ON BOURGAIN’S BOUND FOR SHORT EXPONENTIAL SUMS AND SQUAREFREE NUMBERS

RAMON M. NUNES

ABSTRACT. We use Bourgain’s recent bound for short exponential sums to prove certain independence results related to the distribution of squarefree numbers in arithmetic progressions.

1. Introduction

As usual, let

\[ e(x) := e^{2\pi i x}, \text{ for } x \in \mathbb{R}. \]

In a recent paper, Bourgain [2] proved a non trivial bound for exponential sums such as

\[ \sum_{n \leq N \atop (n, q) = 1} e \left( \frac{a n^2}{q} \right), \]

where \( q > 1 \) is an integer and \( \bar{n} \) denotes the multiplicative inverse of \( n \pmod{q} \), in the range \( N \geq q^\epsilon \), for an arbitrarily small, but fixed, \( \epsilon > 0 \). In his paper, Bourgain was interested in an application related to the size of fundamental solutions \( \epsilon_D > 1 \) to the Pell equation

\[ t^2 - D u^2 = 1. \]

He followed the lead of Fouvry [3], who suggested that such an upperbound could help to improve the lower bounds for the following counting function

\[ S_f(x, \alpha) := \left| \{ (\epsilon_D, D); 2 \leq D \leq x, D \text{ is not a square, and } \epsilon_D \leq D^{1/2 + \alpha} \} \right|, \]

for small values of \( \alpha \). In this article, we are interested in a different application of Bourgain’s result (see Proposition 4.2 below) related to squarefree numbers in arithmetic progressions.

Let \( X \geq 1 \). Let \( a, q \) be integers, with \( q \geq 1 \). We let

\[ (1.1) \quad E(X, q, a) := \sum_{\substack{n \leq X \atop n \equiv a \pmod{q}}} \mu^2(n) - \frac{6}{\pi^2} \left( 1 - \frac{1}{q^2} \right)^{-1} \frac{X}{q}. \]

For fixed \( q \), the last term is known to be equivalent to

\[ \frac{1}{\phi(q)} \sum_{\substack{n \leq X \atop (n, q) = 1}} \mu^2(n) \]

as \( X \to \infty \). So that \( E(X; q, a) \) can be seen as an error term of the distribution of squarefree numbers in arithmetic progressions. One naturally has the trivial bound

\[ (1.2) \quad \left| E(X, q, a) \right| \leq \frac{X}{q} + 1. \]

In a previous article, we [6] proved

Date: July 14, 2014.
Theorem 1.1. There exists an absolute constant $C > 0$, such that, for every $\epsilon > 0$, we have

\[(1.3) \sum_{a \pmod{q}}^{*} E(X, q, a)^2 \sim C \prod_{p|q} \left(1 + 2p^{-1}\right)^{-1} X^{1/2}q^{1/2},\]

for $X \to \infty$, uniformly for $q$ integer satisfying $X^{31/41+\epsilon} \leq q \leq X^{1-\epsilon}$.

This theorem gives the asymptotic variance of the above mentioned distribution.

Inspired by an equivalent problem considered by Fouvry et al [4, Theorem 1.5.], we studied how $E(X, q, a)$ correlates with $E(X, q, \gamma(a))$ for suitable choices of $\gamma : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$. It is natural to choose $\gamma$ to be an affine linear map, i.e.

\[(1.4) \gamma_{r,s}(a) = ra + s,\]

where $r, s \in \mathbb{Z}$, $r \neq 0$ are fixed. Thus our objet of study is the following correlation sum

\[(1.5) C[\gamma_{r,s}](X, q) := \sum_{a \pmod{q}}^{*} E(X, q, a)E(X, q, \gamma_{r,s}(a)),\]

for $q$ prime. In [3], we already considered the case $s = 0$, and we found that correlation always existed for any non zero value of $r$.

In particular, there exists $C_r \neq 0$ such that for $X \to \infty$, $X^{31/41+\epsilon} \leq q \leq X^{1-\epsilon}$, one has

\[(1.6) C[\gamma_{r,0}](X, q) \sim C_r \left(\sum_{a \pmod{q}}^{*} E(X, q, a)^2\right).\]

Our main result is the following theorem which exhibits a certain independence between the functions $a \mapsto E(X, q, a)$ and $a \mapsto E(X, q, \gamma_{r,s}(a))$ considered as random variables on $\mathbb{Z}/q\mathbb{Z}$, which confirms our intuition on this question when $\gamma_{r,s}$ is not an homothety.

Theorem 1.2. There exists an absolute $\delta > 0$ such that

- for every $\epsilon > 0$,
- for every $r$ integer, $r \neq 0$,

there exists $C_{\epsilon, r}$ such that one has the inequality

\[(1.7) \left|C[\gamma_{r,s}](X, q)\right| \leq C_{\epsilon, r} \left(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta} + \frac{X^{5/3+\epsilon}}{q} + \left(\frac{X}{q}\right)^2\right)\]

uniformly for $X \geq 2$, and $q$ prime $\leq X$ such that $q \nmid rs$.

A consequence of Theorems 1.1 and 1.2 (not necessarily with the same $\epsilon$) is the following

Corollary 1.3. For every $\epsilon > 0$ and $r \neq 0$, there exists a function $\Phi_{\epsilon, r} : \mathbb{R}^+ \to \mathbb{R}^+$, tending to zero at infinity, such that for every $X > 1$, for every integer $s$ and for any prime $q$ such that $q \nmid rs$ and $X^{7/9+\epsilon} \leq q \leq X^{1-\epsilon}$, one has the inequality

\[(1.8) \left|C[\gamma_{r,s}](X, q)\right| \leq \Phi_{\epsilon, r}(X) \left(\sum_{a \pmod{q}}^{*} E(X, q, a)^2\right).\]
Inequality (1.8) shows a behavior different from (1.6) corresponding to $s = 0$. In other words, it indicates some independence of the random variables.

Here, as in [6], we give results that are true for a general $r \neq 0$, but in order to simplify the presentation, we give proofs that are only complete when $r$ is squarefree (the case where $\mu^2(r) = 0$ implies a more difficult definition of the $\kappa$ function in (1.10)).

2. Notation

We define the Bernoulli polynomials $B_k(x)$ for $k \geq 1$, on $[0,1)$, in the following recursive way

\[
B_1(x) := x - \frac{1}{2},
\]

\[
\frac{d}{dx} B_{k+1}(x) = B_k(x),
\]

\[
\int_0^1 B_k(x) dx = 0.
\]

We can extend these functions to periodic functions defined in the whole real line by posing

\[
B_k(x) := B_k(\{x\}).
\]

We further notice that $B_1(x)$ satisfy the following relation

(2.1) \[
\lfloor x \rfloor = x - \frac{1}{2} - B_1(x)
\]

and $B_2(x)$ satisfies

(2.2) \[
B_2(x) = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} \quad \text{for } 0 \leq x \leq 1.
\]

In the course of the proof we will make repetitive use of the following multiplicative function

(2.3) \[
h(d) = \mu^2(d) \prod_{p|d} (1 - 2p^{-2})^{-1}.
\]

We also define here the closely related product

(2.4) \[
C_2 = \prod_p \left(1 - \frac{2}{p^2}\right).
\]

We denote, as usual, by $d(n)$, $d_3(n)$ the classical binary and ternary divisor functions, respectively. We write $\omega(n)$ for the number of primes dividing $n$. We write $n \sim N$ as an alternative to $N < n \leq 2N$. If $I \subset \mathbb{R}$ is an interval, $|I|$ denotes its length. We use indistinguishably the notations $f = O(g)$ and $f \ll g$ when there is an absolute constant $C$ such that

\[
|f| \leq Cg,
\]

on a certain domain of the variables which will be clear by the context, and the the same for the symbols $O_\epsilon, O_r, O_{\epsilon,r}$ and $\ll_\epsilon, \ll_r, \ll_{\epsilon,r}$, but with constants that may depend on the subindexed variables.

3. Initial Steps

Let $X > 1$. Let $\gamma = \gamma_{r,s}$ be given by (1.3) and let $q$ be a prime number $\leq X$ such that $q \nmid rs$.

We start by completing the sum defining $C[\gamma](X, q)$ (see (1.5)) and we bound trivially the exceeding terms. We have, in view of (1.2), that

(3.1) \[
C[\gamma](X, q) = \sum_{n=0}^{q-1} E(X, q, a)E(X, q, \gamma(a)) + O\left(\left(\frac{X}{q}\right)^2\right),
\]
In what follows, for simplification, we shall write
\begin{equation}
C(q) = \frac{6}{\pi^2} \left( 1 - \frac{1}{q^2} \right)^{-1}.
\end{equation}

As we develop the first sum on the right-hand side of (3.1), we obtain
\begin{equation}
C[\gamma](X, q) = S[\gamma](X, q) - 2C(q) \frac{X}{q} \sum_{n \leq X} \mu^2(n) + C(q) \frac{X^2}{q} + O \left( \frac{X^2}{q^2} \right),
\end{equation}
where $S[\gamma](X, q)$ is defined by the double sum
\begin{equation}
S[\gamma](X, q) = \sum_{n_1, n_2 \leq X} \mu^2(n_1) \mu^2(n_2).
\end{equation}

We point out that $S[\gamma](X, q)$ is the only difficult term appearing in equation (3.3), since we have the well-known formula
\begin{equation}
\sum_{n \leq X} \mu^2(n) = \frac{6}{\pi^2} X + O(\sqrt{X})
\end{equation}
uniformly for $1 \leq q \leq X$. An asymptotic expansion of $S[\gamma](X, q)$ will be given in Proposition 5.1.

4. USEFUL LEMMATA

We start with a lemma concerning the multiplicative function $h(d)$ which is a simple consequence of [6, lemma 4.2]

**Lemma 4.1.** Let $h(d)$ be as in (2.3) and let $\beta$ be the multiplicative function defined by
\begin{equation}
h(d) = \sum_{mn = d} \beta(m), \ d \geq 1.
\end{equation}

Then $\beta(m)$ satisfies
\begin{equation}
\sum_{m \geq M} \frac{\beta(m)}{m} \ll \frac{(\log 2M)^2}{M},
\end{equation}
\begin{equation}
\sum_{m \leq M} \beta(m) \ll M,
\end{equation}
uniformly for every $M \geq 1$.

**Proof.** By [6, lemma 4.2], we know that $\beta(m)$ is supported on cubefree numbers and, if we write $m = ab^2$ with $a$, $b$ squarefree and relatively prime, then
\begin{equation}
\beta(m) \ll \frac{d(a)}{a^2}.
\end{equation}

In particular, $\beta(m) \ll 1$, which is sufficient to prove (4.2). In order to prove (4.1), we notice that
\begin{equation}
\sum_{m \geq M} \frac{\beta(m)}{m} \ll \sum_{ab^2 \geq M} \frac{d(a)}{n^3b^2} \ll \sum_{n \geq M} \frac{d_3(n)}{n^2} \ll \frac{(\log 2M)^2}{M}.
\end{equation}
\[\square\]
The next proposition is the main result from [2], which is crucial to our proof.

**Proposition 4.2.** (see [2, Proposition 4]) There exist constants $c, C, C'$ such that for every $N, q \geq 2$ and $\frac{1}{\log 2N} < \beta < \frac{1}{10}$, there exist a subset $E_N \subset \{1, 2, \ldots, N\}$ (independent of $q$) satisfying

\[
|E_N| \leq C' \beta \left( \log \frac{1}{\beta} \right)^CN
\]

and such that, uniformly for $(a, q) = 1$, one has

\[
\left| \sum_{n \leq N, (n, q) = 1} e\left(\frac{a\bar{n}^2}{q}\right) \right| \leq C'(\log 2N)^C N^{1-c(\frac{\log N}{\log q})^C}.
\]

**Remark 4.3.** In the statement of his result, Bourgain uses the symbol $\prec$, where one writes $f(x) \prec g(x)$ if there is some $C > 0$ such that $f(x) \leq C g(Cx) + C$.

In our case, it is easy to see that his result implies Proposition 4.2.

In fact we specifically need the following corollary

**Corollary 4.4.** There exists $\delta > 0$ such that for every $\epsilon > 0$, we have

\[
\sum_{n \leq N,(n,q)=1} e\left(\frac{a\bar{n}^2}{q}\right) \ll \epsilon N (\log \log N)^{1-C_{\epsilon}}.
\]

uniformly for $N, q \geq 2$ and $N \geq q^\epsilon$.

**Remark 4.5.** More generally, we may consider the sum

\[
\Sigma(I, q) = \sum_{n \in I, (n,q)=1} e\left(\frac{a\bar{n}^2}{q}\right)
\]

where $I$ is a general interval of length $N$ (mod $q$). By the completion of exponential sums and Weil’s bound for such sums, we know that

\[
\Sigma(I, q) \ll q^{1/2} \log q,
\]

for $q$ prime. Hence, (4.5) is non trivial as soon as $N \geq q^\epsilon$ (for any $\epsilon > 1/2$). Obviously, Bourgain’s result is much stronger than (4.5), but it only applies to intervals containing $0$, roughly speaking.

**Proof.** (of Corollary 4.4) We use Proposition 4.2 and make the choice $\beta = (\log N)^{-\delta_1}$, where $\delta_1 = \min\left(\frac{1}{2}, \frac{1}{2C}\right)$. We add together inequalities (4.3) and (4.4) to obtain

\[
\sum_{n \leq N,(n,q)=1} e\left(\frac{a\bar{n}^2}{q}\right) \ll N \left(\frac{\log \log N}{(\log N)^{\delta_1}}\right)^C + N \exp\left(\frac{(\log N)^C}{\epsilon \exp(\log \log N)^{1/2}}\right).
\]

The corollary now follows by taking, for example, $\delta = \delta_1/2$. \qed

**Remark 4.6.** Corollary 4.4 will be essential to the proof of Proposition 4.7, in which we use it for values of $N$ which are roughly of size $\sqrt{X}$. Since we want to take $q$ as large as $X^{1-\epsilon}$, it is very important that Bourgain’s result holds for $N$ as small as $q^\epsilon$.

The next lemma is very similar in essence to many others to be found in literature, for example [7, Theorem 1], [11, Proposition 1.4] or [5, Theorem 3]. The proof, for instance, follows the lines of [11, Proposition 1.4].
Lemma 4.7. Let $X > 1$ and let $\ell, r$ be integers, $r$ squarefree. Let

\begin{equation}
I(X, \ell, r) := \left\{ u \in \mathbb{R} : u \text{ and } ru + \ell \in (0, X) \right\}
\end{equation}

and

\begin{equation}
S(\ell, r) := \sum_{n \in I(X, \ell, r)} \mu^2(n)\mu^2(rn + \ell).
\end{equation}

Then, for every $r > 0$, we have the equality

\begin{equation}
S(\ell, r) = f(\ell, r)|I(X, \ell, r)| + O_r \left( d_3(\ell)X^{2/3}(|\log 2X|^{7/3}) \right),
\end{equation}

uniformly for $X, \ell \geq 1$. where

\begin{equation}
f(\ell, r) = C_2 \prod_{p|\ell} \left( \frac{p^2 - 1}{p^2 - 2} \right) \prod_{p|\ell} \left( \frac{p^2 - 1}{p^2 - 2} \right) \kappa((\ell, r^2)),
\end{equation}

with

\begin{equation}
\kappa(p^\alpha) = \begin{cases} 
\frac{p^2 - p - 1}{p^2 - 1}, & \text{if } \alpha = 1, \\
\frac{p^2 - p}{p^2 - 1}, & \text{if } \alpha = 2, \\
0, & \text{if } \alpha \geq 3.
\end{cases}
\end{equation}

We recall that $C_2$ and $h(d)$ were already defined in (2.4) and (2.3) respectively.

Proof. We start by defining

\begin{equation}
\sigma(n) = \prod_{p^{|n|} \not| n} p, \ n \neq 0,
\end{equation}

and

\begin{equation}
\xi(n) = \sigma(n)\sigma(rn + \ell).
\end{equation}

Notice that the right-hand side of equation (4.11) above actually depends on $\ell$ and $r$, but since these numbers will be held fixed in the following calculations, we omit this dependency.

Since $\xi(n)$ is an integer $\geq 1$ and since

\[ \mu^2(n)\mu^2(rn + \ell) = 1 \iff \xi(n) = 1, \]

we deduce the equality

\begin{equation}
S(\ell, r) = \sum_{n \in I(X, \ell, r)} \mu(d) = \sum_{d \geq 1} \mu(d)N_d(\ell, r),
\end{equation}

where

\[ N_d(\ell, r) = \left| \left\{ n \in I(X, \ell, r) : \xi(n) \equiv 0 \ (mod \ d) \right\} \right| . \]

Notice that the condition

\[ p \mid \xi(n) \]

only depends on the congruence class of $n (mod \ p^2)$, for fixed values of $\ell$ and $r$. We let

\begin{equation}
u_p(\ell, r) := \left| \left\{ 0 \leq v \leq p^2 - 1; \xi(v) \equiv 0 \ (mod \ p) \right\} \right| ,
\end{equation}

and

\[ U_d(\ell, r) := \prod_{p|d} u_p(\ell, r). \]

Then, by the Chinese Remainder Theorem, we have the equality

\begin{equation}
N_d(\ell, r) = U_d(\ell, r)\left| I(X, \ell, r) \right| \frac{1}{d^2} + O \left( U_d(\ell, r) \right),
\end{equation}
for every positive squarefree integer \(d\).

We also notice that if \((p, r) = 1\), then \(\left| u_p(\ell, r) \right| \leq 2\) and that \(\left| u_p(\ell, r) \right| \leq p^2\) in general. Therefore we have the upper bound

\[
U_d(\ell, r) \ll r^{2\omega(d)}.
\]

Let \(2 \leq y \leq X\) be a parameter, which will be chosen later to be a power of \(X\). As we multiply formula (4.14) by \(\mu(d)\) and sum for \(d \leq y\), we obtain the equality

\[
\sum_{d \leq y} \mu(d) N_d(\ell, r) = \sum_{d \leq y} \mu(d) U_d(\ell, r) \frac{|I(X, \ell, r)|}{d^2} + O_r \left( \sum_{d \leq y} 2^{\omega(d)} \right).
\]

By completing the first sum on the right-hand side of (4.15), we have

\[
\sum_{d \leq y} \mu(d) N_d(\ell, r) = \prod_p \left( 1 - \frac{u_p(\ell, r)}{p^2} \right) |I(X, \ell, r)| + O_r \left( \frac{X \log y}{y} + y \log y \right).
\]

For large values of \(d\), formula (4.14) is useless. Instead of it we will deduce by different means an estimation for

\[
N_{>y}(\ell, r) := \sum_{d > y} \mu(d) N_d(\ell, r)
\]

from which we will deduce the result.

We notice that \(d \mid \xi(n)\) if and only if there exist \(j, k\) such that \(d = jk\), \(j^2 \mid n\) and \(k^2 \mid rn + \ell\). Moreover since \(n, rn + \ell < X\), we have \(j, k < \sqrt{X}\). From this observation we deduce

\[
|N_{>y}(\ell, r)| = \left| \sum_{y < d \leq X} \mu(d) \left\{ n \in I(X, \ell, r); \xi(n) \equiv 0 \pmod{d} \right\} \right|
\]

\[
\leq \sum_{j, k < \sqrt{X}} \left| \left\{ n \in \mathbb{Z}; 0 < n, rn + \ell < X \text{ and } j^2 \mid n, k^2 \mid rn + \ell \right\} \right|
\]

\[
= \sum_{j, k < \sqrt{X}} N(j, k),
\]

by definition.

We shall divide the possible values of \(j\) and \(k\) into sets of the form

\[
\mathcal{B}(J, K) := \left\{ (j, k); j \sim J, k \sim K \right\}.
\]

We can do the division using at most \(O((\log X)^2)\) of these sets, since we are summing over \(j, k \leq X^{1/2}\). Let

\[
\mathcal{N}(J, K) := \sum_{j \sim J, k \sim K} N(j, k)
\]

\[
= \left| \left\{ (j, k, u, v); j \sim J, k \sim K, 0 < j^2u, k^2v < X, \text{ and } k^2v = rj^2u + \ell \right\} \right|
\]

By taking the maximum over all \(J, K\), we obtain a pair \((J, K)\) with \(J, K \leq X^{1/2}\) such that \(JK \geq y/4\) and we have the upper bound

\[
N_{>y}(\ell, r) \ll \mathcal{N}(J, K)(\log X)^2.
\]

At last, we estimate \(\mathcal{N}(J, K)\) in the following way
\[ N(J, K) \leq \sum_{j, k \sim J} \sum_{u \leq XJ^{-2}} \sum_{j \sim J} 1. \]

For \( j, k \) relevant to the sum above, we write \( f = (j, k) \). From the congruence condition in the inner sum, we have that \( f^2 | \ell \). So we write \( j_0 = \frac{j}{f}, k_0 = \frac{k}{f} \) and \( \ell_0 = \frac{\ell}{f^2} \).

The congruence then becomes
\[ j_0^2 ru \equiv -\ell_0 \pmod{k_0^2}. \]

Now, let \( g = (k_0^2, r) \) as above we have \( g | \ell_0 \). We write
\[ k_1 = \frac{k_0^2}{g}, s = \frac{r}{g} \text{ and } t = \frac{\ell_0}{g}. \]

That transforms the congruence into
\[ j_0^2 su \equiv -t \pmod{k_1}. \]

Finally, let \( h = (k_1, t) \). From the considerations above, we must have \( h | u \). We write
\[ k' = \frac{k_1}{h}, t' = \frac{t}{h} \text{ and } u' = \frac{u}{h}. \]

So the congruence becomes
\[ j_0^2 su' \equiv -t' \pmod{k'} \]
and since \((t', k') = 1\), it has at most \( 2,2^{2-(k')} \leq 2d(k_0) \) solutions in \( j_0 \pmod{k'} \). Therefore we have

\[ N(J, K) \leq 2 \sum_{g | r} \sum_{f^2h | \ell k_0 - K/f} \sum_{u \leq XJ^{-2}h^{-1}} \sum_{j_0 \sim J/f} 1 \]
\[ \leq 2 \sum_{g | r} \sum_{f^2h | \ell k_0 - K/f} XJ^{-2}h^{-1} \left\{ \frac{Jgh}{f^2k_0^2} + 1 \right\} d(k_0) \]
\[ \ll r \sum_{f^2h | \ell k_0 - K/f} XJ^{-2} \left\{ \frac{J}{f^2k_0^2} + 1 \right\} d(k_0) \]
\[ \ll \sum_{f^2h | \ell} XJ^{-2} \left\{ \frac{J}{K^2} + \frac{1}{f} \right\} K \log K \]
\[ \ll d_3(\ell) XJ^{-2} \left\{ \frac{J}{K^2} + 1 \right\} K \log X. \]

Hence
\[ N(J, K) \ll_r d_3(\ell) \left\{ Xy^{-1} + XJ^{-2}K \right\} \log X. \]

A similar inequality with the roles of \( J \) and \( K \) interchanged on the right hand side can be obtained in an analogous way. Combining the two formulas, we deduce

\[ N(J, K) \ll_r d_3(\ell) \left\{ Xy^{-1} + X(JK)^{-1/2} \right\} \log X \]
(4.20)

Replacing formula (4.20) in (4.19) and adding the latter to (4.16), it gives

\[ S(\ell, r) = \prod_p \left( 1 - \frac{u_p(\ell, r)}{p^2} \right) \left| I(X, \ell, r) \right| + O_r \left( y \log y + d_3(\ell) Xy^{-1/2}(\log X)^3 \right). \]
We make the choice \( y = X^{2/3}(\log X)^{4/3} \) obtaining

\[
S(\ell, r) = \prod_p \left( 1 - \frac{u_p(\ell, r)}{p^2} \right) |I(X, \ell, r)| + O_r \left( d_3(\ell)X^{2/3}(\log X)^{7/3} \right).
\]

We finish by a study of \( u_p(\ell, r) \). We distinguish five different cases (we recall that \( r \) is squarefree)

- If \( p \mid r, p^2 \nmid \ell \) then \( u_p(\ell, r) = p \),
- If \( p \mid r, p \nmid \ell \) but \( p^2 \nmid \ell \) then \( u_p(\ell, r) = p + 1 \),
- If \( p \mid r, p \nmid \ell \) then \( u_p(\ell, r) = 1 \),
- If \( p \nmid r, p^2 \mid \ell \) then \( u_p(\ell, r) = 1 \),
- If \( p \nmid r, p^2 \nmid \ell \) then \( u_p(\ell, r) = 2 \).

The lemma is now a consequence of formula (4.21) and the different values of \( u_p(\ell, r) \). \( \square \)

4.1. Sums involving the \( B_2 \) function.
In the following we study certain sums involving the Bernoulli polynomials \( B_2(x) \). In the next lemma, we deal with the simplest case

\[
A(Y; q, a) = \sum_{n \geq 1} \left\{ B_2 \left( \frac{Y^2}{n^2} + \frac{an^2}{q} \right) - B_2 \left( \frac{an^2}{q} \right) \right\},
\]

where \( Y \) is a positive real number, \( a, q \) are coprime integers. The sum above will serve as an archetype for more complicated sums appearing in the proof of Proposition 4.10 which in their turn will be central for estimating \( C[\gamma](X, q) \). One elementary bound for \( A(Y; q, a) \) can be given by noticing that we have both

\[
B_2 \left( \frac{Y^2}{n^2} + \frac{an^2}{q} \right) - B_2 \left( \frac{an^2}{q} \right) \ll 1,
\]

since \( B_2 \) is bounded, and

\[
B_2 \left( \frac{Y^2}{n^2} + \frac{an^2}{q} \right) - B_2 \left( \frac{an^2}{q} \right) = \int_{\frac{Y^2}{n^2} + \frac{an^2}{q}} B_1(v)dv
\]

\[
\ll \frac{Y^2}{n^2},
\]

since \( B_1 \) is also a bounded function. Gathering (4.23) and (4.24), we obtain

\[
A(Y; q, a) \ll \sum_{n \leq Y} 1 + \sum_{n > Y} \frac{Y^2}{n^2}
\]

\[
\ll Y.
\]

In the following lemma we give a non-trivial bound for the sum above by means of Bourgain’s bound, in the form of Corollary 4.4. What we obtain is better than trivial by just a small power of \( \log q \), but it is sufficient to obtain Theorem 1.2.
Lemma 4.8. There exists $\delta > 0$ such that for every $\epsilon > 0$, we have the inequality

\begin{equation}
A(Y; q, a) \ll Y (\log q)^{-\delta},
\end{equation}

uniformly for $a$ and $q$ integers satisfying $q \geq 2$ $(a, q) = 1$, and $Y > q^\epsilon$.

Proof. By Corollary 4.4, we know that there exists $\delta_1 > 0$ such that

\begin{equation}
\sum_{n \leq Y, (n, q) = 1} e \left( \frac{an^2}{q} \right) \ll Y (\log q)^{-\delta_1},
\end{equation}

uniformly for $(a, q) = 1$ and $Y > q^{\epsilon/10}$. For simplification, we write

\begin{equation}
\Delta_Y(n; q, a) = B_2 \left( \frac{Y^2}{n^2} + \frac{a\tilde{n}^2}{q} \right) - B_2 \left( \frac{a\tilde{n}^2}{q} \right).
\end{equation}

The sum on the left-hand side of (4.27) appears naturally once we use the Fourier series development for $B_2(x)$

\begin{equation}
B_2(x) = \sum_{h \neq 0} \frac{1}{4\pi^2h^2} e(hx)
\end{equation}

in formula (4.26). Let

\begin{equation}
\theta(q) = (\log q)^{4 \epsilon_1 / 2}.
\end{equation}

By (4.24) and the Fourier decomposition of $B_2(x)$ (4.29), we have

\begin{equation}
\sum_{n \leq Y\theta(q), (n, q) = 1} \Delta_Y(n; q, a) = \sum_{n \leq Y\theta(q), (n, q) = 1} \Delta_Y(n; q, a) + O(Y\theta(q)^{-1})
\end{equation}

\begin{align*}
&= \sum_{h \neq 0} \frac{1}{4\pi^2h^2} \sum_{n \leq Y\theta(q), (n, q) = 1} \left( e \left( \frac{hY^2}{n^2} \right) - 1 \right) e \left( \frac{ah\tilde{n}^2}{q} \right) + O(Y\theta(q)^{-1}) \\
&= \sum_{1 \leq |h| \leq \theta(q)^3} \frac{1}{4\pi^2h^2} \sum_{n \leq Y\theta(q), (n, q) = 1} \left( e \left( \frac{hY^2}{n^2} \right) - 1 \right) e \left( \frac{ah\tilde{n}^2}{q} \right) + O(Y\theta(q)^{-1}).
\end{align*}

Summing by parts, we see that the inner sum of the right-hand side of inequality (4.31) is

\begin{align*}
\ll \sum_{Y\theta(q)^{-1} \leq m \leq Y\theta(q)} \frac{hY^2}{m^3} \left| \sum_{n \leq m, (n, q) = 1} e \left( \frac{ah\tilde{n}^2}{q} \right) \right| + \left| \sum_{Y\theta(q)^{-1} \leq n \leq Y\theta(q)} e \left( \frac{ah\tilde{n}^2}{q} \right) \right|.
\end{align*}

Now, if $q$ is prime and sufficiently large, then any integer $h$ satisfying $1 \leq |h| \leq \theta(q)^3$ is coprime with $q$. Then, by (4.27), the above expression is

\begin{align*}
\ll \sum_{Y\theta(q)^{-1} \leq m \leq Y\theta(q)} \frac{|h|Y^2}{m^2} (\log q)^{-\delta_1} + Y\theta(q)^{-1} \\
\ll |h|Y\theta(q)^{-1}.
\end{align*}

As we insert the upper-bound (4.32) in formula (4.31), we obtain
For the remainder terms we use the trivial upper bound (4.24) to deduce the inequality
\[ \sum_{n > Y^{\theta}(q), (n, q) = 1} \Delta_Y (n; q, a) \ll \sum_{n > Y^{\theta}(q)} \frac{Y^2}{n^2} \ll Y^{\theta}(q)^{-1}. \] (4.34)

We combine the upper bounds (4.33) and (4.34) to conclude. Together they give
\[ \sum_{n \geq 1} \Delta_Y (n; q, a) \ll Y (\log q)^{-\delta + 1/4}. \]

Remark 4.9. Among the hypothesis of lemma (4.8), it is essential that we have \((a, q) = 1\). In the case where \(q \mid a\), one can not improve on (4.25). Indeed, it is possible to show that (see [6, lemma 4.3])
\[ A(Y; q, 0) = -\frac{\varphi(q) \zeta(3/2)}{2\pi} Y + O(d(q)Y^{2/3}) \quad (Y \geq 1). \]

4.2. A consequence of Lemma 4.8.
In order to evaluate \(S[\gamma](X, q)\) (see (3.4)), it is important to consider the following sum which appears in equation (4.33).

Definition 4.1. For \(q, r, s\) integers satisfying \(q \geq 1\), \(q \nmid rs\), let
\[ \mathcal{S}[\gamma](X, q) := \sum_{\ell \equiv s \pmod{q}} f(\ell, r)|I(X, \ell, r)|, \] (4.35)
where \(\gamma = \gamma_{r, s}\).

The purpose of this subsection is to prove the following

Proposition 4.10. There exists \(\delta > 0\), such that for every \(\epsilon > 0\), for every \(r \neq 0\) squarefree, one has
\[ \mathcal{S}[\gamma](X, q) = \left( \frac{6}{\pi^2} \right)^2 \left( 1 + \frac{1}{q^2(q^2-2)} \right)^{-1} X^2/q + O_{\epsilon, r}(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta}), \]
uniformly for \(X > 1\), \(s\) integer and \(q\) prime such that \(q \nmid rs\), with \(C(q)\) as in (3.2).

The special case \(r = 1\) simplifies many of the calculations in the proof below. For instance, the sums over \(\rho, \sigma\) and \(\tau\) disappear. Although, this simpler result is, in fact, equally deep and it shows more clearly the connection between the upper bound (4.26) and the error term in (4.30).

Proof. We start by recalling (4.9)
\[ f(\ell, r) = C_2 \prod_{p|\ell} \left( \frac{p^2-1}{p^2-2} \right) \prod_{p|\ell} \left( \frac{p^2-1}{p^2-2} \right) \kappa((\ell, r^2)), \]
where \(C_2\) is as in (2.4). We notice that the first and second terms on the right-hand side of equation above are independent of \(\ell\), that means that in order to evaluate \(\mathcal{S}[\gamma](X, q)\), we need to study
\[ S'[\gamma](X, q) = \sum_{\ell \equiv s \pmod{q}} |I(X, \ell, r)| \prod_{p^2|\ell \land p \nmid r} \left( \frac{p^2 - 1}{p^2 - 2} \right) \kappa((\ell, r^2)). \]

i.e.

\[ S'[\gamma](X, q) = C_2^{-1} \prod_{p|r} \left( \frac{p^2 - 1}{p^2 - 2} \right)^{-1} S[\gamma](X, q). \] (4.37)

We expand the product \( \prod_{p^2|\ell \land p \nmid r} \left( \frac{p^2 - 1}{p^2 - 2} \right) \) as

\[ \prod_{p^2|\ell \land p \nmid r} \left( \frac{p^2 - 1}{p^2 - 2} \right) = \sum_{d|\ell \land (d, r) = 1} h(d) \frac{d^2}{d^2}, \]

from which we deduce

\[ S'[\gamma](X, q) := \sum_{\rho|\tau^2} \kappa(\rho) \sum_{\ell \equiv s \pmod{q}} |I(X, \ell, r)| \sum_{d^2|\ell \land (d, r) = 1} h(d) \frac{d^2}{d^2} \]

\[ = \sum_{\rho|\tau^2} \kappa(\rho) \mu(\sigma) \sum_{\ell_0 \equiv s \pmod{q}} |I(X, \rho \sigma \ell_0, r)| \sum_{d^2|\ell_0 \land (d, r) = 1} h(d) \frac{d^2}{d^2} \]

\[ = \sum_{\rho|\tau^2} \kappa(\rho) \mu(\sigma) \sum_{(d, q) = 1} h(d) \frac{d^2}{d^2} \sum_{\ell_1 \equiv \rho \sigma d^2 \pmod{q}} |I(X, \rho \sigma d^2 \ell_1, r)| \]

where in the second line we used M"obius inversion formula for detecting the gcd condition and we noticed that the congruence satisfied by \( \ell_0 \) implies \( (d, q) = 1 \).

We write the inner sum as an integral:

\[ \sum_{\ell_1 \equiv \rho \sigma d^2 \pmod{q}} |I(X, \rho \sigma d^2 \ell_1, r)| = \int_0^X \sum_{\ell_1 \equiv \rho \sigma d^2 \pmod{q}} 1_{(0, X)}(ru + \rho \sigma d^2 \ell_1) du, \] (4.39)

where \( 1_{(0, X)} \) is the characteristic function of the interval \((0, X)\). Hence the inner sum above equals

\[ \left| \frac{X - ru}{\rho \sigma d^2 q} - \frac{(\rho \sigma d^2)s}{q} \right| - \left| \frac{-ru}{\rho \sigma d^2 q} - \frac{(\rho \sigma d^2)s}{q} \right| = \frac{X}{\rho \sigma d^2 q} - B_1 \left( \frac{X - ru}{\rho \sigma d^2 q} - \frac{(\rho \sigma d^2)s}{q} \right) + B_1 \left( \frac{-ru}{\rho \sigma d^2 q} - \frac{(\rho \sigma d^2)s}{q} \right), \]

for almost all \( u \in (0, X) \) in the sense of Lebesgue measure.

If we use this formula in equation (4.39), we deduce the equality.
From this point on, we suppose \( r < 0 \). The case \( r > 0 \) requires only minor modifications. With this hypothesis, we have that both
\[
(1-r)X - \rho \sigma d^2 q
\]
and
\[
-rX - \rho \sigma d^2 q
\]
are positive for every \( \rho, \sigma \geq 1 \).

We inject (4.40) above in equation (4.38) and we define
\[
B(D; q, a; r) := \sum_{(d, qr) = 1} h(d) \Delta_D(d, q; a),
\]
where \( \Delta_D(d, q; a) \) is as in (4.28). From (4.38) and (4.40) we deduce the equality
\[
(4.41) \quad \mathcal{G}'[\gamma](X, q) = \lambda(q, r) \frac{X^2}{q} - \frac{\rho \sigma d^2 q}{r} \left\{ B_2 \left( \frac{X^2}{\rho \sigma d^2 q} - \frac{\rho \sigma d^2 s}{q} \right) - B_2 \left( \frac{\rho \sigma d^2 s}{q} \right) 
\right. 

\left. - B_2 \left( \frac{(1-r)X}{\rho \sigma d^2 q} - \frac{\rho \sigma d^2 s}{q} \right) + B_2 \left( \frac{-rX}{\rho \sigma d^2 q} - \frac{\rho \sigma d^2 s}{q} \right) \right\} ,
\]
where
\[
G(Y; q, s; r) = \sum_{\rho \sigma | r^2} \kappa(\rho) \mu(\sigma) \rho \sigma B \left( \sqrt[2]{\frac{Y}{\rho \sigma}}, q, \rho \sigma s; r \right),
\]
and
\[
\lambda(q, r) = \sum_{\rho \sigma | r^2} \kappa(\rho) \mu(\sigma) \rho \sigma + \sum_{(d, qr) = 1} h(d) d^4.
\]

Returning to the function \( \beta(m) \) defined in Lemma 4.1, we observe that for a general \( D > 0 \), one has
\[
B(D; q, a; r) = \sum_{(m, qr) = 1} \beta(m) \sum_{(n, qr) = 1} \Delta_D(mn; q, a)
\]
\[
= \sum_{(m, qr) = 1} \beta(m) \sum_{(n, qr) = 1} \Delta_D/m(n; q, m^2 a)
\]
\[
= \sum_{(m, qr) = 1} \beta(m) \sum_{\tau | r} \mu(\tau) \sum_{(n, q) = 1} \Delta_D/\tau m(n; q, \tau^2 m^2 a)
\]
\[
= \sum_{(m, qr) = 1} \beta(m) \sum_{\tau | r} \mu(\tau) A(D/\tau m, q, \tau^2 m^2 a).
\]

We apply the equality above with \( D = \sqrt{\frac{Y}{\rho \sigma}} \) and \( a = \rho \sigma s \), multiply by \( \kappa(\rho) \mu(\sigma) \rho \sigma \) and sum over \( \rho, \sigma \) such that \( \rho \sigma | r^2 \), we have
\[
(4.42) \quad G(Y; q, s; r) = \sum_{\rho \sigma | r^2} \sum_{\tau | r} \sum_{(m, qr) = 1} \kappa(\rho) \mu(\sigma) \mu(\tau) \rho \sigma \beta(m) A \left( \sqrt[2]{\frac{Y}{\rho \sigma \tau^2 m^2}}, q, \rho \sigma \tau^2 m^2 s \right).
\]
Our discussion depends on the size of $Y$.

- If $Y \leq q^\varepsilon$, we have the trivial bound (see (4.23))

$$A\left(\sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}}, q, \rho \sigma \tau^2 m^2 \right) \ll \sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}} \leq \frac{Y^{1/2}}{m},$$

for every $\rho, \sigma, \tau \geq 1$. Summing over $\rho, \sigma, \tau$ and $m$, it gives

$$G(Y; q, s; r) \ll r \frac{Y^{1/2}}{m} \sum_{m \geq 1} \frac{\beta(m)}{m} \ll q^{\varepsilon/2},$$

(4.43)
as a consequence of upper bound (4.1).

- If $Y > q^\varepsilon$, we separate the quadruple sum on the right-hand side of (4.42) as

$$\sum_{\rho \sigma \tau^2 m^2 \leq Y/q^\varepsilon} \sum_{\rho \sigma \tau^2 m^2 \leq Y/q^\varepsilon} \sum_{\rho \sigma \tau^2 m^2 > Y/q^\varepsilon} \sum_{\rho \sigma \tau^2 m^2 > Y/q^\varepsilon} \sum.$$  

For the first sum we have, again, the trivial bound

$$A\left(\sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}}, q, \rho \sigma \tau^2 m^2 \right) \ll \sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}} \leq q^{\varepsilon/2},$$

(4.44)
The most delicate sum is the second one, since we appeal to (4.26). This gives

$$A\left(\sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}}, q, \rho \sigma \tau^2 m^2 \right) \ll \sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}} (\log q)^{-\delta}.$$

(4.45)
For the third one, we use the trivial bound,

$$A\left(\sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}}, q, \rho \sigma \tau^2 m^2 \right) \ll \sqrt{\frac{Y}{\rho \sigma \tau^2 m^2}},$$

(4.46)

Gathering the inequalities (4.44), (4.45) and (4.46) in (4.42), we obtain

$$G(Y; q, s; r) \ll_{r, \varepsilon} q^{\varepsilon/2} \sum_{m \leq q^{\varepsilon/2}} |\beta(m)| + \sqrt{Y} (\log q)^{-\delta} \sum_{m \leq q^{\varepsilon/2}} \frac{|\beta(m)|}{m} + \sqrt{Y} \sum_{m > q^{\varepsilon/2}} \frac{|\beta(m)|}{m},$$

and finally, by Lemma 4.4

$$G(Y; q, s; r) \ll_{r, \varepsilon} q^{\varepsilon} + \sqrt{Y} (\log q)^{-\delta} \quad (Y > q^\varepsilon).$$

Comparing with (4.43), we have that (4.47) is true for any $Y \geq 1$.

Combining (4.47) and (4.11), one has

$$\Theta'[\gamma](X, q) = \lambda(q, r) \frac{X^2}{q} + O_{r, \varepsilon} (q^{1+\varepsilon} + X^{1/2} q^{1/2} (\log q)^{-\delta}).$$

(4.48)
If we multiply the formula above by $C_2 \prod_{p \mid r} \left(\frac{p^2 - 1}{p^2 - 2}\right)$ (recall formula (4.37)), we deduce

$$\Theta[\gamma](X, q) = A(q, r) \frac{X^2}{q} + O_{r, \varepsilon} (q^{1+\varepsilon} + X^{1/2} q^{1/2} (\log q)^{-\delta}),$$

(4.49)
where
\[
\Lambda(q, r) = C_2 \prod_{p|r} \left( \frac{p^2 - 1}{p^2 - 2} \right) \sum_{\rho \sigma | r^2} \frac{\kappa(\rho)\mu(\sigma)}{\rho \sigma} \sum_{(d, qr) = 1} \frac{h(d)}{d^4}
\]

Since for \( r \) squarefree, we have the equality
\[
\sum_{\rho \sigma | r^2} \frac{\kappa(\rho)\mu(\sigma)}{\rho \sigma} = \prod_{p|r} \left( \frac{p^2 - 1}{p^2 - 2} \right),
\]
then, by some standard calculations, we notice that \( \Lambda(q, r) \) does not depend on \( r \). More precisely, since, \( q \) is prime and \((q, r) = 1\), we have
\[
\Lambda(q, r) = \left( \frac{6}{\pi^2} \right)^2 \left( 1 + \frac{1}{q^2(q^2 - 2)} \right)^{-1}.
\]
As a consequence, formula (4.49) completes the proof of Proposition 4.10.

5. Study of \( S[\gamma](X, q) \)

We rewrite \( S[\gamma](X, q) \) (see (3.31)) as
\[
(5.1) \quad S[\gamma](X, q) = \sum_{\ell \equiv s (\text{mod } q)} \sum_{n \in I(X, \ell, r)} \mu^2(n)\mu^2(\frac{rn + \ell}{q}).
\]
First we notice that the inner sum equals zero if \(|\ell| > 2|q|X\). Hence, by formula (4.8), we have that
\[
(5.2) \quad S[\gamma](X, q) = \sum_{\ell \equiv s (\text{mod } q)} \left| f(\ell, r)I(X, \ell, r) \right| + O_r \left( \frac{X^2}{q} X^{2/3+\varepsilon} \right),
\]
for \( X \geq q \). We notice that if \(|\ell| > 2|q|X\), one also has \( |I(X, \ell, r)| = 0 \), hence we can complete the sum on the right-hand side of (5.2). Thus, we can write (recall definition (4.35))
\[
S[\gamma](X, q) = \mathfrak{S}[\gamma](X, q) + O_r \left( \frac{X^{5/3+\varepsilon}}{q} \right).
\]
From Proposition 4.10, we deduce the equality
\[
S[\gamma](X, q) = \left( \frac{6}{\pi^2} \right)^2 \left( 1 + \frac{1}{q^2(q^2 - 2)} \right)^{-1} X^2 \frac{X^2}{q} + O_r \left( \frac{X^{1+\varepsilon} + X^{1/2} q^{1/2} (\log q)^{-\delta} + X^{5/3+\varepsilon}}{q} \right).
\]
In view of the definition (3.2) of \( C(q) \), it is easy to see that
\[
\left( \frac{6}{\pi^2} \right)^2 \left( 1 + \frac{1}{q^2(q^2 - 2)} \right)^{-1} = C(q)^2 + O \left( \frac{1}{q^2} \right).
\]
In conclusion, we proved

**Proposition 5.1.** There exists \( \delta > 0 \) such that for every \( \varepsilon > 0 \) and every \( r \neq 0 \), one has the asymptotic formula
\[
(5.3) \quad S[\gamma, s](X, q) = C(q)^2 \frac{X^2}{q} + O_{r, \varepsilon} \left( \frac{X^{1+\varepsilon} + X^{1/2} q^{1/2} (\log q)^{-\delta} + X^{5/3+\varepsilon}}{q} + \frac{X^2}{q^3} \right),
\]
uniformly for \( X \geq 2 \), for every integer \( s \) and for any prime \( q \) such that \( q \nmid rs \) and \( q \leq X \).
6. Proof of the main Theorem

We start by recalling the formula (3.3)

\[ C[\gamma](X, q) = S[\gamma](X, q) - 2C(q) \frac{X}{q} \sum_{n \leq X} \mu^2(n) + C(q)^2 \frac{X^2}{q} + O \left( \frac{X}{q^2} \right). \]

By Proposition 5.1 and formula (3.3), we directly obtain the equality

\[ C[\gamma](X, q) = \mathcal{O}_{\varepsilon, r} \left( q^{1+\varepsilon} + X^{1/2} q^{1/2} (\log q)^{-\delta} + \frac{X^{5/3+\varepsilon}}{q} + \frac{X^2}{q^2} \right). \]

The proof of Theorem 1.2 is now complete.

References

[1] V. Blomer: *The average value of divisor sums in arithmetic progressions*, Quart. J. Math. 59 (2007), 275-286.
[2] J. Bourgain: *A remark on solutions of the Pell equation*, to appear, Int. Math. Res. Not.
[3] É. Fouvry: *On the size of the fundamental solution of Pell equation*, to appear, J. Reine Angew. Math. (2014)
[4] É. Fouvry, S. Ganguly, E. Kowalski and Ph. Michel: *Gaussian distribution for the divisor function and Hecke eigenvalues in arithmetic progression*, to appear, Commentarii Mathematici Helvetici.
[5] T. Reuss: *Pairs of k-free Numbers, consecutive square-full Numbers* (arXiv:1212.3150v1 [math.NT])
[6] R. M. Nunes: *Squarefree numbers in arithmetic progressions*, (arXiv:1402.0684v2 [math.NT])
[7] K.-M. Tsang: *The distribution of r-tuples of square-free numbers* Mathematika 32 (1985), 265-275

Université Paris Sud, Laboratoire de mathématiques, Campus d’Orsay, 91405 Orsay Cedex, France
E-mail address: ramon.moreira@math.u-psud.fr