Global existence of the self-interacting scalar field in the de Sitter universe

Karen Yagdjian

School of Mathematical and Statistical Sciences,
University of Texas RGV, 1201 W. University Drive,
Edinburg, TX 78539, USA

Abstract

We present some sufficient conditions for the global in time existence of solutions of the semilinear Klein-Gordon equation for the self-interacting scalar field with complex mass. The estimate for the lifespan is given for the equation with the Higgs potential. The coefficients of the equation depend on spatial variables as well, that makes results applicable, in particular, to the spacetime with the time slices being Riemannian manifolds.

Keywords: de Sitter spacetime; Klein-Gordon equation; semilinear equation; global solution

0 Introduction and Statement of Results

In this paper we present some sufficient conditions for the global in time existence of solutions of the semilinear Klein-Gordon equation for the self-interacting scalar field with complex mass. The estimate for the lifespan is given for the equation with the Higgs potential. The coefficients of the equation depend on spatial variables as well, that makes results applicable, in particular, to the spacetime with the time slices being Riemannian manifolds. The case of equation in the de Sitter spacetime (see, e.g., [9, p.113]) is included.

We consider the equation

\[ \psi_{tt} + n\psi_t - e^{-2t} A(x, \partial_x) \psi + m^2 \psi = F(x, \psi), \]  

(0.1)

where \( A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_{\alpha}(x) \partial_x^\alpha \) is a second order negative uniformly elliptic operator with coefficients \( a_{\alpha} \in B^\infty \), where \( B^\infty \) is the space of all \( C^\infty(\mathbb{R}^n) \) functions with uniformly bounded derivatives of all orders.

We also assume that the mass \( m \) can be a complex number, \( m^2 \in \mathbb{C} \).

In the quantum field theory the description of matter fields is based on the semilinear Klein-Gordon equation generated by the mass \( m \) and the metric \( g \):

\[ \Box_g \psi = m^2 \psi + V'_{\psi}(x, \psi). \]

Here \( \Box_g \) is the Laplace-Beltrami operator. In physical terms this equation describes a local self-interaction for a scalar particle. The special case of the equation (0.1) is the covariant Klein-Gordon equation in the de Sitter spacetime

\[ \psi_{tt} - \frac{e^{-2t}}{\sqrt{|\det \sigma(x)|}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left( \sqrt{|\det \sigma(x)|} \sigma^{ij}(x) \frac{\partial}{\partial x^j} \psi \right) + n\psi_t + m^2 \psi = F(\psi). \]

The metric \( \sigma(x) \) belongs to the time slices. The metric \( g \) in the de Sitter spacetime is as follows, \( g_{00} = g^{00} = -1 \), \( g_{ij} = g^{ij} = 0 \), \( g_{ij}(x, t) = e^{2t} \sigma_{ij}(x) \), \( i, j = 1, 2, \ldots, n \), where \( \sum_{j=1}^n \sigma^{ij}(x) \sigma_{jk}(x) = \delta_{ik} \), and \( \delta_{ij} \) is Kronecker’s delta.
In [31]-[33] the global existence of small data solutions of the Cauchy problem for the semilinear Klein-Gordon equation and systems of equations in the de Sitter spacetime with flat time slices, was proved. The nonlinearity $F$ was assumed Lipschitz continuous with the exponent $\alpha > 0$ (see definition below) while $m \in (0, \sqrt{n^2 - 1}/2) \cup [n/2, \infty)$. The proof of the global existence in [31]-[33] is based on the special integral representations (see Section 1) and $L^p - L^q$ estimates. Later on, in [18] this result for the same range of the parameters $n, m$ and the same nonlinearity was extended on the equation (0.1), that is, from the spatially flat de Sitter spacetime to the de Sitter spacetime with the time slices being, in particular, the Riemannian manifolds. The case of $m \in (\sqrt{n^2 - 1}/2, n/2)$ was left open in [18]. The existence of solution in the energy spaces was not proved in [18]. Another interesting and important case, that is, the case of the complex-valued mass $m$ also was not discussed in [18]. That case contains the Klein-Gordon model of the Higgs boson.

In the present paper we generalize and complete the small data global existence result of [18]. In particular, we study also class of equations containing the Higgs boson equation with the Higgs potential, that is the equation

$$\psi_{tt} - e^{-2t} A(x, D) \psi + m^2 \psi = -\lambda \psi^3,$$

with $\lambda > 0$ and $\mu > 0$, while $n = 3$.

The explicit form of nonlinear term $F$ in this paper is not used. What we use are simply the estimates of the form $\|F(\psi)\|_X < C\|\psi\|_{X'}\|\psi\|_{X''}$, for some function spaces $X, X'$ and $X''$. Furthermore, since we prove the results for small data in the Sobolev space $H_{(s)}(\mathbb{R}^n)$, we are only concerned with the behavior of $F$ at the origin.

**Condition (C).** The smooth in $x$ function $F = F(x, \psi)$ is said to be Lipschitz continuous with exponent $\alpha \geq 0$ in the space $H_{(s)}(\mathbb{R}^n)$ if there is a constant $C \geq 0$ such that

$$\|F(x, \psi_1(x)) - F(x, \psi_2(x))\|_{H_{(s)}} \leq C\|\psi_1 - \psi_2\|_{H_{(s)}} \left(\|\psi_1\|_{H_{(s)}} + \|\psi_2\|_{H_{(s)}} \right) \text{ for all } \psi_1, \psi_2 \in H_{(s)}.$$  \hspace{1cm} (0.2)

The polynomials in $\psi$ are Lipschitz continuous with some exponent $\alpha$ in the space $H_{(s)}(\mathbb{R}^n)$ when $s > n/2$. Moreover, the exponent $\alpha$ is independent of $s$. Another interesting functions $F(x, \psi) = \pm |\psi|^{\alpha + 1}$, $F(\psi) = \pm |\psi|^{n}\psi$ are important examples of the Lipschitz continuous with exponent $\alpha > 0$ in the Sobolev space $H_{(s)}(\mathbb{R}^n)$, $s > n/2$, functions provided that $\alpha$ agrees with $s$ and $n$. More detailed interplay between $\alpha$, $s$, and $n$ of the Condition (C) is an issue interesting in its own right but it is out of the scope of this paper.

Define also the metric space

$$X(R, H_{(s)}, \gamma) := \left\{ \psi \in C([0, \infty); H_{(s)}) \, | \, \| \psi \|_X := \sup_{t \in [0, \infty)} e^{\gamma t} \| \psi(x, t) \|_{H_{(s)}} \leq R \right\},$$

where $\gamma \in \mathbb{R}$, with the metric

$$d(\psi_1, \psi_2) := \sup_{t \in [0, \infty)} e^{\gamma t} \| \psi_1(x, t) - \psi_2(x, t) \|_{H_{(s)}}.$$

We study the Cauchy problem (0.7), (0.8) through the integral equation. To define that integral equation we appeal to the operator

$$G := K \circ \mathcal{EE}$$

($\mathcal{EE}$ stands for the evolution equation) as follows. For the function $f(x, t)$ we define

$$v(x, t; b) := \mathcal{EE}[f](x, t; b),$$

where the function $v(x, t; b)$ is a solution to the Cauchy problem

$$\partial^2_t v - A(x, D)v = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0,$$

$$v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0, \quad x \in \mathbb{R}^n.$$

(0.3)
while $K$ is introduced by

$$
K[v](x, t) := 2e^{-\frac{2}{n}} \int_0^t \int_0^{e^{-\frac{2}{n}} - e^{-t}} dr e^{\frac{2}{n}b} v(x, r; b) E(r, t; 0, b; M).
$$

The kernel $E(r, t; 0, b; M)$ was introduced in [31] and [29] (see also (1.2)). Hence,

$$
G[f](x, t) = 2e^{-\frac{2}{n}} \int_0^t \int_0^{e^{-\frac{2}{n}} - e^{-t}} dr e^{\frac{2}{n}b} \mathcal{E}[f](x, r; b) E(r, t; 0, b; M).
$$

Thus, the Cauchy problem (0.7), (0.8) leads to the following integral equation

$$
\Phi(x, t) = \Phi_0(x, t) + G[F(\cdot, \Phi)](x, t). \tag{0.5}
$$

Every solution to the Cauchy problem (0.7)-(0.8) solves also the last integral equation with some function $\Phi_0(x, t)$, which is a solution for the problem for the linear equation without source term. We define a solution of the Cauchy problem (0.7)-(0.8) via integral equation (0.5). Since only for $m \in (0, \sqrt{n^2 - 1/2}] \cup [n/2, \infty)$ the existence of global in time solution has been proved in [18], in the present paper we consider the more general case of the complex mass $m \in \mathbb{C}$ that includes, in particular, the Higgs boson equation. The principal square root $M := (n^2/4 - m^2)^{1/2}$ is the parameter that controls estimates and solvability. In fact, $M := iM$ is the so-called effective mass or curved mass of the field. The main result of this paper is the next theorem.

**Theorem 0.1** Assume that the nonlinear term $F(x, \Phi)$ is a Lipschitz continuous in the space $H_{(\alpha)}(\mathbb{R}^n)$, $s > n/2 \geq 1$, $F(x, 0) \equiv 0$, and $\alpha > 0$.

(i) Assume also that $0 < \Re M < 1/2$. Then, there exists $\varepsilon_0 > 0$ such that, for every given functions $\varphi_0, \varphi_1 \in H_{(\alpha)}(\mathbb{R}^n)$, such that

$$
\|\varphi_0\|_{H_{(\alpha)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(\alpha)}(\mathbb{R}^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0, \tag{0.6}
$$

there exists a solution $\Phi \in C([0, \infty); H_{(\alpha)}(\mathbb{R}^n))$ of the Cauchy problem

$$
\begin{align*}
\Phi_{tt} + n\Phi_t - e^{-2t} A(x, \partial_x) \Phi + m^2 \Phi &= F(x, \Phi), \tag{0.7} \\
\Phi(x, 0) &= \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x). \tag{0.8}
\end{align*}
$$

The solution $\Phi(x, t)$ belongs to the space $X(2\varepsilon, s, \frac{n-1}{2})$, that is,

$$
\sup_{t \in [0, \infty)} e^{\frac{2}{n}t} \|\Phi(\cdot, t)\|_{H_{(\alpha)}(\mathbb{R}^n)} \leq 2\varepsilon.
$$

(ii) Assume that $M = 1/2$ or $1/2 < \Re M < n/2$ and $\gamma \in (0, \frac{1}{\sqrt{n^2 - 1/2}} - \Re M)$. Then there exists $\varepsilon_0 > 0$ such that for every given functions $\varphi_0, \varphi_1 \in H_{(\alpha)}(\mathbb{R}^n)$, such that (0.6), there exists a solution $\Phi \in X(2\varepsilon, s, \gamma)$ of the Cauchy problem (0.7)-(0.8).

(iii) If $\Re M > n/2$, then the lifespan $T_{is}$ of the solution can be estimated from below as follows

$$
T_{is} \geq \frac{1}{\Re M - \frac{n}{2}} \ln \left(\|\varphi_0\|_{H_{(\alpha)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(\alpha)}(\mathbb{R}^n)}\right) - C(m, n, \alpha)
$$

with some constant $C(m, n, \alpha)$.

In particular, the theorem covers the case of $m \in (\sqrt{n^2 - 1/2}, n/2)$. If

$$
F(\Phi) = \lambda \Phi^3 \quad \text{or} \quad F(\Phi) = \pm |\Phi|^{\alpha} \Phi \quad \text{or} \quad F(\Phi) = \pm |\Phi|^{\alpha+1},
$$

then the small data Cauchy problem is globally solvable for every $\alpha, s$, and $n$ satisfying (L).

Although, there is no conservation of energy due to the dependence on time of the coefficient, the energy estimate provides with the useful tool to prove global existence in the energy space if we impose some restriction on the nonlinearity. The last theorem as well as the results of articles [17, 18] imply global solvability of the problem in the energy space under some conditions on the nonlinear term $F$ and mass $m$.  




Theorem 0.2 Assume that the nonlinear term $F(u)$ is Lipschitz continuous in the space $H^{(s)}(\mathbb{R}^n)$, $s > n/2 \geq 1$, $F(0) = 0$, and $\alpha > \frac{2}{n-1}$. Assume also that either $M^2 \in \mathbb{R}$ and $\Re M \in (0, 1/2)$ or $M = 1/2$. Then, there exists $\varepsilon_0 > 0$ such that, for every given functions $\varphi_0 \in H^{(s+1)}(\mathbb{R}^n)$, $\varphi_1 \in H^{(s)}(\mathbb{R}^n)$, such that
\[
\|\varphi_0\|_{H^{(s+1)}(\mathbb{R}^n)} + \|\varphi_1\|_{H^{(s)}(\mathbb{R}^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,
\]
there exists a global solution $\Phi \in C^1([0, \infty); H^{(s)}(\mathbb{R}^n))$ of the Cauchy problem (0.7)-(0.8). The solution $\Phi(x, t)$ and its time derivative $\partial_t \Phi(x, t)$ belong to the space $X(2\varepsilon, s, \frac{n-1}{2})$, that is,
\[
\sup_{t \in [0, \infty)} e^{\frac{n-1}{2}t} \left(\|\Phi(\cdot, t)\|_{H^{(s)}(\mathbb{R}^n)} + \|\partial_t \Phi(\cdot, t)\|_{H^{(s)}(\mathbb{R}^n)}\right) < 2\varepsilon.
\]
Assume that $M^2 \in \mathbb{R}$ and $\Re M \in (3/2, n/2)$ or $M = 3/2$, then there exists $\varepsilon_0 > 0$ such that, for every given functions $\varphi_0 \in H^{(s+1)}(\mathbb{R}^n)$, $\varphi_1 \in H^{(s+1)}(\mathbb{R}^n)$, such that
\[
\|\varphi_0\|_{H^{(s+1)}(\mathbb{R}^n)} + \|\varphi_1\|_{H^{(s+1)}(\mathbb{R}^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,
\]
there exists a global solution $\Phi \in C^1([0, \infty); H^{(s)}(\mathbb{R}^n))$ of the Cauchy problem (0.7)-(0.8) such that $\Phi(x, t) \in X(2\varepsilon, s, \gamma)$ and its time derivative $\partial_t \Phi(x, t)$ belong to the space $X(2\varepsilon, s, \gamma - 1)$ with $\gamma \in (0, \frac{1}{\alpha+1}(\frac{2}{\alpha} - \Re M))$, that is,
\[
\sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi(\cdot, t)\|_{H^{(s)}(\mathbb{R}^n)} + \sup_{t \in [0, \infty)} e^{(\gamma-1)t} \|\partial_t \Phi(\cdot, t)\|_{H^{(s)}(\mathbb{R}^n)} < 2\varepsilon.
\]

The main tools to study the problem (0.7)-(0.8) are the integral transform from [34] and the standard energy estimate for the finite time interval for the strictly hyperbolic equation. On the other hand, by using the integral transforms given in [34], it is possible to reduce the problem with the infinite time interval to the problem for the hyperbolic equation with time independent coefficients and with the finite time interval due to the fact that the de Sitter spacetime has permanently bounded domain of influence. In this approach the integral transform allows us to push forward the estimates provided that some integrals of the kernel functions lead to the proper estimates. The proof of the estimates for the kernel functions consists of a long sequence of estimates of integrals involving hypergeometric functions. The proof of Theorem 0.1 is concluded by the fixed point arguments.

The estimates derived for the linear equation in Sections 1, 3 include equations of scalar fields considered in [8, 15] with $m^2 < 0$ living on the de Sitter universe. The Klein-Gordon scalar quantum fields on the de Sitter manifold with imaginary mass $m^2 = -k(k + n)$, $k = 0, 1, 2, \ldots$, present a family of tachyonic quantum fields. Epstein and Moschella [15] give a complete study of such family of linear scalar tachyonic quantum fields. The corresponding linear equation is
\[
\psi_{tt} + n\psi_t - e^{-2t}\Delta \psi + m^2 \psi = 0, \quad (0.9)
\]
for which the kernel of the integral transform $K$ is $E(x, t; x_0, t_0; M)$, where $M = k + \frac{n}{2}$, $k = 0, 1, 2, \ldots$. For an odd number $n$, the mass $m$ takes value at knot points set in the sense of [32]. Theorem 0.1 contains an estimate for the lifespan of a self-interacting tachyonic field.

The approach of this paper can be easily modified to obtain estimates for the linear equation and the global solvability for the equation
\[
\Phi_{tt} + \nu\Phi_t - e^{-2t} A(x, \partial_x) \Phi + m^2 \Phi = f(x, t) + F(x, \Phi), \quad (0.10)
\]
where $\nu \in \mathbb{R}$. It is also possible include the derivatives of the field function in the nonlinear term, as well as to apply the approach of the present paper to system of equations similar to [33].

The equation (0.1) in the case of the $x$-independent operator $A(x, \partial_x) = \Delta$ is amenable to the analysis via the Fourier transform and the Bessel functions (see, e.g., [16]). For the equation (0.10) with $A(x, \partial_x) = \Delta$, $f(x, t) = 0$, and $F(x, \Phi) = |\Phi|^p$ it was discussed in [13] and global existence in the energy classes and in the Lebesgue spaces was proved under several restrictions on the nonlinear term. The $x$-independence of the
coefficients allows authors to apply the Fourier transform and to write an explicit form of the solution of the corresponding ordinary differential equation.

Unlike to the case of the operator $A(x, \partial_x) = \Delta$, the linear part of the equation (0.1) is not invariant with respect to de Sitter group (see, e.g., [19, 21]). Nevertheless, the mass intervals $(0, \sqrt{n^2 - 1}/2)$, $[\sqrt{n^2 - 1}/2, n/2)$, $[n/2, \infty)$ appear and play important role also in this case. The first interval $(0, \sqrt{n^2 - 1}/2)$ with $n = 3$ in quantum field theory is known as the Higuchi bound (forbidden mass range). The masses in this range lead to negative norm states, i.e., non-unitarity. In [22] it is shown that for spin-2 fields the forbidden mass range is $0 < m^2 < 2$. The mass $m = \sqrt{n^2 - 1}/2$ is remarkable especially because that is the only mass that makes equation (0.9) huygensian and makes the linear part of the equation conformally invariant [7]. The values $0$ and $\sqrt{n^2 - 1}/2$ are the only values of mass such that the equation obeys incomplete Huygens' principle [32]. In the de Sitter spacetime the existence of two different scalar fields (in fact, with $m = 0$ and $m^2 = (n^2 - 1)/4$), which obey incomplete Huygens' principle, is equivalent to the condition $n = 3$ (Corollary 4 [32]), which is the spatial dimension of the physical world. In fact, Paul Ehrenfest in [14] addressed the question: “Why has our space just three dimensions?”.

Thus, the point $m = \sqrt{2}$ ($n = 3$) is exceptional for the the quantum fields theory in the de Sitter spacetime. In particular, for massive spin-2 fields, it is known [12, 22] that the norm of the helicity zero mode changes sign across the line $m^2 = 2$. The region $m^2 < 2$ is therefore unitarily forbidden. It is noted in [2] that all canonically normalized helicity $-0, \pm 1, \pm 2$ modes of massive graviton on the de Sitter universe satisfy Klein-Gordon equation for a massive scalar field with the same effective mass. Then, it is known (see, e.g., [11]) that, if $n = 3$ and $m = \sqrt{2}$, the action is invariant under the gauge transformation, and that invariance already suggests that there exists some discontinuity in the theory at $m = \sqrt{2}$. For the case of large mass, that is $m^2 \geq n^2/4$, and for the brief review of the bibliography related to that case, one can consult [18, 26, 35] and for the results on the equation in the asymptotically de Sitter spaces see [4, 5, 23, 24, 28]. The waves in spacetimes with a nonvanishing cosmological constant are studied in [3, 10, 25].

Another important value if $n = 3$ is $m = 3/2$. The equation for the scalar field with mass $m$ in de Sitter universe in the physical variables is:

$$\frac{1}{c^2} \psi_{tt} + \frac{1}{c^2} 3H \psi_t - e^{-2tH} \Delta \psi + \left( \frac{cm}{h} \right)^2 \psi = 0.$$  

Here $h = 1.054 \cdot 10^{-27}$ erg·sec, $c \approx 3 \cdot 10^{10}$ cm/sec, $H \approx 10^{-18}$ cm/sec. The following question seems to be natural: For what particle (mass) the equation has the most simple form? In fact, for the scalar field with the mass $m = \frac{3hH}{2c^2}$ the function $u = e^{-\frac{3}{2}Ht} \psi$ solves the equation

$$\frac{1}{c^2} u_{tt} - e^{-2tH} \Delta u = 0.$$  

In the physical units this particle has a mass $m = \frac{3hH}{2c^2} \approx 1.756 \cdot 10^{-66}$ g. The natural question arises: What particle has this mass? In fact, there exists an extensive literature on this topic. The comparison with the estimate $m_g < 1.8 \cdot 10^{-66}$ g from [20] supports the following conjecture (see, e.g., [30]): the mass $m = 3hH/(2c^2)$ is a mass of graviton.

The present paper is organized as follows. In Section 1 we describe the integral transform and the generated by that transform representations (from [34]) for the solutions of the Cauchy problem for the linear equation. Then, we show that the energy estimates for the second order hyperbolic operator with time independent coefficients can be pushed forward via integral transform to the source free equation with time dependent coefficient. In the present paper we prove estimates for the Sobolev spaces only. In fact, the proofs for the Lebesgue, Sobolev and Besov spaces are identical. In Section 2 we obtain similar estimates for the equation with the source term. The last section, Section 3, is devoted to the solvability of the associated integral equation and to the proof of Theorem 0.1 and Theorem 0.2. In the Appendix one can find several useful lemmas concerning hypergeometric function which have been used in the previous sections.

## 1 $H_{(s)}(\mathbb{R}^n)$ Estimates

In this section we consider the linear part of the equation

$$u_{tt} - e^{-2t} A(x, D) u - M^2 u = -e^{\frac{-\bar{R}}{2} t} V'(e^{-\frac{-\bar{R}}{2} t} u),$$  

(1.1)
with $M \in \mathbb{C}$. The equation (1.1) covers two important cases. The first one is the Higgs boson equation, which has $V'(\phi) = \lambda \phi^3$ and $M^2 = \frac{n^2}{4} + \mu^2$ with $\lambda > 0$, $\mu > 0$, and $n = 3$. This includes also equation of tachyonic scalar fields living on the de Sitter universe. (See, e.g. [8, 15].) The second case is the case of the small physical mass (the light scalar field), that is $0 < m \leq \frac{\sqrt{n^2}}{4}$. For the last case $M = \sqrt{\frac{n^2}{4} - m^2}$.

We introduce the kernel functions $E(x, t; x_0, t_0; M)$, $K_0(z, t; M)$, and $K_1(z, t; M)$ (see also [29] and [31]). First, for $M \in \mathbb{C}$ we define the function

$$E(x, t; x_0, t_0; M) = 4^{-M}e^{Mt}t_0+t\left(\left(e^{-t} + e^{-t_0}\right)^2 - (x - x_0)^2\right)^{-\frac{1}{2}+M} \times \mathbb{F}\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t} - e^{-t_0})^2 - (x - x_0)^2}{(e^{-t} + e^{-t_0})^2 - (x - x_0)^2}\right)$$

Next we define also the kernels $K_0(z, t; M)$ and $K_1(z, t; M)$ by

$$K_0(z, t; M) := -\left[\frac{\partial}{\partial b}E(z, t; 0, b; M)\right]_{b=0} = 4^{-M}e^{Mt}(1 + e^{-t})^2 - z^2\left(\frac{1}{(1 + e^{-t})^2 - z^2}\right)^M \times \left[\left(1 + M(e^{-2t} - 1 - z^2)\right)F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right) + (1 - e^{-2t} + z^2)\left(\frac{1}{2} + M\right)F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right)\right]$$

and $K_1(z, t; M) := E(z, t; 0, 0; M)$, that is,

$$K_1(z, t; M) = 4^{-M}e^{Mt}(1 + e^{-t})^2 - z^2\left(\frac{1}{(1 + e^{-t})^2 - z^2}\right)^{-\frac{1}{2}+M} \times \mathbb{F}\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; (1 - e^{-t})^2 - z^2\right) = 0 \leq z \leq 1 - e^{-t},$$

respectively.

The solution $\Phi$ to the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}A(x, \partial_x)\Phi + m^2\Phi = f, \quad \Phi(0, 0) = \varphi_0(x), \quad \Phi_t(0, 0) = \varphi_1(x),$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with $\varphi_0, \varphi_1 \in C^\infty_0(\mathbb{R}^n)$, $n \geq 2$, is given in [34] by the next expression

$$\Phi(x, t) = 2e^{-\frac{t}{2}}\int_0^t dt \int_0^t e^{-t_1} dr e^{\frac{t}{2}t}v(x, r; b)E(r, t; 0, b; M)$$

$$+ e^{-\frac{t}{2}t}v_\varphi(x, t) + e^{-\frac{t}{2}t}\int_0^1 v_{\varphi_0}(x, \phi(t)s)(2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M))\phi(t) ds$$

$$+ 2e^{-\frac{t}{2}t}\int_0^1 v_{\varphi_1}(x, \phi(t)s)K_1(\phi(t)s, t; M)\phi(t) ds, \quad x \in \mathbb{R}^n, \quad t > 0,$$

where the function $v(x, t; b)$ is a solution to the Cauchy problem (0.3)-(0.4), while $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C^\infty_0(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x, t)$ of the Cauchy problem for the equation (0.3) with the initial datum $\varphi(x)$ while the second datum is zero.

The mass $m^2 = (n^2 - 1)/4$, that is, $M = 1/2$, simplifies the hypergeometric functions, as well as, the kernels $E(x, t; x_0, t_0; \frac{1}{2})$, $K_0(z, t; \frac{1}{2})$ and $K_1(z, t; \frac{1}{2})$ (see [32]). In that case

$$E\left(x, t; x_0, t_0; \frac{1}{2}\right) = \frac{1}{2}e^{\frac{t}{2}(t_0+t)}, \quad K_0\left(z, t; \frac{1}{2}\right) = -\frac{1}{4}e^{\frac{t}{2}}, \quad K_1\left(z, t; \frac{1}{2}\right) = \frac{1}{2}e^{\frac{t}{2}}.$$
For the solution of the problem (1.3) it follows
\[
\Phi(x, t) = e^{-\frac{n+1}{2}t} \int_0^t e^{\frac{n+1}{2}b} db \int_0^{e^{b-t}} v(x, r; b) dr + e^{-\frac{n+1}{2}t} v_{\phi_0}(x, 1 - e^{-t}) \tag{1.5}
\]
\[
+ \frac{n-1}{2} e^{-\frac{n-1}{2}t} \int_0^{1-e^{-t}} v_{\phi_0}(x, s) ds + e^{-\frac{n-1}{2}t} \int_0^{1-e^{-t}} v_{\phi_1}(x, s) ds, \quad x \in \mathbb{R}^n, \quad t > 0,
\]
where the functions \(v(x, r; b), \ v_{\phi_0}, \) and \(v_{\phi_1}\) are defined above.

### 1.1 \(H_s(\mathbb{R}^n)\) Estimates for Equation without Source

Let \(A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha\) is a second order negative uniformly elliptic operator with coefficients \(a_\alpha \in \mathcal{B}^\infty, \) where \(\mathcal{B}^\infty\) is the space of all \(C^\infty(\mathbb{R}^n)\) functions with uniformly bounded derivatives of all orders. Let \(u = u(x, t)\) be the solution of
\[
\partial_t^2 u - A(x, D)u = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0,
\]
\[
u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x), \quad x \in \mathbb{R}^n. \tag{1.6}
\]

The following energy estimate is well known. (See, e.g., [27].) For every \(s \in \mathbb{R}\) there is \(C_s\) such that
\[
\|u_t(t)\|_{H_s} + \|u(t)\|_{H_{s+1}} \leq C_s(\|g_1\|_{H_s} + \|g_0\|_{H_{s+1}}), \quad 0 \leq t \leq 1. \tag{1.8}
\]

We note that although in this estimate the time interval is bounded, meanwhile, due to the integral transforms given in [34], it is possible to reduce the problem with infinite time to the problem with the finite time, and to apply (1.8). We must to emphasize that this is possible since the de Sitter spacetime has permanently bounded domain of influence.

**Theorem 1.1** For every given \(s \in \mathbb{R}\), the solution \(\Phi = \Phi(x, t)\) of the Cauchy problem
\[
\Phi_{tt} + n \Phi_t - e^{-2t} A(x, D) \Phi + m^2 \Phi = 0, \quad \Phi(x, 0) = \phi_0(x), \quad \Phi_t(x, 0) = \phi_1(x), \tag{1.9}
\]
with \(\mathcal{R}M = \mathbb{R}(\frac{n^2}{4} - m^2)^{1/2} \in (0, 1/2)\) satisfies the following estimate
\[
\|\Phi(x, t)\|_{H_s} \leq C_{m,n,s} e^{-\frac{n-1}{2}t} \left\{ \|\phi_0\|_{H_s} + (1 - e^{-t})\|\phi_1\|_{H_s} \right\} \quad \text{for all} \quad t \in (0, \infty).
\]

If \(\mathcal{R}M = \mathbb{R}(\frac{n^2}{4} - m^2)^{1/2} > 1/2\) or \(M = 1/2\), then the solution \(\Phi = \Phi(x, t)\) of the Cauchy problem (1.9) satisfies the following estimate
\[
\|\Phi(x, t)\|_{H_s} \leq C e^{(\mathcal{R}M - \frac{n}{2})t} \left\{ \|\phi_0\|_{H_s} + (1 - e^{-t})\|\phi_1\|_{H_s} \right\} \quad \text{for all} \quad t \in (0, \infty).
\]

**Proof.** The case of \(\varphi_1 = 0\) is an evident consequence of (1.8) and the representation (1.5) and in the remaining part of the proof it is not discussed.

First we consider the case of \(\varphi_1 = 0\). Then
\[
\Phi(x, t) = e^{-\frac{n+1}{2}t} v_{\phi_0}(x, \phi(t)) + e^{-\frac{n-1}{2}t} \int_0^1 v_{\phi_0}(x, \phi(t)s)(2K_0(\phi(t)s; t; M) + nK_1(\phi(t)s; t; M)) \phi(t) ds
\]
and, consequently,
\[
\|\Phi(x, t)\|_{H_s} \leq e^{-\frac{n-1}{2}t} \|v_{\phi_0}(x, \phi(t))\|_{H_s} \tag{1.10}
\]
\[
+ e^{-\frac{n-1}{2}t} \int_0^1 \|v_{\phi_0}(x, \phi(t)s)\|_{H_s} |2K_0(\phi(t)s; t; M) + nK_1(\phi(t)s; t; M)| \phi(t) ds.
\]

Then for the solution \(v = v(x, t)\) of the Cauchy problem (1.6)-(1.7) with \(\varphi(x) \in C_0^\infty(\mathbb{R}^n)\) one has the estimate (1.8). Hence,
\[
e^{-\frac{n-1}{2}t} \|v_{\phi_0}(x, \phi(t))\|_{H_s} \leq Ce^{-\frac{n-1}{2}t} \|\varphi_0\|_{H_s} \quad \text{for all} \quad t > 0.
\]
where $\phi(t) := 1 - e^{-t}$. For the second term of (1.10) we obtain
\[
e^{-\frac{3}{2}t}\int_0^1 \|v_{\varphi_0}(x, \phi(t)s)\|_{H(x)} \cdot 2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M)\|\phi(t)\| ds
\leq \|\varphi_0\|_{H(x)} e^{-\frac{3}{2}t}\int_0^1 \left(\|2K_0(\phi(t)s, t; M)\| + n\|K_1(\phi(t)s, t; M)\|\right) \|\phi(t)\| ds.
\]
We have to estimate the following two integrals of the last inequality:
\[
\int_0^1 |K_t(\phi(t)s, t; M)|\|\phi(t)\| ds, \quad i = 0, 1,
\]
where $t > 0$. To complete the estimate of the second term of (1.10) we are going to apply the next two lemmas with $a = 0$.

**Lemma 1.2** Let $a > -1$, $\Re M > 0$, and $\phi(t) = 1 - e^{-t}$. Then
\[
\int_0^1 \phi(t)^a s^a |K_t(\phi(t)s, t; M)|\|\phi(t)\| ds \leq C_M e^{-at}(e^t - 1)^a + 1(e^t + 1)^{\Re M - 1}
\]
for all $t > 0$.

In particular,
\[
\int_0^1 \phi(t)^a s^a |K_t(\phi(t)s, t; M)|\|\phi(t)\| ds \leq C_M e^{\Re M t}
\]
for large $t$.

**Proof.** By the definition of the kernel $K_t$, we obtain
\[
\int_0^1 \phi(t)^a s^a |K_t(\phi(t)s, t; M)|\|\phi(t)\| ds = \int_0^{1 - e^{-t}} r^a |K_t(r, t; M)| dr
\leq 4^{-\Re M} e^{\Re M t} \int_0^{1 - e^{-t}} r^a ((1 + e^{-t})^2 - r^2)^{-\frac{a}{2} + \Re M} \left|F \left(\frac{1}{2} - M, \frac{1}{2} - M, 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right)\right| dr
\leq 4^{-\Re M} e^{\Re M t} \int_0^{e^{-t}} e^{-2\Re M t} e^{-at} y^a ((e^t + 1)^2 - y^2)^{-\frac{a}{2} + \Re M}
\times \left|F \left(\frac{1}{2} - M, \frac{1}{2} - M, 1; \frac{(e^t - 1)^2 - y^2}{(e^t + 1)^2 - y^2}\right)\right| e^{-t} dy,
\]
where the substitution $e^{t} r = y$ has been used. Thus,
\[
\int_0^1 \phi(t)^a s^a |K_t(\phi(t)s, t; M)|\|\phi(t)\| ds \leq 4^{-\Re M} e^{-\Re M t - at} \int_0^{e^{-t}} y^a ((e^t + 1)^2 - y^2)^{-\frac{a}{2} + \Re M}
\times \left|F \left(\frac{1}{2} - M, \frac{1}{2} - M, 1; \frac{(e^t - 1)^2 - y^2}{(e^t + 1)^2 - y^2}\right)\right| dy.
\]
On the other hand, for $\Re M > 0$ we have (see Section A)
\[
\left|F \left(\frac{1}{2} - M, \frac{1}{2} - M, 1; \zeta\right)\right| \leq C_M
\]
for all $\zeta \in [0, 1],$

where
\[
\zeta := \frac{(e^t - 1)^2 - y^2}{(e^t + 1)^2 - y^2} \in [0, 1]
\]
for all $y \in [0, e^t - 1]$ and all $t > 0$.

Hence,
\[
\int_0^1 \phi(t)^a s^a |K_t(\phi(t)s, t; M)|\|\phi(t)\| ds \leq C_M e^{-\Re M t - at} \int_0^{e^{-t}} y^a ((e^t + 1)^2 - y^2)^{-\frac{a}{2} + \Re M} dy.
\]
If we denote $z := e^t$, then for $M > 0$ we have
\[
\int_0^{z^{-1}} y^a((z+1)^2 - y^2)^{\frac{1}{2}+M} \, dy = \frac{1}{1+a} (z - 1)^{1+a}(z + 1)^{2M-1} F\left(\frac{1+a}{2}, \frac{1}{2} - M; \frac{3+a}{2}; \frac{(z-1)^2}{(z+1)^2}\right),
\]
where $a > -1$ and $z \geq 1$. Hence, for $\Re M > 0$ we have
\[
\int_0^1 \phi(t) s^a |K_1(\phi(t)s, t; M)| \phi(t) \, ds \leq C M e^{-\Re M - at} (e^t - 1)^{a+1}(e^t + 1)^{2\Re M - 1} \text{ for all } t > 0.
\]
Thus the lemma is proved.

\[\square\]

**Lemma 1.3** Let $a > -1$, $\Re M > 0$, and $\phi(t) = 1 - e^{-t}$. Then
\[
\int_0^1 \phi(t) s^a |K_0(\phi(t)s, t; M)| \phi(t) \, ds \leq C_{M,a} (e^t - 1)^{a+1} \times \begin{cases}
\frac{e^{-at}(e^t + 1)^{-\frac{1}{2}}}{e^{(\Re M-a)t}} (e^t + 1)^{-1} & \text{if } \Re M < 1/2, \\
1 & \text{if } \Re M > 1/2,
\end{cases}
\]
for all $t > 0$. In particular,
\[
\int_0^1 \phi(t) s^a |K_0(\phi(t)s, t; M)| \phi(t) \, ds \leq C_{M,a} \times \begin{cases}
\frac{e^{\frac{3}{2}t}}{e^{2\Re M t}} & \text{if } \Re M < 1/2, \\
1 & \text{if } \Re M > 1/2,
\end{cases}
\]
for large $t$.

**Proof.** By substituting $K_0$ into integral, we obtain
\[
\int_0^1 \phi(t) s^a |K_0(\phi(t)s, t; M)| \phi(t) \, ds \leq 4^{-\Re M} e^{\Re M} \int_0^{1-e^{-t}} r^a ((1 + e^{-t})^2 - r^2)^{\Re M} \frac{1}{\sqrt{(1 - e^{-t})^2 - r^2}}
\]
\[
\times \left[ (e^{-t} - 1 + M(e^{-2t} - 1 - r^2)) F\left(\frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right)
\]
\[
+ (1 - e^{-2t} + r^2) \left(\frac{1}{2} + M\right) F\left(\frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right) \right] \, dr.
\]
Now we make the change $r = e^{-t}y$ in the last integral and obtain
\[
\int_0^{1-e^{-t}} r^a ((1 + e^{-t})^2 - r^2)^{\Re M} \frac{1}{\sqrt{(1 - e^{-t})^2 - r^2}}
\]
\[
\times \left[ (e^{-t} - 1 + M(e^{-2t} - 1 - r^2)) F\left(\frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right)
\]
\[
+ (1 - e^{-2t} + r^2) \left(\frac{1}{2} + M\right) F\left(\frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right) \right] \, dr
\]
\[
= e^{-2\Re M t} e^{-at} \int_0^{e^t - 1} y^a ((e^t + 1)^2 - y^2)^{\Re M} \frac{1}{((e^t - 1)^2 - y^2) \sqrt{(e^t + 1)^2 - y^2}}
\]
\[
\times \left[ (e^t - e^{2t} + M(1 - e^{2t} - y^2)) F\left(\frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(e^t - 1)^2 - y^2}{(e^t + 1)^2 - y^2}\right)
\]
\[
+ (e^{2t} - 1 + y^2) \left(\frac{1}{2} + M\right) F\left(\frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(e^t - 1)^2 - y^2}{(e^t + 1)^2 - y^2}\right) \right] \, dy.
\]
Then we denote \( z = e^t \) and derive
\[
\int_0^1 \phi(t)^a s^\alpha |K_0(\phi(t)s,t;M)|\phi(t) \, ds \\
\leq z^{-(\Re M + a)} \int_0^{z^{-1}} y^\alpha ((z + 1)^2 - y^2)^{\Re M} \frac{1}{((z - 1)^2 - y^2)^{\Re M} \sqrt{(z + 1)^2 - y^2}} \\
\times \left[ \left( (z - z^2 + M(1 - z^2 - y^2)) F\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \\
+ (z^2 - 1 + y^2) \left( \frac{1}{2} + M \right) F\left( - \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right] \right] \, dy.
\]

To complete the proof of lemma we need the estimate given by the following proposition.

**Proposition 1.4** If \( a > -1 \) and \( \Re M > 0 \), then
\[
\int_0^{z^{-1}} y^\alpha ((z + 1)^2 - y^2)^{\Re M} \frac{1}{((z - 1)^2 - y^2)^{\Re M} \sqrt{(z + 1)^2 - y^2}} \\
\times \left[ \left( (z - z^2 + M(1 - z^2 - y^2)) F\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \\
+ (z^2 - 1 + y^2) \left( \frac{1}{2} + M \right) F\left( - \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right] \right] \, dy \\
\leq C_{\Re M, n, p, q, s}(z - 1)^{1+a} \times \left\{ \begin{array}{ll}
(z + 1)^{\Re M - \frac{1}{2}} & \text{if } \Re M < 1/2, \\
(z + 1)^{2\Re M - 1} & \text{if } \Re M > 1/2.
\end{array} \right.
\]

**Proof.** We follow the arguments have been used in the proof of Lemma 7.4 [29]. For \( \Re M > 0 \) the both hypergeometric functions are bounded. We divide the domain of integration into two zones,
\[
Z_1(\varepsilon,z) := \{(z,r) \mid \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2} \leq \varepsilon, \ 0 \leq r \leq z - 1 \},
\]
\[
Z_2(\varepsilon,z) := \{(z,r) \mid \varepsilon \leq \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}, \ 0 \leq r \leq z - 1 \},
\]
and then split the integral into two parts,
\[
\int_0^{z^{-1}} *dr = \int_{(z,r) \in Z_1(\varepsilon,z)} *dr + \int_{(z,r) \in Z_2(\varepsilon,z)} *dr.
\]

In the first zone \( Z_1(\varepsilon,z) \) we have
\[
F\left( \frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) = 1 + \left( \frac{1}{2} - M \right)^2 \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} + O\left( \left( \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right)^2 \right),
\]
\[
F\left( - \frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) = 1 - \left( \frac{1}{4} - M^2 \right) \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} + O\left( \left( \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right)^2 \right).
\]

We use the last formulas to estimate the term containing hypergeometric functions:
\[
\left| (z - z^2 + M(1 - z^2 - y^2)) F\left( \frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \\
+ (z^2 - 1 + y^2) \left( \frac{1}{2} + M \right) F\left( - \frac{1}{2} - M; \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right|
\]

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\[ \leq \frac{1}{2}((z - 1)^2 - y^2) + \frac{1}{8} |2M - 1||y^2 + 2z(z - 1) + z^2 - 1 + 2M (3y^2 + 2z(z - 1) + z^2 - 1)| \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \]

\[ + \frac{1}{2}((z - 1)^2 - y^2)O\left(\frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2}\right). \]

Hence, we have to consider the following two integrals, which can be easily estimated,

\[ A_1 := \int_{(z,y) \in Z_1(\varepsilon,z)} y^a((z + 1)^2 - y^2)^{\Re M - \frac{a}{2}} dy, \]

\[ A_2 := z^2 \int_{(z,y) \in Z_1(\varepsilon,z)} y^a((z + 1)^2 - y^2)^{\Re M - \frac{a}{2}} dy, \]

for all \( z \in [1, \infty) \). Indeed, for \( A_1 \) we obtain

\[ A_1 \leq \int_0^{z^{-1}} y^a((z + 1)^2 - y^2)^{\Re M - \frac{a}{2}} dy \]

\[ = \frac{1}{1 + a}(z - 1)^{1+a}(z + 1)^{2\Re M - 1} F\left(\frac{1 + a}{2}, \frac{1}{2} - \Re M; \frac{3 + a}{2}, \frac{(z - 1)^2}{(z + 1)^2}\right) \]

\[ \leq C_{M,n,p,q,s}(z - 1)^{1+a}(z + 1)^{2\Re M - 1}. \]

Similarly, if \( \Re M > 0 \), then

\[ A_2 \leq z^2 \int_0^{z^{-1}} y^a((z + 1)^2 - y^2)^{\Re M - \frac{a}{2}} dy \]

\[ = z^2 \frac{1}{1 + a}(z - 1)^{1+a}(z + 1)^{2\Re M - 3} F\left(\frac{1 + a}{2}, \frac{3}{2} - \Re M; \frac{3 + a}{2}, \frac{(z - 1)^2}{(z + 1)^2}\right). \]

Here and henceforth, if \( A \) and \( B \) are two non-negative quantities, we use \( A \lesssim B \) to denote the statement that \( A \leq C B \) for some absolute constant \( C > 0 \).

It suffices to consider the case of real valued \( M \). Then (A.5) and (1.11) in the case of \( M < 1/2 \) imply

\[ A_2 \lesssim z^2 \frac{1}{1 + a}(z - 1)^{1+a}(z + 1)^{2M - 3} z^{\frac{1}{2} - M} \lesssim (z - 1)^{1+a}(z + 1)^{M - \frac{1}{2}}. \]

In the case of \( M \geq 1/2 \) due to (A.4) we derive

\[ A_2 \lesssim z^2(z - 1)^{1+a}(z + 1)^{2M - 3} \lesssim (z - 1)^{1+a}(z + 1)^{2M - 1}. \]

Finally, for the integral over the first zone \( Z_1(\varepsilon, z) \) we obtain

\[ \int_{(z,r) \in Z_1(\varepsilon,z)} * dr \lesssim (z - 1)^{1+a} \times \begin{cases} \frac{(z + 1)^{\Re M - \frac{1}{2}}}{(z + 1)^{2\Re M - 1}} & \text{if } \Re M < 1/2, \\ \frac{(z + 1)^{\Re M - \frac{1}{2}}}{(z + 1)^{2\Re M - 1}} & \text{if } \Re M > 1/2. \end{cases} \]

In the second zone we have

\[ 0 < \varepsilon \leq \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2} < 1 \quad \text{and} \quad \frac{1}{(z - 1)^2 - r^2} \leq \frac{1}{\varepsilon((z + 1)^2 - r^2)}. \]

Then, the hypergeometric functions for \( \Re M > 0 \) obey the estimates

\[ \left| F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \zeta\right)\right| \leq C \quad \text{and} \quad \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \zeta\right)\right| \leq C_M \text{ for all } \zeta \in [\varepsilon, 1). \]
This allows us to estimate the integral over the second zone:

\[
\int_{(x,y)\in Z_2(\varepsilon, z)} y^a((z + 1)^2 - y^2)^{\text{RM}} \left( \frac{1}{(z - 1)^2 - y^2} \right)^{\frac{1}{2}} \frac{1}{\sqrt{(z + 1)^2 - y^2}} \\
\times \left( z - z^2 + M(1 - z^2 - y^2) \right) F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \\
+ (z^2 - 1 + y^2) \left( \frac{1}{2} + M \right) \left( - \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) dy \\
\lesssim z^2 \int_{(x,y)\in Z_2(\varepsilon, z)} y^a((z + 1)^2 - y^2)^{\text{RM} - \frac{2}{a}} dy \\
\lesssim z^2 \int_0^{z^2 - 1} y^a((z + 1)^2 - y^2)^{\text{RM} - \frac{2}{a}} dy.
\]

Then we apply (1.11) and Lemma A.1:

\[
z^2 \int_{(x,y)\in Z_2(\varepsilon, z)} y^a((z + 1)^2 - y^2)^{\text{RM} - \frac{2}{a}} dy \lesssim (z - 1)^{1+a} \times \begin{cases} (z + 1)^{\text{RM} - \frac{2}{a}} & \text{if } \text{RM} < 1/2, \\ (z + 1)^{2\text{RM} - 1} & \text{if } \text{RM} > 1/2, \end{cases}
\]

for all $z \in [1, \infty)$. Finally, for the integral over the second zone $Z_2(\varepsilon, z)$ we obtain

\[
\int_{(x,r)\in Z_2(\varepsilon, z)} * dr \lesssim (z - 1)^{1+a} \times \begin{cases} (z + 1)^{\text{RM} - \frac{2}{a}} & \text{if } \text{RM} < 1/2, \\ (z + 1)^{2\text{RM} - 1} & \text{if } \text{RM} > 1/2, \end{cases}
\]

The rest of the proof is a repetition of the above used arguments. Thus, the proposition is proved. \hfill \Box

**Completion of the proof of Theorem 1.1.** Thus, if $\varphi_1 = 0$, then from (1.10) we derive

\[
\| \Phi(x,t) \|_{H(\varepsilon)} \\
\leq e^{-\frac{\varepsilon^2}{2t}} \| v_{\varphi_0}(x, \phi(t)) \|_{H(\varepsilon)} \\
+ e^{-\frac{\varepsilon^2}{2t}} \int_0^1 \| v_{\varphi_0}(x, \phi(t)s) \|_{H(\varepsilon)} \left[ 2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M) \right] |\phi(t) ds \\
\lesssim C e^{-\frac{\varepsilon^2}{2t}t} (1 - e^{-t}) \| \varphi_0 \|_{H(\varepsilon)} \\
+ \| \varphi_0 \|_{H(\varepsilon)} e^{-\frac{\varepsilon^2}{2t}} \int_0^1 \left( |2K_0(\phi(t)s, t; M)| + n|K_1(\phi(t)s, t; M)| \right) |\phi(t) ds \\
\lesssim e^{-\frac{\varepsilon^2}{2t}} \| \varphi_0 \|_{H(\varepsilon)} \\
+ \| \varphi_0 \|_{H(\varepsilon)} e^{-\frac{\varepsilon^2}{2t}} \left( (e^t - 1)(e^t + 1)^{\text{RM} - 1} + (e^t - 1) \times \begin{cases} (e^t + 1)^{-\frac{1}{2}} & \text{if } \text{RM} < 1/2, \\ e^{\text{RM}t}(e^t + 1)^{-1} & \text{if } \text{RM} > 1/2, \end{cases} \right) \\
\lesssim e^{-\frac{\varepsilon^2}{2t}} \| \varphi_0 \|_{H(\varepsilon)} \\
+ \| \varphi_0 \|_{H(\varepsilon)} e^{-\frac{\varepsilon^2}{2t}} (e^t - 1)^{\text{RM} - 1} + \begin{cases} (e^t + 1)^{-\frac{1}{2}} & \text{if } \text{RM} < 1/2, \\ e^{\text{RM}t}(e^t + 1)^{-1} & \text{if } \text{RM} > 1/2, \end{cases} \right). \]

In particular, for large $t$ we obtain

\[
\| \Phi(x,t) \|_{H(\varepsilon)} \lesssim \| \varphi_0 \|_{H(\varepsilon)} e^{-\frac{\varepsilon^2}{2t}} + \| \varphi_0 \|_{H(\varepsilon)} e^{-\frac{\varepsilon^2}{2t}} e^{t \left( e^{(\text{RM} - 1)t} + \begin{cases} e^{-\frac{t}{2}} & \text{if } \text{RM} < 1/2, \\ e^{\text{RM}t} & \text{if } \text{RM} > 1/2, \end{cases} \right)} \\
\lesssim \| \varphi_0 \|_{H(\varepsilon)} \left( e^{-\frac{\varepsilon^2}{2t}} + e^{(-\frac{1}{2} + 1)t \left( e^{(\text{RM} - 1)t} + \begin{cases} e^{-\frac{t}{2}} & \text{if } \text{RM} < 1/2, \\ e^{\text{RM}t} & \text{if } \text{RM} > 1/2, \end{cases} \right)} \right).
\]

In the case of $\varphi_0 = 0$ we have

\[
\| \Phi(x,t) \|_{H(\varepsilon)} = 2e^{-\frac{\varepsilon^2}{2t}} \int_0^1 v_{\varphi_1}(x, \phi(t)s)K_1(\phi(t)s, t; M)|\phi(t) ds \|_{H(\varepsilon)}.
\]
Theorem is proved. □

1.2 $H_{s}(\mathbb{R}^{n}) - H_{s}(\mathbb{R}^{n})$ Estimate for the time derivative

In this subsection we restrict ourselves to the Sobolev spaces $H_{s}$.

**Theorem 1.5** Consider the problem

$$
\Phi_{tt} - e^{-2t}A(x, \partial_{x})\Phi + n\Phi_{t} + m^{2}\Phi = 0, \quad \Phi(x, 0) = \Phi_{0}(x), \quad \Phi_{t}(x, 0) = \Phi_{1}(x),
$$

where $A(x, \partial_{x}) = \sum_{|\alpha| \leq 2} a_{\alpha}(x)\partial_{x}^{\alpha}$ is a second order negative elliptic partial differential operator, $a_{\alpha} \in B^{\infty}$, and $m^{2} \in \mathbb{R}$. Then, there is a number $C > 0$ such that

$$
\|\Phi_{t}(t)\|_{H_{s}} + e^{-t}\|\Phi(t)\|_{H_{s+1}} \leq C \left( \|\Phi(t)\|_{H_{s}} + e^{-\frac{s}{2}t}\|\Phi_{1}\|_{H_{s}} + e^{-\frac{s}{2}t}\|\Phi_{0}\|_{H_{s+1}} \right) \quad \text{for all } t > 0.
$$

**Proof.** The change of unknown function

$$
\Phi = e^{-\frac{s}{2}t}u, \quad u = e^{\frac{s}{2}t}\Phi
$$

simplifies the equation. Therefore, we consider the problem

$$
u_{tt} - e^{-2t}A(x, D)u - M^{2}u = 0, \quad u(x, 0) = \varphi_{0}(x), \quad u_{t}(x, 0) = \varphi_{1}(x),
$$

with the smooth initial functions $\varphi_{0}(x)$ and $\varphi_{1}(x)$. Here $M^{2} = n^{2}/4 - m^{2}$. The equation leads to the following identity

$$
\frac{1}{2} \frac{d}{dt} \left( (u_{t}, u_{t}) - e^{-2t}(A(x, \partial_{x})u, u) - M^{2}(u, u) \right) = -e^{-2t}(A(x, \partial_{x})u, u) = 0.
$$

Since the operator $A(x, \partial_{x})$ is negative, it follows

$$
\frac{1}{2} \frac{d}{dt} \left( (u_{t}, u_{t}) - e^{-2t}(A(x, \partial_{x})u, u) - M^{2}(u, u) \right) \leq 0.
$$

The integration in time gives

$$
(u_{t}, u_{t}) - e^{-2t}(A(x, \partial_{x})u, u) - M^{2}(u, u) \leq (\varphi_{1}, \varphi_{1}) - e^{-2t}(A(x, \partial_{x})\varphi_{0}, \varphi_{0}) - M^{2}(\varphi_{0}, \varphi_{0}),
$$

and, consequently,

$$
\|u_{t}(t)\|_{L^{2}} + e^{-t}\|u(t)\|_{H_{s}} \leq C(\|u(t)\|_{L^{2}} + \|\varphi_{1}\|_{L^{2}} + \|\varphi_{0}\|_{H_{s}}).
$$

Similarly, using a standard technique (see, e.g., [27]), one can obtain for every $s \in \mathbb{R}$ the following estimate

$$
\|u_{t}(t)\|_{H_{s}} + e^{-t}\|u(t)\|_{H_{s+1}} \leq C_{s}(\|u(t)\|_{H_{s}} + \|u_{t}(0)\|_{H_{s}} + \|u(0)\|_{H_{s+1}}).
$$

Then for the function $\Phi$ we have

$$
\|ne^{\frac{s}{2}t}\Phi(t) + e^{\frac{s}{2}t}\Phi_{t}(t)\|_{H_{s}} + \|\Phi(t)\|_{H_{s}} \leq C_{s} \left(\|\Phi(t)\|_{H_{s}} + \|\Phi_{t}\|_{H_{s}} + \|\Phi_{0}\|_{H_{s+1}}\right),
$$

while

$$
\|\Phi(t)\|_{H_{s+1}} \leq C_{s}e^{-\frac{s}{2}t}(\|u(t)\|_{H_{s}} + \|u_{t}(0)\|_{H_{s}} + \|u(0)\|_{H_{s+1}}).
$$

\[13\]
Thus, the theorem is proved.

According to Theorem 1.1 if $\mathbb{R}M \in (0, 1/2)$, then
\[
\|\Phi(x, t)\|_{H_{(s)}} \leq C(1 - e^{-t})e^{-\frac{n+1}{2}t}\left\{\|\varphi_0\|_{H_{(s+1)}} + (1 - e^{-t})\|\varphi_1\|_{H_{(s)}}\right\}.
\]

Hence
\[
\|\Phi_t(t)\|_{H_{(s)}} \leq ||n\Phi(t)||_{H_{(s)}} + Ce^{-\frac{n+1}{2}t}(e^{\frac{s+1}{2}t}||\Phi(t)||_{H_{(s)}} + ||\Phi_1||_{H_{(s)}} + ||\Phi_0||_{H_{(s+1)}})
\]
\[
\lesssim ||\Phi(t)||_{H_{(s)}} + Ce^{-\frac{n+1}{2}t}(||\Phi_1||_{H_{(s)}} + ||\Phi_0||_{H_{(s+1)}})
\]
\[
\lesssim (1 - e^{-t})e^{-\frac{n+1}{2}t}(||\varphi_0||_{H_{(s+1)}} + (1 - e^{-t})||\varphi_1||_{H_{(s)}})
\]
\[
+ e^{\frac{s+1}{2}t}(||\Phi_1||_{H_{(s)}} + ||\Phi_0||_{H_{(s+1)}})
\]
\[
\lesssim (1 - e^{-t})e^{-\frac{n+1}{2}t}(||\varphi_0||_{H_{(s+1)}} + (1 - e^{-t})||\varphi_1||_{H_{(s)}})
\]
\[
+ e^{\frac{s+1}{2}t}(||\Phi_1||_{H_{(s)}} + ||\Phi_0||_{H_{(s+1)}}).
\]

Finally we have proved the following theorem.

**Theorem 1.6** For $s \in \mathbb{R}$ the solution $\Phi = \Phi(x, t)$ of the Cauchy problem (1.9) for $M^2 \in \mathbb{R}$ and $\mathbb{R}M \in (0, 1/2)$ satisfies the following estimate
\[
\|\Phi(t)\|_{H_{(s)}} \leq Ce^{-\frac{n+1}{2}t}(||\varphi_1||_{H_{(s)}} + ||\varphi_0||_{H_{(s+1)}}).
\]

If $M^2 \in \mathbb{R}$ and $\mathbb{R}M > \frac{1}{2}$ or $M = 1/2$, then
\[
\|\Phi(t)\|_{H_{(s)}} \leq C(\|\Phi(t)\|_{H_{(s)}} + e^{\frac{s+1}{2}t}\|\varphi_1\|_{H_{(s)}} + e^{\frac{s+1}{2}t}\|\varphi_0\|_{H_{(s+1)}})
\]
and
\[
\|\Phi_t(t)\|_{H_{(s)}} \leq Ce^{(\mathbb{R}M - \frac{n+1}{2})t}(\|\varphi_0\|_{H_{(s+1)}} + \|\varphi_1\|_{H_{(s)}}).
\]

## 2 $H_{(s)}(\mathbb{R}^n) - H_{(s)}(\mathbb{R}^n)$ Estimates for Equation with Source

We consider equation with $m \in \mathbb{C}$ and $\frac{n^2}{4} \geq m^2$ although result can be similarly obtained for the case of large mass, that is, for $\frac{n^2}{4} \leq m^2$. Recall $M := (n^2/4 - m^2)^{1/2}$. In fact, for the case of large mass and for the case of $M \in [1/2, n/2]$ (that is $m^2 \in (0, (n^2 - 1)/4)$ one can consult [18]. This is why in the present paper we focus on the case of $M \in (0, 1/2) \cup (n/2, \infty)$ and some complex valued $M$. Thus, we are interested also in the Higgs boson equation, in the massive scalar fields as well as in the tachyons having $m^2 < 0$.

**Theorem 2.1** Let $\Phi = \Phi(x, t)$ be a solution of the Cauchy problem
\[
\dot{\Phi} + n\Phi - e^{-2t}A(x, \partial_x)\Phi + m^2\Phi = f, \quad \Phi(x, 0) = 0, \quad \Phi_t(x, 0) = 0.
\]

Then solution $\Phi = \Phi(x, t)$ for $0 < \mathbb{R}M < 1/2$ satisfies the following estimate:
\[
\|\Phi(x, t)\|_{H_{(s)}} \leq Ce^{-\frac{n+1}{2}t}\int_0^t e^{\frac{n+1}{2}b}||f(x, b)||_{H_{(s)}} db + \text{ for all } t > 0.
\]

If either $\mathbb{R}M > 1/2$ or $M = 1/2$, then
\[
\|\Phi(x, t)\|_{H_{(s)}} \leq CMe^{(\mathbb{R}M - \frac{n+1}{2})t}\int_0^t e^{-(\mathbb{R}M - \frac{n+1}{2})b}||f(x, b)||_{H_{(s)}} db + \text{ for all } t > 0.
\]

Moreover, for the derivative $\partial_t\Phi(x, t)$ if $0 < \mathbb{R}M < 1/2$, then the following estimate holds
\[
\|\partial_t\Phi(x, t)\|_{H_{(s)}} \leq Ce^{-\frac{n+1}{2}t}\int_0^t e^{\frac{n+1}{2}b}||f(x, b)||_{H_{(s)}} db \text{ for all } t > 0.
\]
If $\Re M > 3/2$ or $M = 3/2$, then
\[
\| \partial_t \Phi(x, t) \|_{H(\epsilon)} \leq C e^{(\Re M - \frac{3}{2}) t} \int_0^t e^{(\Re M - \frac{3}{2}) b} \| f(x, b) \|_{H(\epsilon)} \, db \\
+ C e^{\frac{a - 1}{2} t} \int_0^t e^{\frac{a - 1}{2} b} \| f(x, b) \|_{H(\epsilon)} \, db \quad \text{for all} \quad t > 0.
\]

**Proof.** From (1.4) we have
\[
\Phi(x, t) = 2 e^{-\frac{t}{2}} \int_0^t \int_0^{e^{-t} - e^{-t}} \, dr \, e^{\frac{b}{2}} v(x, r; b) a M e^{M(b+t)} \left( (e^{-t} + e^{-t})^2 - r^2 \right)^{\frac{1}{2} + M} \\
\times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right).
\]
According to (1.8) we can write
\[
\| v(x, r; b) \|_{H(\epsilon)} \leq C \| f(x, b) \|_{H(\epsilon)} \quad \text{for all} \quad r \in [0, 1].
\]
Hence,
\[
\| \Phi(x, t) \|_{H(\epsilon)} \leq 2 e^{-\frac{t}{2}} \int_0^t \int_0^{e^{-t} - e^{-t}} \, dr \, e^{\frac{b}{2}} \| v(x, r; b) \|_{H(\epsilon)} a M e^{M(b+t)} \\
\times \left( (e^{-t} + e^{-t})^2 - r^2 \right)^{\frac{1}{2} + M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right) \right| \\
\leq e^{\Re M} e^{-\frac{t}{2}} \int_0^t e^{\frac{b}{2}} a M e^{M(b+t)} \| f(x, b) \|_{H(\epsilon)} \, db \int_0^{e^{-t} - e^{-t}} \left( (e^{-t} + e^{-t})^2 - r^2 \right)^{\frac{1}{2} + M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right) \right| \, dr.
\]
Following the outline of the proof of Lemma 1.2 we set $r = y e^{-t}$ and obtain
\[
\| \Phi(x, t) \|_{H(\epsilon)} \leq C_M e^{-\Re M} e^{-\frac{t}{2}} \int_0^t e^{\frac{b}{2}} a M e^{M(b+t)} \| f(x, b) \|_{H(\epsilon)} \, db \\
\times \int_0^{e^{-b} - 1} \left( (e^{-t} + 1)^2 - y^2 \right)^{\frac{1}{2} + M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - 1)^2 - y^2}{(e^{-b} + 1)^2 - y^2} \right) \right| \, dy.
\]
In order to estimate the second integral we apply Lemma A.5 with $z = e^{-b} > 1$ and $a = 0$. Hence, the estimate (A.7) implies for $0 < \Re M < 1/2$ the following estimate
\[
\| \Phi(x, t) \|_{H(\epsilon)} \leq C_M e^{-\Re M} e^{-\frac{t}{2}} \int_0^t e^{\frac{b}{2}} a M e^{M(b+t)} \| f(x, b) \|_{H(\epsilon)} \, db \\
\times \int_0^{e^{-b} - 1} \left( (e^{-t} + 1)^2 - y^2 \right)^{\frac{1}{2} + M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - 1)^2 - y^2}{(e^{-b} + 1)^2 - y^2} \right) \right| \, dy \\
\leq e^{-\Re M} e^{-\frac{t}{2}} \int_0^t e^{\frac{b}{2}} a M e^{M(b+t)} \| f(x, b) \|_{H(\epsilon)} (e^{-t} - 1) e^{(t-b)(\Re M - \frac{1}{2})} \, db \\
\leq e^{-\frac{a + 1}{2} t} \int_0^t e^{-\frac{a + 1}{2} b} \| f(x, b) \|_{H(\epsilon)} \, db.
\]
while for $\Re M > 1/2$ the estimate (A.8) implies

$$
\|\Phi(x,t)\|_{H(\sigma)} \leq C_M e^{-\Re M t} e^{-\frac{b}{2} t} \int_0^t e^{\frac{b}{2} s} e^{\Re M b} \|f(x, b)\|_{H_{\sigma(t,s)}} \db
\times \int_0^t e^{b-s-1} \left((e^{t-b} + 1)^2 - y^2\right) - \frac{1}{2 + \Re M} \times \left[F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{t-b} - 1)^2}{(e^{t-b} + 1)^2} - y^2\right)\right] \dy
db \lesssim e^{-\Re M t} e^{-\frac{b}{4} t} \int_0^t e^{\frac{b}{2} s} e^{\Re M b} \|f(x, b)\|_{H(\sigma)} (e^{t-b} - 1) (e^{t-b} + 1)^2 e^{2 \Re M - 1} \db
\lesssim e^{-(\Re M - \frac{b}{2}) t} \int_0^t e^{-(\Re M - \frac{b}{2}) s} e^{2 \Re M - 1} \|f(x, b)\|_{H(\sigma)} \db.
$$

In order to estimate a derivative of (1.4) we write

$$
\partial_t \Phi(x,t) = -\frac{n}{2} \Phi(x,t) + 2 e^{-\frac{b}{2} t} \int_0^t \db e^{\frac{b}{2} s} v(x, e^{b} - e^{-t}; b) E(e^{b} - e^{-t}, t; 0, b; M) + 2 e^{-\frac{b}{2} t} \int_0^t \db e^{\frac{b}{2} s} \int_0^t \db dr e^{\frac{b}{2} r} v(x, r; b) \partial_t E(r, t; 0, b; M). \tag{2.12}
$$

Further

$$
E(e^{b} - e^{-t}, t; 0, b; M) = \frac{1}{2} e^{\frac{b}{2} + \frac{t}{2}}
$$

implies

$$
2 e^{-\frac{b}{2} t} \int_0^t e^{\frac{b}{2} s} v(x, e^{b} - e^{-t}; b) E(e^{b} - e^{-t}, t; 0, b; M) \db = e^{-\frac{b}{2} t} \int_0^t e^{\frac{n+1}{2} b} v(x, e^{b} - e^{-t}; b) \db. \tag{2.13}
$$

Due to (1.8) we have

$$
\|v(x, r; b)\|_{H(\sigma)} \leq C \|f(x, b)\|_{H(\sigma)} \quad \text{for all } r \in (0, e^{b} - e^{-t}) \subseteq (0, 1].
$$

Hence (2.13) implies

$$
\|2 e^{-\frac{b}{2} t} \int_0^t \db e^{\frac{b}{2} s} v(x, e^{b} - e^{-t}; b) E(e^{b} - e^{-t}, t; 0, b; M) \|_{H(\sigma)} \leq e^{-\frac{n+1}{2} t} \int_0^t e^{\frac{n+1}{2} b} \|v(x, e^{b} - e^{-t}; b)\|_{H(\sigma)} \db \leq C e^{-\frac{n+1}{2} t} \int_0^t e^{\frac{n+1}{2} b} \|f(x, b)\|_{H(\sigma)} \db. \tag{2.14}
$$

For the last term of the derivative $\partial_t \Phi$ (2.12) we have

$$
\|2 e^{-\frac{b}{2} t} \int_0^t \db \int_0^{e^{b} - e^{-t}} \db dr e^{\frac{b}{2} r} v(x, r; b) \partial_t E(r, t; 0, b; M) \|_{H(\sigma)} \lesssim e^{-\frac{b}{2} t} \int_0^t \db e^{\frac{b}{2} r} \|f(x, b)\|_{H(\sigma)} \int_0^{e^{b} - e^{-t}} \db \|\partial_t E(r, t; 0, b; M)\|.
$$

**Proposition 2.2** If $\Re M > 0$ and $\Re M \neq 1/2$ or $M = 1/2$, then

$$
\int_0^{e^{b} - e^{-t}} \|\partial_t E(r, t; 0, b; M)\| \db \lesssim \begin{cases} e^{-\frac{b}{2} t} e^{-3b} & \text{if } \Re M < 1/2, \\ e^{\Re M (t-b)} e^{-3b} & \text{if } \Re M > 3/2, \end{cases}
$$

for all $t \geq 0$ and $b \geq 0$ such that $b < t$.  

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Proof. We have
\[
\partial_t E(r, t; 0, b; M)
= \left( \partial A e^{M(b+t)} \right) \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right)
+ \frac{4}{e^{M(b+t)}} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \partial_t F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right).
\]  
(2.15)

First we consider the second term of the equation (2.15). If \( \Re M > 1/2 \) and \( M \neq 1/2 \), then
\[
\left| 4^{-M} e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \partial_t F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right) \right|
\leq 4^{-M} e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \frac{1}{(r^2 - e^{2(b+t)}) + 2(e^{b+t} + e^{2b} + e^{2t})} \left( 1 - 2M \right)^2 e^{b+t} \left( r^2 e^{2(b+t)} + e^{2b} - e^{2t} \right)
\times F \left( \frac{1}{2} - M, \frac{3}{2} - M; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right)
\leq e^{(\Re M - 3)(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + \Re M} \left( r^2 e^{2(b+t)} + e^{2b} - e^{2t} \right)
\leq e^{(\Re M - 3)(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + \Re M} \left( e^2 - e^{-2b} \right)
\]  
and for \( \Re M > 3/2 \) we can use (A.6) of Lemma A.4 with \( a = 0 \) to estimate the integral of the last term:
\[
\int_0^{e^{-b} - e^{-t}} \left| e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \partial_t F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right) \right| \, dr
\leq \left( e^{2t} - e^{2b} \right) e^{(\Re M - 3)(b+t)} \int_0^{e^{-b} - e^{-t}} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + \Re M} \, dr
\leq e^{(\Re M - 3)(b+t)} \left( e^{2t} - e^{2b} \right) e^{-2(\Re M - 4)(b+t)} \left( e^t - e^{-b} \right)^{2(\Re M - 5)}
\leq e^{\Re M(t-b)} \quad \text{if} \quad \Re M > 3/2.
\]  
For the case of \( \Re M < 1/2 \) we have
\[
\left| e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \partial_t F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right) \right|
\leq e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \frac{1}{(r^2 - e^{2(b+t)}) + 2(e^{b+t} + e^{2b} + e^{2t})} \left( 1 - 2M \right)^2 e^{b+t} \left( r^2 e^{2(b+t)} + e^{2b} - e^{2t} \right)
\times F \left( M + \frac{1}{2}, M + \frac{1}{2}; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right)
\leq e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} \frac{1}{(e^2(b+t)) \left( (e^{-b} + e^{-t})^2 - r^2 \right)^2} \left( e^{b+t} \left( r^2 e^{2(b+t)} + e^{2b} - e^{2t} \right) \left( \frac{e^{-b+t}}{e^{-b} + e^{-t}} \right)^{M-1} \right)
\]
\[ 
\lesssim \left| e^{-2(b-t)} \left( e^{-t} + e^{-b} \right)^2 - r^2 \right|^{-\frac{1}{2}} \left( r^2 e^{2(b-t)} + e^{2b} - e^{2t} \right) \\
\lesssim \left( e^{-t} + e^{-b} \right)^2 - r^2 \right|^{-\frac{1}{2}} e^{-2b} \text{ for all } r \leq e^{-b} - e^{-t}.
\]

Thus, for the case of \( \Re M < 1/2 \) we obtain
\[
\lesssim \left( e^{-t} + e^{-b} \right)^2 - r^2 \right|^{-\frac{1}{2}} e^{-2b} \text{ for all } r \leq e^{-b} - e^{-t}.
\]

Next we apply Lemma A.2 with \( a = 0 \) and derive
\[
\int_0^{e^{-t} - e^{-t}} e^{-t} M(b) \left( e^{-t} + e^{-b} \right)^2 - r^2 \right|^{-\frac{1}{2}} \left( \partial_t A M \left( e^{-t} + e^{-b} \right)^2 - r^2 \right) \frac{\partial_t F}{\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{e^{-b} - e^{-t}}{(e^{-b} + e^{-t})^2 - r^2} \right)} \, dr \\
\lesssim e^{-t} e^{-b} \text{ if } \Re M < 1/2.
\]

Now we consider the first term of the equation (2.15):
\[
\lesssim \left( e^{-t} + e^{-b} \right)^2 - r^2 \right|^{-\frac{1}{2}} e^{-2b} \text{ if } \Re M < 1/2.
\]

Finally
\[
\lesssim e^{RM(b+t)} \left( e^{-b} + e^{-t} \right)^2 - r^2 \right|^{-\frac{1}{2}} \text{ if } \Re M > 1/2.
\]

and due to Lemmas A.2, A.3 with \( a = 0 \)
\[
\int_0^{e^{-t} - e^{-t}} dr \left( e^{-t} + e^{-b} \right)^2 - r^2 \right|^{-\frac{1}{2}} \left( \partial_t A M \left( e^{-t} + e^{-b} \right)^2 - r^2 \right) \frac{F}{\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{e^{-b} - e^{-t}}{(e^{-b} + e^{-t})^2 - r^2} \right)}
\]

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Thus,

\[ \int_0^{e^{-b}-e^{-t}} dr e^{\Re M(b+t)} \left\{ e^{-2b-\Re M b} \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2}} \right. \text{ if } \Re M < 1/2, \]

\[ \left. (e^{-b} + e^{-t})^2 \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{\Re M - \frac{3}{2}} \right\} \text{ if } \Re M > 1/2. \]

Then estimating the norms in the case of \( \Re \) \( M > 1/2 \), we have

\[ e^{\Re M} \int_0^{e^{-b}-e^{-t}} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + \Re M} \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{\Re M - \frac{3}{2}} \]

\[ \left( (e^{-b} + e^{-t})^2 \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{\Re M - \frac{3}{2}} \right) \]

\[ \text{if } \Re M < 1/2, \]

\[ e^{\Re M} \left( (e^{-b} + e^{-t})^2 \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{\Re M - \frac{3}{2}} \right) \text{ if } \Re M > 1/2. \]

Thus,

\[ \int_0^{e^{-b}-e^{-t}} dr \left( \partial_t A_{-\Re M} e^{\Re M(b+t)} \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2} + \Re M} \right) F \left( \frac{1}{2} - \Re M, \frac{1}{2} - M; \frac{1}{2} - \Re M ; 1, \frac{1}{2} - \Re M, -1 \right) \]

\[ \text{if } \Re M < 1/2, \]

\[ e^{\Re M(b-t)} e^{-3b} \text{ if } \Re M > 1/2. \]

The proposition is proved. □

Then estimating the norms in the case of \( \Re M < 1/2 \) we obtain

\[ \| 2 e^{-\frac{b+t}{2}} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{t}{2} b} v(x,r;b) \partial_t E(r,t;0,b;M) \|_{H_{(a)}} \]

\[ \lesssim e^{-\frac{b+t}{2}} \int_0^t db e^{\frac{t}{2} b} \| f(x,b) \|_{H_{(a)}} \int_0^{e^{-b}-e^{-t}} dr | \partial_t E(r,t;0,b;M) |. \]

Next we apply Proposition 2.2 and obtain

\[ \| 2 e^{-\frac{b+t}{2}} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{t}{2} b} v(x,r;b) \partial_t E(r,t;0,b;M) \|_{H_{(a)}} \lesssim e^{-\frac{b+t}{2}} \int_0^t db e^{\frac{t}{2} b} \| f(x,b) \|_{H_{(a)}}. \]

By collecting estimates (2.14) and (2.16) we obtain the final estimate for \( \| \partial_t \Phi(x,t) \|_{H_{(a)}} \) in the case of \( \Re M < 1/2 \). For the case of \( \Re M > 3/2 \), due to Proposition 2.2 and according to (1.8) we have

\[ \| 2 e^{-\frac{b+t}{2}} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{t}{2} b} v(x,r;b) \partial_t E(r,t;0,b;M) \|_{H_{(a)}} \]

\[ \lesssim e^{-\frac{b+t}{2}} \int_0^t db e^{\frac{t}{2} b} \| v(x,r;b) \|_{H_{(a)}} \| \partial_t E(r,t;0,b;M) | | \]

\[ \lesssim e^{(\frac{b}{2} - \Re M) t} \int_0^t db e^{(\frac{b}{2} - \Re M) b} \| f(x,b) \|_{H_{(a)}}. \]

The last estimate together with (2.14) implies the last statement of the theorem.

**The case of \( M = 3/2 \).** We consider the equation (2.12), where

\[ E \left( r,t;0,b; \frac{3}{2} \right) = \frac{1}{4} e^{-\frac{b+t}{2}} \left( e^{2b} + e^{2t} - r^2 e^{2(b+t)} \right), \]

\[ E \left( e^{-b} - e^{-t}, t;0,b; \frac{3}{2} \right) = \frac{1}{2} e^{b+t}, \]

\[ \partial_t E \left( r,t;0,b; \frac{3}{2} \right) = \frac{1}{8} e^{-\frac{b+t}{2}} \left( 3e^{2t} - 3r^2 e^{2(b+t)} - e^{2b} \right). \]
Consequently, for the first term of (2.12) we have
\[ \| \Phi(x,t) \|_{H^{(s)}} \leq C e^{(\frac{4}{3} - \frac{n}{2})t} \int_0^t e^{-(\frac{4}{3} - \frac{n}{2})b} \| f(x,b) \|_{H^{(s)}} \, db, \]
while for the second term the following estimate
\[ \| 2e^{-\frac{2}{3}t} \int_0^t e^{\frac{2}{3}b} v(x,e^{-b} - e^{-t};b) E(e^{-b} - e^{-t},t;0,b;\frac{3}{2}) \, db \|_{H^{(s)}} \]
\[ \lesssim e^{-\frac{3}{2}t} \int_0^t e^{\frac{3}{2}b} \| v(x,e^{-b} - e^{-t};b) \|_{H^{(s)}} \, db \]
\[ \lesssim e^{-\frac{3}{2}t} \int_0^t e^{\frac{3}{2}b} \| f(x,b) \|_{H^{(s)}} \, db \]
holds. For the last term of (2.12) we obtain
\[ \| 2e^{-\frac{2}{3}t} \int_0^t db \int_0^{e^{-b} - e^{-t}} dr e^{\frac{2}{3}b} v(x,r;b) \partial_t E(r,t;0;b;M) \|_{H^{(s)}} \]
\[ \lesssim e^{-\frac{2}{3}t} \int_0^t db \int_0^{e^{-b} - e^{-t}} dr \| v(x,r;b) \|_{H^{(s)}} \left( 3e^{2b(b+t)} + e^{2b} - 3e^{2t} \right) \]
\[ \lesssim e^{-\frac{2}{3}t} \int_0^t db \int_0^{e^{-b} - e^{-t}} \| f(x,b) \|_{H^{(s)}} \left( 3e^{2b(b+t)} + e^{2b} - 3e^{2t} \right) \, dr \]
\[ \lesssim e^{-\frac{2}{3}t} \int_0^t db \int_0^{e^{-b} - e^{-t}} \| f(x,b) \|_{H^{(s)}} \left( e^{-(b+t)} (e^t - e^b) \right) \left( e^t + e^b \right)^2 \, db \]
\[ \lesssim e^{-\frac{2}{3}t} \int_0^t e^{\frac{2}{3}b} \| f(x,b) \|_{H^{(s)}} \, db. \]

The final estimate for this case is
\[ \| \partial_t \Phi(x,t) \|_{H^{(s)}} \lesssim e^{-\frac{n-3}{2}t} \int_0^t e^{\frac{n-3}{2}t} \| f(x,b) \|_{H^{(s)}} \, db + e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}t} \| f(x,b) \|_{H^{(s)}} \, db. \]
The theorem is proved. \(\square\)

3 Global Existence. Small Data Solutions

We are going to apply the Banach fixed-point theorem. In order to estimate nonlinear term we use the Lipschitz condition (L). First we consider the integral equation (0.5), where the function \( \Phi_0 \in C([0, \infty); L^q(\mathbb{R}^n)) \) is given. Every solution to the equation (0.7) solves also the last integral equation with some function \( \Phi_0 = \Phi_0(x,t) \). We note here that any classical solution to the equation (0.7) solves also the integral equation (0.5) with some function \( \Phi_0(t,x) \), which is classical solution to the Cauchy problem for the linear equation (1.9).

The operator \( G \) and the structure of the nonlinear term determine the solvability of the integral equation (0.5). For the operator \( G \) generated by the linear parts of the equation (0.1) with \( m^2 < 0 \) the global solvability of the integral equation (0.5) was studied in [30]. For the case of \( m^2 < 0 \) and the nonlinearity \( F(\Phi) = c|\Phi|^{p+1}, \ c \neq 0 \), the results of [30] imply the nonexistence of the global solution even for arbitrary small function \( \Phi_0(x,0) \) under some conditions on \( n, \alpha \), and \( M \in \mathbb{C} \).

Consider the problem in the Sobolev space \( H^{(s)}(\mathbb{R}^n) \) with \( s > n/2 \), which is an algebra. In the next theorem operator \( \mathcal{K} \) is generated by linear part of the equation (0.7).
Theorem 3.1 Assume that $F(x,u)$ is Lipschitz continuous in the space $H_s^r(R^n)$, $s > n/2$, $F(0) = 0$, and also that $\alpha > 0$.

(i) Suppose that $0 < \Re M < 1/2$ and $\gamma \in [0, \frac{n-1}{2}]$. Then for every given function $\Phi_0(x,t) \in X(\varepsilon, s, \gamma)$ such that

$$\sup_{t \in [0, \infty)} e^{\gamma t} \| \Phi_0(\cdot, t) \|_{H_s^r(R^n)} < \varepsilon,$$

and for sufficiently small $\varepsilon$, the integral equation (0.5) has a unique solution $\Phi(x,t) \in X(2\varepsilon, s, \gamma)$. For the solution one has

$$\sup_{t \in [0, \infty)} e^{\gamma t} \| \Phi(\cdot, t) \|_{H_s^r(R^n)} < 2\varepsilon.$$  \hspace{1cm} (3.1)

(ii) Suppose that $\Re M \in [1/2, n/2)$. Then for every given function $\Phi_0(x,t) \in X(\varepsilon, s, \gamma_0)$, $\gamma_0 > 0$, such that

$$\sup_{t \in [0, \infty)} e^{\gamma_0 t} \| \Phi_0(\cdot, t) \|_{H_s^r(R^n)} < \varepsilon,$$

for every $\gamma$ such that $\gamma \leq \gamma_0$, $\gamma < (n/2 - \Re M)/(\alpha + 1)$, and for sufficiently small $\varepsilon$, the integral equation (0.5) has a unique solution $\Phi(x,t) \in X(2\varepsilon, s, \gamma)$. For the solution one has (3.1).

(iii) Suppose that $\Re M > n/2$. Then for the function $\Phi_0(x,t) \in X(\varepsilon, s, \gamma)$, $\gamma < \frac{1}{n+1}(\frac{n}{2} - \Re M)$, a unique solution $\Phi(x,t)$ of the integral equation (0.5) has the lifespan $T_{\alpha}$ that can be estimated from below by

$$T_{\alpha} \geq \frac{1}{|\gamma|} \ln \left( \sup_{\tau \in [0, \infty)} e^{\gamma \tau} \| \Phi_0(\cdot, \tau) \|_{H_s^r(R^n)} \right) - C(M, n, \alpha, \gamma).$$

with some constant $C(M, n, \alpha, \gamma)$.

Proof. (i) Consider the mapping

$$S[\Phi](x,t) := \Phi_0(x,t) + G[F(\cdot, \Phi)](x,t).$$

We are going to prove that $S$ maps $X(R, s, \gamma)$ into itself and that $S$ is a contraction, provided that $\varepsilon$ and $R$ are sufficiently small. Consider the case of $\Re M = (\frac{n^2}{4} - m^2)^{1/2} < 1/2$. Theorem 2.1 implies

$$\|S[\Phi](\cdot, t)\|_{H_s^r(R^n)} \leq \|\Phi_0(\cdot, t)\|_{H_s^r(R^n)} + \|G[F(\cdot, \Phi)](\cdot, t)\|_{H_s^r(R^n)}$$

$$\leq \|\Phi_0(\cdot, t)\|_{H_s^r(R^n)} + C_M e^{\frac{n-1}{2} t} \int_0^t e^{\frac{n-1}{2} b} \|F(\cdot, \Phi)(\cdot, b)\|_{H_s^r(R^n)} db.$$

Taking into account the Condition (L) we arrive at

$$\|S[\Phi](\cdot, t)\|_{H_s^r(R^n)} \leq \|\Phi_0(\cdot, t)\|_{H_s^r(R^n)} + C_M e^{\frac{n-1}{2} t} \int_0^t e^{\frac{n-1}{2} b} \|\Phi(\cdot, b)\|_{H_s^r(R^n)}^{\alpha + 1} db.$$

Then, for $\gamma \in R$ we have

$$e^{\gamma t} \|S[\Phi](x,t)\|_{H_s^r(R^n)}$$

$$\leq e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_s^r(R^n)} + C_M e^{\gamma t - \frac{n-1}{2} t} \int_0^t e^{\frac{n-1}{2} b} e^{-\gamma(\alpha+1)b} \left( e^{\gamma b} \|\Phi(\cdot, b)\|_{H_s^r(R^n)} \right)^{\alpha + 1} db$$

$$\leq e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_s^r(R^n)} + C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi(\cdot, \tau)\|_{H_s^r(R^n)} \right)^{\alpha + 1} e^{\gamma t - \frac{n-1}{2} t} \int_0^t e^{\frac{n-1}{2} b} e^{-\gamma(\alpha+1)b} db.$$

For $\gamma \in [0, \frac{n-1}{2}]$ and $\alpha > 0$, the following function is bounded

$$e^{\gamma t - \frac{n-1}{2} t} \int_0^t e^{\frac{n-1}{2} b} e^{-\gamma(\alpha+1)b} db \leq C \text{ for all } t \in [0, \infty).$$  \hspace{1cm} (3.2)
Consequently,

$$\sup_{t \in [0, \infty)} e^{\gamma t} \| S[\Phi](x, t) \|_{H^s(\mathbb{R}^n)}$$

$$\leq \sup_{t \in [0, \infty)} e^{\gamma t} \| \Phi_0(\cdot, t) \|_{H^s(\mathbb{R}^n)} + C_M \left( \sup_{t \in [0, \infty)} e^{\gamma t} \| \Phi(\cdot, \tau) \|_{H^s(\mathbb{R}^n)} \right)^{\alpha+1}.$$

Thus, the last inequality proves that the operator $S$ maps $X(R, s, \gamma)$ into itself if $\varepsilon$ and $R$ are sufficiently small, namely, if $\varepsilon + CR^{\alpha+1} < R$.

It remains to prove that $S$ is a contraction mapping. As a matter of fact, we just apply the estimate (0.2) and get the contraction property from

$$e^{\gamma t} \| S[\Phi](\cdot, t) - S[\Psi](\cdot, t) \|_{H^s(\mathbb{R}^n)} \leq CR(t)^{\alpha} d(\Phi, \Psi),$$

where $R(t) := \max\{ \sup_{0 \leq \tau \leq t} e^{\gamma \tau} \| \Phi(\cdot, \tau) \|_{H^s(\mathbb{R}^n)}, \sup_{0 \leq \tau \leq t} e^{\gamma \tau} \| \Psi(\cdot, \tau) \|_{H^s(\mathbb{R}^n)} \} \leq R$. Indeed, we have

$$\| S[\Phi](\cdot, t) - S[\Psi](\cdot, t) \|_{H^s(\mathbb{R}^n)} = \| G( F(\cdot, \Phi) - F(\cdot, \Psi) ) (\cdot, t) \|_{H^s(\mathbb{R}^n)}$$

$$\leq C_M e^{-\frac{\alpha+1}{2} t} \int_0^t e^{\frac{\alpha+1}{2} b} \| F(\cdot, \Phi) - F(\cdot, \Psi) (\cdot, b) \|_{H^s(\mathbb{R}^n)} \ db$$

$$\leq C_M e^{-\frac{\alpha+1}{2} t} \int_0^t e^{\frac{\alpha+1}{2} b} \| \Phi(\cdot, b) - \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)} \left( \| \Phi(\cdot, b) \|_{H^s(\mathbb{R}^n)}^\alpha + \| \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)}^\alpha \right) \ db.$$

Thus, taking into account (3.2), the last estimate, and the definition of the metric $d(\Phi, \Psi)$, we obtain

$$e^{\gamma t} \| S[\Phi](\cdot, t) - S[\Psi](\cdot, t) \|_{H^s(\mathbb{R}^n)}$$

$$\leq C_M e^{\gamma t} e^{-\frac{\alpha+1}{2} t} \int_0^t e^{\frac{\alpha+1}{2} b} \| \Phi(\cdot, b) - \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)} \left( \| \Phi(\cdot, b) \|_{H^s(\mathbb{R}^n)}^\alpha + \| \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)}^\alpha \right) \ db$$

$$\leq C_M e^{\gamma t} e^{-\frac{\alpha+1}{2} t} \int_0^t e^{\frac{\alpha+1}{2} b} \| \Phi(\cdot, b) - \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)} \left( \max_{0 \leq \tau \leq b} e^{\gamma \tau} \| \Phi(\cdot, \tau) - \Psi(\cdot, \tau) \|_{H^s(\mathbb{R}^n)} \right)$$

$$\times \left( \max_{0 \leq \tau \leq b} e^{\gamma \tau} \| \Phi(\cdot, \tau) \|_{H^s(\mathbb{R}^n)} \right)^\alpha + \left( \max_{0 \leq \tau \leq b} e^{\gamma \tau} \| \Psi(\cdot, \tau) \|_{H^s(\mathbb{R}^n)} \right)^\alpha \ db$$

$$\leq C_M e^{\gamma t} e^{-\frac{\alpha+1}{2} t} \int_0^t e^{\frac{\alpha+1}{2} b} \| \Phi(\cdot, b) - \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)} \ db$$

$$\leq C_M e^{\gamma t} e^{-\frac{\alpha+1}{2} t} \int_0^t e^{\frac{\alpha+1}{2} b} \| \Phi(\cdot, b) - \Psi(\cdot, b) \|_{H^s(\mathbb{R}^n)} \ db.$$
Then, for \( \Phi_0 \in X(R, s, \gamma_0) \) and \( \gamma \geq 0 \) we have

\[
e^{\gamma t} \| S[\Phi](x, t) \|_{H(\alpha)(\mathbb{R}^n)} \leq e^{\gamma t} \| \Phi_0(\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} + C_M e^{\gamma t + (\Re M - \frac{\gamma}{2}) t} \int_0^t e^{-(\Re M - \frac{\gamma}{2}) s} \left( e^{-\gamma(\alpha+1)b} \left( e^{\gamma \| \Phi(\cdot, b) \|_{H(\alpha)(\mathbb{R}^n)} t \right) ^{\alpha+1} ds \right.
\]

If \( \gamma = \frac{1}{\alpha+1}(\frac{\alpha}{2} - \Re M - \delta) > 0, \gamma \leq \gamma_0, \) and \( \delta > 0, \) then

\[
e^{\gamma t} \| S[\Phi](x, t) \|_{H(\alpha)(\mathbb{R}^n)} \leq e^{\gamma t} \| \Phi_0(\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} + C_M e^{\gamma t + (\Re M - \frac{\gamma}{2}) t} \int_0^t e^{\delta b} \, db \leq e^{\gamma t} \| \Phi_0(\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} + C_M e^{-\beta t} \delta^{-1} \left( \sup_{\tau \in [0, t]} e^{\gamma \| \Phi(\cdot, \tau) \|_{H(\alpha)(\mathbb{R}^n)} t \right) ^{\alpha+1} \sup_{\tau \in [0, t]} e^{\gamma \| \Phi(\cdot, \tau) \|_{H(\alpha)(\mathbb{R}^n)} t \right) ^{\alpha+1} \cdot
\]

In follows \( \Phi \in X(R, s, \gamma) \) provided that \( R \) and \( \epsilon \) are sufficiently small. We skip the remaining part of the proof since it is similar to the case (i).

(iii) Consider now the case of \( \Re M \geq n/2 > 1/2 \) and \( \Phi_0 \in X(R, s, \gamma) \). Theorem 2.1 implies

\[
\| S[\Phi](\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} \leq \| \Phi_0(\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} + \| G[F(\Phi)](\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} \leq \| \Phi_0(\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} + C_M e^{(\Re M - \frac{\beta}{2}) t} \int_0^t e^{-(\Re M - \frac{\beta}{2}) s} \| G[F(\Phi)](x, b) \|_{H(\alpha)} \, db.
\]

Taking into account the Condition (L) we arrive at

\[
e^{\gamma t} \| S[\Phi](\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} \leq e^{\gamma t} \| \Phi_0(\cdot, t) \|_{H(\alpha)(\mathbb{R}^n)} + C_M e^{(\gamma + \Re M - \frac{\beta}{2}) t} \int_0^t e^{-(\Re M - \frac{\gamma}{2}) s} \left( e^{-\gamma(\alpha+1)b} \left( e^{\gamma \| \Phi(\cdot, b) \|_{H(\alpha)(\mathbb{R}^n)} t \right) ^{\alpha+1} ds \right)
\]

If \( \gamma < \frac{1}{\alpha+1}(\frac{\alpha}{2} - \Re M) \leq 0, \) then for the given \( \Phi_0 \in X(T, s, \gamma) \) the lifespan of the solution \( \Phi \) can be estimated from below. Set

\[
T_\varepsilon := \inf \{ T : \sup_{\tau \in [0, T]} e^{\gamma \| \Phi(x, \tau) \|_{H(\alpha)(\mathbb{R}^n)} \geq 2\varepsilon \}, \quad \varepsilon := \sup_{\tau \in [0, \infty]} e^{\gamma \| \Phi_0(\cdot, \tau) \|_{H(\alpha)(\mathbb{R}^n)} \}
\]

Then

\[
2\varepsilon \leq \varepsilon + C_M e^{-\alpha T_\varepsilon} \varepsilon^{\alpha+1}
\]

implies

\[
T_\varepsilon \geq \frac{1}{|\varepsilon|} \ln \frac{\varepsilon}{C_M}.
\]

Thus, the theorem is proved. \( \square \)
Theorem 2.1: Every initial function \( \varphi \) defines a global solution \( \Phi \).

Remark 3.2: By the arguments have been used in the proof of the last theorem it is easy to derive the existence of the local in time solution for the arbitrary size initial data.

Proof of Theorem 0.1. (i) The case of \( \Re M \in (0, 1/2) \). For the function \( \Phi_0 \), that is, for the solution of the Cauchy problem (1.9) and for \( s > \frac{n}{2} \), according to Theorem 1.1 we have the estimate

\[
\|\Phi_0(x, t)\|_{H^{(s)}(\mathbb{R}^n)} \leq C_{M,n,s} e^{-\frac{n-s}{2}t} \left\{ \|\varphi_0\|_{H^{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H^{(s)}(\mathbb{R}^n)} \right\}.
\]

For every \( T > 0 \) we have \( \Phi_0 \in C([0, T]; H^{(s)}(\mathbb{R}^n)) \cap C^1([0, T]; H^{(s-1)}(\mathbb{R}^n)) \). According to Theorem 3.1, for every initial functions \( \varphi_0 \) and \( \varphi_1 \) the function \( \Phi_0 \) belongs to the space \( X(R, s, \frac{n-1}{2}) \), where the operator \( S \) is a contraction.

(ii) In the case of \( \Re M \in [1/2, n/2) \) for the function \( \Phi_0 \), that is, for the solution of the Cauchy problem (1.9) and for \( s > \frac{n}{2} \), according to Theorem 1.1 we have the estimate

\[
\|\Phi_0(x, t)\|_{H^{(s)}(\mathbb{R}^n)} \leq Ce^{(\Re M - \frac{n}{2})t} \left\{ \|\varphi_0\|_{H^{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H^{(s)}(\mathbb{R}^n)} \right\}.
\]

According to Theorem 3.1, for every initial functions \( \varphi_0 \) and \( \varphi_1 \) the function \( \Phi_0 \) belongs to the space \( X(R, s, \frac{n-1}{2}) \).

(iii) If \( \Re M > n/2 \), then according to Theorem 1.1 for the solution of (1.9) we have the estimate (3.4) and, consequently, \( \Phi_0 \in X(R, s, \gamma) \) with \( \gamma = n/2 - \Re M < 0 \) for some \( R > 0 \). On the other hand,

\[
e^\gamma \|\Phi(\cdot, t)\|_{H^{(s)}(\mathbb{R}^n)} \leq e^\gamma \|\Phi_0(\cdot, t)\|_{H^{(s)}(\mathbb{R}^n)} + C_M \left( \max_{\tau \in [0, t]} e^{\gamma \tau} \|\Phi(\cdot, \tau)\|_{H^{(s)}(\mathbb{R}^n)} \right) \frac{e^{-(\gamma - \alpha)T_x} - 1}{-\gamma \alpha}.
\]

Next we define (3.3). Then

\[
2 \varepsilon \leq \varepsilon + C_M \varepsilon^{\alpha + 1} \frac{e^{-\gamma \alpha T_x} - 1}{-\gamma \alpha}
\]

implies \( T_x \geq -\frac{1}{\Re M - \frac{n}{2}} \ln (\varepsilon) - C(M, n, \alpha) \). Theorem is proved.

Proof of Theorem 0.2. First consider the case of \( \Re M \in (0, 1/2) \). According to Theorem 0.1 there is a global solution \( \Phi \in X(R, s, \frac{n-1}{2}) \). Then, Theorem 1.6 implies \( \partial_t \Phi_0 \in X(R, s, \frac{n-1}{2}) \). In order to check that \( \partial_t F(\Phi) \in X(R, s, \frac{n-1}{2}) \) we apply the Condition (L) and the property of the operator \( G \) proved in Theorem 2.1:

\[
\|\partial_t F(x, \Phi)\|_{H^{(s)}} \leq Ce^{-\frac{n-1}{2}t} \left\lVert e^{\frac{n-1}{2}b} \|F(x, \Phi)\|_{H^{(s)}(\mathbb{R}^n)} dB \right\lVert_{H^{(s)}} \leq Ce^{-\frac{n-1}{2}t} \left\lVert e^{\frac{n-1}{2}b} \|\Phi(x, b)\|_{H^{(s)}} \right\lVert_{H^{(s)}} \frac{\alpha + 1}{H^{(s)}} dB \right\lVert_{H^{(s)}}.
\]

Hence,

\[
e^{\gamma_1 t} \|\partial_t G[F(x, \Phi)](x, t)\|_{H^{(s)}} \leq Ce^{(\gamma_1 - \frac{n-1}{2})t} \int_0^t e^{\frac{n-1}{2}b} e^{-(\alpha + 1)\frac{n-1}{2}b} e^{\frac{n-1}{2}b} \|\Phi(x, b)\|_{H^{(s)}} \right\lVert_{H^{(s)}} \frac{\alpha + 1}{H^{(s)}} \int_0^t e^{1 - \frac{\gamma}{2}(n-1)b} dB \right\lVert_{H^{(s)}}.
\]

If \( \alpha > 2/(n-1) \) we can set \( \gamma_1 = \frac{n-1}{2} \) and derive

\[
e^{\frac{n-1}{2}t} \|\partial_t G[F(\Phi)](x, t)\|_{H^{(s)}} \leq C \left( \max_{0 \leq \tau \leq t} e^{\frac{n-1}{2}b} \|\Phi(x, b)\|_{H^{(s)}} \right)^{\alpha + 1}.
\]
Assume now that $\Re M > 3/2$ or $M = 3/2$, then Theorem 1.6 implies $\partial_t \Phi_0 \in X(R, s, \frac{a}{2} - \Re M)$. Furthermore,
\[
\|\partial_t G[F(x, \Phi)](x, t)\|_{H^{(\alpha)}} \leq C e^{(\Re M - \frac{a}{2})t} \int_0^t e^{- (\Re M - \frac{a}{2})b} \|\Phi(x, b)\|_{H^{(\alpha)}}^n \, db \\
+ C e^{- \frac{a-1}{2}t} \int_0^t e^{\frac{a-1}{2}b} \|\Phi(x, b)\|_{H^{(\alpha)}}^n \, db,
\]
and
\[
e^{-\gamma t} \|\partial_t G[F(x, \Phi)](x, t)\|_{H^{(\alpha)}} \leq C \left( \max_{0 \leq \tau \leq t} e^{\gamma \tau} \|\Phi(x, b)\|_{H^{(\alpha)}} \right)^{\alpha+1} \left( e^{(\gamma_1 + \Re M - \frac{a}{2})t} \int_0^t e^{(\Re M - \frac{a}{2})b - \gamma(a+1)b} \, db + e^{\gamma_1} \int_0^t e^{\frac{a-1}{2}b - \gamma(a+1)b} \, db \right).
\]
Here $\frac{a}{2} - \Re M - \gamma(a+1) > 0$ and $\frac{a-1}{2} - \gamma(a+1) > 0$ since $\gamma < \frac{a+1}{\alpha+1}(\frac{a}{2} - \Re M)$. The last factor is bounded if $\gamma_1 \leq \gamma(a+1) - 1$. We set $\gamma_1 = \gamma - 1 < \min\{\frac{a}{2} - \Re M, \gamma(a+1) - 1\}$. The theorem is proved.

□

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A Appendix

There is a formula (see 15.3.6 of Ch.15[1] and [6]) that ties together points $z = 0$ and $z = 1$:
\[
F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z) + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z), \ |\arg(1-z)| < \pi. \tag{A.1}
\]
Here $a, b, c \in \mathbb{C}$. It follows
\[
F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{if} \quad \Re (c-a-b) > 0. \tag{A.2}
\]
Each term of the formula (A.1) has a pole when $c = a+b \pm k$, $(k = 0, 1, 2, \ldots)$; this case is covered by 15.3.10 of Ch.15[1]
\[
F(a, b; a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \left(2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(1-z)\right) (1-z)^n, \tag{A.3}
\]
\[
|\arg(1-z)| < \pi, \quad |1-z| < 1.
\]

Lemma A.1 If $a > -1$ and $M \in \mathbb{C}$ satisfies either $\Re M > 1/2$ or $\Re M = 1/2 & \Im M \neq 0$, then
\[
\lim_{z \to \infty} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) = \frac{\Gamma\left(\frac{a+3}{2}\right)\Gamma\left(M - \frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2} + \frac{1}{2}\right)} . \tag{A.4}
\]
If $a > -1$ and $M = 1/2$, then
\[
\lim_{z \to \infty} \frac{1}{\ln z} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) = \frac{1+a}{2} .
\]
If $a > -1$ and $\Re M < 1/2$, then
\[
\lim_{z \to \infty} z^{M - \frac{1}{2}} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) = 2^{2M-1} \frac{1+a}{1-2M} . \tag{A.5}
\]

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Proof. The statement (A.4) follows from (A.2). Now consider the case of \( \Re M < 1/2 \). According to [6, (29) Sec.2.1.5],
\[
F \left( \frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; x \right) = \left( 1 - x \right)^{-\left(\frac{1}{2} - M\right)} F \left( 1, \frac{a}{2} + M; \frac{a+3}{2}; x \right)
\]
while
\[
(1 - x)^{-\left(\frac{1}{2} - M\right)} = \left( \frac{4z}{(z+1)^2} \right)^{-\left(\frac{1}{2} - M\right)}.
\]
(A.2), and \( 1 + \frac{a}{2} + \Re M < \frac{a+3}{2} \) yield
\[
\lim_{z \to \infty} z^{M - \frac{1}{2}} F \left( \frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}, (z-1)^2 \right) = 4^{-\left(\frac{1}{2} - M\right)} \lim_{z \to \infty} F \left( 1, \frac{a}{2} + M; \frac{a+3}{2}, (z-1)^2 \right) = 4^{-\left(\frac{1}{2} - M\right)} \lim_{z \to \infty} F \left( 1, \frac{a}{2} + M; \frac{a+3}{2}, (z-1)^2 \right)
\]
\[
= 4^{-\left(\frac{1}{2} - M\right)} \frac{\Gamma \left( \frac{a+3}{2} \right) \Gamma \left( \frac{1}{2} - M \right)}{\Gamma \left( \frac{a+3}{2} - 1 \right) \Gamma \left( \frac{1}{2} - M \right)} = 4^{-\left(\frac{1}{2} - M\right)} \frac{\frac{a+1}{2}}{\left( \frac{1}{2} - M \right)}
\]
If \( M = 1/2 \), then we apply (A.3) to \( F \left( \frac{a+1}{2}, \frac{a+1}{2}, (z-1)^2 \right) \) with
\[
\zeta = \frac{(z-1)^2}{(z+1)^2}, \quad 1 - \zeta = \frac{4z}{(z+1)^2}, \quad \lim_{z \to \infty} \frac{\ln(1 - \zeta)}{\ln z} = -1.
\]
Hence
\[
F \left( \frac{a+1}{2}, \frac{a+1}{2} + 1; \zeta \right) = \frac{\Gamma \left( \frac{a+3}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \sum_{n=0}^{\infty} \frac{(a+1)_n (1)_n}{(n!)^2} \left[ \psi(n+1) - \psi \left( \frac{a+1}{2} + n \right) - \ln(1 - \zeta) \right] (1 - \zeta)^n
\]
\[
= \frac{\Gamma \left( \frac{a+3}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \left[ \psi(1) - \psi \left( \frac{a+1}{2} \right) - \ln(1 - \zeta) \right]
\]
\[+ \frac{\Gamma \left( \frac{a+3}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \sum_{n=1}^{\infty} \frac{(a+1)_n (1)_n}{(n!)^2} \left[ \psi(n+1) - \psi \left( \frac{a+1}{2} + n \right) - \ln(1 - \zeta) \right] (1 - \zeta)^n
\]
and
\[
\lim_{z \to \infty} \frac{1}{\ln z} F \left( \frac{a+1}{2}, \frac{a+1}{2} ; \frac{(a+3)(z-1)^2}{2} \right) = \lim_{z \to \infty} \frac{1}{\ln z} \left[ \frac{\Gamma \left( \frac{a+3}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \left[ \psi(1) - \psi \left( \frac{a+1}{2} \right) - \ln(1 - \zeta) \right]
\]
\[+ \frac{\Gamma \left( \frac{a+3}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \sum_{n=1}^{\infty} \frac{(a+1)_n (1)_n}{(n!)^2} \left[ \psi(n+1) - \psi \left( \frac{a+1}{2} + n \right) - \ln(1 - \zeta) \right] (1 - \zeta)^n
\]
\[\left( 1 - \zeta \right)^n
\]
\[= \frac{a + 1}{2} \lim_{z \to \infty} \frac{\ln(1 - \zeta)}{\ln z}
\]
\[= \frac{1 + a}{2}.\]

The lemma is proved. \(\square\)

**Lemma A.2** For \(a > -1\) and \(t > b > 0\) we have
\[\int_0^{e^{-b}-e^{-t}} r^a (e^{-t} + e^{-b})^2 - r^2)^{\frac{3}{2}} e^{-2b} dr \leq C e^{-\frac{1}{2} t} e^{-(a + 1)b}.\]

**Proof.** We have
\[\int_0^{e^{-b}-e^{-t}} r^a (e^{-t} + e^{-b})^2 - r^2)^{\frac{3}{2}} e^{-2b} dr \leq C e^{-\frac{1}{2} t} e^{-(a + 1)b}.\]

for \(z > 1\). Then with \(z = e^{t-b}\) we obtain
\[\int_0^{e^{-b}-e^{-t}} r^a (e^{-t} + e^{-b})^2 - r^2)^{\frac{3}{2}} e^{-2b} dr = e^{-2b} e^{-at-t+3/2t} \int_0^{z^{-1}} y^a ((z + 1)^2 - y^2)^{\frac{3}{2}} dy \leq C(1 + t - b)^{-1 - \sgn(\Re M - 1/2)} (e^{t-b})^a (e^b + e^t)^2 (e^{t-b} - e^{-b}) \left( e^{-a + (2\Re M - 2)(b + t)} \right).\]

**Lemma A.3** Assume that \(a > -1\), \(t > b > 0\), \(M \in \mathbb{C}\), and \(\Re M \geq 1/2 \& M \neq 1/2\). Then
\[\left| \int_0^{e^{-b}-e^{-t}} r^a (e^{-t} + e^{-b})^2 - r^2)^{\frac{3}{2}} + M dr \right| \leq C(1 + t - b)^{-1 - \sgn(\Re M - 1/2)} (e^{t-b})^a (e^b + e^t)^2 (e^{t-b} - e^{-b}) \left( e^{-a + (2\Re M - 2)(b + t)} \right).\]

**Proof.** We have with \(r = e^{-t} y\)
\[\int_0^{e^{-b}-e^{-t}} r^a (e^{-t} + e^{-b})^2 - r^2)^{\frac{3}{2}} + M dr = e^{-a + (a - (3 + 2M)t) \int_0^{e^{t-b}} y^a ((1 + e^{t-b})^2 - y^2)^{\frac{3}{2}} dy.\]

If \(z := e^{t-b} > 1\), then we, in fact, can be evaluate the last integral
\[\int_0^{z^{-1}} y^a ((1 + z)^2 - y^2)^{\frac{3}{2}} dy = \frac{1}{a + 1} (z - 1)^a (a + 1)^2 (z + 1)^{2M-3} F \left( \frac{a + 1}{2}, \frac{3}{2} - M; \frac{a + 3}{2}; (z - 1)^2 \right).\]

Hence
\[\left| \int_0^{e^{-b}-e^{-t}} r^a (e^{-t} + e^{-b})^2 - r^2)^{\frac{3}{2}} + M dr \right| = \frac{1}{a + 1} (e^{t-b})^a (e^b + e^t)^2 (e^{t-b} - e^{-b}) \left( e^{-a + (2\Re M - 2)(b + t)} \right) \left| F \left( \frac{a + 1}{2}, \frac{3}{2} - M; \frac{a + 3}{2}; (e^{t-b} - 1)^2 \right) \right| \leq C(1 + t - b)^{-1 - \sgn(\Re M - 1/2)} (e^{t-b})^a (e^b + e^t)^2 (e^{t-b} - e^{-b}) \left( e^{-a + (2\Re M - 2)(b + t)} \right).\]

Lemma is proved. \(\square\)

We skip the proof of the next lemma.
Lemma A.4 Assume that $a > -1$, $t > b > 0$, $M \in \mathbb{C}$, $\Re M \geq 3/2$ and $M \neq 3/2$. Then
\[
\int_0^{e^{-b}-e^{-t}} r^a \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2} + M} dr \leq C(1 + (t - b)^{1 - \text{sgn} [\Re M - 3/2]} e^{-(a + 2M - 4)(b + t)} (e^t - e^b)^{1+a} (e^b + e^t)^{2M - 5}.
\]

Lemma A.5 If $\Re M > 0$, $z > 1$ and $a > -1$, then
\[
\int_0^{z-1} y^a \left( (z + 1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right| dy \leq \begin{cases} 
(z - 1)^{1+a} \Re M - \frac{1}{2} & \text{if } 0 < \Re M < 1/2, \\
(z - 1)^{1+a}(z + 1)^{2\Re M - 1} & \text{if } \Re M > 1/2.
\end{cases}
\]

Proof. Since $\Re M > 0$, then we have
\[
\int_0^{z-1} y^a \left( (z + 1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right| dy \leq C_M \frac{1}{1+a} (z - 1)^{1+a}(z + 1)^{1+2\Re M} F \left( \frac{1+a}{2}, \frac{1}{2} - \Re M; \frac{3+a}{2}, \frac{(z - 1)^2}{(z + 1)^2} \right).
\]

Now we use Lemma A.1 (for $\Re M$), that is,
\[
\lim_{z \to \infty} z^{M - \frac{1}{2}} F \left( \frac{a+1}{2}, \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2} \right) = \frac{\pi (a+1) 4^{M-1} \sec(\pi M)}{\Gamma \left( \frac{a}{2} - M \right) \Gamma \left( M + \frac{1}{2} \right)}
\]
and obtain for $0 < \Re M < 1/2$
\[
\int_0^{z-1} y^a \left( (z + 1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right| dy \leq C_M \frac{1}{1+a} (z - 1)^{1+a}(z + 1)^{1+2\Re M} z^{\frac{1}{2} - \Re M}
\]
while for $\Re M > 1/2$ we obtain
\[
\int_0^{z-1} y^a \left( (z + 1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} \left| F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z - 1)^2 - y^2}{(z + 1)^2 - y^2} \right) \right| dy \leq C \int_0^{z-1} y^a \left( (z + 1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} dy = \frac{1}{1+a} (z - 1)^{1+a}(z + 1)^{2\Re M - 1} \left| F \left( \frac{1+a}{2}, \frac{1}{2} - \Re M; \frac{3+a}{2}, \frac{(z - 1)^2}{(z + 1)^2} \right) \right| \leq (z - 1)^{1+a}(z + 1)^{2\Re M - 1}.
\]

Lemma is proved. \hfill \Box

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