MAXIMAL FUNCTION ASSOCIATED TO THE BOUNDED LAW OF THE ITERATED LOGARITHMS VIA ORTHOMARTINGALE APPROXIMATION

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ABSTRACT. We give sufficient conditions for the bounded law of the iterated logarithms for strictly stationary random fields when the summation is done on rectangle. The study is done by the control of an appropriated maximal function. The case of orthomartingales is treated. Then results on projective conditions are derived.

1. INTRODUCTION, GOAL OF THE PAPER

1.1. Bounded law of the iterated logarithms for random fields. Before we present the scope of the paper, let us introduce the following notations.

(1) In all the paper, $d$ is an integer greater or equal to one.
(2) For any integer $N$, we denote by $[N]$ the set $\{k \in \mathbb{Z}, 1 \leq k \leq N\}$.
(3) The element of $\mathbb{Z}^d$ whose coordinates are all 0 (respectively 1) is denoted by $0$ (resp. 1).
(4) We denote by $\preceq$ the coordinatewise order on the elements of $\mathbb{Z}^d$, that is, we write for $i = (i_q)_{q=1}^d$ and $j = (j_q)_{q=1}^d$ that $i \preceq j$ if $i_q \leq j_q$ for all $q \in [d]$. Similarly, we write $i \succeq j$ if $i_q \geq j_q$ for all $q \in [d]$.
(5) For a family of numbers $(a_n)_{n \geq 1}$, we define $\limsup_{n \to +\infty} a_n := \lim_{n \to +\infty} \sup_{n \geq 1} a_n$.
(6) Let $L: (0, +\infty) \to \mathbb{R}$ be defined by $L(x) = \max\{\ln x, 1\}$ and $LL: (0, +\infty) \to \mathbb{R}$ by $LL(x) = L \circ L(x)$.

Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary random field and denote for $n \geq 1$ the partial sum

$$S_n := \sum_{1 \leq i \leq n} X_i. \quad (1.1.1)$$

We are interested in finding a family of positive numbers $(a_n)_{n \geq 1}$ with the smallest possible growth as $\max n \to \infty$ such that the quantity

$$\left\| \sup_{n \geq 1} \frac{1}{a_n} |S_n| \right\|_p < +\infty, 1 \leq p < 2, \quad (1.1.2)$$

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is finite. It has been shown in [Wic73] that for an i.i.d. collection of centered random variables \( \{X_i, i \in \mathbb{Z}^d\} \), (with \( d > 1 \)) satisfying \( \mathbb{E} \left[ X_0^2 (L (|X_0|))^{d-1} / LL (|X_0|) \right] < +\infty \), then

\[
\limsup_{n \to +\infty} \frac{1}{\sqrt{|n| LL(|n|)}} S_n = \|X_0\|_2 \sqrt{d} = - \liminf_{n \to +\infty} \frac{1}{\sqrt{|n| LL(|n|)}} S_n. \tag{1.1.3}
\]

In particular, the moment condition as well as the \( \limsup/\liminf \) depend on the dimension \( d \) and the choice of \( c_n = \sqrt{|n| LL(|n|)} \) is the best possible among those guaranting the finiteness of the random variable involved in (1.1.2).

In this paper, we will be concentrated in the following questions. First, we would like to give bound on the quantity involved in (1.1.2). Results in the one dimensional case are known in the i.i.d. setting [Pis76] and martingales [Cun15], but to the best of our knowledge, it seems that no results are available in dimension greater than one. Nevertheless, the question of giving the limiting points of \( \left( \frac{S_n}{\sqrt{|n| LL(|n|)}} \right)_{n \geq 1} \) has been investigated in dimension 2 in [Jia99].

A first objective is to deal with the case of orthomartingales. Approximations by the latter class of random fields lead to results for the central limit theorem and its functional version (see [CDV15, Gir18, PZ18a, PZ18b]). Therefore, a reasonable objective is to try to establish similar results in the context of the bounded law of the iterated logarithms. Therefore, the second objective is to deal with projectives conditions in order to extend the results for orthomartingales to larger classes of random fields.

1.2. Stationary random fields.

**Definition 1.1.** We say that the random field \( \{X_i\}_{i \in \mathbb{Z}^d} \) is strictly stationary if for all \( j \in \mathbb{Z}^d \), all \( N \geq 1 \) and all \( i_1, \ldots, i_N \), the vectors \( (X_{i_1+j}, \ldots, X_{i_N+j}) \) and \( (X_{i_1}, \ldots, X_{i_N}) \) have the same distribution.

It will be convenient to represent strictly stationary random field via dynamical systems. Let \( \{X_i\}_{i \in \mathbb{Z}^d} \) be a strictly stationary random field on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Then there exists a probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \), \( f: \Omega \to \mathbb{R} \) and maps \( T_q: \Omega' \to \Omega' \) which are invertible, bi-measurable and measure preserving and commuting such that \( \{X_i\}_{i \in \mathbb{Z}^d} \) has the same distribution as \( \{f \circ T^i\}_{i \in \mathbb{Z}^d} \), where \( T^i = T_1^{i_1} \circ \cdots \circ T_d^{i_d} \).

Since the behaviour of supremum of the weighted partial sums depends only on the law of the random field, we will assume without loss of generality that the involved stationary random field is of the form \( \{f \circ T^i\}_{i \in \mathbb{Z}^d} \) and use the notations \( U^i(f)(\omega) = f(T^i\omega) \) and

\[
S_n(f) = \sum_{0 \leq i < n-1} U^i(f).
\tag{1.2.1}
\]

2. Orthomartingale case

In order to use martingale methods for the law of iterated logarithms, we need to introduce the concept of orthomartingales, which can be viewed as a generalization of martingales. Orthomartingales are also an adapted tool in order to treat the summations on rectangles. However, the theory of orthomartingales bumps into obstacles which are not only technical. Indeed, the extension of the notion of stopping time is not clear. Moreover, in most of the exponential inequality for martingales [FGL17, FGL15, BT08], the sum of square of increments and conditional variances plays a key role. A multidimensional equivalent does not seem obvious.
2.1. Definition of orthomartingales. We start by defining the meaning of filtrations in the multi-dimensional setting.

**Definition 2.1.** We call the collection of sub-$\sigma$-algebras $(F_i)_{i \in \mathbb{Z}^d}$ of $\mathcal{F}$ a filtration if for all $i, j \in \mathbb{Z}^d$ such that $i \leq j$, the inclusion $F_i \subset F_j$ holds.

In order to have filtrations compatible with the map $T$, we will consider filtrations of the form $F_i := T^{-i}F_0$. These are indeed filtrations provided that $T_qF_0 \subset F_0$ holds for all $q \in [d]$.

We will also impose commutativity of the involved filtrations, that is, for each integrable random variable $Y$, the following inequalities should hold for all $i$ and $j \in \mathbb{Z}^d$:

$$E \left[ E \left[ Y | F_i \right] | F_j \right] = E \left[ E \left[ Y | F_j \right] | F_i \right] = E \left[ Y | F_{\min(i,j)} \right],$$

where $\min \{i, j\}$ is the coordinatewise minimum, that is, $\min \{i, j\} = (\min \{i_q, j_q\})_{q=1}^d$.

**Definition 2.2.** Let $(T^{-i}F_0)_{i \in \mathbb{Z}^d}$ be a commuting filtration. We say that $(m \circ T^i)_{i \in \mathbb{Z}^d}$ is an orthomartingale differences random field if the function $m$ is integrable, $F_0$-measurable and for all $q \in [d]$, the equality $E \left[ m | T_qF_0 \right] = 0$ holds.

Strictly stationary orthomartingale differences random fields are a convenient class of random fields to deal with, especially from the point of view of limit theorems. If $(m \circ T^i)_{i \in \mathbb{Z}^d}$ is a martingale differences random field and one of the maps $T^q$ is ergodic, then $(S_n(m)/|n|)_{n \geq 1}$ converges to a normal distribution as $\max n$ goes to infinity (see [Vol15]). Under these conditions, a functional central limit theorem has also been established in Theorem 1 of [CDV15].

It turns out that a central limit theorem still holds without the assumption of ergodicity of one of the marginal transformations $T^q$ (see Theorem 1 in [Vol18]). However, it seems that there is no result regarding the law of the iterated logarithms for orthomartingale differences random fields.

2.2. Definition of the maximal function. Consider one of the most simple example of orthomartingale differences random fields in dimension two defined in the following way: let $\Omega := \Omega_1 \times \Omega_2$, where $(\Omega_1, A_1, \mu_1, T_1)$ and $(\Omega_2, A_2, \mu_2, T_2)$ are dynamical systems, where $A_1$ and $A_2$ are generated respectively by $c_1 \circ T_1^{i_1}$, $i_1 \in \mathbb{Z}$ and $c_2 \circ T_2^{i_2}$, $i_2 \in \mathbb{Z}$ and $c_1$, $c_2$ are bounded centered functions such that the sequences $(c_1 \circ T_1^{i_1})_{i_1 \in \mathbb{Z}}$ and $(c_2 \circ T_2^{i_2})_{i_2 \in \mathbb{Z}}$ are both i.i.d.

Define $X_{i_1,i_2} := c_1 \circ T_1^{i_1} \cdot c_2 \circ T_2^{i_2}$ and let $F_{i_1,i_2} := \sigma \{X_{j_1,j_2}, j_1 \leq i_1 \text{ and } j_2 \leq i_2\}$. Then $F_{i_1,i_2} = T_1^{-i_1}T_2^{-i_2}F_{0,0}$ and $(F_{i_1,i_2})_{i_1,i_2 \in \mathbb{Z}}$ is a commuting filtration. Moreover, $(X_{0,0} \circ T_1^{i_1})_{i_1 \in \mathbb{Z}}$ is an orthomartingale difference random field and $X_{0,0}$ is bounded. Observe that for all $n_1, n_2 \geq 1$, the following inequality holds

$$\frac{1}{\sqrt{n_1n_2LL(n_1n_2)}}|S_{n_1,n_2}| = \frac{1}{\sqrt{n_1LL(n_1n_2)}} \left| \sum_{i_1=0}^{n_1-1} c_1 \circ T_1^{i_1} \right| \frac{1}{\sqrt{n_2}} \left| \sum_{i_2=0}^{n_2-1} c_2 \circ T_2^{i_2} \right|,$$

which can be rewritten as

$$\frac{1}{\sqrt{n_1n_2LL(n_1n_2)}}|S_{n_1,n_2}| = \frac{1}{\sqrt{n_1LL(n_1)}} \sum_{i_1=0}^{n_1-1} c_1 \circ T_1^{i_1} \frac{\sqrt{LL(n_1)}}{\sqrt{n_1n_2}} \frac{1}{\sqrt{n_2}} \left| \sum_{i_2=0}^{n_2-1} c_2 \circ T_2^{i_2} \right|.$$

(2.2.1)
Consequently, for any fixed $n_2 \geq 1$, it holds, from the classical law of the iterated logarithms and the fact that
\[
\frac{\sqrt{LL(n_1)}}{\sqrt{LL(n_1 n_2)}} \to 1 \quad (2.2.3)
\]
that
\[
\sup_{n_1 \geq 1} \frac{1}{\sqrt{n_1 n_2 LL(n_1 n_2)}} |S_{n_1, n_2}| \geq \frac{1}{\sqrt{2}} \|e_1\|_2 \frac{1}{\sqrt{n_2}} \left| \sum_{|i_2| = 0}^{n_2-1} e_2 \circ T_{i_2} \right| \quad (2.2.4)
\]
hence the same maximal function as in the Bernoulli case (see [Gir19]) would be almost surely infinite. This lead to a alternative definition, namely,
\[
M(f) := \sup_{n \in \mathbb{N}^d} \frac{|S_n(f)|}{|n|^{1/2} \left( \prod_{i=1}^d LL(n_i) \right)^{1/2}}. \quad (2.2.5)
\]

This definition is coherent with the previous example of orthomartingale and its generalization to the dimension $d$. In this case, $M(X_{0,0})$ is simply the product of the 1-dimensional maximal function associated to bounded i.i.d. sequences, hence is almost surely finite.

### 2.3. Result

It turns out that for a stationary orthomartingale difference sequence, the maximal function is almost surely finite provided that $m$ belongs to $\mathbb{L}_{2,2(d-1)}$. The next result gives also a control the moments of the maximal function.

**Theorem 2.3.** Let $d \geq 1$ be an integer. For all $1 \leq p < 2$, there exists a constant $C_{p,d}$ depending only on $p$ and $d$ such that for all strictly stationary orthomartingale differences random field $(m \circ T_i)_{i \in \mathbb{Z}^d}$, the following inequality holds:
\[
\|M(m)\|_p \leq C_{p,d} \|m\|_{2,2(d-1)}. \quad (2.3.1)
\]
Moreover, for all $r \geq 0$,
\[
\|M(m)\|_{2,r} \leq C_{p,d,r} \|m\|_{2,r+2d}. \quad (2.3.2)
\]

**Remark 2.4.** When $d = 1$, we recover the result Theorem 2.3 in [Cum15] in the real-valued case. Moreover, (2.3.2) gives $\|M(m)\|_{2,r} \leq C_{p,d,r} \|m\|_{2,r+2d}$ hence in particular a control on the $\mathbb{L}^2$-norm of $M(m)$.

**Remark 2.5.** The condition $m \in \mathbb{L}_{2,2(d-1)}$ is sufficient for the bounded law of the iterated logarithms. However, we are not able to determine whether the parameter $2(d-1)$ is optimal.

### 3. Projective conditions

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a measure preserving action $T$ and a commuting filtration $(T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d}$, a projective condition is a requirement on a function $f : \Omega \to \mathbb{R}$ involving the functions $\mathbb{E} [f \circ T^i | \mathcal{F}_0]$, $i \in \mathbb{Z}^d$. In the orthomartingale case, the function $\mathbb{E} [f \circ T^i | \mathcal{F}_0]$ is identically equal to zero if $i \geq 0$ and $i \neq 0$. Therefore, projective conditions can be intuitively seen as a measure of the distance with respect to the martingale case.
3.1. Hannan-type condition. If \((F_i)_{i \in \mathbb{Z}^d}\) is a commuting filtration and \(J \subset [d]\), we denote by \(\mathcal{F}_{\infty, 1, +}\) the \(\sigma\)-algebra generated by the union of \(\mathcal{F}_j\) where \(j\) runs over all the elements of \(\mathbb{Z}^d\) such that \(j_q \leq i_q\) for all \(q \in [d]\). Let \((U^i f)_{i \in \mathbb{Z}^d}\) be a strictly stationary random field.

Assume first that \(d = 1\), \(T : \Omega \to \Omega\) is a bijective bimeasurable measure preserving map and \(\mathcal{F}_0\) is a sub-\(\sigma\)-algebra such that \(T \mathcal{F}_0 \subset \mathcal{F}_0\). Assume that \(f : \Omega \to \mathbb{R}\) is measurable with respect to the \(\sigma\)-algebra generated by \(\bigcup_{k \in \mathbb{Z}} T^k \mathcal{F}_0\) and such that \(E[f | \bigcap_{k \in \mathbb{Z}} T^k \mathcal{F}_0] = 0\) and let us consider the condition

\[
\sum_{i \in \mathbb{Z}} \|E[f \circ T^i | \mathcal{F}_0] - E[f \circ T^i | T \mathcal{F}_0]\|_2 < +\infty. \tag{3.1.1}
\]

The generalization of condition (3.1.1) to random field has been considered by Volný and Wang. Let us recall the notations and results of [VW14]. The projection operators with respect to a commuting filtration \((F_i)_{i \in \mathbb{Z}^d}\) are defined by

\[
\pi_j := \prod_{q=1}^d \pi_j^{(q)}, \quad j \in \mathbb{Z}^d, \tag{3.1.2}
\]

where for \(\ell \in \mathbb{Z}\), \(\pi^{(q)}_\ell : L^1(\mathcal{F}) \to L^1(\mathcal{F})\) is defined for \(f \in L^1\) by

\[
\pi^{(q)}_\ell(f) = E^{(q)}_\ell[f] - E^{(q)}_{\ell-1}[f] \tag{3.1.3}
\]

and

\[
E^{(q)}_\ell[f] = E[f | \bigvee_{i \in \mathbb{Z}^d} \mathcal{F}_i], \quad q \in [d], \ell \in \mathbb{Z}. \tag{3.1.4}
\]

The natural extension of (3.1.1) to the dimension \(d\) is

\[
\sum_{j \in \mathbb{Z}^d} \|\pi_j (f)\|_2 < +\infty. \tag{3.1.5}
\]

Under (3.1.5), the functional central limit holds (Theorem 5.1 in [VW14] and Theorem 8 in [CDV15]) and its quenched version [ZRP18]. Therefore, it is reasonable to look for a condition in this spirit for the bounded law of the iterated logarithms. The obtained result is as follows.

**Theorem 3.1.** Let \((F_i)_{i \in \mathbb{Z}^d} := (T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d}\) be a commuting filtration. Let \(f\) be a function such that for each \(q \in [d]\), \(E[f | T_q^i \mathcal{F}_0] \to 0\) as \(\ell \to +\infty\), measurable with respect to the \(\sigma\)-algebra generated by \(\bigcup_{i \in \mathbb{Z}^d} T^i \mathcal{F}_0\). Then for all \(1 < p < 2\),

\[
\|M(f)\|_p \leq C_{p,d} \sum_{j \in \mathbb{Z}^d} \|\pi_j (f)\|_2^{2(d-1)}. \tag{3.1.6}
\]

3.2. Maxwell and Woodroofe type condition. In order to extend the results obtained for orthomartingales to a larger class of strictly stationary random fields, we need an extension of the following almost sure maximal inequality (Proposition 4.1 in [Cun17]).
Proposition 3.2. Let \((\Omega, \mathcal{F}, \mu, T)\) be a dynamical system and let \(\mathcal{F}_0\) be a sub-\(\sigma\)-algebra of \(\mathcal{F}\) such that \(T\mathcal{F}_0 \subset \mathcal{F}_0\). Denote \(E_j[Y] := E[Y | T^{-j}\mathcal{F}_0]\). Then for all integer \(n \geq 0\) and all \(\mathcal{F}_0\)-measurable function \(f\), the following inequality holds almost surely:

\[
\max_{1 \leq i \leq 2^n} \left| \sum_{j=0}^{i-1} f \circ T^j \right| \leq \max_{1 \leq i \leq 2^n} \left| \sum_{j=0}^{i-1} (f - E_{-1}[f]) \circ T^j \right| + \sum_{k=0}^{n-1} \max_{1 \leq i \leq 2^{n-k-1}} \left| \sum_{\ell=0}^{i-1} d_k \circ T^{2^{k+1} \ell} \right| + |u_n| + \sum_{k=0}^{n-1} \max_{1 \leq i \leq 2^{n-k-1}} |u_k| \circ T^{2^{k+1} \ell},
\]

where

\[
u_k = E_{-2^k} \left[ \sum_{j=0}^{2^k-1} f \circ T^j \right],
\]

\[
d_k = u_k + u_k \circ T^{2^k} - u_{k+1}.
\]

We can observe that for all fixed \(k\), the sequence \(\left(d_k \circ T^{2^{k+1} \ell}\right)_{\ell \geq 0}\) is a martingale differences sequence, while for each fixed \(k\), the contribution of \(u_k\) is analogous as that of a coboundary.

The goal of the next proposition is to extend the previous almost sure inequality to the dimension \(d\). It turns out that an analogous inequality can be established, where the decomposition while involve orthomartingale differences random fields in some coordinates and coboundary in the other one. In order to formalize this, we need the following notation. If \(T\) is a measure preserving \(\mathbb{Z}^d\)-action on \((\Omega, \mathcal{F}, \mu)\), \(i \in \mathbb{N}^d\), \(I \subset [d]\) and \(h: \Omega \to \mathbb{R}\), we define

\[
S^I_k(T, h) := \sum_{0 \leq j \leq t_q - 1} h \circ T^{\sum_{q \in I, j_q \in I} q_q + \sum_{q \in [d] \setminus I, j_q \in I} q_q}.
\]

In other words, the summation is done on the coordinates of the set \(I\) and the coordinates of \([d] \setminus I\) are equal to the corresponding ones of \(i\). In particular, for \(I = [d]\), this is nothing but the classical partial sums. We will need also the following notations: for \(k \in \mathbb{Z}^d\), we denote by \(Z(k)\) the set of the elements \(q \in [d]\) such that \(k_q = 0\). Moreover, given a commuting filtration \((T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}\) and an integrable random variable \(X\), we define the operator \(E_i[X]\) by

\[
E_i[X] := E[X | T^{-i}\mathcal{F}_0].
\]

We are now in position to state the following almost sure inequality for stationary random fields.

Proposition 3.3. Let \(T\) be a measure preserving \(\mathbb{Z}^d\)-action on a probability space \((\Omega, \mathcal{F}, \mu)\). Let \(\mathcal{F}_0 \subset \mathcal{F}\) be a sub-\(\sigma\)-algebra such that \(T^{-q}\mathcal{F}_0 \subset \mathcal{F}_0\) for all \(q \in [d]\) and the filtration \((T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}\) is commuting. For each \(\mathcal{F}_0\)-measurable function \(f\), the following inequality takes place almost surely:

\[
\max_{1 \leq i \leq 2^n} |S_i(f)| \leq \sum_{0 \leq k \leq n} \sum_{I \subset [d]} \max_{1 \leq i \leq 2^{n-k}} \left| S^I_k \left( T^{2^k}, d_{k, I} \right) \right|,
\]

where

\[
d_{k, I} := \sum_{I'' \subset I \setminus \text{Z}(k)} \sum_{I' \subset I \setminus \text{Z}(k)} (-1)^{|I'| + |I''|} E_{-2^k-1} \left[ S_{2^{k-1}Z(k)-1}(f) \right].
\]
Observe that for each $I \subset [d]$ and for all $k$ such that $0 \less k \less n$, the random field $(d_{k,1} \circ T^{2^{k_1}})_{j \in 2^{|I|}}$ is an orthomartingale differences random field. In particular, taking the $L^2$-norm (resp. $L^p$) on both sides of the inequality allows us to recover Proposition 2.1 in [Gir18] (resp. Proposition 7.1 of [WW13]) in the adapted case.

In order to have a better understanding of the terms involved in the right hand side of (3.2.6), we will write this inequality in dimension 2. This becomes

\[
\max_{1 \leq i_1 \leq 2^{n_1}} \max_{1 \leq i_2 \leq 2^{n_2}} |S_{i_1,i_2}(f)| \leq \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \max_{1 \leq i_1 \leq 2^{n_1-k_1}} \max_{1 \leq i_2 \leq 2^{n_2-k_2}} \left| S_{i_1,i_2} \left( T^{2^{k_1},2^{k_2}}, d_{k_1,k_2,[2]} \right) \right|
\]

\[+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \max_{1 \leq i_1 \leq 2^{n_1-k_1}} \left| S_{i_1,i_2} \left( T^{2^{k_1},0}, d_{k_1,k_2,\{1\}} \right) \right| \circ T^{0,2^{k_2}i_2}
\]

\[+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \max_{1 \leq i_1 \leq 2^{n_1-k_1}} \left| S_{i_1,i_2} \left( T^{0,2^{k_2}}, d_{k_1,k_2,\{2\}} \right) \right| \circ T^{2^{k_1}i_1,0}
\]

\[+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \max_{0 \leq i_1 \leq 2^{n_1-k_1}} \max_{0 \leq i_2 \leq 2^{n_2-k_2}} \left| d_{k_1,k_2,0} \right| \circ T^{2^{k_1+1},2^{k_2}i_2}, \quad (3.2.8)
\]

where for $k_1, k_2 \geq 1$,

\[d_{0,0,[2]} = f - \mathbb{E}_{0,-1}[f] - \mathbb{E}_{-1,0}[f] + \mathbb{E}_{-1,-1}[f], \quad (3.2.9)
\]

\[d_{k_1,0,[2]} = \mathbb{E}_{-2^{k_1}-1,0} \left[ S_{2^{k_1}-1,1}(f) \right] - \mathbb{E}_{-2^{k_1},0} \left[ S_{2^{k_1}-1,1}(f) \right]
\]

\[- \mathbb{E}_{-2^{k_1}-1} \left[ S_{2^{k_1}-1,1}(f) \right] + \mathbb{E}_{-2^{k_1}-1,-1} \left[ S_{2^{k_1}-1,1}(f) \right], \quad (3.2.10)
\]

\[d_{0,k_2,[2]} = \mathbb{E}_{0,-2^{k_2}-1} \left[ S_{1,2^{k_2}-1}(f) \right] - \mathbb{E}_{0,-2^{k_2}} \left[ S_{1,2^{k_2}-1}(f) \right]
\]

\[- \mathbb{E}_{-1,-2^{k_2}-1} \left[ S_{1,2^{k_2}-1}(f) \right] + \mathbb{E}_{-1,-2^{k_2}} \left[ S_{1,2^{k_2}-1}(f) \right], \quad (3.2.11)
\]

\[d_{k_1,k_2,[2]} = \mathbb{E}_{2^{k_1}-1,-2^{k_2}-1} \left[ S_{2^{k_1}-1,2^{k_2}-1}(f) \right] - \mathbb{E}_{2^{k_1}-1,-2^{k_2}} \left[ S_{1,2^{k_2}-1}(f) \right]
\]

\[- \mathbb{E}_{-2^{k_1}-1,-2^{k_2}-1} \left[ S_{1,2^{k_2}-1}(f) \right] + \mathbb{E}_{-2^{k_1}-1,-2^{k_2}} \left[ S_{1,2^{k_2}-1}(f) \right], \quad (3.2.12)
\]

a similar expression for $d_{k_1,k_2,\{1\}}$ by switching the roles of $T_1$ and $T_2$ and

\[d_{0,0,\{1\}} = \mathbb{E}_{-2^{k_1}-1,-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}+1}(f) \right] - \mathbb{E}_{-2^{k_1}-1,-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}+1}(f) \right]; \quad (3.2.13)
\]

\[d_{k_1,0,\{1\}} = \mathbb{E}_{-2^{k_1},-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}}(f) \right] - \mathbb{E}_{-2^{k_1},-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}}(f) \right], \quad (3.2.14)
\]

\[d_{0,k_2,\{1\}} = \mathbb{E}_{-2^{k_1}-1,-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}+1}(f) \right] - \mathbb{E}_{-2^{k_1}-1,-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}+1}(f) \right]; \quad (3.2.15)
\]

\[d_{k_1,k_2,\{1\}} = \mathbb{E}_{-2^{k_1},2^{k_2}} \left[ S_{2^{k_1},2^{k_2}}(f) \right] - \mathbb{E}_{-2^{k_1}-1,2^{k_2}} \left[ S_{2^{k_1},2^{k_2}}(f) \right], \quad (3.2.16)
\]

\[d_{0,0,0} = \mathbb{E}_{-2^{k_1}-1,-2^{k_2}} \left[ S_{2^{k_1}+1,2^{k_2}+1}(f) \right], \quad (3.2.17)
\]

\[d_{k_1,0,0} = \mathbb{E}_{-2^{k_1},-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}+1}(f) \right], \quad (3.2.18)
\]

\[d_{0,k_2,0} = \mathbb{E}_{-2^{k_1},-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}}(f) \right], \quad (3.2.19)
\]

\[d_{k_1,k_2,0} = \mathbb{E}_{-2^{k_1},-2^{k_2}} \left[ S_{2^{k_1},2^{k_2}}(f) \right]. \quad (3.2.20)
\]
We are now in position to state a result for the law of the iterated logarithms under a condition in the spirit of the Maxwell and Woodroofe condition, that is, involving the norm in some space of $E[S_n(f) | \mathcal{F}_0]$.

**Theorem 3.4.** Let $T$ be a $\mathbb{Z}^d$-measure preserving action on a probability space $(\Omega, \mathcal{F}, \mu)$. Let $\mathcal{F}_0$ be a sub-$\sigma$-algebra of $\mathcal{F}$ such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ is a commuting filtration. Let $1 < p < 2$. There exists a constant $c_{p,d}$ such that for all $\mathcal{F}_0$-measurable function $f : \Omega \to \mathbb{R}$, the following inequality holds:

$$
\|M(f)\|_p \leq c_{p,d} \sum_{n \geq 1} \frac{1}{|n|^{3/2}} \|E[S_n(f) | \mathcal{F}_0]\|_{2,2(d-1)}. 
$$

(3.2.21)

3.3. Application. The previous conditions can be checked for linear processes whose innovations are orthomartingale differences random fields.

**Corollary 3.5.** Let $(m \circ T^i)_{i \in \mathbb{Z}^d}$ be a strictly stationary orthomartingale differences random field with $m \in L_{2,2(d-1)}$, let $(a_i)_{i \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ and let $(f \circ T^i)_{i \in \mathbb{Z}^d}$ be the causal linear random field defined by

$$
f \circ T^i = \sum_{j \geq 0} a_j m \circ T^j - i.
$$

(3.3.1)

Then for all $1 < p < 2$, the following inequalities take place:

$$
\|M(f)\|_p \leq C_{p,d} \sum_{i \geq 0} |a_i| \|m\|_{2,2(d-1)}; 
$$

(3.3.2)

$$
\|M(f)\|_p \leq C_{p,d} \sum_{n \geq 1} \frac{1}{|n|^{3/2}} \left( \sum_{\ell \geq 0} \left( \sum_{0 \leq i \leq n-1} a_{i+\ell} \right)^2 \right)^{1/2} \|m\|_{2,2(d-1)},
$$

(3.3.3)

where $C_{p,d}$ depends only on $p$ and $d$.

**Remark 3.6.** In [Gir19], linear processes were also investigated but with the assumption that the innovations are i.i.d. In this case, the normalization in the definition of maximal function is weaker.

One of the points of considering orthomartingale innovations is the decomposition of a stationary process as a sum of linear process. More precisely, let $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ be a commuting filtration. Define the subspaces

$$
V_d := \{ f \in L^1, f \text{ is } \mathcal{F}_0 \text{-measurable and for all } q \in [d], E[f | T^{eq}\mathcal{F}_0] = 0 \} 
$$

(3.3.4)

$$
W_d = V_d \cap L_{2,2(d-1)}.
$$

(3.3.5)

**Corollary 3.7.** Assume that there exists a sequence $(e_k)_{k \geq 1}$ of elements of $W_d$ such that each element $f$ of $W_d$ can be written as $\sum_{k=1}^{+\infty} c_k e_k$, where the limit is taken with respect to the $L_{2,2(d-1)}$-norm and $\|e_k\|_{2,2(d-1)} \leq 1$. Let $f$ be an $\mathcal{F}_0$-measurable function such that for each $q \in [d], E[f | T^l_q\mathcal{F}_0] \to 0$ as $l \to +\infty$. Then $f$ admits the representation

$$
f = \sum_{j \geq 0} a_{k,j} (f) U^{-j} e_k
$$

(3.3.6)
and for all $1 < p < 2$, the following inequalities hold:

$$\|M(f)\|_p \leq C_{p,d} \sum_{k \geq 1} \sum_{i \geq 0} |a_{k,i}(f)|;$$  \hfill (3.3.7)

$$\|M(f)\|_p \leq C_{p,d} \sum_{k \geq 1} \sum_{n \geq 1} \frac{1}{|n|^{3/2}} \left( \sum_{\ell \geq 0} \left( \sum_{0 \leq i \leq n-1} |a_{k,i+\ell}(f)| \right)^2 \right)^{1/2}. \hfill (3.3.8)$$

4. Proofs

4.1. Tools for the proofs.

4.1.1. Global ideas of proofs. Let us explain the main steps in the proof of the results.

Let us first focus on orthomartingale differences. The maximal function is defined as a supremum over all the $n \in \mathbb{N}^d$. However, due to the lack of exponential inequalities for the maximal of partial sums on rectangles, we will instead work with other maximal functions, where the supremum is restricted to the elements of $\mathbb{N}^d$ whose components are powers of two. The martingale property helps to shows that the moments of the former maximal function are bounded up to a constants by those of the later.

We then have control the deviation probability of the sum on a rectangle. It is convenient to control the latter probability intersected with the event where the sum (in one direction) of squares and conditional variances of the random field is bounded by some $y$. The contribution of this term can be controlled by an application of the maximal ergodic theorem and we are left to control moment of maximal functions in lower dimension. Then we use an induction argument.

For result concerning projective conditions, there are consequences of the result for orthomartingales after an appropriated decomposition of the involved random field.

4.1.2. Weak $L^p$-spaces. The results of the paper involve all a control of the $L^p$ norm of a maximal function. However, it will sometimes be more convenient to work directly with tails. To this aim, we will consider weak $L^p$-spaces.

**Definition 4.1.** Let $p > 1$. The weak $L^p$-space, denoted by $L^p,w$, is the space of random variables $X$ such that $\sup_{t > 0} t^p \mathbb{P}\{|X| > t\}$ is finite.

These spaces can be endowed with a norm.

**Lemma 4.2.** Let $1 < p \leq 2$. Define the following norm on $L^{p,w}$

$$\|X\|_{p,w} := \sup \left\{ \mathbb{P}\left( A \right)^{1/p - 1} \mathbb{E}[|X|] \right\}.$$  \hfill (4.1.1)

For all random variable $X \in L^{p,w}$, the following inequality holds:

$$c_p \|X\|_{p,w} \leq \left( \sup_{t > 0} t^p \mathbb{P}\{|X| > t\} \right)^{1/p} \leq C_p \|X\|_{p,w} \leq C_p \|X\|_p,$$  \hfill (4.1.2)

where $c_p$ and $C_p$ depend only on $p$. 
4.1.3. *Deviation inequalities.* The following deviation inequality is consequence of Theorem 2.1 in [BT08].

**Proposition 4.3.** Let \((d_j)_{j \geq 1}\) be a square integrable martingale differences sequence with respect to the filtration \((\mathcal{F}_j)_{j \geq 0}\). Then for all positive numbers \(x\) and \(y\), the following inequality holds:

\[
P \left( \left\{ \sum_{j=1}^{n} d_j > x \right\} \cap \left\{ \sum_{j=1}^{n} (d_j^2 + \mathbb{E} [d_j^2 \mid \mathcal{F}_{j-1}]) \leq y \right\} \right) \leq 2 \exp \left( -\frac{x^2}{2y} \right).
\]  

(4.1.3)

**Lemma 4.4.** Assume that \(X\) and \(Y\) are two non-negative random variables such that for each positive \(x\), we have

\[
x \mathbb{P} \{X > x\} \leq \mathbb{E} [Y \mathbb{1} \{X \geq x\}].
\]  

(4.1.4)

Then for each \(t\), the following inequality holds:

\[
P \{X > 2t\} \leq \int_1^{+\infty} \mathbb{P} \{Y > st\} \, ds.
\]  

(4.1.5)

4.1.4. *Facts on Orlicz spaces.*

**Lemma 4.5** (Lemma 3.7 in [Gir19]). Let \(p \geq 1\) and \(r \geq 0\). Let \(\varphi := \varphi_{p,q}\) and let \(a > 0\) be a constant. There exists a constant \(c\) depending only on \(a\), \(p\) and \(q\) such that for all random variable \(X\),

\[
\|X\|_{\varphi} \leq c \|X\|_{a\varphi}.
\]  

(4.1.6)

**Lemma 4.6** (Lemma 3.8 in [Gir19]). Let \(r \geq 0\). There exists a constant \(c_r\) such that for any random variable \(X\),

\[
\|X^2\|_{1,r} \leq c_r \|X\|_{2,r}^2;
\]  

(4.1.7)

\[
\|X^{1/2}\|_{2,r} \leq c_r \|X\|_{1,r}^{1/2}.
\]  

(4.1.8)

**Lemma 4.7** (Lemma 3.11 in [Gir19]). For all \(p > 1\) and \(r \geq 0\), there exists a constant \(c_{p,r}\) such that if \(X\) and \(Y\) are two non-negative random variables satisfying for each positive \(x\),

\[
x \mathbb{P} \{X > x\} \leq \mathbb{E} [Y \mathbb{1} \{X \geq x\}],
\]  

(4.1.9)

then \(\|X\|_{p,r} \leq c_{p,r} \|Y\|_{p,r}\).

**Lemma 4.8.** For any non-negative random variable \(X\), \(p \geq 1\) and \(q \geq 0\), the following inequalities hold,

\[
\sum_{k=1}^{+\infty} 2^k k^{q/p} \mathbb{P} \{X > \frac{2^k}{\sqrt{k}}\} \leq c_{p,q} \mathbb{E} \left[ X^p (\ln X)^{q/p/2} \mathbb{1} \{X > 1\} \right],
\]  

(4.1.10)

\[
\sum_{k=1}^{+\infty} 2^k k^{q/2} \mathbb{P} \{X > \frac{2^{k/2}}{\sqrt{k}}\} \leq c_q \mathbb{E} \left[ X^2 (\ln X)^{q/2} \mathbb{1} \{X > 1\} \right],
\]  

(4.1.11)

where \(c_{p,q}\) depends only on \(p\) and \(q\) and \(c_q\) only on \(q\).
Proof. Let \( a_k := 2^k/\sqrt{k} \) and \( A_j \) be the event \( \{ a_k < X \leq a_{k+1} \} \). Since for all \( k \geq 1 \), the set \( \{ X > a_k \} \) is the disjoint union of \( A_j, j \geq k \), we have

\[
A := \sum_{k=1}^{+\infty} 2^k p^q \mathbb{P} \left( X > \frac{2^k}{\sqrt{k}} \right) = \sum_{k=1}^{+\infty} 2^k p^q \sum_{j \geq k} \mathbb{P} (A_j) = \sum_{j=1}^{+\infty} \mathbb{P} (A_j) \sum_{k=1}^{j} 2^k p^q. \quad (4.1.12)
\]

Since there exists a constant \( K_{p,q} \) such that for all \( j \geq 1 \), \( \sum_{k=1}^{j} 2^k p^q \leq K_{p,q} 2^j p^q \), it follows that

\[
A \leq K_{p,q} \sum_{j=1}^{+\infty} \mathbb{P} (A_j). \quad (4.1.13)
\]

Writing

\[
2^j p^q \mathbb{P} (A_j) = a^p j^{q+p/2} \mathbb{P} (a_j < X < a_{j+1}) \leq \mathbb{E} \left[ X^p j^{q+p/2} 1 \{ a_j < X < a_{j+1} \} \right], \quad (4.1.14)
\]

the previous estimate becomes

\[
A \leq K_{p,q} \sum_{j=1}^{+\infty} \mathbb{E} \left[ X^p j^{q+p/2} 1 \{ a_j < X < a_{j+1} \} \right]. \quad (4.1.15)
\]

For \( x \in (a_j, a_{j+1}] \), we have in view of \( 2^j \geq j \) that \( 2^j \leq x \sqrt{j} \) hence \( 2^j \leq x^{2/j} \) which implies that \( j \ln 2 \leq 2 \ln x \). We end the proof by letting \( c_{p,q} := (2/\ln 2)^{q+p/2} \) and by noticing that \( \bigcup_{j \geq 1} A_j \subset \{ X > 1 \} \).

The proof of (4.1.11) is analogous hence omitted.

\[\square\]

4.2. Reduction to dyadics. Let \( d \) be a fixed integer and for \( 0 \leq i \leq d-1 \) define by \( \mathbb{N}_i \) the elements of \( (\mathbb{N} \setminus \{0\})^d \) whose coordinates \( i+1, \ldots, d \) are dyadic numbers. More formally,

\[
\mathbb{N}_i := \left\{ n \in \mathbb{N}^d, \min_{1 \leq q \leq d} n_q \geq 1 \text{ and for all } i+1 \leq j \leq d, \exists k_j \in \mathbb{N} \text{ such that } n_j = 2^{k_j} \right\}. \quad (4.2.1)
\]

We also define \( \mathbb{N}_d \) as \( \mathbb{N}^d \). Notice that \( \mathbb{N}_0 \) is the set of all the elements of \( \mathbb{N}^d \) such that all the coordinates are powers of 2. The goal of this subsection is to show that it suffices to prove Theorem 2.3 where the supremum over \( \mathbb{N}^d \) is replaced by the corresponding one over \( \mathbb{N}_0 \).

**Proposition 4.9.** Let \( (m \circ T^i)_{i \in \mathbb{Z}^d} \) be an orthonormal differences random field with respect to a commuting filtration \( (T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d} \). Then for all \( 1 < p < 2 \), the following inequality holds

\[
\left\| \sup_{n \in \mathbb{N}^d} \left| S_n (m) \right| \right\|_p \leq c_{p,d} \left\| \sup_{n \in \mathbb{N}_0} \left| S_n (m) \right| \right\|_p, \quad (4.2.2)
\]

where \( c_{p,d} \) depends only on \( p \) and \( d \).

**Lemma 4.10.** Let \( (a_n)_{n \in \mathbb{N}^d} \) be a family of positive numbers such that \( a_n \leq a_{n'} \) if \( n \preceq n' \) and

\[
c := \sup_{n \in \mathbb{N}_d} \max_{1 \leq i \leq d} a_{n+e_i} < +\infty. \quad (4.2.3)
\]

Assume that \( (m \circ T^i)_{i \in \mathbb{Z}^d} \) is an orthonormal differences random field with respect to a commuting filtration \( (T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d} \). Let

\[
M_i := \sup_{n \in \mathbb{N}_i} \left| S_n (m) \right|/a_n. \quad (4.2.4)
\]
Then for any real number number $x$ and any $i \in \{0, \ldots, d\}$,
\[
P \{M_i > x \} \leq \int_1^{+\infty} p \left\{ M_{i-1} > \frac{\ln t}{2e} \right\} dt.
\] (4.2.5)

**Proof.** Let $0 \leq i \leq d-1$. Define the random variables
\[
Y_N := \frac{1}{a_{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d}} \sup_{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d} \sup_{m} |S_{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d}(m)|,
\] (4.2.6)
and the following events
\[
Y'_N := \frac{a_{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d}}{a_{n_1, \ldots, n_{i-1}, 2^{n+1}, n_{i+1}, \ldots, n_d}} Y_N, \quad 2^n + 1 \leq N \leq 2^{n+1}.
\] (4.2.7)

and the following events
\[
A_N := \{Y_N > x \}, B_0 = \emptyset, B_N := A_N \setminus \bigcup_{i=0}^{N-1} A_i,
\] (4.2.8)
\[
C_{N,n} := \begin{cases} \bigcup_{i=2^{n+1}}^N B_i, & \text{if } 2^n + 1 \leq N \leq 2^{n+1}; \\ \emptyset, & \text{if } N \leq 2^n \text{ or } N > 2^{n+1}. \end{cases}
\] (4.2.9)

In this way, the set $\{M_i > x\}$ can be expressed as the disjoint union $\bigcup_{N \geq 1} B_N$ hence
\[
P \{M_i > x\} \leq \sum_{N \geq 1} P(B_N) = \sum_{n=0}^{+\infty} \sum_{N=2^n+1}^{2^{n+1}-1} P(B_N).
\] (4.2.10)

Since $x \mathbf{1}(B_N) \leq Y_N \mathbf{1}(B_N)$, we infer that
\[
xP \{M_i > x\} \leq \sum_{n=0}^{+\infty} \sum_{N=2^n+1}^{2^{n+1}-1} E[Y_N \mathbf{1}(B_N)].
\] (4.2.11)

By definition of $c$ in (4.2.3), we get that
\[
xP \{M_i > x\} \leq c \sum_{n=0}^{+\infty} \sum_{N=2^n+1}^{2^{n+1}-1} E[Y'_N \mathbf{1}(B_N)].
\] (4.2.12)

Let $n \geq 0$ be fixed. Since $\mathbf{1}(B_N) = \mathbf{1}(C_{N,n}) - \mathbf{1}(C_{N-1,n})$ for all $n$ such that $2^n + 1 \leq N \leq 2^{n+1}$,
\[
\sum_{N=2^n+1}^{2^{n+1}-1} E[Y'_N \mathbf{1}(B_N)] = \sum_{N=2^n+1}^{2^{n+1}-1} E[Y'_N \mathbf{1}(C_{N,n}) - \mathbf{1}(C_{N-1,n})]
\]
\[
= E \left[ \sum_{N=2^n+1}^{2^{n+1}-1} Y'_N \mathbf{1}(C_{N,n}) - \sum_{N=2^n}^{2^{n+1}-1} Y'_{N+1} \mathbf{1}(C_{N,n}) \right]
\]
\[
\leq E \left[ Y'_{2^n+1} \mathbf{1}(C_{2^n+1,n}) \right] + E \left[ \sum_{N=2^n+1}^{2^{n+1}-1} (Y'_N - Y'_{N+1}) \mathbf{1}(C_{N,n}) \right].
\]

The set $\mathbf{1}(C_{N,n})$ is measurable with respect to the $\sigma$-algebra $\mathcal{G}_N := \mathcal{F}_{\infty \setminus \{i\} + N} e_i$ and by the orthomartingale difference property of $(m \circ T^i)_{i \in \mathbb{Z}^d}$ the random variable $E \left[ Y'_{N+1} - Y'_N \mid \mathcal{G}_N \right]$ is non-negative and consequently,
\[
E \left[ (Y'_N - Y'_{N+1}) \mathbf{1}(C_{N,n}) \right] = E \left[ E \left[ (Y'_N - Y'_{N+1}) \mathbf{1}(C_{N,n}) \mid \mathcal{G}_N \right] \right]
\] (4.2.13)
\[
= E \left[ \mathbf{1}(C_{N,n}) E \left[ Y'_N - Y'_{N+1} \mid \mathcal{G}_N \right] \right] \leq 0,
\] (4.2.14)
from which it follows that
\[
\sum_{N=2^n+1}^{2^{n+1}} \mathbb{E} |Y'_N 1(B_N)| \leq \mathbb{E} |Y'_{2^{n+1}} 1(C_{N,n})|. \tag{4.2.15}
\]

The latter inequality combined with (4.2.12) allows to deduce that
\[
x \mathbb{P} \{ M_i > x \} \leq c \sum_{n=0}^{+\infty} \mathbb{E} [Y'_{2^{n+1}} 1(C_{2^{n+1},n})]. \tag{4.2.16}
\]

Observe that for all \( n \geq 0 \), the random variable \( Y_{2^{n+1}} \) is bounded by \( M_{i-1} \). Combining this with the definition of \( C_{N,n} \) given by (4.2.9), we derive that
\[
x \mathbb{P} \{ M_i > x \} \leq c \sum_{n=0}^{+\infty} \mathbb{E} [M_{i-1} 1(B_{k})]. \tag{4.2.17}
\]

Since the family \( \{ B_k, k \geq 1 \} \) is pairwise disjoint, so is the family \( \{ \bigcup_{k=2^{n+1}}^{2^{n+1}} B_k, n \geq 0 \} \). Therefore, using again the fact that \( M_i > x \) can be expressed as the disjoint union \( \bigcup_{N \geq 1} B_N \), we establish the inequality
\[
x \mathbb{P} \{ M_i > x \} \leq c \mathbb{E} [M_{i-1} 1(M_i > x)]. \tag{4.2.18}
\]

We estimate the right hand side of the previous inequality in the following way:
\[
\mathbb{E} [M_{i-1} 1(M_i > x)] = \int_0^{+\infty} \mathbb{P} \{ (M_i > x) \cap (M_{i-1} > t) \} \, dt \\
\leq \int_0^{\frac{x}{2c}} \mathbb{P} \{ M_i > x \} \, dt + \int_{\frac{x}{2c}}^{+\infty} \mathbb{P} \{ M_{i-1} > t \} \, dt \\
= \frac{x}{2c} \mathbb{P} \{ M_i > x \} + \frac{x}{2c} \int_{1}^{+\infty} \mathbb{P} \{ M_{i-1} > \frac{x}{2c} u \} \, du,
\]
from which (4.2.4) follows. This ends the proof of Lemma 4.10. \( \Box \)

4.3. Proof of Theorem 2.3. We will use the following notations. We define the random variables
\[
Y_n := \frac{|S_{2^n}(m)|}{|2^{n}|^{1/2} \prod_{q=1}^d L(n_i)^{1/2}}, \tag{4.3.1}
\]
\[
Y_{n,i} := \frac{S_{2^n-(n_i-1)}e_i(m)}{|2^{n-n_ie_i}|^{1/2} \prod_{q=1,q\neq i}^{d} L(n_q)^{1/2}}, \tag{4.3.2}
\]
\[
Z_i := \sup_{n \in \mathbb{N}^d} |Y_{n,i}|. \tag{4.3.3}
\]

Lemma 4.11. Let \((m \circ T^t, T^{-1}T_0)_{i \in \mathbb{Z}^d}\) be a strictly stationary orthomartingale differences random field. For all integer \( N \) and all \( x \geq e^{2d+2} \), the following inequality holds:
\[
\mathbb{P} \left\{ \sup_{n \in \mathbb{N}^d} Y_n > x \right\} \leq 2^{Nd} \mathbb{P} \{|X_1| > x2^{-Nd}\} + dN^{d/2}x^{-\ln N}
+ 8 \sum_{i=1}^{d} \int_{1/\sqrt{2}}^{+\infty} v \mathbb{P} \left\{ Z_i > \frac{x}{\sqrt{\ln x}} \right\} \, dv. \tag{4.3.4}
\]
Proof. Define the events
\[ A_n := \{ Y_n > x \}, \]  
(4.3.5)  
\[ B_{n,i} := \left\{ \frac{1}{2n_i} \sum_{j=1}^{2n_i} U^{j \varepsilon_i} \left( Y_{n_i}^2 + \mathbb{E} \left[ Y_{n_i}^2 \mid F_{\infty [1]} \setminus \{\varepsilon_i\} \right] \right) \leq \frac{x^2}{\ln x} \right\}, B_n := \bigcap_{i=1}^d B_{n,i}. \]  
(4.3.6)  
Denoting by \( J_N \) the set of elements of \( \mathbb{N}^d \) such that at least one coordinate is bigger than \( N + 1 \), we have
\[ \mathbb{P} \left\{ \bigcup_{n \in \mathbb{N}^d} A_n \right\} \leq \mathbb{P} \left( \bigcup_{1 \leq n \leq N1} A_n \right) + \sum_{n \in J_N} \mathbb{P} (A_n \cap B_n) + \sum_{i=1}^d \mathbb{P} \left( \bigcup_{n \in \mathbb{N}^d} B_{n,i}^c \right). \]  
(4.3.7)  
Observe that for \( 1 \ll n \ll N1 \), the following inclusions hold
\[ A_n \subset \left\{ \sum_{1 \leq i \leq 2^n} |m \circ T^i| > x \right\}, \]  
(4.3.8)  
\[ \subset \left\{ \sum_{1 \leq i \leq 2^{N1}} |m \circ T^i| > x \right\}, \]  
(4.3.9)  
\[ \subset \bigcup_{1 \leq i \leq 2^{N1}} \left\{ |m \circ T^i| > x2^{-Nd} \right\}. \]  
(4.3.10)  
\[ \mathbb{P} \left( \bigcup_{1 \leq n \leq N1} A_n \right) \leq 2^{Nd} \mathbb{P} \left\{ |m| > x2^{-Nd} \right\}. \]  
(4.3.11)  
Let us control \( \mathbb{P} (A_n \cap B_n) \). First observe that
\[ \mathbb{P} (A_n \cap B_n) \leq \min_{1 \leq i \leq d} \mathbb{P} (A_n \cap B_{n,i}). \]  
(4.3.12)  
Then, in order to control \( \mathbb{P} (A_n \cap B_{n,i}) \) for a fixed \( i \in [d] \), we apply Proposition 4.3 in following setting:
\[ d_j := U^{j \varepsilon_i} Y_{n,i}, \bar{F}_j := F_{1 \setminus \{\varepsilon_i\} + (j-1)\varepsilon_i}, \bar{x} := x2^{n_i/2}L(n_i)^{1/2}, \bar{y} := 2^n x^2 / \ln x. \]  
(4.3.13)  
This leads to the following estimate
\[ \mathbb{P} (A_n \cap B_{n,i}) \leq 2 \exp \left( -\frac{1}{2} L(n_i) \ln x \right) \]  
(4.3.14)  
and plugging this into (4.3.12) gives
\[ \mathbb{P} (A_n \cap B_n) \leq 2 \left( \max_{1 \leq i \leq d} n_i \right)^{-\ln x / 2}. \]  
(4.3.15)  
For a fixed positive integer \( \ell \), the number of elements of \( \mathbb{N}^d \) such that \( \max_{1 \leq i \leq d} n_i = \ell \) do not exceed \( d\ell^{d-1} \) hence
\[ \sum_{n \in J_N} \mathbb{P} (A_n \cap B_n) \leq \sum_{\ell \geq N} d \ell^{d-1} \ell^{-\ln x / 2} \]  
(4.3.16)  
and the latter sum can be estimated by
\[ d \frac{1}{\ln x} - d N^{d-\ln x} = d \frac{1}{\ln x} - d N^{d/2} x^{-\ln N}, \]  
(4.3.17)
hence
\[ \sum_{n \in J_N} \mathbb{P}(A_n \cap B_n) \leq d \frac{1}{\ln x - d} N^{d/2} x^{-\ln N}. \] (4.3.18)

Since \( x \geq e^{2d+2} \), we deduce the estimate
\[ \sum_{n \in J_N} \mathbb{P}(A_n \cap B_n) \leq dN^{d/2} x^{-\ln N}. \] (4.3.19)

In order to bound the third term of the right hand side of (4.3.7), we need the following inequality, valid for any map \( Q : \mathbb{L}^\infty \to \mathbb{L}^\infty \) such that \( \|Qf\|_1 \leq \|f\|_1 \) and \( \|Qf\|_\infty \leq \|f\|_\infty \) for all \( f \in \mathbb{L}^\infty \):
\[ \mathbb{P} \left\{ \sup_{N \geq 1} \frac{1}{N} \sum_{j=1}^{N} Q^j f \geq y \right\} \leq \int_{y/2}^{+\infty} \mathbb{P} \{ |f| > yu \} \, du \] (4.3.20)
(see Lemma 6.1 in [Kre85]). We apply it for any \( i \in [d] \) to the following setting: \( f = Y_i^2 \), \( Q : g \mapsto (U^{e_i} g + U^{e_i} \mathbb{E} \{ g \mid \mathcal{F}_{\infty [d] \setminus \{i\}} - e_i \}) / 2 \), and \( y := \frac{x^2}{\ln x} \).

This leads to
\[ \mathbb{P} \left( \bigcup_{n \in \mathbb{N}^d} B_{c,n,i}^c \right) \leq 4 \int_{y/2}^{+\infty} \mathbb{P} \left\{ Z_i^2 > \frac{x^2}{\ln x} u / 4 \right\} \, du, \] (4.3.21)
and after the substitution \( v = \sqrt{u} \), the latter term becomes
\[ 8 \int_{y/2}^{+\infty} v \mathbb{P} \left\{ Z_i > \frac{x}{\sqrt{\ln x}} v / 2 \right\} \, dv \] (4.3.22)
hence
\[ \sum_{i=1}^{d} \mathbb{P} \left( \bigcup_{n \in \mathbb{N}^d} B_{c,n,i}^c \right) \leq 8 \sum_{i=1}^{d} \int_{y/2}^{+\infty} v \mathbb{P} \left\{ Z_i > \frac{x}{\sqrt{\ln x}} v / 2 \right\} \, dv. \] (4.3.23)

We end the proof of Lemma 4.11 by combining (4.3.7), (4.3.11), (4.3.19) and (4.3.23).

\[ \square \]

**Lemma 4.12.** Let \( Z := \sup_{n \in \mathbb{N}_0} Y_n \), where \( Y_n \) is defined by (4.3.1) and \( Z_i \) given by (4.3.3). Let \( 1 \leq p < 2 \). There exists a constant \( C_{p,d} \) depending only on \( p \) and \( d \) such that for all strictly stationary orthomartingale differences random field \( (m \circ T^i, T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d} \), the following inequality holds:
\[ \|Z\|_p \leq C_{p,d} \max_{1 \leq i \leq d} \|Z_i\|_2. \] (4.3.24)

**Proof.** The proof follows the same lines as that of Lemma 4.11. Let \( x \) be such that \( x^p / 2 - d \geq pd / (2 - p) \). Define the events
\[ A_n := \{ Y_n > x \}, \] (4.3.25)
\[ B_{n,i} := \left\{ \frac{1}{2^{n_i}} \sum_{j=1}^{2^{n_i}} U^{e_j} (Y_{n,i}^2 + \mathbb{E} \{ Y_{n,i}^2 \mid \mathcal{F}_{\infty [d] \setminus \{i\}} - e_i \}) \leq x^p \right\}, B_n := \bigcap_{i=1}^{d} B_{n,i}. \] (4.3.26)

Denoting by \( J_N \) the set of elements of \( \mathbb{N}^d \) such that at least one coordinate is bigger than \( N + 1 \), we have
\[ \mathbb{P} \left( \bigcup_{n \in \mathbb{N}^d} A_n \right) \leq \sum_{1 \leq n \leq N} \mathbb{P}(A_n) + \sum_{n \in J_N} \min_{1 \leq i \leq d} \mathbb{P}(A_n \cap B_{n,i}) + \sum_{i=1}^{d} \mathbb{P} \left( \bigcup_{n \in \mathbb{N}^d} B_{n,i}^c \right). \] (4.3.27)
Lemma 4.13. Let $\sum get the general case. Using inequality $x_p/2 - d \geq pd/(2-p)$ yields

$$
\sum_{n \in J_N} \min_{1 \leq i \leq d} P (A_n \cap B_{n,i}) \leq d^{d-1} e^{-x^{2-q}/2} \leq d \frac{1}{x^{r/2} - d} N^{d-x^q/2}.
$$

Using inequality $x^q/2 - d \geq pd/(2-p)$ yields

$$
\sum_{n \in J_N} \min_{1 \leq i \leq d} P (A_n \cap B_{n,i}) \leq d \frac{2 - p}{p} N^{-d/p}.
$$

Moreover, an application of the maximal ergodic theorem gives that

$$
\sum_{i=1}^{d} P \left( \bigcup_{n \in \mathbb{N}^d} B_{n,i}^c \right) \leq x^{-p} \sum_{i=1}^{d} \mathbb{E} \left[ Z_i^2 \right].
$$

The combination of the previous three estimates gives that for all positive $x$

$$
P \{ Z > x \} \leq c_{p,d} \min \left\{ 1, N^{d-x^{-2}} + N^{p(d/2 - p)} + x^{2-q} \sum_{i=1}^{d} \mathbb{E} \left[ Z_i^2 \right] \right\}.
$$

We choose $N := \left[ x^{2-q} \right]$; in this way

$$
P \{ Z > x \} \leq c_{p,d} \min \left\{ 1, x^{-p} + x^{-p} \sum_{i=1}^{d} \mathbb{E} \left[ Z_i^2 \right] \right\}.
$$

If $\sum_{i=1}^{d} \mathbb{E} \left[ Z_i^2 \right] \leq 1$, this gives that $\| Z \|_{p,w} \leq c_{p,d}$ and we replace $m$ by $m/\sqrt{\sum_{i=1}^{d} \mathbb{E} \left[ Z_i^2 \right]}$ to get the general case.

This ends the proof of Lemma 4.12.

Lemma 4.13. Let $Z := \sup_{n \in \mathbb{N}_0} Y_n$, where $Y_n$ is defined by (4.3.1) and $Z_i$ given by (4.3.3).

There exists a constant $C_{r,d}$ depending only on $r$ and $d$ such that for all strictly stationary orthomartingale differences random field $(m \circ T^i, T^{-i} F_0)_{t \in \mathbb{Z}^d}$, the following inequality holds:

$$
\| Z \|_{2,r} \leq C_{r,d} \max_{1 \leq i \leq d} \| Z_i \|_{2,r+2}.
$$

Proof. In this proof, $c$ will denote a constant depending only on $r$ and $d$ and which may change from line to line. We start from the equality

$$
\mathbb{E} \left[ \phi_{2,r} (Z) \right] = \int_0^{+\infty} \phi_{2,r} (x) P \{ Z > x \} dx.
$$

Since $0 \leq \phi_{2,r} (x) \leq cx (\log (1 + x))^r$ and if $x \leq e^{2(d+1)}$, $\phi_{2,r} (x) \leq c$, we derive that

$$
\mathbb{E} \left[ \phi_{2,r} (Z) \right] \leq c + c \int_{e^{2(d+1)}}^{+\infty} x (\log (1 + x))^r P \{ Z > x \} dx.
$$

For $1 \leq n \leq N1$, we control $P (A_n)$ by using Chebyshev’s and Doob’s inequality in order to get

$$
\sum_{1 \leq n \leq N1} P (A_n) \leq 2^d N^{d-x^{-2}}.
$$

(4.3.28)
An application of Lemma 4.11 with $N$ such that $\ln N > 2$ (for example $N = 10$) gives that for some constant $c$,

$$
\mathbb{E} [\varphi_{2,r} (Z)] \leq c \left( 1 + \int_{e^{2(d+1)}}^{+\infty} x (\log (1 + x))^r \mathbb{P} \{ |m| > x \} \, dx \
+ \int_{e^{2(d+1)}}^{+\infty} x^{1-\ln 10} (\log (1 + x))^r \, dx \\
+ \sum_{i=1}^{d} \int_{e^{2(d+1)}}^{+\infty} x (\log (1 + x))^r \int_{1/\sqrt{T}}^{+\infty} v \mathbb{P} \left\{ Z_i > \frac{x}{\sqrt{\ln x}} \right\} \, dv \right). \quad (4.3.37)
$$

The second term does not exceed $\mathbb{E} [\varphi_{2,r} (|m|)]$. The third term of (4.3.37) is a constant depending on $p$ and $r$. Therefore,

$$
\mathbb{E} [\varphi_{2,r} (Z)] \leq c \left( 1 + \mathbb{E} [\varphi_{2,r} (|m|)] \\
+ \sum_{i=1}^{d} \int_{1/\sqrt{T}}^{+\infty} \int_{e^{2(d+1)}}^{+\infty} x (\log (1 + x))^r v \mathbb{P} \left\{ Z_i > \frac{x}{\sqrt{\ln x}} \right\} \, dx \, dv \right). \quad (4.3.38)
$$

Bounding the integral over $x$ by the corresponding one on $(2, +\infty)$, cutting this interval into intervals of the form $\langle 2^{k}, 2^{k+1} \rangle$, we end up with the inequality

$$
\mathbb{E} [\varphi_{2,r} (Z)] \leq c \left( 1 + \mathbb{E} [\varphi_{2,r} (|m|)] \\
+ \sum_{i=1}^{d} \int_{1/\sqrt{T}}^{+\infty} \sum_{k=1}^{+\infty} 2^{k} \, v \mathbb{P} \left\{ Z_i > \frac{2^{k} v}{\sqrt{k}} \right\} \, dv \right). \quad (4.3.39)
$$

We apply Lemma 4.8 for a fixed $i \in \{1, \ldots, d\}$ and $v > 1/\sqrt{T}$ to $X := 2Z_i/v$ in order to obtain

$$
\mathbb{E} [\varphi_{2,r} (Z)] \leq c \left( 1 + \mathbb{E} [\varphi_{2,r} (|m|)] \\
+ \sum_{i=1}^{d} \int_{1/\sqrt{T}}^{+\infty} \mathbb{E} \left\{ (2Z_i/v)^2 (\ln (1 + 2Z_i/v))^{r+1} 1 \{2Z_i/v > 1\} \right\} \, dv \right). \quad (4.3.40)
$$

Switching the integral and the expectation, we first have to bound the random variable

$$
Z_i' := \int_{1/\sqrt{T}}^{+\infty} v (2Z_i/v)^2 (\ln (1 + 2Z_i/v))^{r+1} 1 \{2Z_i/v > 1\} \, dv. \quad (4.3.41)
$$

A first observation is that if $2Z_i < 1/\sqrt{T}$, the random variable $Z_i'$ vanishes hence

$$
Z_i' \leq 1 \left\{ 2Z_i > 1/\sqrt{T} \right\} \int_{1/\sqrt{T}}^{2Z_i} v (2Z_i/v)^2 (\ln (1 + 2Z_i/v))^{r+1} \, dv. \quad (4.3.42)
$$

$$
Z_i' \leq 1 \left\{ 2Z_i > 1/\sqrt{T} \right\} \frac{(2Z_i)^2}{1/\sqrt{T}} \int_{1/\sqrt{T}}^{2Z_i} v^{-1} (\ln (1 + 2Z_i/v))^{r+1} \, dv \quad (4.3.43)
$$

and the integral is bound by a constant depending on $r$ times $(\ln (1 + Z_i))^{r+2}$ hence

$$
Z_i' \leq c Z_i^2 (\ln (1 + Z_i))^{r+2}. \quad (4.3.44)
$$

We obtain

$$
\mathbb{E} [\varphi_{2,r} (Z)] \leq c \left( 1 + \mathbb{E} [\varphi_{2,r} (|m|)] + \max_{1 \leq i \leq d} \mathbb{E} [\varphi_{2,r+2} (Z_i)] \right). \quad (4.3.45)
$$
Consider $\lambda \geq \max_{1 \leq i \leq d} \|Z_i\|_{2,r+2}$ and also greater than $\|m\|_{2,r}$. Replacing $m$ by $m/\lambda$ in (4.3.45) yields
\[
\mathbb{E}[\varphi_{2,r}(Z/\lambda)] \leq 3c
\] (4.3.46)
Letting $\varphi := \varphi_{2,r}/(3c)$ gives that
\[
\|Z\|_\varphi \leq \max_{1 \leq i \leq d} \|Z_i\|_{2,r+2} + \|m\|_{2,r}.
\] (4.3.47)
Then an application of Lemma 4.5 gives
\[
\|Z\|_{2,r} \leq c \left( \max_{1 \leq i \leq d} \|Z_i\|_{2,r+2} + \|m\|_{2,r} \right).
\] (4.3.48)
Finally, noticing that $\|m\|_{2,r} \leq \|Z_1\|_{2,r+2}$ gives (4.3.45). This ends the proof of Lemma 4.13.

\textbf{End of the proof of Theorem 2.3.} We start by proving (2.3.2) by induction on the dimension. For $d = 1$, this follows from Lemma 4.13.

Assume that (2.3.2) holds for all stationary orthomartingale differences $d - 1$-dimensional random fields (with $d \geq 2$) and all $r \geq 0$. Using Lemma 4.13, we get that for all $d$ dimensional strictly stationary orthomartingale differences random fields, $\|M\|_{2,r} \leq c_{r,d} \max_{1 \leq i \leq d} \|Z_i\|_{2,r+2}$.

By the induction hypothesis applied with $\bar{r} := r + 2$, we get that $\|Z_i\|_{2,r+2} \leq \|m\|_{2,r+2 + 2(d-1)}$, which gives (2.3.2).

Let us show (2.3.1). By Lemma 4.12, we derive that $\|M\|_p \leq c_{p,d} \max_{1 \leq i \leq d} \|Z_i\|_2$. Using (2.3.2), we derive that $\|Z_i\|_2 \leq \|m\|_{2,2(d-1)}$, from which (2.3.1) follows.

This ends the proof of Theorem 2.3. \hfill \Box

\subsection*{4.4. Proof of Theorem 3.1.} The conditions of theorem imply that $f = \lim_{N \to +\infty} \sum_{-N \leq i \leq N} \pi_i (f)$ almost surely. Therefore, for each $n \in \mathbb{N}^d$, inequality
\[
|S_n (f)| \leq \sum_{-N \leq i \leq N} |S_n (\pi_i (f))| \leq \sum_{i \in \mathbb{Z}^d} |S_n (\pi_i (f))| \tag{4.4.1}
\]
holds almost surely hence
\[
\|M (f)\|_p \leq \sum_{i \in \mathbb{Z}^d} \|M (\pi_i (f))\|_p. \tag{4.4.2}
\]
Since $(U^j \pi_i (f))_{j \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the completely commuting filtration $(T^{-j} \mathcal{F}_0)_{i \in \mathbb{Z}^d}$, an application of Theorem 2.3 ends the proof of Theorem 3.1.

\subsection*{4.5. Proof of the results of Section 3.2.}

\textit{Proof of Proposition 3.3.} The proof with be done by induction on the dimension $d$.

In dimension 1, the right hand side of (3.2.6) reads
\[
\sum_{k=0}^{n} \max_{1 \leq i \leq 2^n - k} \left| \sum_{\ell=0}^{i-1} d_{k,\ell} \circ T^{2^k} \right| + \sum_{k=0}^{n} \max_{0 \leq \ell \leq 2^n - k} \left| d_{k,0} \circ T^{2^k} \right|, \tag{4.5.1}
\]
where
\[
d_{k,\ell} = E_{-2^k} [S_{2^k} (f)], \tag{4.5.2}
\]
\[
d_{0,1} = f - E_{-1} [f], \tag{4.5.3}
\]
\[ d_{k,1} = E_{-2^{k-1}} [S_{2^k}(f)] - E_{-2^k} [S_{2^k}(f)], \]  
and the term in (4.5.1) is greater than the right hand side of (3.2.1).

Let \( d \geq 2 \) and suppose that Proposition 3.3 holds for all \( d' \)-dimensional random fields where \( 1 \leq d' \leq d - 1 \). Let \( T \) be a measure preserving \( \mathbb{Z}^d \)-action on a probability space \((\Omega, \mathcal{F}, \mu)\). Let \( \mathcal{F}_0 \subset \mathcal{F} \) be a sub-\( \sigma \)-algebra such that \( T^q \mathcal{F}_0 \subset \mathcal{F}_0 \) for all \( q \in [d] \) and the filtration \((T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d}\) is commuting. Finally let \( f \) be an \( \mathcal{F}_0 \)-measurable function and \( n \in \mathbb{N}^d \). Let \( j \) be such that \( 1 \leq j \leq 2^n \). Observe that

\[ |S_j(f)| \leq \max_{1 \leq i \leq 2^{n-k}} |S_{i_1, \ldots, i_{d-1}, i_d}(f)|. \]

We apply the \( d - 1 \) dimensional case in the following setting:

- \( \bar{n} := \sum_{q=1}^{d-1} n_q e_q; \)
- \( \bar{T}^i = T \sum_{q=1}^{d-1} n_q e_q, \ i \in \mathbb{Z}^{d-1}; \)
- \( \mathcal{F}_0 := \bigvee_{l \in \mathbb{Z}} \mathcal{F}_0; \)
- \( \bar{f} := \sum_{l=0}^{j_d-1} f \circ T^{l e_d}. \)

In view of (4.5.5), we obtain that

\[ |S_j(f)| \leq \sum_{0 \leq k \leq \bar{n}} \sum_{I \subset [d-1], 2^{k} \leq |I| \leq 2^{n-k}} \max_{0 \leq k \leq d-1} |S_{1, \ldots, 1, k}(f)|. \]

where

\[ \bar{d}_{k,I} := \sum_{I' \subset I \setminus \mathcal{Z}(k)} \sum_{I'' \subset I \cap \mathcal{Z}(k)} (-1)^{|I'| + |I''|} E_{-2^{k-1}, I'' - 1_{I'' + 1_{d}}(4.5.6)} \left[ \bar{S}_{2^{k+1}2^{(k-1)} - 1_{I}} \left( f \right) \right]. \]

Observe that for all \( \ell_1, \ldots, \ell_{d-1} \in \mathbb{Z} \), the \( \sigma \)-algebra \( \mathcal{F}_{\ell_1, \ldots, \ell_{d-1}, \infty} \) is invariant by \( T^{e_d} \). Consequently, we can write

\[ \bar{d}_{k,I} = \sum_{\ell_d=0}^{j_d-1} \bar{f}_{k,1} \circ T^{\ell_d e_d}, \]

where

\[ \bar{f}_{k,1} = \sum_{I'' \subset I \setminus \mathcal{Z}(k)} \sum_{I'' \subset I \cap \mathcal{Z}(k)} (-1)^{|I'| + |I''|} E_{-2^{k-1}, I'' - 1_{I''} + 1_{d}} \left[ \bar{S}_{2^{k+1}2^{(k-1)} - 1_{I}} \left( f \right) \right]. \]

Using commutativity of the filtration \((T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d}\) and \( \mathcal{F}_0 \)-measurability of \( f \), we derive that

\[ \bar{f}_{k,I} = \bar{S}_{2^{k+1}2^{(k-1)} - 1_{I}} \left( f \right) \]

hence

\[ \bar{f}_{k,1} = \sum_{I'' \subset I \setminus \mathcal{Z}(k)} \sum_{I'' \subset I \cap \mathcal{Z}(k)} (-1)^{|I'| + |I''|} E_{-2^{k-1}, I'' - 1_{I''} \circ T^{\ell_d e_d}} \left[ \bar{S}_{2^{k+1}2^{(k-1)} - 1_{I}} \left( f \right) \right]. \]

Since for all \( I \subset [d - 1] \) and \( i \in \mathbb{Z}^{d-1} \), the operators \( S^I \) and \( T^{e_d} \) commute, the previous rewriting of \( \bar{d}_{k,I} \) combined with (4.5.6) yields

\[ |S_j(f)| \leq \sum_{0 \leq k \leq \bar{n}} \sum_{I \subset [d-1], 2^{k} \leq |I| \leq 2^{n-k}} \max_{0 \leq k \leq d-1} \left| \sum_{l_d = 0}^{j_d-1} \left( S^I \left( \bar{T}^{2^k}, \bar{f}_{k,1} \right) \right) \circ T^{\ell_d e_d} \right|. \]
Fix $k \in \mathbb{Z}^d$ such that $0 \leq k \leq n$, $\sum_{I \subset [d-1]} i \in \mathbb{Z}^d$ such that $1 - 1_I \leq i \leq 2^{n-k}$. We apply the result of Proposition 3.3 to the one dimensional case in the following setting:

- $\tilde{n} = n_d$,
- $\tilde{T}_{\ell_d} = T^{\ell_d e_d}$,
- $\tilde{F}_0 := F_{\infty 1_{[d-1]}}$,
- $\tilde{f} := S_{i}^{f} \left( \tilde{T}^{2^k}, f_{k,l} \right)$.

We get

$$\left| \sum_{\ell_d = 0}^{|d| - 1} \left( S_{i}^{f} \left( \tilde{T}^{2^k}, f_{k,l} \right) \right) \right| \leq \max_{k_d = 0}^{n_d} \left| \sum_{\ell_d = 0}^{|d| - 1} d_{k_d,d} \right| + \max_{k_d = 0}^{n_d} \left| \sum_{0 \leq \ell_d \leq 2^{n-k}} d_{0,d} \right|,$$

where

$$d_{k_d,d} = E_{\infty 1_{[d-1]}} \left( \left| \tilde{f} \right| - E_{\infty 1_{[d-1]}}^{e_d} \left| \tilde{f} \right| \right),$$

$$d_{0,d} = E_{\infty 1_{[d-1]}} \left( S_{2^k e_d} \left( \left| \tilde{f} \right| \right) - E_{\infty 1_{[d-1]}}^{e_d} \left( S_{2^k e_d} \left( \left| \tilde{f} \right| \right) \right) \right).$$

Simplifying the expression gives the wanted result. \hfill \square

**Proof of Theorem 3.4.** Starting from Proposition 3.3, we derive that

$$M(f) \leq c_d \sup_{n > 0} \left| \sum_{q=1}^{2n} \left( L(n_q) \right)^{1/2} \right| \max_{1 - 1_I \leq i \leq 2^{n-k}} \left| S_{i}^{f} \left( T^{2^k}, d_{k,l} \right) \right|,$$

where $d_{k,l}$ is given by (3.2.7). For each $I \subset [d]$ and each $k$ such that $0 \leq k \leq n$, the following inequalities take place:

$$\left| \sum_{q=1}^{2n} \left( L(n_q) \right)^{1/2} \right| \max_{1 - 1_I \leq i \leq 2^{n-k}} \left| S_{i}^{f} \left( T^{2^k}, d_{k,l} \right) \right| \leq \frac{1}{2^k} \sup_{m > 0} \left| \sum_{q=1}^{2m} \left( L(m_q) \right)^{1/2} \right| \max_{1 - 1_I \leq i \leq 2^{m-k}} \left| S_{i}^{f} \left( T^{2^k}, d_{k,l} \right) \right|,$$

and combining with (4.5.17), we derive that

$$M(f) \leq c_d \sum_{I \subset [d]} \sum_{k > 0} \left| \sum_{q=1}^{2k} \sup_{m > 0} \left| \sum_{q=1}^{2m} \left( L(m_q) \right)^{1/2} \right| \max_{1 - 1_I \leq i \leq 2^{m-k}} \left| S_{i}^{f} \left( T^{2^k}, d_{k,l} \right) \right|.$$

Consequently,

$$\left\| M(f) \right\|_{p,w} \leq c_d \sum_{I \subset [d]} \sum_{k > 0} \left| \sum_{q=1}^{2k} \sup_{m > 0} \left| \sum_{q=1}^{2m} \left( L(m_q) \right)^{1/2} \right| \max_{1 - 1_I \leq i \leq 2^{m-k}} \left| S_{i}^{f} \left( T^{2^k}, d_{k,l} \right) \right|,$$

(4.5.19)
where
\[ c_{k,I} := \left\| \sup_{m > 0} \frac{1}{|2^m|^{1/2}} \prod_{q=1}^d (L(m_q))^{1/2} \max_{1 \leq i < 2^m} \left| \mathcal{S}_i^I \left( T^{2^k}, d_{k,I} \right) \right| \right\|_{p,w}. \] (4.5.21)

We cannot directly apply Theorem 2.3 because of the particular partial sums \( S_i^I \) defined in (3.2.4). First, it suffices to control \( c_{k,I} \) when \( I = [d] \setminus [i] \), \( 0 \leq i \leq d \). The general case can be deduced form this one by permuting the roles of the operators \( T_c^q \).

We first consider the case where \( i = d \). Then \( I \) is the empty set and due to the definition given by (3.2.4), there is no summation. Therefore,
\[ c_{k,0} = \left\| \sup_{m > 0} \frac{1}{|2^m|^{1/2}} \prod_{q=1}^d (L(m_q))^{1/2} \max_{1 \leq i < 2^m} \left| d_{k,0} \circ T^{2^k,i} \right| \right\|_{p,w}, \] (4.5.22)
and for a fixed \( x \),
\[ \mathbb{P} \left\{ \sup_{m > 0} \frac{1}{|2^m|^{1/2}} \prod_{q=1}^d (L(m_q))^{1/2} \max_{1 \leq i < 2^m} \left| d_{k,0} \circ T^{2^k,i} \right| > x \right\} \leq \sum_{m > 0} \left| 2^m \right| \mathbb{P} \left\{ \left| d_{k,0} \right| > x \left| 2^m \right|^{1/2} \right\}. \] (4.5.23)

Now, taking into account the fact for a fixed \( k \), the number of elements of \( \mathbb{N}^d \) whose sum is \( N \) is bounded by \( C_d N^{d-1} \), an application of (4.1.11) in Lemma 4.8 gives that
\[ c_{k,0} \leq \left\| d_{k,0} \right\|_{2,d-1}. \] (4.5.24)

Assume now that \( 1 \leq i \leq d - 1 \). Let
\[ Y := \sup_{m_i+1 \leq m_q \leq m} 2^{-\frac{1}{2}} \sum_{q=i+1}^d m_q \prod_{q=i+1}^d (L(m_q))^{-1/2} \max_{j_{i+1}, \ldots, j_d} \left| \sum_{\ell_{i+1}=0}^{j_{i+1}-1} \sum_{\ell_d=0}^{j_d-1} U_{\sum_{u=i+1}^{d} 2^{u} \ell_u} d_{k,I} \right|, \] (4.5.25)

With this notation, the inequality
\[ c_{k,I} \leq \left\| \sup_{m_1, \ldots, m_d \geq 0} \frac{1}{|2^m|^{1/2}} \prod_{q=1}^d (L(m_q))^{-1/2} \max_{0 \leq \ell_q \leq 2^{m_q}, 1 \leq q \leq i} \left| Y \circ T^{\sum_{q=1}^{i} \ell_q} \right| \right\|_{p,w}, \] (4.5.26)
holds and with the same arguments as before, it follows that
\[ c_{k,I} \leq C_{p,d} \left\| Y \right\|_{2,i-1}. \] (4.5.27)

Now, we can apply Theorem 2.3 to \( \tilde{d} := d - i \), \( m = d_{k,I} \), \( T^I = T^{2^k} \) and \( r = i - 1 \) to get that
\[ c_{k,I} \leq C_{p,d} \| d_{k,I} \|_{2,i-1 + 2(d - i)} = C_{p,d} \| d_{k,I} \|_{2,2d - i - 1} \leq C_{p,d} \| d_{k,I} \|_{2,2(d - 1)}. \] (4.5.28)

When \( I = [d] \) we can directly apply Theorem 2.3. In total, we got that
\[ \| M(f) \|_{p,w} \leq c_{p,d} \sum_{k > 0} \sum_{I \subset [d]} \left| 2^k \right|^{-1/2} \| d_{k,I} \|_{2,2(d - 1)}. \] (4.5.29)
Now, keeping in mind the definition of \( d_{k,I} \) given by (3.2.7), the following inequality takes place
\[ \| d_{k,I} \|_{2,2(d - 1)} \leq |I|^2 \left\| \mathbb{E}_0 \left[ S_{2^{i+1}Z(k)}^{-1} (x) \right] \right\|_{2,2(d - 1)}. \] (4.5.30)
Now, using the fact that 
\[ \|E_0 \left[ S_{2^{k+1}2^{-1}(k-1)} (f) \right] \|_{2,2(d-1)} \leq c_d \|E_0 \left[ S_{2^k} (f) \right]\|_{2,2(d-1)} + c_d \|f\|_{2,2(d-1)}, \]
we derive that
\[ \|M (f)\|_{p,w} \leq c_{p,d} \sum_{k > 0} |2^k|^{-1/2} \|E_0 \left[ S_{2^k} (f) \right]\|_{2,2(d-1)}. \tag{4.5.31} \]

Now, we have to bound the series in the right hand side of the previous equation in terms of right hand side of (3.2.21). To this aim, we define for fixed \( k_1, \ldots, k_{d-1} \geq 1 \) the quantity
\[ V_n^{(d)} := \|E_0 \left[ S_{2^{k_1}, \ldots, 2^{k_{d-1}}, n} (f) \right]\|_{2,2(d-1)}. \tag{4.5.32} \]

Then the sequence \( \left( V_n^{(d)} \right)_{n \geq 1} \) is subadditive. Therefore, by Lemma 2.7 in [PU05],
\[ \sum_{k > 0} |2^k|^{-1/2} \|E_0 \left[ S_{2^k} (f) \right]\|_{2,2(d-1)} \leq C_d \sum_{k_1, \ldots, k_{d-1}, n \geq 1} 2^{-\frac{3}{2}(k_1 + \cdots + k_{d-1})} \frac{1}{n_{d/2}} \|E_0 \left[ S_{2^{k_1}, \ldots, 2^{k_{d-1}}, n_d} (f) \right]\|_{2,2(d-1)}. \tag{4.5.33} \]

Then defining for a fixed \( n_d \) and fixed \( k_1, \ldots, k_{d-2} \) the sequence
\[ V_n^{(d-1)} := \|E_0 \left[ S_{2^{k_1}, \ldots, 2^{k_{d-2}}, n_{d-1}, n_d} (f) \right]\|_{2,2(d-1)}, \tag{4.5.34} \]
we get an other subadditive sequence. By repeating this argument, we end the proof of Theorem 3.4. \( \square \)

**Proof of Corollary 3.5.** Observe that \( \pi_j (f) = a_j m \) hence (3.3.2) follows.

In order to prove (3.3.3), we first have to simplify \( E_0 [S_n (f)] \). First,
\[ S_n (f) = \sum_{0 \leq i<n-1} \sum_{j \geq 0} a_j m \circ T_i^{-j} = \sum_{0 \leq i<n-1} \sum_{\ell \leq i} a_{i-\ell} m \circ T_{\ell}. \tag{4.5.35} \]

By conditioning with respect to \( \mathcal{F}_0 \), only the terms with index \( \ell \ll 0 \) remain hence
\[ E_0 [S_n (f)] = \sum_{0 \leq i<n-1} \sum_{\ell \leq 0} a_{i-\ell} m \circ T_{\ell}. \tag{4.5.36} \]

By using a combination of Lemmas 3.1 and 6.1 in [Bur73], we derive that for all martingale differences sequence \( (d_j)_{j \geq 1} \),
\[ \left\| \sum_{j=1}^n d_j \right\|_{2,2(d-1)}^2 \leq C_d \sum_{j=1}^n \left\| d_j \right\|_{2,2(d-1)}^2. \tag{4.5.37} \]

By induction on the dimension, this can be extended to sum of orthomartingales differences on a rectangle, then by the use of Fatou’s lemma, we can apply this to summation on \( \mathbb{Z} \). This gives
\[ \|E_0 [S_n (f)]\|_{2,2(d-1)} \leq C_d \left( \sum_{\ell \ll 0} \left\| \sum_{0 \leq i<n-1} a_{i-\ell} m \circ T_{\ell} \right\|_{2,2(d-1)}^2 \right)^{1/2}. \tag{4.5.38} \]
Since $T^\ell$ is measure preserving, the following inequality holds
\[
\left\| \sum_{0 \leq i < n-1} a_i - \ell m \circ T^\ell \right\|_{2,2(d-1)}^2 = \left( \sum_{0 \leq i < n-1} a_i - \ell \right)^2 \|m\|_{2,2(d-1)}^2.
\] (4.5.39)

We can conclude from Theorem 3.4. This ends the proof of Corollary 3.5. □

**Proof of Corollary 3.7.** It suffices to prove the representation of $f$ given by (3.3.6); then inequalities (3.3.7) and (3.3.8) follow from an application of Corollary 3.5 to each linear process involved in (3.3.6). The assumptions imply that
\[
f = \sum_{j \geq 0} \pi_{-j}(f),
\] (4.5.40)
where $\pi_{-j}$ is defined by (3.1.2). Observe that $U^j \pi_{-j}(f)$ belongs to the space $W_d$ defined by (3.3.5). Therefore, this function admits the representation
\[
U^j \pi_{-j}(f) = \sum_{k=1}^{+\infty} a_{k,j}(f) e_k.
\] (4.5.41)
Plugging this equality in (4.5.40) gives (3.3.6). □

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