DUALITY OF ANDERSON $t$-MOTIVES HAVING $N \neq 0$

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Abstract. This paper extends the main result of the paper "Duality of Anderson $t$-motives", that the lattice of the dual of a $t$-motive $M$ is the dual lattice of $M$, to the case when the nilpotent operator $N$ of $M$ is non-zero.

0. Introduction.

Paper [GL07] gives a definition of the dual $M'$ of an Anderson $t$-motive $M$, and proves that the lattice of $M'$ is the dual of the lattice of $M$ — for the case of $M$ with the nilpotent operator $N = 0$. It also contains some explicit formulas for equations defining $M'$.

The present paper extends this result to the case of arbitrary $M$ (Theorem 4.5), i.e. of $M$ having the nilpotent operator $N \neq 0$. The theorem is proved not for all $M$ but for "almost all", namely, only for $M$ satisfying a technical Condition 4.2. Conjecturally, it holds for all $M$. We also demand that $M$ is defined over the affine line $\mathbb{A}^1$.

The same result, and even for more general objects — "virtual Anderson $t$-motives" — was proved in [HJ]. Nevertheless, the result of [HJ] is proved only for mixed t-motives, while the proof of the present paper is free from this restriction. Non-mixed t-motives having dual exist, their explicit equations are described for example in [GL07], Section 11. They are called standard t-motives, results of [GL07] show that many of them have dual, are non-pure and without submotives, hence non-mixed.

Notations of [HJ] differ from the ones of the present paper. We give in 3.5a the relations between these notations.

The proof is in coordinates: we consider Siegel objects — analogs of Siegel matrices for $M$ and for $M'$, and we show that they are — in some sense — mutually dual. A Siegel object of a $t$-motive $M$ of degree $m$ ($= \text{minimal number such that } N^m = 0$) is a set of matrices $S$, depending on 3 parameters, see (3.15). The

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set of variation of these parameters is the set of integer points of a 3-dimensional
tetrahedron such that each edge contains \( m \) points, hence the total quantity of
matrices \( S_* \) is the tetrahedral number \( T_m = \binom{m+2}{3} \).

Also, these Siegel objects can be interpreted as representatives of unilateral cosets
in the maximal Schubert cycle in the corresponding flag variety. See for example
[B] for the theory of flag varieties. Equivalently, they can be interpreted as representatives of unilateral cosets in elements of Hecke algebra. See 3.5b, 3.15a, 3.15b
for more details. It is not too difficult to generalize this description for the case of
non-maximal Schubert cycles.

The dual Siegel object is a set of matrices \( P_* \) depending on the same parameters.
Each of these \( P_* \) is a polynomial in some \( S_* \). For \( P_* \) belonging to a face of the
tetrahedron, these polynomials appear in the formula for elements of the inverse
of an unitriangular matrix (Remark 3.24B). We do not know any interpretation of
other polynomials.

The whole proof is mainly combinatorial: we work with quadruple sums involving
entries of \( S_* \), \( P_* \). In fact, the only links relating this combinatorics of Siegel objects
with Anderson t-motives, are Propositions 2.2, 2.4 and Lemma 4.1.

0.1. For example, same Siegel objects appear for the general situation. Let \( F_s \)
be a “small” subfield of the “big” field \( F_b \), \( V \) a \( n \)-dimensional vector space over \( F_b \),
\( N \) a nilpotent operator on \( V \) such that \( N^m = 0 \), \( \theta \in F_b \) a transcendental element
over \( F_s \). We denote \( R := F_s[\theta \cdot Id_V + N] \). Let \( L \subset V \) be a free \( R \)-module
of dimension \( r \). It is clear that the Siegel objects describe \( L \) in \((V,N)\).

Here we describe some possible generalizations of this subject.

1. Discrete invariants of \( N \) are numbers \( k_1, \ldots, k_{m+1} \) (see (3.3)), they come
from sizes of Jordan blocks of \( N \). For the case \( N = 0 \) ( \( \iff \) \( m = 1 \)) we have
\( (k_1, k_2) = (r - n, n) \). Anderson t-motives of dimension \( n \), rank \( r \), having \( N = 0 \)
are analogs of abelian varieties of dimension \( r \) with multiplication by an imaginary
quadratic field, of signature \((r - n, n)\). Hence, the reductive group associated to
Anderson t-motives of dimension \( n \), rank \( r \), having \( N = 0 \) is \( GU(r - n, n) \). We can
expect that to an Anderson t-motive with invariants \( k_1, \ldots, k_{m+1} \) we can associate
some object which is a generalization of the reductive group \( GU(r - n, n) \).

2. We have \( GU(r - n, n) = GL_r \times G_m \) over \( \mathbb{C} \). We can define analogs of Anderson
t-motives for other reductive groups, for example \( GSp \). What is the analog of the
results of the present paper for them? For example, what is the set of Siegel objects?

3. We can apply the methods of the paper to tensor products of Anderson
t-motives, i.e. to prove Theorem 1.4.

4. It is known that if \( A_1, A_2 \) are abelian varieties then \( A_1 \otimes A_2 \) is a mixed motive.
Is it possible to get an analog of our results for it?

1. Definitions. Let \( q \) be a power of a prime, \( \theta \) a transcendental element, and
\( \mathbb{C}_\infty \) the completion of an algebraic closure of \( \mathbb{F}_q((1/\theta)) \) (topology: \( \theta^{-n} \to 0 \)). Let
\( \mathbb{C}_\infty[T,\tau] \) be the Anderson ring, i.e. the ring of non-commutative polynomials in
two variables \( T, \tau \) satisfying the following relations (here \( a \in \mathbb{C}_\infty \)):

\[
    Ta = aT, \ T\tau = \tau T, \ \tau a = a^q \tau
\]  

(1.1)
and $\mathbb{C}_\infty[T]$, resp. $\mathbb{C}_\infty\{\tau\}$ its subrings of polynomials in $T$, resp. $\tau$.

**Definition 1.2.** ([G], 5.4.2, 5.4.18, 5.4.16). An Anderson t-motive\(^2\) $M$ is a left $\mathbb{C}_\infty[T, \tau]$-module which is free and finitely generated both as $\mathbb{C}_\infty[T]$- and $\mathbb{C}_\infty\{\tau\}$-module, and such that

$$\exists m \text{ depending on } M, \text{ such that } (T - \theta)^m M/\tau M = 0 \tag{1.2.1}$$

The dimension of $M$ over $\mathbb{C}_\infty\{\tau\}$ (resp. $\mathbb{C}_\infty[T]$) is denoted by $n$ (resp. $r$); this number is called the dimension (resp. rank) of $M$.

We shall need an explicit matrix description of t-motives. We denote by $s^t$ the transposition. Let $e_* = (e_1, ..., e_n)^t$ be the vector column of elements of a basis of $M$ over $\mathbb{C}_\infty\{\tau\}$. There exists a matrix $\mathcal{A} \in M_n(\mathbb{C}_\infty\{\tau\})$ (depending on $e_*$) such that

$$Te_* = \mathcal{A}e_*, \quad \mathcal{A} = \sum_{i=0}^l \mathcal{A}_i \tau^i \text{ where } \mathcal{A}_i \in M_n(\mathbb{C}_\infty). \tag{1.3}$$

Condition (1.2.1) is equivalent to the condition

$$\mathcal{A}_0 = \theta I_n + N, \tag{1.3.1}$$

where $N$ (depending on $e_*$) is a nilpotent matrix, and the condition \{we can choose $m(M) = 1\}$ is equivalent to the condition $N = 0$ — because the $\mathbb{C}_\infty$-linear span of $\{e_*\}$ is identified with $M/\tau M$.

The Carlitz module $\mathcal{C}$ is an Anderson t-motive with $r = n = 1$. Let $\{e\} = \{e_1\}$ be the only element of a basis of $M$ over $\mathbb{C}_\infty\{\tau\}$. It is defined uniquely up to the multiplication by an element of $\mathbb{F}_q^*$. $\mathcal{C}$ is given by the equation $Te = \theta e + \tau e$. We have: $e$ also is the only element of a basis of $\mathcal{C}$ over $\mathbb{C}_\infty[T]$, and the multiplication by $\tau$ is given by $\tau e = (T - \theta)e$.

The tensor product of Anderson t-motives $M_1$, $M_2$ is defined by $M_1 \otimes_{\mathbb{C}_\infty[T]} M_2$ where the action of $\tau$ is given by $\tau(m_1 \otimes m_2) = \tau(m_1) \otimes \tau(m_2)$. It is known that $M_1 \otimes M_2$ is really a t-motive of rank $r_1 r_2$, of dimension $n_1 r_2 + n_2 r_1$. $M_1 \otimes M_2$ has $N \neq 0$ even if $M_1$, $M_2$ have $N = 0$.

The $m$-th tensor power of $\mathcal{C}$ is denoted by $\mathcal{C}^m$. Its rank $r$ is 1 and its dimension is $m$.

Let $M$ be a t-motive. We fix $m$ from (1.2.1). We define the $m$-dual of $M$ (denoted by $M'^m$, or simply by $M'$ if $m$ is fixed) by the formula

$$M'^m = \text{Hom}_{\mathbb{C}_\infty[T]}(M, \mathcal{C}^m)$$

where for $\varphi \in M'^m$ (i.e., $\varphi : M \rightarrow \mathcal{C}^m$) the action of $\tau$ on $\varphi$ is defined in the standard manner:

$$(\tau(\varphi))(m) = \tau(\varphi(\tau^{-1}(m)))$$

$M'$ exists not for all t-motives $M$. See [GL07] for a proof that $M'$ exists for all pure $M$ and for large classes of $M$ called standard t-motives ([GL07], Section 11).\(^2\)

\(^2\)Goss calls these objects abelian t-motives.
Remark. Hartl and Juschka in [HJ] consider more general objects (they can be called virtual t-motives). All virtual t-motives have dual.

Let us consider the following exact sequence of left $\mathbb{C}_\infty[T,\tau]$-modules

$$0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow Z_3 \rightarrow 0$$

from [G], (5.9.22). $Z_1$ is defined as follows:

$$Z_1 = \mathbb{C}_\infty\{T\} := \left\{ \sum_{i=0}^{\infty} a_i T^i \mid \lim_{i \rightarrow \infty} a_i = 0 \right\}$$

For any t-motive $M$ of dimension $n$ and rank $r$ we have an exact sequence of $\mathbb{F}_q[T]$-modules

$$0 \rightarrow \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1) \xrightarrow{\zeta} \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_2) \xrightarrow{\exp} \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_3) \quad (1.3.1a)$$

whose terms are denoted by $L(M)$, $\text{Lie}(M)$, $E(M)$ respectively (see [G], 5.9.25, 5.9.17). A fixed basis $e_*$ defines a structure of a $\mathbb{C}_\infty$-vector space of dimension $n$ on $E(M)$; this structure depends on $e_*$. Analogically, $e_*$ defines a structure of a $\mathbb{C}_\infty$-vector space of dimension $n$ on $\text{Lie}(M)$, but it does not depend on a choice of $e_*$, i.e. $\text{Lie}(M)$ has a canonical structure of $\mathbb{C}_\infty$-vector space.

Further, $T$ acts on $\text{Lie}(M)$ as a $\mathbb{C}_\infty$-linear operator. We have $N = T - \theta I_n$ acts on $\text{Lie}(M)$, it can be identified with the dual of $N$ of (1.3.1), because there is an isomorphism of $\mathbb{C}_\infty[T]$-modules $(M/\tau M)^*$ and $\text{Lie}(M)$. Finally, $L(M)$ is a free $\mathbb{F}_q[T]$-module of dimension $\leq r$ (a lattice), and the inclusion $\zeta : L(M) \rightarrow \text{Lie}(M)$ is a homomorphism of $\mathbb{F}_q[T]$-modules.

Condition $\{\exp\text{ is surjective}\}$ is equivalent to the condition $\dim_{\mathbb{F}_q[T]} L(M) = r$ (Anderson; see [G], (5.9.14)). t-motives $M$ satisfying these conditions are called uniformizable. Further on, we consider only uniformizable $M$.

$\zeta$ defines an epimorphism $L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[T - \theta]] \rightarrow \text{Lie}(M)$. Its kernel is denoted by $q_M$; the exact sequence

$$0 \rightarrow q_M \rightarrow L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[T - \theta]] \rightarrow \text{Lie}(M) \rightarrow 0 \quad (1.3.2)$$

is a particular case of the Hodge-Pink structure. It gives us information on the lattice $L(M) \xrightarrow{\zeta} \text{Lie}(M)$.

**Theorem 1.4** (Anderson; see also [P]). Let $M_1$, $M_2$ be two uniformizable t-motives. Then $M_1 \otimes M_2$ is also uniformizable, and

$$q_{M_1 \otimes M_2} = q_{M_1} \otimes_{\mathbb{C}_\infty[[T - \theta]]} q_{M_1} \quad (1.4.1)$$

**Remark 1.5.** Uniformizability of $M_1 \otimes M_2$ follows from [G], Corollary 5.9.38. A proof of Theorem 1.4 for the case: $\{\text{Both } M_1, M_2 \text{ have } N = 0\}$ is given in [GL07], Theorem 6. A proof of Theorem 1.4 can be easily obtained combining the methods of the proof of [GL07], Theorem 6, and of the proof of the present paper.
Theorem 1.4 describes the lattice of $M_1 \otimes M_2$ in terms of the lattices of $M_1$, $M_2$. Clearly there exists an analog of this theorem for $M'$. To formulate it, we need to define $q'_M$ — the generalization of the notion of the dual lattice (see [GL07], Definition 2.3; Section 3) to the case $N \neq 0$. Let $L(M')$ be a free $\mathbb{F}_q[T]$-module dual to $L(M)$, i.e. such that the $\mathbb{F}_q[T]$-pairing $<L(M), L(M')>$ is perfect. We define $q'_M = q_M \subset L(M')' \otimes \mathbb{C}_\infty[[T - \theta]]$ as follows:

$$q'_M = \{ x \in L(M')' \otimes \mathbb{C}_\infty[[T - \theta]] : \forall y \in q_M \text{ we have } <x, y> \in (T - \theta)^m \mathbb{C}_\infty[[T - \theta]] \} \quad (1.6)$$

Roughly speaking, we should have: If $M$ is uniformizable and has dual $M'$ then $M'$ is uniformizable, $L(M')$ is canonically isomorphic to $L(M)'$, and $q'_M = q_M'$. We prove this theorem only for $M$ satisfying Condition 4.2.

2. Results from the theory of $t$-motives.

Let $M$ be an uniformizable $t$-motive. We fix its $\mathfrak{m}$. We use notations $H^1(M)$, $H_1(M)$ from [G], 5.9.11. Suppose that $M$ has the dual $M'$.

Lemma 2.0. $M'$ is uniformizable.

Proof. There exists the canonical (up to multiplication by an element of $\mathbb{F}_q^*$) isomorphism between $H^1(M)$ and $H_1(M') = L(M')$, see [GL21], (1.9). Further on, we have (see [G], (5.9.14)):

$M$ is uniformizable $\iff h_1(M) = r \iff h_1(M) = r \iff h_1(M') = r \iff M'$ is uniformizable. $\square$

Proposition 2.1. There is the canonical (up to multiplication by an element of $\mathbb{F}_q^*$) perfect pairing $L(M) \otimes_{\mathbb{F}_q[T]} L(M') \rightarrow \mathbb{F}_q[T]$.

Proof. This is [GL07], Lemmas 5.3.6, 5.3.7. We can also prove existence of this pairing using the composition of the perfect pairing between $H^1(M)$ and $H_1(M) = L(M)$ (see [G], (5.9.35); Goss uses $E = E(M)$ instead of $M$), and the above isomorphism between $H^1(M)$ and $H_1(M') = L(M')$. $\square$

2.1a. Let $l_1, \ldots, l_r$ be a basis of $L(M)$ over $\mathbb{F}_q[T]$ and $\varphi_1, \ldots, \varphi_r$ the dual basis of $L(M')$ with respect to this pairing. Further, let $f_1, \ldots, f_r$ be a basis of $M$ over $\mathbb{C}_\infty[T]$. Since $M' = \text{Hom}_{\mathbb{C}_\infty[T]}(M, \mathbb{C}^m)$, there exists the dual basis $f'_1, \ldots, f'_r$ of $M'$ over $\mathbb{C}_\infty[T]$ (we fix a basis element of $\mathbb{C}^m$ which is defined up to multiplication by $\mathbb{F}_q^*$). We denote these bases by $\hat{l}, \hat{\varphi}, \hat{f}, \hat{f}'$ respectively.

Since $L(M) = \text{Hom}_{\mathbb{C}_\infty[T]}(M, Z_1)$, for any $l \in L(M)$ and any $f \in M$ there exists $l(f) \in Z_1 = \mathbb{C}_\infty\{T\}$. We denote $l(f)$ by $< l, f >_1$ (first pairing). For given bases $\hat{l}, \hat{f}$, their scattering matrix $\Psi \in M_r(\mathbb{C}_\infty\{T\})$ ([A], p. 486) is defined as follows: $\Psi_{ij} := < l_j, f_i >_1$.

Let us consider $\Xi = \sum_{i=0}^{\infty} a_i T^i \in \mathbb{C}_\infty\{T\}$ of [G], p. 172, line 1; recall that it is the only element (up to multiplication by $\mathbb{F}_q^*$) satisfying

$$\Xi = (T - \theta)\Xi^{(1)}, \text{ i.e. } \Xi = (T - \theta) \sum_{i=0}^{\infty} a_i T^i, \lim_{i \rightarrow \infty} a_i = 0, \ |a_0| > |a_i| \ \forall i > 0$$
(see [G], p. 171, (*)). Ξ satisfies the following condition: let ℂ be the Carlitz module and \( e = f \in ℂ \) satisfies

\[
\tau(f) = (T - \theta)f
\]

Then \( l := Ξf \) is a basis element of the lattice of ℂ, i.e. \( \tau(l) = l \).

Equivalently,

\[
Ξ = c(1 - \theta^{-1}T)(1 - \theta^{-qT})(1 - \theta^{-q^2T})(1 - \theta^{-q^3T})... \quad (2.1.2)
\]

where \( c \in ℂ_∞ \) satisfies \( c^{q-1} = -1/\theta \).

**Proposition 2.2.** Let \( Ψ' \) be the scattering matrix of \( M' \) corresponding to the bases \( \hat{ϕ}, \hat{f}' \). We have \( Ψ' = Ξ^{-m}(Ψ^t)^{-1} \).

**Proof.** For \( m = 1 \) this is [GL07], Lemma 5.4.2. For \( m > 1 \) the proof is similar. Let us give it. We should prove that

\[
Ψ^tΨ' = Ξ^{-m}I_r \quad (2.2.1)
\]

Let \( l_i \) be an element of \( \hat{l} \). We associate it a matrix column \( X = X(l_i, \hat{f}) \), see [GL21], lines above (1.6). \( X \) is the \( i \)-th column of \( Ψ \). Hence, the equality \( Ψ^tΨ' = Ξ^{-m}I_r \) is equivalent to the equality \( X(l_i, \hat{f})X(\varphi_j, \hat{f}') = Ξ^{-m}\delta_{ij} \).

Analogically, for \( z \in H^1(M) \) we associate it a matrix line \( Y(z) \), see [GL21], lines above (1.6).

Let us consider the explicit formula of the isomorphism \( ι : H_1(M') \to H^1(M) \), see [GL21], proof of Proposition 1.9. Namely, let \( \varphi_j \in H_1(M') \). We have \( Y(ι(\varphi_j)) = Ξ^{-m}X(\varphi_j, \hat{f}')^t \) (in [GL21] only the case \( m = 1 \) is considered). The pairing \( H_1(M) \otimes H^1(M) \to ℱ_q[T] \) (see [G], 5.9.35; [GL21], 1.7.1) is defined by \( T Ψ^tΨ' = Ξ^{-m}\delta_{ij} \).

[GL21], (1.7.2). Taking into consideration that \( \hat{l}, \hat{ϕ} \) are mutually dual bases with respect to this pairing, we get immediately (2.2.1). \( □ \)

**Remark 2.3.** Both \( Ξ \) and the dual bases \( \hat{ϕ}, \hat{f}' \) are defined up to multiplication by \( ℱ_q^* \). Proposition 2.2 should be understood in the form that there is a concordant choice of \( Ξ, \hat{ϕ}, \hat{f}' \) such that the formula for \( Ψ' \) is valid.

Now we need a definition of \( θ \)-shift. Let \( ψ = \sum_{i=0}^\infty y_i T^i \in ℂ_∞ \{T\} \). We substitute \( T = N + θ \) (here \( N \) is an abstract symbol). For some (clearly not for all) \( ψ \), as a result of this substitution, we get \( \sum_{j=\kappa}^\infty z_{-j}N^j \in ℂ_∞ \{T\} \) for some \( \kappa \in ℤ \), \( z_\ast \in ℂ_∞ \). In this case, we denote this series \( \sum_{j=\kappa}^\infty z_{-j}N^j \) by \( ψ_N \). Clearly \( θ \)-shift is compatible with the multiplication of series. For a scattering matrix \( Ψ \) we denote by \( Ψ_N \) the result of application of \( θ \)-shift to all entries of \( Ψ \).

An elementary calculation shows that \( (Ξ^{-1})_N \) exists and that its \( \kappa \) is equal to 1. Really, (2.1.2) shows that \( Ξ^{-1} \) has a simple pole at \( T = θ \).

For \( x \in \text{Lie}(M) \) and \( f \in M \) there exists an element \( ∂_x(f) \in ℂ_∞ \), it is canonically defined. See [A] for a definition and [GL20] for explicit formulas in coordinates.

**Proposition 2.4.** For any \( l \in L(M), f \in M \) we have:

\[
<l, f>_N = ∑_{j=\kappa}^\infty z_{-j}N^j \in ℂ_∞ \{T\}
\]

exists, \( \kappa \leq m \), and for \( i \geq 1 \)
\[ z_i = -\partial_{N^{i-1}(\zeta(l))}(f). \]

**Proof.** This is [A], 3.3.2 - 3.3.4. For \( m = 1 \) this is [GL07], Lemma 5.6; for \( m > 1 \) the proof is similar. Let us give it. We consider two pairings. First, for \( l \in L, f \in M \) let \( < l, f > \) be the value of \( l(f) \), where \( l \) is considered as an element of \( \text{Hom}_{C_\infty}(M, Z_1) \), see (1.3.1a).

Second, for \( x \in E(M), f \in M \) let \( < x, f > \) be the value of \( x(f) \), where \( x \) is considered as an element of \( E = \text{Hom}_{C_\infty}(M, C_\infty) \), see [G], proof of 5.6.3, or more explicitly [GL20], Definition 6.2A.

According [G], 5.9.25, we have \( \text{Lie}(M) = \text{Hom}_{C_\infty}(\tau)(M, \mathbb{F}_q((T^{-1}))) \). Hence, for \( z \in \text{Lie}(M), f \in M \) an element \( z(f) \in \mathbb{F}_q((T^{-1})) \) is defined. If \( z = \zeta(l) \) for \( l \in L(M) \) then \( \zeta(l)(f) = < l, f > \in Z_1 \subset \mathbb{F}_q((T^{-1})) \). The explicit formula for \( < l, f > \) is the following:

\[ < l, f > = \sum_{j=0}^{\infty} < \exp(T^{-(j+1)}\zeta(l)), f > T^j \]

(2.4.1)

this follows immediately from the results of [G], 5.9; see also [A], p. 486, the first formula of (3.2), or the first formula of the proof of Lemma 3.2.1. Further on, for \( z \in \text{Lie}(M) \) we denote \( \exp(z) - \alpha(z) \) by \( \varepsilon(z) \), hence \( \sum_{j=0}^{\infty} < \exp(T^{-(j+1)}\zeta(l)), f > T^j = A + E \), where

\[ A = \sum_{j=0}^{\infty} < \alpha(T^{-(j+1)}\zeta(l)), f > T^j; \quad E = \sum_{j=0}^{\infty} < \varepsilon(T^{-(j+1)}\zeta(l)), f > T^j \]

Let us calculate their \( \theta \)-shifts. First, the \( \theta \)-shift of \( E \) has no terms \( N^j \) for \( j < 0 \).

In fact, we can identify \( \text{Lie}(M), E(M) \) with \( C_\infty^n \) in such manner that \( \exp(z) = \sum_{i=0}^{\infty} C_i z^{(i)} \) where \( C_0 = I_n \). Hence, we get that \( \varepsilon(z) = \sum_{i=1}^{\infty} C_i z^{(i)} \). This means that for large \( j \) the element \( \varepsilon(T^{-(j+1)}\zeta(l)) \) is small, and hence \( \kappa(E) = 0 \), because finitely many terms having small \( j \) do not contribute to the pole of the \( \theta \)-shift of \( E \) (the reader can prove easily the exact estimations himself, or to look [A], p. 491).

Now, let us consider \( A_N \). We have

\[ T^{-j} = \sum_{i=0}^{m-1} (-1)^i \left( \begin{array}{c} j + i - 1 \end{array} \right) \theta^{-(j+i)} N^i \]

hence

\[ A_N = \sum_{i=0}^{m-1} (-1)^i \left( \sum_{j=0}^{\infty} \left( \begin{array}{c} j + i \end{array} \right) \theta^{-(j+i+1)} T^j \right) < \alpha(N^i\zeta(l)), f > 2 \]

We have

\[ (-1)^i \sum_{j=0}^{\infty} \left( \begin{array}{c} j + i \end{array} \right) \theta^{-(j+i+1)} T^j = (T - \theta)^{-(i+1)}, \]

hence

\[ A_N = - \sum_{i=0}^{m-1} < \alpha(N^i\zeta(l)), f > 2 N^{-(i+1)} \]
This formula implies the proposition. □

3. Elementary lemmas.

We consider the objects \( F_s, F_b, V, N, \theta, m, L \) from (0.1) for the case \( F_b = \mathbb{C}_\infty \), \( f_s = \mathbb{F}_q \). Let \( l_1, \ldots, l_r \) be a \( \mathbb{F}_q[\theta \cdot I_n + N] \)-basis of \( L \). Hence, at the moment we consider \( V, N, L \) etc. as abstract objects, and not as objects coming from a \( t \)-motive \( M \). Further on, we consider the case when they satisfy the following condition 3.1 (see 3.37A below):

3.1. Elements \( N^i l_j, i = 0, \ldots, m - 1, j = 1, \ldots, r \), generate \( V \) as a \( \mathbb{C}_\infty \)-vector space.

We define numbers \( k_i = k_i(L), i = 2, \ldots, m+1 \) (analog of \( n \) for \( m = 1 \)) as follows:

\[
k_i := \dim(\ker N^{i-1}/\ker N^{i-2}) - \dim(\ker N^i/\ker N^{i-1})
= \dim(\im N^{i-2}/\im N^{i-1}) - \dim(\im N^{i-1}/\im N^i)
\]

(3.3)

Equivalently, let

\[
n = d_1 + \ldots + d_{\alpha}
\]

(3.4)

where \( d_1 \geq d_2 \geq \ldots \geq d_{\alpha} > 0 \), be the partition of \( n \) corresponding to the Jordan form of \( N \), i.e. the Jordan form of \( N \) consists of \( \alpha \) 0-Jordan blocks of sizes \( d_1, d_2, \ldots, d_{\alpha} \). We have \( m \geq d_1, \alpha \leq r \). We shall call a partition with zeroes of length \( r \) of a number \( n \) a representation of \( n \) as a sum

\[
n = \varnothing_1 + \varnothing_2 + \ldots + \varnothing_r
\]

where \( \varnothing_i \in \mathbb{Z} \) and \( \varnothing_1 \geq \varnothing_2 \geq \ldots \geq \varnothing_r \geq 0 \), i.e. a partition with zeroes is a partition plus several zeroes at its end. We extend the partition \( n = d_1 + \ldots + d_{\alpha} \) to a partition with zeroes of length \( r \) denoted by \( p = p(L) \):

\[
n = d_1 + \ldots + d_{\alpha} + d_{\alpha+1} + \ldots + d_r
\]

(3.4.1)

where \( d_{\alpha+1} = \ldots = d_r = 0 \). Let \( n = c_1 + \ldots + c_{d_1} + c_{d_1+1} + \ldots + c_m \) be the partition with zeroes of length \( m \) dual to \( p \) (the definition of the dual partition with zeroes of a given length is clear). We have \( \alpha = c_1 \geq c_2 \geq \ldots \geq c_{d_1} > 0, c_{d_1+1} = \ldots = c_m = 0 \). We have \( \dim \ker N^i = c_1 + \ldots + c_i \), hence (for \( i = m + 1 \) let \( c_{m+1} = 0 \))

\[
k_i = c_{i-1} - c_i
\]

(3.4.2)

We have \( k_i \geq 0 \) (we let \( c_{m+1} = 0 \)), and

\[
n = \sum_{i=1}^{m} ik_{i+1}
\]

(3.5)

We have \( \alpha = c_1 = \sum_{i=2}^{m+1} k_i \), hence \( r \geq \sum_{i=2}^{m+1} k_i \). Let \( k_1 := r - \sum_{i=2}^{m+1} k_i = r - c_1 \). Particularly, (3.4.2) is valid for \( i = 1 \), if we let \( c_0 = r \). For \( m = 1 \) the pair \((k_1, k_2)\) is \((r-n, n)\).
For the reader’s convenience, we indicate the relations between the objects and notations of [HJ] and the present paper. First, instead of

\[ L = [\text{Hom}_{\mathbb{C}_\infty[T]}(M, Z_1)]^r \]

Hartl and Juschka consider the dual \( \mathbb{F}_q[T] \)-module \( \Lambda := [M \otimes_{\mathbb{C}_\infty[T]} Z_1]^r \). They consider virtual Anderson t-motives; the t-motives considered in the present paper are called effective in [HJ]. Further, for an effective \( M \) the lattice \( q \) of [HJ] is the dual of \( q \) of the present paper. Particularly, it contains \( \Lambda \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \) while in the present paper \( q \) is contained in \( L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \). Finally, \( N \) of [HJ] is \( T^{-1} - \theta^{-1} \) and not \( T - \theta \) like in the present paper.

Let \( k_1, \ldots, k_{m+1} \) be from the present paper and \( q = q_M \) from (1.3.2). The Hodge-Pink weights (see [HJ], below Remark 2.4) for this \( q \): \( \omega_1, \omega_2, \ldots, \omega_r \) are

\[-m, \ldots, -m, \ -m + 1, \ldots, -m + 1, \ \ldots, \ 0, \ldots, 0\]

where the number \(-i\) occurs \( k_{i+1} \) times. The Hodge-Pink filtration spaces \( F^i \subset L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \) are the following:

- \( F^{-m} \) is the whole \( L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \);
- \( \dim F^{-i} = k_1 + k_2 + \ldots + k_{i+1} \) \( (i = 0, \ldots, m) \); \( F^1 = 0 \);
- \( \dim Gr_{F^{-i}} = k_{i+1} \).

3.5b. We have

\[ r = \sum_{i=1}^{m+1} k_i. \]

The below theory has an analogy with the theory of partial flag varieties, see for example [B]: our \( m + 1 \), resp. \( r \), resp. \( k_i \) are \( m \), resp. \( n \), resp. \( d_i \) of [B]. Unlike the case of partial flag varieties, in our case some \( k_i \) can be 0.

To simplify exposition, we consider only the maximal Schubert cycle. Hence, using (3.1), we arrange elements \( l_1, \ldots, l_r \) in \( m + 1 \) segments as follows. First, elements \( N^{m-1}l_j, j = 1, \ldots, r \), generate \( N^{m-1}V \) as a \( \mathbb{C}_\infty \)-vector space. Its dimension is \( k_{m+1} \), hence (first step of the process) we can choose \( k_{m+1} \) elements from \( l_1, \ldots, l_r \) (we denote them by \( l_{m+1,1}, \ldots, l_{m+1,k_{m+1}} \), respectively) such that

\[ (3.6) \quad N^{m-1}(l_{m+1,i}), \ i = 1, \ldots, k_{m+1}, \ \text{form a } \mathbb{C}_\infty \text{-basis of } N^{m-1}V. \]

Further on (second step), elements \( N^{m-2}l_j, N^{m-1}l_j, j = 1, \ldots, r \), generate \( N^{m-2}V \) as a \( \mathbb{C}_\infty \)-vector space. Elements \( N^{m-2}(l_{m+1,i}), N^{m-1}(l_{m+1,i}), i = 1, \ldots, k_{m+1} \), are linearly independent over \( \mathbb{C}_\infty \). Indeed, let

\[ \sum_{\alpha_2=1}^{k_{m+1}} c_{\alpha_2} N^{m-2}(l_{m+1,\alpha_2}) + \sum_{\alpha_1=1}^{k_{m+1}} c_{\alpha_1} N^{m-1}(l_{m+1,\alpha_1}) = 0 \quad (3.7) \]

be a non-trivial dependence relation. Applying \( N \) to (3.7) we get

\[ \sum_{\alpha_2=1}^{k_{m+1}} c_{\alpha_2} N^{m-1}(l_{m+1,\alpha_2}) = 0. \]
This contradicts (3.6). Hence, all \( c_{\alpha_2} \) are 0, and this fact also contradicts (3.6).

We have \( \dim N^{m-2}V = 2k_{m+1} + k_m \); this follows immediately from (3.3). Hence, we get:

Among \( l_1, \ldots, l_r \) there exist \( k_m \) elements (we denote them by \( l_{m,1}, \ldots, l_{m,k_m} \), respectively) such that their intersection with \( l_{m+1,1}, \ldots, l_{m+1,k_{m+1}} \) is empty and such that

\[
N^{m-2}(l_{m,i}), \ i = 1, \ldots, k_m, N^{m-2}(l_{m+1,i}), \ i = 1, \ldots, k_{m+1}, N^{m-1}(l_{m+1,i}), \ i = 1, \ldots, k_{m+1}, \ \text{form a } C_\infty\text{-basis of } N^{m-2}V.
\]

Third step of the process: elements \( N^{m-3}l_j, N^{m-2}l_j, N^{m-1}l_j, j = 1, \ldots, r, \) generate \( N^{m-3}V \) as a \( C_\infty \)-vector space. Elements \( N^{m-3}(l_{m,i}), N^{m-2}(l_{m,i}), i = 1, \ldots, k_m, \) and \( N^{m-3}(l_{m+1,i}), N^{m-2}(l_{m+1,i}), N^{m-1}(l_{m+1,i}), i = 1, \ldots, k_{m+1}, \) are linearly independent over \( C_\infty \) (the proof is exactly the same as that of the second step). Hence, we get:

Among \( l_1, \ldots, l_r \) there exist \( k_{m-1} \) elements (we denote them by \( l_{m-1,1}, \ldots, l_{m-1,k_{m-1}} \), respectively) such that their intersection with \( l_{m,1}, \ldots, l_{m,k_m}, l_{m+1,1}, \ldots, l_{m+1,k_{m+1}} \) is empty and such that

\[
N^{m-3}(l_{m-1,i}), \ i = 1, \ldots, k_{m-1}, N^{m-3}(l_{m,i}), N^{m-2}(l_{m,i}), i = 1, \ldots, k_m, \) and
\[
N^{m-3}(l_{m+1,i}), N^{m-2}(l_{m+1,i}), N^{m-1}(l_{m+1,i}), i = 1, \ldots, k_{m+1}, \ \text{form a } C_\infty\text{-basis of } N^{m-3}V.
\]

Continuing this process, we represent \( r \) as an ordered partition

\[
r = k_1 + \ldots + k_{m+1}
\]

(recall that some \( k_* \) can be 0), and we represent the set \( \{l_1, \ldots, l_r\} \) (after some permutation of its elements if necessary) as a union of segments

\[
\{l_1, \ldots, l_r\} = \bigcup_{u=1}^{m+1} \{l_{u1}, \ldots, l_{uk_u}\}
\]

(the union is ordered and disjoint) such that \( \forall \ u = 0, \ldots, m - 1 \) we have:

(3.12) A \( C_\infty \)-basis of \( N^uV \) is formed by elements \( N^{\alpha}(l_{\beta\gamma}) \), where \( \alpha \in [u, \ldots, m - 1], \beta \in [\alpha + 2, \ldots, m + 1], \gamma \in [1, \ldots, k_\beta] \).

This implies that for any

\[
u = u - 1
\]

there exist matrices \( S_{uvyz} \) of size \( k_u \times k_y \) with entries in \( C_\infty \) (analogs of the Siegel matrix for the case \( m = 1 \)) such that \( \forall \ i = 1, \ldots, k_u \) the following holds:

\[
N^{u-1}l_{ui} = - \sum_{z=u-1}^{m-1} \sum_{y=z+2}^{m+1} \sum_{j=1}^{k_y} (S_{uvyz})_{ij} N^z y_j
\]
(if some $k_*$ are 0 then the corresponding $S_{****}$ do not exist). The whole set $S_{****}$ is called a Siegel object for $(V, N, L)$.

3.15a. Let us indicate an analogy with the theory of flag varieties. Let $G(d, n)$ be a Grassmann variety and $I \subset \{1, \ldots, n\}$ of order $d$. There is a subgroup $U^I$ of $GL_n(\mathbb{C})$ (notations of [B], below Definition 1.1.2) describing elements of Schubert cell $C_I$. Clearly the set of our $S_{****}$ is an analog of the group $U^I$ for the case of partial flag varieties and maximal Schubert cells.

3.15b. There exists a closer analogy with the theory of Hecke correspondences. Let $\Gamma = GL_r(\mathbb{C}[[N]])$ (here $N$ is an abstract variable). The double coset

$$\Gamma \ \text{diag} \ (1, \ldots, 1, N, \ldots, N, \ldots, N) \ \Gamma$$

(where $N^i$ appears $k_{i+1}$ times) is a union of left cosets. The set of these left cosets is partitioned to the set of Schubert cells of the corresponding flag variety, and the set of left cosets corresponding to the maximal Schubert cell is exactly the set of the above $S_{****}$.

To simplify the formulas, below, for any $\alpha$ we consider $\hat{l}_\alpha := l_{\alpha^*}$ as matrix columns. (3.15) becomes a matrix equality

$$N^{u-1} \hat{l}_u = - \sum_{z=u-1}^{m-1} \sum_{y=z+2}^{m+1} S_{uvyz} N^z \hat{l}_y$$

(3.16)

**Remark 3.17.** Since always $v = u - 1$, in fact, the matrices $S_{uvyz}$ depend on 3 parameters $u, y, z$ satisfying (3.13). Their set is the set of integer points in a tetrahedron. Number $v$ indicates the exponent of $N$ in the left hand side of (3.15), by analogy with $z$, which indicates the exponent of $N$ in the right hand side of (3.15). This notation is convenient to define a symmetry between $S_*$ and $P_*$, see below.

3.18. Example for $m = 3, u = 1$:

$$l_{1i} = - (\sum_{j=1}^{k_2} (S_{1020})_{ij} l_{2j} + \sum_{j=1}^{k_3} (S_{1030})_{ij} l_{3j} + \sum_{j=1}^{k_4} (S_{1040})_{ij} l_{4j}$$

$$+ \sum_{j=1}^{k_3} (S_{1031})_{ij} N(l_{3j}) + \sum_{j=1}^{k_4} (S_{1041})_{ij} N(l_{4j})$$

$$+ \sum_{j=1}^{k_4} (S_{1042})_{ij} N^2(l_{4j})$$

(terms of a fixed column of this formula correspond to a fixed $y$ and different $z$ of (3.15), and terms of a fixed row of this formula correspond to a fixed $z$ and different $y$ of (3.15)).

Applying powers of $N$ to (3.16), for any $v \in [0, \ldots, m - 1]$, $u \in [1, \ldots, m + 1]$ we can represent $N^v(\hat{l}_u)$ as a linear combination of $N^z(\hat{l}_u)$ where for a fixed $v$ the numbers $z, y$ satisfy

$$z \in [v, \ldots, m - 1], \ y \in [z + 2, \ldots, m + 1],$$

(3.19)

Namely, there exist polynomials in $S_{****}$ denoted by $P_{uvyz}$ such that (matrix notations)
Clearly for $v = u - 1$ we have $P_{uvyz} = S_{uvyz}$.

3.21. The domain $v \geq u - 1 \land \{z, y \text{ satisfy } (3.19)\}$ is called the non-trivial domain of definition of $P_{****}$. 

For $v < u - 1$ (trivial domain) we have:

$$P_{u,v,y,z} = -I_\ast, \text{ resp. } P_{u,v,y,z} = 0$$

for $y, z$ satisfying (3.19), $(y, z) = (u, v)$, resp. $(y, z) \neq (u, v)$.

3.23. Example for $m = 3$:

$$N^2 \hat{l}_2 = (S_{2131}S_{3242} - S_{2141})N^2 \hat{l}_4, \text{ i.e. } P_{2242} = -S_{2131}S_{3242} + S_{2141};$$

$$N^2 \hat{l}_1 = (-S_{1020}S_{2131}S_{3242} + S_{1020}S_{2141} + S_{1030}S_{3242} - S_{1040})N^2 \hat{l}_4, \text{ i.e. } P_{1242} = S_{1020}S_{2131}S_{3242} - S_{1020}S_{2141} - S_{1030}S_{3242} + S_{1040};$$

$$N\hat{l}_1 = (S_{1020}S_{2131} - S_{1030})N\hat{l}_3 + (S_{1020}S_{2141} - S_{1040})N\hat{l}_4 +$$

$$+ (S_{1020}S_{2142} + S_{1031}S_{3242} - S_{1041})N^2 \hat{l}_4, \text{ i.e. } P_{1131} = -S_{1020}S_{2131} + S_{1030}, \text{ } P_{1141} = -S_{1020}S_{2141} + S_{1040},$$

$$P_{1142} = -S_{1020}S_{2142} - S_{1031}S_{3242} + S_{1041}.$$

Remark 3.24. A. Matrices $P_{****}$ are used to form a Siegel object of $M'$. In fact, not all $P_{uvyz}$ form it, but only $P_{u,v,y,y-2}$, see Definition 3.27 below. Hence, the set of these "essential" $P_{****}$ is also a tetrahedron.

B. Although we do not need this fact, let us give a formula for some $P_{****}$. Let us define a block unitriangular matrix $\mathfrak{S}$ whose $(i, j)$-th block is $S_{i,i-1,j,i-1}$ for $j > i$, $I_{k_i}$ for $i = j$ and 0 for $j < i$. Further on, we define a block unitriangular matrix $\mathfrak{B}$ whose $(i, j)$-th block is $-P_{i,j-2,i,j-2}$ for $j > i$, $I_{k_i}$ for $i = j$ and 0 for $j < i$. We have $\mathfrak{B} = \mathfrak{S}^{-1}$ (a proof follows immediately from the lemmas below).

Some $P_{****}$ that enter in the below formula for $\bar{B}$ are not of the form of the elements of the inverse unitriangular matrix, for example $P_{1142}, m = 3$.

For the proof of Lemmas 3.32, 3.39, we need

Lemma 3.25. For all $i, j, \psi, \xi$ satisfying $i \in [2, \ldots, m + 1], j \in [1, \ldots, m - i + 2], \xi \in [m - j, \ldots, m - 1], \psi \in [\xi + 2, \ldots, m + 1]$ we have

$$\sum_{\beta=0}^{j+\xi-m} \sum_{\alpha=i+\beta}^{m+1-j+\beta} S_{i-1,i-2,\alpha,i-2+\beta} P_{\alpha,m-j+\beta,\psi,\xi} - S_{i-1,i-2,\psi,i-2+\xi+j-m} + P_{i-1,m-j,\psi,\xi} = 0$$

(A recurrent formula for $P_{****}$).

Proof. First, we rewrite (3.16) for $u = i - 1$:
\[ N^{i-2}l_{i-1} = - \sum_{z=i-2}^{m+1} \sum_{y=z+2}^{m+1} S_{i-1,i-2,y,z} N^z l_y \] (3.25.2)

Now, for any
\[ j = 1, \ldots, m - i + 2 \] (3.25.3)
we apply \( N^{m-i+2-j} \) to (3.25.2):
\[ N^{m-j}l_{i-1} = - \sum_{z=i-2}^{m+1} \sum_{y=z+2}^{m+1} S_{i-1,i-2,y,z} N^{z+m-i+2-j} l_y \] (3.25.4)

(since \( N^m = 0 \), we get that \( z \leq i + j - 3 \)).

We change the summation variables: \( z \to i - 2 + \beta \), and \( y \to \alpha \), we get
\[ N^{m-j}l_{i-1} = - \sum_{\beta=0}^{j-1} \sum_{\alpha=i+\beta}^{m+1} S_{i-1,i-2,\alpha,i-2+\beta} N^{\alpha+i+3-2j} l_\alpha \] (3.25.5)

Now we use (3.20) making the following variable change:
\[ u \to \alpha \quad y \to \psi \quad v \to m - j + \beta \quad z \to \xi. \]

We get
\[ N^{m-j+\beta}l_\alpha = - \sum_{\xi=m-j+\beta}^{m+1} \sum_{\psi=\xi+2}^{m+1} P_{\alpha,m-j+\beta,\psi,\xi} N^{\xi} l_\psi \] (3.20a)

We substitute (3.20a) in (3.25.5):
\[ N^{m-j}l_{i-1} = \sum_{\beta=0}^{j-1} \sum_{\alpha=i+\beta}^{m+1} \sum_{\xi=m-j+\beta}^{m+1} \sum_{\psi=\xi+2}^{m+1} S_{i-1,i-2,\alpha,i-2+\beta} P_{\alpha,m-j+\beta,\psi,\xi} N^{\xi} l_\psi \] (3.25.6)

We change the order of summation in (3.25.6):
\[ N^{m-j}l_{i-1} = \sum_{\xi=m-j}^{m+1} \sum_{\psi=\xi+2}^{m+1} (\sum_{\beta=0}^{j-1} \sum_{\alpha=i+\beta}^{m+1} S_{i-1,i-2,\alpha,i-2+\beta} P_{\alpha,m-j+\beta,\psi,\xi}) N^{\xi} l_\psi \] (3.25.7)

We rewrite (3.20) making changes:
\[ u \to i - 1 \quad y \to \psi \quad v \to m - j \quad z \to \xi. \]

We get
\[ N^{m-j}l_{i-1} = - \sum_{\xi=m-j}^{m+1} \sum_{\psi=\xi+2}^{m+1} P_{i-1,m-j,\psi,\xi} N^{\xi} l_\psi \] (3.25.8)

For \( \psi \geq \xi + 2 \) elements \( N^{\xi} l_\psi, i = 1, \ldots, k_\psi \), are linearly independent over \( \mathbb{C}_\infty \).
Hence, (3.25.7), (3.25.8) imply
Taking into consideration (3.22) we can rewrite (3.25.9) as follows:

\[ P_{i-1,m-j,\psi,\xi} = -\sum_{\beta=0}^{\psi-m} \sum_{\alpha=i+\beta}^{m+1} S_{i-1,i-2,\alpha,i-2+\beta} P_{\alpha,m-j+\beta,\psi,\xi} \]  \hspace{1cm} (3.25.9)

Here the domain of \( \xi, \psi \) is:

\[ \xi \in [m-j, \ldots, m-1], \quad \psi \in [\xi+2, \ldots, m+1] \]

Taking into consideration (3.22) we can rewrite (3.25.9) as follows:

\[ P_{i-1,m-j,\psi,\xi} = -\left( \sum_{\beta=0}^{\psi-m} \sum_{\alpha=i+\beta}^{m+1} S_{i-1,i-2,\alpha,i-2+\beta} P_{\alpha,m-j+\beta,\psi,\xi} \right) + \]

\[ + S_{i-1,i-2,\psi,i-2+\xi+j-m} \]  \hspace{1cm} (3.25.10)

with the same domain of \( \xi, \psi \). This is (3.25.1). Because of (3.25.3), this formula is valid for \( m-j \geq i-2 \) (the non-trivial case of the definition of \( P_{***} \)). □

Let us consider the symmetry \( s : \mathbb{Z}^4 \to \mathbb{Z}^4 \) defined as follows: \( s(\alpha, \beta, \gamma, \delta) = (m+2-\gamma, m-1-\delta, m+2-\alpha, m-1-\beta) \).

**Remark 3.26.** \( s \) has the following geometric interpretation. Let us consider a matrix \( NL \) whose \((i, j)\)-th entry is a symbol \( N^{i-1}l_j \). We interpret a quadruple \((\alpha, \beta, \gamma, \delta)\) as a vector from \( N^{\delta}l_\alpha \) to \( N^{\delta}l_\gamma \) in \( NL \). \( s \) is the reflection of this vector with respect to the center of \( NL \) and the inversion of its direction.

**Definition 3.27.** \( \tilde{S}_{uvyz} := -P_{s(uvyz)}^t \) (defined if \( P_{s(uvyz)} \) has meaning).

We consider block \( r \times r \)-matrices having the following block structure: their block size is \((m+1) \times (m+1)\), quantities of columns in blocks are \( k_{m+1}, k_m, \ldots, k_1 \) (counting from the left to the right), and quantities of lines in blocks are \( k_1, k_2, \ldots, k_{m+1} \) (counting from up to down). Hence, the \((\alpha, \beta)\)-th block of this matrix is a \( k_\alpha \times k_{m+2-\beta} \)-matrix. These matrices will be called skew \( k_r \)-block matrices.

\( \forall i = 0, \ldots, m \) we define skew \( k_r \)-block matrices \( C_i = C_i(S_{***}) \) as follows:

The \((\alpha, \beta)\)-th block of \( C_i \) is \( S_{m+2-\beta,m+1-\beta,\alpha,i} \) if the quadruple \((m+2-\beta, m+1-\beta, \alpha, i)\) satisfies (3.13, 3.14) \(^3\) (i.e. if \( S_{m+2-\beta,m+1-\beta,\alpha,i} \) exists); the \((i+1, m+1-i)\)-th block of \( C_i \) is \( I_{k_{i+1}} \), all other blocks of \( C_i \) are 0. Namely,

\[ (C_i)_{\alpha\beta} = S_{m+2-\beta,m+1-\beta,\alpha,i}^t \]  \hspace{1cm} (3.28.1)

\[ (C_i)_{i+1,m+1-i} = I_{k_{i+1}} \]  \hspace{1cm} (3.28.2)

\( \forall i = 0, \ldots, m \) we define skew \( k_r \)-block matrices \( \tilde{C}_i = \tilde{C}_i(S_{***}) \) as follows:

The \((\alpha, \beta)\)-th block of \( \tilde{C}_i \) is given by the formula

\[ (\tilde{C}_i)_{\alpha\beta} = -P_{\alpha,m-1-i,m+2-\beta,m-\beta} = S_{\beta,\beta-1,m+2-\alpha,i}^t \]  \hspace{1cm} (3.29)

if the quadruple \((\alpha, m-1-i, m+2-\beta, m-\beta)\) belongs to the non-trivial domain of \( P_{***} \);

\(^3\)Obviously it always satisfies (3.14).
For \( i = 0, \ldots, m \)
\[
(C_i)_{m+1-i, i+1} = I_{k_{m+1-i}} \tag{3.30}
\]

other block entries of \( \bar{C}_i \) are 0.

**Remark 3.31.** Formula (3.30) is concordant with (3.29), if we consider \( P_{****} \) from (3.22). Nevertheless, some 0-blocks of \( \bar{C}_i \) correspond to \( P_{**yz} \) where \( (y, z) \) do not satisfy (3.19), and hence this \( P_{****} \) is not defined.

Finally, we define elements \( B(S_{****}) := \sum_{i=0}^{m} C_i N^i \in M_r(\mathbb{C}_\infty)[N] \) and \( \bar{B}(S_{****}) := \sum_{i=0}^{m} \bar{C}_i N^i \in M_r(\mathbb{C}_\infty)[N] \).

Example for \( m = 3 \):

\[
B(S_{****}) = \begin{pmatrix} 0 & 0 & 0 & I_{k_1} \\ 0 & 0 & 0 & S_{1020}^t \\ 0 & 0 & 0 & S_{1030}^t \\ 0 & 0 & 0 & S_{1040}^t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_2} & 0 \\ 0 & 0 & S_{2131}^t & S_{1031}^t \\ 0 & 0 & S_{2141}^t & S_{1041}^t \end{pmatrix} N^2 \]
\[
+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{k_3} & 0 & 0 \\ 0 & S_{3242}^t & S_{2142}^t & S_{1042}^t \end{pmatrix} N^3
\]

\[
\bar{B}(S_{****}) = \begin{pmatrix} S_{1040}^t & 0 & 0 & 0 \\ S_{1030}^t & 0 & 0 & 0 \\ S_{1020}^t & 0 & 0 & 0 \\ I_{k_4} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{1041}^t & S_{2141}^t & 0 & 0 \\ S_{1031}^t & S_{2131}^t & 0 & 0 \\ 0 & 0 & I_{k_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} N^2 \]
\[
+ \begin{pmatrix} \bar{S}_{1042}^t & \bar{S}_{2142}^t & \bar{S}_{3242}^t & 0 \\ 0 & 0 & I_{k_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} N^3 = \]
\[
= \begin{pmatrix} -P_{1242} & 0 & 0 & 0 \\ -P_{2242} & 0 & 0 & 0 \\ -P_{3242} & 0 & 0 & 0 \\ I_{k_4} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -P_{1142} & -P_{1131} & 0 & 0 \\ -P_{2142} & -P_{2131} & 0 & 0 \\ 0 & 0 & I_{k_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} N^2 \]
\[
+ \begin{pmatrix} -P_{1042} & -P_{1031} & -P_{1020} & 0 \\ 0 & 0 & I_{k_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} N^3
\]

**Lemma 3.32.** \( B(S_{****})^t \cdot \bar{B}(S_{****}) = I_r N^m \in M_r(\mathbb{C}_\infty)[N] \).

**Proof.** We denote \( B(S_{****})^t \cdot \bar{B}(S_{****}) \) by \( \sum_{\mu} C_\mu N^\mu \). The fact that \( C_m = I_r \) is obvious: the only non-zero factors that enter in the sum \( \sum_{\gamma=0}^{m} C_\gamma \bar{C}_{m-\gamma} \) are products of blocks of \( C_\gamma \), \( \bar{C}_\gamma \) containing \( I_r \), and they form \( I_r \). Also it is obvious that for \( \mu > m \) we have \( C_\mu = 0 \), because all products whose sum is \( C_\mu \), have at least
one factor 0. We need to consider $C_\mu$ for $\mu < m$. (3.28.1), (3.28.2), (3.29), (3.30) give us (here and below $(C^t_\gamma)_{\nu\delta}$ is the $(\nu\delta)$-th block of $C^t_\gamma$, i.e. $(C^t_\gamma)_{\nu\delta} = ((C_\gamma)_{\delta\nu})^t$)

$$(C_\mu)_{\nu\pi} = \sum_{\gamma=0}^{\mu} \sum_{\delta=1}^{\mu+1}(C^t_\gamma)_{\nu\delta}(C_\mu-\gamma)_{\delta\pi}$$

(3.32.1.1)

$$=-\sum_{\gamma,\delta} S_{m+2-\nu,m+1-\nu,\delta,\gamma} P_{\delta,m-1-\mu+\gamma,m+2-\pi,m-\pi} + \sum_{\gamma=0}^{\mu} \sum_{\delta=1}^{\mu+1}(C^t_\gamma)_{\nu\delta}(C_\mu-\gamma)_{\delta\pi}$$

(3.32.1.2)

$$-P_{m+2-\nu,2m-\mu-\nu,m+2-\pi,m-\pi} + S_{m+2-\nu,m+1-\nu,m+2-\pi,m-\pi} + \sum_{\gamma=0}^{\mu} \sum_{\delta=1}^{\mu+1}(C^t_\gamma)_{\nu\delta}(C_\mu-\gamma)_{\delta\pi}$$

(3.32.1.3)

where (3.32.1.2) corresponds to the products $(C^t_\gamma)_{\nu\delta}(C_\mu-\gamma)_{\delta\pi}$ where both terms $\neq 0$, $I_\ast$, and (3.32.1.3) corresponds to the products where one of the terms is $I_\ast$.

Let us find the relations satisfied by $\mu$, $\nu$, $\pi$ and the domain of summation by $\gamma$, $\delta$ in (3.32.1.2). We have

$$(C^t_\gamma)_{\nu\delta} \neq 0, I_{k_\ast} \iff \nu \geq m + 1 - \gamma \land \delta \geq \gamma + 2$$

(3.32.2)

$$C_\mu(\gamma)_{\delta\pi} \neq 0, I_{k_\ast} \iff \delta \leq m - (\mu - \gamma) \land \pi \leq \mu - \gamma + 1$$

The set of $\gamma$, $\delta$ is non-empty $\iff \mu \leq m - 2$ and $\mu + \nu - \pi \geq m$. In this case the conditions (3.32.2) on $\gamma$, $\delta$ become

$$\mu + 1 - \pi \geq \gamma \geq m + 1 - \nu$$

(3.32.3.1)

$$m + \gamma - \mu \geq \delta \geq \gamma + 2$$

(3.32.3.2)

Now we use Proposition 3.25 for

$$i = m + 3 - \nu$$

$$j = \mu + \nu - m$$

$$\psi = m - \pi + 2$$

$$\xi = m - \pi$$

$$\mu = j + i - 3$$

$$\nu = m - i + 3$$

$$\pi = m - \xi = m - \psi + 2$$

(3.32.4)

and summation variables $\alpha$, $\beta$ in (3.25.1) are

$$\alpha = \delta$$

$$\beta = \gamma - i + 2$$

(3.32.5)

Under this variable change, (3.32.1.2) becomes the double sum in (3.25.10), and (3.32.1.3) becomes

$$-P_{i-1,m-j,\psi,\xi} + S_{i-1,i-2,\psi,\xi-2+\xi+j-\mu}$$

hence the desired. □

Let for $i = 0, \ldots, m - 1$ $X_i$ be skew $k_\ast$-matrices having the following property:

If $(\alpha, \beta)$ are such that the $(\alpha, \beta)$-block of $\bar{C}_i$ is 0 or $I_\ast$ then $(X_i)_{\alpha\beta} = (\bar{C}_i)_{\alpha\beta}$;

If $(\alpha, \beta)$ are such that the $(\alpha, \beta)$-block of $\bar{C}_i$ is $\neq 0$, $I_\ast$ then $(X_i)_{\alpha\beta}$ is arbitrary.

We denote $X := \sum_{i=0}^{m-1} X_i N_i$.

**Lemma 3.33.** If $B(S_{\ast\ast\ast})^t \cdot X \in N^m M_r(C_{\infty}[N])$ then $X = \bar{B}(S_{\ast\ast\ast})$. 
Proof. For any fixed $\mu, \nu, \pi$ (3.32.1.2), (3.32.1.3) become
\[
\sum_{\gamma, \delta} S_{m+2-\nu, m+1-\nu, \delta, \gamma} (X_{\mu-\gamma})_{\delta, \pi} \]
\[
+(X_{\mu+\nu-m-1})_{m+2-\nu, m+1-\nu, m+2-\pi, \mu+1-\pi} = 0
\] (3.33.1b)
This is system of linear equations with unknowns $(X_i)_{\alpha\beta}$ where
\[
0 \leq i \leq m - 1 \quad 1 \leq \beta \leq i + 1 \quad 1 \leq \alpha \leq m - i
\] (3.33.2)
(for other values of $i, \alpha, \beta$ we have $(X_i)_{\alpha\beta} = 0$ or $I_*$). We arrange $(X_i)_{\alpha\beta}$ in decreasing order of $i + \alpha$ (for $(X_i)_{\alpha\beta}$ having equal $i + \alpha$ their ordering is arbitrary), and we arrange equations (3.33.1) in decreasing order of $\mu$ (for equations having equal $\mu$ the order of equations corresponds to the order of $(X_i)_{\alpha\beta}$ having equal $i + \alpha$). Under this arrangement of unknowns and equations, the matrix of the system (3.33.1) becomes unitriangular. Really, for any $i, \alpha, \beta$ satisfying (3.33.2) there is exactly one values of $\mu, \nu, \pi$ — namely, $\mu = i + \alpha - 1 \quad \nu = m + 2 - \alpha \quad \pi = \beta$
such that the first term of (3.33.1b) is $(X_i)_{\alpha\beta}$. Other terms of (3.33.1) for these $\mu, \nu, \pi$ — namely, the terms that enter in (3.33.1a) — contain $(X_{\mu-\gamma})_{\delta, \pi}$ such that $\mu - \gamma + \delta > i + \alpha$ hence unitriangularity.

Lemma 3.32 affirms that
\[
(X_i)_{\delta, \pi} = -P_{\delta, m-1-i, m+2-\pi, m-\pi}
\]
is a solution to this system. Unitriangularity implies that this solution is unique. □

Let us consider (1.3.2) for the present setting. We have (notations of (0.1)) $R = \mathbb{F}_q[\theta \cdot I_n + N]$. (1.3.2) becomes
\[
0 \to q_L \to L \otimes \mathbb{C}_\infty[[N]] \to V \to 0
\] (3.33a)
By abuse of notations, we shall denote elements $l_i \in L \subset V$ as elements of $V$, and elements $l_i = l_i \otimes 1 \in L \otimes \mathbb{C}_\infty[[N]]$, by the same symbol: there will be no confusion. For example, $N^m l_i \neq 0$ in $L \otimes \mathbb{C}_\infty[[N]]$ while $N^m l_i = 0$ in $V$.

Lemma 3.34. $\forall u = 1, \ldots, m + 1$, for $v = u - 1, \forall i = 1, \ldots, k_u$ the elements
\[
\omega_{ui} := N^v l_{ui} + \sum_{y=u+1}^{m+1} \sum_{z=u-1}^{y-2} \sum_{j=1}^{k_y} (S_{uvyz})_{ij} N^z l_{yj}
\] (3.34.1)
form a basis of $q_L$. □

We denote the set of elements $\omega_{ui}$ ($u$ is fixed, $i$ varies) by $\hat{\omega}_u$ (matrix columns). So, (3.34.1) becomes ($v = u - 1$)
\[
\hat{\omega}_u = N^v \hat{l}_u + \sum_{z=u-1}^{m-1} \sum_{y=z+2}^{m+1} S_{uvyz} N^z \hat{l}_y
\] (3.35)
3.35.1. Let $L' = \text{Hom}_R(L, R)$ be the dual module, and let $\lambda_i$, $i = 1, \ldots, r$ be the basis of $L'$ dual to $\hat{t}$, i.e. $\lambda_i(\hat{t}_j) = \delta_{ij}$. We shall need the dual numbers $k'_i := k_{m+2-i}$ (inverse order of $k_s$). We consider the analogous two-subscript notation of $\lambda_i$, but the order of segments of the partition of $\lambda_i$ is opposite, namely:

$$(\lambda_{m+1,1}, \ldots, \lambda_{m+1,k'_{m+1}}, \lambda_{m,1}, \ldots, \lambda_{m,k'_m}, \ldots, \lambda_{11}, \ldots, \lambda_{1,k'_1}) := (\lambda_1, \ldots, \lambda_r)$$

(order of elements $\lambda_s$ is the same in both sides of this equality). Further, let $q'_L$ be defined by the same formula (1.6), $L'$ instead of $L(M)'$.

Lemma 3.36. $\forall \ u = 1, \ldots, m+1$, for $v = u-1$, $\forall \ i = 1, \ldots, k'_u$ the elements

$$\chi_{ui} := N^v \lambda_{ui} + \sum_{y=u+1}^{m+1} \sum_{z=u-1}^{y-2} \sum_{j=1}^{k'_y} (\bar{S}_{uvyz})_{ij} N^z \lambda_{yj}$$

(3.36.1)

form a basis of $q'_L$.

Proof. As above we denote the set of elements $\lambda_{ui}$, resp. $\chi_{ui}$ ($u$ is fixed, $i$ varies) by $\hat{\lambda}_u$, resp. $\hat{\chi}_u$ (matrix columns). (3.35), (3.36.1) can be written in terms of blocks of $C'_i$, $\bar{C}_i$:

$$\hat{\omega}_u = \sum_{z=0}^{m+1} \sum_{y=1}^{m+1} (C'_t)_{m+2-u,y} N^z \hat{I}_y$$

$$\hat{\chi}_u = \sum_{z=0}^{m+1} \sum_{y=1}^{m+1} (C'_t)_{uy} N^z \hat{\lambda}_{m+2-y}$$

We must prove that $\forall \ u_1, \ u_2$ we have $\hat{\omega}_{u_1} \hat{\chi}_{u_2}^t = \delta_{u_2}^u I_{k_{u_1}} N^m$ (product is pairing).

This is immediate:

$$\hat{\omega}_{u_1} \hat{\chi}_{u_2}^t = \sum_{z=0}^{m} \sum_{y_1=1}^{m+1} \sum_{y_2=0}^{m} \sum_{z_1=0}^{m+1} (C'_{z_1})_{m+2-u_1,y_1} N^{z_1} \hat{I}_{y_1} N^{z_2} \hat{\lambda}_{m+2-y_2} (C'_{z_2})_{y_2,u_2}$$

We have $\hat{I}_{y_1} \hat{\lambda}_{m+2-y_2} = \delta_{y_2}^{y_1} I_{k_{y_1}}$, hence

$$\hat{\omega}_{u_1} \hat{\chi}_{u_2}^t = \sum_{z=0}^{m} \sum_{y_1=1}^{m+1} \sum_{y_2=0}^{m} \sum_{z_2=0}^{m+1} (C'_{z_2})_{m+2-u_1,y_1} N^{z_1} + z_2 (C'_{z_2})_{y_2,u_2} = \sum_{z=0}^{2m} (C_z)_{m+2-u_1,u_2} N^z$$

Lemma 3.32 implies the desired. □

**Corollary 3.37.** Matrices $S_{uvyz}(L')$ for the dual lattice $L'$ are $\bar{S}_{uvyz}(L)$ (order of segments of $\lambda_s$, and hence of numbers $k_s$, is inverse).

3.37A. Until now we considered abstract $L$ and $V$. Now we consider $L$, $V$ coming from $M$. Namely, let $M$ be an uniformizable t-motive such that $N^m = 0$. We let $L = L(M)$, $V = \text{Lie}(M)$, and we consider $\hat{t}$, $S_s$ etc. for them. (3.1) holds for this case, because $L(M) \otimes \mathbb{C}_\infty[[T - \theta]] \rightarrow \text{Lie}(M)$ is an epimorphism, see (1.3.2).
Let
\[ \Psi_N = \sum_{u=-m}^{\infty} D_{-u} N^u \]
be the \( \theta \)-shift of \( \Psi(M) \) (the fact that the series \( \Psi_N \) exists and starts from \( D_m N^{-m} \) follows from Proposition 2.4). We represent each \( D_i \) \( (i = 1, \ldots, m) \) as a union of \( r \times k_j \)-blocks \( D_{ij} \), \( j = 1, \ldots, m+1 \), namely \( D_{i1} \) is a submatrix of \( D_i \) formed by its first \( k_1 \) columns, \( D_{i2} \) is a submatrix of \( D_i \) formed by its next \( k_2 \) columns, etc, until \( D_{i,m+1} \) is a submatrix of \( D_i \) formed by its last \( k_{m+1} \) columns.

Proposition 2.4 implies that \( (D_{ij})_{\alpha\beta} = - < N^{i-1}(l_{j\beta}), f_{\alpha} > \). This means that there are relations between \( D_{ij} \) coming from (3.20), namely:

\[ D_{v+1,u} = - \sum_{z=v}^{m-1} \sum_{y=z+2}^{m+1} D_{z+1,y} P_{uvyz} \]  

(3.38)

Like in (3.21), for \( (z, y) \) satisfying \( y \geq z + 1 \) (resp. \( y < z + 1 \)) we shall call the corresponding \( D_{zy} \) as belonging to the trivial (resp. non-trivial) domain.

**Lemma 3.39.** \( \Psi_N B(S_{\ast\ast\ast\ast}) \in M_r(\mathbb{C}_\infty[[N]]) \).

**Proof.** For any \( \mu = 0, \ldots, m-1 \) we must prove that \( \mathcal{C}_\mu := \sum_{\delta=0}^{\mu} D_{m-\delta} C_{\mu-\delta} \) is 0. The \( \nu \)-th block \( (\nu = 1, \ldots, m+1) \) of this matrix is

\[ (\mathcal{C}_\mu)_\nu := \sum_{\delta=0}^{\mu} \sum_{\gamma=1}^{m+1} D_{m-\delta,\gamma}(C_{\mu-\delta})_{\gamma\nu} \]  

(3.39.1)

We have

\[ (C_{\mu-\delta})_{\gamma\nu} \neq 0, I_{k_{\ast}} \iff \delta \leq \nu + \mu - m - 1 \wedge \gamma \geq \mu - \delta + 2 \]

\[ (C_{\mu-\delta})_{\gamma\nu} = I_{k_{\ast}} \iff \delta = \nu + \mu - m - 1 \wedge \gamma = m + 2 - \nu \]

hence (3.39.1) becomes

\[ (\mathcal{C}_\mu)_\nu = \sum_{\delta=0}^{\nu+\mu-m-1} \sum_{\gamma=\mu-\delta+2}^{m+1} D_{m-\delta,\gamma} S_{m+2-\nu,m+1-\nu,\gamma,\mu-\delta}^t + D_{2m+1-\nu-\mu,m+2-\nu} \]  

(3.39.2.1)

(3.39.2.2)

where (3.39.2.1) is non-empty if \( \nu + \mu \geq m + 1 \).

Terms \( D_{2m+1-\nu-\mu,m+2-\nu} \) always belong to the non-trivial domain. We separate the terms of (3.39.2.1) in terms of trivial and non-trivial domain:

\[ (\mathcal{C}_\mu)_\nu = \sum_{\delta=0}^{\nu+\mu-m-1} \sum_{\gamma=\mu-\delta+2}^{m-\delta} D_{m-\delta,\gamma} S_{m+2-\nu,m+1-\nu,\gamma,\mu-\delta}^t + \sum_{\delta=0}^{\nu+\mu-m-1} \sum_{\gamma=m-\delta+1}^{m+1} D_{m-\delta,\gamma} S_{m+2-\nu,m+1-\nu,\gamma,\mu-\delta}^t \]  

(3.39.3.1)

(3.39.3.2)
\[ +D_{2m+1-\nu-\mu,m+2-\nu} \]  

(3.39.3.3)

Now we substitute non-trivial \( D_{**} \) by linear combinations of the trivial ones, using (3.38):

\[
(\mathcal{E}_\mu)_\nu = - \sum_{\delta=0}^{\nu+\mu-m-1} \sum_{\gamma=\mu-\delta+2}^{m-\delta} \sum_{m=1}^{m-1} \sum_{m=1}^{m+1} D_{z+1,y} P_\gamma, m-\delta-1, y, z S_{m+2-\nu,m+1-\nu, \gamma, \mu-\delta}^t \]

(3.39.4.1)

\[ + \sum_{\delta=0}^{\nu+\mu-m-1} \sum_{\gamma=m-\delta+1}^{m+1} D_{m-\delta, \gamma} S_{m+2-\nu, m+1-\nu, \gamma, \mu-\delta}^t \]

(3.39.4.2)

\[
- \sum_{m=1}^{m-1} \sum_{m=1}^{m+1} D_{z+1,y} P_{m+2-\nu, 2m-\nu-\mu, y, z}^t \]

(3.39.4.3)

Now we change variables in (3.39.4.2):

\[
\delta = m - z - 1
\]

\[
\gamma = y
\]

interchange the order of summation and transpose:

\[
(\mathcal{E}_\mu)_\nu = \sum_{\gamma=2m-\nu-\mu, y = z+2}^{m-1} \sum_{m=1}^{m+1} \mathcal{R}(\mu, \nu, z, y) D_{z+1,y}^t
\]

where

\[
\mathcal{R}(\mu, \nu, z, y) = -\left( \sum_{\delta=m-1}^{\nu+\mu-m-1} \sum_{\gamma=\mu-\delta+2}^{m-\delta} S_{m+2-\nu, m+1-\nu, \gamma, \mu-\delta}^t P_\gamma, m-\delta-1, y, z \right) +
\]

\[
+S_{m+2-\nu, m+1-\nu, \gamma, \mu-\delta}^t P_{m+2-\nu, 2m-\nu-\mu, y, z}
\]

(3.39.5)

Change of variables in (3.39.5):

\[
y = \psi \quad \nu = m + 3 - i \quad \gamma = \alpha
\]

\[
z = \xi \quad \mu = i + \alpha - 3 \quad \delta = \alpha - \beta - 1
\]

transforms (3.39.5) to (3.25.1), hence all \( \mathcal{R}(\mu, \nu, z, y) \) are 0. □

4. Proof.

Let \( M \) be such that \( M' = M'^m \) — its \( m \)-dual — exists and satisfies \( N'^m = 0 \), where \( N \) is of \( M' \). We denote numbers \( k_i, k'_i \), for \( L(M) \), \( k_i(M) \), \( k'_i(M) \).

Lemma 4.1. For \( i = 1, \ldots, m+1 \) we have \( k_i(M') = k'_i(M) \).

Proof. For \( m = 1 \) this is [GL07], Lemma 10.2. For any \( m \) the proof is analogous, let us give it. We denote numbers \( d_i, c_i \) (see (3.4.1) and below) for \( M' \) by \( d'_i, c'_i \) respectively. We need an explicit formula for the action of \( \tau \) on \( M' \). Namely, there exists a matrix \( Q = Q(M, \hat{f}) \in M_r(\mathbb{C}_\infty[T]) \) such that \( Q\hat{f} = \tau\hat{f} \). The matrix \( Q \)
defines \( M \) uniquely. We denote \( Q(M', \hat{\varphi}') \) by \( Q' \). We have (this follows immediately from the definition of \( M' \); see also [GL07], (1.10.1)):

\[
Q' = (T - \theta)^m Q^{t-1}
\]

(here and below \( \ast^{t-1} \) means \( (\ast^t)^{-1} \)).

The module \( \tau M \) is a \( \mathbb{C}_\infty[T] \)-submodule of \( M \) (because \( a \tau x = \tau a^{1/q} x \) for \( x \in M \)), hence there are \( \mathbb{C}_\infty[T] \)-bases \( f_* = (f_1, \ldots, f_r) \), \( g_* = (g_1, \ldots, g_r) \) of \( M \), \( \tau M \) respectively such that \( g_i = P_i f_i \), where \( P_1 P_{r-1} \ldots P_1, \ P_i \in \mathbb{C}_\infty[T] \). Condition (1.2.1) means that \( \forall i \ (T - \theta)^m f_i \in \tau M \), i.e. \( P_i | (T - \theta)^m \). There exists an isomorphism of \( \mathbb{C}_\infty[T] \)-modules \( \text{Lie}(M) \) and \( M/\tau M \), hence \( \forall i \ P_i = (T - \theta)^{d_i} \), where \( d_i \) are from (3.4.1) (numbers \( d_i \) are \( m_{r+1-i} \) from [GL07], Lemma 10.2). There exists a matrix \( \Omega = \{ \xi_{ij} \} \in M_r(\mathbb{C}_\infty[T]) \) such that

\[
\tau f_i = \sum_{j=1}^{r} \xi_{ij} g_j = \sum_{j=1}^{r} \xi_{ij} P_j f_j
\]

(4.1.1)

Although \( \tau \) is not a linear operator, it is easy to see that \( \Omega \in GL_r(\mathbb{C}_\infty[T]) \) (really, there exists \( C = \{ c_{ij} \} \in M_r(\mathbb{C}_\infty[T]) \) such that \( g_i = P_i f_i = \tau(\sum_{j=1}^{r} c_{ij} f_j) \), we have \( C(1) \Omega = I_r \).

We denote the matrix \( \text{diag} (P_1, P_2, \ldots, P_r) \) by \( P \), so (4.1.1) means that \( Q = \Omega P \).

This implies that \( Q' = Q(M') \) satisfies

\[
Q' = \Omega^{t-1} \text{diag} ((T - \theta)^{m-d_r}, \ldots, (T - \theta)^{m-d_1})
\]

This means that \( \forall i = 1, \ldots, r \ d_i' = m - d_{r+1-i} \) and hence \( \forall i = 1, \ldots, m \ c_i' = r - c_{m+1-i} \). The lemma follows from (3.4.2).

We have \( L(M') = L(M') \), see (2.1), (2.1a). Hence, we identify \( \lambda_* \) from (3.3.1) with \( \varphi_* \) from (2.1a). We shall consider another basis \( \hat{\eta} \) of \( L(M') \) obtained by a permutation matrix from the basis (3.3.2). Namely, we let \( \eta_{ij} := \varphi_{ij} \ (i = 1, \ldots, m + 1, \ j = 1, \ldots, k'_j) \), but the order of elements \( \eta_{ij} \) is the following (\( \hat{\eta} \) is a matrix column):

\[
\hat{\eta} = (\eta_{11}, \ldots, \eta_{1,k'_1}, \eta_{21}, \ldots, \eta_{2,k'_2}, \ldots, \eta_{m+1,1}, \ldots, \eta_{m+1,k'_{m+1}})^t
\]

We shall consider only \( M \) satisfying the following (compare with 3.12)

**Condition 4.2.** \( M' \) exists, satisfies \( N^m = 0 \), and for any \( u \) the elements \( N^\alpha(\eta_{\beta \gamma}) \) (where \( \alpha \in [u, \ldots, m - 1], \ \beta \in [\alpha + 2, \ldots, m + 1], \ \gamma \in [1, \ldots, k'_{\beta}] \) are linearly independent over \( \mathbb{C}_\infty \).

According the general principle that almost all \( n \)-uples \( (v_1, \ldots, v_n) \) of vectors in \( n \)-dimensional vector space form a basis of this space, we can guess that almost all \( M \) satisfy 4.2. Really, Lemma 4.1 affirms that the dimension of \( N^u \text{Lie}(M') \) is exactly the quantity of elements \( N^\alpha(\eta_{\beta \gamma}) \) mentioned in 4.2. Again by Lemma 4.1, we see that Condition 4.2 implies that these elements are a basis of \( N^u \text{Lie}(M') \).

**Remark 4.3.** We can use ideas of [GL17] in order to prove rigorously that a large class of Anderson \( t \)-motives \( M \) satisfies Condition 4.2. Namely, the subject of
[GL17] is consideration of Anderson t-motives $M(A)$ of dimension $n$ and rank $2n$ given by the equation
\[ Te_* = \theta e_* + A\tau e_* + \tau^2 e_* \quad (4.3.1) \]

([GL17], (0.8)) where $A \in M_n(\mathbb{C}_\infty)$ is near 0. A Siegel matrix of $M(0)$ is $\omega I_n$ where $\omega \in \mathbb{F}_{q^\infty} - \mathbb{F}_q$. It is shown in [GL17] that any matrix near $\omega I_n$ is a Siegel matrix of some $M(A)$.

We can apply methods of [GL17] to standard-1 t-motives (see [GL07], (11.3); (11.2.1)) having $N \neq 0$. Let us give here its definition. Let $N = N_{ij}$ be the matrix of $N$, i.e. the Jordan matrix with 0-blocks of sizes $d_1, d_2, ..., d_\alpha$ from (3.4). We consider a function (see [GL07], (11.2): $\mathfrak{t} : (1, ..., n) \rightarrow \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the set of integers $\geq 1$. A standard-1 t-motive $M(N, \mathfrak{t}, a_{**})$ is an Anderson t-motive of dimension $n$ given by the formulas ($i = 1, ..., n$):

\[ Te_i = (\theta e_i + \sum_{\alpha=1}^{n} N_{i\alpha}e_\alpha) + \sum_{\alpha=1}^{n} a_{j,i,\alpha} \tau^j e_\alpha + \tau^{\mathfrak{t}(i)} e_i \quad (4.3.2) \]

where $a_{j,i,\alpha} \in \mathbb{C}_\infty$ is the $(i, \alpha)$-th entry of the matrix $\mathfrak{A}_j$ (see (1.3)). This is a simplified version of [GL07], (11.2.1). As an initial t-motive (analog of $M(0)$ of [GL17]) we choose $M(N, \mathfrak{t}, a_{***})$ for $a_{***} = 0$. We denote the set of its $S_{****}$ by $S(N, \mathfrak{t}, 0)$.

Methods of [GL17] show that for any $S_{****}$ near $S(N, \mathfrak{t}, 0)$ there exist $a_{***}$ near 0 such that a Siegel set of $M(N, \mathfrak{t}, a_{**})$ is the given $S_{****}$. It is clear that for almost all $a_{***}$ near 0 the Condition 4.2 holds for $M(N, \mathfrak{t}, a_{**})$.

Really, we have

**Conjecture 4.4.** All $M$ (uniformizable, having dual) satisfy Condition 4.2.

**Theorem 4.5.** Let $M$ satisfy Condition 4.2. Then $q'_{M} = q_{M'}$.

**Proof.** We denote the basis $\varphi_*$ of $L(M')$ by $\hat{\varphi}$ (matrix column). We have $\hat{\eta} = \hat{J} \cdot \hat{\varphi}$ where $\hat{J}$ is a matrix of the change of bases. It is a skew $k'_*$-block anti-

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & I_{k_{m+1}} \\
0 & 0 & \ldots & I_{k_m} & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & I_{k_2} & \ldots & 0 & 0 \\
I_{k_1} & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

identity matrix, i.e. its antidiagonal block entries are identity matrices: the $(m + 2 - i, i)$-block is $I_{k_i}$, and other block entries are 0.

We denote by $\Psi'_{N}$, resp. $\Psi'_{N,\eta}$ the $\theta$-shift of the scattering matrix of $M'$ in bases $\hat{\varphi}$, resp. $\hat{\eta}$ (as a base of $M'$ over $\mathbb{C}_\infty[T]$ we use $\hat{J}'$ in both cases). We have $\Psi'_{N} = \Psi'_{N}^{-1} \Xi_{N}^{-m}$ (Proposition 2.2), $\Psi'_{N,\eta} = \Psi'_{N} \hat{J}'$. Since $M$ satisfies Condition 4.2, there exists a set of Siegel matrices for $M'$ with respect to the basis $\hat{\eta}$. We denote it by $U_{****}$. It defines $B(U_{****})$ — the corresponding $B$ in $M_{r}(\mathbb{C}_{\infty})[N]$.

We denote $\Psi_{N} \cdot B(S_{****})$, resp. $\Psi_{N,\eta} \cdot B(U_{****})$ by $\mathfrak{z}(M)$, resp. $\mathfrak{z}(M^d)$. We have $\mathfrak{z}(M) \in M_{r}(\mathbb{C}_{\infty}[[N]])$, $\mathfrak{z}(M^d) \in M_{r}(\mathbb{C}_{\infty}[[N]])$ (Lemma 3.39), hence

\[ \mathfrak{z}(M)^t \cdot \mathfrak{z}(M^d) = B(S_{****})^t \cdot \Psi_{N}^t \cdot \Psi'_{N} \cdot \hat{J}' \cdot B(U_{****}) = \]

\[ B(S_{****})^t \cdot \hat{J}' \cdot B(U_{****}) \cdot \Xi_{N}^{-m} \in M_{r}(\mathbb{C}_{\infty}[[N]]) \]
and hence

\[ B(S_{****})^t \cdot \mathcal{J}^t \cdot B(U_{****}) \cdot \mathcal{J}^t \in \Xi^m_N M_r(\mathbb{C}_\infty[[N]]) \]

We have \( \mathcal{J}^t \cdot B(U_{****}) \cdot \mathcal{J}^t \) is of the form \( X \) of Lemma 3.33. Further, \( \Xi^m_N \in N^m M_r(\mathbb{C}_\infty[[N]]) \), hence \( \mathcal{J}^t \cdot B(U_{****}) \cdot \mathcal{J}^t = \tilde{B}(S_{****}) \) (Lemma 3.33). This means that \( U_{uvyz} = \tilde{S}_{uvyz} \). The theorem follows from Lemma 3.36. □

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