**Theorem 2.1.** Suppose that parameters \(\beta, \alpha, r, m\) are positive, \(d_1 = d_2 = d\) and \(\Omega \in \mathbb{R}^N\) is a bounded domain with smooth boundary. Then for \(u_0(x) \geq 0, v_0(x) \geq 0\), system (4) admits a unique solution \((u(x, t), v(x, t))\) satisfying \(u(x, t) > 0, v(x, t) > 0\) for \(t > 0\) and \(x \in \Omega\). Furthermore, for any given positive constant \(\epsilon_0\), one yields

\[
\limsup_{t \to \infty} \max_{x \in \Omega} (\alpha u(x, t) + v(x, t)) \leq \frac{\alpha + (1 + \epsilon_0)\beta}{\min\{1, r\}}.
\]
Proof. Define

\[ f(u, v) := 1 - ru + \frac{u^2v}{1 + mv}, \quad g(u, v) := \beta - v - \frac{\alpha u^2v}{1 + mv}. \]

Then we have \( f_v(u, v) = \frac{u^2}{1 + mv} \geq 0 \) and \( g_u(u, v) = -\frac{2\alpha uv}{1 + mv} \leq 0 \) in \( \mathbb{R}^2_+ = \{ u(x, t) \geq 0, v(x, t) \geq 0 \} \). We thus claim that system (4) is a mixed quasi-monotone system. Let \((u_1(x, t), v_1(x, t)) = (0, 0)\) and \((u_2(x, t), v_2(x, t)) = \( (\bar{u}(t), \bar{v}(t)) \)\), where we denote by \((\bar{u}(t), \bar{v}(t))\) the unique solution to system

\[
\begin{cases}
\frac{du(t)}{dt} = \frac{u^2v}{1 + mv} - ru + 1, \\
\frac{dv(t)}{dt} = \beta - v, \\
u(0) = u_0^* = \sup_{x \in \Omega} u_0(x), \\
v(0) = v_0^* = \sup_{x \in \Omega} v_0(x).
\end{cases}
\]

Then \((u_1(x, t), v_1(x, t)) = (0, 0)\) and \((u_2(x, t), v_2(x, t)) = \( (\bar{u}(t), \bar{v}(t)) \)\) are lower and upper solutions to system (4), since a simple calculation shows that

\[
\frac{\partial u_2(x, t)}{\partial t} - d_1 \Delta u_2(x, t) = \frac{u_2^2(x, t)v_2(x, t)}{1 + mv_2(x, t)} + ru_2(x, t) - 1 = 0
\]

and

\[
\frac{\partial v_2(x, t)}{\partial t} - d_2 \Delta v_2(x, t) - \beta + v_2(x, t) + \frac{\alpha u_2^2(x, t)v_2(x, t)}{1 + mv_2(x, t)} = 0
\]

On the other hand, we can see that

\[ 0 \leq u_0(x) \leq \sup_{x \in \Omega} u_0(x), \quad 0 \leq v_0(x) \leq \sup_{x \in \Omega} v_0(x). \]

In light of [23], system (4) is a mixed quasi-monotone system and has a unique globally defined solution \((u(x, t), v(x, t))\) satisfying \( 0 \leq u(x, t) \leq u^*(t) \) and \( 0 \leq v(x, t) \leq v^*(t) \) for \( t > 0 \) and \( x \in \bar{\Omega} \). The strong maximum principle gives \( u(x, t) > 0 \) and \( v(x, t) > 0 \) for \( t > 0 \) and \( x \in \bar{\Omega} \).

In what follows, we define a auxiliary function

\[ U_1(x, t) = \alpha u(x, t) + (1 + \epsilon_0)v(x, t), \]

then for any given constant \( \epsilon_0 \geq 0 \) we obtain

\[
\begin{aligned}
\frac{\partial U_1(x, t)}{\partial t} - \Delta U_1(x, t) &= \alpha + (1 + \epsilon_0)\beta - ruu(x, t) - v(x, t) \\
&\quad - \frac{\alpha \epsilon_0 u^2(x, t)v(x, t)}{1 + mv(x, t)} \\
&\quad \leq \alpha + (1 + \epsilon_0)\beta - ru(x, t) - v(x, t) \\
&\quad \leq \alpha + (1 + \epsilon_0)\beta - \min\{1, r\} U_1(x, t) \\
&\quad = \min\{1, r\} \left( \frac{\alpha + (1 + \epsilon_0)\beta}{\min\{1, r\}} - U_1(x, t) \right).
\end{aligned}
\]
We thus obtain
\[
\begin{aligned}
&\frac{\partial U_1(x,t)}{\partial t} - d \Delta U_1(x,t) \leq \min\{1,r\} \left( \frac{\alpha + (1 + \epsilon_0)\beta}{\min\{1,r\}} - U_1(x,t) \right), \quad x \in \Omega, \ t > 0, \\
&\frac{\partial U_1(x,t)}{\partial n} = 0, \quad x \in \partial \Omega, \ t \geq 0, \\
&U_1(x,0) = au_0(x) + (1 + \epsilon_0)v_0(x) \geq 0, \quad x \in \Omega.
\end{aligned}
\]

Then the standard comparison principle of parabolic equations gives that
\[
\limsup_{t \to \infty} \max_{x \in \Omega} (au(x,t) + v(x,t)) \leq \frac{\alpha + (1 + \epsilon_0)\beta}{\min\{1,r\}}.
\]

This completes the proof. \(\Box\)

**Theorem 2.2.** Suppose that parameters \(\beta, \alpha, r, m, d_1, d_2\) are positive in (4), then any non-negative solution \((u(x,t), v(x,t))\) of system (4) satisfies
\[
\limsup_{t \to \infty} \max_{x \in \Omega} u(\cdot,t) \leq C, \quad \limsup_{t \to \infty} \max_{x \in \Omega} v(\cdot,t) \leq \beta,
\]
where \(C\) is a positive constant independent of \(u_0(x), v_0(x), d_1\) and \(d_2\), but only dependent on a lower bound of \(d_1\).

**Proof.** From (4), \(v(x,t)\) satisfies
\[
\begin{aligned}
&\frac{\partial v}{\partial t} - d_2 \Delta v \leq \beta - v, \quad x \in \Omega, \ t > 0, \\
&\frac{\partial v}{\partial n} = 0, \quad x \in \partial \Omega, \ t \geq 0, \\
&v(x,0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{aligned}
\]

As a result, by the comparison principle of parabolic equations, for any \(\epsilon_1 > 0\), there exists \(T_1 > 0\), such that \(v(x,t) \leq \beta + \epsilon_1 := \tilde{v}\) for all \(x \in \Omega\) and \(t > T_1\). Now define \(P(t) = \int_{\bar{\Omega}} au(x,t)dx, Q(t) = \int_{\bar{\Omega}} v(x,t)dx\). Obviously, both \(P(t) > 0\) and \(Q(t) > 0\) due to Theorem 2.1. Then by virtue of (4) we yield
\[
\frac{d(P(t) + Q(t))}{dt} = \int_{\Omega} (\alpha + \beta - rau(x,t) - v(x,t))dx \\
= (\alpha + \beta)|\Omega| - rP(t) - Q(t) \\
\leq (\alpha + \beta)|\Omega| - \min\{1,r\}(P(t) + Q(t)).
\]

As a result, for any \(\epsilon_2 > 0\), there exists \(T_2 > 0\), such that
\[
\int_{\Omega} au(x,t)dx = P(t) < P(t) + Q(t) \leq \frac{(\alpha + \beta)|\Omega|}{\min\{1,r\}} + \epsilon_2,
\]
this means \(\limsup_{t \to \infty} \int_{\Omega} u(x,t)dx \leq \frac{(\alpha + \beta)|\Omega|}{\min\{1,r\}}\). Combining this with the similar argument in [28], we know that there is a positive constant \(C\) independent of \(u_0(x), v_0(x), d_1\) and \(d_2\), but dependent only on a lower bound of \(d_1\) such that \(u(x,t) \leq C + \epsilon_2 := \hat{v}\) is valid for \(x \in \Omega\) and \(t > T_2\). This completes the proof. \(\Box\)

Now one has the following result.

**Theorem 2.3.** Suppose that parameters \(\beta, \alpha, r, m, d_1, d_2\) are positive in (4) and \(\Omega \in \mathbb{R}^N\) is a bounded domain with smooth boundary. Then any non-negative solution \((u(x,t), v(x,t))\) of system (4) satisfies
\[
\liminf_{t \to \infty} \min_{x \in \Omega} u(\cdot,t) \geq \frac{4 - \vartheta^2 C_0}{4(r - \vartheta C_0)}, \quad \liminf_{t \to \infty} \min_{x \in \Omega} v(\cdot,t) \geq \frac{\beta}{1 + \alpha C^2},
\]
where \(\vartheta\) is any given positive constant satisfying \(\vartheta \leq \min\left\{\frac{\alpha}{\vartheta_0}, \frac{\beta}{\vartheta_0}\right\}\) and \(C_0 = \frac{\beta}{(1 + m\beta)(1 + \alpha C^2)}\).
Proof. From (4) and (5), it is noted that \( v(x, t) \) satisfies
\[
\begin{cases}
\frac{\partial v}{\partial t} - d_2 \Delta v \geq \beta - v - \alpha v(C + \varepsilon_2)^2, & x \in \Omega, \ t > T_2, \\
\frac{\partial v}{\partial n} = 0, & x \in \partial \Omega, \ t \geq T_2, \\
v(x, 0) = v_0(x) \geq 0, & x \in \Omega.
\end{cases}
\]
Hence, from the comparison principle of parabolic equations, for any \( \varepsilon_3 > 0 \), there exists \( T_3 > 0 \), such that
\[
v(x, t) \geq \frac{\beta}{1 + \alpha(C + \varepsilon_2)^2} + \varepsilon_3 := v^{-},
\]
for all \( x \in \bar{\Omega} \) and \( t > T_3 \).

From the first equation of (4), we have
\[
\begin{cases}
\frac{\partial u}{\partial t} - d_1 \Delta u \geq C_0 \left( \vartheta u - \frac{\vartheta^2}{4} \right) - ru + 1, & x \in \Omega, \ t > T_3, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, \ t \geq T_3, \\
u(x, 0) = u_0(x) \geq 0, & x \in \Omega,
\end{cases}
\]
where we denote by \( C_0 = \frac{u}{1 + \varepsilon_2} \), and \( \vartheta \) is any given positive constant satisfying \( \vartheta \leq \min \left\{ \frac{r}{C_0}, \frac{2}{\sqrt{C_0}} \right\} \). By the comparison principle of parabolic equations one deduces that for any \( \varepsilon_4 > 0 \), there exists \( T_4 > 0 \), such that
\[
u(x, t) \geq \frac{4 - \vartheta^2 C_0^e}{4(r - \vartheta C_0^e)} + \varepsilon_4 := u^{-},
\]
for all \( x \in \bar{\Omega} \) and \( t > T_4 \). This completes the proof. \( \Box \)

Remark 1. The obtained results exhibited in Theorem 2.3 indicate that the substances \( u(x, t) \) and \( v(x, t) \) will always coexist at any time \( t \) and any location \( x \) in the bounded domain \( \Omega \), no matter what the diffusion coefficients are. That means that the system has the persistence property for any non-negative initial concentration \((u_0(x), v_0(x))\) with \( u_0(x) \neq 0 \) and \( v_0(x) \neq 0 \).

2.2. Constant steady state and its stability. In this subsection, we discuss the existence and stability of the constant steady state solution of system (4). To this end, let
\[
\begin{align*}
f(u, v) &:= 1 - ru + \frac{u^2}{1 + mv}, \\
g(u, v) &:= \beta - v - \frac{\alpha u^2}{1 + mv}.
\end{align*}
\]
Consequently, system (4) has constant steady state solution if and only if \( f(u, v) = g(u, v) = 0 \). From \( f(u, v) = 0 \), one has
\[
v = \frac{ru - 1}{u^2 - rmu + m}.
\]
Putting it into \( g(u, v) = 0 \), we have
\[
\beta - \alpha (ru - 1) - \frac{ru - 1}{u^2 - rmu + m} = 0.
\]
It is equivalent to
\[
\psi(u) := \alpha ru^3 - (\beta + \alpha^2 m + \alpha)u^2 + (rm\beta + 2arm + r)u - (m\beta + am + 1) = 0.
\]
For convenience, denote by $\gamma_1 = \alpha r$, $\gamma_2 = - (\beta + \alpha r^2 m + \alpha)$, $\gamma_3 = r m \beta + 2 \alpha r m + r$, $\gamma_4 = -(m \beta + \alpha m + 1)$. Then, $\psi(u) = \gamma_1 u^3 + \gamma_2 u^2 + \gamma_3 u + \gamma_4 = 0$ has no negative real roots due to $\gamma_1 > 0$, $\gamma_2 < 0$, $\gamma_3 > 0$ and $\gamma_4 < 0$. Now set

$$A = \gamma_2^2 - 3 \gamma_1 \gamma_3 = \alpha m r^2 (\alpha m r^2 - 4 \alpha - \beta) - (3 r^2 + \alpha + 2 \beta) \alpha + \beta^2,$$

$$B = \gamma_2 \gamma_3 - 9 \gamma_1 \gamma_4 = m r (7 \alpha^2 - 6 \alpha \beta - \beta^2) - \alpha m^2 (2 \alpha m + \beta m + 1) + (8 \alpha - \beta),$$

$$C = \gamma_3^2 - 3 \gamma_2 \gamma_4 = \alpha m r^2 (\alpha m + \beta m + 1) + r^2 (\beta m + 1)^2 - 3 m (\alpha + \beta)^2 - 3 (\alpha + \beta),$$

and the discriminant of cubic equation is

$$\Delta = 27 \gamma_1^2 \gamma_3^2 - \gamma_2^2 \gamma_3^2 + 4 \gamma_1^3 \gamma_4 + 4 \gamma_1 \gamma_3^3 - 18 \gamma_1^2 \gamma_3 \gamma_4.$$ We have the following result about the root distribution of cubic equation $\psi(u) = 0$.

**Lemma 2.4.** Assume that $m, r, \alpha$ and $\beta$ are positive constants in (4), then the root distribution of cubic equation $\psi(u) = 0$ can be described as follows

(i) if $A = B = 0$, then $\psi(u) = 0$ has three equal positive real roots $u^*_1 = u^*_2 = u^*_3 = - \frac{\gamma_2}{3 \gamma_1} = - \frac{\gamma_3}{\gamma_2} = - \frac{3 \gamma_4}{\gamma_3}$.

(ii) if $\Delta > 0$, then $\psi(u) = 0$ has a unique positive real root

$$u^*_1 = - \frac{\gamma_2 - (\sqrt{R_1} + \sqrt{R_2})}{3 \gamma_1},$$

with

$$R_{1,2} = \gamma_2 (\gamma_2^2 - 3 \gamma_1 \gamma_3) + 3 \gamma_1 \left( - \frac{\gamma_2 \gamma_3 + 9 \gamma_1 \gamma_4 \pm \sqrt{\Delta}}{2} \right).$$

(iii) if $\Delta = 0$, then $\psi(u) = 0$ has two different positive real roots $u^*_1 = \frac{\gamma_2 \gamma_3 - 9 \gamma_1 \gamma_4}{\gamma_2^2 - 3 \gamma_1 \gamma_3} - \frac{\gamma_2}{\gamma_1}$, $u^*_2 = u^*_3 = - \frac{\gamma_2 \gamma_3 - 9 \gamma_1 \gamma_4}{2 (\gamma_2^2 - 3 \gamma_1 \gamma_3)}$.

(iv) if $\Delta < 0$, then $\psi(u) = 0$ has three different positive real roots

$$u^*_1 = - \frac{\gamma_2 - 2 \sqrt{7 \gamma_2^2 - 3 \gamma_1 \gamma_3 \cos(\theta)}}{3 \gamma_1},$$

$$u^*_{2,3} = - \frac{\gamma_2 + \sqrt{7 \gamma_2^2 - 3 \gamma_1 \gamma_3 (\cos(\theta) \pm \sqrt{3} \sin(\theta))}}{3 \gamma_1},$$

where

$$\theta = \arccos \left( \frac{2 \gamma_2 (\gamma_2^2 - 3 \gamma_1 \gamma_3) - 3 \gamma_1 (\gamma_2 \gamma_3 - 9 \gamma_1 \gamma_4)}{2 \sqrt{(\gamma_2^2 - 3 \gamma_1 \gamma_3)^3}} \right).$$

**Proof.** Shengjin formulas [6] can be used directly to obtain above results. \hfill \square

**Remark 2.** From Lemma 2.4, the root distribution of $\psi(u) = 0$ is given and the roots $u^*_i$ are positive for $i = 1, 2, 3$, however, we do not know the existence interval of positive equilibria of system (4). Note that $v = \beta - \alpha (ru - 1)$ by setting $f(u, v) = g(u, v) = 0$, that implies that if $E^*_1 = (u^*_1, v^*_1)$ is a positive constant steady state solution, then one has $v^*_1 = \beta - \alpha (ru^*_1 - 1)$. Therefore, if $0 < u^*_1 < \frac{\alpha + \beta}{r \alpha}$, then $v^*_1 > 0$. Furthermore, note that if $0 < u^*_1 < \frac{1}{r}$, then it always holds that $f(u^*_1, v^*_1) = 1 - ru^*_1 + \frac{(u^*_1)^2 v^*_1}{1 + m v^*_1} > 0$, this is a contradiction with $f(u^*_1, v^*_1) = 0$. Hence, it is necessary that $\frac{1}{r} < u^*_1 < \frac{\alpha + \beta}{r \alpha}$, so that system (4) admits positive equilibria.

The above analysis could be summarized as the following statement.
Theorem 2.5. Assume that m, r, α and β are positive constants in (4), u_i^*, A, B, θ and Δ are defined as in Lemma 2.4, then

(i) if A = B = 0 or Δ > 0, then system (4) has a unique positive equilibrium \( E_1^* = (u_1^*, v_1^*) \) with \( v_1^* = \beta - \alpha (ru_1^* - 1) \);

(ii) if Δ = 0, then system (4) has two positive equilibria \( E_i^* = (u_i^*, v_i^*) \), with \( v_i^* = \beta - \alpha (ru_i^* - 1) (i = 1, 2) \);

(iii) if Δ < 0, then system (4) has three positive equilibria \( E_i^* = (u_i^*, v_i^*) \), with \( u_i^* \) satisfying \( \frac{1}{r} < u_i^* < \frac{\alpha + \beta}{r \alpha} \).

Since the similar technique could be used to study the dynamics of different positive equilibria, for the sake of convenience we always assume that \( A = B = 0 \) or \( Δ > 0 \) such that system (4) has a unique positive equilibrium, denote this unique positive equilibrium by \( E_* = (\lambda, v_\lambda) \) and denote by \( \pi_u = \frac{1}{r} \) and \( \pi_v = \frac{\alpha + \beta}{r \alpha} \) such that \( \lambda \in (\pi_u, \pi_v) \).

Let \( \Xi := \{ \lambda_j \}_{j=0}^\infty \) be the complete set of eigenvalues of the operator \(-\Delta\) under the homogeneous Neumann boundary conditions in \( \Omega \), then one could arrange eigenvalues \( \lambda_i \in \Xi \) such that

\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \quad \text{and} \quad \lim_{i \to \infty} \lambda_i = \infty.
\] (8)

Define \( X_i \) to be the subspace generated by the eigenfunctions corresponding to eigenvalues \( \lambda_i \) for \( i = 0, 1, 2, \cdots \). \( X_i := \{ c \phi_{ij} : c \in \mathbb{R} \} \), where \( \{ \phi_{ij} \}_{j=1}^{m_j} \) is the orthonormal basis of subspace \( X_i \) in \( L^2(\Omega) \) for \( j = 1, 2, \cdots, m_j \) with \( m_j = \dim X_i \). Let \( X := \{ u, v \} \in \mathbb{C}[1(\Omega)] \times \mathbb{C}[1(\Omega)] \), \( \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \) on \( \partial \Omega \), then \( X = \bigoplus_{i=1}^\infty X_i \) with \( X_i = \bigoplus_{j=1}^{m_i} X_{ij} \). Particularly, if \( \Omega = (0, 1 \pi) \), then the eigenvalue \( \lambda_i \) is \( \frac{1}{r^2} \) and simple, with eigenfunction \( \cos \frac{1}{r} x \) and \( X_i = \text{span} \{ \cos \frac{1}{r^2} x \} \).

The linearization form of system (4) at \( E_* = (\lambda, v_\lambda) \) can be written as follows

\[
\dot{U} = D\Delta U + JU,
\] (9)

where \( D = \text{diag}(d_1, d_2) \), \( U = (u, v)^T \) and

\[
J = \begin{pmatrix}
\frac{2\lambda \nu}{1 + m \lambda} - r & \frac{\lambda^2}{(1 + m \lambda^2)} - 1 \\
-\frac{2a \lambda \nu}{1 + m \lambda} & -\frac{a \lambda^2}{(1 + m \lambda^2)} - 1
\end{pmatrix} := \begin{pmatrix}
f_u & f_v \\
g_u & g_v
\end{pmatrix},
\]

where

\[
f_u = \frac{2\lambda \nu}{1 + m \lambda} - r, \quad f_v = \frac{\lambda^2}{(1 + m \lambda^2)} > 0,
\]

\[
g_u = -\frac{2a \lambda \nu}{1 + m \lambda} < 0, \quad g_v = -\frac{a \lambda^2}{(1 + m \lambda^2)} - 1 < 0.
\]

Suppose that \((\Phi(x), \Psi(x))\) is an eigenfunction corresponding to the eigenvalue \( \mu \) of operator \( D \Delta + J \), where

\[
D \Delta + J = \begin{pmatrix}
f_u + d_1 \Delta & f_v \\
g_u & g_v + d_2 \Delta
\end{pmatrix},
\]

and

\[
\Phi = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_j} a_{ij} \phi_{ij} \quad \text{and} \quad \Psi = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_j} b_{ij} \phi_{ij}.
\]

Then we have

\[
\begin{pmatrix}
f_u + d_1 \Delta & f_v \\
g_u & g_v + d_2 \Delta
\end{pmatrix} \begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix} = \mu \begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix}.
\] (10)
It follows that
\[
\sum_{0 \leq i \leq \infty, 1 \leq j \leq m_j} \begin{pmatrix} f_u - d_1 \lambda_i & f_v \\ g_u & g_v - d_2 \lambda_i \end{pmatrix} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \phi_{ij} = \mu \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_j} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \phi_{ij}.
\]

Clearly, eigenvalue \( \mu \) needs to satisfy the following characteristic equation
\[
\mu^2 - T_i(\lambda_i, \lambda) \mu + D_i(\lambda_i, \lambda) = 0, \tag{11}
\]
where
\[
\begin{align*}
T_i(\lambda_i, \lambda) &= -(d_1 + d_2) \lambda_i + f_u + g_v, \\
D_i(\lambda_i, \lambda) &= d_1 d_2 \lambda_i^2 - (d_2 f_u + d_1 g_v) \lambda_i + f_u g_v - f_v g_u.
\end{align*}
\]

Therefore, for \( E_s = (\lambda, v_\lambda) = (\lambda, \beta - \alpha(r \lambda - 1)) \), we have
\[
T_i(\lambda_i, \lambda) = -(d_1 + d_2) \lambda_i + f_u + g_v = \frac{\lambda}{(1 + m v_\lambda)^2} h_1(\lambda) - (d_1 + d_2) \lambda_i - r - 1,
\]
with \( h_1(\lambda) = 2 m v_\lambda^2 + 2 v_\lambda - \alpha \lambda \), and
\[
D_i(\lambda_i, \lambda) = d_1 d_2 \lambda_i^2 + h_2(\lambda) \left( d_1 - \frac{d_2 h_3(\lambda)}{h_2(\lambda)(1 + m v_\lambda)} \right) \lambda_i + \frac{\lambda}{(1 + m v_\lambda)^2} [\alpha \lambda (r - 1) - h_1(\lambda)] + r,
\]
with
\[
h_2(\lambda) = \frac{\alpha \lambda^2 + (1 + m v_\lambda)^2}{(1 + m v_\lambda)^2} \quad \text{and} \quad h_3(\lambda) = 2 \alpha v_\lambda - r (1 + m v_\lambda).
\]

Now, differentiating \( h_1(\lambda) \) with respect to \( \lambda \), we have
\[
h_1'(\lambda) = -4 r m v_\lambda - 2 r \alpha - \alpha < 0.
\]

It is noticed that when \( 0 < \alpha < 2 r \beta (m \beta + 1) \), one yields
\[
\begin{align*}
h_1(\pi_u) &= h_1 \left( \frac{1}{r} \right) = \frac{1}{r} \left( 2 r m \beta^2 + 2 r \beta - \alpha \right) > 0, \\
\end{align*}
\]
\[
\begin{align*}
h_1(\pi_v) &= h_1 \left( \frac{\alpha + \beta}{r \alpha} \right) = - \frac{\alpha + \beta}{r} < 0.
\end{align*}
\]

Hence, there exists a unique \( \tilde{\lambda} \) satisfying \( \pi_u < \tilde{\lambda} < \pi_v \) such that \( h_1(\tilde{\lambda}) = 0 \). So, when \( \lambda \in (\pi_u, \tilde{\lambda}) \) we have \( h_1(\lambda) > 0 \) and \( h_1(\lambda) < 0 \) when \( \lambda \in (\tilde{\lambda}, \pi_v) \).

Differentiating \( h_3(\lambda) \) with respect to \( \lambda \), we obtain
\[
h_3'(\lambda) = 2 (\alpha + \beta) + \alpha m r^2 - 4 r \alpha \lambda,
\]
and \( h_3''(\lambda) = -4 r \alpha < 0 \). We know that \( h_3(\lambda) \) has its maximum point \( \lambda_2 := \frac{2 (\alpha + \beta) + m r^2 \alpha}{4 r \alpha} \), and \( \lambda_2 \in (\pi_u, \pi_v) \) if \( \beta > \frac{\alpha (2 - m r^2)}{2} \) and \( 0 < m r^2 < 2 \). That is to say, \( h_3'(\lambda) > 0 \) when \( \lambda \in (\pi_u, \lambda_2) \) and \( h_3'(\lambda) < 0 \) when \( \lambda \in (\lambda_2, \pi_v) \). Moreover, if
As a result, one claims that there exists a unique 
\( r \) and if \( \beta > \alpha \)

Assume that

that

have

\( b \)

\( \lambda \)

\( \tilde{\lambda} \)

Thus when

\( h \)

that

\( h \)

\( h(\bar{1}) \)

have no positive real parts. Hence, the positive equilibrium \( E \) is locally asymptotically stable. Namely, the positive equilibrium \( E^* = (\lambda, v^*_\lambda) \) is locally asymptotically stable.

On the other hand, through direct observation it is found that if \( f_u < 0 \) holds, then the positive equilibrium \( E^* \) is also locally asymptotically stable. To this end, differentiating \( f_u \) with respect to \( \lambda \), we have

\[
 f'_u(\lambda) = \frac{2mv^2_\lambda + 2v_\lambda - 2r\alpha \lambda}{(1 + mv_\lambda)^2}.
\]

Letting \( b(\lambda) := 2mv^2_\lambda + 2v_\lambda - 2r\alpha \lambda \) and differentiating it with respect to \( \lambda \), one yields

\[
 b'(\lambda) = -4r\alpha (1 + mv_\lambda) < 0,
\]

and if \( \beta > \alpha \), we have

\[
 b(\pi_u) = 2m\beta^2 + 2(\beta - \alpha) > 0,
\]

\[
 b(\pi_v) = -2(\beta + \alpha) < 0.
\]

Then there exists a unique \( \lambda^f \in (\pi_u, \pi_v) \) such that \( b(\lambda) = 0 \) when \( \lambda = \lambda^f \). We have \( b(\lambda) > 0 \) when \( \lambda \in (\pi_u, \lambda^f) \), and \( b(\lambda) < 0 \) when \( \lambda \in (\lambda^f, \pi_v) \). This means that \( f'_u(\lambda) > 0 \) when \( \lambda \in (\pi_u, \lambda^f) \), and \( f'_u(\lambda) < 0 \) when \( \lambda \in (\lambda^f, \pi_v) \). In addition, if \( \beta > \frac{r^2}{2mr^2} \) and \( 0 < mr^2 < 2 \), we have

\[
 f_u(\lambda^f) > f_u(\pi_u) = \frac{2r^2(1 + m\beta)}{r(1 + m\beta)} > 0,
\]

\[
 f_u(\pi_v) = -r < 0.
\]

As a result, one claims that there exists a unique \( \lambda^f \in (\lambda^f, \pi_v) \) such that \( f_u(\lambda^f) = 0 \), and \( f_u(\lambda) > 0 \) when \( \lambda \in (\lambda^f, \lambda^f) \) and \( f_u(\lambda) < 0 \) when \( \lambda \in (\lambda^f, \pi_v) \).

Summarizing the above analysis, we can get the following results.

**Theorem 2.6.** Assume that \( d_1, d_2, m, r \) and \( \beta \) are positive constants in (4) such that \( \pi_u < \lambda < \pi_v \) and \( 0 < mr^2 < 2 \) hold,
Corollary 1. For system (4) without diffusion,

(i) if $r \geq 1, 0 < \alpha < 2r\beta(m\beta + 1)$ and $\lambda \in (\bar{\lambda}, \pi_v)$, then $E_\ast = (\lambda, v_\lambda)$ is locally asymptotically stable;

(ii) if $0 < r < 1, 0 < \alpha < 2r\beta(m\beta + 1)$, $\beta > \max \left\{ \frac{\alpha(2-mr^2)}{2}, \frac{r^2}{2-mr^2} \right\}$ and $\lambda \in (\bar{\lambda}, \pi_v) \cap (\tilde{\lambda}, \pi_v)$, then $E_\ast = (\lambda, v_\lambda)$ is locally asymptotically stable;

(iii) if $\beta > \max \left\{ \alpha, \frac{r^2}{2-mr^2} \right\}$, then there exists a unique $\tilde{\lambda} \in (\lambda^1, \pi_v)$ satisfying $f_u(\lambda) < 0$ for $\lambda \in (\lambda^1, \pi_v)$, so that $E_\ast = (\lambda, v_\lambda)$ is locally asymptotically stable, where $\lambda, \tilde{\lambda}, \bar{\lambda}, \lambda^1$ and $\lambda^2$ are unique solutions of $h_1(\lambda) = 0, h_3(\lambda) = 0, h(\lambda^1) = 0, \Lambda(\tilde{\lambda}) = 0$ and $f_u(\lambda^2) = 0$, respectively.

From Theorem 2.6, when diffusion in system (4) disappears, the stability of the constant steady state solution $E_\ast = (\lambda, v_\lambda)$ can be presented as follows.
The upper panel of Figure 2 shows the existence of $\lambda$ in the plane of $\lambda - h_1(\lambda)$. By choosing suitable parameters, we find $h_1(\lambda)$ is strictly monotone decreasing in the interval $(\pi_u, \pi_v)$, and there exists a unique $\tilde{\lambda}$ in $(\pi_u, \pi_v)$ such that $h_1(\tilde{\lambda}) = 0$. Also, when $\lambda \in (\pi_u, \tilde{\lambda})$ we have $h_1(\lambda) > 0$, and $h_1(\lambda) < 0$ when $\lambda \in (\tilde{\lambda}, \pi_v)$. The lower left panel of Figure 2 shows the existence of $\tilde{\lambda}$ when all conditions in (i) of Theorem 2.6 are satisfied. It is found that the positive constant steady state solution $E^* = (u(x, t), v(x, t))$ is always strictly monotone decreasing in the interval $(\pi_u, \pi_v)$, and there exists a unique $\lambda^* \in (\pi_u, \pi_v)$ such that $b(\lambda) = 0$ when $\lambda = \lambda^*$, it follows that $b(\lambda) > 0$ when $\lambda \in (\pi_u, \lambda^*)$ and $b(\lambda) < 0$ when $\lambda \in (\lambda^*, \pi_v)$. All parameters can be found in the bottom of Figure 2. Choose parameters $d_1 = 1.5, d_2 = 0.5, r = 1.15, m = 0.02, \alpha = 0.1, \beta = 0.05$ and the initial condition is $(u(x, 0), v(x, 0)) = (0.902568 - 0.01 \cos(2x), 0.0462763 - 0.01 \cos(2x))$, so all conditions in (i) of Theorem 2.6 are satisfied. It is found that the positive constant steady state solution $E^* = (0.9022568, 0.0462763)$ is locally asymptotically stable, see Figure 3. This agrees with our theoretical analysis.

![Figure 3](image)

**Figure 3.** Positive constant steady state $E^* = (0.9022568, 0.0462763)$ is locally asymptotically stable. Here $d_1 = 1.5, d_2 = 0.5, r = 1.15, m = 0.02, \alpha = 0.1, \beta = 0.05$.

A global stability result of the constant steady state solution $E^* = (\lambda, v(\lambda))$ is presented below.
Theorem 2.7. Assume that $d_1, d_2, \beta, r$ constants in (4) are positive and $0 < m < \frac{1}{2\pi}$. If $(H_1)$ is satisfied, then $E_3 = (\lambda, v_\lambda)$ is globally asymptotically stable, where

$$(H_1) \max \left\{ 1 + \frac{2\alpha}{\lambda}, \frac{1}{1 - 2m\lambda} \right\} < \alpha < \frac{\lambda}{(mv_\lambda + 1)(\hat{u} + \lambda)},$$

and $\hat{u}, \hat{v}$ are defined in Theorem 2.2.

Proof. Define the Lyapunov function

$$V(t) = \int_\Omega \left( u - \lambda - \lambda \ln \frac{u}{\lambda} \right) dx + \int_\Omega (v - v_\lambda - v_\lambda \ln \frac{v}{v_\lambda}) dx.$$

Then the derivative of $V(t)$ along the any positive solution $(u, v)$ of system (4) is

$$\frac{dV(t)}{dt} = \int_\Omega \frac{\partial u}{\partial t} \left( 1 - \frac{\lambda}{u} \right) dx + \int_\Omega \frac{\partial v}{\partial t} \left( 1 - \frac{v_\lambda}{v} \right) dx$$

$$= \int_\Omega d_1 \frac{\partial^2 u}{\partial x^2} \left( 1 - \frac{\lambda}{u} \right) dx + \int_\Omega d_2 \frac{\partial^2 v}{\partial x^2} \left( 1 - \frac{v_\lambda}{v} \right) dx$$

$$+ \int_\Omega (u - \lambda) \left( \frac{1}{u} - r + \frac{uv}{1 + mv} \right) dx$$

$$+ \int_\Omega (v - v_\lambda) \left( \frac{\beta}{v} - 1 - \frac{\alpha u^2}{1 + mv} \right) dx$$

$$= I_1 + I_2 + I_3,$$

where

$$I_1 = \int_\Omega d_1 \frac{\partial^2 u}{\partial x^2} \left( 1 - \frac{\lambda}{u} \right) dx + \int_\Omega d_2 \frac{\partial^2 v}{\partial x^2} \left( 1 - \frac{v_\lambda}{v} \right) dx$$

$$= -d_1 \int_\Omega \nabla u \left( 1 - \frac{\lambda}{u} \right) dx - d_2 \int_\Omega \nabla v \left( 1 - \frac{v_\lambda}{v} \right) dx$$

$$= -d_1 \lambda \int_\Omega \frac{|
abla u|^2}{u^2} dx - d_2 \lambda \int_\Omega \frac{|
abla v|^2}{v^2} dx$$

$$\leq 0,$$

and

$$I_2 = \int_\Omega (u - \lambda) \left( \frac{1}{u} - r + \frac{uv}{1 + mv} \right) dx$$

$$= \int_\Omega (u - \lambda) \left( \frac{1}{u} + \frac{uv}{1 + mv} - \frac{\lambda}{1 + mv} \right) dx$$

$$= \int_\Omega (u - \lambda) \left[ \frac{\lambda - u}{\lambda u} + \frac{v(mv_\lambda + 1)(u - \lambda) + \lambda(v - v_\lambda)}{(1 + mv)(1 + mv_\lambda)} \right] dx$$

$$= -\int_\Omega \frac{(u - \lambda)^2}{\lambda u} dx + \int_\Omega \frac{v(mv_\lambda + 1)(u - \lambda)^2}{(1 + mv)(1 + mv_\lambda)} dx + \int \frac{\lambda(u - \lambda)(v - v_\lambda)}{(1 + mv)(1 + mv_\lambda)} dx.$$

For $I_3$, in a similar manner one gets

$$I_3 = \int_\Omega (v - v_\lambda) \left( \frac{\beta}{v} - 1 - \frac{\alpha u^2}{1 + mv} \right) dx$$

$$= \int_\Omega (v - v_\lambda) \left( \frac{\beta}{v} - \frac{\alpha u^2}{1 + mv} - \frac{\beta}{v_\lambda} + \frac{\alpha \lambda^2}{1 + mv_\lambda} \right) dx.$$
3. Properties of non-constant steady state solutions. In this section, we investigate the properties of the non-constant steady states of a local system. The corresponding steady state problem of (4) is the elliptic system and governed by

\[ -d_1 \Delta u(x) = \frac{u^2(x)\omega(x)}{1+mv(x)} - ru(x) + 1, \quad x \in \Omega, \]

\[ -d_2 \Delta v(x) = \beta - v(x) - \frac{\alpha u^2(x)\omega(x)}{1+mv(x)}, \quad x \in \Omega, \]

\[ \frac{\partial u(x)}{\partial n} = \frac{\partial v(x)}{\partial n} = 0, \quad x \in \partial \Omega. \]

3.1. Some estimates. To exhibit our results, we first state a lemma due to Lou and Ni [15].

**Lemma 3.1.** (Maximum Principle) Suppose that \( F(x, \omega(x)) \in C(\bar{\Omega} \times \mathbb{R}) \).
(i) If \( \omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) satisfies

\[
\Delta \omega(x) + F(x, \omega(x)) \geq 0, \text{ in } \Omega, \quad \frac{\partial \omega(x)}{\partial n} \leq 0 \text{ on } \partial \Omega,
\]
and \( \omega(x_0) = \max_{x \in \bar{\Omega}} \omega(x) \), then \( F(x_0, \omega(x_0)) \geq 0 \).

(ii) If \( \omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) satisfies

\[
\Delta \omega(x) + F(x, \omega(x)) \leq 0, \text{ in } \Omega, \quad \frac{\partial \omega(x)}{\partial n} \geq 0 \text{ on } \partial \Omega,
\]
and \( \omega(x_0) = \min_{x \in \bar{\Omega}} \omega(x) \), then \( F(x_0, \omega(x_0)) \leq 0 \).

**Theorem 3.2.** (A priori estimates) Assume that \((u(x), v(x))\) is an arbitrary positive solution of elliptic system (12), then \((u(x), v(x))\) satisfies

\[
\frac{r + \sqrt{r^2 - 4\beta}}{2c_\ast \beta} \leq u(x) \leq \frac{\alpha + \beta}{r \alpha} + \frac{d_2 \beta}{d_1 \alpha},
\]
and

\[
\frac{\beta}{1 + \alpha \left( \frac{\alpha + \beta}{r \alpha} + \frac{d_2 \beta}{d_1 \alpha} \right)^2} \leq v(x) \leq \beta
\]

for \( r \geq 2\sqrt{\beta} \) and \( c_\ast \) is a positive satisfying Harnack-type inequality.

**Proof.** Define

\[
u(x_0) = \min_{x \in \Omega} u(x), \quad u(x_1) = \max_{x \in \Omega} u(x), \quad v(y_0) = \min_{x \in \Omega} v(x), \quad v(y_1) = \max_{x \in \Omega} v(x).
\]

Then from the second equation of system (12) and Lemma 3.1, we have

\[
\beta - v(y_1) - \frac{\alpha u^2(y_1) v(x_1)}{1 + mv(y_1)} \geq 0,
\]

it follows that \( v(y_1) \leq \beta \), i.e., \( v(x) \leq \beta \) for \( x \in \Omega \). Now define

\[
\omega(x) = \alpha d_1 u(x) + d_2 v(x).
\]

A straightforward computation shows that

\[
\Delta \omega(x) = r c_\ast u(x) + v(x) - \alpha - \beta, \quad x \in \Omega, \quad \frac{\partial \omega(x)}{\partial n} = 0 \text{ on } \partial \Omega.
\]

Let \( \omega(z_1) = \max_{x \in \bar{\Omega}} \omega(x) \), then by Lemma 3.1 we obtain

\[
r c_\ast u(z_1) + v(z_1) - \alpha - \beta \leq 0,
\]

this leads to

\[
u(z_1) \leq \frac{\alpha + \beta}{r \alpha} - \frac{1}{r \alpha} v(z_1).
\]

From (13), one yields

\[
\alpha d_1 u(x_1) \leq \omega(x_1) \leq \omega(z_1) \leq \frac{(\alpha + \beta) d_1}{r} + \left( d_2 - \frac{d_1}{r} \right) v(z_1),
\]

namely

\[
u(x) \leq u(x_1) \leq \frac{\alpha + \beta}{r \alpha} + \frac{v(z_1)}{\alpha} \left( \frac{d_2}{d_1} - \frac{1}{r} \right) \leq \frac{\alpha + \beta}{r \alpha} + \frac{d_2 \beta}{d_1 \alpha}, \quad \forall x \in \Omega.
\]

Reconsider the first equation of (12), then one can give

\[
0 \leq \frac{u^2(x_1) v(x_1)}{1 + mv(x_1)} - ru(x_1) + 1 \leq \beta u^2(x_1) - ru(x_1) + 1.
\]
Thereby, one obtains
\[ u(x_1) = \max_{x \in \Omega} u(x) \geq \frac{r + \sqrt{r^2 - 4\beta}}{2\beta} \]
for \( r \geq 2\sqrt{\beta} \). Then by well-known Harnack-type inequality (for example, see [13]), we know that there is a positive constant \( c_* \) such that
\[ u(x_0) \geq \frac{1}{c_*} \max_{x \in \Omega} u(x) \geq \frac{r + \sqrt{r^2 - 4\beta}}{2c_*\beta}, \]
i.e., \( u(x) \geq \frac{r + \sqrt{r^2 - 4\beta}}{2c_*\beta} \) for \( x \in \bar{\Omega} \). Again using the second equation of system (12) and Lemma 3.1, we have
\[ \beta - v(y_0) - \frac{\alpha u^2(y_0)v(y_0)}{1 + mv(y_0)} \leq 0, \]
it leads to
\[ \beta - v(y_0) - \alpha u^2(x_1)v(y_0) \leq \beta - v(y_0) - \frac{\alpha u^2(y_0)v(y_0)}{1 + mv(y_0)} \leq 0. \]
Therefore \( v(y_0) \geq \frac{\beta}{1 + \alpha\left(\frac{\alpha^2 + 4\beta}{4\alpha^2}\right)} \), i.e., \( v(x) \geq \frac{\beta}{1 + \alpha\left(\frac{\alpha^2 + 4\beta}{4\alpha^2}\right)} \) for \( x \in \bar{\Omega} \). The proof is completed. \( \square \)

Let us discuss some properties of positive solution \((u(x), v(x))\) of system (12). To this end, one first denotes their averages over \( \Omega \) by
\[ \bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u(x)dx \text{ and } \bar{v} = \frac{1}{|\Omega|} \int_{\Omega} v(x)dx, \]
where \(|\Omega|\) is the volume of \( \Omega \). In addition, Theorem 3.2 tells us there exist positive constants \( C_1, C_2, C_3 \) and \( C_4 \) such that \( C_1 \leq u(x) \leq C_2, C_3 \leq v(x) \leq C_4 \). Keeping this in mind, we present some results as follows.

**Proposition 1.** If \((u(x), v(x))\) is the non-constant solution of system (12), then
\[
(i) \quad \int_{\Omega} (u - \bar{u})(v - \bar{v})dx = -b_0 \left( \int_{\Omega} |
abla \omega|^2 dx + r\alpha^2 d_1 \int_{\Omega} (u - \bar{u})^2 dx \right.
\]
\[ + d_2 \int_{\Omega} (v - \bar{v})^2 dx \left. \right) < 0. \]
\[ (ii) \quad \int_{\Omega} \nabla (u - \bar{u}) \cdot \nabla (v - \bar{v})dx = \frac{-r\alpha}{d_2} \int_{\Omega} (u - \bar{u})^2 dx \frac{1}{d_2} \int_{\Omega} (u - \bar{u})(v - \bar{v})dx \]
\[ - \frac{\alpha d_1}{d_2} \int_{\Omega} |
abla (u - \bar{u})|^2 dx, \]
where \( b_0 = \frac{1}{\alpha(rd_2 + d_1)} > 0. \)

**Proof.** If \((u(x), v(x))\) is the non-constant solution of elliptic system (12), then it is not difficult to derive
\[ \int_{\Omega} (u - \bar{u})dx = \int_{\Omega} (v - \bar{v})dx = 0, \]
and integrating (14) over \( \Omega \), we have
\[ \int_{\Omega} (r\alpha u(x) + v(x) - \alpha - \beta)dx = \int_{\Omega} \Delta \omega(x)dx = 0, \]
it then follows that

\[ r\alpha\bar{u} + \bar{v} = \alpha + \beta. \]

Hence, we have

\[ \Delta \omega(x) = r\alpha(u(x) - \bar{u}) + (v(x) - \bar{v}), \ x \in \Omega, \ \frac{\partial \omega(x)}{\partial n} = 0 \text{ over } \partial \Omega. \quad (15) \]

Multiplying (15) by (13) and integrating by parts yields

\[
-\int_{\Omega} |\nabla \omega|^2 dx = \int_{\Omega} [r\alpha(u - \bar{u}) + (v - \bar{v})][\alpha d_1 u + d_2 \bar{v}] dx \\
= r\alpha^2 d_1 \int_{\Omega} u(u - \bar{u}) dx + r\alpha d_2 \int_{\Omega} v(u - \bar{u}) dx \\
+ d_2 \int_{\Omega} v(v - \bar{v}) dx + \alpha d_1 \int_{\Omega} u(v - \bar{v}) dx \\
= r\alpha^2 d_1 \int_{\Omega} (u - \bar{u})^2 dx + r\alpha d_2 \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx \\
+ d_2 \int_{\Omega} (v - \bar{v})^2 dx + \alpha d_1 \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx \\
= r\alpha^2 d_1 \int_{\Omega} (u - \bar{u})^2 dx + d_2 \int_{\Omega} (v - \bar{v})^2 dx \\
+ \alpha (d_2 r + d_1) \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx.
\]

Therefore, we have

\[
\int_{\Omega} (u - \bar{u})(v - \bar{v}) dx = -b_0 \left( \int_{\Omega} |\nabla \omega|^2 dx + r\alpha^2 d_1 \int_{\Omega} (u - \bar{u})^2 dx + d_2 \int_{\Omega} (v - \bar{v})^2 dx \right).
\]

Now, we verify (ii). Multiplying (15) by \((u - \bar{u})\) and integrating by parts over \(\Omega\) leads to

\[
r\alpha \int_{\Omega} (u - \bar{u})^2 dx + \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx \\
= -\int_{\Omega} \nabla \omega \cdot \nabla (u - \bar{u}) dx \\
= -\int_{\Omega} \nabla (u - \bar{u}) \cdot \nabla (\alpha d_1 u + d_2 \bar{v}) dx - \alpha d_1 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx \\
- d_2 \int_{\Omega} \nabla (u - \bar{u}) \cdot \nabla (v - \bar{v}) dx,
\]

it implies that

\[
\int_{\Omega} \nabla (u - \bar{u}) \cdot \nabla (v - \bar{v}) dx \\
= -\frac{r\alpha}{d_2} \int_{\Omega} (u - \bar{u})^2 dx - \frac{1}{d_2} \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx - \frac{\alpha d_1}{d_2} \int_{\Omega} |\nabla (u - \bar{u})|^2 dx,
\]

the second part of the proposition is true. \(\square\)

**Corollary 2.** From (i) in Proposition 1 we know that \(\int_{\Omega} (u - \bar{u})(v - \bar{v}) dx < 0\). Hence, the sign of \(\int_{\Omega} \nabla (u - \bar{u}) \cdot \nabla (v - \bar{v}) dx\) has two cases.
Applying the same way to (ii) and (iii), we have
\[ r\alpha \int |\nabla (u - \bar u)|^2 dx + \alpha d_1 \int |\nabla (u - \bar u)|^2 dx < - \int (u - \bar u)(v - \bar v) dx. \]

Proof. For (i), it is noted that
\[ \int \omega^2 dx + \int |\nabla \omega|^2 dx \leq \frac{(1 + \lambda_1)(\alpha + \beta + r\alpha C_2 + C_4)^2|\Omega|}{\lambda_1^2}, \]
\[ \int (u - \bar u)^2 dx + \int |\nabla (u - \bar u)|^2 dx \leq \frac{(1 + \lambda_1)(1 + \frac{C_2^2}{m})^2|\Omega|}{d_1^2 \lambda_1^2}, \]
\[ \int (v - \bar v)^2 dx + \int |\nabla (v - \bar v)|^2 dx \leq \frac{(1 + \lambda_1)\beta^2|\Omega|}{d_2^2 \lambda_1^2}. \]

Proposition 2. Let \((u(x), v(x))\) be the non-constant solution to system (12), then
\[ (i) \int \omega^2 dx + \int |\nabla \omega|^2 dx \leq \frac{(1 + \lambda_1)(\alpha + \beta + r\alpha u - v)|\Omega|}{\lambda_1^2}, \]
\[ (ii) \int (u - \bar u)^2 dx + \int |\nabla (u - \bar u)|^2 dx \leq \frac{(1 + \lambda_1)(1 + \frac{C_2^2}{m})^2|\Omega|}{d_1^2 \lambda_1^2}, \]
\[ (iii) \int (v - \bar v)^2 dx + \int |\nabla (v - \bar v)|^2 dx \leq \frac{(1 + \lambda_1)\beta^2|\Omega|}{d_2^2 \lambda_1^2}. \]

In view of Cauchy-Schwarz inequality, system (12) and Theorem 3.2, one obtains
\[ -\int |\nabla \omega|^2 dx = \int (ad_1 \Delta u + d_2 \Delta v)\omega dx = -\int (\alpha + \beta - r\alpha u - v)\omega dx. \]

Applying the same way to (ii) and (iii), we have
\[ d_1 \int |\nabla (u - \bar u)|^2 dx = \int (u - \bar u) \left( 1 - r u + \frac{u^2 v}{1 + m v} \right) dx \]
\[ \leq \left( 1 + \frac{C_2^2}{m} \right) \int |u - \bar u| dx \]
\[ \leq \left( 1 + \frac{C_2^2}{m} \right) \sqrt{|\Omega|} \left( \int |u - \bar u|^2 dx \right)^{\frac{1}{2}}, \]
and
\[ d_2 \int |\nabla (v - \bar v)|^2 dx = \int (v - \bar v) \left( \beta - v - \frac{\alpha u^2 v}{1 + m v} \right) dx \]
\[ \leq \beta \int |v - \bar v| dx \]
\[ \leq \beta \sqrt{|\Omega|} \left( \int |v - \bar v|^2 dx \right)^{\frac{1}{2}}. \]
The Poincaré’s inequality shows the following facts
\[
\int_{\Omega} \omega^2 dx \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla \omega|^2 dx,
\]
and
\[
\int_{\Omega} (u - \bar{u})^2 dx \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla (u - \bar{u})|^2 dx, \quad \int_{\Omega} (v - \bar{v})^2 dx \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla (v - \bar{v})|^2 dx,
\]
where \(\lambda_1\) is the first non-zero eigenvalue of the operator \(-\Delta\) under the homogeneous Neumann boundary conditions in \(\Omega\), see (8). Then, we have
\[
\int_{\Omega} |\nabla \omega|^2 dx \leq \frac{(\alpha + \beta + r\alpha C_2 + C_4)\sqrt{\lambda_1|\Omega|}}{\lambda_1} \left( \int_{\Omega} |\nabla \omega|^2 dx \right)^{\frac{1}{2}},
\]
\[
\int_{\Omega} |\nabla (u - \bar{u})|^2 dx \leq \frac{(1 + \frac{c_2^2}{m})^2 \sqrt{\lambda_1|\Omega|}}{d_1^2 \lambda_1} \left( \int_{\Omega} |\nabla (u - \bar{u})|^2 dx \right)^{\frac{1}{2}},
\]
\[
\int_{\Omega} |\nabla (v - \bar{v})|^2 dx \leq \frac{\beta \sqrt{\lambda_1|\Omega|}}{d_2^2 \lambda_1} \left( \int_{\Omega} |\nabla (v - \bar{v})|^2 dx \right)^{\frac{1}{2}}.
\]
This leads to the following results
\[
\int_{\Omega} |\nabla \omega|^2 dx \leq \frac{(\alpha + \beta + r\alpha C_2 + C_4)^2|\Omega|}{\lambda_1},
\]
\[
\int_{\Omega} |\nabla (u - \bar{u})|^2 dx \leq \frac{(1 + \frac{c_2^2}{m})^2 |\Omega|}{d_1^2 \lambda_1},
\]
\[
\int_{\Omega} |\nabla (v - \bar{v})|^2 dx \leq \frac{\beta^2|\Omega|}{d_2^2 \lambda_1}.
\]
Again employing the Poincaré’s inequality, we have
\[
\int_{\Omega} \omega^2 dx + \int_{\Omega} |\nabla \omega|^2 dx \leq \frac{(1 + \lambda_1)(\alpha + \beta + r\alpha C_2 + C_4)^2|\Omega|}{\lambda_1^2},
\]
\[
\int_{\Omega} (u - \bar{u})^2 dx + \int_{\Omega} |\nabla (u - \bar{u})|^2 dx \leq \frac{(1 + \lambda_1)(1 + \frac{c_2^2}{m})^2 |\Omega|}{d_1^2 \lambda_1^2},
\]
\[
\int_{\Omega} (v - \bar{v})^2 dx + \int_{\Omega} |\nabla (v - \bar{v})|^2 dx \leq \frac{(1 + \lambda_1)\beta^2 |\Omega|}{d_2^2 \lambda_1^2}.
\]
The results in the proposition are true. \(\square\)

3.2. Non-existence of non-constant steady state solutions. In this subsection, some results about the nonexistence of nonconstant steady state solutions of system (12) are given, which are stated in the following three theorems.

**Theorem 3.3.** Suppose that \(r, m, \alpha, \beta, d_1, d_2\) are positive constants and \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary, then system (12) has no non-constant steady state if one of following conditions holds,
(i) if \(d_1 > d_1^*\) and \(d_2 > d_2^*\), where
\[
d_1^* := \frac{C_2 C_4 (4 + mC_2 + 2\alpha^2 C_2 C_4)}{2\lambda_1}, \quad d_2^* := \frac{mC_2^2 C_4 (1 + 2\alpha)}{2\lambda_1};
\]
(ii) if \(\frac{C_2^2 C_4 (2r \lambda_1 d_1^2 + d_1)}{2r \lambda_1 d_1^2 (d_2 \lambda_1 - m\alpha C_2 C_4)} < 1\) and \(d_2 > \frac{m\alpha C_2 C_4}{\lambda_1}\).
Proof. Let \((u(x), v(x))\) be a non-negative steady state of system (12). Then, multiplying the first equation of (12) by \((u - \bar{u})\) and integrating by parts over domain \(\Omega\), one obtains

\[
d_1 \int_\Omega |\nabla (u - \bar{u})|^2 dx = \int_\Omega (u - \bar{u}) \left(1 - ru + \frac{u^2 v}{1 + mv}\right) dx
\]

\[
= \int_\Omega (u - \bar{u}) dx - r \int_\Omega u(u - \bar{u}) dx + \int_\Omega \frac{u^2 v}{1 + mv} (u - \bar{u}) dx
\]

\[
= -r \int_\Omega (u - \bar{u})^2 dx + \int_\Omega \left(\frac{u^2 v}{1 + mv} - \frac{u^2 \bar{v}}{1 + m\bar{v}}\right) (u - \bar{u}) dx
\]

\[
+ \int_\Omega \left(\frac{u^2 \bar{v}}{1 + m\bar{v}} - \frac{u^2 \bar{v}}{1 + m\bar{v}}\right) (u - \bar{u}) dx,
\]

\[
= -r \int_\Omega (u - \bar{u})^2 dx + \int_\Omega \bar{v}(u + \bar{u}) (u - \bar{u})^2 dx
\]

\[
- \int_\Omega \frac{m u^2 \bar{v}}{(1 + mv)(1 + m\bar{v})} (u - \bar{u})(v - \bar{v}) dx
\]

\[
+ \int_\Omega \frac{u^2}{1 + mv} (u - \bar{u})(v - \bar{v}) dx.
\]  

(16)

Similarly, one has

\[
d_2 \int_\Omega |\nabla (v - \bar{v})|^2 dx = \int_\Omega (v - \bar{v}) \left(\beta - v - \frac{\alpha u^2 v}{1 + mv}\right) dx
\]

\[
= \beta \int_\Omega (v - \bar{v}) dx - \int_\Omega v(v - \bar{v}) dx - \int_\Omega \frac{\alpha u^2 v}{1 + mv} (v - \bar{v}) dx
\]

\[
= -\int_\Omega (v - \bar{v})^2 dx - \int_\Omega \left(\frac{\alpha u^2 v}{1 + mv} - \frac{\alpha u^2 \bar{v}}{1 + m\bar{v}}\right) (v - \bar{v}) dx
\]

\[
- \int_\Omega \left(\frac{\alpha u^2 \bar{v}}{1 + m\bar{v}} - \frac{\alpha u^2 \bar{v}}{1 + m\bar{v}}\right) (v - \bar{v}) dx
\]

\[
= -\int_\Omega (v - \bar{v})^2 dx - \int_\Omega \frac{\alpha u^2}{1 + mv} (v - \bar{v})^2 dx
\]

\[
- \int_\Omega \frac{\alpha \bar{v}(u + \bar{u})}{1 + m\bar{v}} (u - \bar{u})(v - \bar{v}) dx
\]

\[
+ \int_\Omega \frac{\alpha m u^2 \bar{v}}{(1 + mv)(1 + m\bar{v})} (v - \bar{v})^2 dx.
\]  

(17)

From Theorem 3.2, we know

\[
\frac{\bar{v}(u + \bar{u})}{1 + m\bar{v}} \leq 2C_2C_4, \quad \left|\frac{m u^2 \bar{v}}{(1 + mv)(1 + m\bar{v})}\right| \leq mC_2^2C_4.
\]

It then follows that

\[
d_1 \int_\Omega |\nabla (u - \bar{u})|^2 dx = -r \int_\Omega (u - \bar{u})^2 dx + \int_\Omega \frac{\bar{v}(u + \bar{u})}{1 + m\bar{v}} (u - \bar{u})^2 dx
\]

\[
- \int_\Omega \frac{m u^2 \bar{v}}{(1 + mv)(1 + m\bar{v})} (u - \bar{u})(v - \bar{v}) dx
\]

\[
+ \int_\Omega \frac{u^2}{1 + mv} (u - \bar{u})(v - \bar{v}) dx
\]
\[ \leq 2C_2C_4 \int_{\Omega} (u - \bar{u})^2 dx + mC_2^2C_4 \int_{\Omega} |u - \bar{u}| |v - \bar{v}| dx, \]

and
\[
\begin{aligned}
d_2 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx &= - \int_{\Omega} (v - \bar{v})^2 dx - \int_{\Omega} \frac{\alpha u^2}{1 + m\bar{v}} (v - \bar{v})^2 dx \\
& \quad - \int_{\Omega} \frac{\alpha u^2}{1 + m\bar{v}} (u - \bar{u})(v - \bar{v}) dx \\
& \quad + \int_{\Omega} \frac{\alpha m\bar{v}u^2}{(1 + m\bar{v})(1 + m\bar{v})} (v - \bar{v})^2 dx \\
& \leq (m\alpha C_2^2 C_4 - 1) \int_{\Omega} (v - \bar{v})^2 dx \\
& \quad + 2\alpha C_2 C_4 \int_{\Omega} |u - \bar{u}| |v - \bar{v}| dx.
\end{aligned}
\]

The Cauchy-Schwarz inequality, Poincaré’s inequality and Young’s inequality give
\[
\begin{aligned}
d_1 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx &\leq 2C_2C_4 \int_{\Omega} (u - \bar{u})^2 dx + mC_2^2C_4 \int_{\Omega} |u - \bar{u}| |v - \bar{v}| dx \\
&\leq 2C_2C_4 \int_{\Omega} (u - \bar{u})^2 dx + mC_2^2C_4 \left( \int_{\Omega} (u - \bar{u})^2 dx \int_{\Omega} (v - \bar{v})^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{C_2C_4(4 + mC_2)}{2} \int_{\Omega} (u - \bar{u})^2 dx + \frac{mC_2^2C_4}{2} \int_{\Omega} (v - \bar{v})^2 dx \\
&\leq \frac{C_2C_4(4 + mC_2)}{2\lambda_1} \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + \frac{mC_2^2C_4}{2\lambda_1} \int_{\Omega} |\nabla (v - \bar{v})|^2 dx, \tag{18}
\end{aligned}
\]
and
\[
\begin{aligned}
d_2 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx &\leq (m\alpha C_2^2 C_4 - 1) \int_{\Omega} (v - \bar{v})^2 dx + 2\alpha C_2 C_4 \int_{\Omega} |u - \bar{u}| |v - \bar{v}| dx \\
&\leq (m\alpha C_2^2 C_4 - 1) \int_{\Omega} (v - \bar{v})^2 dx + 2\alpha C_2 C_4 \left( \int_{\Omega} (u - \bar{u})^2 dx \int_{\Omega} (v - \bar{v})^2 dx \right)^{\frac{1}{2}} \\
&\leq \alpha^2 C_2^2 C_4 \int_{\Omega} (u - \bar{u})^2 dx + mC_2^2 C_4 \int_{\Omega} (v - \bar{v})^2 dx \\
&\leq \frac{\alpha^2 C_2^2 C_4}{\lambda_1} \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + \frac{m\alpha C_2^2 C_4}{\lambda_1} \int_{\Omega} |\nabla (v - \bar{v})|^2 dx. \tag{19}
\end{aligned}
\]

We therefore have an inequality as below
\[
\begin{aligned}
d_1 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + d_2 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx &\leq \frac{C_2C_4(4 + mC_2 + 2\alpha^2 C_2 C_4)}{2\lambda_1} \int_{\Omega} |\nabla (u - \bar{u})|^2 dx \\
&\quad + \frac{mC_2^2 C_4(1 + 2\alpha)}{2\lambda_1} \int_{\Omega} |\nabla (v - \bar{v})|^2 dx \\
= d_1^0 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + d_2^0 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx,
\end{aligned}
\]
where
\[ d_1^* := \frac{C_2C_4(4 + mC_2 + 2\alpha^2 C_2C_4)}{2\lambda_1}, \quad d_2^* := \frac{mC_2C_4(1 + 2\alpha)}{2\lambda_1}. \]

Therefore, if \( d_1 > d_1^* \) and \( d_2 > d_2^* \), then one has \( \nabla(u - \bar{u}) = \nabla(v - \bar{v}) = 0 \), this means \((u(x), v(x))\) must be a positive constant steady state of system (12).

Now, let us consider case (ii). First, multiplying equation (15) by \((u - \bar{u})\) and integrating by parts, one yields
\[
- \int_\Omega \nabla \omega \cdot \nabla (u - \bar{u})dx = r \alpha \int_\Omega (u - \bar{u})^2 dx + \int_\Omega (u - \bar{u})(v - \bar{v})dx,
\]
it then follows that
\[
\alpha d_1 \int_\Omega |\nabla (u - \bar{u})|^2 dx + d_2 \int_\Omega \nabla (u - \bar{u}) \cdot \nabla (v - \bar{v})dx
+ r \alpha \int_\Omega (u - \bar{u})^2 dx + \int_\Omega (u - \bar{u})(v - \bar{v})dx = 0.
\]

Then, from equation (13), that is \( \omega(x) = \alpha d_1 u(x) + d_2 v(x) \) we get
\[
\int_\Omega |\nabla \omega|^2 dx = \alpha^2 d_1^2 \int_\Omega |\nabla (u - \bar{u})|^2 dx + d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx
+ 2\alpha d_1 d_2 \int_\Omega \nabla (u - \bar{u}) \cdot \nabla (v - \bar{v})dx
= d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx - \alpha^2 d_1^2 \int_\Omega |\nabla (u - \bar{u})|^2 dx
- 2r \alpha^2 d_1 \int_\Omega (u - \bar{u})^2 dx - 2\alpha d_1 \int_\Omega (u - \bar{u})(v - \bar{v})dx.
\]

We notice that \( \int_\Omega |\nabla \omega|^2 dx \geq 0, \) then
\[
\alpha^2 d_1^2 \int_\Omega |\nabla (u - \bar{u})|^2 dx + 2r \alpha^2 d_1 \int_\Omega (u - \bar{u})^2 dx
\leq d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx + 2\alpha d_1 \int_\Omega |u - \bar{u}|v - \bar{v}|dx
\leq d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx + 2\alpha d_1 \left( \int_\Omega |u - \bar{u}|^2 dx \int_\Omega (v - \bar{v})^2 dx \right)^{1/2}
\leq d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx + 2r \alpha^2 d_1 \int_\Omega (u - \bar{u})^2 dx + \frac{d_1}{2r} \int_\Omega (v - \bar{v})^2 dx.
\]
The Poincaré’s inequality gives
\[
\alpha^2 d_1^2 \int_\Omega |\nabla (u - \bar{u})|^2 dx \leq d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx + \frac{d_1}{2r} \int_\Omega (v - \bar{v})^2 dx
\leq d_2^2 \int_\Omega |\nabla (v - \bar{v})|^2 dx + \frac{d_1}{2r \lambda_1} \int_\Omega |\nabla (v - \bar{v})|^2 dx.
\]
Accordingly,
\[
\int_\Omega |\nabla (u - \bar{u})|^2 dx \leq \frac{2r \lambda_1 d_2^2 + d_1}{2r \lambda_1 \alpha^2 d_1^2} \int_\Omega |\nabla (v - \bar{v})|^2 dx.
\]
From (19), one obtains
\[
\int_\Omega |\nabla (v - \bar{v})|^2 dx \leq \frac{\alpha^2 C_2^2 C_4^2}{d_2^2 \lambda_1 - m\alpha C_2^2 C_4} \int_\Omega |\nabla (u - \bar{u})|^2 dx.
\]
so inequality (20) becomes
\[
\int_{\Omega} |\nabla (u - \bar{u})|^2 dx \leq \frac{C_2^2 C_4^2 (2r \lambda_1 d_2^2 + d_1)}{2r \lambda_1 d_1^2 (d_2 \lambda_1 - \alpha C_2^2 C_4)} \int_{\Omega} |\nabla (u - \bar{u})|^2 dx.
\]
Therefore, if
\[
\frac{C_2^2 C_4^2 (2r \lambda_1 d_2^2 + d_1)}{2r \lambda_1 d_1^2 (d_2 \lambda_1 - \alpha C_2^2 C_4)} < 1 \quad \text{and} \quad d_2 > \frac{\alpha C_2^2 C_4}{\lambda_1},
\]
we have
\[
\int_{\Omega} |\nabla (u - \bar{u})|^2 dx = \int_{\Omega} |\nabla (v - \bar{v})|^2 dx = 0,
\]
namely \(\nabla (u - \bar{u}) = \nabla (v - \bar{v}) = 0\), that is to say, \((u(x), v(x))\) must be a positive constant steady state of system (12). The proof is completed. \(\square\)

**Theorem 3.4.** Suppose that \(r, m, \alpha, \beta, d_1, d_2\) are positive constants and \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary. Let \(\epsilon^{(1)} \ll 1\) be an arbitrary positive constant, then system (12) has no non-constant steady state solution when \(d_1 > d_1^{(1)}\) and \(d_2 > \frac{(\alpha + \epsilon^{(1)}) C^2_2}{\lambda_1}\), where \(d_1^{(1)}\) depends on \(r, m, \alpha, \beta, \epsilon^{(1)}\) and domain \(\Omega\).

**Proof.** Let \((u(x), v(x))\) be a positive solution of system (12). Then from equations (16), (17) and the \(\epsilon\)-Young’s inequality, one has
\[
d_1 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + d_2 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx
\]
\[
= \int_{\Omega} \left[ \frac{\bar{v}(u + \bar{u})}{1 + m \bar{v}} - r \right] (u - \bar{u})^2 dx + \int_{\Omega} \left[ \frac{\alpha u \bar{v}^2}{1 + m \bar{v}}(1 + m \bar{v}) - \frac{\alpha u^2}{1 + m \bar{v}} - 1 \right] (v - \bar{v})^2 dx
\]
\[
+ \int_{\Omega} \left[ \frac{u^2}{1 + m \bar{v}} - \frac{m u^2 \bar{v}}{(1 + m \bar{v})(1 + m \bar{v})} - \frac{\alpha \bar{v}(u + \bar{u})}{1 + m \bar{v}} \right] (u - \bar{u})(v - \bar{v}) dx
\]
\[
\leq \int_{\Omega} \left[ \frac{\bar{v}(u + \bar{u})}{1 + m \bar{v}} - r \right] (u - \bar{u})^2 dx + \int_{\Omega} \left[ \frac{\alpha u \bar{v}^2}{1 + m \bar{v}}(1 + m \bar{v}) \right] (v - \bar{v})^2 dx
\]
\[
+ \int_{\Omega} \left[ \frac{u^2}{1 + m \bar{v}} - \frac{m u^2 \bar{v}}{(1 + m \bar{v})(1 + m \bar{v})} \right] (u - \bar{u})(v - \bar{v}) dx
\]
\[
\leq \left( 2C_2 C_4 + C_2^2 C^{(1)} \right) \int_{\Omega} (u - \bar{u})^2 dx + \left( \alpha + \epsilon^{(1)} \right) C_2^2 \int_{\Omega} (v - \bar{v})^2 dx.
\]
Namely
\[
d_1 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + d_2 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx
\]
\[
\leq \left( 2C_2 C_4 + C_2^2 C^{(1)} \right) \int_{\Omega} (u - \bar{u})^2 dx + \left( \alpha + \epsilon^{(1)} \right) C_2^2 \int_{\Omega} (v - \bar{v})^2 dx. \tag{21}
\]
By the Poincaré’s inequality (21) becomes
\[
d_1 \int_{\Omega} |\nabla (u - \bar{u})|^2 dx + d_2 \int_{\Omega} |\nabla (v - \bar{v})|^2 dx
\]
where \( C(\varepsilon^{(1)}) \) large enough. Let \( d_1^{(1)} = \frac{2C_2C_4 + C_2^2C(\varepsilon^{(1)})}{\lambda_1} \), which depends on \( r, m, \alpha, \beta, \varepsilon^{(1)} \) and domain \( \Omega \). It is clear that if \( d_1 > d_1^{(1)} \) and \( d_2 > \frac{(\alpha + \varepsilon^{(1)})C_2^2}{\lambda_1} \), then one obtains \( \nabla(u - \bar{u}) = \nabla(v - \bar{v}) = 0 \). Thereby, \((u(x), v(x))\) must be a positive constant steady state of system (12).

**Theorem 3.5.** Suppose that \( r, m, \alpha, \beta, d_1, d_2 \) are positive constants and \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary, let \( \varepsilon^{(2)} \ll 1 \) be an arbitrary positive constant, then system (12) has no non-constant steady state solution when \( d_2 > d_2^{(1)} \) and \( d_1 > \frac{2C_2C_4 + C_2^2\varepsilon^{(2)}}{\lambda_1} \), where \( d_2^{(1)} \) depends on \( r, m, \alpha, \beta, \varepsilon^{(2)} \) and domain \( \Omega \).

**Proof.** Let \((u(x), v(x))\) be a positive solution of system (12). Then from equations (16), (17) and the \( \varepsilon \)-Young’s inequality, one has

\[
d_1 \int_{\Omega} |\nabla(u - \bar{u})|^2 dx + d_2 \int_{\Omega} |\nabla(v - \bar{v})|^2 dx \\
\leq \frac{2C_2C_4 + C_2^2C(\varepsilon^{(1)})}{\lambda_1} \int_{\Omega} |\nabla(u - \bar{u})|^2 dx + \frac{(\alpha + \varepsilon^{(1)})C_2^2}{\lambda_1} \int_{\Omega} |\nabla(v - \bar{v})|^2 dx,
\]

From the Poincaré’s inequality (22) becomes

\[
d_1 \int_{\Omega} |\nabla(u - \bar{u})|^2 dx + d_2 \int_{\Omega} |\nabla(v - \bar{v})|^2 dx \\
\leq \frac{2C_2C_4 + C_2^2\varepsilon^{(2)}}{\lambda_1} \int_{\Omega} |\nabla(u - \bar{u})|^2 dx + \frac{(\alpha + C(\varepsilon^{(2)}))C_2^2}{\lambda_1} \int_{\Omega} |\nabla(v - \bar{v})|^2 dx.
\]
3.3. Existence of non-constant steady state solutions. In this subsection, the existence of non-constant positive steady state to system (12) will be discussed by using the Leray-Schauder degree theory. Denote by \( E_* = (\lambda, v_\lambda) := u_* \), then elliptic system (12) becomes
\[
\begin{cases}
- D \Delta u(x) = w(u(x)), & x \in \Omega, \\
\frac{\partial u(x)}{\partial n} = 0, & x \in \partial \Omega,
\end{cases}
\]
where \( D = \text{diag}(d_1, d_2) \). It should be noticed that \( u \) is a solution of system (23) if and only if \( u \) satisfies
\[
F(d_1, d_2; u) := u - (I - \Delta)^{-1}\{D^{-1}w(u) + u\} = 0,
\]
where
\[
w(u(x)) = \begin{pmatrix}
\frac{u^*(x)v^*(x)}{1 + mu^*(x)} - ru(x) + 1 \\
\beta - v(x) - \frac{m^2(x)v(x)}{1 + mv(x)}
\end{pmatrix},
\]
and \((I - \Delta)^{-1}\) is the inverse operator of \( I - \Delta \) with respect to the homogeneous no-flux boundary conditions.

A straightforward calculation reveals that
\[
D_u F(d_1, d_2; u_*) = I - (\Delta)^{-1}\{D^{-1}w(u_*) + I\} = 0.
\]
It is noticed that \( \xi \) is an eigenvalue of \( D_u F(d_1, d_2; u_*) \) on \( X_j \) if and only if \( \xi(1 + \lambda_j) \) is an eigenvalue of the matrix
\[
L_j = \lambda_j I - D^{-1}w_u(u_*) = \begin{pmatrix}
\lambda_j - \frac{f_*}{\partial_1} & -\frac{g_*}{\partial_2} \\
-\frac{g_*}{\partial_1} & \lambda_j - \frac{f_*}{\partial_2}
\end{pmatrix}.
\]
We thus obtain the following results
\[
\begin{align*}
\text{Tr}(L_j) & = d_1^{-1}d_2^{-1}(2d_1d_2\lambda_j - d_2f_u - d_1g_v), \\
\text{Det}(L_j) & = d_1^{-1}d_2^{-1}[d_1d_2\lambda_j^2 - (d_2f_u + d_1g_v)\lambda_j + f_u g_v - f_v g_u].
\end{align*}
\]
Define
\[
M(d_1, d_2; \lambda_j) = d_1d_2\lambda_j^2 - (d_2f_u + d_1g_v)\lambda_j + f_u g_v - f_v g_u
\]
such that \( M(d_1, d_2; \lambda_j) = d_1d_2\text{Det}(L_j) \).

On the other hand, it is found that \( f_v > 0, g_u < 0, g_v < 0, g_u = -\alpha f_u - \rho a \) and \( g_v = -\alpha f_v - 1 \). Thereby, if \( 0 < f_u < \rho a f_v \) we get \( f_u g_v - f_v g_u > 0 \). As a result for large \( d_2 \), one has
\[
d_2 f_u + d_1 g_v > 2\sqrt{d_1d_2 (f_u g_v - f_v g_u)}.
\]
Then, we know \( M(d_1, d_2; \lambda_j) = 0 \) has two positive real roots
\[
\lambda_+(d_1, d_2) = \frac{d_2 f_u + d_1 g_v + \sqrt{(d_2 f_u + d_1 g_v)^2 - 4d_1d_2 (f_u g_v - f_v g_u)}}{2d_1d_2} > 0,
\]
\[
\lambda_-(d_1, d_2) = \frac{d_2 f_u + d_1 g_v - \sqrt{(d_2 f_u + d_1 g_v)^2 - 4d_1d_2 (f_u g_v - f_v g_u)}}{2d_1d_2} > 0.
\]
Therefore, for the large \( d_2 \), one obtains the following result.
Lemma 3.6. If \( d_1 \) is fixed and \( 0 < f_u < r a f_v \), then
\[
\lim_{d_2 \to \infty} \lambda_+(d_1, d_2) = \frac{f_u}{d_1} > 0 \quad \text{and} \quad \lim_{d_2 \to \infty} \lambda_-(d_1, d_2) = 0.
\]

Now define the set
\[
B(d_1, d_2) = \{ \lambda \geq 0 : \lambda_-(d_1, d_2) < \lambda < \lambda_+(d_1, d_2) \}.
\]
In order to compute the index of \( F(d_1, d_2; u) \), we introduce the following result [22].

Lemma 3.7. Assume that \( M(d_1, d_2; \lambda_j) \neq 0 \) for all \( \lambda_j \in \Xi \). Then
\[
\text{index}(F(d_1, d_2; u), u_*) = (-1)^\sigma,
\]
where
\[
\sigma = \begin{cases} 
\lambda_j \in \Xi \setminus B(d_1, d_2) & \text{if } \Xi \cap B(d_1, d_2) \neq \emptyset, \\
0 & \text{if } \Xi \cap B(d_1, d_2) = \emptyset.
\end{cases}
\]
In particular, if \( M(d_1, d_2; \lambda_j) > 0 \) for all \( \lambda > 0 \), then \( \sigma = 0 \).

We thus have the existence result of non-constant steady states to system (12).

Theorem 3.8. Suppose that \( 0 < f_u < r a f_v \) and \( d_1 > 0 \) is fixed. If \( \frac{d k}{d_1} \in (\lambda_k, \lambda_{k+1}) \) for some \( k \geq 1 \), and \( \sigma_k = \sum_{j=0}^{k} m_j \) is odd, then there exists a positive constant \( d_0 \) such that system (12) has at least one non-constant positive solution for all \( d_2 \geq d_0 \).

Proof. From Lemma 3.6 and the assumption \( \frac{d k}{d_1} \in (\lambda_k, \lambda_{k+1}) \) for some \( k \geq 1 \), there exists \( \hat{d}_2 > 1 \) such that (25) holds, and there exists \( d_0 > \hat{d}_2 \gg 1 \) such that
\[
0 < \lambda_-(d_1, \hat{d}_2) < \lambda_1, \quad \lambda_k < \lambda_+(d_1, \hat{d}_2) < \lambda_{k+1},
\]
for all \( d_2 \geq d_0 \). Hence, there is a positive constant satisfying \( \hat{d}_2 > d_0 \) such that
\[
0 < \lambda_-(d_1, \hat{d}_2) < \lambda_1, \quad \lambda_k < \lambda_+(d_1, \hat{d}_2) < \lambda_{k+1},
\]
for a small positive constant \( \varepsilon > 0 \). As a result, for a positive constant \( \hat{d}_1 \) satisfying \( \hat{d}_1 > \hat{d}_2 \), we obtain \( \frac{d k}{d_1} < \lambda_1 \) for all \( d_1 > \hat{d}_1 \). Hence, one deduces
\[
0 < \lambda_-(\hat{d}_1, \hat{d}_2) < \lambda_+(\hat{d}_1, \hat{d}_2) < \lambda_1.
\]

Now, we shall verify for any \( d_2 \geq d_0 \), the elliptic system (12) has at least one non-constant positive steady state solution. If not, that is, results in Theorem 3.8 are not true, then there exists a suitable \( d_2 \) such that the elliptic system (12) has no non-constant positive steady state solution for some \( d_2 \geq d_0 \). Therefore, for the above fixed \( d_1, d_2, d_1, d_2 \) and \( 0 \leq t \leq 1 \), we let
\[
D(t) = \begin{pmatrix} td_1 + (1-t)\hat{d}_1 & 0 \\ 0 & td_2 + (1-t)\hat{d}_2 \end{pmatrix},
\]
and consider the following problem
\[
\begin{cases}
-D(t)\Delta u(x) = w(u(x)), & x \in \Omega, \\
\frac{\partial u(x)}{\partial n} = 0, & x \in \partial\Omega,
\end{cases}
\]
where \( w(u(x)) \) can be found in (23).

It is clear that problem (28) is equivalent to the following problem
\[
\Psi(u; t) := u - (I - \Delta)^{-1}D^{-1}(t)w(u) + u = 0, \quad u \in X,
\]
Note that \( \Psi(u; 0) = F(d_1, d_2; u) \), \( \Psi(u; 1) = F(d_1, d_2; u) \),
and
\[
D_u F(d_1, d_2; u_*) = I - (I - \Delta)^{-1}\{\text{diag}(d_1, d_2)^{-1}w_u(u_*) + I\} = 0.
\]
From the above arguments we know that \( \Psi(u; 0) = 0 \) and \( \Psi(u; 1) = 0 \) have no non-constant solution. On the other hand, by (26) and (27), one has
\[
\Xi \cap B(d_1, d_2) = \emptyset, \quad \Xi \cap B(d_1, d_2) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\},
\]
this follows from Lemma 3.7 that
\[
\text{index}(\Psi(\cdot, 0), u_*) = \text{index}(F(d_1, d_2; \cdot), u_*) = (-1)^0 = 1,
\]
\[
\text{index}(\Psi(\cdot, 1), u_*) = \text{index}(F(d_1, d_2; \cdot), u_*) = (-1)^{\sigma_k}.
\]
Thus if \( \sigma_k \) is odd, one obtains
\[
\text{index}(\Psi(\cdot, 0), u_*) = \text{index}(F(d_1, d_2; \cdot), u_*) = 1,
\]
\[
\text{index}(\Psi(\cdot, 1), u_*) = \text{index}(F(d_1, d_2; \cdot), u_*) = -1.
\]

From Theorem 3.2, we deduce that there are positive constants \( C_1, C_2, C_3 \) and \( C_4 \) such that
\[
\Theta := \{(u, v)^T \in X : C_1 \leq u(x) \leq C_2, \quad C_3 \leq v(x) \leq C_4, \quad x \in \bar{\Omega}\}.
\]
Then for all \( u \in \partial \Theta \) and \( 0 \leq t \leq 1 \), we have \( \Psi(u; t) \neq 0 \). It then follows from the homotopy invariance of the Leray-Schauder degree that
\[
\text{deg}(\Psi(\cdot, 0), \Theta, 0) = \text{deg}(\Psi(\cdot, 1), \Theta, 0).
\] (29)
Note that the fact \( \Psi(u; 0) = 0 \) and \( \Psi(u; 1) = 0 \) have only the positive constant solution \( u_* \) in \( \Theta \), and hence
\[
\text{deg}(\Psi(\cdot, 0), \Theta, 0) = \text{index}(\Psi(\cdot, 0), u_*) = 1,
\]
\[
\text{deg}(\Psi(\cdot, 1), \Theta, 0) = \text{index}(\Psi(\cdot, 1), u_*) = -1,
\]
this leads to a contradiction to (29). The proof is finished. \( \square \)

4. Bifurcations. In this section, we shall derive the existence of the steady state bifurcation and the Hopf bifurcation for system (4). Assume that all eigenvalues \( \lambda_i \) are simple with the corresponding eigenfunction \( \phi_i(x), i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). As just stated before, such assumption is always satisfied when the domain \( \Omega = (0, \ell \pi) \), where \( \ell > 0 \) is a scalar. Denote by \( \lambda^S \) and \( \lambda^H \) the steady state bifurcation and the Hopf bifurcation critical value, respectively. To identify the bifurcation critical values, we recall the following conditions.

For the steady state bifurcation.

\((H_2)\) There exists some \( i \in \mathbb{N}_0 \) and some values \( \lambda^S \) such that
\[
D_i(\lambda^S) = 0, \quad T_i(\lambda^S) \neq 0 \quad \text{and} \quad D_j(\lambda^S) \neq 0, \quad T_j(\lambda^S) \neq 0, \quad i \neq j,
\]
and satisfying \( \frac{\partial}{\partial \lambda} D_i(\lambda) \neq 0 \).

For the Hopf bifurcation.
There exists some \( i \in \mathbb{N}_0 \) and some values \( \lambda^H \) such that
\[
T_i(\lambda^H) = 0, \quad D_i(\lambda^H) > 0 \quad \text{and} \quad T_j(\lambda^H) \neq 0, \quad D_j(\lambda^H) \neq 0, \quad i \neq j,
\]
and the unique pair of complex eigenvalues \( \alpha(\lambda) \pm i\omega(\lambda) \) near the imaginary axis satisfying
\[
\alpha'(\lambda^H) \neq 0, \quad \omega(\lambda^H) > 0.
\]

Before studying the steady state bifurcation and the Hopf bifurcation, we first derive some properties of bifurcation curves.

### 4.1. Properties of bifurcation curves
From Section 2, the stability of the unique positive constant steady state \( E^* = (\lambda, v_\lambda) \) is determined by the sign of \( T_i(\lambda_i, \lambda) \) and \( D_i(\lambda_i, \lambda) \). As is well known, the Hopf bifurcation and the steady state bifurcation are also determined by the trace \( T_i(\lambda_i, \lambda) \) and the determinant \( D_i(\lambda_i, \lambda) \), respectively. For the sake of convenience, let \( p := \lambda_i \), then
\[
T_i(p, \lambda) = \frac{\lambda h_1(\lambda)}{(1 + mv_\lambda)^2} - (d_1 + d_2)p - r - 1, \tag{30}
\]
and
\[
D_i(p, \lambda) = d_1d_2p^2 + h_2(\lambda) \left( d_1 - \frac{d_2h_3(\lambda)}{h_2(\lambda)(1 + mv_\lambda)} \right) p + \frac{\lambda}{(1 + mv_\lambda)^2} [\alpha \lambda(r - 1) - h_1(\lambda)] + r, \tag{31}
\]
with
\[
h_1(\lambda) = 2mv_\lambda^2 + 2v_\lambda - \alpha \lambda, \quad h_2(\lambda) = \frac{\alpha \lambda^2 + (1 + mv_\lambda)^2}{(1 + mv_\lambda)^2},
\]
\[
h_3(\lambda) = 2\lambda v_\lambda - r(1 + mv_\lambda).
\]

Now define two sets
\[
\Sigma_1 = \{(p, \lambda) \in \mathbb{R}^2_+ : T_i(p, \lambda) = 0\},
\]
\[
\Sigma_2 = \{(p, \lambda) \in \mathbb{R}^2_+ : D_i(p, \lambda) = 0\}
\]
to be the possible Hopf bifurcation curve and the steady state bifurcation curve, respectively.

For \( (p, \lambda) \in \Sigma_1 \), we have \( T_i(p, \lambda) = 0 \). Solving \( p \) from \( T_i(p, \lambda) = 0 \), one has
\[
p = \frac{\lambda h_1(\lambda) - (r + 1)(1 + mv_\lambda)^2}{(d_1 + d_2)(1 + mv_\lambda)^2} := \frac{\phi(\lambda)}{(d_1 + d_2)(1 + mv_\lambda)^2},
\]
where
\[
\phi(\lambda) = \lambda(2mv_\lambda^2 + 2v_\lambda - \alpha \lambda) - (r + 1)(1 + mv_\lambda)^2.
\]
The property of \( \phi(\lambda) \) with respect to \( \lambda \) will be explored. Differentiating \( \phi(\lambda) \) with respect to \( \lambda \) gives
\[
\phi'(\lambda) = 2mv_\lambda^2 + 2rma(m(r + 1) - 2\lambda)v_\lambda + 2v_\lambda - 2\alpha \lambda + 2ra(m(r + 1) - \lambda).
\]

It follows that
\[
\phi''(\lambda) = -2r^2a^2m(m(r + 1) - 2\lambda) - 8rma v_\lambda - 4ra - 2a,
\]
then if \( \alpha > \frac{2\beta}{mr(r+1)-2} \), we have \( \phi' (\lambda) < 0 \). In addition, if \( 1 - mr^2 > 0 \), \( 0 < \alpha < r \beta \) and \( mr(r+1) > 2 \), one yields

\[
\phi' (\pi_u) = \frac{2(r\beta - \alpha)}{r} + 2m\beta^2 + 2m\alpha (mr(r+1) - 2)\beta + 2\alpha (mr(r+1) - 1) > 0,
\]

\[
\phi' (\pi_v) = -\frac{2(r+1)[\beta + \alpha(1 - mr^2)]}{r} < 0.
\]

**Figure 4.** Graph of the curve \( \phi (\lambda) \). There exists a unique \( \lambda < \lambda < \pi_v \) such that \( \phi (\lambda) = 0 \). Thus, \( \phi (\lambda) > 0 \) when \( \lambda \in (\lambda, \lambda) \) and \( \phi (\lambda) < 0 \) when \( \lambda \in (\lambda, \pi_v) \), and \( \phi (\lambda) \) achieves its maximum \( \phi (\lambda) = \phi_* \) at \( \lambda = \lambda \) for \( \lambda \in (\pi_u, \pi_v) \). Here \( r = 0.25, m = 2.7, \alpha = 0.098, \beta = 0.5 \).

Then it is noticed that \( \phi' (\lambda) \) is a decreasing function with respect to \( \lambda \) when \( \alpha > \frac{2\beta}{mr(r+1)-2} \). So there exists a unique \( \pi_u < \lambda < \pi_v \) such that \( \phi' (\lambda) = 0 \). It then follows that \( \phi' (\lambda) > 0 \) when \( \lambda \in (\pi_u, \lambda) \) and \( \phi' (\lambda) < 0 \) when \( \lambda \in (\lambda, \pi_v) \), and \( \phi (\lambda) \) can achieve its maximum value at \( \lambda = \lambda \). Define \( \phi (\lambda) = \phi_{\text{max}} (\lambda) := \phi_* \) for \( \pi_u < \lambda < \pi_v \). In addition, if

\[
(H_4) \quad 0 < \alpha < (1 + m\beta)[2r\beta - r^2(r+1)(1+ m\beta)]
\]

holds, then one has

\[
\phi (\lambda) > \phi (\pi_u) = \frac{1}{r^2} [(1 + m\beta)(2r\beta - r^2(r+1)(1+ m\beta)) - \alpha] > 0,
\]

\[
\phi (\pi_v) = -\alpha \pi_v^2 - (r+1) < 0.
\]

It is found that \( \phi' (\lambda) < 0 \) when \( \lambda \in (\lambda, \pi_v) \), which implies \( \phi (\lambda) \) is a decreasing function. Then there exists a unique \( \lambda < \lambda < \pi_v \) such that \( \phi (\lambda) = 0 \). So \( \phi (\lambda) > 0 \) when \( \lambda \in (\lambda, \lambda) \), and \( \phi (\lambda) < 0 \) when \( \lambda \in (\lambda, \pi_v) \), for more details refer to Figure 4.

Summarizing the above analysis, we have the following result.

**Theorem 4.1.** Suppose that \( r, m, \alpha, \beta, d_1, d_2 \) are positive constants and \( (H_4) \) is satisfied, if

\[
1 - mr^2 > 0, \quad \frac{2\beta}{mr(r+1)-2} < \alpha < r \beta, \quad mr(r+1) > 2,
\]

then there exist \( \lambda \) and \( \overline{\lambda} \) satisfy \( \pi_u < \lambda < \overline{\lambda} < \pi_v \) such that

(i) when \( \lambda \in (\pi_u, \lambda) \), then \( \phi (\lambda) \) is increasing and decreasing for \( \lambda \in (\overline{\lambda}, \pi_v) \).
(ii) when \( \lambda \in (\pi_u, \bar{\lambda}) \), then \( \phi(\lambda) > 0 \) and \( \phi(\lambda) < 0 \) for \( \lambda \in (\bar{\lambda}, \pi_v) \), where \( \bar{\lambda} \) is the unique solution of \( \phi(\lambda) = 0 \).

Moreover, \( \phi(\lambda) \) achieves its maximum \( \phi(\bar{\lambda}) = \phi_* \) at \( \lambda = \bar{\lambda} \) for \( \lambda \in (\pi_u, \pi_v) \).

**Figure 5.** Left: there exist two positive constants \( \lambda_* \) and \( \lambda^* \) satisfying \( \pi_u < \lambda_* < \lambda^* < \pi_v \), such that \( \Delta(\lambda_*) = \Delta(\lambda^*) = 0 \) and \( \Delta(\lambda) > 0 \) when \( \lambda \in (\pi_u, \lambda_*) \cup (\lambda^*, \pi_v) \); Right: the existence of \( p_+(\lambda) \) and \( p_-(\lambda) \) in \( C_1 \), it is found that \( p_+(0) < 0, p_-(0) < 0 \) and \( \lim_{\lambda \to \lambda_{\min}} p_+(\lambda) = \lim_{\lambda \to \lambda_{\min}} p_-(\lambda) > 0 \). Here \( d_1 = 0.4, d_2 = 1, r = 0.6, m = 0.5, \alpha = 0.2, \beta = 1.25 \).

Now consider the steady state bifurcation curve \( D_i(p, \lambda) = 0 \). For \( (p, \lambda) \in \Sigma_2 \), we have \( D_i(p, \lambda) = 0 \). Solving \( p \) from \( D_i(p, \lambda) = 0 \), one yields

\[
p_+ (\lambda) := \frac{d_2 f_u + d_1 g_v + \sqrt{(d_2 f_u + d_1 g_v)^2 - 4d_1 d_2 (f_u g_v - f_v g_u)}}{2d_1 d_2},
\]

and

\[
p_-(\lambda) := \frac{d_2 f_u + d_1 g_v - \sqrt{(d_2 f_u + d_1 g_v)^2 - 4d_1 d_2 (f_u g_v - f_v g_u)}}{2d_1 d_2},
\]

where

\[
d_2 f_u + d_1 g_v = h_2(\lambda) \left( \frac{d_2 h_3(\lambda)}{h_2(\lambda)(1 + m v_\lambda)} - d_1 \right),
\]

\[
\Delta(\lambda) := (d_2 f_u + d_1 g_v)^2 - 4d_1 d_2 (f_u g_v - f_v g_u)
\]

\[
=h_2^2(\lambda) \left( d_1 - \frac{d_2 h_3(\lambda)}{h_2(\lambda)(1 + m v_\lambda)} \right)^2

- 4d_1 d_2 \left[ \frac{\lambda}{(1 + m v_\lambda)^2} (\alpha \lambda (r - 1) - h_1(\lambda)) + r \right].
\]

Note that positive \( p_+(\lambda) \) and \( p_-(\lambda) \) exist if and only if \( \Delta(\lambda) > 0 \) and \( d_2 f_u + d_1 g_v > 0 \). It could be found that there exist two positive constants \( \lambda_* \) and \( \lambda^* \) satisfying \( \pi_u < \lambda_* < \lambda^* < \pi_v \), such that \( \Delta(\lambda_*) = \Delta(\lambda^*) = 0 \) and \( \Delta(\lambda) > 0 \) when \( \lambda \in (\pi_u, \lambda_*) \cup (\lambda^*, \pi_v) \), see the left panel of Figure 5. Now it will be showed that \( d_2 f_u + d_1 g_v > 0 \). It is clear that \( h_2(\lambda) > 0 \) for any \( \lambda \in (\pi_u, \pi_v) \). Then define

\[
\varphi(\lambda) = \frac{d_2 h_3(\lambda)}{h_2(\lambda)(1 + m v_\lambda)} - d_1. \]

From the properties of \( h_3(\lambda) \) in Section 2, there exists a unique \( \bar{\lambda} \) satisfying \( \lambda_* < \bar{\lambda} < \pi_v \) such that \( h_3(\bar{\lambda}) = 0, h_3(\bar{\lambda}) > 0 \) when \( \lambda \in (\lambda_*, \bar{\lambda}) \).
and $h_3(\lambda) < 0$ when $\lambda \in (\bar{\lambda}, \pi_v)$. Also note that $\varphi(\bar{\lambda}) = -d_1 < 0$. On the other hand, differentiating $\varphi(\lambda)$ with respect to $\lambda$, we have

$$
\varphi'(\lambda) = \frac{d_2 h_2'(\lambda) h_2(\lambda)(1 + mv_\lambda) - d_2 h_3(\lambda)[h_2'(\lambda)(1 + mv_\lambda) + mh_2(\lambda)v'_\lambda]}{h_2(\lambda)(1 + mv_\lambda)^2}.
$$

Now one claims that $h_2'(\lambda)(1 + mv_\lambda) + mh_2(\lambda)v'_\lambda > 0$ if $\lambda \geq \frac{mr(1 + m(\alpha + \beta))}{m r^2 \alpha + 2} := \lambda^* 
\in (\lambda^*, \bar{\lambda})$. In fact,

$$
h_2'(\lambda)(1 + mv_\lambda) + mh_2(\lambda)v'_\lambda
= \frac{2\alpha \lambda(1 + mv_\lambda) + m r a^2 \lambda^2 - m r a(1 + mv_\lambda)^2}{(1 + mv_\lambda)^2}
= \frac{m r a^2 \lambda^2 + \alpha(1 + mv_\lambda)[(2 + m^2 r^2 \alpha)\lambda - m r (m(\alpha + \beta) + 1)]}{(1 + mv_\lambda)^2}
\geq \frac{m r a^2 \lambda^2}{(1 + mv_\lambda)^2} > 0.
$$

From analysis in Section 2, it holds that $h_0'(\lambda) < 0$ when $\lambda \in (\lambda^*, \pi_v)$, and $h_3(\lambda) > 0$ when $\lambda \in (\lambda^*, \bar{\lambda})$. That leads to $\varphi'(\lambda) < 0$, so $\varphi(\lambda)$ is strictly monotone decreasing for $\lambda \in (\lambda^*, \bar{\lambda})$. Moreover, if assuming that $d_2 > d_1$ and $\frac{h_3(\lambda^*)}{h_2(\lambda^*)(1 + mv_\lambda^*)} \geq 1$, then $\varphi(\lambda^*) = \frac{h_3(\lambda^*)}{h_2(\lambda^*)(1 + mv_\lambda^*)} d_2 - d_1 \geq d_2 - d_1 > 0$. Consequently, there must exist $\lambda_* \in (\lambda^*, \bar{\lambda})$ such that $\varphi(\lambda_*) = 0$, $\varphi(\lambda) > 0$ when $\lambda \in (\lambda^*, \lambda_*)$, and $\varphi(\lambda) < 0$ when $\lambda \in (\lambda_*, \bar{\lambda})$. In short, $p_+(\lambda)$ and $p_-(\lambda)$ exist and are positive when $\lambda \in (\pi_u, \lambda_u) \cap (\lambda^*, \lambda_*) := C_1$ or $\lambda \in (\pi_u, \pi_v) \cap (\lambda^*, \lambda_*) := C_2$. $\Delta(\lambda) = 0$ has at least two roots, denote the minimum root by $\lambda_{min}$. Now one has the following result.

**Theorem 4.2.** Suppose that $r, m, \alpha, \beta$ are positive, $d_2 > d_1$ and $\frac{h_3(\lambda^*)}{h_2(\lambda^*)(1 + mv_\lambda^*)} \geq 1$ are satisfied, $\lambda^*, \lambda_{min}, \lambda_*$, $p_+(\lambda), C_1$ and $C_2$ are defined as before, then there are some $\lambda \in C_1$ or $\lambda \in C_2$ such that $p_+(\lambda)$ exists. Moreover,

$$
\lim_{\lambda \rightarrow \lambda_{min}} p_+(\lambda) = \lim_{\lambda \rightarrow \lambda_{min}} p_-(\lambda) = \frac{h_2(\lambda_{min})/\varphi(\lambda_{min})}{2d_1 d_2} > 0,
$$

and

$$
p_+(0) = -\frac{r}{d_1} < 0, \quad p_-(0) = -\frac{1}{d_2} < 0.
$$

Take $d_1 = 0.4, d_2 = 1, r = 0.6, m = 0.5, \alpha = 0.2, \beta = 1.25$. Then $\lambda_u = \lambda_{min} = \lambda_\pi$. The second panel of Figure 5 shows the existence of $p_+(\lambda)$ and $p_-(\lambda)$ in the interval $(\lambda^*, \lambda_*) = (\lambda^*, \lambda_{min}) \subseteq C_1$. It is found that $p_+(\lambda) < 0$ and $p_-(\lambda) < 0$ when $\lambda = 0$. Furthermore, one could find $p_+(\lambda)$ first undergoes the monotonic increase and then the monotonic decrease in the interval $(\pi_u, \lambda_{min})$, and it is always monotonically decreasing in the interval $(\lambda^*, \lambda_{min})$. Meanwhile, $p_-(\lambda)$ is always monotonically increasing in the interval $(\lambda^*, \lambda_{min})$.

**4.2. The Hopf bifurcation.** In this subsection, the Hopf bifurcation of reaction-diffusion system (4) will be considered. To this end, condition $(H_3)$ should be satisfied.

From Theorem 4.1 we know that $\phi(\lambda) > 0$ when $\lambda \in (\pi_u, \bar{\lambda})$ and $\phi(\lambda) < 0$ for $\lambda \in (\bar{\lambda}, \pi_v)$. Therefore, $T_i(\lambda_i, \lambda) < 0$ when $\lambda \in (\bar{\lambda}, \pi_v)$. It means that the critical value of the Hopf bifurcation must be located in the interval $(\pi_u, \bar{\lambda})$. Define
\( \bar{\lambda} := \lambda_0^H \), then \( T_0(\lambda_0, \lambda_0^H) = \phi(\lambda_0^H) = 0 \) and \( D_0(\lambda_0, \lambda_0^H) = \frac{\alpha(r-1)(\lambda_0^H)^2}{(1+m\nu_\alpha \lambda_0^H)^2} - 1 > 0 \) if \( \alpha > \frac{(1+m\nu_\alpha \lambda_0^H)^2}{(r-1)(\lambda_0^H)^2} \) and \( r > 1 \). Now it is necessary to verify whether \( D_i(\lambda_i, \lambda^H_0) > 0 \) for \( i \geq 1 \). Recall that \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \) are the complete set of eigenvalues of the operator \( -\Delta \) under the homogeneous Neumann boundary conditions in \( \Omega \). Thus if \( (H_5) \) is true

\[
(H_5) \quad d_1 - \frac{d_2 h_3(\lambda^H_0)}{h_2(\lambda^H_0)(1+m\nu_\alpha \lambda^H_0)} > 0,
\]

one obtains \( D_0(\lambda_0, \lambda^H_0) < D_1(\lambda_1, \lambda^H_0) < D_2(\lambda_2, \lambda^H_0) < \cdots < D_i(\lambda_i, \lambda^H_0) < \cdots \).

Hence, if \( \alpha > \frac{(1+m\nu_\alpha \lambda^H_0)^2}{(r-1)(\lambda_0^H)^2} \), \( r > 1 \) and \( (H_5) \) are satisfied, then \( \lambda = \lambda^H_0 \) must be the spatially homogeneous Hopf bifurcation value.

Next we shall search for the Hopf bifurcation point with spatial inhomogeneity for \( i \geq 1 \). Again from Theorem 4.1, we know that \( \phi(\lambda) \) is decreasing and \( \phi(\lambda) > 0 \) for \( \lambda \in (\underline{\lambda}, \bar{\lambda}) \), so that one could find that \( \phi(\underline{\lambda}) = \phi_{\text{max}}(\lambda) = \phi_\ast > 0 \) and \( \phi(\lambda) = \phi(\lambda^H_0) = 0 \). The spatially inhomogeneous Hopf bifurcation point \( \lambda^H_0 \) is defined to be the solution of \( T_i(\lambda_1, \lambda) = 0 \) for \( \lambda \in (\underline{\lambda}, \lambda^H_0) \). Assume that \( n \) is the largest integer such that \( 0 < \lambda_n = p < \frac{\phi_\ast}{(d_1+d_2)(1+m\nu_\alpha \lambda^H_0)^2} \), then one could have \( n \) Hopf bifurcation points \( \lambda^H_0, \ldots, \lambda^H_n \) satisfying \( T_i(\lambda_i, \lambda^H_i) = 0, i = 0, 1, \ldots, n \). Now we will check that \( D_i(\lambda_i, \lambda^H_i) > 0 \) is true under certain conditions. Again from analysis in Section 2, \( h_3(\lambda) \) admits its maximum value at \( \lambda = \lambda_2 \in (\pi_u, \pi_v) \). Thus, if \( \lambda_2 \in (\underline{\lambda}, \lambda^H_0), h_3(\lambda) \)

also has its maximum value and denote by \( h_3(\lambda_2) := M \). In addition,

\[
h_2(\lambda^H_0) = \frac{\alpha(\lambda^H_0)^2 + (1+m\nu_\alpha \lambda^H_0)^2}{(1+m\nu_\alpha \lambda^H_0)^2} = 1 + \frac{\alpha(\lambda^H_0)^2}{(1+m\nu_\alpha \lambda^H_0)^2} > 1,
\]

and

\[
\lambda^H_0 h_1(\lambda^H_0) \frac{\lambda^H_0}{(1+m\nu_\alpha \lambda^H_0)^2} = (d_1 + d_2)\lambda_i + r + 1,
\]

from \( T_i(p, \lambda^H_i) = T_i(\lambda_i, \lambda^H_i) = 0 \). Hence if \( r > 1 \), from (30)-(33) one has

\[
D_i(\lambda_i, \lambda^H_i) = d_1d_2\lambda_i^2 + h_2(\lambda^H_0) \left( d_1 - \frac{d_2 h_3(\lambda^H_0)}{h_2(\lambda^H_0)(1+m\nu_\alpha \lambda^H_0)} \right) \lambda_i
\]

\[
+ \frac{\lambda^H_0}{(1+m\nu_\alpha \lambda^H_0)^2} \left[ \alpha \lambda_i^H(r-1) - h_1(\lambda^H_i) \right] + r
\]

\[
=d_1d_2\lambda_i^2 + h_2(\lambda^H_0) \left( d_1 - \frac{d_2 h_3(\lambda^H_0)}{h_2(\lambda^H_0)(1+m\nu_\alpha \lambda^H_0)} \right) \lambda_i
\]

\[
+ \frac{\alpha(r-1)(\lambda_i^H)^2}{(1+m\nu_\alpha \lambda^H_0)^2} - (d_1 + d_2)\lambda_i - 1
\]

\[
\geq d_1d_2\lambda_i^2 + (d_1 - d_2 h_3(\lambda^H_0)) \lambda_i - (d_1 + d_2)\lambda_i - 1
\]

\[
=d_1d_2\lambda_i^2 - d_2(1 + h_3(\lambda^H_0)) \lambda_i - 1
\]

\[
\geq d_1d_2\lambda_i^2 - d_2(1 + M)\lambda_i - 1.
\]
Obviously, if

\[(H_6) \ D_1(\lambda_1, \lambda_H^i) \geq d_1d_2\lambda_i^2 - d_2(1 + M)\lambda_1 - 1 > 0\]

holds, then \(D_i(\lambda_1, \lambda_H^i) > 0\) for all \(i \geq 1\). In fact, since \(0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots\) and \(\lim_{i \to \infty} \lambda_i = \infty\), this allows us to assume that \(\frac{\lambda_i}{\lambda_0} = \chi_0 > 1\) for \(i > 1\), i.e., \(\lambda_i = \chi_0\lambda_1(\chi_0 > 1)\). This then follows that

\[
D_i(\lambda_1, \lambda_H^i) \geq d_1d_2\chi_0(\lambda_1)^2 - d_2(1 + M)(\chi_0\lambda_1) - 1
\]

\[
= \chi_0(d_1d_2\lambda_1^2 - d_2(1 + M)\lambda_1) - 1
\]

\[
\geq \chi_0 - 1
\]

> 0.

From the above analysis and assumptions, one has \(T_i(\lambda_i, \lambda_H^i) = 0\), \(D_i(\lambda_i, \lambda_H^i) > 0\) and \(T_j(\lambda_i, \lambda_H^i) \neq 0\), \(D_j(\lambda_i, \lambda_H^i) \neq 0\) for any \(i \neq j\). As a result, let \(\Omega \subset \mathbb{R}^N\) be a bounded domain with sufficiently smooth boundary, the spectral set \(\Xi = \{\lambda_i\}_{i=0}^{\infty}\) satisfies

\[(R_1)\text{ there exists } n \in \mathbb{N}_0 \text{ such that}
\]

\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \frac{\phi_\alpha}{(d_1 + d_2)(1 + m\nu\lambda)^2} < \lambda_{n+1}.
\]

\[\text{Figure 6. There exist spatially homogeneous periodic solutions of system (4)}\]

From the above analysis, assumptions and [32], the main result is stated as follows.

**Theorem 4.3.** Suppose that \(m, \alpha, \beta, d_1, d_2\) are positive constants in (4). If \(r > 1\), \(\alpha > \frac{(1 + m\nu\lambda)^2}{(r-1)(\lambda_0^2)}\), \((H_5), (H_6)\) and \((R_1)\) are satisfied, then
\( \lambda = \xi \), such that its spectral set
\( \text{Theorem 4.4.} \) as follows.

(i) the periodic orbits bifurcating from \( (\lambda^H_i, \lambda^H_i, v_{\lambda^H_i}) \) are spatially homogeneous;
(ii) system (4) undergoes the Hopf bifurcation with spatial inhomogeneity at \( \lambda = \lambda^H_i \) for each \( 1 \leq i \leq n \), and there exists a smooth curve \( \Upsilon_i \) of positive periodic orbits of system (4) bifurcating from \( (\lambda, u, v) = \left( \lambda^H_i, \lambda^H_i, v_{\lambda^H_i} \right) \), and \( \Upsilon_i \) contained in a global bifurcation \( \mathcal{P}_i \) of positive nontrivial periodic orbits of system (4);
(iii) if all eigenvalues \( \lambda_i \) are simple, then near the point \( \lambda = \lambda^H_i \) the spatially inhomogeneous periodic orbits are in form of

\[
(\lambda, u, v) = \\
\left( \lambda^H_i + o(s), \lambda^H_i + s a_i \cos(\omega(\lambda^H_i)t)\psi_i(x) + o(s), v_{\lambda^H_i} + s b_i \cos(\omega(\lambda^H_i)t)\phi_i(x) + o(s) \right),
\]

for \( s \in (0, \delta), \) where \( \omega(\lambda^H_i) = \sqrt{D_i(\lambda_i, \lambda^H_i)} > 0 \) is the corresponding time frequency, \( (a_i, b_i) \) is the corresponding eigenvector, and \( \phi_i(x) \) is the corresponding spatial eigenfunction.

In order to verify that the existence of the periodic solutions of the reaction-diffusion system (4), the spatial domain is taken as \( \Omega = (0, d) \) and parameters \( d_1 = 1.5, d_2 = 0.362, r = 0.0621, m = 1.2, \alpha = 4.5, \beta = 1.5 \). It is found that a spatially homogeneous periodic solution will emerge, see Figure 6.

4.3. Steady state bifurcation. From the above analysis and Figure 5, we know that the steady state bifurcation curves exist only for \( \lambda \in C_1 \) or \( \lambda \in C_2 \) and \( p_+(\lambda) \) can reach its maximum, denoted by \( \bar{\rho} \). Now define the set

\[
\Sigma_3 = \{ \lambda \in C_1 \text{ or } \lambda \in C_2 : (H_2) \text{ is satisfied for some } i \in \mathbb{N}_0 \}.
\]

Then there exists some potential steady state bifurcation point \( \lambda^S_i \in \Sigma_3 \) such that \( D_i(\lambda_i, \lambda^S_i) = 0 \). However, it is possible that \( p_-(\lambda^S_i) = p_+(\lambda^S_i) \) such that \( D(p, \lambda) = 0 \) for some \( i \neq j \). For this case, Yi et al. [32] showed that there are only countably many \( \ell \) such that such situation holds for \( \Omega = (0, \ell \pi) \) and some \( i \neq j \) in one dimensional space. Moreover, it is easy to verify that \( \frac{\partial}{\partial \lambda} D_i(\lambda_i, \lambda)|_{\lambda^S_i} \neq 0 \) if \( \lambda^S_i \neq \lambda_{\text{min}} \). Therefore, from [32, 25] the steady state bifurcation result could be presented as follows.

**Theorem 4.4.** Let \( \Omega \subseteq \mathbb{R}^N \) be a bounded domain with sufficiently smooth boundary such that its spectral set \( \Xi = \{ \lambda_i \}_{i=0}^\infty \) satisfies

\( \text{(R2) there exists } n \in \mathbb{N} \text{ such that} \)

\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \bar{\rho} < \lambda_{n+1}.
\]

If \( \lambda \in C_1 \) or \( \lambda \in C_2 \), then there must exist some \( \lambda^S_i \in C_1 \) or \( \lambda^S_i \in C_2 \) such that \( D_i(\lambda_i, \lambda^S_i) = 0 \). Furthermore, if

\[
\lambda_i^S \neq \lambda_j^S, \ i \neq j \leq n; \ \lambda_i^S \neq \lambda_{\text{min}}, \ i \leq n,
\]

are satisfied, then

(i) there is a smooth curve \( \Gamma_i \) of the positive solutions of system (12) bifurcating from \( (\lambda, u, v) = (\lambda^S_i, \lambda^S_i, v_{\lambda^S_i}) \) with \( \Gamma_i \) contained in a global branch \( \mathcal{T}_i \) of the positive nontrivial solutions of system (12).

(ii) if all eigenvalues \( \lambda_i \) are simple, then near \( (\lambda, u, v) = (\lambda^S_i, \lambda^S_i, v_{\lambda^S_i}) \), \( \Gamma_i = \{(\lambda_i(s), u_i(s), v_i(s)) : s \in (-\varepsilon, \varepsilon)\}, \) where \( u_i(s) = \lambda_i^S + s a_i \phi_i(x) + sv_{\lambda^S_i}(s), v_i(s) = \lambda_i^S + s b_i \phi_i(x) + sv_{\lambda^S_i}(s) \) for some \( C^\infty \) smooth functions \( \lambda_i(s), \psi_{1,i}(s), \psi_{2,i}(s) \) such that
\( \lambda_i(0) = \lambda_i^S \) and \( \psi_{1,i}(0) = \psi_{2,i}(0) = 0 \), and \( (a_i, b_i) \) satisfies \( L(\lambda_i^S)((a_i, b_i)^T \phi_i(x)) = (0, 0)^T \).

Numerical simulations are also carried out to verify the existence of the steady state solution of system (4), here the spatial domain \( \Omega = (0, 1.5\pi) \) and the parameters \( d_1 = 8, d_2 = 0.5, r = 0.058, m = 0.15, \alpha = 4.0, \beta = 1.35 \), we therefore obtain \( E_* = (16.8013, 1.4521) \). It is found that with the evolution of time \( t \) the substances \( u(x, t) \) and \( v(x, t) \) will exhibit the steady state solution. Here we choose the initial values \( u(x, 0) = 16.8013 + 0.01 \cos(1.5x) \) and \( v(x, 0) = 1.4521 + 0.01 \cos(1.5x) \), see Figure 7 for details.

![Figure 7](image.png)

**Figure 7.** There exist steady state solutions of system (4). Here \( d_1 = 8, d_2 = 0.5, r = 0.058, m = 0.15, \alpha = 4.0, \beta = 1.35 \).

5. **Conclusions.** In this work, we research a depletion type Gierer-Meinardt model with Langmuir-Hinshelwood reaction scheme and the homogeneous Neumann boundary conditions. It is showed that the model introduces such Langmuir-Hinshelwood law can exhibit many complex dynamical behaviors. We first establish some boundedness results and find the persistence property of the parabolic system. For the positive constant steady state solutions, it is found that it is not easy to solve them from the cubic polynomial, their exact expressions can not be given by the Descarte’s rule of signs. Therefore, we use the Shengjin formulas to overcome such difficulty and give the expressions of the positive constant steady state solutions. Moreover, the local and global stability conditions of the corresponding positive constant steady state solution have been established by rigorous mathematical analysis and computations.

Then, a priori estimates are given and some properties of the non-constant steady state solution of the elliptic system are explored. The non-existence and existence of the non-constant steady state solutions of the model have been carried out by some estimates and the Leray-Schauder degree theory, respectively. The non-existence results of the non-constant steady state solution show that the diffusion rates \( d_1 \) and \( d_2 \) of substances \( u \) and \( v \) have an essential influence on their non-existence.

Finally, the Hopf bifurcation and the steady state bifurcation of the system are investigated, respectively. In fact, some properties of the Hopf bifurcation and the steady state bifurcation curves are first studied to obtain the existence interval of these bifurcations, and the spatially homogeneous periodic solution and the non-constant steady state solution can be observed in numerical simulations. Theoretical results show that such reaction-diffusion system can exhibit rich dynamical behaviors. However, it is worthy of further study in the future whether more complex Hopf-steady state bifurcation phenomena will appear in the system.
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