Kirchhoff-Type Problems Involving Logarithmic Nonlinearity with Variable Exponent and Convection Term

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Abstract. In the present article, we deal with a class of Kirchhoff-type equations involving a logarithmic nonlinearity and a convection term. Due to the lack of a variational structure, the well-known variational methods are not applicable. Using a topological approach based on the Galerkin method together with fixed point theorem, we obtain the existence of the finite-dimensional approximate solutions, generalized solutions, and strong generalized solutions. One of the main difficulties and innovations of present article is that we consider both convective term and logarithmic nonlinearity with variable exponents, another one is the weak assumptions on nonlocal term $M_{p(x)}$ and nonlinear term $g$, and finally, we discuss the existence of solutions for discontinuous Kirchhoff-type equations.

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1. Introduction

In recent decades, Kirchhoff-type equations or systems have been some of the classical topics in the qualitative analysis of partial differential equations, and the existence of solutions to these problems has been obtained under various assumptions on nonlinearity and Kirchhoff functions.

Kirchhoff in [12] introduced the following model, which came to be known as the Kirchhoff-type equation:

$$\rho \frac{\partial^2 \eta(x)}{\partial t^2} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial \eta(x)}{\partial t} \right|^2 \, dx \right) \frac{\partial^2 \eta(x)}{\partial x^2} = 0,$$

where parameters $\rho, p_0, h, E,$ and $L$ are real positive constants. Equation (1.1) is nonlocal problems which contains a nonlocal coefficient $\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial \eta(x)}{\partial t} \right|^2 \, dx$,
and can be used to model some physical systems in concrete real-world applications, such as, anomalous diffusion, ultra-relativistic of quantum mechanics and water waves. Since then the literature on Kirchhoff-type problems are quite large, here we just list a few, such as, see \[4,5,10,13,14,23,26,31,33\] for more details on mathematical theories and its applications.

The purpose of the present article is to study the existence of solutions for the following Kirchhoff-type equations driven by the $p(x)$-Laplacian:

$$(-\Delta)^{M_{p(x)}} \eta = \lambda |\eta|^{p(x)-2} \eta \ln |\eta| + g(x, \eta, \nabla \eta), \text{ in } \Omega, \quad \eta|_{\partial \Omega} = 0, \quad (1.2)$$

where $\lambda$ is a parameter, and $\Omega$ is an open bounded domain in $\mathbb{R}^N$ with smooth boundary.

Here, Kirchhoff function $M_{p(x)}$ is the following form:

$$M_{p(x)}(\cdot) = \int_{\Omega} |\nabla \eta|^{p(x)} \, dx$$

and $\Delta_{p(x)}$ is a $p(x)$-Laplace operator, which can be defined by

$$\Delta_{p(x)} \eta = \text{div}(|\nabla \eta|^{p(x)-2} \nabla \eta) = \sum_{i=1}^{N} \left( |\nabla \eta|^{p(x)-2} \frac{\partial \eta}{\partial x_i} \right), \quad (1.4)$$

for all $x \in \Omega$ and $\eta \in C^\infty_0(\mathbb{R}^N)$.

Hence, $\Delta_{M_{p(x)}}$ denotes the $p(x)$-Kirchhoff-type operator expressed as

$$(-\Delta)^{M_{p(x)}} \eta = -M_{p(x)} \Delta_{p(x)} \eta, \text{ for all } \eta \in W_0, \quad (1.5)$$

where $M_{p(x)} : \mathbb{R}^+_{0} \to \mathbb{R}^+$ and $p(x) : \mathbb{R}^N \to (1, +\infty)$ satisfy the following conditions

$M_1$: There exists a constant $m_0 = m_0(\tau) > 0$ for all $\tau > 0$, such that

$$M_{p(x)}(t) \geq m_0, \text{ for any } t > \tau.$$

$P$: The condition that we impose on the continuous function $p(x)$ is as follows:

$$1 < p^- := \inf_{x \in \mathbb{R}^N} p(x) \leq p^+ := \sup_{x \in \mathbb{R}^N} p(x) < +\infty.$$

For the framework of variable exponential, there have been some papers on this topic, for instance, see \[1,8,15,16,27,28,30,32\]. Most of these papers deal with problems for power-type nonlinearities; however, few papers consider the existence of solutions for both a logarithmic nonlinearity and a convection term.

One of the main features of the problem (1.2) is the presence of convection term $g(x, \eta, \nabla \eta)$, depending on the function $\eta$ and on its gradient $\nabla \eta$, which plays an important role in science and technology fields and is widely used to describe physical phenomena. For example, due to convection and diffusion processes, particles or energy are converted and transferred inside a physical system. For the work related to this topic, we cite the interesting work \[7,17,20,34\] and their references.
The work in [2] focuses on the $p$-Kirchhoff-type equations with gradient dependence in the reaction, that is
\[
\begin{cases}
-M\left(\int_{\Omega}|\nabla \eta(x)|^p \, dx\right) \Delta_p \eta(x) = g(x, \eta(x), \nabla \eta(x)), & \text{in } \Omega, \\
\eta(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. The existence of solutions for the problem (1.6) was obtained using Galerkin’s approach.

One more reference on convection is Vetro [6], which was devoted to the study of the following $p$-Kirchhoff-type equations:
\[
-\Delta^K_{p(x)} \eta(x) = g(x, \eta(x), \nabla \eta(x)), \quad \text{in } \Omega, \quad \eta|_{\partial \Omega} = 0.
\]
(1.7)
The existence of weak solutions and generalized solutions for the problem (1.7) with gradient dependence was gotten via applying a topological method.

The nonlinearity $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, satisfying
\[
\begin{align*}
\mathcal{G}_1: & \text{ There exists a positive function } \phi(x) \in L^{(p^*(x))'}(\Omega) \text{ and some positive constants } a \text{ and } b, \text{ such that} \\
& |g(x, \mu, \nu)| \leq \phi(x) + a|\mu|^{p^*(x)-1} + b|\nu|^{p(x)-1}, \forall (x, \mu, \nu) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \\
\mathcal{G}_2: & \text{ There exist some constants } c < 1, d > 0 \text{ and a function } \alpha \in [1, p^-), \text{ such that} \\
& g(x, \mu, \nu)\mu \leq c|\nu|^{p(x)} + d(|\mu|^{\alpha(x)} + 1), \forall (x, \mu, \nu) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.
\end{align*}
\]
Remark 1.1 Notice that the convection term $g(x, \eta, \nabla \eta)$ makes the problem (1.2) non-variational. To overcome this difficulty and tricky challenge, our approach is the Galerkin method combined with Brouwer’s fixed-point theorem.

Another significant characteristic of the problem (1.2) is the presence of logarithmic nonlinearity. The interest in studying problems with logarithmic nonlinearity is motivated not only by the purpose of describing mathematical and physical phenomena but also by their application in practical models. For instance, in the biological population, we use the function $\eta(x)$ to represent the density of the population, and the logarithmic nonlinear term $|\eta|^{p(x)-2}\eta \ln |\eta|$ to denote external influencing factors.

Many scholars make efforts to investigate logarithmic nonlinearity, and indeed, some important results were obtained, for example, see [3,18,25,36]. Peculiarly, Xiang et al. in [21] considered the following equations:
\[
\begin{cases}
M\left(\left|\eta\right|^{s,p}_\eta\right)(-\Delta)^{s,p} \eta = h(x)|\eta|^{\theta p-2}\eta \ln |\eta| + \lambda|\eta|^{q-2}\eta, & x \in \Omega, \\
\eta(x) = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
(1.8)
where $M\left(\left|\eta\right|^{s,p}_\eta\right) = \left|\eta\right|^{p(\theta-1)}_{s,p}$ and $h(x)$ is a sign-changing function. The existence of least energy solutions (1.8) was obtained by utilizing the Nehari manifold method.

Until now, there are few papers to handle the equations involving logarithmic nonlinearity with variable exponent. Recently, Boudjeriou in [29]
studied the following initial value problem:
\[
\begin{aligned}
\eta_t(x) - \Delta_{p(x)} \eta(x) &= |\eta(x)|^{s(x)-2} \eta(x) \log(|\eta(x)|), \quad \text{in } \Omega, \ t > 0, \\
\eta(x) &= 0, \quad \text{in } \partial \Omega, \ t > 0, \\
\eta(x, 0) &= \eta_0(x), \quad \text{in } \Omega.
\end{aligned}
\] (1.9)

The weak solutions of Eq. (1.9) were obtained under suitable conditions and an appropriate space of functions. Moreover, Zeng et al. in [11] were devoted to the study of equations with logarithmic nonlinearity and variable exponent by applying the logarithmic inequality.

**Remark 1.2** Since the logarithmic nonlinearity does not satisfy the monotonicity condition or Ambrosetti–Rabinowitz condition, the problem (1.2) becomes extremely complex. Therefore, we do need more careful analysis.

To the best of my knowledge, there is no result for the Kirchhoff-type equations, which combines with variable exponent, logarithmic nonlinearity, and convection term. Therefore, motivated by the previous and above-cited works, we will investigate the existence of solutions for this kind of equation, which are different from the work of [2, 6, 21, 29]. Under weak conditions on the nonlocal term $M_{p(x)}$ and the nonlinearities $g$, we prove the existence of solutions with the help of the Galerkin method. Our study extends previous results, such as from the elliptic problem with logarithmic nonlinearity or convection term to $p(x)$-Kirchhoff-type equations both logarithmic nonlinearity with variable exponent and convection term.

**Definition 1.1.** We say that $\eta \in W_0$ is a weak solution of Kirchhoff-type problem (1.2), if
\[
\langle (-\Delta)^{M_{p(x)}} \eta, \varphi \rangle = \lambda \int_{\Omega} \left( |\eta|^{p(x) - 2} \eta \ln |\eta| \right) \varphi dx + \int_{\Omega} g(x, \eta, \nabla \eta) \varphi dx,
\]
for all $\varphi \in W_0$, and $(-\Delta)^{M_{p(x)}} \eta$ is defined as (1.5).

The paper is divided into eight sections. Aside from Sect. 1, we have Sect. 2 given some preliminary notions and results about Lebesgue spaces and Sobolev spaces and Sect. 3 proved some technical lemmas. The finite-dimensional approximate solutions, generalized solutions, strong generalized solutions are obtained in Sects. 4, 5, 6, respectively. Section 7 discuss the discontinuous case of kirchhoff functions and we make a conclusion in Sect. 8.

### 2. Preliminary Results

In this subsection, we briefly review some basic knowledge, lemmas, and propositions of generalized Lebesgue spaces and Sobolev spaces with variable exponents.

For any real-valued function $H$ defined on a domain $\overline{\Omega}$, we denote
\[
C_+(\overline{\Omega}) := \left\{ H(x) \in C(\overline{\Omega}, \mathbb{R}) : 1 < H^- := \inf_{x \in \overline{\Omega}} H(x) \leq H(x) \leq H^+ := \sup_{x \in \overline{\Omega}} H(x) < +\infty \right\}.
\]
Let $\vartheta(x) \in C_+(\Omega)$, we define the generalized Lebesgue spaces with variable exponents as

$$L^{\vartheta(x)}(\Omega) := \left\{ \eta : \eta \text{ is a measurable function and } \int_\Omega |\eta|^{\vartheta(x)} \, dx < \infty \right\},$$

provided with the Luxemburg norm:

$$\|\eta\|_{\vartheta(x)} = \|\eta\|_{L^{\vartheta(x)}(\Omega)} := \inf \left\{ \chi > 0 : \int_\Omega \frac{\|\eta\|^{\vartheta(x)}}{\chi^{\vartheta(x)}} \, dx \leq 1 \right\},$$

then $(L^{\vartheta(x)}(\Omega), \|\cdot\|_{\vartheta(x)})$ is a separable and reflexive Banach spaces, see [24,35].

**Lemma 2.1.** (See, [24]) Let $\vartheta(x)$ be the conjugate exponent of $\vartheta(x) \in C_+(\Omega)$, that is

$$\frac{1}{\vartheta(x)} + \frac{1}{\vartheta(x)} = 1, \text{ for all } x \in \Omega.$$

Suppose that $\eta \in L^{\vartheta(x)}(\Omega)$ and $\xi \in L^{\tilde{\vartheta}(x)}(\Omega)$, then

$$\left| \int_{\mathbb{R}^N} \eta \xi \, dx \right| \leq \left( \frac{1}{\vartheta^-} + \frac{1}{\vartheta^+} \right) \|\eta\|_{\vartheta(x)} \|\xi\|_{\tilde{\vartheta}(x)} \leq 2 \|\eta\|_{\vartheta(x)} \|\xi\|_{\tilde{\vartheta}(x)}.$$

**Proposition 2.1.** (See, [19]) The modular of $L^{\vartheta(x)}(\Omega)$, which is the mapping $\rho_{\vartheta(x)} : L^{\vartheta(x)}(\Omega) \to \mathbb{R}$, is defined by

$$\rho_{\vartheta(x)}(\eta) := \int_\Omega |\eta|^{\vartheta(x)} \, dx.$$

Suppose that $\eta_n, \eta \in L^{\vartheta(x)}(\Omega)$, then the following properties hold

1. $\|\eta\|_{\vartheta(x)} > 1 \Rightarrow \|\eta\|_{\vartheta(x)}^{\vartheta^-} \leq \rho_{\vartheta(x)}(\eta) \leq \|\eta\|_{\vartheta(x)}^{\vartheta^+}$,
2. $\|\eta\|_{\vartheta(x)} < 1 \Rightarrow \|\eta\|_{\vartheta(x)}^{\vartheta^+} \leq \rho_{\vartheta(x)}(\eta) \leq \|\eta\|_{\vartheta(x)}^{\vartheta^-}$,
3. $\|\eta\|_{\vartheta(x)} < 1$ (resp. $= 1 > 1$) $\iff \rho_{\vartheta(x)}(\eta) < 1$ (resp. $= 1 > 1$),
4. $\|\eta_n\|_{\vartheta(x)} \to 0$ (resp. $\to +\infty$) $\iff \rho_{\vartheta(x)}(\eta_n) \to 0$ (resp. $\to +\infty$),
5. $\lim_{n \to \infty} |\eta_n - \eta|_{\vartheta(x)} = 0 \iff \lim_{n \to \infty} \rho_{\vartheta(x)}(\eta_n - \eta) = 0$.

Now, we consider the following generalized Sobolev spaces with variable exponents:

$$W = W^{1,\vartheta(x)}(\Omega) := \left\{ \eta \in L^{\vartheta(x)}(\Omega) : |\nabla \eta| \in L^{\vartheta(x)}(\Omega) \right\},$$

endowed with the norm:

$$\|\eta\|_W := \|\eta\|_{\vartheta(x)} + \|\nabla \eta\|_{\vartheta(x)},$$

then $(W, \|\cdot\|_W)$ is a separable and reflexive Banach spaces, see [35].

**Lemma 2.2.** (see, [35]) Assume that $q(x) \in C_+(\Omega)$ fulfill

$$1 < q^- = \min_{x \in \Omega} q(x) \leq q(x) < q^+ = \frac{Nq(x)}{N-\vartheta(x)},$$

for any $x \in \Omega$.

Then, there exists $C_q = C_q(N, \vartheta, q, \Omega) > 0$ such that

$$\|\eta\|_{q(x)} \leq C_q \|\eta\|_W,$$
for any $\eta \in W$. Moreover, the embedding $W \hookrightarrow L^q(x)(\Omega)$ is compact.

Let $W_0$ denote the closure of $C_0^\infty(\Omega)$ in $W$ with respect to the norm $\|\eta\|_{W_0}$, which is the subspace of $W$. Thus, the spaces $(W_0, \| \cdot \|_{W_0})$ is also a uniformly convex and reflexive Banach spaces.

**Remark 2.1** According to the Poincaré inequality, we know that $\|\nabla \eta\|_{\varrho(x)}$ and $\|\eta\|_{W_0}$ are equivalent norms in $W_0$. From now on, we work on $W_0$ and replace $\|\eta\|_{W_0}$ by $\|\nabla \eta\|_{\varrho(x)}$, that is,

$$\|\eta\|_{W_0} = \|\nabla \eta\|_{\varrho(x)}, \text{ for all } \eta \in W_0.$$

**Remark 2.2** To simplify the presentation, we will denote the norm of $W_0$ by $\| \cdot \|$ instead of $\| \cdot \|_{W_0}$. $W_0^*$ denotes the dual space of $W_0$.

Our technique of proof is based on Galerkin methods together with the fixed point theorem, whose proof may be found in Lions [9].

**Lemma 2.3.** Let $W_0$ be a finite dimensional space with the norm $\| \cdot \|$ and let $G : W_0 \rightarrow W_0^*$ be a continuous mapping. Assume that there is a constant $R > 0$, such that

$$\langle G(\eta), \eta \rangle \geq 0, \text{ for all } \eta \in W_0 \text{ with } \|\eta\| = R,$$

and then there exists $\eta \in W_0$ with $\|\eta\| \leq R$ satisfying $G(\eta) = 0$.

3. Some Technical Lemmas

The following Lemma 3.1 provides an useful growth estimate, related to the reaction term $g(x, \eta(x), \nabla \eta(x))$, and is proved using the Hölder inequality.

**Lemma 3.1.** Suppose that condition $G_1$ holds, then the following inequality holds

$$\left| \int_{\Omega} g(x, \eta, \nabla \eta) \xi dx \right| \leq C \left( \|\phi\|_{L((p^*(x))'(\Omega))} + aM_1 + bM_2 \right) \|\xi\|_{p^*(x)},$$

where

$$M_1 = \max \left\{ \|\eta\|_{(p^*(x))'(\Omega)}^{p^*(x)+1}, \|\eta\|_{(p^*(x))'(\Omega)}^{-1} \right\}, \quad M_2 = \max \left\{ \|\eta\|_{p(x)}^{p(x)+1}, \|\eta\|_{p(x)}^{-1} \right\}$$

for all $\eta, \xi \in W_0$, and $C > 0$.

**Proof.** From hypothesis $G_1$ and Lemma 2.1, we have

$$\left| \int_{\Omega} g(x, \eta, \nabla \eta) \xi dx \right| \leq \left| \int_{\Omega} (\phi(x) + a|\mu|^{p^*(x)-1} + b|\nu|^{p(x)-1}) \xi dx \right|$$

$$\leq \int_{\Omega} |\phi(x)||\xi| dx + a \int_{\Omega} |\mu|^{p^*(x)-1}||\xi| dx + b \int_{\Omega} |\nu|^{p(x)-1}||\xi| dx$$

$$\leq ||\phi||_{(p^*(x))'} ||\xi||_{p^*(x)} + aM_1 ||\xi||_{p^*(x)} + bM_2 ||\xi||_{p(x)},$$

where

$$M_1 = \max \left\{ ||\mu||_{(p^*(x))'}^{p^*(x)+1}, ||\mu||_{(p^*(x))'}^{-(p^*(x)-1)} \right\}, \quad M_2 = \max \left\{ ||\nu||_{p(x)}^{p(x)+1}, ||\nu||_{p(x)}^{-(p(x)-1)} \right\}.$$
Since \( p(x) < p^*(x) \) then using the continuous embedding \( L^{p^*}(\Omega) \hookrightarrow L^{p(x)}(\Omega) \), we obtain
\[
\int_\Omega g(x, \eta, \nabla \eta) \xi \, dx \leq C \left( \| \phi \|_{(p^*)'} + aM_1 + bM_2 \right) \| \xi \|_{p^*(x)},
\]
where \( C \) is some positive constant.

Notice that the problem (1.2) contains logarithmic nonlinear terms, we need to establish the following two estimates, which play an important role during our proof process.

**Lemma 3.2.** Suppose that \( h(x) \in C_+(\overline{\Omega}) \), then we have the following estimate
\[
\ln t \leq \frac{1}{eh(x)} t^{h(x)} \leq \frac{1}{eh(x)} t^{-h(x)}, \quad \text{for all } t \in [1, \infty).
\]

**Proof.** Suppose that \( h(x) \in C_+(\overline{\Omega}) \), we construct the following function
\[
f(t) = \ln t - \frac{1}{eh(x)} t^{h(x)}, \quad \text{for all } t \in [1, \infty).
\]

With respect to \( t \), just by taking a simple derivative, we deduce
\[
f'(t) = \frac{1}{t} - \frac{1}{e^{h(x)} t^{-1}}, \quad \text{for all } t \in [1, \infty),
\]
and let \( f'(t) = 0 \), then \( t^* = e^{h^{-1}(x)} \).

It is obvious that \( t^* \) is the unique maximum point of the function \( f(t) \), so, \( f(t) \leq f(t^*) = 0 \) for all \( t \in [1, \infty) \). Therefore, based on the above discussion, we can obtain the stated conclusion. \( \square \)

**Lemma 3.3.** Suppose that for all \( \eta_n, \eta \in W_0 \) and \( h(x), p(x) \in C_+(\overline{\Omega}) \), then the following properties hold
\[
(i) \int_\Omega |\eta|^{p(x)} \ln |\eta| \, dx \leq C_{\Omega_1} |\Omega| + \frac{1}{eh} \max \left\{ C_{h^++p^+} \| \eta \|^{h^++p^+}, C_{h^-+p^-} \| \eta \|^{h^-+p^-} \right\},
\]
\[
(ii) \lim_{n \to \infty} \int_\Omega (\eta_n - \eta)|\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \, dx = 0,
\]
\[
(iii) \int_\Omega |\eta_n|^{p(x)} \ln |\eta_n| \, dx \to \int_\Omega |\eta|^{p(x)} \ln |\eta| \, dx, \quad \text{as } n \to \infty.
\]

where \( C_{\Omega_1}, C_{h^++p^+}, C_{h^-+p^-} \) are some positive constants and \( p^+ \leq h(x) + p(x) \leq h^+ + p^+ < p^*(x) = \frac{Np(x)}{N-p(x)} \).

**Proof.** (i) Let \( \Omega_1 = \{ x \in \Omega : |\eta(x)| \leq 1 \} \) and \( \Omega_2 = \{ x \in \Omega : |\eta(x)| \geq 1 \} \), then
\[
\int_\Omega |\eta|^{p(x)} \ln |\eta| \, dx = \int_{\Omega_1} |\eta|^{p(x)} \ln |\eta| \, dx + \int_{\Omega_2} |\eta|^{p(x)} \ln |\eta| \, dx.
\]

By a simple calculation, we obtain
\[
\int_{\Omega_2} |\eta|^{p(x)} \ln |\eta| \, dx < C_{\Omega_1} |\Omega|,
\]
\[ (3.1) \]
where $|\Omega|$ denotes the Lebesgue measure of $\Omega$ and $C_{\Omega} > 0$. Using Lemma 3.2 with $p^+ \leq h(x) + p(x) \leq h^+ + p^+ < p^*(x)$, we deduce

$$\int_{\Omega_2} \eta |p(x)| \ln |\eta| dx \leq \frac{1}{eh} \int_{\Omega_2} |\eta|^{p(x)+h}(x) dx$$

$$\leq \frac{1}{eh^+} \max \left\{ \|\eta\|_{h^+(x)+p(x)}^{h^+ + p^+}, \|\eta\|_{h^+(x)+p(x)}^{h^+ - p^-} \right\},$$

in view of Lemma 2.2, there exist some constants $C_{h^+ + p^+} > 0$ and $C_{h^- + p^-} > 0$, such that

$$\int_{\Omega_2} |\eta|^{p(x)} \ln |\eta| dx \leq \frac{1}{eh^+} \max \left\{ C_{h^+ + p^+} \|\eta\|_{h^+ + p^+}, C_{h^- + p^-} \|\eta\|_{h^- + p^-} \right\}. \quad (3.2)$$

It follows from (3.1) and (3.2) that

$$\int_{\Omega} |\eta|^{p(x)} \ln |\eta| dx \leq C_{\Omega_1} |\Omega| + \frac{1}{eh^+} \max \left\{ C_{h^+ + p^+} \|\eta\|_{h^+ + p^+}, C_{h^- + p^-} \|\eta\|_{h^- + p^-} \right\}.$$ 

This yields the stated conclusion.

(ii) Going if necessary up to a subsequence, we suppose there exists $\eta \in W_0$, such that

$$\eta_n \rightharpoonup \eta, \text{ weakly in } W_0,$$

$$\eta_n \to \eta, \text{ strongly in } L^{p(x)}(\Omega),$$

$$\eta_n \to \eta, \text{ a.e. in } \Omega. \quad (3.3)$$

In fact, by a simple calculation for the logarithmic nonlinear term, we deduce

$$\int_{\Omega} \left| \eta_n |p(x) - 2\eta_n \ln |\eta_n| \right|^{p^+} dx = \int_{\Omega_1} \left| \eta_n |p(x) - 2\eta_n \ln |\eta_n| \right|^{p^+} dx$$

$$+ \int_{\Omega_2} \left| \eta_n |p(x) - 2\eta_n \ln |\eta_n| \right|^{p^+} dx$$

$$\leq C_{\Omega_1} |\Omega| + \int_{\Omega_2} \left| \eta_n |p(x) - 2\eta_n \ln |\eta_n| \right|^{p^+} dx.$$ 

Since $p^+ < p^*(x)$ then using the continuous embedding $L^{p^*(x)}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ and combining Lemma 3.2, we deduce

$$\int_{\Omega} \left| \eta_n |p(x) - 2\eta_n \ln |\eta_n| \right|^{p^+} dx \leq C_{\Omega_1} |\Omega| + \int_{\Omega} |\eta_n|^{p^+} dx$$

$$\leq C_{\Omega_1} |\Omega| + C_{\Omega_2} \|\eta_n\|_{p^*(x)}. \quad (3.4)$$

where $C_{\Omega_2} > 0$. From Lemma 2.1, we obtain

$$\left| \int_{\Omega} (\eta_n - \eta) |p(x) - 2\eta_n \ln |\eta_n| dx \right|$$

$$\leq \|\eta_n - \eta\|_{L^{p^+}(\Omega)} \left\| |p(x) - 2\eta_n \ln |\eta_n| \right\|_{L^{p^+}(\Omega)}. \quad (3.5)$$
Notice that the relation (3.4) implies that

\[ \left\| |\eta_n|^{p(x)-1} \ln |\eta_n| \right\|_{L^{p^+}_{p^+ - 1}(\Omega)} \leq C \frac{p^+}{p^+ - 1}, \quad (3.6) \]

where \( C \frac{p^+}{p^+ - 1} > 0 \). Therefore, it follows from (3.3), (3.5) and (3.6) that

\[ \left| \int_{\Omega} (\eta_n - \eta) |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \, dx \right| \to 0, \text{ as } n \to \infty. \quad (3.7) \]

This yields the stated conclusion.

(iii) To prove the third result, we are entitled to use all the formulas in (ii). Hence, from (3.3), it yields

\[ |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \to |\eta|^{p(x)-2} \eta \ln |\eta|, \text{ a.e. in } \Omega, \]

which implies that

\[ \left| \int_{\Omega} \eta \left( |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| - |\eta|^{p(x)-2} \eta \ln |\eta| \right) \, dx \right| \to 0, \text{ as } n \to \infty. \quad (3.8) \]

We find that

\[ \left| \int_{\Omega} \left( |\eta_n|^{p(x)} \ln |\eta_n| - |\eta|^{p(x)} \ln |\eta| \right) \, dx \right| \leq \left| \int_{\Omega} (\eta_n - \eta) |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \, dx \right| \]

\[ + \left| \int_{\Omega} \eta \left( |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| - |\eta|^{p(x)-2} \eta \ln |\eta| \right) \, dx \right|. \quad (3.9) \]

It follows from (3.7) and (3.8) that

\[ \int_{\Omega} |\eta_n|^{p(x)} \ln |\eta_n| \, dx \to \int_{\Omega} |\eta|^{p(x)} \ln |\eta| \, dx, \text{ as } n \to \infty. \]

This yields the stated conclusion.

\[ \square \]

4. Finite Dimensional Approximate Solutions

Since \( W_0 \) is a reflexive and separable Banach space, there exists an orthonormal basis \( \{e_1, ..., e_n, ...\} \) in \( W_0 \), such that

\[ W_0 = \text{span}\{e_1, ..., e_n\}. \]

Define \( X_n = \text{span}\{e_1, ..., e_n\} \), which means a sequence of vector \( X_n \) subspaces of \( W_0 \), satisfying

\[ \dim(X_n) < \infty \text{ for all } n \geq 1, X_n \subset X_{n+1} \text{ for all } n \geq 1, \text{ and } \bigcup_{n=1}^\infty X_n = W_0. \]

It is known that \( X_n \) and \( \mathbb{R}^N \) are isomorphic and for \( \eta \in \mathbb{R}^N \), we have an unique \( \xi \in X_n \) by the identification:

\[ \eta \to \Sigma_{i=1}^N \xi_i e_i = \xi, \quad \|\eta\| = |\xi|, \]

where \( |\cdot| \) is the Euclidean norm in \( \mathbb{R}^N \).

**Theorem 4.1.** Suppose that conditions \( \mathcal{P}, \mathcal{M}_1 \), and \( \mathcal{G}_2 \) are satisfied, then
• if $2p^- > p^+$ and $p^- > \alpha^+$, the problem (1.2) admits a approximate solution for all $\lambda \leq 0$,
• if $2p^- > p^+ + h^+$ and $p^- > \alpha^+$, the problem (1.2) admits a approximate solution for all $\lambda > 0$,

that is, for all $n \geq 1$ and $\varphi \in X_n$, there exists $\eta_n \in X_n$, such that

$$\langle (-\Delta)^{M_{p(x)}} \eta_n, \varphi \rangle = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-2}\eta_n \ln |\eta_n| \right) \varphi dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \varphi dx. \quad (4.1)$$

**Proof.** Inspired by the literature [2,6,7], Theorem 4.1 was proved using Galerkin’s method. For all $\eta \in X_n$, we consider the mapping $G = (G_1, G_2, ..., G_N) : \mathbb{R}^N \to \mathbb{R}$ by

$$G_i = \langle (-\Delta)^{M_{p(x)}} \eta, e_i \rangle - \lambda \int_{\Omega} \left( |\eta|^{p(x)-2}\eta \ln |\eta| \right) e_i dx - \int_{\Omega} g(x, \eta, \nabla \eta)e_i dx.$$

The following work shows that, for each $n \geq 1$, the problem (1.2) has an approximate solution $\eta_n$ in $X_n$, namely

$$\langle (-\Delta)^{M_{p(x)}} \eta_n, e_i \rangle = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-2}\eta_n \ln |\eta_n| \right) e_i dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n)e_i dx. \quad (4.2)$$

For $\eta \in X_n$, we have

$$\langle G, \eta \rangle = \langle (-\Delta)^{M_{p(x)}} \eta, \eta \rangle - \int_{\Omega} |\eta|^{p(x)} \ln |\eta| dx - \int_{\Omega} g(x, \eta, \nabla \eta)\eta dx,$$

$$\geq \left( \int_{\Omega} |\nabla \eta|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \left( |\eta|^{p(x)} \ln |\eta| \right) dx - \int_{\Omega} g(x, \eta, \nabla \eta)\eta dx.$$

From $G_2$ and Lemma 3.3, we have the following estimate

$$\langle G, \eta \rangle \geq \left( \int_{\Omega} |\nabla \eta|^{p(x)} dx \right)^2 - c \int_{\Omega} |\nabla \eta|^{p(x)} dx - d \int_{\Omega} (|\eta|^{\alpha(x)} + 1) dx$$

$$- \frac{\lambda}{eh^-} \max \left\{ C_{h^+ + p^+} \|\eta\|^{h^+ + p^+}, C_{h^- + p^-} \|\eta\|^{h^- + p^-} \right\} - \lambda C_{\Omega_1} |\Omega|.$$ 

According to Remark 2.1 and Lemma 2.2, there exist some positive constants $C_{\alpha^+}$ and $C_{\alpha^-}$, such that

$$\langle G, \eta \rangle \geq \min \left\{ \|\eta\|^{2p^+}, \|\eta\|^{2p^-} \right\} - c \max \left\{ \|\eta\|^{p^+}, \|\eta\|^{p^-} \right\}$$

$$- d \max \left\{ C_{\alpha^+} \|\eta\|^{\alpha^+}, C_{\alpha^-} \|\eta\|^{\alpha^-} \right\}$$

$$- \frac{\lambda}{eh^-} \max \left\{ C_{h^+ + p^+} \|\eta\|^{h^+ + p^+}, C_{h^- + p^-} \|\eta\|^{h^- + p^-} \right\} - (\lambda C_{\Omega_1} + d)|\Omega|.$$ 

If $\|\eta\| > 1$, then

$$\langle G, \eta \rangle \geq \|\eta\|^{2p^-} - c\|\eta\|^{p^+} - dC_{\alpha} \|\eta\|^{\alpha^+}$$

$$- \frac{\lambda C_{h^+ + p^+}}{eh^-} \|\eta\|^{h^+ + p^+} - (\lambda C_{\Omega_1} + d)|\Omega|.$$ 

Combined with the above analysis, we deduce that
**Case 1:** If \(2p^- > p^+\) and \(p^- > \alpha^+\) with \(\lambda \leq 0\), there exists some positive constant \(R\), provided sufficiently large, such that

\[
\langle G, \eta \rangle \geq 0, \text{ for all } \eta \in X_n, \text{ with } \|\eta\| = R.
\]

**Case 2:** If \(2p^- > p^+ + h^+\) and \(p^- > \alpha^+\) with \(\lambda > 0\), there exists some positive constant \(R\), provided sufficiently large, such that

\[
\langle G, \eta \rangle \geq 0, \text{ for all } \eta \in X_n, \text{ with } \|\eta\| = R.
\]

In both cases, \(G\) is continuous, so, in view of Lemma 2.3, the problem (1.2) admits a approximate solution \(\eta_n\) in \(X_n \subset W_0\) with \(\|\eta_n\| \leq R\).

\[\Box\]

**Corollary 4.1.** Suppose that the conditions of Theorem 4.1 are satisfied, then the sequence \(\{\eta_n\}_{n \geq 1}\) with \(\eta_n \in X_n\) constructed in Theorem 4.1, is bounded in \(W_0\).

**Proof.** If \(\|\eta_n\| \leq 1\) for all \(n \in \mathbb{N}\), then the sequence \(\{\eta_n\}_{n \in \mathbb{N}}\) is bounded in \(W_0\).

If \(\|\eta_n\| > 1\) for all \(n \in \mathbb{N}\), with \(\eta_n\) in place of \(\phi\) in (4.1), we have

\[
\left\langle (-\Delta)^{M_{p(x)}} \eta_n, \eta_n \right\rangle = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \right) \eta_n dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx.
\]

On the basis of condition \(\mathcal{G}_2\) and Lemma 3.3, it gives

\[
\left( \int_{\Omega} |\nabla \eta_n|^{p(x)} dx \right)^2 \leq C \int_{\Omega} |\nabla \eta_n|^{p(x)} dx + d \int_{\Omega} (|\eta_n|^{\alpha(x)} + 1) dx + \lambda C_{\Omega_1} |\Omega|
\]

\[
+ \frac{\lambda}{eh^-} \max \left\{ C_{h^+} + p^+, \eta_n \| h^+ + p^+ \| \eta_n \| h^- + p^- \| \eta_n \| h^- + p^- \right\}.
\]

**Case 1:** Utilizing that \(2p^- > p^+ + h^+\) and \(p^- > \alpha^+\) with \(\lambda > 0\), and by Lemma 2.1 and Lemma 2.2, we deduce

\[
\|\eta_n\|^{2p^-} \leq \frac{\lambda C_{h^+} + p^+}{eh^-} \|\eta_n\|^{h^+ + p^+} + c \|\eta_n\|^{p^+} + d C_{\alpha^+} \|\eta_n\|^{\alpha^+} + \left( \lambda C_{\Omega_1} + d \right) |\Omega|.
\]

**Case 2:** Using that \(2p^- > p^+\) and \(p^- > \alpha^+\) with \(\lambda \leq 0\), and combining Lemma 2.1 and Lemma 2.2, we derive

\[
\|\eta_n\|^{2p^-} \leq c \|\eta_n\|^{p^+} + d C_{\alpha^+} \|\eta_n\|^{\alpha^+} + d |\Omega|.
\]

In both cases, we conclude that the sequence \(\{\eta_n\}_{n \geq 1}\) is bounded in \(W_0\).

\[\Box\]

5. Existence of Generalized Solutions

Before stating our theorem, we make the following definition of the generalized solution.
**Definition 5.1.** Suppose that condition $G_1$ holds, we say that a function $\eta \in W_0$ is a generalized solution to problem (1.2), if

(i) $\eta_n \rightharpoonup \eta$ in $W_0$ as $n \to \infty$,

(ii) $(-\Delta)^{M_p(x)}_{p(x)} \eta_n - \lambda |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| - g(x, \eta_n, \nabla \eta_n) \to 0$, in $W_0$ as $n \to \infty$,

(iii) $\lim_{n \to \infty} \left[ \langle (-\Delta)^{M_p(x)}_{p(x)} \eta_n, \eta_n - \eta \rangle - \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-1} \ln |\eta_n| \right) (\eta_n - \eta) dx \right.\nonumber - \left. \int_{\Omega} g(x, \eta_n, \nabla \eta_n)(\eta_n - \eta) dx \right] = 0.$

**Theorem 5.1.** Suppose that conditions $P$, $M_1$, $G_1$, and $G_2$ are satisfied, then, for $\lambda \in \mathbb{R}$, there exists a generalized solution to problem (1.2) in the sense of Definition 5.1.

**Proof.** Corollary 4.1 shows that the sequence $\{\eta_n\}_{n \geq 1}$ is bounded in $W_0$. Since the space $W_0$ is reflexive, we have

$$\eta_n \rightharpoonup \eta, \text{ in } W_0, \quad (5.1)$$

for some $\eta$ in $W_0$.

Define the Nemytskii operator $N_g : W_0 \to W_0^*$ with respect to the function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, given by

$$N_g(\eta) = g(x, \eta(x), \nabla \eta(x)), \text{ for all } \eta \in W_0.$$ 

Lemma 3.1 shows that the Nemytskii operator $N_g$ is well-defined. Moreover, there exists some positive constant $C'$ such that

$$\|N_g(\eta_n)\|_{W_0^*} \leq C' (\|\phi\|_{(p_2^*(x))'} + aM_1 + bM_2), \text{ for all } \eta_n \in W_0.$$ 

Hence, the Nemytskii operator $N_g$ is bounded. In combination with (5.1), we obtain

$$\{N_g(\eta_n)\}_{n \geq 1} \text{ is a bounded sequence in } W_0^*. \quad (5.2)$$

Define another Nemytskii operator $N_f : W_0 \to W_0^*$ with respect to the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$N_f(\eta) = f(x, \eta(x)) = |\eta|^{p(x)-2} \eta \ln |\eta|, \text{ for all } \eta \in W_0,$$ 

The Nemytskii operator $N_f$ is also well-defined, and the following estimate holds

$$\|N_f(\eta_n)\|_{W_0^*} \leq C_{\Omega_1} |\Omega| + C_f \max \left\{ C_{h^+ + p^+ - 1} \|\eta_n\|_{h^+ + p^+ - 1}, \right.\nonumber C_{h^- + p^- - 1} \|\eta_n\|_{h^- + p^- - 1} \left. \right\},$$ 

for all $\eta_n \in W_0$ and $C_f, C_{h^+ + p^+ - 1}, C_{h^- + p^- - 1} > 0$. Thus, the Nemytskii operator $N_f$ is also bounded. In combination with (5.1), we deduce

$$\{N_f(\eta_n)\}_{n \geq 1} \text{ is a bounded sequence in } W_0^*. \quad (5.3)$$

Since, the sequence $\eta_n$ constructed in Theorem 4.1 is bounded in $W_0$ and $M_{p(x)}$ is continuous, $(-\Delta)^{M_{p(x)}}_{p(x)} \eta_n : W_0 \to W_0^*$ is also a bounded operator,
which join with (5.2) and (5.3), we obtain
\[
\{(-\Delta)^{M_p(x)}_{p(x)} \eta_n - \lambda N_f(\eta_n) - N_g(\eta_n)\}_{n \geq 1} \text{ is bounded in } W_0^*.
\] (5.4)

Thanks to the reflexivity of \(W_0^*\), we get
\[
(-\Delta)^{M_p(x)}_{p(x)} \eta_n - \lambda N_f(\eta_n) - N_g(\eta_n) \rightharpoonup \zeta, \text{ in } W_0^* \text{ as } n \to \infty,
\] (5.5)
for some \(\zeta\) in \(W_0\).

Let \(\varphi \in \bigcup_{n=1}^\infty X_n\). There is an integer \(m \geq 1\), such that \(\varphi \in X_m\). Applying Theorem 4.1, we see that equality (4.1) holds true for all \(n \geq m\). Letting \(n \to \infty\) in (4.1) entails
\[
\langle \zeta, \psi \rangle = 0, \text{ for all } \varphi \in \bigcup_{n \geq 1} X_n.
\]
Since \(\bigcup_{n \geq 1} X_n\) is dense in \(W_0^*\), we obtain that \(\zeta = 0\). As a result, the expression (5.5) becomes
\[
(-\Delta)^{M_p(x)}_{p(x)} \eta_n - \lambda N_f(\eta_n) - N_g(\eta_n) \to 0, \text{ as } n \to \infty.
\] (5.6)

Insert \(\varphi = \eta_n\) in (4.1), that is
\[
\left\langle (-\Delta)^{M_p(x)}_{p(x)} \eta_n, \eta_n \right\rangle = \lambda \int_{\Omega} \left|\eta_n\right|^{p(x)} \ln \left|\eta_n\right| dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx,
\] (5.7)
for all \(n \geq 1\). Taking into account (5.6), we have
\[
\left\langle (-\Delta)^{M_p(x)}_{p(x)} \eta_n - \lambda N_f(\eta_n) - N_g(\eta_n), \eta \right\rangle \to 0, \text{ as } n \to \infty.
\] (5.8)

In combination with (5.7) and (5.8), we derive
\[
\lim_{n \to \infty} \left[ \left\langle (-\Delta)^{M_p(x)}_{p(x)} \eta_n, \eta_n - \eta \right\rangle - \lambda \int_{\Omega} \left|\eta_n\right|^{p(x)-2} \eta_n \ln \left|\eta_n\right| (\eta_n - \eta) dx \right.
\]
\[
- \left. \int_{\Omega} g(x, \eta_n, \nabla \eta_n)(\eta_n - \eta) dx \right] = 0.
\] (5.9)
As a result, it follows from (5.1), (5.6) and (5.9) that there exists a generalized solution to problem (1.2) in the sense of Definition 5.1.

\[\square\]

6. Existence of Strong Generalized Solutions

Before we state our theorem, we give the definition of the generalized solution and the new hypothesis about \(g\).

\(\mathcal{G}'_1\): Let \(\tau_1(x), \tau_1(x), \tau_2(x) \in [1, p^*(x))\) and \(\phi(x) \in L^{(p^*(x))'}(\Omega)\) with \(\phi(x) > 0\).

There exist constants \(a_1, b_2 > 0\), such that
\[
|g(x, \mu, \nu)| \leq \phi(x) + a_1|\mu|^{p^*(x)}_{\tau_1(x)} + b_2|\nu|^{p(x)}_{\tau_2(x)}, \text{ for all } (x, \mu, \nu) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.
\]

Remark 6.1 Hypothesis \(\mathcal{G}'_1\) implies hypothesis \(\mathcal{G}_1\). If \(\tau_1(x), \tau_2(x) \in [1, p^*(x))\), then \(\tau_1(x) > (p^*(x))'\) and \(\tau_2(x) > (p^*(x))'\), which yields
\[
\frac{p^*(x)}{\tau_1(x)} < \frac{p(x)}{\tau_1(x)} = p^*(x) - 1 \text{ and } \frac{p(x)}{\tau_2(x)} < \frac{p(x)}{\tau_2(x)} = p(x) - 1.
\]
**Definition 6.1.** Suppose that condition $G'_1$ holds, we say that a function $\eta \in W_0$ is a generalized solution to problem (1.2), if

(i) $\eta_n \rightharpoonup \eta$ in $W_0$ as $n \to \infty$,

(ii) $(-\Delta)_{p(x)}^{M_{p(x)}} \eta_n - \lambda|\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| - g(x, \eta_n, \nabla \eta_n) \rightharpoonup 0$, in $W_0$ as $n \to \infty$,

(iii) $\lim_{n \to \infty} \left( (-\Delta)_{p(x)}^{M_{p(x)}} \eta_n. \eta_n - \eta \right) = 0.$

**Theorem 6.1.** Suppose that conditions $\mathcal{P}$, $\mathcal{M}_1$, $G'_1$, and $G_2$ are satisfied, then, for $\lambda \in \mathbb{R}$, there exists a generalized solution to problem (1.2) in the sense of Definition 6.1.

**Proof.** Since the proofs of (i) and (ii) of this theorem are quite similar to Theorem 5.1, we omit the details. Next, we only give the proof of (iii).

According to Remark 6.1 and $G'_1$, we have

$$\left| \int_{\Omega} g(x, \eta_n, \nabla \eta_n)(\eta_n - \eta)dx \right| \leq \int_{\Omega} |g(x, \eta_n, \nabla \eta_n)(\eta_n - \eta)|dx$$

$$\leq \int_{\Omega} |\phi(x) + a_1|\mu|^p(x) + b_2|\nu|^p(x)\|\eta_n - \eta|dx.$$ 

Combining this with Lemma 2.1, it follows that

$$\left| \int_{\Omega} g(x, \eta, \nabla \eta)(\eta_n - \eta)dx \right| \leq \|\phi\|_{r(x)}\|\eta_n - \eta\|_{r(x)} + a_1M_{r_1(x)}\|\eta_n - \eta\|_{r_1(x)}$$

$$+b_1M_{r_2(x)}\|\eta_n - \eta\|_{r_2(x)}. \quad (6.1)$$

For all $\eta_n, \eta \in W_0$ and $n \geq 1$, where

$$M_{r_1(x)} = \max \left\{ \left\| \eta_n \right\|_{(p^*(x))'}^\frac{(p^*)'}{(r_1(x))'}, \left\| \eta_n \right\|_{(p^*)'}^\frac{(p^*)}{(r_1(x))'} \right\} , \quad M_{r_2(x)}$$

$$= \max \left\{ \left\| \nabla \eta_n \right\|_{p(x)}^{\frac{p^*}{(p^*)'}}, \left\| \nabla \eta_n \right\|_{p(x)}^{\frac{p^*}{(p^*)'}} \right\} .$$

Since the sequence $\{\eta_n\}_{n \geq 1}$ is bounded in $W_0$, which implies that the sequence $\{\eta_n\}_{n \geq 1}$ is bounded in $L^{p^*(x)}(\Omega)$ and the sequence $\{\nabla \eta_n\}_{n \geq 1}$ is bounded in $(L^{p(x)}(\Omega))^N$, by the joint relation (6.1), we deduce that

$$\left| \int_{\Omega} g(x, \eta, \nabla \eta)(\eta_n - \eta)dx \right| \leq C_M \left( \|\eta_n - \eta\|_{r(x)} + \|\eta_n - \eta\|_{r_1(x)} + \|\eta_n - \eta\|_{r_2(x)} \right), \quad (6.2)$$

for all $\eta_n, \eta \in W_0$ and $n \geq 1$ and $C_M = \max \{1, \|\phi\|_{r(x)} + a_1M_{r_1(x)} + b_1M_{r_2(x)}\}$.

By the Rellich-Kondrachov theorem we know that $\tau(x), \tau_1(x)$, and $\tau_2(x) \in [1, p^*(x))$ in $G'_1$ and (5.1) ensure the strong convergence $\eta_n \rightharpoonup \eta$ in $L^{r(x)}(\Omega)$, $L^{r_1(x)}(\Omega)$, and $L^{r_2(x)}(\Omega)$ as $n \to \infty$. Then, from (6.2) we derive

$$\lim_{n \to \infty} \int_{\Omega} g(x, \eta_n, \nabla \eta_n)(\eta_n - \eta)dx = 0. \quad (6.3)$$

On the basis of Lemma 3.3 and (6.3), we have

$$\lim_{n \to \infty} \left( (-\Delta)_{p(x)}^{M_{p(x)}} \eta_n. \eta_n - \eta \right) = 0. \quad (6.4)$$
As a result, it follows from (5.1), (5.6) and (6.4) that there exists a generalized solution to problem (1.2) in the sense of Definition 6.1.

**Corollary 6.1.** Suppose that conditions $\mathcal{P}$, $\mathcal{M}_1$, $\mathcal{G}_1$, and $\mathcal{G}_2$ are satisfied, and $\eta \in W_0$ is a strong generalized solution to problem (1.2), then $\eta \in W_0$ is a weak solution to problem (1.2) for $\lambda \in \mathbb{R}$.

**Proof.** Since the sequence $\{\eta_n\}_{n \in \mathbb{N}}$ is bounded in $W_0$, passing to a subsequence, we suppose that

$$
\int_\Omega |\nabla \eta_n|^{p(x)} dx \to t_0 \geq 0 \text{ as } n \to \infty.
$$

**Case 1:** If $t_0 = 0$, then $\eta_n$ strongly converges to $\eta = 0$ in $W_0$.

**Case 1:** If $t_0 > 0$. Since the function $M_{p(x)}$ is continuous, in conjunction with hypothesis $\mathcal{M}_1$, there exist $C_s, C_w > 0$ as $n \to \infty$, such that

$$
0 < C_s \leq \int_\Omega |\nabla \eta_n|^{p(x)} dx \leq C_w, \tag{6.5}
$$

Considering that $\{\int_\Omega |\nabla \eta_n|^{p(x)} dx\}_{n \in \mathbb{N}}$ is bounded, and $\eta \in W_0$ is a strong generalized solution to problem (1.2), we obtain

$$
\lim_{n \to \infty} \left\langle (-\Delta)^{M_{p(x)}} \eta_n, \eta_n - \eta \right\rangle = 0, \tag{6.6}
$$

this means that

$$
\lim_{n \to \infty} \int_\Omega |\nabla \eta_n|^{p(x)} dx \left\langle (-\Delta)^{p(x)} \eta_n, \eta_n - \eta \right\rangle = 0, \tag{6.7}
$$

thus, it follows from (6.5) and (6.7) that

$$
\lim_{n \to \infty} \left\langle (-\Delta)^{p(x)} \eta_n, \eta_n - \eta \right\rangle = 0. \tag{6.8}
$$

Therefore, the operator $(-\Delta)^{p(x)}$ on $W_0$ fulfills the $(S_\pm)$-property, we finally achieve the strong convergence of $\eta_n \to \eta$ as $n \to \infty$ in $W_0$.

Using again Definition 6.1, we deduce that

$$
(-\Delta)^{M_{p(x)}} \eta_n - \lambda |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| - g(x, \eta_n, \nabla \eta_n) \to 0, \text{ in } W_0 \text{ as } n \to \infty, \tag{6.9}
$$

which implies that

$$
(-\Delta)^{M_{p(x)}} \eta - \lambda |\eta|^{p(x)-2} \eta \ln |\eta| - g(x, \eta, \nabla \eta) = 0, \tag{6.10}
$$

Thus, the sequence $\eta \in W_0$ is a weak solution to problem (1.2). \hfill \Box

**7. The Discontinuous Case of Kirchhoff Function**

In this subsection, our interest is devoted to the existence of solutions for problem (1.2), where Kirchhoff function $M_{p(x)}$ is discontinuous. In other words, we consider the problem (1.2) with $M_{p(x)} : \mathbb{R} \setminus \varrho \to \mathbb{R}$ continuous, such that

$\mathcal{M}_2$: $\lim_{t \to \varrho^+} M_{p(x)}(t) = \lim_{t \to \varrho^-} M_{p(x)}(t) = +\infty.$

$\mathcal{M}_3$: $\limsup_{t \to +\infty} M_{p(x)}(t^{p(x)}) t = +\infty$, and $\mathcal{M}_1$ is satisfied for some $t_\infty > \varrho$.

$\mathcal{M}_4$: $\lim_{n \to \infty} M^n_{p(x)}(t^{p(x)}) \neq 0, \|t\|^{p(x)} \geq \varrho - \varepsilon''_n$ or $\|t\|^{p(x)} \leq \varrho - \varepsilon'_n$. 
Theorem 7.1. Suppose that conditions $\mathcal{P}$, $\mathcal{M}_1 - \mathcal{M}_4$, $\mathcal{G}_1$, and $\mathcal{G}_2$ are satisfied, then, for $\lambda \in \mathbb{R}$, the problem (1.2) possesses a solution $\eta \in W_0$.

Proof. To prove Theorem 7.1, we first construct the sequence of functions $M^n_{p(x)}(t) : \mathbb{R} \to \mathbb{R}$ as follows:

$$M^n_{p(x)}(t) = \begin{cases} n, & \rho - \varepsilon_n' \leq t \leq \rho - \varepsilon_n'', \\ M_{p(x)}(t), & t \leq \rho - \varepsilon_n' \text{ or } t \geq \rho - \varepsilon_n'', \end{cases} \tag{7.1}$$

for $n > m_0$, where $\rho, \varepsilon_n', \varepsilon_n''$, $\rho - \varepsilon_n'$, $\rho - \varepsilon_n''$ are some positive constants with $\varepsilon_n', \varepsilon_n'' \to 0$ as $n \to \infty$, and

$$M_{p(x)}(\rho - \varepsilon_n') = M_{p(x)}(\rho - \varepsilon_n'') = n.$$

Take $n > m_0$ and observe that the horizontal lines $y = n$ cross the graph of $M_{p(x)}(t)$. Therefore, the sequence of Kirchhoff functions $M^n_{p(x)}(t)$ is continuous and satisfy condition $\mathcal{M}_1$ for each $n > m_0$, then there exists $\eta_n \in W_0$, such as for all $\varphi \in W_0$:

$$\frac{\langle (-\Delta)^{p(x)}M^n_{p(x)} \eta_n, \varphi \rangle}{\int_{\Omega} |\eta_n|^{p(x)} - 2 |\eta_n| dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \varphi dx} = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} - 2 |\eta_n| \ln |\eta_n| \right) \varphi dx. \tag{7.2}$$

Inset $\varphi = \eta_n$ in (7.2), that is

$$M^n_{p(x)}(\|\eta_n\|^{p(x)} \|\eta_n\|^{p(x)}) = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} - 2 |\eta_n| \ln |\eta_n| \right) \eta_n dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx, \tag{7.3}$$

which implies that

$$M^n_{p(x)}(\|\eta_n\|^{p(x)}) \leq \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} - 2 |\eta_n| \ln |\eta_n| \right) \eta_n dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx. \tag{7.4}$$

From $\mathcal{M}_3$, the sequence $\|\eta_n\|$ is bounded. Thus

$$\eta_n \rightharpoonup \eta, \text{ in } W_0, \eta_n \to \eta, \text{ in } L^{p(x)}(\Omega), \|\eta_n\|^{p(x)} \to \varrho_0, \text{ for some } \varrho_0.$$

Due to $\mathcal{M}_4$, the sequence of Kirchhoff functions $M^n_{p(x)}(\|\eta_n\|^{p(x)})$ converges and its limit is not equal to zero. Assume that $\|\eta_n\|^{p(x)} \to \varrho_0$.

If $\|\eta_n\|^{p(x)} \geq \varrho - \varepsilon_n''$ or $\|\eta_n\|^{p(x)} \leq \varrho - \varepsilon_n'$ as $n \to \infty$, we have $M^n_{p(x)}(\|\eta_n\|^{p(x)}) = M_{p(x)}(\|\eta_n\|^{p(x)})$, for all $\eta_n \in W_0$, such that

$$\langle (-\Delta)^{p(x)} \eta_n, \eta_n \rangle = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} - 2 |\eta_n| \ln |\eta_n| \right) \eta_n dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx, \tag{7.5}$$

this means

$$\infty = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} - 2 |\eta_n| \ln |\eta_n| \right) \eta_n dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx, \tag{7.6}$$

which is a contradiction.

If $\varrho - \varepsilon_n' \leq \|\eta_n\|^{p(x)} \leq \varrho - \varepsilon_n''$ as $n \to \infty$, we obtain $M^n_{p(x)}(\|\eta_n\|^{p(x)}) = n$, such that

$$n \|\eta_n\|^{p(x)} = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} - 2 |\eta_n| \ln |\eta_n| \right) \eta_n dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n dx. \tag{7.7}$$
it follows that
\[ \infty = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \right) \eta_n \, dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n \, dx, \]  
(7.8)
and we arrive again in a contradiction.

Thus, \( \| \eta_n \|^{p(x)} \to \varrho \neq \varrho \), which implies
\[ \| \eta_n \|^{p(x)} \geq \varrho - \varepsilon''_n \text{ or } \| \eta_n \|^{p(x)} \leq \varrho - \varepsilon'_n, \text{ as } n \to \infty, \]
and so
\[ M^n_{p(x)}(\| \eta_n \|^{p(x)}) = M_{p(x)}(\| \eta_n \|^{p(x)}), \]
which yields, for all \( \varphi \in W_0 \):
\[ \langle (-\Delta)^{M_{p(x)}(\eta_n, \varphi)} = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \right) \varphi \, dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \varphi \, dx. \]  
(7.9)
Fixing \( i \) in (4.2) and letting \( n \to +\infty \), we have
\[ M_{p(x)}(\varrho) \int_{\Omega} |\nabla \eta|^{p(x)-1} \nabla e_i \, dx = \lambda \int_{\Omega} \left( |\eta|^{p(x)-2} \eta \ln |\eta| \right) e_i \, dx + \int_{\Omega} g(x, \eta, \nabla \eta) e_i \, dx, \]
and since \( (e_i) \) is an orthonormal basis of \( W_0 \), thus
\[ M_{p(x)}(\varrho) \int_{\Omega} |\nabla \varphi|^{p(x)-1} \nabla \varphi \, dx = \lambda \int_{\Omega} \left( |\eta|^{p(x)-2} \eta \ln |\eta| \right) \varphi \, dx + \int_{\Omega} g(x, \eta, \nabla \eta) \varphi \, dx. \]  
(7.10)
Inset \( \varphi = \eta \) in (7.11), we obtain
\[ M_{p(x)}(\varrho) \| \eta \|^{p(x)} = \lambda \int_{\Omega} \left( |\eta|^{p(x)} \ln |\eta| \right) \, dx + \int_{\Omega} g(x, \eta, \nabla \eta) \eta \, dx. \]  
(7.11)
Similarly, from (4.2), we deduce
\[ M_{p(x)}(\| \eta_n \|^{p(x)}) \int_{\Omega} |\nabla \eta_n|^{p(x)-1} \nabla \varphi \, dx = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)-2} \eta_n \ln |\eta_n| \right) \varphi \, dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \varphi \, dx, \]
with \( \varphi \) replaced by \( \eta_n \) in (7.12), we have
\[ M_{p(x)}(\| \eta_n \|^{p(x)}) \| \eta_n \|^{p(x)} = \lambda \int_{\Omega} \left( |\eta_n|^{p(x)} \ln |\eta_n| \right) \, dx + \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n \, dx, \]
letting \( n \to +\infty \) for the above expression, we get
\[ M_{p(x)}(\varrho) \varrho = \lambda \int_{\Omega} \left( |\eta|^{p(x)} \ln |\eta| \right) \, dx + \int_{\Omega} g(x, \eta, \nabla \eta) \eta \, dx. \]  
(7.13)
Indeed, we have only to justify the limit:
\[ \int_{\Omega} g(x, \eta_n, \nabla \eta_n) \eta_n \, dx \to \int_{\Omega} g(x, \eta, \nabla \eta) \eta \, dx, \text{ as } n \to \infty. \]  
(7.14)
We find that
\[ \left| \int_{\Omega} (g(x, \eta_n, \nabla \eta_n) \eta_n - g(x, \eta, \nabla \eta) \eta) \, dx \right| \leq \int_{\Omega} |g(x, \eta_n, \nabla \eta_n)(\eta_n - \eta)| \, dx \]
\[ + \int_{\Omega} |(g(x, \eta_n, \nabla \eta_n) - g(x, \eta, \nabla \eta)) \eta| \, dx. \]
From hypothesis $G_1$, we have
\[ |g(x, \mu, \nu)| \leq \phi(x) + a_1|\mu|^{\frac{p(x)}{\tau_1(x)}} + b_2|\nu|^{\frac{p(x)}{\tau_2(x)}}, \]
thus, as in [22], we obtain
\[ |g(x, \mu, \nu)|^{p(x)} \leq 2^{2(p^+ - 1)} \left( \phi(x)^{p(x)} + a_1^{p(x)}|\mu|^{\frac{p(x)p(x)}{\tau_1(x)}} + b_2^{p(x)}|\nu|^{\frac{p(x)p(x)}{\tau_2(x)}} \right). \]

It follows from (3.3) and $g$ is a carathéodory function that
\[ |g(x, \eta_n, \nabla \eta_n)|^{p'(x)} \to |g(x, \eta, \nabla \eta)|^{p'(x)}, \text{ a.e. in } \Omega, \]
\[ |g(x, \eta_n, \nabla \eta_n)|^{p'(x)} \leq 2^{2(p^+ - 1)} \left( \phi(x)^{p'(x)} + a_1^{p'(x)}|\eta_n|^{\frac{p(x)p'(x)}{\tau_1(x)}} + b_2^{p'(x)}|\nabla \eta_n|^{\frac{p(x)p'(x)}{\tau_2(x)}} \right), \]
and
\[ \left( \phi(x)^{p'(x)} + a_1^{p'(x)}|\eta_n|^{\frac{p(x)p'(x)}{\tau_1(x)}} + b_2^{p'(x)}|\nabla \eta_n|^{\frac{p(x)p'(x)}{\tau_2(x)}} \right) \to \left( \phi(x)^{p'(x)} + a_1^{p'(x)}|\eta|^{\frac{p(x)p'(x)}{\tau_1(x)}} + b_2^{p'(x)}|\nabla \eta|^{\frac{p(x)p'(x)}{\tau_2(x)}} \right). \]

Thanks to Vitali’s theorem, we obtain
\[ g(x, \eta_n, \nabla \eta_n) \to g(x, \eta, \nabla \eta), \text{ in } L^{p'(x)}(\Omega), \]
which means that
\[ \int_{\Omega} \left| (g(x, \eta_n, \nabla \eta_n) - g(x, \eta, \nabla \eta)) \eta \right| dx \to 0, \text{ as } n \to \infty \] (7.15)

Using (6.3) and (7.15), we justify (7.14). Thus, it follows from Lemma 3.3 and (7.14) that (7.13) holds.

As a result, (7.11) and (7.13) imply that
\[ M_{p(x)}(\varrho_0)||\eta||^{p(x)} = M_{p(x)}(\varrho_0)\varrho_0. \]
Reasoning as before $M_{p(x)}(\varrho_0) \neq 0$, we deduce $||\eta||^{p(x)} = \varrho_0$ and so $\eta_n \to \eta$ strongly in $W_0$. The proof of the theorem is complete. \qed

8. Conclusions

In this article, we study a kind of Kirchhoff-type elliptic problems, which combine with variable exponent, logarithmic nonlinearity, and convection term. Under some reasonable assumptions and with the help of the Galerkin method, we obtain the existence of finite-dimensional approximate solutions, generalized solutions, and strong generalized solutions, and we also discuss the existence of solutions for discontinuous Kirchhoff-type equations. Our study extends previous results, such as, from the elliptic problem with logarithmic nonlinearity or convection term to Kirchhoff-type equations both logarithmic nonlinearity with variable exponent and convection term. Finally, we consider that it will be a new field to study such problems (1.2) in
fractional sobolev spaces with variable exponents and in sobolev spaces with variable exponents and variable fractional order.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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