On the Wess–Zumino term in high energy QCD

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Abstract

Recently, Kovner and Lublinsky proposed new small–$x$ QCD evolution equations valid when the gluon density inside the target is low. The key element of their construction is the Wess–Zumino term which ensures the non-commutativity of valence charges. In this paper we clarify the origin and significance of this term by showing that it can be naturally incorporated in the effective theory of Color Glass Condensate. We also reexamine the renormalization group description in the high density (JIMWLK) regime.

1 Introduction

Over the past few years, considerable effort has been made [1–13] towards understanding the high energy QCD evolution equation beyond the Balitsky–Kovchegov (BK) equation [14, 15], or the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) equation [16, 17]. Interest in the problem was sparked by several seemingly unrelated observations [1–4, 18] regarding the insufficiency of the BK–JIMWLK equation, but they all point to a necessity to include new types of Feynman diagrams called Pomeron loop diagrams.

Fig. 1(a) is a typical diagram summed by the JIMWLK equation. The upper blob represents an energetic hadron (the “target”) moving in the positive $z$ direction. At very high energy, the target behaves like a weakly coupled many body system of small–$x$ gluons commonly dubbed the Color Glass Condensate (CGC). In one step of quantum evolution towards smaller $x$, arbitrarily many $t$–channel gluons inside the target recombine into two gluons. This is the gluon saturation phenomenon [19–22] which plays an important role for the unitarization of the BFKL Pomeron [23].

Attention is currently focused on diagrams with arbitrarily many gluon legs as represented by Fig 1(b). Viewed from the target side, these diagrams describe the gluon splitting or the gluon number fluctuation inside the target [4]. They are particularly important when the target is dilute, and are responsible for the eventual formation of a dense system. On
the other hand, the same diagrams, when viewed from the projectile side, clearly describe the gluon recombination process in the projectile. In either interpretation, Fig. 1(b) is the leading diagram in the kinematic regime which is complementary to that considered in the JIMWLK equation. A set of evolution equations including the vertex of Fig. 1(b) with four gluon legs below the rung has been proposed as the first step beyond the JIMWLK equation [4, 5, 12]. [See also earlier works [24].] These results were obtained in the frameworks of Mueller’s dipole model [25] and its alternative formulation as a color glass [26]. In this model, the relevant gluon splitting process (or better be called the Pomeron splitting process in the large–$N_c$ approximation) is naturally described by the dipole splitting and subsequent gluon emissions. Combining the dipole model with the JIWMMLK formalism in the dipole sector [27], one can investigate more general problems which involve both the Pomeron splitting and the Pomeron recombination (Fig. 1(c)), i.e., the Pomeron loops [9, 24].

Meanwhile, in a very interesting paper [6] Kovner and Lublinsky have derived evolution equations in the dilute regime to all orders in the gluon legs without using the large–$N_c$ approximation. The kernel of the evolution equation was shown to be dual to the JIMWLK kernel [7, 10, 11] essentially reflecting the symmetry between Figs. 1(a) and 1(b). However, identification of states (“observables”) on which this kernel acts does not follow from simple duality considerations. The nature of the evolution equation in the dilute regime is peculiar; one has to treat color charges as non-commutative operators. Kovner and Lublinsky overcame this problem by introducing a Wess–Zumino term and an extra coordinate (“ordering variable”), after which the charges can be treated as commutative. The emergence of this term at first sight is somewhat mysterious, as it is rarely discussed in the context of high energy QCD. Still, the fact that the duality and the relationship between the wave function approach [6] and the effective action/Hamiltonian approaches [10, 11, 21] become manifest only in this “commutative world” calls for a formulation which explicitly includes the Wess–Zumino term. [See, however, [8].]

In this paper we fully clarify the origin and significance of the Wess–Zumino term in the high density (JIMWLK) regime. This may be unexpected as the issue of non-commutativity

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1 As shown in [12], this problem disappears in the dipole model in the large–$N_c$ approximation, and one can check the consistency with the previous results in [4].
seems least important in this regime. On the contrary, we shall show that the Wess–Zumino term is quite useful and can be naturally incorporated in the JIMWLK formalism as a part of the source term. This observation has led us to more detailed treatment of the renormalization group description than in the literature [16, 17] in the sense that we discuss renormalization of the source term. It should be emphasized that, although in this paper we focus our attention to the JIMWLK regime, our approach is general and therefore applicable to both the dilute regime (Fig. 1(b)) and the Pomeron loop regime (Fig. 1(c)) where the importance of the Wess–Zumino term has been first recognized. Indeed, an essential difference between effective theories in these regimes and the current JIMWLK formalism [16, 17] is that one has to introduce the light–cone time coordinate $x^+ (=\text{ordering variable})$ in the former [11]. By reformulating the latter with $x^+$–dependent classical charges, we can address the high energy evolution in different regimes in a single framework. In this context the Wess–Zumino term necessarily arises as a consequence of gauge invariance.

In Section 2, after a brief review of the JIMWLK formalism we propose a new source term which explicitly includes a Wess–Zumino term. We show that our source term correctly reproduces the induced charge upon quantum evolution. In Section 3, we perform the renormalization group evolution directly at the level of the JIMWLK functional integral. We conclude with Section 4.

2 JIMWLK formalism with the Wess–Zumino term

2.1 The Color Glass Condensate

The color glass condensate (CGC) formalism [16, 17, 20, 22] is an effective theory of gluon saturation at a small Bjorken parameter $x = \Lambda^+/P^+$, where $P^+$ and $\Lambda^+$ are the light–cone momenta of the right–moving parent hadron (the target) and the small–$x$ gluon of interest, respectively. Partons with momentum fraction larger than $\Lambda^+$ are described as the external source $\rho$, while the gluons with momenta $\Lambda^+$ are described as the classical field $A^\mu$ created by the source according to the Yang–Mills equation

$$D_\nu F^{\nu\mu} = \delta^{+\mu} \rho(\vec{x}).$$ (2.1)

We use the notation $\vec{x} \equiv (x^-, x^i)$ where $x^i (i = 1, 2)$ is the two–dimensional transverse coordinate. In the saturated regime (JIMWLK regime), $\rho$ is effectively static (i.e., $x^+$–independent) and parametrically of order $\sim \mathcal{O}(1/g)$. In the light cone gauge $A^+ = 0$, the solution to Eq. (2.1) is given by

$$A^\mu = \delta^{\mu i} B^i, \quad B^i = \frac{i}{g} U \partial^i U^\dagger,$$ (2.2)
where $U$ is a Wilson line in the $x^-$ direction

$$U^\dagger(\vec{x}) = P \exp \left( ig \int_{-\infty}^{x^-} dz^- \alpha(z^-, x^i) \right). \quad (2.3)$$

$\alpha$ is the classical solution in the Coulomb gauge

$$\tilde{A}^\mu \equiv \delta^{\mu+} \alpha, \quad (2.4)$$

and is related to the Coulomb gauge charge $\tilde{\rho} \equiv U^\dagger \rho U$ via the plus component of the Yang–Mills equation

$$-\nabla_\perp^2 \alpha = \tilde{\rho}. \quad (2.5)$$

Observables $\mathcal{O}$ (e.g., scattering amplitudes) are first calculated for a given background field Eq. (2.2), $\mathcal{O}[\rho]$, and then averaged over $\rho$ with the weight function $W_\tau[\rho]$

$$\langle \mathcal{O} \rangle_\tau = \int D\rho W_\tau[\rho] \mathcal{O}[\rho], \quad (2.6)$$

where $\tau$ is the rapidity $\tau = \ln 1/x = \ln P^+/\Lambda^+$. $W_\tau$ satisfies the JIMWLK equation [16, 17] which is a renormalization group equation in rapidity

$$\frac{\partial}{\partial \tau} W_\tau[\rho] = -H_{\text{JIMWLK}} \left[ \alpha, \frac{\delta}{\delta \rho} \right] W_\tau[\rho]. \quad (2.7)$$

The precise form of the JIMWLK Hamiltonian $H_{\text{JIMWLK}}$ is irrelevant here. [See, however, Eq. (3.34).] Suffice it to say that it is quadratic in the functional derivative $\delta/\delta \rho$ (corresponding to the two gluon legs in Fig. 1(a)) and all orders in $\alpha$ (corresponding to the merging gluons above the rung in Fig. 1(a)) in the form of the Wilson line Eq. (2.3).

The derivation of Eq. (2.7) proceeds with the following steps [16, 17]: (i) Start with the QCD functional integral in the light–cone gauge

$$\int D\rho W_\tau[\rho] \int_\tau DA^\mu \delta(A^+) e^{iS_{\text{YM}}[A^\mu]} + iS_{\text{SW}}[A^\mu, \rho], \quad (2.8)$$

where the subscript $\tau$ in the $A^\mu$ integral means that the gauge fields contain only modes with $p^+ < \Lambda^+ = e^{-\tau} P^+$. Modes with $p^+ > \Lambda^+$ have already been integrated out. $S_{\text{SW}}$ is the source term which gives the right hand side of the Yang–Mills equation Eq. (2.1). (ii) Expand the gauge field around the solution to the Yang–Mills equation

$$A^\mu = \delta a^\mu B^i + a^\mu + \delta A^\mu, \quad (2.9)$$

where $a^\mu$ is the semihard field with the momentum fraction $\Lambda^+ > p^+ > b\Lambda^+ (b \ll 1)$ and $\delta A^\mu$ is the soft field with $p^+ < b\Lambda^+$. (iii) Functionally integrate out the semihard field $a^\mu$. (iv) Show that, in the leading logarithmic approximation (LLA), i.e., keeping terms proportional to $\alpha_s \ln 1/b \equiv \alpha_s \delta \tau$ ($\alpha_s = g^2/4\pi$), the effect of the integration is absorbed by the renormalization of the weight function $W_\tau[\rho] \rightarrow W_{\tau+\delta \tau}[\rho + \delta \rho]$, where $\delta \rho$ is the
additional charge induced by the semihard field. In this paper we perform this program in the most direct way, including the renormalization of the source term $S_W$.

For this purpose, first we must know $S_W$. The original definition in [16] is

$$S_W = \frac{i}{N_c} \int d\vec{x} \text{tr}[\rho(\vec{x})\bar{W}(\vec{x})], \quad (2.10)$$

where

$$\bar{W}(\vec{x}) \equiv \text{P} \exp \left( ig \int_{-\infty}^{\infty} dx^+ A^+_a(x^+, \vec{x}) T^a \right), \quad (2.11)$$

with $T^a$ the color matrices in the adjoint representation. [Hereafter $W$ and $\bar{W}$ denote Wilson lines in the fundamental and adjoint representation, respectively.] Eq. (2.10) is invariant under $x^+$-independent gauge transformations. Upon quantum evolution, one shifts $A^- \to a^- + \delta A^-$ [Note that $A^- = 0$ for the classical solution.] and expands the exponential. The coefficient of $-\delta A^-$ defines the induced charge $\delta \rho[a^-]$ to be included in the classical theory at rapidity $\tau + \delta \tau = \ln P^+/b\Lambda^+$. Several different choices of $S_W$ can be found in the literature [17, 32]. The non-uniqueness of $S_W$ simply means that they are effective actions. Although they all lead to the renormalization group equation Eq. (2.7), it is not clear whether intermediate calculations go hand-in-hand with the corresponding perturbative QCD calculation. Moreover, when we consider the all order effect of $A^-$ in the dilute regime, the choice of $S_W$ is crucial to obtain correct evolution equations.

Below we construct another $S_W$ that is most suited for our purpose. This is not in the form of an effective action, but contains its own dynamics due to the presence of the Wess–Zumino term. For the sake of simplicity and clarity, throughout this paper we consider the color SU(2) gauge group.\(^2\) We describe the target hadron as a bunch of energetic “valence partons” having longitudinal momenta $p^+ > e^{-\tau}\Lambda^+$. They carry various representations of SU(2) ($J = \frac{1}{2}, 1, \frac{3}{2}, \cdots$) at various transverse coordinates $x_i$, and are also distributed in a narrow strip in the $x^-$ direction with the width $|\Delta x^-| \sim 1/\Lambda^+$ (Fig. 2). By a deliberate choice of the light–cone prescription, one can restrict the support of the charges to the region $x^- > 0$ [17].

Let us focus on a single quark at the coordinate $\vec{x}$ inside the target. In the eikonal approximation, the propagation of this quark is described by the amplitude

$$Z = \langle \bar{\psi} a W^{ab}(\vec{x}) \psi b \rangle, \quad (2.12)$$

where $W$ is the Wilson line Eq. (2.11) in the fundamental representation $T^a \to \tau^a/2$ ($\tau^a a = 1, 2, 3$ are the Pauli matrices) and $\psi$ is the Dirac spinor of the quark. With the use of a formula for an open Wilson line $W^{ab}$ derived by Diakonov and Petrov [29], Eq. (2.12) takes the form

\(^2\) The form of the Wess–Zumino term is a bit simpler for SU(2) than for SU(3). The latter can be found in the literature. See, for example, Ref. [29].
Fig. 2. Eikonal propagation of the valence charges. The charges have support only at positive $x^-$.

\[
Z = \int DS_{\pm \infty} W[S_{\infty}, S_{-\infty}] \int_{S_{-\infty}}^{S_{\infty}} DS(x^+) \int DA^u \\
\times \exp \left( iS_{YM} + igJ \int dx^+ \text{tr} [\tau_3 S A^- S^+] + iS_{WZ}[S(x^+)] \right) \\
= \int D\rho_{\pm \infty} W[\rho_{\infty}, \rho_{-\infty}] \int_{\rho_{-\infty}}^{\rho_{\infty}} D\rho(x^+) \int DA^u \\
\times \exp \left( iS_{YM} - i \int dx^+ \rho(x^+) A^- (x^+) + iS_{WZ}[\rho(x^+)] \right), \tag{2.13}
\]

where $S(x^+)$ is a SU(2) matrix ($S_{\pm \infty} \equiv S(x^\mp \infty)$), $J = 1/2$ for the quark fundamental representation. $S_{WZ}$ is the Wess–Zumino term (also called a geometric phase, Berry’s phase, Polyakov’s spin factor) \cite{28,29}

\[
S_{WZ} = iJ \int_{-\infty}^{\infty} dx^+ \text{tr} [\tau^3 S \partial^- S^+], \tag{2.14}
\]

where $\partial^- = \partial_+ = \partial/\partial x^+$. In the second equality in Eq. (2.13), we switched to the $\rho$-representation defined by ($A^- = A^-_a \tau_a^z$)

\[
\rho^a(x^+) = -\frac{gJ}{2} \text{tr} [\tau^3 S \tau^a S^+]. \tag{2.15}
\]

The constraint $\rho^a \rho^a = g^2 J^2$ is implicit in the measure $D\rho$. For a Wilson loop, $S_{WZ}$ can be written as a functional of $\rho$ by introducing yet another coordinate $u$ ($1 \geq u \geq 0$) (which is the radial coordinate of the disc spanned by the loop) and extrapolating $\rho(x^+) \rightarrow \rho(x^+, u)$ such that $\rho(x^+, u = 1) = \rho(x^+)$. But for an open Wilson line, this is not possible in
general.\textsuperscript{3} Nevertheless, we use the notation \(S_{\text{WZ}}[\rho]\) with \(\rho\) and \(S\) related by Eq. (2.15). The “weight function” \(W[\rho_\infty, \rho_{-\infty}]\) (not to be confused with the Wilson line \(W\)) contains Wigner rotation matrices \([29]\) and quark spinors. It is in general complex and depends on \(S\) (or \(\rho\)) only at \(x^+ = \pm \infty\). Note that the sum of the last two terms in the exponential of Eq. (2.13)

\[
g J \text{tr}[\tau_3 S \partial^- S^\dagger] + i J \text{tr} [\tau^3 S \partial^- S^\dagger]
\]

is invariant under the following gauge transformation

\[
\tilde{A}^- = U^\dagger A^- U + \frac{i}{g} U^\dagger \partial^- U, \quad \tilde{S} = SU
\]

Under this transformation, \(\rho\) transforms as

\[
\tilde{\rho} = U^\dagger \rho U.
\]

In fact, Eq. (2.13) is a path integral formula for a spin \([28]\). The exponential factor

\[
H(x^+) = -g A_a^- (x^+) \frac{\tau^a}{2},
\]

of the fundamental Wilson line in Eq. (2.12) may be regarded as a Hamiltonian for a non-relativistic “spin” \(\tau^a\) immersed in a time-dependent “magnetic field” \(A_a^- (x^+)\). The Wess–Zumino term is nothing but the kinetic term (the \(p \dot{q}\) term in Eq. (2.20)) which arises when one goes from the Hamiltonian to the Lagrangian\textsuperscript{4}

\[
P \exp \left( -i \int dx^+ H \right) \sim \int Dp Dq \exp \left( i \int dx^+ (p \dot{q} - H) \right).
\]

Generalization to an arbitrary set of valence partons (with longitudinal momenta \(p^+ > e^{-\tau} A^+\)) can be done simply by replacing

\[
J \rightarrow \int d\bar{x} J(\bar{x}),
\]

and endowing \(W_{\tau}[\rho_{\pm \infty}(\bar{x})]\) with the information about the distribution of partons and their group representations. This can be done along the lines suggested in [33]. In this way we are led to the following source term \((x = (x^+, \bar{x}))\)

\[
S_W[A^-, \rho] = - \int dx \rho(x) A^-(x) + S_{\text{WZ}}[\rho(x)],
\]

\textsuperscript{3} We could consider an overall color–singlet target and make closed loops by connecting Wilson lines at different transverse coordinates at \(x^+ = \pm \infty\). However, this leads to unnecessary complications because in the light–cone gauge in which we are working, there is nonzero transverse field \(B^i\) at infinity.

\textsuperscript{4} One can make the correspondence explicit by introducing angular coordinates \(\rho \propto (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\). Under this parametrization, \(p \sim \cos \theta\), \(q \sim \phi\) and \(D \rho \sim Dp Dq\).
with $\rho(x)$ defined by Eq. (2.15) with $J \to J(\vec{x})$. Correspondingly, Eq. (2.13) is generalized to

$$Z = \int D\rho_{\pm\infty}(\vec{x}) W_{\tau}[\rho_{\infty}, \rho_{-\infty}] \int_{\rho_{-\infty}}^{\rho_{\infty}} D\rho(x^+, \vec{x}) \int_{\tau} D\tilde{A}^\mu \exp \left(iS_{YM}[A^\mu] + iS_W[A^-\rho]\right).$$  

(2.23)

Eq. (2.23) is the starting point of quantum evolution. By extending the definition of gauge transformation to the charge sector, Eq. (2.18), one can explicitly maintain the gauge invariance of the theory. Namely, Eq. (2.23) can also be written as

$$Z = \int D\tilde{\rho}_{\pm\infty} W_{\tau}[\tilde{\rho}_{\infty}, \tilde{\rho}_{-\infty}] \int_{\tilde{\rho}_{-\infty}}^{\tilde{\rho}_{\infty}} D\tilde{\rho}(x^+) \int_{\tau} D\tilde{A}^\mu \exp \left(iS_{YM}[\tilde{A}^\mu] + iS_W[\tilde{A}^-\tilde{\rho}]\right).$$  

(2.24)

### 2.2 Induced charge from the source term

In order for Eq. (2.22) to be an acceptable source term, first we have to show that it gives the correct induced charge $\delta \rho[a^-]$ under one step of quantum evolution $A^- = a^- + \delta A^-$. Namely, $S_W[a^- + \delta A^-, \rho] \sim - (\rho + \delta \rho[a^-]) \delta A^-$. At first sight this looks impossible because $S_W$ is linear in $A^-$ (so $a^-$ and $\delta A^-$ do not couple). However, we observe that the corresponding coupling $\bar{\psi} \gamma^+ t^a \psi A^-_a \sim \rho^a A^-_a$ in the QCD Lagrangian is also linear in $A^-$. As with the perturbative QCD calculation, we expand the exponential $\exp[i\rho A^-]$ in powers of $A^-$. To quadratic order, we get

$$-\frac{1}{2} \int D\rho(x^+) \int dx^+ dy^+ \rho_a(x^+) \rho_b(y^+) e^{iS_{WZ}}(\delta A^-_a(x^+) + a^-_a(x^+))(\delta A^-_b(y^+) + a^-_b(y^+)), $$

(2.25)

Hereafter we suppress the spacial coordinate $\vec{x} = (x^-, x^+).$ The crucial step of our approach is that we rewrite the $a^-\delta A^-$ coupling in Eq. (2.25) as

$$-\int D\rho(x^+) \int dx^+ dy^+ \rho_a(x^+) \rho_b(y^+) \delta A^-_a(x^+) a^-_b(y^+) e^{iS_{WZ}}$$

$$= -\int dx^+ dy^+ \left(\theta(x^+ - y^+) \langle \hat{\rho}^a \hat{\rho}^b \rangle + \theta(y^+ - x^+) \langle \hat{\rho}^b \hat{\rho}^a \rangle \right) \delta A^-_a(x^+) a^-_b(y^+),$$

(2.26)

where $\hat{\rho}^a (a = 1, 2, 3)$ are non-commutative charge operators. In the zero dimensional problem, $\hat{\rho}^a = -g t^a/2.$ In the three-dimensional case $\hat{\rho}^a$ are functions of $\vec{x}$ and satisfy local commutation relations

$$[\hat{\rho}^a(\vec{x}), \hat{\rho}^b(\vec{y})] = -ig\varepsilon^{abc} \rho^c(\vec{x} - \vec{y}).$$

(2.27)

Eq. (2.26) is the ‘magic’ of the Wess–Zumino term which can be mathematically justified [28]. A path integral of commutative $\rho$’s endowed with a Wess–Zumino term is equal to a matrix element of non-commutative charges $\hat{\rho}.$ The ordering of $\hat{\rho}$’s follows from the ordering of the $x^+$ coordinate of $\rho(x^+)$ under the path integral.
By decomposing the product of two $\hat{\rho}$’s into the symmetric and anti-symmetric parts
\[
\hat{\rho}^a \hat{\rho}^b = \frac{1}{2} [\hat{\rho}^a, \hat{\rho}^b] + \frac{1}{2} \{\hat{\rho}^a, \hat{\rho}^b\},
\]
we write the second line of Eq. (2.26) as
\[
-\frac{1}{2} \int dx^+ dy^+ \left(-igf^{abc}\theta(x^+ - y^+) - \theta(y^+ - x^+)\right)\langle \hat{\rho}^c \rangle + \langle \{\hat{\rho}^a, \hat{\rho}^b\} \rangle \delta A_a^-(x^+)a_b^-(y^+).
\] (2.29)

The symmetric term $\propto \langle \{\hat{\rho}^a, \hat{\rho}^b\} \rangle$ in Eq. (2.29) vanishes because it is proportional to
\[
\int dy^+ a_b^-(y^+) = a_b^-(p^- = 0) = 0.
\] (2.30)

Eq. (2.30) is valid since the semihard field $a^a$ is nearly an on–shell excitation [17] having $\Lambda^+ > p^+ > b\Lambda^+$ and $p_\perp^2/2b\Lambda^+ > p^- > p_\perp^2/2\Lambda^+$, where $p_\perp$ is a typical transverse momentum. [In the LLA, the precise value of $p_\perp$ does not matter.] Returning to the path integral representation $\langle \hat{\rho}^c \rangle \to \int D\rho \rho^c e^{iS_{WZ}}$, we re-exponentiate Eq. (2.29) and read off (a part of) the induced charge from the coefficient of $-i\delta A^-$
\[
\delta \rho^{(1)}_a(x^+) \equiv -g^{abc}f_{\rho^c} \int dy^+ \left(\theta(x^+ - y^+) - \theta(y^+ - x^+)\right) a_b^-(y^+).
\] (2.31)

The $x^+$ coordinate of $\rho^c$ can be chosen freely, since there is only one factor of $\hat{\rho}$ left in Eq (2.29). This really does not matter; as we shall argue at the beginning of Section 3, $\rho$ is almost static in the JIMWLK regime. In the above, we have set $x^+ = -\infty$. Eq. (2.31) agrees with the literature [16, 17], although the derivation here is very different. As remarked above, our derivation is closely tied to the corresponding pQCD calculation. Indeed, pQCD, the $a^-\delta A^-$ coupling arises due to second order perturbation
\[
-\frac{g^2}{2} (\bar{\psi} \gamma^+ t^a \psi \delta A_a^-)(\bar{\psi} \gamma^+ t^b \psi a^c_b) \sim -\frac{1}{2} \rho^a \delta A^- \rho^b a^c_b,
\] (2.32)

We see that the Wess–Zumino term is a convenient trick to replace the dynamical fermionic field with classical charges while correctly retaining the color commutator which in pQCD was implied by the quantum commutator.

Next we consider the virtual contribution to the induced charge coming from the $\sim \rho a^- a^- \delta A^-$ coupling. This time we expand up to the cubic order
\[
\frac{(-i)^3}{3!} \int D\rho \left(\rho(a^- + \delta A^-)\right)^3 e^{iS_{WZ}}
\]
\[
\sim \frac{i}{2} \int D\rho \int dx^+ dy^+ dz^+ \rho^a(x^+)\rho^b(y^+)\rho^c(z^+)\delta A_a^-(x^+)a_b^-(y^+)a_c^-(z^+) e^{iS_{WZ}}
\]
\[
= \frac{i}{2} \int dx^+ dy^+ dz^+ \delta A_a^-(x^+)a_b^-(y^+)a_c^-(z^+) \left(\theta_{xyz}\langle \hat{\rho}_a \hat{\rho}_b \hat{\rho}_c \rangle + \theta_{zyx}\langle \hat{\rho}_a \hat{\rho}_b \hat{\rho}_c \rangle + \theta_{xyz}\langle \hat{\rho}_a \hat{\rho}_b \hat{\rho}_c \rangleight),
\] (2.33)
\[ \theta_{xyz} \equiv \theta(x^+ - y^+)\theta(y^+ - z^+). \]  

(2.34)

The coefficient of \(-i\delta A\) is another contribution to the induced charge \(\delta \rho^{(2)}\). For this part of the induced charge, all we need in the later developments is the expectation value \(\langle \delta \rho^{(2)} \rangle \) where the averaging \(\langle \ldots \rangle\) is computed in the Gaussian approximation using the background field propagator Eq. (3.23). Anticipating this, we perform the replacement

\[ a^-_b (y^+) a^-_c (z^+) \rightarrow \delta_{bc} iG^-(y^+ - z^+) \]  

(2.35)

already at this point. Note that this component of the propagator is diagonal in color indices. This is certainly correct for the free propagator (which is the case for the BFKL regime). In the JIMWLK regime, the background field modifies the propagator. Nevertheless, Eq. (2.35) is valid in practice. See, Appendix C of the third paper in Ref. [17]. Since \(\hat{\rho}_b \hat{\rho}_b\) is a group Casimir, it commutes with \(\hat{\rho}_a\). Using this fact and Eq. (2.30), we obtain after some algebra

\[ \frac{-i g^2 N_c}{4} \int dx^+ dy^+ dz^+ \delta A^- (x^+) iG^- (y^+ - z^+) \rho_a (\theta(y^+ - x^+)\theta(x^+ - z^+) + \theta(z^+ - x^+)\theta(x^+ - y^+)). \]  

(2.36)

From this we can read off the (expectation value of the) induced charge

\[ \langle \delta \rho^{(2)}_a \rangle \equiv \frac{g^2 N_c}{4} \rho^a_{-\infty} \int dy^+ dz^+ iG^- (y^+ - z^+) (\theta(y^+ - x^+)\theta(x^+ - z^+) + \theta(z^+ - x^+)\theta(x^+ - y^+)), \]  

(2.37)

in agreement with the literature.

Summarizing, we expanded \(e^{-i\rho (a^- + \delta A^-)}\) and re-exponentiated the relevant (in the sense of the LLA) couplings between \(a^-\) and \(\delta A^-\). Effectively, we have replaced

\[ \exp \left( -i \int dx^+ \rho (x^+) (a^- (x^+) + \delta A^- (x^+)) \right) \rightarrow \exp \left( -i \int dx^+ \rho_{-\infty} a^- - i \int dx^+ (\rho(x^+) + \delta \rho^{(1)} + \langle \delta \rho^{(2)} \rangle) \delta A^- + \frac{i}{2} \int a^- \Pi a^- \right). \]  

(2.38)

where we again chose \(\rho(x^+ = -\infty)\) in the linear term in \(a^-\) (see the remark below Eq. (2.31)). The last term in Eq. (2.38) is a correction to the background field propagator (c.f., Eq. (3.23)) coming from the \(a^- a^-\) term in Eq. (2.25)

\[ \frac{i}{2} \int a^- \Pi a^- \equiv \frac{igf^{abc}}{4} \int dx^+ dy^+ (\theta(x^+ - y^+) - \theta(y^+ - x^+))\rho^c_{-\infty} a^- (x^+) a^- (y^+). \]  

(2.39)
3 Quantum evolution and renormalization group at high density

3.1 Recovering the CGC picture

In this section, we revisit the renormalization group evolution in the high density regime [16, 17] with the aim of showing the renormalization of the source term. We start with Eq. (2.23) in the light–cone gauge $A^+ = 0$ and expand around the solution to the classical equations of motion which are obtained by varying the field $A^{i-}$

$$D_\mu F^{\mu+} = \rho(x^+), \quad D_\mu F^{\mu i} = 0. \quad (3.1)$$

Eq. (3.1) was solved in [11], with the solution again in the same form as Eq. (2.2)\(^5\)

$$A^\mu(x^+) = \delta^{\mu i} B^i(x^+), \quad B^i = \frac{i}{g} U \partial^i U^\dagger(x^+). \quad (3.2)$$

Crucial simplification in the JIMWLK regime is that we can neglect the $x^+$–dependence in Eqs. (3.1) and (3.2). This is because the important paths in the path integral $\int D\rho(x^+)$ are those with weak $x^+$ dependence. To illustrate this, consider the solution to the saddle point equation

$$D^- \rho(x^+) = 0. \quad (3.3)$$

This can be obtained by performing infinitesimal rotation

$$S \rightarrow Se^{i\omega}, \quad \omega = \omega^a \frac{\tau^a}{2}, \quad (3.4)$$

in Eq. (2.14) and Eq. (2.15). Under this rotation, $\rho$ and $S_{WZ}$ change by

$$\delta \rho^a = \epsilon^{abc} \omega^b \rho^c, \quad \delta S_{WZ} = -\frac{1}{g} \int dx^+ \rho^a \partial^- \omega^a. \quad (3.5)$$

Eq. (3.3) immediately follows by imposing $-\int \delta \rho^a A_+^a + \delta S_{WZ} = 0$. The solution to Eq. (3.3) is

$$\rho^a(x^+) = \tilde{W}_{ab}(x^+)\rho^b_{-\infty} = \left(W(x^+)\rho_{-\infty} W^\dagger(x^+)\right)^a, \quad (3.6)$$

where

$$W(x^+) = P \exp \left(ig \int_{-\infty}^{x^+} dz^+ A^-(z^+) \right). \quad (3.7)$$

\(^5\) In [11], the $x^+$–dependence of $\rho$ was assumed to be of the form Eq. (3.6). However, the solution Eq. (3.2) is more general and is valid for arbitrary $x^+$–dependence of $\rho(x^+)$.\[11\]
In Appendix A, we derive an interesting property of the saddle point solution Eq. (3.6). Since $A^-$ field is weak in the JIMWLK regime, at the saddle point

$$\rho(x^+) \approx \rho_{-\infty}. \quad (3.8)$$

At high density where $\rho \sim \mathcal{O}(1/g) \gg 1$, the path integral receives important contributions from the neighborhood of the saddle point. Therefore, it is legitimate to approximate $\rho(x^+) \approx \rho_{-\infty}$ when solving Eq. (3.1). Moreover,

$$W_\tau[\rho_\infty, \rho_{-\infty}] \approx W_\tau[\rho_{-\infty}, \rho_{-\infty}] \equiv W_\tau[\rho_{-\infty}]. \quad (3.9)$$

This is a very important approximation which allows us to recover the CGC picture from our $x^+$-dependent formulation. Namely, as a result of Eq. (3.9), $W_\tau$ becomes real and positive, therefore it literally serves as a weight function. In the high density regime, at fixed rapidity, the above argument about the saddle point is generally correct and leads to a static, classical theory characterized by the averaging Eq. (2.6). However, the saddle point solution and the corresponding approximation Eq. (3.8) have only a limited sense when we consider quantum evolution, namely, when the semihard field $a^\mu$ is introduced.

Indeed, it was essential to keep track of the $x^+$-dependence of $\rho(x^+)$ and maintain the full (not saddle point) path integral $D\rho(x^+)$ with the Wess–Zumino term in order to correctly derive the induced charge $\delta \rho^{(1,2)}[a^-]$. Therefore, we use the approximation $\rho(x^+) \approx \rho_{-\infty}$ only when it is safe to do so. Our criterion of ‘safe’ is that the semihard field $a^\mu$ is not involved. For example, $W_\tau[\rho_\infty, \rho_{-\infty}]$ is safe but the term $-\rho(x^+)a^-(x^+)$ in $S_W$ is not: $\rho(x^+)a^-(x^+) \neq \rho_{-\infty}a^-(x^+)$. In this way we can maximally exploit the approximately static nature of the charges in the JIMWLK regime without tampering the precise $x^+$ structure of the source term $S_W[\rho(x^+)]$.

### 3.2 Renormalization group

With these caveats in mind, we expand around the static solution $B^i[\rho_{-\infty}]$

$$Z = \int D\rho_{-\infty} W_\tau[\rho_{-\infty}] \int_{\rho_{-\infty}}^{\rho_{-\infty}} D\rho(x^+) \int_\tau D\delta A^i_{-} \times \exp \left( i S_{\text{YM}}[B^i + \delta A^i, \delta A^-] - i \int dx^+ \rho \delta A^- + i S_{\text{WZ}}[\rho(x^+)] \right). \quad (3.10)$$

In the above, the field $\delta A^i_{-}$ contains modes with $p^+ < \Lambda^+$. In order to derive the effective theory at rapidity $\tau + \delta \tau$, we decompose the soft field $\delta A^i_{-} \rightarrow a^i_{-} + \delta A^i_{-}$ and functionally integrate over $a^i_{-}$

$$Z = \int D\rho_{-\infty} W_\tau[\rho_{-\infty}] \int_{\rho_{-\infty}}^{\rho_{-\infty}} D\rho(x^+) \int_{\tau}^{\tau + \delta \tau} D\delta A^i_{-} \int_{\tau}^{\tau + \delta \tau} Da^i_{-} \times \exp \left( i S_{\text{YM}}[B^i + a^i + \delta A^i, a^- + \delta A^-] - i \int dx^+ (a^- + \delta A^-) + i S_{\text{WZ}}[\rho(x^+)] \right) \quad (3.11)$$
In the source term we perform the replacement Eq. (2.38). The Yang–Mills action gives other contributions [16, 17] to the induced charge

\[-\delta \rho_{YM}[a^i] \delta A^-,\]  

(3.12)

to be added to the induced charge from the source term

\[\delta \rho_{YM} + \delta \rho^{(1)} + \delta \rho^{(2)} \equiv \delta \rho[a].\]  

(3.13)

As was observed in [17], \(\delta \rho\) has a support in a narrow strip \(1/b\Lambda^+ > x^- > 1/\Lambda^+\), while the original charges \(\rho\) sit on \(1/\Lambda^+ > x^- > 0\) (Fig. 2). Therefore, each step of quantum evolution piles up a layer of new classical charges at larger positive values of \(x^-\).

The expansion of

\[S_{YM}[B^i + a^i + \delta A^i, a^- + \delta A^-]\]  

(3.14)

requires a care because in the presence of quantum fluctuations the average field is not just \(B^i\), but \(B^i + \langle \delta A^i_{\text{ind}}[\delta \rho] \rangle [17]\) where \(\delta A^i_{\text{ind}}[\delta \rho]\) is that part of the soft field \(\delta A^\mu\) induced by \(\delta \rho\) and obeys the Yang–Mills equation with the renormalized charge

\[\frac{\delta S_{YM}}{\delta A^\mu} \bigg|_{B^i + \delta A^i_{\text{ind}}} = D_\nu F^{\nu\mu} \bigg|_{B^i + \delta A^i_{\text{ind}}} = \delta^{\mu i}(\rho_{-\infty} + \delta \rho[a]).\]  

(3.17)

For given \(\delta \rho\), the solution of Eq. (3.17) is again a pure gauge (see Eq. (2.2))

\[B^i + \delta A^i_{\text{ind}} = \frac{i}{g} \bar{U} \partial^i U^\dagger, \quad U^\dagger \equiv \text{P exp} \left( ig \int dx^- (\alpha + \delta \alpha) \right),\]  

(3.18)

where \(\delta \alpha\) is the induced field in the Coulomb gauge. Taking this into account, we write

\[\delta A^i \rightarrow \delta A^i_{\text{ind}}[\delta \rho] + \delta A^i,\]  

(3.19)

and expand Eq. (3.14) around \(B^i + \delta A^i_{\text{ind}}[\delta \rho]\). To linear order in small fields, we get

6 In the LLA, only the transverse \(\mu = i\) components are important. Indeed, the one– and the connected two–point functions of \(\delta A^i_{\text{ind}}\) are logarithmically enhanced [17] just like the one– and the two–point functions of \(\delta \rho[a]\). By expanding the left hand side of Eq. (3.17) to quadratic order in \(\delta A^i_{\text{ind}}\), we obtain relations between the correlation functions:

\[\sigma \equiv \langle \delta \rho[a] \rangle = \frac{\delta^2 S_{YM}}{\delta A^- \delta A^i} \bigg|_B \langle \delta A^i_{\text{ind}} \rangle + \frac{1}{2} \frac{\delta^3 S_{YM}}{\delta A^- \delta A^i \delta A^j} \bigg|_B \langle \delta A^i_{\text{ind}} \delta A^j_{\text{ind}} \rangle \propto \alpha_s \delta \tau,\]  

(3.15)

from the \(\mu = +\) component of Eq. (3.17) and

\[\frac{\delta^2 S_{YM}}{\delta A^i \delta A^j} \bigg|_B \langle \delta A^j_{\text{ind}} \rangle + \frac{1}{2} \frac{\delta^3 S_{YM}}{\delta A^i \delta A^j \delta A^k} \bigg|_B \langle \delta A^j_{\text{ind}} \delta A^k_{\text{ind}} \rangle = 0,\]  

(3.16)

from the \(\mu = i\) components.
\[ S_{\text{YM}}[B^i + \delta A^i_{\text{ind}} + a^i + \delta A^i, a^- + \delta A^-] \\
= S_{\text{YM}}[B^i + \delta A^i_{\text{ind}}] + D_\mu F^{\mu i}|_{B^i + \delta A^i_{\text{ind}}} (a^- + \delta A^-) + D_\mu F^{\mu i}|_{B^i + \delta A^i_{\text{ind}}} (a^i + \delta A^i) + \cdots \\
= (\rho_{-\infty} + \delta \rho[a]) (\delta A^- + a^-) + \cdots, \tag{3.20} \]

where we used
\[ S_{\text{YM}}[B^i + \delta A^i_{\text{ind}}] = 0, \tag{3.21} \]

because \( B^i + \delta A^i_{\text{ind}} \) is a pure gauge. Note that there is a mismatch between Eq. (3.20) and Eq. (2.38) (with the total induced charge Eq. (3.13)). The term \( \delta \rho[a]a^- \) in Eq. (3.20) is not cancelled by the source term. This term will play an important role below.

Collecting all factors, we are now prepared to integrate over the semihard field. In the Gaussian approximation, Eq. (3.11) becomes
\[
Z = \int D\rho_{-\infty} W_\tau[\rho_{-\infty}] \int D\rho(x^+) D\delta A^{i-} Da^{i-} \\
\times \exp \left( i S_{\text{YM}}[B^i + \delta A^i_{\text{ind}}] \delta \rho[a] + \delta A^i, \delta A^- + \frac{i}{2} a^\mu G^{-1}_{\mu \nu} a^\nu \\
+ i \delta \rho a^- - i (\rho + \delta \rho) \delta A^- + i S_{\text{WZ}}[\rho] \right), \tag{3.22} \]

where \( G^{\mu \nu} \) is the background field propagator
\[
G^{-1}_{\mu \nu} \equiv \left. \frac{\delta S_{\text{YM}}}{\delta A^\mu \delta A^\nu} \right|_{B^i + \delta A^i_{\text{ind}}} + \delta_{\mu \nu} \Pi. \tag{3.23} \]

We introduce the following trick.
\[
Z = \int D\rho_{-\infty} D\delta \rho_{-\infty} W_\tau[\rho_{-\infty}] \int D\rho(x^+) D\delta \rho(x^+) D\delta A^{i-} Da^{i-} \delta (\delta \rho - \delta \rho[a]) \\
\times \exp \left( i S_{\text{YM}}[B^i + \delta A^i_{\text{ind}}] \delta \rho[a] + \delta A^i, \delta A^- + \frac{i}{2} a^\mu G^{-1}_{\mu \nu} a^\nu \\
+ i \delta \rho a^- - i (\rho + \delta \rho) \delta A^- + i S_{\text{WZ}}[\rho] \right) \\
= \int D\rho_{-\infty} D\delta \rho_{-\infty} W_\tau[\rho_{-\infty}] \int D\rho(x^+) D\delta \rho(x^+) D\delta A^{i-} Da^{i-} D\pi \\
\times \exp \left( i S_{\text{YM}}[B^i + \delta A^i_{\text{ind}}] \delta \rho[a] + \delta A^i, \delta A^- + \frac{i}{2} a^\mu G^{-1}_{\mu \nu} a^\nu \\
+ i \delta \rho a^- + i \pi (\delta \rho - \delta \rho[a]) - i (\rho + \delta \rho) \delta A^- + i S_{\text{WZ}}[\rho] \right). \tag{3.24} \]

The integration over \( a^{i-} \) can be done. To the order of interest (retaining terms enhanced by \( \delta \tau \)), we obtain [16]
\[ Z = \int D\rho_{-\infty} D\delta\rho_{-\infty} W_i[\rho_{-\infty}] \int D\rho(x^+) D\delta\rho(x^+) D\delta A^i_{-\infty} D\pi \]
\[
\times \exp \left( i S_{YM} [B^i + \delta A^i_{\text{ind}}[\delta\rho] + \delta A^i, \delta A^-] + i \pi (\rho - \sigma) - \frac{1}{2} \pi \chi \pi \right)
\]
\[-i(\rho + \delta\rho)\delta A^- + i S_{WZ}[\rho] + i S_{WZ}[\delta\rho], \tag{3.25} \]

where \( \sigma = \langle \delta\rho[a] \rangle \propto \alpha_s \delta\tau \) and \( \chi = \langle \delta\rho[a]\delta\rho[a] \rangle \propto \alpha_s \delta\tau \). Note that, in addition to the usual building blocks of the JIMWLK kernel, the \( \delta\rho \) contribution is lost. This suggests that in Eq. (3.24) we have to perform an integral with the Mills action in the Gaussian approximation, invariance under gauge transformations. Therefore, its origin must have to do with gauge invariance. Once we truncate the Yang–Mills action, the Wess–Zumino term comes from the large \( a^i_{-\infty} \) region of the integral Eq. (3.27). To see this, consider the following change of integration variables in Eq. (3.27) [See [30] for more details.]

\[
f[\delta\rho] \equiv \int D\rho \exp \left( i S_{YM}[\rho] + i \int dx^+ \delta\rho a^- \right), \tag{3.27} \]

where we neglected the external fields \( B^i \) and \( \rho \). By this we formally treat the fields \( \rho \) as large and assume that the external fields are not essential for the generation of \( S_{WZ}[\delta\rho] \). \([S_{WZ}[\delta\rho] \) does not depend on \( B^i \) or \( \rho \).] Indeed, it is known [30] that the Wess–Zumino term comes from the large \( a^i_{-\infty} \) region of the integral Eq. (3.27). To see this, consider the following change of integration variables in Eq. (3.27) [See [30] for more details.]

\[
a^i_{-\infty} \rightarrow \tilde{a}^i_{-\infty} = V^+ a^i_{-\infty} V + \frac{i}{g} V^+ \tilde{\sigma}^i_{-\infty} V, \tag{3.28} \]

where \( V \) does not depend on \( x^- \). This is a gauge transformation which preserves the light–cone gauge condition \( a^+ = 0 \). Using the gauge invariance of the Yang–Mills action, we can write explicitly as a functional of \( \delta\rho \) and \( \rho \). Therefore, the \( p^- \) integral (\( p^- \neq 0 \)) in the Fourier representation of \( G^{--}(x, y) \) contains an oscillating phase \( e^{i(x^+ - y^+) p^-} \) which prohibits a logarithm. This is in contrast to Eq. (2.37) where the integration over \( y^+ \) and \( z^+ \) can be explicitly performed and a logarithm does not arise.

\[ \text{Note that the } x^+ \text{ dependence of } \delta\rho \text{ and } G^{--} \text{ originates from that of the semihard field } a^i. \]
one can easily check that
\[ f[\delta \rho] = f[V \delta \rho V^\dagger] e^{-\delta \rho V^\dagger \partial V}. \] (3.29)

A general solution for Eq. (3.29) is
\[ f[\delta \rho] = g[\delta \rho] e^{iS_{WZ}[\delta \rho]}, \] (3.30)
where \( g[\delta \rho] \) is a gauge invariant function
\[ g[\delta \rho] = g[V \delta \rho V^\dagger]. \] (3.31)

In Eq. (3.30), the Wess–Zumino term arises as a special solution to Eq. (3.29). The relevance of this term is easy to understand. The extra phase factor in Eq. (3.29) due to the non-gauge invariance of \( \int \delta \rho a^- \) is precisely generated by the variation of the Wess–Zumino term. The function \( g[\delta \rho] \) cannot be calculated exactly, but since the Gaussian approximation Eq. (4.2) (with \( G^{-} \) the free or dressed propagator) is already unimportant (i.e., not enhanced by a logarithm), we can simply set \( g = 1 \) in the leading logarithmic approximation. This is the reasoning of our addition of the Wess–Zumino term in Eq. (3.25).

Returning to Eq. (3.25), we observe that
\[ S_{WZ}[\rho] + S_{WZ}[\delta \rho] = S_{WZ}[\rho + \delta \rho]. \] (3.32)

This holds because \( \rho \) and \( \delta \rho \) have different supports in \( x^- [17] \): \( 1/\Lambda^+ > x^- > 0 \) for \( \rho \), and \( 1/b \Lambda^+ > x^- > 1/\Lambda^+ \) for \( \delta \rho \). Moreover, from Eq. (3.25), we can easily deduce the evolution of \( W_{\tau} (\rho_{-_\infty} \equiv \rho_{-_\infty} + \delta \rho_{-_\infty}) \)
\[ W_{\tau+\delta \tau}[\rho_{-_\infty}] = \int D\pi \exp \left( i\pi (\delta \rho_{-_\infty} - \sigma) - \frac{1}{2} \pi \chi \pi \right) W_{\tau}[\rho_{-_\infty}], \] (3.33)
in agreement with the path integral formula previously derived in [31].\(^8\)\(^9\) The infinitesimal evolution Eq. (3.33) is equivalent to the JIMWLK equation
\[ \frac{\partial}{\partial \tau} W_{\tau}[\rho] = \frac{1}{2} \frac{\delta}{\delta \rho_{\tau}} \frac{\delta}{\delta \rho_{\tau}} \left( \chi W_{\tau}[\rho] \right) - \frac{\delta}{\delta \rho_{\tau}} \left( \sigma W_{\tau}[\rho] \right), \] (3.34)
where the subscript \( \tau \) in \( \rho \) means that the derivatives are taken at the highest value of \( x^-; x^- = 1/\Lambda^+ \) [17]. [Remember that the support of \( \delta \rho_{-_\infty} \) in Eq. (3.33) is \( 1/b \Lambda^+ > x^+ > 1/\Lambda^+ \).] Using Eq. (3.32) and Eq. (3.33), we finally arrive at
\(^8\) Here we neglect the \( x^+ \) dependence of \( \delta \rho \). See the remarks following Eq. (3.9). Eq. (3.33) shows that \( \delta \rho_{-_\infty} \) is Gaussian distributed with the mean \( \sigma \) and the variance \( \chi \) (which are both static).
\(^9\) See, Eq. (4.11) of Ref. [31]. There the authors work in the Coulomb gauge Eq. (2.4). There is a subtlety in going from the light–cone gauge to the Coulomb gauge. The gauge function which realizes this rotation is not Eq. (2.3), but Eq. (3.18). See the discussion in Section 3 of the first paper in Ref. [17].
\[ Z = \int D\rho_{-\infty} W_{\tau+\delta\tau}[\rho_{-\infty}] \int D\rho'(x^+) \int_{\tau+\delta\tau} D\delta A^{i-} \]
\[ \times \exp \left( i\text{SYM}[B^i + \delta A^i, \delta A^-] - i\rho'\delta A^- + i\text{WZ}[\rho'(x^+)] \right) , \]  

(3.35)

where \( B^i = B^i + \delta A^i_{\text{ind}}[\delta \rho] \). Eq. (3.35) is exactly the same form as our starting formula Eq. (3.10). This is the JIMWLK renormalization group picture. Namely, the quantum effects at one step of evolution is completely absorbed by the change of the weight function and the form of the effective theory is preserved.

4 Conclusions

In this paper, we have investigated the role of the Wess–Zumino term in the high energy limit of QCD where the target becomes a Color Glass Condensate. We have shown that the Wess–Zumino term can be naturally incorporated in the JIMWLK formalism as a part of the source term \( S_{\text{W}} \). The simple eikonal coupling between the gauge field and the source \(-\rho\delta A^-\) makes the renormalization group description more transparent. We have argued that after one step of quantum evolution the functional integral Eq. (3.10) remains the same form including the source term. This last point has not been discussed in the existing proofs of renormalization group [16, 17].

Finally, we expect that the real strength of our approach is that it can be straightforwardly applied to other limits of high energy QCD, namely, the dilute regime (Fig. 1(b)) and the Pomeron loop regime (Fig. 1(c)), where the importance of the Wess–Zumino term has been first recognized. The color glass averaging in the dilute regime proposed in [6] is (see also [8])

\[ \langle \rho^a \rho^b \rho^c \cdots \rangle_{\tau} = \int D\rho(t) W_{\tau}[\rho(t)] \rho^a(t_1) \rho^b(t_2) \rho^c(t_3) \cdots , \]  

(4.1)

where the ordering variables \( t_i \) are constrained such that \( t_1 > t_2 > t_3 > \cdots \), but otherwise arbitrary. The “weight function” \( W_{\tau}[\rho(t)] \) contains a complex phase \( e^{iS_{\text{WZ}}[\rho(t)]} \) which ensures the color commutators of \( \hat{\rho} \). It follows from our analysis in Section 2 (see Eq. (2.23)) that Eq. (4.1) should read

\[ \langle \hat{\rho}^a \hat{\rho}^b \hat{\rho}^c \cdots \rangle_{\tau} = \int D\rho_{\pm\infty} W_{\tau}[\rho_{\pm\infty}] \int_{\rho_{-\infty}}^{\rho_{\infty}} D\rho(x^+) e^{iS_{\text{WZ}}[\rho(x^+)]} \rho^a(x_1^+) \rho^b(x_2^+) \rho^c(x_3^+) \cdots , \]

(4.2)

with the same Wess–Zumino term appearing in Eq. (2.22).\(^{10}\) Therefore, the JIMWLK formalism with our new source term allows us to treat different limits of high energy QCD in a single framework. The renormalization group description of quantum evolution in the dilute regime is subtler than in the JIMWLK regime because the division between

\(^{10}\) Note that the approximation Eq. (3.9) is invalid in the dilute regime. This means that \( W_{\tau}[\rho_{\pm\infty}] \) is complex even without the Wess–Zumino term.
the “quantum” and “classical” theories is shifted towards the quantum side. [Compare Eq. (4.2) with Eq. (2.6).] We leave this problem for future works.

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A Remarks on the saddle point solution Eq. (3.6)

For possible future applications, in this appendix we note a curious property of the saddle point solution Eq. (3.6). First we observe that the source term vanishes at the saddle point

\[
S_W = - \int dx^+ \rho(x^+) \delta A^- + S_{WZ}[\rho(x^+)] \bigg|_{\text{saddle point}} = 0. \tag{A.1}
\]

To show this, return to Eq. (2.15)

\[
\rho^a(x^+) = - \frac{g J}{2} \text{Tr} \tau^3 S \tau^a S^\dagger = (W \rho_{-\infty} W^\dagger)^a, \quad \rho^a_{-\infty} = - \frac{g J}{2} \text{Tr} \tau^3 S_{-\infty} \tau^a S_{-\infty}^\dagger. \tag{A.2}
\]

We see that \(S(x^+)%2\) and \(S_{-\infty}\) are related by

\[
S(x^+) = S_{-\infty} W^\dagger(x^+). \tag{A.3}
\]

Therefore, the Wess–Zumino term Eq. (2.14) becomes

\[
S_{WZ}[\rho(x^+)] \bigg|_{\text{saddle point}} = i J \int dx^+ \text{tr} \left[ \tau^3 S_{-\infty} W^\dagger \partial^- W S_{-\infty}^\dagger \right]
= - g J \int dx^+ \text{tr} \left[ \tau^3 S_{-\infty} W^\dagger \tau^a W S_{-\infty}^\dagger \right] \frac{\delta A^-}{2} = \int dx^+ \rho^a(x^+) \delta A^- a. \tag{A.4}
\]

Since \(S_W\) is gauge invariant, Eq. (A.1) holds in any gauge. Then let us consider the exponential factors in Eq. (3.10) in the Coulomb gauge Eq. (2.4)

\[
S_{YM}[\alpha, \delta \tilde{A}^-] = \int \tilde{\rho} \delta \tilde{A}^- + S_W[\tilde{\rho}(x^+)], \tag{A.5}
\]
where we neglected $\delta A^i$. In the context of [11], Eq. (A.5) should be called the tree level effective action. Since the only non-vanishing components of the field strength are $F^{\pm i}$, the Yang–Mills action becomes

$$\frac{-1}{4} \int \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} \approx \int \delta^i \partial^i \delta A^- = -\int (\nabla^2 \alpha) \delta A^- = \int \delta A^-.$$  \hspace{1cm} (A.6)

Therefore, in the Coulomb gauge, at the saddle point Eq. (3.6), all the three terms in Eq. (A.5) are equal up to a sign. In particular,

$$S_{\text{YM}} = S_{\text{WZ}}.$$  \hspace{1cm} (A.7)

Eq. (A.7) was conjectured in [6] as a possible origin of the Wess–Zumino term. Although at the moment we do not see a connection between our derivation and the argument in [6], in any case it is interesting to pursue physical implications of Eq. (A.7).

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