Some group theory problems

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Dedicated to Boris Isaakovich Plotkin

Abstract

This is a survey of some problems in geometric group theory which I find interesting. The problems are from different areas of group theory. Each section is devoted to problems in one area. It contains an introduction where I give some necessary definitions and motivations, problems and some discussions of them. For each problem, I try to mention the author. If the author is not given, the problem, to the best of my knowledge, was formulated by me first.

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I assume that the reader knows basic definitions of group theory, as well as the definitions of amenable group, and the growth function of a group.

There are currently several methods of constructing groups with unusual properties (say, Tarski monsters, i.e. groups with all proper subgroups cyclic, or non-Amenable groups without free non-Abelian subgroups, or finitely generated infinite groups with finitely many conjugacy classes). The methods were originated in works by Adian, Novikov, Olshanskii [Ad, Ol3], Gromov [Gr2, Gr3], and developed by their students. The latest advance in the “theory of monsters” was due to Osin who constructed an infinite finitely generated group with exactly two conjugacy classes [Osin]. Each of these constructions proceeds as follows: we start with a free non-Abelian group (or a non-Abelian hyperbolic group in general), and then add relations one by one, at each step solving a little portion of the problem. It is important that at each step, we get a hyperbolic (or relatively hyperbolic) group, so that we can proceed by induction. In the limit, we get a group satisfying the desired property. For example, if we construct an infinite finitely generated group satisfying the law \( x^n = 1 \), then (following the method from [Ol3]), at each step we add one relation of the form \( u^n = 1 \), so that in the limit the law \( x^n = 1 \) is satisfied for all \( x \).

The groups constructed this way are usually infinitely presented: relations at each step are chosen so that they “do not interfere” with the relations from the previous step (we usually need some kind of small cancelation condition to achieve that). Hence relations added later do not follow from the previous relations. In this section, by “monsters”, I understand finitely generated infinite groups satisfying any of the following properties. These are only samples of possible properties, one can easily find generalizations or similar properties which are also of great interest.

1. The group is non-amenable but contains no non-Abelian free subgroups (solving the so-called Day-von Neumann problem [Ol3]);
2. The group satisfies the law \( x^n = 1 \) (Burnside problem, [Ad], [Ol3]);
3. Every proper subgroup is cyclic (Tarski problem [Ol3]);
4. Every proper subgroup is finite and cyclic of the same fixed prime order (Tarski problem [Ol3]);
5. The group is complete (i.e. every element has roots of any degree, Guba, [Guba1], Mikhaljovskii, Olshanskii [MO]);
6. The number of conjugacy classes is finite (S. Ivanov, [Ol3]), preferably 2 [Osin].

I would add to this list the following property which so far is dealt with by completely different methods (Grigorchuk, [Gri2], [GNS]), but all known examples are also infinitely presented. The property is very important:
The group has intermediate – not bounded above by a polynomial and not bounded below by an exponent – growth function (Milnor problem).

The following problem is very old, and probably was first formulated by Kurosh for torsion groups in the 50s, or even before that by somebody else. I heard the problem for other types of monsters from Adian, Grigorchuk, Olshanskii and others.

**Problem 1.1.** Find finitely presented monsters of types (1)-(7) or prove that they do not exist.

So far only finitely presented monsters of type (1) (non-Amenable but without non-Abelian free subgroups) have been found [OS1]. There are some ideas how to construct monsters of type (2). In [OS1], Olshanskii and I constructed a finitely presented bounded torsion group G with finite endomorphic presentation (S. Ivanov later constructed another example in [Iv2]): the group is generated by a finite set S and there exists an injective endomorphism \( \phi \) of the free group with basis S and a finite list of relations \( r_1, ..., r_n \) of the group such that the words \( \phi^m(r_i) \) for a presentation of G \((m = 0, 1, ..., i = 1, ..., n)\). Finite endomorphic presentations of groups are interesting because such a group G has a finitely presented ascending HNN extension \( \text{HNN}_{\phi}(G) \), and ascending HNN extensions preserve many properties of the group including amenability and elementary amenability. Ascending HNN extensions and endomorphic presentations have been used to produce finitely presented solvable groups by Remeslennikov [Rem] and Baumslag [Bau]. Lysenok discovered such a presentation for Grigorchuk’s group. Many other groups acting on locally finite rooted trees have such presentations (see the survey Grigorchuk-Nekrashevich-Sushchanskii [GNS]).

Although I am reasonably sure that a finitely presented infinite bounded torsion group exists (note that unbounded torsion infinite finitely presented groups are not known also), I am not at all sure that finitely presented monsters of types (3)-(7) exist. There are results which show that finitely presented groups satisfy nicer properties than arbitrary finitely generated groups. One example of such a result is the theorem of M. Kapovich and Kleiner: if one asymptotic cone of a finitely presented group is a tree, then the group is hyperbolic (see the appendix of [OOS]). In [OOS], Olshanskii, Osin and I showed that for finitely generated groups this is is extremely far from being true.

It is interesting that (as I have mentioned above) Grigorchuk’s group has also a finite endomorphic presentation, and is torsion. But it does not have bounded torsion. It is very interesting to know whether there are infinite bounded torsion groups of intermediate growth, or, more generally, whether there are finitely generated infinite amenable bounded torsion groups. Note that all torsion automata groups (groups constructed by Sushchanskii, Grigorchuk, Gupta-Sidki and others [GNS]) are residually finite by construction (they act faithfully on rooted locally finite trees), but there are no bounded torsion residually finite groups. That follows from Zelmanov’s solution of the restricted Burnside problem [Ze1, Ze2].

The following even more general problem is again a part of old folklore.

**Problem 1.2.** Are there (finitely generated non-virtually cyclic) amenable monsters of types (2)-(6).

Note that groups of intermediate growth are all amenable by the Fölner criterium (Fölner sets are simply balls in that case).

## 2 Asymptotic cones of groups

Asymptotic cones were introduced by Gromov in [Gr1], a definition via ultrafilters was given by van den Dries and Wilkie [VDW]. An asymptotic cone of a metric space is, roughly speaking,
what one sees when one looks at the space from infinitely far away.

Here is a more precise definition of asymptotic cones. A non-principal ultrafilter \(\omega\) is a finitely additive measure defined on all subsets \(S\) of \(\mathbb{N}\), such that \(\omega(S) \in \{0, 1\}\), \(\omega(\mathbb{N}) = 1\), and \(\omega(S) = 0\) if \(S\) is a finite subset. An ultrafilter allows one to define the limit of any bounded sequence of numbers \(x_n, n \in \mathbb{N}\): the limit \(\lim^\omega x_n\) with respect to \(\omega\) is the unique real number \(a\) such that \(\omega\{i \in \mathbb{N} : |x_i - a| < \epsilon\} = 1\) for every \(\epsilon > 0\). Similarly, \(\lim^\omega x_n = \infty\) if \(\omega\{i \in \mathbb{N} : |x_i| > M\}\) = 1 for every \(M > 0\).

Given two infinite sequences of real numbers \((a_n)\) and \((b_n)\) we write \(a_n = o_\omega(b_n)\) if \(\lim^\omega a_n/b_n = 0\). Similarly \(a_n = \Theta_\omega(b_n)\) (respectively \(a_n = O_\omega(b_n)\)) means that \(0 < \lim^\omega (a_n/b_n) < \infty\) (respectively \(\lim^\omega (a_n/b_n) = \infty\)).

Let \((X_n, \text{dist}_n), n \in \mathbb{N}\), be a metric space. Fix an arbitrary sequence \(e = (e_n)\) of points \(e_n \in X_n\). Consider the set \(\mathcal{F}\) of sequences \(g = (g_n), g_n \in X_n\), such that \(\text{dist}_n(g_n, e_n) \leq c\) for some constant \(c = c(g)\). Two sequences \((f_n)\) and \((g_n)\) of this set \(\mathcal{F}\) are said to be equivalent if \(\lim^\omega (\text{dist}_n(f_n, g_n)) = 0\). The equivalence class of \((g_n)\) is denoted by \((g_n)^\omega\). The \(\omega\)-limit \(\lim^\omega ((X_n)_e)\) is the quotient space of equivalence classes where the distance between \((f_n)^\omega\) and \((g_n)^\omega\) is defined as \(\lim^\omega (\text{dist}(f_n, g_n))\).

An asymptotic cone \(\text{Con}^\omega(X, e, d)\) of a metric space \((X, \text{dist})\) where \(e = (e_n), e_n \in X\), and \(d = (d_n)\) is an unbounded non-decreasing scaling sequence of positive real numbers, is the \(\omega\)-limit of spaces \(X_n = (X, \text{dist}/d_n)\). The asymptotic cone is a complete space; it is a geodesic metric space if \(X\) is a geodesic metric space (\([\text{Gr}3, \text{Dr}1]\)). Note that \(\text{Con}^\omega(X, e, d)\) does not depend on the choice of \(e\) if \(X\) is homogeneous (say, if \(X\) is a finitely generated group with a word metric), so in that case, we shall omit \(e\) in the notation of an asymptotic cone.

Asymptotic cones of groups capture global geometric properties of Cayley graphs of these groups. For example, a hyperbolic group “globally” looks like a tree because all asymptotic cones of them are trees. In fact all asymptotic cones of all non-virtually cyclic hyperbolic groups are isometric (\([\text{DP}\]).

Note that every asymptotic cone of a finitely generated group is a complete homogeneous geodesic metric space. The asymptotic cones of a group do not depend on the choice of observation points. So we shall omit it from the notation. The subgroup \(G^e\) consisting of all sequences \((g_i)\) in the ultrafilter \(G^\omega\) with \(|g_i| = O(d_i)\) acts isometrically and transitively on the asymptotic cone \(\text{Con}^\omega(G, (d_n))\) by left multiplication.

There are several basic questions about asymptotic cones that are still unanswered (see below).

### 2.1 The number of asymptotic cones

Having unique asymptotic cone or at least asymptotic cones that do not differ much is very convenient because in the applications (see below), one never knows which asymptotic cone of a group one gets because the scaling constants \(d_n\) are almost never defined explicitly. Unfortunately, the situation with the number of distinct asymptotic cones of a group is far from ideal.

In (\([\text{KST}1]\)), it is proved (answering a question of Gromov) that any uniform lattice \(\Gamma\) in \(\text{SL}_n(\mathbb{R}), n \geq 3\), has \(2^{2^{20}}\) pairwise non-homeomorphic asymptotic cones provided the Continuum Hypothesis (CH) is not true. If CH is true, then \(\Gamma\) has only one asymptotic cone. Moreover if CH is true then any group has at most continuum asymptotic cones (roughly, that is because by a theorem of Shelah, every asymptotic cone is determined by its “elementary theory”, and there are at most continuum different elementary theories). In (\([\text{DS}2]\), we constructed a group with continuum pair-wise non-\(\pi_1\)-equivalent asymptotic cones (without any assumption about CH).
The group is given by an infinite presentation satisfying a small cancelation condition (similar to the construction of “monsters” from the previous section). So it is not finitely presented, not amenable, etc. The following problem can be probably attributed to Gromov.

**Problem 2.1.** *Is there a finitely presented group with continuum asymptotic cones (assuming CH)?*

The answer is probably “yes”. The group from [DS2] is recursively presented, so by Higman’s theorem, it can be embedded into a finitely presented group. There are many different versions of Higman’s embedding [BORS], [OS1], [OS2], etc. One can try to prove that there exists an embedding preserving the property of having continuum non-$\pi_1$-equivalent asymptotic cones.

Note that the first (independent of the Continuum Hypothesis) example of a finitely presented group with two non-$\pi_1$-equivalent cones is constructed in [OS3]. Elementary amenable (infinitely presented) groups need not have unique asymptotic cone [OOS].

Note that having different asymptotic cones means that geometrically, a group looks different at different scales. Thus it is possible that a “less extreme” group always has unique asymptotic cone (assuming CH). For example, Pansu [Pa] proved that every nilpotent group has unique asymptotic cone.

**Problem 2.2.** *Is there a CAT(0)-group (a group acting properly co-compactly on a finite dimensional CAT(0)-space) having several non-homeomorphic asymptotic cones?*

**Problem 2.3.** *Let $F$ be the R. Thompson group, $G$ be one of the Grigorchuk groups of intermediate growth or any other self-similar group. How many non-homeomorphic asymptotic cones does $F$ (resp. $G$) have?*

Paper [OOS] gives large classes of groups all of whose asymptotic cones are locally isometric, but not all of them are isometric. Similar methods can be used to show that the groups from [TV] and from [DS2] Section 7 also satisfy this property. However all these groups are infinitely presented. Moreover, in all our examples, asymptotic cones are locally isometric to an $\mathbb{R}$–tree, which implies hyperbolicity for finitely presented groups [OOS]. However the following problem is still open.

**Problem 2.4 (Olshanskii, Osin, Sapir).** *Does there exist a finitely presented group all of whose asymptotic cones are locally isometric, but not all of them are isometric?*

**Problem 2.5 (Olshanskii, Osin, Sapir).** *Is there a non-virtually cyclic amenable group all of whose asymptotic cones are locally isometric to a tree?*

An elementary amenable group with one asymptotic cone an $\mathbb{R}$-tree was constructed in [OOS] (that answered a question of B. Kleiner).

### 2.2 Fundamental groups of asymptotic cones

Fundamental groups of asymptotic cones provide some important information about the structure of the group. For example, if all asymptotic cones are simply connected, then the group is finitely presented, has polynomial Dehn function, and linear isodiametric function (Gromov, [Gr3]). Since any asymptotic cone is homogeneous, uncountable, and its isometry group has usually uncountable point stabilizers, every non-trivial loop in the asymptotic cone has uncountably many copies having the same base point. This was probably the motivation of Gromov’s problem: Is there an asymptotic cone of finitely generated group, whose fundamental group is
non-trivial and not of order continuum? Such a group has been constructed in [OOS]. In fact the asymptotic cone of that group is homeomorphic to the direct product of a $\mathbb{R}$-tree and a circle. So its fundamental group is $\mathbb{Z}$. Note that the group in [OOS] is not finitely presented, finding a finitely presented example would be very interesting.

Since the asymptotic cones of a direct product are isometric to direct products of asymptotic cones of the factors, one can realize any finitely generated free Abelian group as the fundamental group of $\text{Con}^\omega(G,d)$ for a suitable $G$ by taking direct products of groups. It is quite possible that similarly one can construct an asymptotic cone with a finite Abelian fundamental group.

**Problem 2.6 (Olshanskii, Osin, Sapir).** Does there exist a finitely generated group $G$ such that $\pi_1(\text{Con}^\omega(G,d))$ is countable (or, better, finitely generated) and non-Abelian for some (any) $d$ and $\omega$? Can $\pi_1(\text{Con}^\omega(G,d))$ be finite and non-Abelian?

Note that for every countable group $C$ there exists a finitely generated group $G$ and an asymptotic cone $\text{Con}^\omega(G,d)$ such that $\pi_1(\text{Con}^\omega(G,d))$ is isomorphic to the uncountable free power of $C$ [DS2, Theorem 7.33].

### 3 Actions of groups on tree-graded spaces

One of the most important applications of asymptotic cones is the following observation due to Bestvina and Paulin: if a group $\Gamma$ has infinitely many pairwise non-conjugate in a group $G$ homomorphisms $\phi: \Gamma \to G$, then $\Gamma$ acts non-trivially by isometries on an asymptotic cone of $G$. The action is the following:

$$\gamma \circ (x_i) = (\phi_i(\gamma)x_i).$$

More generally, if a group admits “many” actions by isometries on a metric space $X$, then it acts non-trivially on an asymptotic cone of $X$. Thus, for example, if a hyperbolic group $G$ has infinite $\text{Out}(G)$, then $G$ acts non-trivially on an $\mathbb{R}$-tree.

If $G$ is not hyperbolic, then asymptotic cones of $G$ need not be trees. But they often are tree-graded spaces.

Recall [DS3] that a geodesic metric space $T$ is *tree-graded with respect to a collection of geodesic subsets* $\mathcal{P}$ (called *pieces*) if any two subsets in $\mathcal{P}$ intersect by at most a point and every simple loop in $T$ is contained in one of the pieces. General properties of tree-graded spaces can be found in [DS2].

In [DS3], it is proved that a group acting on a tree-graded space non-trivially (i.e. not stabilizing a piece and not fixing a point) also acts on an $\mathbb{R}$-tree without a global fixed points. Moreover, the stabilizers of points and arcs of the $\mathbb{R}$-tree are described.

Groups with tree-graded asymptotic cones occur much more often than groups whose asymptotic cones are $\mathbb{R}$-trees. It is proved in [DS2] that a group $G$ is relatively hyperbolic with respect to proper subgroups $H_1, \ldots, H_n$ if and only if is asymptotic cones are tree-graded with respect to the collection of all $\omega$-limits of sequence of cosets of the subgroups $H_1, \ldots, H_n$. Also it is noticed in [DS2], that a geodesic metric space is tree-graded with respect to a collection of proper pieces if and only if it has a global cut-point. Such a situation occurs very often. For example, if a group is given by a graded small cancelation presentation [Ol3, Ol4, OOS], then its asymptotic cones are *circle-trees*, that is tree-graded with respect to embedded circles of diameters bounded from below and from above [OOS]. Asymptotic cones of mapping class groups are tree-graded [Be]. So are the asymptotic cones of acylindrical amalgamated products of groups [DMS]. See more examples in [Be, DS3, DMS].
Using the result of Bestvina and Paulin, and the Rips theory of groups acting on trees (developed further by Bestvina, Feighn, Levitt, Sela, Guirardel and others), we found \[ DS_3 \] many properties of subgroups of relatively hyperbolic groups that are similar to properties of subgroups of hyperbolic groups. For example, if a subgroup \( H \) of a relatively hyperbolic group \( G \) has property (T), then \( \text{Aut}(H) \) is commensurable with the normalizer of \( H \) in \( G \). For other properties of subgroups of relatively hyperbolic groups obtained by using actions on tree-graded spaces see \[ DS_3 \].

**Problem 3.1.** *Can one extend results of \([DS_3]\) to subgroups of other groups with tree-graded asymptotic cones?*

According to \[ DS_3 \], in order to do that, one needs to:

- Describe the pieces of the tree-graded structure of the asymptotic cones of \( G \);
- Describe the stabilizers of the action of the isometry group \( G_\infty \)
  - a piece;
  - a pair of distinct pieces;
  - a pair (a piece and a point in it);
  - a pair of points not in the same piece.

It certainly can be done for a much larger class than the class of relatively hyperbolic groups. For example, Olshanskii, Osin and I found results similar to the results in \([DS_3]\) for subgroups of groups given by graded small cancelation presentations \([OOS]\). That class includes torsion groups, groups with all proper subgroups cyclic, and other monsters, in particular.

It would be extremely interesting to extend results of \([DS_3]\) to subgroups of mapping class groups. The pieces of the asymptotic cones of the mapping class groups have been described by Behrstock, Kleiner, Minsky, and Mosher. The work by Behrstock, Drutu and myself on completing the program is currently in progress.

## 4 Lacunary hyperbolic groups

Lacunary hyperbolic groups were introduced in \([OOS]\). A group is called *lacunary hyperbolic* if one of its asymptotic cones is an \( \mathbb{R} \)-tree. By \([OOS]\), a finitely generated group \( G = \langle S \rangle \) is lacunary hyperbolic if and only if it is a direct limit of groups \( G_i = \langle S \rangle \) and surjective homomorphisms \( \alpha_i : G_i \to G_{i+1} \) that is an identity on \( S \), such that each \( G_i \) is \( \delta_i \)-hyperbolic, each \( \alpha_i \) is injective on a ball of radius \( r_i \) and \( \delta_i = o(r_i) \).

Every finitely presented lacunary hyperbolic group is hyperbolic (Kapovich-Kleiner, \([OOS]\)). But that class is much larger than the class of hyperbolic groups: Tarski monsters, torsion groups, and other “monsters” discussed above, can be (and many known examples are by construction) lacunary hyperbolic. There are amenable non-virtually cyclic lacunary hyperbolic groups (I have already mentioned that above) \([OOS]\).

Although the class of lacunary hyperbolic groups is very large, groups in that class share many common properties with hyperbolic groups (see \([OOS]\)). For example,

- an undistorted subgroup of a lacunary hyperbolic group is lacunary hyperbolic itself,

- a lacunary hyperbolic group cannot contain a copy of \( \mathbb{Z}^2 \), an infinite finitely generated subgroup of bounded torsion and exponential growth, a copy of the lamplighter group, etc.,
• every lacunary hyperbolic group is embedded into a relatively hyperbolic 2-generated lacunary hyperbolic group as a peripheral subgroup,
• any group that is hyperbolic relative to a lacunary hyperbolic subgroup is lacunary hyperbolic itself.

There are probably more such properties. Here are two concrete problems.

**Problem 4.1** (Olshanskii, Osin, Sapir). Is it true that the growth of every non-elementary lacunary hyperbolic group is (a) exponential? (b) uniformly exponential?

**Problem 4.2** (Olshanskii, Osin, Sapir). Can a finitely generated non virtually cyclic subgroup of exponential growth of a lacunary hyperbolic group satisfy a non-trivial law?

**Problem 4.3.** Is there an analog of Bestvina-Feighn combination theorem [BF] for lacunary hyperbolic groups?

In [OOS], it is proved that a subgroup with a non-trivial law of a lacunary hyperbolic group cannot have relative exponential growth.

The answer to Problem 4.2 is “no” for lacunary hyperbolic groups for which, using the notation of the definition of lacunary hyperbolic groups, the injectivity radii \( r_i \) are “much larger” than the hyperbolicity constants \( \delta_i \). More precisely it is enough to assume that \( r_i = \exp \exp(\delta_i) \) [OOS].

All known “monsters” mentioned in Section 1 can be constructed in such a way that this growth condition is satisfied.

It is also interesting to study linearity of lacunary hyperbolic groups. We do not know the answer to the following basic question.

**Problem 4.4** (Olshanskii, Osin, Sapir). Is every linear lacunary hyperbolic group hyperbolic?

## 5 Linear and residually finite hyperbolic groups

It is known (Kapovich, [Kap]) that a hyperbolic group can be non-linear. But the existing examples are very implicit (although their presentations can be, in principle, written down). I think that non-linear hyperbolic groups may appear very often. One of the simple sources of hyperbolic groups are ascending HNN extensions of free groups. Let \( F_n = \langle x_1, \ldots, x_n \rangle, n \geq 2 \), be the free group, \( \phi: F_n \to F_n \) be an injective homomorphism, \( \text{HNN}_\phi(F_n) \) be the corresponding HNN extension.

It is known that \( \text{HNN}_\phi(F_n) \) is always coherent [FH] (every finitely generated subgroup is finitely presented) and residually finite [BS]. Some of these groups are not linear (example: \( \langle x, y, t \mid txt^{-1} = x^2, tjt^{-1} = y^2 \rangle \) [DS]). But known examples all contain Baumslag-Solitar subgroups, so these examples are not hyperbolic. Note that most groups \( \text{HNN}_\phi(F_n) \) are small cancelation, hence hyperbolic.

**Problem 5.1.** Are there non-linear hyperbolic groups of the form \( \text{HNN}_\phi(F_n) \) for non-surjective \( \phi \)? In particular, is the group \( \langle x, y, t \mid txt^{-1} = xy, tjt^{-1} = yx \rangle \) linear?

It is known (Minasyan) that this group is hyperbolic. If \( \phi \) is surjective (i.e. an automorphism), then \( \text{HNN}_\phi(F_2) \) is always linear. This follows from the linearity of the braid group \( B_4 \) (see [DFG]). It is quite possible (but still unknown) that \( \text{HNN}_\phi(F_n) \) is always linear if \( \phi \) is an automorphism.
The conjectural answers to the first question in Problem 5.1 is “yes” and to the second question “no”. Note that some groups of the form $HNN\varphi(F_n)$ are even inside $SL_2(\mathbb{R})$. The first example is of course the Baumslag-Solitar group $BS(1,n)$ for every $n$. But there are more complicated examples found in [CD]. All known examples are non-hyperbolic (contain Baumslag-Solitar subgroups), have $n \geq 5$, and correspond to reducible endomorphisms $\varphi$. Are there hyperbolic ascending HNN extensions $HNN\varphi(F_2)$ with non-surjective but injective $\varphi$ inside $SL_2(\mathbb{R})$?

If one considers two injective but not surjective homomorphisms $\varphi, \psi : F_n \to F_n$ then one can form the double HNN extension $HNN\varphi,\psi(F_n)$ which is also quite often hyperbolic. Methods from [BS] (based on studying periodic points of polynomial maps over finite fields) do not give a proof that such groups are residually finite. To the contrary, they indicate that these groups, generically, are not residually finite because pairs of independent polynomial maps over a finite field should not have common periodic points.

Thus it is possible that a double HNN extension of a free group can be hyperbolic and non-residually finite.

**Problem 5.2.** Are there non-residually finite hyperbolic groups of the form $HNN\varphi,\psi(F_n)$? In particular, is the group

$$\langle x, y, t, u \mid txt^{-1} = xy, tyt^{-1} = yx, uxu^{-1} = [x, y], uyu^{-1} = [x^2y, y^2x] \rangle$$

residually finite (this is just a random choice of “independent” endomorphisms $\varphi, \psi$)?

The conjectural answer is “yes” to the first question and “no” to the second.

## 6 R. Thompson group and other diagram groups

### 6.1 Diagram groups

One of the definitions of diagram groups is the following (see [GS0]). Consider an alphabet $X$ and a set $S$ of cells, each cell is a polygon whose boundary is subdivided into two directed paths (the top path and the bottom path having common initial and terminal points) labeled by positive words $u$ (the top path) and $v$ (the bottom path) in the alphabet $X$. One can consider the cell as a rewriting rule $u \to v$. Each cell $\pi$ is an elementary $(u,v)$-diagram with top path labeled by $u$, bottom path labeled by $v$, and two distinguished vertices $\iota$ and $\tau$: the common starting and ending points of the top and bottom paths. For every $x \in X$, there exists also the trivial $(x,x)$-diagram: an edge labeled by $x$. Its top path and bottom path coincide. There are four operations allowing to construct more complicated diagrams from the elementary ones. These are defined as follows.

- The addition: $\Delta_1 + \Delta_2$ is obtained by identifying the terminal vertex $\tau$ of $\Delta_1$ with the initial vertex $\iota$ of $\Delta_1$. The top and the bottom paths of $\Delta_1 + \Delta_2$ are defined in a natural way.
• The multiplication: If the label of the bottom path of $\Delta_1$ coincides with the label of the top path $\Delta_2$, then $\Delta_1\Delta_2$ is defined by identifying the bottom path of $\Delta_1$ with the top path of $\Delta_2$.

• The inversion: $\Delta^{-1}$ is obtained from $\Delta$ by switching the top and the bottom paths of the diagram.

• Dipole cancelation: if $\pi$ is an $(u,v)$-cell, then we identify $\pi\pi^{-1}$ with the trivial $(u,u)$-diagram. Thus we can always replace a subdiagram $\pi\pi^{-1}$ by the trivial $(u,u)$-subdiagram.

For every word $u$, the set of all $(u,u)$-diagrams forms a group under the product operation, the *diagram group* with base word $u$ and the given collection of cells. The $(u,u)$-diagrams are called spherical.

**Example 6.1** ([GS3]). Here are some examples of diagram groups.

• The R. Thompson group $F$ is the diagram group of all $(x,x)$-diagrams corresponding to the 1-letter alphabet $\{x\}$ and one cell $x^2 \rightarrow x$.

• The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is the diagram group of $(ac,ac)$-diagrams over the alphabet $\{a,b_1,b_2,b_3,c\}$ corresponding to cells $ab_1 \rightarrow a,b_1 \rightarrow b_2,b_2 \rightarrow b_3,b_3 \rightarrow b_1,b_1c \rightarrow c$.

• The free group $F_2$ is the diagram group of $(a,a)$-diagrams over the alphabet $\{a,a_1,a_2,a_3,a_4\}$ and cells $a \rightarrow a_1,a_1 \rightarrow a_2,a_2 \rightarrow a,a \rightarrow a_3,a_3 \rightarrow a_4,a_4 \rightarrow a$.

• The direct product $\mathbb{Z} \times \mathbb{Z}$ is the diagram group of $(ab,ab)$-diagram over the alphabet $\{a,a_1,a_2,b,b_1,b_2\}$ and cells $a \rightarrow a_1,a_1 \rightarrow a_2,a_2 \rightarrow a,b \rightarrow b_1,b_1 \rightarrow b_2,b_2 \rightarrow b$.

• Many right angled Artin groups are diagram groups [GS3].

The class of diagram groups is closed under direct and free products [GS1], each diagram group is linearly orderable [GS3] (and so it is torsion-free). One can view a diagram group as a 2-dimensional analog of a free group (a free group is the group of 1-paths of a graph; the diagram groups are groups of 2-paths on directed 2-complexes [GS2]). The word problem in any subgroup of a diagram group is very easy to decide. In many important cases (including the Thompson group $F$), the conjugacy problem in a diagram group has also an easy diagrammatic solution.

Although the class of diagram groups is large, some basic facts are still missing.

**Problem 6.1** (Guba, Sapir). *Is there a hyperbolic non-free diagram group?*
It is not even easy to construct a hyperbolic non-free subgroup of a diagram group (i.e. representable by diagrams). Nevertheless it is proved in [CSS] that fundamental groups of the orientable surface of genus 2 and the non-orientable surface of genus 4 are representable by diagrams. The fundamental group of a non-orientable surface cannot be a diagram group because all integral homology groups of all diagram groups are free Abelian [GS2].

There is a reason to believe that the answer to problem 6.1 is negative: by [GS0], the centralizer of a diagram in a diagram group is always a direct product of cyclic groups and diagram groups (over the same collection of cells but with different base words). The number of factors is equal to the maximal number of summands in a representation of a conjugate of the diagram (by \((u,v)\)-diagrams) as a sum of spherical summands. Hence if a diagram group does not contain \(\mathbb{Z} \times \mathbb{Z}\), then all diagrams in that group and all their conjugates are indecomposable into sums of spherical diagrams. So far the only diagram groups satisfying this property were free.

6.2 The R. Thompson group \(F\) and its subgroups

The R. Thompson group \(F\), corresponding to the “simplest” collection of cells \(\{xx \rightarrow x\}\), plays an important role in the class of diagram groups. It is proved in [GS3] that every diagram group is a subgroup of the so-called universal diagram group \(U\). The group \(U\) is finitely presented, and is explicitly constructed as a split extension of a right angled Artin group and the group \(F\).

Subgroups of R.Thompson’s group \(F\) have been of great interest after the first result of Thompson and Brin-Squier [BrSq85] that \(F\) contains no free non-Abelian subgroups. There are several Tits-like dichotomies concerning subgroups of \(F\):

- (Brin-Squier) A subgroup of \(F\) is either Abelian or contains a copy of \(\mathbb{Z}^\infty\).
- (Guba-Sapir) A subgroup of \(F\) is either Abelian or contains a copy of \(\mathbb{Z} \wr \mathbb{Z}\).
- (Bleak) A subgroup of \(F\) is either solvable or contains a copy of the (restricted) direct product \(W\) of wreath products \(\mathbb{Z} \wr \mathbb{Z} \cdots \wr \mathbb{Z}\) \((n\ times)\), \(n = 1, 2, \ldots\).

Bleak’s papers [Bl1, Bl2] contain, in a sense, a complete description of all solvable subgroups of \(F\). Note that nilpotent subgroups of diagram groups are free Abelian [GS0].

The following problem of Brin (also suggested by Guba and me) is still open and very interesting.

**Problem 6.2** (Brin). Is it true that every subgroup of \(F\) either contains a copy of \(F\) or is elementary amenable.

It is easy to see that \(F\) is not elementary amenable itself (since \(F'\) is simple). Note that Brin’s examples [Br1] show that even finitely generated elementary amenable subgroups subgroups of \(F\) can be very complicated. The most promising methods related to Problem 6.2 seems to be those developed by Brin [Br1, Br2] and Bleak [Bl1, Bl3]. But the diagram group methods may prove fruitful too.

One of the strongest results about the group \(F\) is Guba’s theorem that \(F\) has quadratic Dehn function [Guba2]. (Hence by Papasoglu’s result [Pap], the asymptotic cones of \(F\) are simply connected.) The following question remains open and even more intriguing after [Guba2].

**Problem 6.3** (Guba, Sapir). Is \(F\) automatic?
6.3 Amenability, property A, and embeddings into Hilbert spaces

Of course, I should also mention one of the central problems about R. Thompson’s group:

**Problem 6.4** (R. Thompson, R. Geoghegan). Is $F$ amenable?

The closest property to amenability that $F$ provably has is a-T-menability: existence of a proper action on a Hilbert space (Farley, [Far00]). It is not known even if $F$ satisfies the much weaker than amenability G. Yu’s property A.

A metric space $X$ satisfies property A if for every $\epsilon > 0$, and every $R$, there exists a collection of finite subsets $A_x \subseteq X \times \mathbb{N}$, $x \in X$, and a number $S > 0$ such that

- For every $x, y \in X$ with $\operatorname{dist}(x, y) < R$, we have
  \[
  \frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon,
  \]
  where $\Delta$ denotes the symmetric difference,

- For every $x \in X$, the diameter of the projection of $A_x$ onto $X$ is at most $S$.

G. Yu proved that if a Cayley graph $X$ of a group $G$ satisfies property A, then it coarsely (uniformly) embeds into a Hilbert space $H$, i.e. there exists a function $f : X \to H$ and an increasing unbounded compression function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

\[
\rho(\operatorname{dist}(x, y)) \leq \operatorname{dist}(f(x), f(y)) \leq \operatorname{dist}(x, y)
\]

for every $x, y \in X$.

Guentner and Kaminker [GK] proved that, “conversely”, if such an embedding exists and, in addition

\[
\lim_{n \to \infty} \frac{\rho(n)}{\sqrt{n}} = \infty
\]

(i.e. $\sqrt{n} \ll \rho(n)$), then $X$ satisfies property A. If, moreover, the embedding $f$ is equivariant, that is for some proper action of $G$ on $H$, we have $f(gx) = gf(x)$ (for all $g, x \in G$), then $G$ is amenable. In [AGS], we constructed (using actions of diagram groups on the so-called 2-trees, and a result of Burillo [Bur2]) an equivariant embedding of $F$ into a Hilbert space with

\[
\lim \inf_{n \to \infty} \frac{\rho(n)}{\sqrt{n}} > 0
\]

and proved that for every (not necessary equivariant) coarse embedding of $F$ into a Hilbert space,

\[
\lim \sup_{n \to \infty} \frac{\rho(n)}{\sqrt{n} \log n} < \infty.
\]

Thus there is a gap $[\sqrt{n}, \sqrt{n} \log n]$ where a possible compression function of $F$ may belong. Hence there is still hope that the answer to the following problem is positive and $F$ has property $A$ or even is amenable.

**Problem 6.5** (Arzhantseva, Guba, Sapir). Is there (equivariant or not) coarse embedding of the R. Thompson group $F$ into a Hilbert space with compression function $\rho$ satisfying $\lim_{n \to \infty} \frac{\rho(n)}{\sqrt{n}} = \infty$?
In [ADS], we introduced the notion of a Hilbert space compression gap of a metric space $X$.

**Definition 6.6.** Let $(X, \text{dist})$ be a metric space. Let $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be two increasing functions such that $f \ll g$ and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$.

We say that $(f, g)$ is a Hilbert space compression gap of $(X, \text{dist})$ if

1. there exists an embedding of $X$ into a Hilbert space with compression function at least $f$;
2. for every embedding of $X$ into a Hilbert space, its compression function $\rho$ satisfies $\rho \ll g$.

The quotient $g/f$ is called the size of the compression gap. We showed in [ADS], in particular, that for every real $\alpha$ between 0 and 1, there exists a finitely generated group of asymptotic dimension at most 2 such that for every $\epsilon > 0$, $\left(\frac{x^\alpha}{\log^{1+\epsilon}(x+1)}, x^\alpha\right)$ is a Hilbert space compression gap of that group (that answered a question of Guentner and Niblo). Results of [ADS] show that Thompson’s group $F$ has compression gap of logarithmic size. Similar results for free groups, lattices in Lie groups, etc. follow from results of Bourgain [Bou] and Tessera [Te].

**Problem 6.7.** Is there any group with compression gap less than $\log x$ (say, $\log \log x$)?

The answer would be positive (for the R. Thompson group) if Problem 6.5 had positive answer.

Note that every finitely generated diagram group such as $F$, $\mathbb{Z} \wr \mathbb{Z}$, etc. has two natural metrics: the word metric and the diagram metric. The distance between two diagrams $\Delta_1$ and $\Delta_2$ in the diagram metric is the number of cells in the diagram $\Delta_1^{-1}\Delta_2$ after removing all the dipoles. Translated into the language of diagram groups, the result of Burillo [Bur2] used in [AGS] is the following: for the group $F$, the word metric and the diagram metric are quasi-isometric. In [AGS], we proved that the same is true for $\mathbb{Z} \wr \mathbb{Z}$ and one of the universal diagram groups.

**Problem 6.8** (Arzhantseva, Guba, Sapir). Does every finitely generated diagram group satisfy the Burillo property?

If the answer is “yes”, then every finitely generated diagram group would have an embedding into a Hilbert space with compression function $\rho(n) \succ \sqrt{n}$.

### 6.4 The membership problem and distortion of subgroups

Many algorithmic problems of $F$ are known to be solvable. Those are the word problem, the conjugacy problem [GS0], the multiple conjugacy problem [KM] and others. One of the most important problems that are still open is the generalized word problem, that is the membership problem in the finitely generated subgroups.

**Definition 6.9.** Recall that an increasing function $f(n) : \mathbb{N} \rightarrow \mathbb{N}$ is called the distortion function of a finitely generated subgroup $H = \langle Y \rangle$ of a group $G = \langle X \rangle$ if every element $h \in H$ that has length $n$ in $G$ (relative to $X$) has length at most $f(n)$ in $H$ (relative to $Y$), and $f$ is the smallest function with this property.

If we change generating sets $X$ and $Y$ (keeping them finite), the new distortion function is equivalent to the old one. We say that increasing functions $f$ and $g$ are equivalent if

$$\frac{1}{C} f\left(\frac{n}{C}\right) - C < g(n) < C f(Cn) + C$$
for some $C \geq 1$ and all sufficiently large $n$.

It is well known (probably Farb was the first who noticed that) that the generalized word problem is solvable for a given finitely generated subgroup $H$ of a group $G$ if and only if the distortion of $H$ is recursive. Very little is known about the distortion of subgroups of $F$. It is known that $F$ and its “brothers” $F_{n}$, direct powers $F^{n}$ for every $n$, $F \wr \mathbb{Z}$, etc. (see [GS0, GS1]) can be embedded into $F$ without distortion. It is much more complicated to find distorted subgroups of $F$. There are known examples of subgroups with non-linear distortion [GS1]. But all these subgroups are solvable and the distortion is at most quadratic.

**Problem 6.10** (Guba-Sapir).  

a) Does $F$ contain a distorted copy of itself (note that by Brin [Br2], $F$ contains lots of copies of itself).

b) Is there a finitely generated subgroup of $F$ with bigger than polynomial (or even bigger than quadratic) distortion?

c) Is there a finitely generated subgroup of $F$ with non-recursive distortion?

Problem 6.10.b) has been also suggested by Brin.

7 Surface subgroups of right angled Artin groups

7.1 The main problem

This is another kind of representation problems. We want to represent closed hyperbolic surface groups (and more generally hyperbolic non-free groups) as subgroups of right angled Artin groups. Right angled Artin groups have so many nice properties (surveyed in Charney [Ch]) that embedding a group into one of them is certainly interesting. There is an industry developed mostly by D. Wise and F. Haglund designed to embed a group into a right angled Artin group. Here we are interested in a different kind of problems: what subgroups a given right angled Artin group can have?

**Problem 7.1** (Crisp-Sageev-Sapir). Is there an algorithm to decide given a finite graph $K$ whether the right angled Artin group $A(K)$ contains the fundamental group of a closed hyperbolic surface?

There are several partial results related to this problem [DSS], [Kim], [CSS]. Clearly if $K_{1}$ is a full subgraph of $K_{2}$, then $A(K_{1}) \leq A(K_{2})$, so if $A(K_{1})$ contains $\pi_{1}(S)$ for some surface $S$ then so does $A(K_{2})$. Kim [Kim] proved that if $K_{1}$ is obtained from $K$ by the co-contraction of a pair of non-adjacent vertices, then $A(K_{1})$ is inside $A(K)$. The co-contraction of a pair $(a, b)$ amounts to replacing $a, b$ by a new vertex $c$ connected to all the common neighbors of $a, b$ in $K$.

Hence it is enough to describe graphs $P$ such that no proper subgraph $P'$ of $P$ and no graph $P'$ obtained from $P$ by co-contraction is such that $A(P')$ contains a hyperbolic closed surface subgroup. Let us call such graphs *extremal*. There is one series of extremal graphs: the $n$-cycles for $n \geq 5$ [DSS]. There are also eight exceptional extremal graphs $P_{1}(6), P_{2}(6), P_{1}(7), P_{2}(7), P_{1}(8) - P_{1}(8)$ (the number in parentheses is the number of vertices in the graph) found in [CSS] (Kim also found $P_{1}(6)$, the triangular prism). Is it the full collection of extremal graphs? We know [CSS] that this collection contains all extremal graphs with up to 8 vertices.

There are also several reduction moves $P \rightarrow P'$ found in [CSS] such that if $A(P)$ contains a hyperbolic closed surface subgroup, then so does $P'$ (and $P'$ contains fewer vertices or edges than $P$). We do not know if this collection of reduction moves is complete, namely we do not know whether the following statement is true: If $A(P)$ does not contain a hyperbolic closed surface subgroup, then $A(P')$ does not as well, for one of our reduction moves $P \rightarrow P'$. This collection of moves is complete for 8-vertex graphs [CSS].
7.2 Dissection diagrams

Let $S$ be a surface (possibly with boundary). Let $G = \langle X \mid R \rangle$ be a finitely presented group. Let $\Psi$ be a van Kampen diagram over the presentation of $G$ drawn on $S$. Roughly speaking it is a graph drawn on $S$ with edges labeled by letters from $X$, such that each connected component of $S \setminus \Psi$ is a polygon with boundary path labeled by a word from $R^\pm$ (see more details in [Ol3]).

Given a van Kampen diagram $\Psi$ on $S$, one can define a homomorphism $\psi: \pi_1(S) \to G$ as follows. As a base-point, pick a vertex $v$ of $\Psi$. Let $\gamma$ be any loop at $v$. Since all cells in the tessellation $\Psi$ are polygons, $\gamma$ is homotopic to a curve that is a composition of edges of $\Psi$. Then $\psi(\gamma)$ is the word obtained by reading the labels of edges of $\Psi$ along $\gamma$. Since the label of the boundary of every cell in $\Psi$ is equal to 1 in $G$, the words corresponding to any two homotopic loops $\gamma, \gamma'$ represent the same element in $G$. Hence $\psi$ is indeed well-defined. The fact that $\psi$ is a homomorphism is obvious.

Conversely, the standard argument involving $K(\,.,1)$-complexes gives that every injective homomorphism $\psi: \pi_1(S) \to G$ corresponds in the above sense to a van Kampen diagram over $G$ on $S$.

If $G = A(K)$ is a right angled Artin group, then every cell in a van Kampen diagram is a square, and instead of a van Kampen diagram on $S$, it is convenient to consider its dual picture (pick a point inside every cell, connect the points in neighbor cells by an edge labeled by the label of the common edge of the cells). It is called the $K$-dissection diagram of the surface, and was introduced by Crisp and Wiest in [CW]. The edges having the same labels form collections of pairwise disjoint simple closed orientation preserving curves and arcs connecting points on the boundary of $S$. Each of these curves has a natural transversal direction. Each curve is labeled by a vertex of $K$, two curves intersect only if their labels are adjacent in $K$.

If $\Delta$ is the $K$-dissection diagram corresponding to a van Kampen diagram $\Psi$ on $S$, then the corresponding homomorphism $\psi: \pi_1(S) \to A(K)$ takes any loop $\gamma$ based at $v$ to the word of labels of the dissection curves and arcs of $\Delta$ crossed by $\gamma$ (a letter in the word can occur with exponent 1 or $-1$ according to the direction of the dissection curve crossed by $\gamma$). There are several partial algorithms allowing to check whether a homomorphism $\psi$ corresponding to the $K$-dissection diagram is injective [CSS]. But the answer to the next question is still unknown.

**Problem 7.2** (Crisp-Sageev-Sapir). Is there an algorithm which given a $K$-dissection diagram on a surface $S$, decides whether the corresponding homomorphism $\psi: \pi_1(S) \to A(K)$ is injective?

8 Surface subgroups of hyperbolic groups and ascending HNN extensions of free groups

A very interesting problem by Gromov (on the Bestvina problem list) asks whether every non-virtually free hyperbolic group contains a hyperbolic closed surface subgroups. Here is a concrete partial case of this problem.

**Problem 8.1.** Does the group $\langle x, y, t \mid txt^{-1} = xy, tyt^{-1} = yx \rangle$ contain a hyperbolic closed surface subgroup?

Of course a negative answer is more interesting than a positive one.

According to the previous section, to solve this problem, one would need to study van Kampen diagrams over this presentation on surfaces. Here is a more general algorithmic problem.
Problem 8.2. For which injective but not surjective endomorphisms $\phi: F_k \to F_k$, the HNN extension $\text{HNN}_\phi(F_k)$ contains a hyperbolic closed surface subgroup?

Note that subgroups of such HNN extensions have been described in details by Feighn and Handel in [FH].

9 Percolation in Cayley graphs of groups
by Iva Kozáková and Mark Sapir

9.1 Basic properties of percolation

Geometry of random structures associated with Cayley graphs of groups is very interesting in general. There is a lot of work related, say, to random walks on Cayley graphs (starting with the classical cases of Cayley graphs of $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ with the natural generators), see Woess, [Woess]. Here we shall formulate some problems related to random subgraphs of Cayley graphs and percolation. For more problems, motivation and discussion, see the survey of Benjamini and Schramm [BeSch].

Definition 9.1. A Bernoulli bond percolation on $G$ is a product probability measure $P_\rho$ on the space $\Omega = \{0, 1\}^E$, the subsets of the edge set $E$. For $0 \leq \rho \leq 1$ the product measure is defined via $P_\rho(\omega(e) = 1) = \rho$ for all $e \in E$.

An element $\omega$ of the probability space $\Omega$ is called configuration or realization of percolation. For any $\omega \in \Omega$, the bond $e \in E$ is called open if $\omega(e) = 1$ and closed otherwise. Thus each bond is open with probability $\rho$ independently of all other bonds.

We write $\mathbb{E}_\rho$ for the expected value with respect to $P_\rho$. For any configuration $\omega$, open edges form a random subgraph of $G$.

Definition 9.2. An (open) cluster is a connected component of such subgraph $\omega$. An open cluster containing the origin is denoted by $C$ and the number of vertices in $C$ by $|C|$. The percolation function is defined to be the probability that the origin is contained in an infinite cluster, i.e. $\theta(\rho) = P_\rho(|C| = \infty)$.

The connectivity function $\tau_\rho(o, x) = P_\rho(o \leftrightarrow x)$ is the probability that there is an open path connecting vertices $o$ and $x$. We use $\chi(\rho) = \mathbb{E}_\rho(|C|)$ for the mean size of the open cluster at the origin.

As $\rho$ grows from 0 to 1, the behavior of the percolation process changes. We can distinguish several phases of similar characteristic.

Definition 9.3. For any graph, we define three critical probabilities:

$$p_c = \sup\{p : \theta(p) = 0\},$$
$$p_u = \inf\{p : \text{There is a unique infinite cluster } P_\rho\text{-almost surely}\},$$
$$p_{\text{exp}} = \sup\{p : \exists_{C, \gamma > 0} \forall_{x, y \in V} \tau_\rho(x, y) \leq C e^{-\gamma \text{dist}(x, y)}\}.$$

Häggström, Peres and Schramm [HPS] showed that for any homogeneous graph, for every $p$ such that $0 \leq p < p_c$ all clusters are finite, there are infinitely many infinite clusters if $p_c < p < p_u$, and if $p_u < p \leq 1$, then there is unique infinite cluster $P_\rho$-almost surely.

It is known [Sch] that $p_c \leq p_{\text{exp}} \leq p_u$ for every Cayley graph. It is interesting to characterize all groups (resp. Cayley graphs) such that $p_c$ and $p_u$ are different. It is known that $p_c = p_u$ for all amenable groups. The converse has not been proved yet.
Problem 9.4 (Benjamini-Schramm [BeSch]). Is it true that $p_c \neq p_a$ for every Cayley graph of a non-amenable group?

The conjectural answer is “yes” [BeSch]. There exist partial results in this direction. Pak and Smirnova-Nagnibeda [PS-N] proved that for any nonamenable (finitely generated) group there exists a finite symmetric set of generators $S$ in $G$ such that for the Cayley graph of $G$ with respect to $S$, $p_c < p_a$. Schonnmann [Sch] showed that $p_c < p_a$ for highly nonamenable groups (i.e. such that the edge-isoperimetric (Cheeger) constant is bigger than $(\sqrt{2d^2 - 1} - 1)/2$, where $d$ is the vertex degree). Gaboriau [Gab] proved that the first $\ell^2$-Betti number of a group does not exceed $\frac{1}{2}(p_a - p_c)$ for every Cayley graph of $G$, therefore as soon as $\ell^2$-Betti number is nonzero, $p_a > p_c$.

The study of percolation started with the square lattice $\mathbb{Z}^2$. Using the self-duality of this lattice it can be shown that $p_c = \frac{1}{4}$ [Ke], although the proof is very non-trivial. The exact value of $p_c$ for $d$-dimensional cubic lattices $\mathbb{Z}^d$, $d \geq 3$, is unknown, only numerical approximations are available.

A well understood situation is the case of a regular tree, i.e. the Cayley graph of a free group with respect to the free generating set. For a regular tree, the critical probability $p_c = \frac{1}{d-1}$, where $d$ is the vertex degree.

Using the tree-graded structure of Cayley graphs of free products Kozáková [Ko] proved that for a free product of groups $G_1 * G_2$ with the natural generating set, the $p_c$ is a root of $(\chi_1(p) - 1)(\chi_2(p) - 1) = 1$, where $\chi_i(p)$ is the expected cluster size in the Cayley graph of $G_i$, [Ko]. For example for $\text{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, $p_c$ is 0.5199..., the unique root of the polynomial $2p^5 - 6p^4 + 2p^3 + 4p^2 - 1$ in the interval $(0, 1)$.

A degenerated case is the Cayley graph of the infinite cyclic group $\mathbb{Z}$, or any finite extension of it. For these graphs, $p_c = 1$. And again the converse is not known in general. The question can be formulated in the following way using the growth function.

Problem 9.5 (Benjamini, Schramm [BeSch]). Assume that $G$ has growth function faster than linear, is $p_c < 1$?

The conjectural answer is “yes” [BeSch]. Only amenable case is of interest, since for every nonamenable graph, the edge-isoperimetric (Cheeger) constant $i_E$ is positive and $p_c < \frac{1}{i_E + 1}$ [BeSch]. It has been shown by Lyons (see [BeSch2]) that for groups with polynomial or exponential growth $p_c < 1$. The same is true for all finitely presented groups [BI]. All Grigorchuk groups of intermediate growth have subgroups that are direct products of two proper infinite subgroups, therefore $p_c < 1$ (noticed by Muchnik and Pak [MP]).

It is also not known whether the properties $p_c = 1$ or $p_c = p_a$ are invariant under the change of generators of a group or even the same for any graph that is quasi-isometric to the Cayley graph of a group (physicists believe that it is the case at least for $\mathbb{Z}^d$).

Problem 9.6 (Smirnova-Nagnibeda). What are $p_c, p_a, p_{exp}$ for $\text{SL}_2(\mathbb{Z})$, the $R$. Thompson group $F$, Grigorchuk groups (and other automata groups)?

It may be difficult to find precise values, but even observations whether $p_c, p_a, p_{exp}$ are all different are interesting.

On planar graphs a very useful property is that any bi-infinite simple path splits the graph into two components (for examples of applications of this see [BeSch3] and [BB]). Can a similar property hold for other graphs?

Problem 9.7. Is it true that for a hyperbolic group an infinite cluster ($p > p_c$) separates the graph $P_p$-a.s.?
An open cluster as a subgraph of the Cayley graph carries some properties of the original graph, but its geometry can be also different. Gaboriau [Gab] related harmonic Dirichlet functions on a graph to those on the infinite clusters in the uniqueness phase. One way to describe the change in the geometry of a subgraph is the distortion (compare with Definition 6.9).

**Definition 9.8.** The distortion function of a connected subgraph \( C \) (containing origin \( O \)) of a graph \( G \) is given by

\[
D(n) = \max_{y \in C \text{ dist}_G(O,y) \leq n} \text{dist}_C(O,y)
\]

where \( \text{dist}_C \) (resp. \( \text{dist}_G \)) is the graph metric in \( C \) (resp. in \( G \)). In the context of percolation, \( D(n) \) gives the distortion of an open cluster \( C \) (at the origin) for any given realization \( \omega \in \Omega \), and we define the expected distortion as \( E_\rho(D(n)) \).

Note that different definitions of distortion of clusters related to random walks in random environments are also useful (see [BLS], for example).

For \( p < p_c \) the distortion is bounded by the cluster size, therefore the expected distortion is a constant (asymptotically). On trees the distance in a subgraph does not differ from the original one, therefore if \( p > p_c \) the distortion of an infinite open cluster on a tree is linear. The result by Antal and Pisztora [AP] suggests that the expected distortion for the \( d \)-dimensional cubic lattice \( \mathbb{Z}^d \) should be at most linear as well, they proved that there exists \( \rho = \rho(d,p) \) such that \( P_{p_c}(|C| = n) \approx n^{-1 - \frac{1}{d}} \).

**Problem 9.9.** What is the possible expected distortion of an open cluster in a Cayley graph of a group? Same question for hyperbolic groups is also open.

The most interesting percolation phenomena occur, when the parameter \( p \) is near its critical value \( p_c \).

The principal hypothesis of the percolation theory is that various quantities (like the probability of the cluster at the origin being infinite, or the mean size of this cluster) have certain specific asymptotic behavior near the critical point \( p_c \).

**Definition 9.10.** Assume \( \theta(p) \) is continuous at \( p_c \) and that

\[
\theta(p) \approx (p - p_c)^\beta \quad \text{as } p \searrow p_c,
\]

\[
\chi(p) \approx (p_c - p)^-\gamma \quad \text{as } p \nearrow p_c,
\]

\[
P_{p_c}(|C| = n) \approx n^{-1 - \frac{1}{\delta}}.
\]

Then we say that \( \beta, \gamma \) and \( \delta \) are critical exponents of the graph.

For a regular tree it is possible to compute these exponents directly. The critical exponents have the following so called mean-field values:

\[
\beta = 1, \gamma = 1, \delta = 2.
\]

Physicists believe that the numerical values of critical exponents depend only on the underlying space and not on the structure of the particular lattice.
It was proved by Hara and Slade [HaSl] that the critical exponents of a $d$-dimensional cubic lattice take their mean-field values for $d \geq 19$. It is believed to hold even for $d > 6$. In fact it might hold for all Cayley graphs of groups with growth function $\geq n^6$ (compare with Problem 9.11 below).

A well known conjecture in percolation theory claims that the critical exponents of all lattices in $\mathbb{R}^2$ are the same ($\beta = 5/36$, $\gamma = -43/18$) as proved for triangular lattices by Smirnov and Werner [SmWe]). The motivation for this conjecture is that for every lattice $L$ in $\mathbb{R}^2$ the Gromov-Hausdorff limit of rescaled copies of $L$, $L/2$, $L/3$, ... is isometric to $\mathbb{R}^2$ with the $L_1$-metric, i.e. any two lattices in $\mathbb{R}^2$ have the same asymptotic cones ([Gr3]). Sapir conjectured that this is true in general: if two groups have isometric asymptotic cones corresponding to the same ultrafilters and the same scaling constants, then their critical exponents should coincide. In particular, since all asymptotic cones of all non-elementary Gromov-hyperbolic groups are isometric (they are isometric to the universal $\mathbb{R}$-tree of degree continuum by a result of [DP]), every non-elementary hyperbolic group should have mean-field valued critical exponents. Thus we formulate

**Problem 9.11.** Is it true that the critical exponents of all Cayley graphs of groups with isometric asymptotic cones are equal? In particular, is it true that every Cayley graph of a non-elementary hyperbolic group has mean-field valued critical exponents?

By [Sch], the critical exponents take their mean-field values for all non-amenable planar graphs with one end, and for unimodular graphs with infinitely many ends (in particular, for all Cayley graphs of groups with infinitely many ends).

### 9.2 Scale invariant groups

One of the methods used by physicists in studying percolation in $\mathbb{Z}^2$ (and $\mathbb{Z}^d$ in general) is rescaling or renormalization. For some $n > 1$, we consider a tessellation of $\mathbb{Z}^d$ by cubes of size $n$. That tessellation can be viewed as a new (rescaled) copy $\Gamma = n\mathbb{Z}$ of $\mathbb{Z}^d$. Given a percolation realization on $\mathbb{Z}^d$, we say that an edge $e$ of the $n \times n$-cube in $\Gamma$ is open if one can cross the cube in the direction of the edge using open edges of $\mathbb{Z}^d$ (i.e. there is an open path connecting the $(d-1)$-faces of the cube that are orthogonal to $e$).

This collection of open and closed edges in $\Gamma$ can be viewed (under certain assumptions that are obvious to physicists) as a realization of percolation on $\Gamma$ (see Grimmett [Grim]). If the original realization had an infinite cluster, the new one would have infinite clusters too. This way, starting with a super-critical percolation ($p > p_c$) we get a new supercritical percolation with a bigger $p$. Hence one is able to deduce information about $p_c$ and critical exponents [Grim].

The rescaling method is based on the fact that $\mathbb{Z}^d$ contains lots of copies of itself of finite index. In general, following Benjamini, we call a finitely generated group $G$ scale invariant if it has subgroups of finite index that are isomorphic to $G$ and the intersection of all such subgroups of finite index is finite.

**Problem 9.12** (Benjamini, [http://www.math.weizmann.ac.il/~itai/]). Is it true that every scale invariant finitely generated group is virtually nilpotent?

One way to deal with this problem would be to consider the actions of the group on itself induced by the finite index embeddings of the group into itself. That gives an action of the group on its asymptotic cone (as in [DS3] and Section 3). This should imply, in particular, that a relatively hyperbolic group is scale invariant only if it is elementary.

Note that R. Thompson group $F$ has many copies of itself as subgroups of finite index. But $F$ is not residually finite, hence not scale invariant. Grigorchuk’s groups have many subgroups
of finite index which are isomorphic to finite direct powers of the ambient group. But these subgroups are not isomorphic to the ambient group, so Grigorchuk groups are not scale invariant too.

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