Hall viscosity and nonlocal conductivity of “gapped graphene”

Mohammad Sherafati and Giovanni Vignale∗

Department of Physics & Astronomy, University of Missouri, Columbia, Missouri 65211, USA

(Dated: February 28, 2018)

Abstract

We calculate the Hall viscosity and the nonlocal (i.e., dependent on wave vector \( q \)) Hall conductivity of “gapped graphene” (a non-topological insulator with two valleys) in the presence of a strong perpendicular magnetic field. Although the final formulas are similar to the ones previously obtained for gapless graphene, the derivation reveals a significant difference between the two systems. First of all, both the Hall viscosity and the Hall conductivity vanish when the Fermi level lies in the gap that separates the lowest Landau level in the conduction band from the highest Landau level in the valence band. It is only when the Fermi level is not in the gap that the familiar formulas of gapless graphene are recovered. Second, in the case of gapped graphene, it is not possible (at least within our present approach) to define a single-valley Hall viscosity: this quantity diverges with a strength proportional to the magnitude of the gap. It is only when both valleys are included that the diverging terms, having opposite signs in the two valleys, cancel out and the familiar result is recovered. In contrast to this, the nonlocal Hall conductivity is finite in each valley. These results indicate that the Hoyos-Son formula connecting the Hall viscosity to the coefficient of \( q^2 \) in the small-\( q \) expansion of the \( q \)-dependent Hall conductivity cannot be applied to each valley, but only to the system as a whole. The problem of defining a “valley Hall viscosity” remains open.

PACS numbers: 73.43.Cd, 72.80.Vp, 66.20.Cy, 71.70.Di
I. INTRODUCTION

Graphene-based materials have attracted much attention because of their intriguing electronic properties\(^1\). As long as both time-reversal and inversion symmetries are respected, graphene remains a gapless system with massless quasiparticles. Addition of a “mass” term to the Hamiltonian of pristine graphene creates a gap in the energy spectrum. There are 36 possible gap-opening instabilities in graphene associated with spin, valley, and superconducting channels\(^2\). In particular, the addition of the so-called “Semenoff mass”\(^3\) \(\Delta \hat{\sigma}_z\) where \(\Delta\) is the strength of the staggered onsite potential between \(A\)– and \(B\)–sublattices and \(\hat{\sigma}_z\) is the \(z\)-component of the pseudospin operator has widely been investigated. Semenoff mass opens a band gap of \(2\Delta\) at the Dirac points while preserving the time-reversal symmetry (in the absence of a magnetic field) and breaking the inversion symmetry of pristine graphene. Such a gapped graphene system with broken inversion symmetry can be realized by growing graphene on substrates such as SiC\(^4\) and hexagonal boron-nitride (BN)\(^5\). In addition, one can show that the system is non-topological with vanishing Chern number when both valleys are considered.

Gapped graphene is host to a unique valley Hall effect even in the absence of a magnetic field\(^6\). Similar to the gapless graphene in the presence of a perpendicular external magnetic field\(^7\), the broken time-reversal symmetry in gapped graphene leads to the formation of Landau levels (LLs)\(^8\) with respective wave functions whose two-component spinor nature have recently been confirmed\(^9\). In addition, ultraclean samples of graphene have also been shown to support a hydrodynamic regime for the electron flow\(^10\), where the resistance arises from the viscosity when adjacent parts of the fluid move with different velocities. Being the coefficient that controls the transport of momentum in a fluid, viscosity is a fourth-rank tensor that connects the stress tensor with the rate of change of the strain tensor according to the formula

\[
P_{ij} = \sum_{kl} \eta_{ijkl} v_{kl},
\]

where \(i, j, k, l\) are cartesian indices, \(v_{kl} = \frac{1}{2}(\partial_k v_l + \partial_l v_k)\) is the symmetrized gradient of the velocity field \(\mathbf{v}\), and the stress tensor \(P_{ij}\) is obtained from the derivative of the Hamiltonian with respect to the metric tensor. The viscosity tensor in \(d = 2\) dimensions is then given by

\[
\eta_{ijkl} = \zeta \delta_{ij} \delta_{kl} + \eta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl})
+ \frac{1}{2} \eta_H (\epsilon_{ik} \delta_{jl} + \epsilon_{il} \delta_{jk} + \epsilon_{jk} \delta_{il} + \epsilon_{jl} \delta_{ik}),
\]

where \(\epsilon_{ij}\) is the rank-2 Levi-Civita tensor and \(\eta\) and \(\zeta\) are the two transport coefficients, the shear and the bulk viscosities, respectively, which are both dissipative in rotationally-invariant systems with preserved time-
reversal symmetry. The coefficient in the third term denoted by \( \eta_H = \eta_{xx,xy} \), produces a force density

\[
\mathbf{f} = \eta_H \nabla^2 \mathbf{v} \times \hat{z},
\]

where \( \hat{z} \) is the unit vector perpendicular to the plane of the two-dimensional fluid. Therefore, \( \eta_H \) is a non-dissipative component of the viscosity tensor for any fluid. As a first attempt for an electron fluid, Avron et al. obtained an expression for \( \eta_H \) for a two-dimensional electron gas (2DEG) in a perpendicular magnetic field. They revealed that \( \eta_H \) is proportional to the filling factor of the LLs, suggesting a connection with topology. Later, \( \eta_H \) was coined as the Hall viscosity, denoted by \( \eta_H \), and shown to have a topological significance as, for a rotationally-invariant system, it is proportional to average orbital spin known as Wen-Zee shift, a topological feature a quantum Hall system on a curved space possesses, which stems from a “gravitational coupling” to the intrinsic geometry of the real-space manifold on which the electrons reside. This topological quantity is represented as an excess carrier number density that can be measured. The relationship between the Hall viscosity and the Wen-Zee shift suggests an interesting possibility to experimentally determine the Hall viscosity.

Interestingly, such relationship has recently been verified for a synthetic photonic quantum Hall system put on the tip of a cone.

Over the past decade, the Hall viscosity has been studied for various quantum Hall systems. It was pointed out by Hoyos and Son that for Galilean-invariant systems, the Hall viscosity can be extracted from the \( q^2 \) term in the expansion of the nonlocal Hall conductivity in powers of wave vector \( q \), which suggests an experimental access to the Hall viscosity from the electromagnetic response of the electronic quantum Hall system. Due to its two-dimensional structure and the possibility to place it on a curved surface, graphene is an excellent material for the experimental realization of the Hall viscosity for an electronic quantum Hall system. Recently, Sherafati et al. demonstrated that a relation similar to the one derived by Hoyos and Son holds between the Hall viscosity and the Hall conductivity in monolayer graphene, in spite of the absence of the Galilean invariance. They also obtained an analytical expression for the Hall viscosity in a continuum model of graphene using the linear-response formulation of the Hall viscosity suggested by Tokatly and Vignale.

In brief, the Hall viscosity, \( \eta_H = \eta_{xx,xy} \), is an instance of a class of “anomalous transport coefficients” – of which the Hall conductivity is the best known example – which are given by the imaginary part of an off-diagonal linear response function, in this case

\[
\eta_H = \lim_{\omega \to 0} \Im \langle \langle P_{xx}; P_{xy} \rangle \rangle_{\omega},
\]
FIG. 1. Schematic diagram of the Landau levels in gapless graphene. The highest occupied and the lowest empty levels are labeled as $N_L - 1$ and $N_L$, respectively, compared to the Fermi level which lies within the gap between these two levels. Here for definiteness we assume electron doping, i.e., $N_L \geq 1$. However, the final formulas will be given in a form that is invariant under the electron-hole transformation $N_L \rightarrow -N_L + 1$.

where $\langle\langle P_{xx}; P_{xy}\rangle\rangle_\omega$ is a shorthand for the off-diagonal stress-stress response function.

Starting from the Kubo formula of Eq. (4) we obtained\textsuperscript{18}

\begin{equation}
\eta_H = g_{sv} \frac{\hbar}{4\pi\ell^2} \left[ N_L^2 + (N_L - 1)^2 \right] \text{sgn} \left( N_L - \frac{1}{2} \right),
\end{equation}

($g_{sv} = 4$ for graphene) where the Fermi level falls in a gap between the LLs with indices $N_L - 1$ and $N_L$ (see Fig. 1). $N_L$ can be zero or negative, corresponding to the possibility of hole doping\textsuperscript{1} and the result exhibits full electron-hole antisymmetry, that is to say, the viscosity changes sign under the transformation $N_L \rightarrow -N_L + 1$.

In addition, we found that the expansion of the $q$-dependent Hall conductivity in graphene can be cast in a form very similar to that of found by Hoyos and Son\textsuperscript{17} for 2DEG, namely

\begin{equation}
\sigma_{xy}(q) \approx g_{sv} \left( N_L - \frac{1}{2} \right) \frac{e^2}{\hbar} \left\{ 1 + q^2 \ell^2 \left[ \frac{|\eta_H|}{4\hbar n} - \left| N_L - \frac{1}{2} \right| \right] \right\},
\end{equation}

where $n = g_{sv} \frac{|N_L-1/2|}{2\pi\ell^2}$ is the carrier density (electron density for $N_L \geq 1$, hole density for $N_L \leq 0$). Furthermore, we will show that the second term in Eq (6) retains the physical interpretation proposed by Hoyos and Son, i.e., can be expressed in terms of the orbital magnetic susceptibility with the appropriate effective mass.
The conclusion is that electrons in doped graphene behave (not unexpectedly) like a Galilean-invariant 2DEG with an effective mass $m_c$ which is given by

$$m_c \equiv \frac{\hbar k_F}{v_F} = \frac{\hbar \sqrt{2|N_L - 1/2|}}{v_F \ell}. \quad (7)$$

The goal of this work is to extend the calculation of the Hall viscosity and nonlocal Hall conductivity to the case of “gapped graphene” (defined in the next section) in a strong magnetic field, and to obtain expressions for the Hall viscosity and Hall conductivity of this system. Our main results can be summarized as follows:

1. Both the Hall viscosity and the Hall conductivity vanish when the Fermi level lies in the gap that separates the lowest Landau level in the conduction band from the highest Landau level in the valence band.

2. When the Fermi level is not in the gap, the familiar formulas of gapless graphene are recovered.

3. It is not possible, within our present approach, to define a single-valley Hall viscosity: this quantity diverges with a strength proportional to the magnitude of the gap. It is only when both valleys are included that the diverging terms, having opposite signs in the two valleys, cancel out and the familiar result is recovered.

4. In contrast to the above, the nonlocal Hall conductivity is finite in each valley. These results indicate that the Hoyos-Son formula connecting the Hall viscosity to the coefficient of $q^2$ in the small-q expansion of the q-dependent Hall conductivity cannot be applied to each valley, but only to the system as a whole.

5. The problem of defining a “valley Hall viscosity” remains open.

II. MODEL AND FORMULATION

In this work we consider graphene in the presence of a perpendicular magnetic field being gapped due to a Semenoff mass term added to the low-energy Hamiltonian of both valleys. The single-particle Hamiltonian for small momenta $p$ in $K$- or $K'$-valley is given by

$$\hat{H}_0 = \begin{cases} 
  v_F \hat{\Pi} \cdot \hat{\sigma} + \Delta \hat{\sigma}_z & \text{For } K\text{-valley} \\
  v_F \hat{\Pi} \cdot \hat{\sigma}^* + \Delta \hat{\sigma}_z & \text{For } K'\text{-valley,}
\end{cases} \quad (8)$$
where $\hat{\Pi} = \hat{p} + \frac{\xi}{e} A(\hat{r})$ is the kinetic momentum operator with vector potential $A = Bx\hat{y}$ in the Landau gauge corresponding to a perpendicular magnetic field $B = B\hat{z}$. The pseudospin operators for $K$- and $K'$-valley operators are denoted by $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y)$ and $\hat{\sigma}^* = (-\hat{\sigma}_x, \hat{\sigma}_y)$, respectively.

The Landau levels (LLs) associated with the $K$-valley Hamiltonian in Eq. (8) are labeled by an integer index $n = \ldots, -2, -1, 0, 1, 2, \ldots$ including the Zeroth-Landau Level (ZLL) and they are characterized by the following eigenstates and energies for the valence ($\lambda = 1$) and conduction ($\lambda = -1$) bands:

$$E^K_{\lambda, n} = \lambda E_n \equiv \lambda \hbar \omega_0 \sqrt{\gamma^2 + n}, \quad |\lambda; n, k_y\rangle_K = \begin{pmatrix} \lambda a_{\lambda, n} |n - 1, k_y\rangle \\ a_{-\lambda, n} |n, k_y\rangle \end{pmatrix}, \quad (n \geq 1)$$

$$E^K_{\text{ZLL}} (n = 0) = -E_0 \equiv -\Delta, \quad |1, 0, k_y\rangle_K = \begin{pmatrix} 0 \\ |0, k_y\rangle \end{pmatrix}$$

where the magnetic length associated with the magnetic field $B$ is given by $\ell = \sqrt{\hbar c/eB}$ with characteristic frequency $\omega_0 = \sqrt{2v_F/\ell}$ and $\psi_{n, k_y}(x, y) = \langle r|n, k_y\rangle = (2^n n! \sqrt{\pi \ell})^{-1/2} e^{ik_y y} H_n \left(\frac{x}{\ell} + k_y \ell\right) e^{-\frac{(x + ik_y \ell)^2}{2\ell^2}}$ are the Landau-gauge wave functions with $H_n(x)$ being the $n$-th order Hermite polynomial. We have used the following definitions:

$$\gamma = \frac{\Delta}{\hbar \omega_0}$$

$$a_{\lambda, n} = \sqrt{\frac{1}{2} \left(1 + \lambda \frac{\Delta}{E_n}\right)} = \frac{\lambda \gamma}{\sqrt{2\gamma^2 + n}}$$

(10)

We note that the amplitude $a_{\lambda, n}$ can also uniquely identify ZLL wavefunction as $a_{-1,0} = 0$ and $a_{1,0} = 1$. The degeneracy per unit area for each Landau level is $(2\pi\ell^2)^{-1}$.

For $K'$-valley the LLs and their corresponding energies are given by:

$$E^{K'}_{\lambda, n} = E^K_{\lambda, n} \equiv \lambda \hbar \omega_0 \sqrt{\gamma^2 + n}, \quad |\lambda; n, k_y\rangle_{K'} = \begin{pmatrix} a_{\lambda, n} |n, k_y\rangle \\ -\lambda a_{-\lambda, n} |n - 1, k_y\rangle \end{pmatrix}, \quad (n \geq 1)$$

$$E^{K'}_{\text{ZLL}} (n = 0) = E_0 \equiv \Delta, \quad |1, 0, k_y\rangle_{K'} = \begin{pmatrix} 0, k_y\rangle \\ 0 \end{pmatrix}$$

(11)

Similarly, the amplitudes $a_{1,0} = 1$ and $a_{-1,0} = 0$ uniquely specify ZLL wavefunction in this valley.
FIG. 2. Schematic diagram of the Landau levels in “gapped graphene”. There are two valleys \( K \) and \( K' \). The “zero-th” Landau level is at the top of the valence band in the \( K \) valley and at the bottom of the conduction band in the \( K' \) valley. The lowest unoccupied Landau level in the \( K \) valley is labeled as \( N_L \). This coincides with the lowest unoccupied level in the \( K' \) valley except when the Fermi level is in the gap \(-\Delta < E_F < \Delta\), in which case \( N_L = 1 \) for \( K \)-valley and \( N_L = 0 \) for the \( K' \)-valley. Here for definiteness we assume electron doping, i.e., \( N_L \geq 1 \) (for the \( K \)-valley). However, the final formulas will be given in a form that is invariant under the electron-hole transformation \( N_L \to -N_L + 1 \).

A. Hall viscosity

The geometric Hall viscosity can be expressed in terms of the matrix elements of the stress tensor

\[
\eta_H = \lim_{\omega \to 0} \Im \langle \langle P_{xx}; P_{xy} \rangle \rangle, \tag{12}
\]

where \( \langle \langle P_{xx}; P_{xy} \rangle \rangle \) is a shorthand for the off-diagonal stress-stress response function. The stress tensor operator is defined as

\[
\hat{P}_{ij}[g^{ij}] = \frac{2}{\sqrt{g}} \frac{\partial \hat{H}}{\partial g^{ij}}, \tag{13}
\]

where \( \hat{H} \) is the many-body Hamiltonian and \( g \) is the determinant of the metric tensor. Evaluating the derivative and setting \( g^{ij} = \delta_{ij} \) we arrive at the Euclidean stress tensor

\[
\hat{P}_{ij} = \begin{cases} 
    v_F \sum \left( \hat{\Pi}_i \sigma_j + \sigma_i \hat{\Pi}_j \right) & \text{For } K \text{-valley} \\
    v_F \sum \left( \hat{\Pi}_i \sigma_j^* + \sigma^*_i \hat{\Pi}_j \right) & \text{For } K' \text{-valley},
\end{cases} \tag{14}
\]
where summation runs over all noninteracting electrons. From this point on, the metrics will be fixed to Euclidean. The single-particle states are two-component pseudo-spinors of the form presented in Eqs. (9) and (11).

Plugging Eq. (14) into Eq. (4) for the matrix elements of the stress tensor between the LLs we obtain the central starting expression for the Hall viscosity in terms of these matrix elements which reads

$$\eta_H(\omega) = \frac{2}{\hbar A} \sum_{l,k} \Im \left( \frac{[\hat{P}_{xx}]_{kl}[\hat{P}_{xy}]_{lk}}{\omega^2 - \omega_{lk}^2} \right)$$

(15)

where \(A\) is the area of graphene, \(\omega_{lk} = \frac{\hbar}{2} (E_{\text{occ},l}^K - E_{\text{occ},k}^K)\) for the occupied LLs labeled by \(k\) and empty levels labeled by \(l\).

We note that for a typical experimentally accessible strength of magnetic fields of \(B = 10\) T the magnetic length \(\ell \approx \frac{257[\text{Å}]}{\sqrt{\text{Tesla}}}\) is approximated to be \(\ell \approx 81\ \text{Å} \gg a\) where \(a \approx 1.42\ \text{Å}\) is the carbon-carbon bond length. Therefore, the present continuum model for gapped graphene electrons in magnetic field is legitimate. In addition, for \(B = 10\) T, we obtain \(\hbar\omega_0 = 0.61\) eV, which corresponds to a temperature \(T = \frac{\hbar\omega_0}{k_B} \approx 7100\) K, while the Zeeman splitting temperature, \(2\mu_B B/k_B \approx 13\) K is relatively negligible. Finally, for a graphene epitaxially grown on SiC substrate, a gap of \(\Delta \approx 0.26\) eV is produced. Therefore, in our continuum model we have \(\gamma = \frac{\Delta}{\hbar\omega_0} < 1\).

B. Nonlocal Hall conductivity

The nonlocal Hall conductivity \(\sigma_{xy}(q, \omega)\) can be expressed for both valleys in terms of the current-current response function \(\chi_{xy}(q, \omega)\) corresponding to that valley, namely

$$\sigma_{xy}(q, \omega) = -\frac{1}{\omega} \Im [\chi_{xy}(q, \omega)]$$

(16)

The response function is given by

$$\chi_{xy}(q, \omega) = \frac{1}{A} \sum_{\lambda, \lambda', n, n', k_y, k'_y} \frac{f(E_{\lambda,n}) - f(E_{\lambda',n'})}{\hbar\omega + E_{\lambda,n} - E_{\lambda',n'} + i\hbar 0^+} \langle \lambda'; n', k_y | \hat{j}_x(q) | \lambda; n, k_y \rangle \langle \lambda; n, k_y | \hat{j}_y(-q) | \lambda'; n', k'_y \rangle$$

(17)
where the wave vector is chosen to be along the $y$-direction ($\mathbf{q} = q\hat{y}$), $f(E_{\lambda,n})$ is the Fermi distribution function, and the current density operator is given by

$$\hat{j}(q) = \begin{cases} -ev_F\hat{\sigma}e^{-iq\hat{r}} & \text{For } K\text{-valley} \\ -ev_F\hat{\sigma}^*e^{-iq\hat{r}} & \text{For } K'\text{-valley}. \end{cases}$$

(18)

III. HALL VISCOSITY FOR K-VALLEY

From Eq. (14) for the $K$-valley the matrix elements of the single-particle stress tensor are given by

$$\hat{p}^K_{xx} = \frac{\hbar}{2} \left( \hat{\Pi}_+ \hat{\sigma}_+ + \hat{\Pi}_- \hat{\sigma}_- \right) + \hat{H}_0^K - \Delta \hat{\sigma}_z$$

$$\hat{p}^K_{xy} = \frac{\hbar}{2i} \left( \hat{\Pi}_+ \hat{\sigma}_- - \hat{\Pi}_- \hat{\sigma}_+ \right),$$

(19)

where $\hat{\sigma}_\pm \equiv \hat{\sigma}_x \pm i\hat{\sigma}_y$ and

$$\hat{\Pi}_\pm = \frac{\ell \hat{\Pi}_x \pm i\hat{\Pi}_y}{\hbar} \sqrt{2}$$

(20)

The expression for the Hall viscosity in Eq. (15) is then simplified to

$$\eta^K_H(\omega) = -\frac{\hbar\omega_0^2}{4\pi\ell^2} \sum_{k,l} \frac{\omega^2 - \omega^2_{il}}{\omega^2 - \omega^2_{lk}} \left| [\Pi_+ \sigma_+]_{lk} \right|^2 - \left| [\Pi_- \sigma_-]_{lk} \right|^2$$

(21)

A. Electron-doped case

For $E_F > -\Delta$ and having the Fermi level pinned within the gap or the two consecutive LLs, the index of the lowest unoccupied Landau level, $N_L$, starts from one ($N_L \geq 1$). The $K$-valley Hall viscosity is found by taking into account the contribution from three types of transitions:

- **Case ①**: Both empty and occupied levels belong to the positive sector of the LLs
- **Case ②**: Occupied levels belong to the negative sector and unoccupied levels belong to the positive sector
- **Case ③**: Contribution from the occupied ZLL to the unoccupied levels in the positive sector
The contribution of each case to the $K$-valley Hall viscosity is then given by:

$$\eta^K_{H,1}(\omega) = \begin{cases} 
-\frac{\hbar \omega_0^2}{\pi \ell^2} \sum_{k=\text{Max}(1,N_L-2)}^{\min(N_L-1,N_C-2)} \frac{a_{+,-,2}^2(k+1)}{\omega^2 - \omega_0^2 \left(\sqrt{\gamma^2 + k + 2} + \sqrt{\gamma^2 + k}\right)^2}, & \text{if } N_L \geq 2 \\
0, & \text{otherwise}
\end{cases}$$

where $N_C$ is the index of the highest-lying Landau level in the positive or negative sector within the linear Dirac bands for which the Hamiltonian is valid. The other two contributions are given by

$$\eta^K_{H,2}(\omega) = -\frac{\hbar \omega_0^2}{\pi \ell^2} \sum_{k=\text{Max}(1,N_L-2)}^{N_C-2} \frac{a_{+,-,2}^2(k+1)}{\omega^2 - \omega_0^2 \left(\sqrt{\gamma^2 + k + 2} + \sqrt{\gamma^2 + k}\right)^2} + \frac{\hbar \omega_0^2}{\pi \ell^2} \sum_{k=N_L}^{N_C-2} \frac{a_{+,-,2}^2(k+1)}{\omega^2 - \omega_0^2 \left(\sqrt{\gamma^2 + k + 2} + \sqrt{\gamma^2 + k}\right)^2},$$

and

$$\eta^K_{H,3}(\omega) = \begin{cases} 
-\frac{\hbar \omega_0^2}{\pi \ell^2} \frac{\sqrt{\gamma^2 + 2 + \gamma}}{2\sqrt{\gamma^2 + 2 + \gamma}} \frac{1}{\omega^2 - \omega_0^2 \left(\sqrt{\gamma^2 + 2 + \gamma}\right)^2}, & \text{if } N_L = 1, 2 \\
0, & \text{otherwise}
\end{cases}$$

The final expression of the $K$-valley Hall viscosity for $\omega = 0$ can be simplified to

$$\eta^K_H(\omega = 0) = \eta^K_{H,1}(\omega = 0) + \eta^K_{H,2}(\omega = 0) + \eta^K_{H,3}(\omega = 0) = \frac{\hbar}{4\pi \ell^2} \left[ \theta(N_L - 1) \sum_{k=\text{Max}(1,N_L-2)}^{N_L-1} F^K_1(k, \gamma) + \sum_{k=N_L}^{N_C-2} F^K_2(k, \gamma) + F^K_3(\gamma)(\delta_{N_L,1} + \delta_{N_L,2}) \right],$$

where $N_L \geq 1$ and the kernel functions corresponding to the positive, negative, and the zeroth LLs are given by:

$$F^K_1(k, \gamma) = \left[ (k+1)^2 - \frac{\gamma(k+1)}{\sqrt{\gamma^2 + k + 2}} \right],$$

$$F^K_2(k, \gamma) = \frac{2\gamma(k+1)}{\sqrt{\gamma^2 + k} \sqrt{\gamma^2 + k + 2} \left(\sqrt{\gamma^2 + k + 2} + \sqrt{\gamma^2 + k}\right)},$$

$$F^K_3(\gamma) = \frac{2}{\sqrt{\gamma^2 + 2} \left(\sqrt{\gamma^2 + 2 + \gamma}\right)}.$$
and the step function $\theta(x)$ is defined by

$$
\theta(x) = \begin{cases} 
1, & (x > 0) \\
0, & (x < 0) 
\end{cases}
$$

(27)

Three remarks are worth making here:

1. Eq. (25) is valid for the Fermi level pinned either within the band gap ($N_L = 1$) or the gap between any two consecutive LLs in the positive sector. The step function assures the vanishing contribution from the first term for $N_L = 1$.

2. For the infinite number of occupied LLs in the negative sector ($N_C \to \infty$), the second summation in Eq. (25) diverges. This can be verified by the fact that $F^K_2(k, \gamma)$ is positive and monotonically decreasing function of $k$ for any $\gamma < 1$. Then, the integral test for the convergence of series guarantees the divergence of the second summation.

3. For $\gamma = 0$, we have $F^K_2(k, 0) = 0$ and $F^K_3(0) = 1$ for which we will eventually recover the result for the $K$-valley Hall viscosity of the gapless graphene as reported in Eq. (6) of our paper.\textsuperscript{18}

IV. HALL VISCOSITY FOR K’-VALLEY

From Eq. (14), the two nonzero components of the stress tensor are given by:

$$
\hat{P}^{K'}_{xx} = -\frac{\hbar \omega}{2} \left( \hat{\Pi}_+ \hat{\sigma}_- + \hat{\Pi}_- \hat{\sigma}_+ \right) + \hat{H}^{K'}_0 - \Delta \hat{\sigma}_z \\
\hat{P}^{K'}_{xy} = -\frac{\hbar \omega}{2i} \left( \hat{\Pi}_+ \hat{\sigma}_- - \hat{\Pi}_- \hat{\sigma}_+ \right),
$$

(28)

where $\hat{\sigma}_\pm \equiv \hat{\sigma}_x \pm i \hat{\sigma}_y$.

Then, Eq. (15) can again be used for the valley Hall viscosity in this case. The closed expression for the $K'$-valley Hall viscosity is given by

$$
\eta^{K'}_H(\omega) = -\frac{\hbar \omega^2}{4\pi \ell^2} \sum_{k,l} \frac{\left| [\Pi_+ \sigma_-]_{lk} \right|^2 - \left| [\Pi_- \sigma_+]_{lk} \right|^2}{\omega^2 - \omega^2_{lk}},
$$

(29)
A. Electron-doped case

Again, the Fermi levels with $E_F > \Delta$ corresponds to $N_L \geq 1$ for which the ZLL is occupied, and for Fermi levels pinned within the band gap ($|E_F| < \Delta$) we have the ZLL unoccupied. For the latter case, we must calculate the transitions from all occupied LLs in the negative sector to this ZLL. Similar to the $K$-valley Hall viscosity, we consider three contributions to the $K'$-valley Hall viscosity each of which are given by:

\[
\eta_{H}^{K',1}(\omega) = \begin{cases} 
- \frac{\hbar \omega^2}{\pi \ell^2} \sum_{k=\text{Max}(1,N_L-2)}^{N_L-1} \frac{a_+^{2,k+2}(k+1)}{\omega^2 - \omega_0^2 (\sqrt{\gamma^2 + k + 2} - \sqrt{\gamma^2 + k})^2}, & \text{if } N_L \geq 2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\eta_{H}^{K',2}(\omega) = - \frac{\hbar \omega^2}{\pi \ell^2} \sum_{k=\text{Max}(1,N_L-2)}^{N_L-1} \frac{a_+^{2,k+2}a_+^{2,-k}(k+1)}{\omega^2 - \omega_0^2 (\sqrt{\gamma^2 + k + 2} + \sqrt{\gamma^2 + k})^2} + \frac{\hbar \omega^2}{\pi \ell^2} \sum_{k=N_L}^{N_L-2} \frac{a_+^{2,k+2}a_+^{2,-k}(k+1)}{\omega^2 - \omega_0^2 (\sqrt{\gamma^2 + k + 2} - \sqrt{\gamma^2 + k})^2},
\]

and

\[
\eta_{H}^{K',3}(\omega) = \begin{cases} 
- \frac{\hbar \omega^2}{\pi \ell^2} \sqrt{\frac{\gamma^2 + 2 - \gamma}{\gamma^2 + 2}} \frac{1}{\omega^2 - \omega_0^2 (\sqrt{\gamma^2 + 2} - \gamma)}^2, & \text{if } N_L = 1, 2 \\
0, & \text{otherwise}
\end{cases}
\]

We note that for $E_F > \Delta$ the ZLL is occupied whereas for $|E_F| < \Delta$ it is empty and we must consider all nonzero matrix elements of the stress tensor between this level and occupied LLs in the negative sector.

The final expression of the $K'$-valley Hall viscosity for $\omega = 0$ can be simplified to

\[
\eta_{H}^{K'}(\omega = 0) = \eta_{H}^{K',1}(\omega = 0) + \eta_{H}^{K',2}(\omega = 0) + \eta_{H}^{K',3}(\omega = 0)
= \frac{\hbar}{4\pi \ell^2} \theta(E_F - \Delta) \left[ \sum_{k=N_L}^{N_L-1} F_1^{K'}(k, \gamma) + \sum_{k=N_L}^{N_L-2} F_2^{K'}(k, \gamma) + F_3^{K'}(\gamma) (\delta_{N_{L,1}} + \delta_{N_{L,2}}) \right] + \frac{\hbar}{4\pi \ell^2} \theta(\Delta - |E_F|) \left[ \sum_{k=N_L}^{N_L-2} F_2^{K'}(k, \gamma) + F_4^{K'}(\gamma) \right],
\]

where $N_L \geq 1$ and the kernel functions corresponding to the positive, negative, and the zeroth LLs are given by:
\[ F^{K'}_1(k, \gamma) = F^K_1(k, -\gamma), \]
\[ F^{K'}_2(k, \gamma) = -F^K_2(k, \gamma), \]
\[ F^{K'}_3(\gamma) = F^K_3(-\gamma), \]
\[ F^{K'}_4(\gamma) = -F^K_3(\gamma). \] (34)

Notice that Eq. (33) gives \( K' \)-valley Hall viscosity for a Fermi level either being pinned within the gap (the second term) or larger than the gap (the first term for \( E_F > \Delta \)). Similar to the \( K \)-valley Hall viscosity, there is a diverging summation coming from contributions from an infinite number of LLs in the negative sector.

V. TOTAL HALL VISCOSITY: ELECTRON DOPING

A. \( E_F > \Delta \)

For \( \omega = 0 \), the addition of the valley Hall viscosities given in Eqs. (25) and (33) yield the total Hall viscosity for the integer quantum Hall states of the gapped graphene in the presence of a perpendicular magnetic field.

\[
\eta^{\text{doped}}_H(\omega = 0) = \eta^K_H(\omega = 0) + \eta^{K'}_H(\omega = 0)
\]

\[
= \frac{\hbar}{2\pi \ell^2} \theta(E_F - \Delta) \left[ \sum_{k=N_L-2}^{N_L-1} F^K_1(k, 0) + \delta_{N_L,1} + \delta_{N_L,2} \right]
\]

\[
= \frac{\hbar}{2\pi \ell^2} \theta(E_F - \Delta) \left[ (N_L - 1)^2 \theta(N_L - 2) + N_L^2 \theta(N_L - 1) + \delta_{N_L,1} + \delta_{N_L,2} \right] \] (35)

Eq. (35) indicates that if the Fermi level is pinned within the band gap the total Hall viscosity is zero and for the Fermi level in the positive sector the total Hall viscosity is identical to the value we have already obtained for the gapless graphene\(^{18}\), namely

\[
\eta_H(\omega = 0) = g_{sv} \frac{\hbar}{4\pi \ell^2} \left[ N_L^2 + (N_L - 1)^2 \right] \text{sgn} \left( N_L - \frac{1}{2} \right), \quad \text{(Gapless graphene)} \] (36)

where \( g_{sv} = 4 \) for the valley and spin multiplicity. The equivalence between Eqs. (35) and (36) is clear for \( \Delta = 0 \).

B. \( E_F > \Delta \)

The total Hall viscosity vanishes if the Fermi level is pinned within the gap. In this case, only the diverging summation and the contribution from the ZLL exist in \( \eta^K_H(\omega = 0) \) while for \( K' \)-valley we must consider the terms in the second bracket of Eq. (33). On the other hand it can easily be seen from Eq. (34) that all these contributions cancel each other, despite the divergence of the individual valley Hall viscosities.
VI. TOTAL HALL VISCOSITY: HOLE DOPING

The hole doping case corresponds to the negative Fermi levels such that \( E_F < -\Delta \). This corresponds the shift of \( N_L \) index for the electron-doped case to \(-N_L + 1\) for the hole-doped one. In other words, for the latter case, all the LLs in the negative sector below the highest empty one labelled as \(-N_L\) are now occupied. As before we will consider three cases:

- **Case ①**: Interband transitions for which both empty and occupied levels belong to the negative sector. In this case we have \(-N_L + 1 \leq l \leq -1\) for empty levels and \(-N_C \leq k \leq -N_L\) for the occupied ones.

- **Case ②**: Interband transitions for which the occupied levels belong to the negative sector and the unoccupied ones belong to the positive sector. In this case we have \(1 \leq l \leq N_C\) for empty levels and \(-N_C \leq k \leq -N_L\) for the occupied ones.

- **Case ③**: Contribution from the occupied LLs in the negative sector to the unoccupied ZLL. In this case we have \(l = 0\) and \(-N_C \leq k \leq -N_L\).

It turns out that the contribution to the hole-doped \(K\)-valley Hall viscosity for each of the above cases is exactly the opposite of the corresponding case for the electron-doped \(K'\)-valley Hall viscosity, and vice versa. In other words, we obtain the following relationships:

\[
\eta_{H,K}^{①}(\omega) = -\eta_{H,K'}^{①}(\omega), \\
\eta_{H,K}^{②}(\omega) = -\eta_{H,K'}^{②}(\omega), \\
\eta_{H,K}^{③}(\omega) = -\eta_{H,K'}^{③}(\omega),
\]

(37)

where the left-hand side is the contribution for the electron doing and the right-hand sign is that for the hole doping. We conclude that for \(N_L \leq -1\) the total Hall viscosity changes sign under transformation \(N_L \rightarrow -N_L + 1\) and we must have

\[
\eta_{H}^{e\text{-doped}}(\omega) = -\eta_{H}^{h\text{-doped}}(\omega).
\]

(38)

VII. NONLOCAL HALL CONDUCTIVITY FOR K-VALLEY

Using the LL eigenstates from Eq. (9) and the \(K\)-valley current density operator from Eq. (18) we can finally obtain the matrix elements of \(x\)- and \(y\)-components the current density in Eq. (17) as
\[ K \langle \lambda'; n', k'_y \mid \hat{J}^K(q) \mid \lambda; n, k_y \rangle_K = -e v_F \left[ \lambda' a_{-\lambda,n} a_{\lambda,n'} I(n' - 1, n, q) + \lambda a_{\lambda,n} a_{-\lambda,n'} I(n', n - 1, q) \right] \\
K \langle \lambda; n, k_y \mid \hat{J}^K(-q) \mid \lambda'; n', k'_y \rangle_K = -e v_F \left[ -i \lambda a_{\lambda,n} a_{-\lambda,n'} I^*(n', n - 1, q) + i \lambda' a_{-\lambda,n} a_{\lambda,n'} I^*(n' - 1, n, q) \right], \quad (39) \]

where the integral \( I(n', n, q) \) is given by

\[
I(n', n, q) = \int d\mathbf{r} \psi^*_n, k_y(x, y) e^{-i q \cdot \mathbf{r}} \psi_n, k_y(x, y) \\
= \delta_{q, k_y - k'_y} \begin{cases} 
G_{n'n}(q) & n \geq n' \\
G_{nn'}(-q) & n < n' 
\end{cases}, \quad (40)
\]

where the function \( G_{n'n}(q) \) is given by

\[
G_{n'n}(q) = \sqrt{\frac{n!}{n'}!} \left( -\frac{\ell q}{\sqrt{2}} \right)^{n-n'} e^{-\ell q^2/4} L_n^{n-n'}(q^2 \ell^2/2), \quad (41)
\]

with the property of \( G_{n'n}(q) = [G_{nn'}(-q)]^* \), \( L_n^m(x) \) is the associated Laguerre polynomial and we have also made use the integral\(^{21}\)

\[
\int_{-\infty}^{\infty} e^{-x^2} H_m(x + y) H_n(x + z) dx = 2^n \pi^{1/2} m! y^{n-m} L_{n-m}^m(-2yz). \quad [m < n]
\]

The following properties of \( I(n', n, q) \) integral can easily be verified and are going to be used in the subsequent calculations:

\[
I^*(n', n, q) = I(n, n', -q) \\
|I(n', n, q)|^2 = |I(n, n', -q)|^2 = |I(n, n', q)|^2 \quad (42)
\]

Using Eq. (16), the nonlocal Hall conductivity is then given by:

\[
\sigma_{xy}^K(q, \omega) = \frac{e^2 v_F^2}{A h \omega} \sum_{\lambda, \lambda'} \frac{\mathcal{F}(\lambda, n; \lambda', n')}{} \mathcal{E}(\lambda, n; \lambda', n') \mathcal{T}^K(\lambda, n; \lambda', n', q), \quad (43)
\]

where the three main functions in Eq. (43) with their symmetry with respect to the interchange of \( \lambda \leftrightarrow \lambda' \)
and \( n \leftrightarrow n' \) are defined as

\[
\mathcal{F}(\lambda, n; \lambda', n') = f(E_{\lambda,n}) - f(E_{\lambda',n'}) = -\mathcal{F}(\lambda', n'; \lambda, n)
\]

\[
\mathcal{E}(\lambda, n; \lambda', n') = \omega + (E_{\lambda,n} - E_{\lambda',n'})/\hbar = \mathcal{E}(-\lambda', n'; \lambda, n)
\]

\[
\mathcal{I}^K(\lambda, n; \lambda', n', q) = \lambda^2 a_{\lambda,n}^2 a_{\lambda',n'}^2 |I(n', n - 1, q)|^2 - \lambda^2 a_{\lambda,n}^2 a_{\lambda',n'}^2 |I(n' - 1, n, q)|^2 = -\mathcal{I}^K(\lambda', n'; \lambda, n, q).
\]

The last relationship for function \( \mathcal{I}^K(\lambda, n; \lambda', n', q) \) is found using the symmetry of \( I(n', n, q) \) integrals expressed in Eq. (42).

Using the properties expressed in Eq. (43) and considering the degeneracy of LLs we can eventually simplify the multiple summations in Eq. (43) into separate summations over LLs in the conduction and valence bands. The expression for the nonlocal Hall conductivity is then given by

\[
\sigma_{xy}^K(q, \omega) = \frac{e^2 \omega_0^2}{\hbar} \sum_{\lambda,n,E_{\lambda,n}<E_F} \sum_{\lambda',n',E_{\lambda',n'}>E_F} \frac{\mathcal{I}^K(\lambda, n; \lambda', n', q)}{\omega^2 - \omega_{\lambda,n,\lambda'n'}^2}, \quad (45)
\]

where \( \omega_{\lambda,n,\lambda'n'} = (E_{\lambda,n}^K - E_{\lambda',n'}^K)/\hbar \) and the function in the numerator is given by Eqs. (44), (41), and (40).

Now, similar to the electron-doped case of valley Hall viscosity we calculate the contributions to the nonlocal Hall conductivity in Eq. (45) from three cases. Adding these three contributions we obtain the total \( K \)-valley nonlocal Hall conductivity to be given by

\[
\sigma_{xy}^K(q, \omega) = \frac{e^2 \omega_0^2}{\hbar} \sum_{n'=N_L}^{N_C} \left[ \sum_{n=1}^{N_L-1} \frac{\mathcal{I}^K(+, n; +, n', q)}{\omega^2 - \omega_0^2(\sqrt{\gamma^2 + n - \sqrt{\gamma^2 + n'}^2})^2} + \sum_{n=1}^{N_C} \frac{\mathcal{I}^K(-, n; +, n', q)}{\omega^2 - \omega_0^2(\sqrt{\gamma^2 + n + \sqrt{\gamma^2 + n'}^2})^2} \right. \\
\left. + \frac{\mathcal{I}^K(-, 0; +, n', q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n'}^2)^2} \right]. \quad (46)
\]

Now we note that for an antisymmetric function \( f(j, j') \) such that \( f(j', j) = -f(j, j') \) we can simply show by swapping the summation indices that an identical double sum over the two indices is zero, namely \( \sum_{j=n}^m \sum_{j'=n}^m f(j, j') = 0 \). This point can be applied to the second term in the bracket of Eq. (46) which allows us to simplify the Hall conductivity to the following final form:

\[
\sigma_{xy}^K(q, \omega) = \frac{e^2 \omega_0^2}{\hbar} \sum_{n'=N_L}^{N_C} \left[ \sum_{n=1}^{N_L-1} \left( \frac{\mathcal{I}^K(+, n; +, n', q)}{\omega^2 - \omega_0^2(\sqrt{\gamma^2 + n - \sqrt{\gamma^2 + n'}^2})^2} + \frac{\mathcal{I}^K(-, n; +, n', q)}{\omega^2 - \omega_0^2(\sqrt{\gamma^2 + n + \sqrt{\gamma^2 + n'}^2})^2} \right) \right. \\
\left. + \frac{\mathcal{I}^K(-, 0; +, n', q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n'}^2)^2} \right]. \quad (47)
\]
VIII. NONLOCAL HALL CONDUCTIVITY K’-VALLEY

The current density operator for $K'$-valley is given by
\[ \hat{j}^{K'}(q) = -e v_F \hat{\sigma}^* e^{-i q \cdot \hat{r}} \] with the pseudo-spin operator to be $\hat{\sigma}^* = (-\hat{\sigma}_x, \hat{\sigma}_y)$. The nonlocal valley Hall conductivity in this case is calculated from the current-current response function similar to the one expressed in Eq. (17) using the matrix elements of the current density operator with respect to the LLs in this valley represented in Eq. (11). We finally obtain the following expression for the nonlocal Hall conductivity:

\[ \sigma_{xy}^{K'}(q, \omega) = \frac{e^2 \omega_0^2}{h} \sum_{\lambda, n; E_{\lambda, n} < E_F, \lambda', n'; E_{\lambda', n'} > E_F} \frac{\mathcal{I}^{K'}(\lambda, n; \lambda', n', q)}{\omega^2 - \omega_{\lambda n, \lambda' n'}^2}, \] (48)

where the function in the numerator of the summand is defined as

\[ \mathcal{I}^{K'}(\lambda, n; \lambda', n', q) = \lambda^2 a_{\lambda, n}^2 a_{\lambda', n'}^2 |I(n', n - 1, q)|^2 - \lambda'^2 a_{\lambda, n}^2 a_{\lambda', n'}^2 |I(n' - 1, n, q)|^2 \]
\[ = \mathcal{I}^{K}(-\lambda, n; -\lambda', n', q) \]
\[ = -\mathcal{I}^{K'}(\lambda', n'; \lambda, n, q). \] (49)

Similar to the result of the $K$-valley Hall conductivity given in Eq. (47), further simplification of Eq. (48) yields

\[ \sigma_{xy}^{K'}(q, \omega) = \frac{e^2 \omega_0^2}{h} \sum_{n=N_L}^{N_C} \sum_{n=1}^{N_L-1} \left[ \frac{\mathcal{I}^{K'}(+, n; +, n', q)}{\omega^2 - \omega_0^2 (\sqrt{\gamma^2 + n} - \sqrt{\gamma^2 + n'}^2)} + \frac{\mathcal{I}^{K'}(-, n; +, n', q)}{\omega^2 - \omega_0^2 (\sqrt{\gamma^2 + n} + \sqrt{\gamma^2 + n'}^2)} \right] \]
\[ + \frac{\mathcal{I}^{K'}(+, 0; +, n', q)}{\omega^2 - \omega_0^2 (\gamma - \sqrt{\gamma^2 + n'}^2)}. \] (50)

IX. TOTAL NONLOCAL HALL CONDUCTIVITY: K+K'.

We obtain the nonlocal Hall conductivity of the electron-doped system here for two cases of the Fermi level one being larger that the gap energy and another being pinned within the gap.

A. $E_F > \Delta$

Adding the contributions from both valleys given by Eqs. (47) and (50) we can obtain the total nonlocal Hall conductivity
\[ \sigma_{xy}(q, \omega) = \sigma_{xy}^K(q, \omega) + \sigma_{xy}^{K'}(q, \omega) \]
\[ = \frac{e^2 \omega_0^2}{h} \sum_{n'=N_L}^{N_C} \sum_{n=0}^{N_L-1} \frac{\omega^2 - (n + n')^2_0 \omega_0^2}{\omega^4 - 2(n + n' + 2\gamma^2)\omega^2 \omega_0^2 + (n' - n)^2_0 \omega_0^4}. \tag{51} \]

We note that the gap energy only appears in the denominator in Eq. (51) in the coefficient of \( \omega^2 \). Now we define \( n' = n + k \) and write the summations in the expression of the conductivity in terms of \( n \) and \( k \) variable, namely
\[ \sigma_{xy}(q, \omega) = \frac{e^2 \omega_0^2}{h} \sum_{k=1}^{N_C-1} \sum_{n=\text{Max}(0,N_L-k)}^{\text{min}(N_L-1,N_C-k)} \frac{\omega^2 - (2n + k)^2_0 \omega_0^2}{\omega^4 - 2(2n + k + 2\gamma^2)\omega^2 \omega_0^2 + k^2 \omega_0^4}. \tag{52} \]

At zero frequency (\( \omega = 0 \)) Eq. (52) yields the final expression for the total nonlocal Hall conductivity
\[ \sigma_{xy}(q, \omega = 0) = \frac{e^2 \omega_0^2}{h} \sum_{k=1}^{N_C-1} \sum_{n=\text{Max}(0,N_L-k)}^{\text{min}(N_L-1,N_C-k)} \frac{(2n + k + \sqrt{\gamma^2 + n')^2}}{k^2_0 \omega_0^2} \left[ G_{n,n+k-1}^2(q) - G_{n,n+k}^2(q) \right]. \tag{53} \]

B. \( |E_F| < \Delta \)

In this case, we see that the ZLL for the \( K \)-valley in occupied (\( N_L = 1 \)) but it is empty for the \( K' \)-valley. The contributions from the ZLL and the transitions from all the LLs in the negative sector are then given by
\[ \sigma_{xy}^K(q, \omega) = \frac{e^2 \omega_0^2}{h} \sum_{n'=1}^{N_C} \sum_{n=1}^{N_C} \frac{I^K(-, n; +, n', q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n')^2}} + \frac{I^K(-, 0; +, n', q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n')^2}} \]
\[ = \frac{e^2 \omega_0^2}{h} \sum_{n'=1}^{N_C} \frac{I^K(-, 0; +, n', q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n')^2}}. \tag{54} \]

To obtain the second line in Eq. (54) we again note that the double sum of an antisymmetric function \[ \frac{I^K(-, n; +, n', q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n')^2}} \] [look at Eq. (49)] is zero. Similarly the only nonzero contribution to the Hall conductivity of \( K' \)-valley is coming from the ZLL, namely
\[ \sigma_{xy}^{K'}(q, \omega) = \frac{e^2 \omega_0^2}{h} \sum_{n'=1}^{N_C} \frac{I^{K'}(-, n'; +, 0, q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n')^2}}. \tag{55} \]

Adding the contributions from both valleys given by Eqs. (54) and (55) we can obtain the total nonlocal Hall conductivity
\[ \sigma_{xy}(q, \omega) = \sigma_{xy}^K(q, \omega) + \sigma_{xy}^{K'}(q, \omega) \]
\[ = \frac{e^2 \omega_0^2}{\hbar} \sum_{n'=1}^{N_C} \frac{\mathcal{I}^K(-, 0, +; n', q) + \mathcal{I}^{K'}(-, n'; +, 0, q)}{\omega^2 - \omega_0^2(\gamma + \sqrt{\gamma^2 + n'})^2}, \]  
(56)

Now using Eqs. (44) and (49) we can simplify the numerator of Eq. (56), namely

\[ \mathcal{I}^K(-, 0, +; n', q) = -a_{+,-n'}^2 |I(n' - 1, 0, q)|^2 \]
\[ \mathcal{I}^{K'}(-, n'; +, 0, q) = a_{+,-n'}^2 |I(0, n' - 1, q)|^2 \]  
(57)

But from symmetry property of the integral \( I(n', n, q) \) given in Eq. (42) we conclude that \( \mathcal{I}^K(-, 0, +; n', q) = \mathcal{I}^{K'}(-, n'; +, 0, q) \); hence \( \sigma_{xy}(q, \omega) = 0 \) for any Fermi level within the gap.

**X. NONLOCAL HALL CONDUCTIVITY AND HALL VISCOSITY**

We emphasize that the expression for zero-frequency nonlocal Hall conductivity given in Eq. (53) is valid for both gapped and gapless graphene. We can examine this by expanding the expression in Eq. (53) for various filling factors \( N_L \) for small values of wave vector \( q \). In addition, in this section we show that, similar to the case of gapless graphene, the Hall viscosity explicitly appears in the small-\( q \) expansion of the zero-frequency nonlocal Hall conductivity in the coefficient of \( O(q^2) \) term.

For example, for \( N_L = 1 \) the small-\( q \) expansion of the Hall conductivity for gapless graphene is simplified to \( e^2(1 - (q\ell)^2/4)/(2\hbar) \) while the conductivity expression for the gapped graphene in Eq. (53) yields

\[ \sigma_{xy}(q, \omega = 0) = \frac{e^2}{\hbar} \sum_{k=1}^{N_C-1} \frac{G^2_{0,k-1}(-q)}{k}. \]  
(58)

Using Eq. (41) we note that \( G^2_{0,k-1}(-q) = 2^{1-k} e^{-q^2\ell^2/2} (q\ell)^{-2+2k} \). Up to order \( O(q^2) \) the only contributing terms correspond to the values \( k = 1, 2 \). Therefore the small-\( q \) expansion of the conductivity for the gapped graphene will be identical to that of for gapless graphene for \( N_L = 1^{18} \).

We note from the definition of \( G_{nn'}(q) \) [Eq. (41)] that for keeping terms up to \( O(q^2) \), the index \( k = n' - n \) can only be \( k = 1, 2 \). Evaluating the summations in Eq. (53) for \( k = 1, 2 \) we arrive at a simplified expression
for the conductivity as

$$
\sigma_{xy}(q, \omega = 0) = \frac{e^2}{h} \left\{ \frac{(2N_L - 2 + 1)}{2} \left[ G^2_{N_L-1,N_L-1}(-q) - G^2_{N_L-2,N_L}(-q) \right] + \right.
\left. \frac{N_L}{2} \left[ G^2_{N_L-1,N_L}(-q) - G^2_{N_L-2,N_L+1}(-q) \right] + \frac{N_L - 1}{2} \left[ G^2_{N_L-2,N_L-1}(-q) - G^2_{N_L-3,N_L}(-q) \right] \right\}. 
$$

(59)

Now we expand the expression in Eq. (59) up to $O(q^2)$ which yields

$$
\sigma_{xy}(q, \omega = 0) \simeq -\frac{e^2}{4h} \left\{ 4 - 8N_L + (q\ell)^2 \right\}. 
$$

(60)

We then re-write the coefficient of $(q\ell)^2$ term as

$$
1 + 6N_L(N_L - 1) = -8 \left( \frac{N_L^2 + (N_L - 1)^2}{8} - \left( N_L - \frac{1}{2} \right)^2 \right) 
= -8 \left( N_L - \frac{1}{2} \right) \left[ g_v \left( \frac{\hbar^2}{4\pi^2} \right) \left( N_L^2 + (N_L - 1)^2 \right) - \left( N_L - \frac{1}{2} \right) \right] 
= -8 \left( N_L - \frac{1}{2} \right) \left[ \frac{\eta H}{4\hbar n} - \left( N_L - \frac{1}{2} \right) \right], 
$$

(61)

where $n = g_v \frac{|N_L - 1/2|}{2\pi^2}$ is the carrier density (electron density for $N_L \geq 1$, hole density for $N_L \leq 0$) for gapless graphene and the Hall viscosity is given by Eq. (36). Plugging the expression in Eq. (61) into Eq. (60) and taking the valley and spin multiplicity into account we finally obtain

$$
\sigma_{xy}(q, \omega = 0) \simeq g_v \left( \frac{N_L - 1}{2} \right) \frac{e^2}{h} \times \left\{ 1 + q^2\ell^2 \left[ \frac{\eta H}{4\hbar n} - \left| N_L - \frac{1}{2} \right| \right] \right\}, \quad \text{(Gapped and gapless graphene)} 
$$

(62)

which is identical to the result obtained for gapless graphene in Eq. (8) of our earlier work.$^{18}$

XI. SUMMARY AND DISCUSSION

We have shown by explicit calculation and by theoretical argument that the Hall viscosity and the Hall conductivity of “gapped graphene” in a uniform perpendicular magnetic field are both zero (at zero temperature) when the Fermi level lies in the gap $|E_F| < \Delta$, where $\Delta$ is a “Semenoff mass”. In all other cases the two quantities are given by the same formulas that were previously derived for gapless graphene$^{18}$. Perhaps the
most intriguing result of the analysis is that it is not possible, at least with our present method, to calculate the Hall viscosity for a single valley (K or K’). The sum over Landau levels diverges with a strength proportional to the gap $\Delta$. It is only when the contributions of K and K’ are combined that the diverging terms cancel, leaving us with the familiar finite result. This unexpected result seems to point to a fundamental limitation of the continuum model, when applied to the calculation of the valley-filtered viscosity. In contrast to this, the results for the nonlocal conductivity remain well-behaved, even when they are calculated in a single valley. An immediate consequence of this phenomenon is that the Hoyos-Son relation between nonlocal conductivity and viscosity breaks down in a single valley. The origin of this singular behavior remains unclear at the time of this writing.

ACKNOWLEDGEMENTS

The work was supported by the grant No. DE- FG02-05ER46203 funded by the U.S. Department of Energy, Office of Science.

* vignaleg@missouri.edu

1 A. H. Castro Neto, F. Guinea, N. M. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009); D. S. L. Abergel, V. Apalkov, J. Berashevich, K. Ziegler, and T. Chakraborty, Adv. Phys. 59, 261 (2010).
2 C. Chamon1, C.-Y Hou, C. Mudry, S. Ryu, and L. Santos, Phys. Scr. T146, 014013 (2012).
3 G. W. Semenoff, Phys. Rev. Lett. 53, 2449 (1984); A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B 50, 7526 (1994).
4 S. Y. Zhou, G.-H. Gweon, A. V. Fedorov, P. N. First, W. A. de Heer, D.-H. Lee, F. Guinea, A. H. Castro Neto, and A. Lanzara, Nat. Mater. 6, 770 (2007); G. Giovannetti, P. A. Khomyakov, G. Brocks, P. J. Kelly, and J. van den Brink, Phys. Rev. B 76, 073103 (2007).
5 B. Hunt, J. D. Sanchez-Yamagishi, A. F. Young, M. Yankowitz, B. J. LeRoy, K. Watanabe, T. Taniguchi, P. Moon, M. Koshino, P. Jarillo-Herrero, and R. C. Ashoori, Massive Dirac fermions and Hofstadter butterfly in a van der Waals heterostructure, Science 340, 1427 (2013).
6 D. Xiao, W. Yao, and Q. Niu, Phys. Rev. Lett. 99, 236809 (2007); R. V. Gorbachev, J. C. W. Song, G. L. Yu, A. V. Kretinin, F. Withers, Y. Cao, A. Mishchenko, I. V. Grigorieva, K. S. Novoselov, L. S. Levitov, and A. K. Geim, Science
346, 448 (2014); M. Sui, G. Chen, L. Ma, W.-Y. Shan, D. Tian, K. Watanabe, T. Taniguchi, X. Jin, W. Yao, Di Xiao, and Y. Zhang, Nat. Phys. 11, 1027 (2015); Y. Shimazaki, M. Yamamoto, I. V. Borzenets, K. Watanabe, T. Taniguchi, and S. Tarucha, Nat. Phys. 11, 1032 (2015); Y. D. Lensky, J. C. W. Song, P. Samutpraphoot, and L. S. Levitov, Phys. Rev. Lett. 114, 256601 (2015); T. Ando, J. Phys. Soc. Japan 84, 114705 (2015).

7 M. O. Goerbig, Rev. Mod. Phys. 83, 1193 (2011).
8 M. Koshino and T. Ando, Phys. Rev. B 81, 195431 (2010).
9 W. -X. Wang, L.-J. Yin, J.-B. Qiao, T. Cai, S.-Y. Li, R. -F. Dou, J.-C. Nie, X. Wu, and L. He, Phys. Rev. B 92, 165420 (2015).
10 A. Tomadin, G. Vignale, and M. Polini, Phys. Rev. Lett. 113, 235901 (2014); I. Torre, A. Tomadin, A. K. Geim, and M. Polini, Phys. Rev. B 92, 165433 (2015); D. A. Bandurin, I. Torre, R. Krishna Kumar, M. Ben Shalom, A. Tomadin, A. Principi, G. H. Auton, E. Khestanova, K. S. Novoselov, and I. V. Grigorieva et al., Science 351, 1055 (2016); J. Crossno, J. K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas, S. Sachdev, P. Kim, T. Taniguchi, and K. Watanabe et al. 351, 1058 (2016); L. Levitov and G. Falkovich, Nat. Phys. (London) 12, 672 (2016); A. Principi, G. Vignale, M. Carrega, and M. Polini, Phys. Rev. B 93, 125410 (2016).
11 J. E. Avron, R. Seiler, and P. G. Zograf, Phys. Rev. Lett. 75, 697 (1995); J. E. Avron, J. Stat. Phys. 92, 543 (1998)
12 N. Read, Phys. Rev. B 79, 045308 (2009)
13 X. G. Wen and A. Zee, Phys. Rev. Lett. 69, 953 (1992).
14 R. R. Biswas and D. T. Son, Proc. Nat. Acad. Sci. 113, 8636 (2016);
15 N. Schine, A. Ryou, A. Gromov, A. Sommer, and J. Simon, Nature (London) 534, 671 (2016).
16 N. Read, Phys. Rev. B 79, 045308 (2009); I. V. Tokatly and G. Vignale, J. Phys. Condens. Matter 21, 275603 (2009); N. Read and E. H. Rezayi, Phys. Rev. B 84, 085316 (2011); B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B 86, 245309 (2012); C. Hoyos, Int. J. Mod. Phys. B 28, 1430007 (2014).
17 C. Hoyos and D. T. Son, Phys. Rev. Lett. 108, 066805 (2012).
18 M. Sherafati, A. Principi, and G. Vignale, Phys. Rev. B 94, 125427 (2016).
19 I. V. Tokatly and G. Vignale, 76, 161305 (2007); (E) 79, 199903 (2009).
20 P. Gusynin, S. G. Sharapov, and J. Carbotte, Int. J. Mod. Phys. B 21, 4611 (2007).
21 I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and products, Eighth edition (Academic Press, Burlington, 2007), 8.97, 7.377.