ON WEAK-STRONG UNIQUENESS FOR COMPRESSIBLE NAVIER-STOKES SYSTEM WITH GENERAL PRESSURE LAWS

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Abstract. The goal of the present paper is to study the weak–strong uniqueness problem for the compressible Navier–Stokes system with a general barotropic pressure law. Our results include the case of a hard sphere pressure of Van der Waals type with a non–monotone perturbation and a Lipschitz perturbation of a monotone pressure. Although the main tool is the relative energy inequality, the results are conditioned by the presence of viscosity and do not seem extendable to the Euler system.

1. Introduction

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded domain. We consider the compressible Navier-Stokes equation in time-space cylinder $(0,T) \times \Omega$:

\begin{align}
\frac{\partial \rho}{\partial t} + \text{div}_x(\rho \mathbf{u}) &= 0, \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \text{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho) &= \text{div}_x S(\nabla_x \mathbf{u}).
\end{align}

Here $S(\nabla_x \mathbf{u})$ is Newtonian stress tensor defined by

\begin{equation}
S(\nabla_x \mathbf{u}) = \mu \left( \frac{\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}}{2} - \frac{1}{d} (\text{div}_x \mathbf{u}) \mathbb{I} \right) + \lambda (\text{div}_x \mathbf{u}) \mathbb{I},
\end{equation}

where $\mu > 0$ and $\lambda > 0$ are the shear and bulk viscosity coefficients, respectively. An external force $f$ can be included in the momentum equation (1.2).

We focus on two basic types of boundary conditions:(i) the no slip boundary condition:

\begin{equation}
\mathbf{u}|_{\partial \Omega \times (0,T)} = 0,
\end{equation}

or (ii) the periodic boundary conditions, where the domain $\Omega$ is identified with the flat torus,

\begin{equation}
\Omega = ([−1,1] \times \mathbb{R}^{d−1})^d.
\end{equation}

We denote $C^{0,1}[0,\infty)$ the space of globally Lipschitz functions. Here we consider the following pressure laws.

- **[Barotropic Law]** In a perturbation of the isentropic setting, the pressure $p$ and the density $\rho$ of the fluid are interrelated by :

\begin{equation}
p(\rho) = \rho^{\gamma} + q(\rho), \text{ with } \gamma \geq 1, \ a > 0 \text{ and } q \in C^{0,1}[0,\infty) \text{ globally Lipschitz}.
\end{equation}

As a matter of fact, the hypothesis on $\gamma$ will reflect the growth of $q$ as $\rho \to \infty$. Note that our goal is not to show existence of solutions but stability of strong solutions in a larger class of weak/measure valued solutions.
• [General pressure] Instead of considering $a\varrho^\gamma$ we can take a more general barotropic equation of state,
\[
p(\varrho) = h(\varrho) + q(\varrho), \quad \text{with } q \in C^{0,1}[0, \infty) \text{ globally Lipschitz},
\]
(1.7)
\[
h \in C^1[0, \infty), \quad h(0) = 0, \quad h' > 0, \quad \text{in } (0, \infty), \quad \text{and } \liminf_{\varrho \to \infty} \frac{h'(\varrho)}{\varrho^\gamma} > 0 \quad \text{with } \gamma \geq 1.
\]

The above hypotheses on the equation of state are motivated by the recent work of Bresch and Jabin [3].

• [Hard-Sphere Law] Finally, we consider a singular pressure law, where the pressure $p$ and the density $\varrho$ of the fluid are interrelated by a non-monotone hard-sphere equation of state in the interval $[0, \bar{\varrho})$:
\[
p \in C^1[0, \bar{\varrho}), \quad p(\varrho) = h(\varrho) + q(\varrho), \quad h(0) = 0,
\]
(1.8)
\[
h' > 0 \text{ on } (0, \bar{\varrho}), \quad \lim_{\varrho \to \bar{\varrho}} h(\varrho) = +\infty, \quad q \in C^1_c(0, \bar{\varrho}).
\]

Compressible Navier–Stokes system has been widely studied by many people in the last few decades.

• If $q \equiv 0$, the relation (1.6) reduces to standard isentropic equation of state, for which the problem (1.1)-(1.3) admits global in time weak solutions for any finite energy initial data, see Antontsev et al. [1] for $d = 1$, Lions [17] for $d = 2, \gamma \geq \frac{3}{2}$, $d = 3$, $\gamma \geq \frac{9}{5}$, and [7] for $d = 2$, $\gamma > 1$, $d = 3$, $\gamma > \frac{3}{2}$. One can go further and introduce the more general class of measure–valued solutions in the spirit of the pioneering work of DiPerna [5]. Regarding compressible Navier-Stokes for $\gamma \geq 1$, Feireisl et al. [9] proved the existence of dissipative measure valued solution.

• If $q \neq 0$, the pressure need not be a monotone function of density. A weak solution however still exists for $\gamma > \frac{3}{2}$ and $N = 3$, see [6]. Recently instead of compactly supported $q$, Bresch-Jabin [3] proved existence for more general pressure.

• The pressure law (1.8) is motivated by two famous models for viscous fluids, namely Van Der Waal’s equation of state and Hard sphere law modeled by Carnahan-Sterling. Now for some constant temperature Van Der Waal’s equation of state gives,
\[
p(\varrho) = C \frac{\tilde{p}(\varrho)}{\tilde{\varrho} - \varrho},
\]
where $\tilde{p}$ is some polynomial and $\tilde{\varrho}$ is a positive constant. The Carnahan-Sterling model reflects the hard sphere model and is given by,
\[
p(\varrho) = C \frac{\tilde{p}(\varrho)}{(\tilde{\varrho} - \varrho)^3},
\]
with $\tilde{p}$ polynomial and $\tilde{\varrho}$ positive constant. The main difference for these models from isentropic setting is here $p(\varrho) \to +\infty$ when $\varrho \to \tilde{\varrho}$. In [11] Feireisl et. al. and [15] Feireisl and Zhang, the existence of global weak solution for similar models was shown.

The weak–strong uniqueness principle asserts that a weak and the strong solution emanating for the same initial data coincide as long as the strong solution exists. The leading idea is based on the concept of relative entropy that goes back to the pioneering paper by Dafermos [4], and that was later exploited in different context by
Berthelin and Vasseur [2], Mellet and Vasseur [18], or Saint–Raymond [19] to name a few examples.

The available results for the compressible Navier–Stokes system are as follows.

- Germain [16] showed the weak–strong uniqueness in a class of weak solutions enjoying extra regularity properties. Unfortunately, the existence of weak solutions in his class is still an open problem.
- Feireisl, Novotný and Sun [13] and Feireisl, Jin and Novotný [10] showed the weak–strong uniqueness result in the existence class for a isentropic (barotropic) pressure equation of state with strictly increasing pressure. These results were extended to the class of the so–called dissipative measure–valued solutions by Feireisl et al. [9].
- Feireisl, Lu and Novotný [12] extended the weak–strong uniqueness principle to the hard–sphere pressure type equation of state, still with strictly monotone pressure–density relation.
- Recently, Feireisl [8] proved weak–strong uniqueness in the class of weak solutions, with a non–monotone compactly supported perturbation of the isentropic equation of state.

Our goal in this paper is to extend the weak–strong uniqueness principle in two directions:

1. To extend the results of [8] to a more general class of non–monotone Lipschitz perturbations.
2. To consider non–monotone compact perturbations of the hard sphere model in the context of weak solutions.

The plan for this paper is as follows:

- In the first part we will discuss about weak–strong uniqueness where pressure is given by (1.6) and (1.7).
- For the later part of the paper we discuss weak strong uniqueness when pressure is given by (1.8).

Part 1. Pressure following equation of state (1.6) and (1.7)

In this part, we focus on the problem with the no-slip boundary conditions (1.4).

2. Dissipative Weak Solution, Main Result

Before going to our formal discussion, define pressure potential as:

- For $h(\rho) = a \rho^\gamma$,

$$P(\rho) = H(\rho) + Q(\rho), \quad \text{where } H(\rho) = \frac{a}{\gamma - 1} \rho^\gamma \text{ and } Q(\rho) = \rho \int_1^\rho \frac{q(z)}{z^2} \, dz. \quad (2.1)$$

- When $p$ is given by the more general formula (1.7),

$$P(\rho) = H(\rho) + Q(\rho) \text{ where }$$

$$H(\rho) = \rho \int_1^\rho \frac{h(z)}{z^2} \, dz, \quad Q(\rho) = \rho \int_1^\rho \frac{q(z)}{z^2} \, dz. \quad (2.2)$$
As a trivial consequence of the above we obtain,
\begin{equation}
\varrho H'(\varrho) - H(\varrho) = h(\varrho) \text{ and } \varrho H''(\varrho) = h'(\varrho) \text{ for } \varrho > 0,
\end{equation}
\begin{equation}
\varrho Q'(\varrho) - Q(\varrho) = q(\varrho) \text{ and } \varrho Q''(\varrho) = q'(\varrho) \text{ for } \varrho > 0.
\end{equation}

We impose some hypothesis on the initial data as,
\begin{equation}
\varrho(0, \cdot) = \varrho_0(\cdot), \quad \varrho_0 \geq 0 \text{ a.e. in } \Omega \text{ and } \varrho_0 \in L^7(\Omega),
\end{equation}
\begin{equation}
\int_\Omega \left( \frac{|\varrho u_0|^2}{\varrho_0} + H(\varrho_0) \right) < \infty.
\end{equation}

The definition of Dissipative Weak Solution is as follows:

**Definition 2.1.** We say that \([\varrho, u]\) is a dissipative weak solution in \((0, T) \times \Omega\) to the system of equations (1.1)-(1.3), with the no-slip condition (1.4), supplemented with initial data (2.4) and pressure follows the law (2.2) and (2.1) if:

- **Regularity Class:** \(\varrho \in C_w([0, T]; L^7(\Omega))\), \(p(\varrho) \in L^1((0, T) \times \Omega)\), \(u \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d))\), \(\varrho u \in C_w([0, T]; L^{\frac{2r}{2-r}}(\Omega; \mathbb{R}^d))\), \(\varrho |u|^2 \in L^\infty(0, T; L^1(\Omega))\).

- **Renormalized equation of Continuity:** For any \(\tau \in (0, T)\) and any \(\varphi \in C^1_c([0, T] \times \Omega)\) it holds
\begin{equation}
\left[ \int_\Omega (\varrho + b(\varrho)) \varphi \; dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ (\varrho + b(\varrho)) \partial_t \varphi + (\varrho + b(\varrho)) u \cdot \nabla_x \varphi + (b(\varrho) - \varrho \varrho' b'(\varrho)) \text{div}_x u \varphi \right] \; dx \; dt,
\end{equation}
where, \(b \in C^1([0, \infty))\), \(\exists r_b > 0\) such that \(b'(x) = 0\), \(\forall x > r_b\).

- **Momentum equation:** For any \(\tau \in (0, T)\) and any \(\varphi \in C^1_c([0, T] \times \Omega; \mathbb{R}^d)\), it holds
\begin{equation}
\left[ \int_\Omega \varrho u(\tau, \cdot) \cdot \varphi(\tau, \cdot) \; dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \varrho \partial_t \varphi + (\varrho u \otimes u : \nabla_x \varphi + p(\varrho) \text{div}_x \varphi) - S(\nabla_x u) : \nabla_x u \right] \; dx \; dt,
\end{equation}

- **Energy inequality:** For a.e. \(\tau \in (0, T)\), the energy inequality holds:
\begin{equation}
\left[ \int_\Omega \left( \frac{1}{2} |u|^2 + P(\varrho) \right)(t, \cdot) \; dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega S(\nabla_x u) : \nabla_x u \; dx \; dt \leq 0.
\end{equation}

2.1. **Discussion of Definition:** Now from (2.2) we have \(H(\varrho) \approx \varrho^\gamma\), and \(Q(\varrho) \approx \varrho \log(\varrho)\) for all \(\varrho\) large enough. In particular, there is a constant \(c > 0\) such that
\begin{equation}
P(\varrho) \geq \frac{1}{2} H(\varrho) - c \text{ for all } \varrho \geq 0
\end{equation}

With help of a limiting procedure in (2.5) we have,
\begin{equation}
\left[ \int_\Omega b(\varrho)(t, \cdot) \; dx \right]_{t=0}^{t=\tau} = -\int_0^\tau \int_\Omega (\varrho b'(\varrho) - b(\varrho)) \text{div}_x u \; dx \; dt.
\end{equation}

The class of functions \(b\) in (2.8) can be extended to those for which both \(b'(\varrho)\varrho\) and \(b(\varrho)\) belong to \(L^2(0, \infty)\). As our goal is to apply (2.8) to the globally Lipschitz
perturbation $q$, we have to assume $\gamma \geq 2$ in the pressure law. Note that similar hypothesis is also used by Bresch and Jabin [3]. Accordingly, we have

$$\left[ \int_{\Omega} Q(\varrho) \, dx \right]_{t=0}^{t=T} = - \int_{0}^{T} \int_{\Omega} q(\varrho) \text{div}_x u \, dx \, dt$$

as long as $q$ is a globally Lipschitz function and $\gamma \geq 2$. Consequently

$$\left[ \hat{\Omega} Q(\varrho) \right]_{t=0}^{t=T} = - \hat{\tau} \int_{0}^{T} \int_{\Omega} q(\varrho) \text{div}_x u \, dx \, dt.$$

2.2. Main Result. Our goal is to show the following result.

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded Lipschitz domain. Let the pressure be given by (1.6) or (1.7), with $\gamma \geq 2$. Suppose that $[\rho, u]$ is a dissipative weak solution and $[r, U]$ a classical solution of the problem (1.1)-(1.3) with no slip boundary condition (1.4) on the time interval $[0, T]$ such that,

$$\rho(0, \cdot) = r(0, \cdot) > 0, \quad \rho u(0, \cdot) = r(0, \cdot) U(0, \cdot).$$

Then

$$\rho = r, \quad u = U \text{ in } (0, T) \times \Omega.$$

**Remark 2.3.** Hypothesis $\gamma \geq 2$ is related to the growth of the perturbation $q$ when $\rho \to \infty$. The result remains valid for any $\gamma \geq 1$ as soon as

$$q'(\rho) \approx \rho^\alpha \text{ for } \rho \to \infty, \quad \text{where } \alpha + 1 \leq \frac{\gamma}{2}.$$

In the next section we will prove the result.

3. Relative Energy and Weak Strong Uniqueness

3.1. Relative Energy. Following [10] and [8] (cf. the standard reference material by Dafermos [4]) we introduce relative energy functional:

$$\mathcal{E}(t) = \mathcal{E}(\rho, u|r, U)(t) := \int_{\Omega} \frac{1}{2} \rho |u - U|^2 + (H(\rho) - H(r) - H'(r)(\rho - r))(t, \cdot) \, dx,$$

where $r, U$ are arbitrary test functions and $[\rho, u]$ in (3.1) is weak solution of (1.1)-(1.3) as in (2.1). By direct calculation we can show that,

$$\mathcal{E}(\tau) = \int_{\Omega} \frac{1}{2} q |u|^2 + H(s) \, dx - \int_{\Omega} \rho u \cdot U \, dx$$

$$+ \int_{\Omega} \frac{1}{2} \rho |U|^2 \, dx - \int_{\Omega} \rho H'(r) \, dx + \int_{\Omega} h(r) \, dx = \Sigma_{i=1}^{5} K_i$$

Now we look for the terms $K_i$ for $i = 1(1)5$. First we note that $K_1$ can be evaluated by means of (2.10). To compute $K_2$ we use (2.6) and for $K_3, K_4$ we use
(2.5) Calculating above terms we get,
\[ [\mathcal{E}(t)]_{t=0}^{\infty} + \int_0^\tau \int_\Omega S(\nabla_x u) : (\nabla_x u - \nabla_x U) \, dx \, dt \]
\[ \leq - \int_0^\tau \int_\Omega \rho u \cdot \partial_t U \, dx \, dt \]
\[ - \int_0^\tau \int_\Omega [\rho u \otimes u : \nabla_x U + h(\rho) \text{div}_x U] \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega \rho U \cdot \partial_t U + \rho u \cdot (U \cdot \nabla_x U) \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega \left[ (1 - \frac{\rho}{\rho(r)} \right] \partial r - \rho u \cdot \frac{h'(r)}{r} \nabla x r \right] \, dx \, dt \]
\[ - \int_0^\tau \int_\Omega q(\rho) \text{div}_x U \, dx \, dt + \int_0^\tau \int_\Omega q(\rho) \text{div}_x u \, dx \, dt \]
(3.3)

Now if we assume \([r, U]\) satisfies (1.1)-(1.3), and these are smooth solution with \(r > 0\) then we have,
\[ [\mathcal{E}(t)]_{t=0}^{\infty} + \int_0^\tau \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt \]
\[ \leq - \int_0^\tau \int_\Omega S(\nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt \]
\[ - \int_0^\tau \int_\Omega (\rho u - \rho U) \cdot (-(U \cdot \nabla_x U - \frac{1}{r} \nabla_x p(r) + \frac{1}{r} \text{div}_x S(\nabla_x U))) \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega h(\rho) \text{div}_x U \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega \left[ (1 - \frac{\rho}{\rho(r)} \right] h'(r) \partial r - \rho u \cdot \frac{h'(r)}{r} \nabla x r \right] \, dx \, dt \]
\[ - \int_0^\tau \int_\Omega q(\rho) \text{div}_x U \, dx \, dt + \int_0^\tau \int_\Omega q(\rho) \text{div}_x u \, dx \, dt \]

Thus we have,
\[ [\mathcal{E}(t)]_{t=0}^{\infty} + \int_0^\tau \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt \]
\[ \leq \int_0^\tau \int_\Omega \left( \frac{\rho}{r} - 1 \right) (U - u) \, (\text{div}_x S(\nabla_x U) - \nabla x q(r)) \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega \rho (u - U) \cdot ((U - u) \cdot \nabla x U) \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega (-h(\rho) + h(r) + h'(r)(\rho - r)) \text{div}_x U \, dx \, dt \]
\[ + \int_0^\tau \int_\Omega (\text{div}_x u - \text{div}_x U)(q(\rho) - q(r)) \, dx \, dt \]
\[ = \Sigma_{i=1}^{4} \mathcal{L}_i. \]

Here \(\mathcal{L}_i\) for \(i = 1(1)4\) have been termed as *remainder terms*.

We know that for our choice of interrelation between pressure and density we have,
Remark 3.3. In our case we choose \( r \) lies on a compact subset of \((0, \infty)\) then we have,

\[
H(q) - H(r) - H'(r)(q - r) \geq c(r) \left\{ \begin{array}{ll}
(g - r)^2 & \text{for } r_1 \leq q \leq r_2, \\
(1 + g) & \text{otherwise}
\end{array} \right.
\]

where, \( c(r) \) is uniformly bounded for \( r \) belonging to compact subsets of \((0, \infty)\).

Hence for (1.6) and (17) we have,

Lemma 3.2. For \( q \geq 0 \),

\[
|h(q) - h(r) - h'(r)(q - r)| \leq C(r)(H(q) - H(r) - H'(r)(q - r)),
\]

where \( C(r) \) is uniformly bounded if \( r \) lies in some compact subset of \((0, \infty)\).

Proof. The proof of both lemmas have been discussed in [13] and [10]. \( \square \)

Remark 3.3. In our case we choose \( r_1, r_2 \) such that they satisfy, \( r_1 < \frac{\inf_{(x,t) \in (0,T) \times \Omega} r(x,t)}{2} \), \( r_2 > 2 \times \sup_{(x,t) \in (0,T) \times \Omega} r(x,t) \) and \( 1 + q \geq \max\{q, q^2\}, \forall q \geq r_2 \).

3.2. Weak strong uniqueness. Now we want to compute remainder terms i.e. \( \mathcal{L}_i \) for \( i = 1, 2, 3, 4 \). For \( \mathcal{L}_2 \) we have,

\[
\int_0^T \int_{\Omega} g(u - U) \cdot ((U - u) \cdot \nabla_x) U \ dx \ dt \leq \|\nabla_x U\|_{C([0,T] \times \Omega)} \int_0^T \mathcal{E}(t) \ dt.
\]

Next for \( \mathcal{L}_3 \) we use lemma (5.2) and obtain,

\[
\int_0^T \int_{\Omega} (-h(q) + h(r) + h'(r)(q - r)) \ div_x U \ dx \ dt
\]

\[
\leq \|\nabla_x U\|_{C([0,T] \times \Omega)} \int_0^T \mathcal{E}(t) \ dt.
\]

Now we focus on \( \mathcal{L}_1 \) and \( \mathcal{L}_4 \). Since \( q \) is globally Lipschitz by Rademacher Theorem \( q \) is almost everywhere differentiable and its derivative is less than the Lipschitz constant \( L_q \). Hence we obtain,

\[
\left| \frac{1}{r} \nabla_x q(r) \right| \leq \frac{L_q}{\inf r} \|r\|_{C^1}.
\]

Consider \( \psi \in C_c^\infty(0, \infty) \), \( 0 \leq \psi \leq 1 \), \( \psi(s) = 1 \) for \( s \in (r_1, r_2) \). Then we have,

\[
(q - r)(U - v) = \psi(q)(q - r)(U - v) + (1 - \psi(q))(q - r)(U - v).
\]

Consequently, we obtain

\[
\psi(q)(q - r)(U - v) \leq \frac{1}{2} \psi^2(q) (q - r)^2 + \frac{1}{2} \psi^2(q) q |U - u|^2.
\]

Now using that \( \psi \) is compactly supported in \((0, \infty)\) and lemma (3.1) we control both the terms by \( \mathcal{E}(\cdot) \). Thus we have,

\[
\int_0^T \int_{\Omega} \psi(q)(q - r)(U - u) \cdot \frac{1}{r} (\text{div}_x S(\nabla_x U) - \nabla_x q(r)) \ dx \ dt
\]

\[
\leq \left( \|\frac{1}{r} (\text{div}_x S(\nabla_x U))\|_{C([0,T] \times \Omega; R^d)} + \frac{L_q}{\inf r} \|r\|_{C^1([0,T] \times \Omega; R^d)} \right) \int_0^T \mathcal{E}(t) \ dt.
\]
We rewrite $1 - \psi(\varrho) = w_1(\varrho) + w_2(\varrho)$, where $\text{supp}(w_1) \subset [0, r_1)$ and $\text{supp}(w_2) \subset (0, r_2]$,

$$(1 - \psi(\varrho))(\varrho - r)(U - u) = (w_1(\varrho) + w_2(\varrho))(\varrho - r)(U - u).$$

For $\delta > 0$ we obtain,

$$w_1(\varrho)(\varrho - r)(U - u) \leq C(\delta)w_1^2(\varrho)(\varrho - r)^2 + \delta|U - u|^2.$$ 

Thus using Poincaré inequality we have,

$$\int_0^T \int_\Omega w_1(\varrho)(\varrho - r)(U - v) \cdot \frac{1}{r}(\text{div}_x S(\nabla_x U) - \nabla_x q(r)) \, dx \, dt \leq C(\|\frac{1}{r}(\text{div}_x S(\nabla_x U))\|_{C([0,T] \times \hat{\Omega};\mathbb{R}^d)} + \frac{L_q}{\inf r}\|r\|_{C^1([0,T] \times \hat{\Omega};\mathbb{R}^d)}), \delta) \int_0^T \mathcal{E}(t) \, dt$$

$$+ \delta \int_0^T \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt .$$

Next,

$$w_2(\varrho)(\varrho - r)(U - v) \leq C(r)(\varrho + \varrho|U - u|^2).$$

Using remark of Lemma [3.3] we obtain

$$\int_0^T \int_\Omega w_2(\varrho)(\varrho - r)(U - u) \cdot \frac{1}{r}(\text{div}_x S(\nabla_x U) - \nabla_x q(r)) \, dx \, dt \leq C(\|\frac{1}{r}(\text{div}_x S(\nabla_x U))\|_{C([0,T] \times \hat{\Omega};\mathbb{R}^d)} + \frac{L_q}{\inf r}\|r\|_{C^1([0,T] \times \hat{\Omega};\mathbb{R}^d)}) \int_0^T \mathcal{E}(t) \, dt$$

So combining (3.11), (3.12) and (3.13) we obtain

$$\int_0^T \int_\Omega (\varrho - r)(U - v) \cdot \frac{1}{r}(\text{div}_x S(\nabla_x U) - \nabla_x q(r)) \, dx \, dt \leq C(\delta, r, U, q) \int_0^T \mathcal{E}(t) \, dt + \delta \int_0^T \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt .$$

Next is term $I_4$,

$$\int_0^T \int_\Omega (\text{div}_x u - \text{div}_x U)(q(\varrho) - q(r)) \, dx \, dt \leq C(\delta)L_q \int_0^T \int_\Omega (\varrho - r)^2 \, dx \, dt + \delta \int_0^T \int_\Omega |\text{div}_x U - \text{div}_x u|^2 \, dx \, dt .$$

Note that, by virtue of our choice of the no-slip boundary conditions (1.4) the last integral is controlled by

$$\int_0^T \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt .$$
Using a similar argument as in the earlier case we can say that,
\[
\int_0^\tau \int_\Omega (\text{div}_x u - \text{div}_x U)(q(\varrho) - q(r)) \ dx \ dt \\
\leq C(\delta, r, q) \int_0^\tau E(t) \ dt + \delta \int_0^\tau \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \ dx \ dt.
\]
Thus combining (3.8), (3.9), (3.14) and (3.15) and choosing \(\delta\) small we obtain,
\[
[\mathcal{E}(t)]_{t=0}^{t=\tau} + \frac{1}{2} \int_0^\tau \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \ dx \ dt \leq C(r, U, q) \int_0^\tau \mathcal{E}(t) \ dt.
\]

3.3. End of the proof.

\textit{Proof of Theorem 3.2.} As \(C(r, U, q)\) in (3.16) is uniformly bounded in \([0, T]\). Hence we apply Grönwall’s inequality and using hypothesis on initial condition we obtain \(\mathcal{E} = 0\) a.e. in \((0, T)\). \(\square\)

Part 2. Pressure following equation of state \(1.8\)

In this part, we focus on the problem endowed with the periodic boundary conditions \(1.5\). Accordingly, the domain \(\Omega\) is here and hereafter identified with the flat torus \(\Omega = ([-1, 1]^d)\). Now considering density-pressure interrelation \(1.8\) we will define weak solution and study the weak-strong uniqueness.

4. Dissipative Weak solution, Main result

We impose some hypothesis on the initial data as,
\[
\varrho(0, \cdot) = \varrho_0(\cdot) \text{ with } 0 \leq \varrho_0 < \bar{\varrho} \text{ in } \Omega, \int_\Omega H(\varrho_0) \ dx < \infty,
\]
\[
u(0, \cdot) = u_0(\cdot), \int_\Omega |u_0|^2 \varrho_0 < \infty.
\]

Weak solution are defined as follows:

\textbf{Definition 4.1.} We say that \((\varrho, \nu)\) is a dissipative weak solution in \((0, T) \times \Omega\) to the system of equations \((1.1 - 1.3)\), with the periodic boundary conditions \((1.5)\), supplemented with initial data \((4.1)\), if:

- \(0 \leq \varrho < \bar{\varrho}\) a.e. in \((0, T) \times \Omega, \varrho \in C_w([0, T]; L^\gamma(\Omega))\) for any \(\gamma > 1, p(\varrho) \in L^1((0, T) \times \Omega), \nu \in L^2([0, T]; W^{1,2}(\Omega; \mathbb{R}^d)), \varrho \nu \in C_w([0, T]; L^2(\Omega; \mathbb{R}^d)), \varrho|\nu|^2 \in L^\infty(0, T; L^1(\Omega)).\)

- For any \(\tau \in (0, T)\) and any test function \(\varphi \in C^\infty([0, T] \times \Omega)\), one has
\[
\int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \varrho \nu \cdot \nabla_x \varphi] \ dx \ dt = \int_\Omega \varrho(\tau, \cdot) \varphi(\tau, \cdot) - \int_\Omega \varrho_0 \varphi(0, \cdot) \ dx.
\]

- For any \(\tau \in (0, T)\) and any test function \(\varphi \in C^\infty([0, T] \times \Omega; \mathbb{R}^d)\), one has
\[
\int_0^\tau \int_\Omega [\varrho \nu \cdot \partial_t \varphi + (\varrho \nu \otimes \nu) : \nabla_x \varphi + p(\varrho) \text{div}_x \varphi - S(\nabla_x \nu) : \nabla_x \varphi] \ dx \ dt
\]

\[
= \int_\Omega \varrho(\tau, \cdot) \cdot \varphi(\tau, \cdot) \ dx - \int_\Omega \varrho_0 u_0 \cdot \varphi(0, \cdot) \ dx.
\]
• The continuity equation also holds in the sense of renormalized solutions:
\[
\left[ \int_{\Omega} (b(\rho)) \varphi \, dx \right]_{t=0}^{t=r} = \int_0^r \int_{\Omega} \left[ b(\rho) \partial_t \varphi + b(\rho) u \cdot \nabla \varphi + (b(\rho) - \rho b'(\rho)) \text{div}_x u \varphi \right] \, dx \, dt ,
\]
where, \( \varphi \in C^\infty([0, T] \times \Omega) \) for any \( b \in C^1[0, \bar{\rho}] \) satisfying
\[
|b(s)|^2 + |b'(s)|^2 \leq C(1 + h(s)) \quad \text{for some constant } C \text{ and any } s \in [0, \bar{\rho}].
\]

• For a.e. \( \tau \in (0, T) \), the energy inequality holds:
\[
\int_{\Omega} \left[ \frac{1}{2} q|u|^2 + P(\rho) \right] (\tau, \cdot) \, dx + \int_0^\tau \int_{\Omega} \mathcal{S}(\nabla_x u) : \nabla_x u \, dx \, dt \leq \int_{\Omega} \left[ \frac{1}{2} q_0|u_0|^2 + P(q_0) \right] \, dx ,
\]
where \( P \) is given by,
\[
P(q) = q \int_{\frac{\hat{q}}{2}}^q \frac{p(z)}{z^2} \, dz .
\]

**Remark 4.2.** We denote,
\[
H(q) = q \int_{\frac{\hat{q}}{2}}^q \frac{h(z)}{z^2} \, dz \quad \text{and} \quad Q(q) = q \int_{\frac{\hat{q}}{2}}^q \frac{q(z)}{z^2} \, dz .
\]

**Remark 4.3.** Throughout our discussion we have the following assumption near \( \bar{\rho} \):
\[
\lim_{\theta \to \bar{\rho}} \frac{h(\theta)}{\theta - \bar{\rho}}^\beta > 0 , \quad \text{for some } \beta > \frac{5}{2} .
\]
This assumption is possibly technical but necessary for the analysis. Note that similar assumption also ensures global existence, cf. Feireisl, Lu, Novotný [14].

### 4.1. Discussion of Definition
Since \( s \in \text{supp}(q) = [s_0, s_1] \subset (0, \bar{\rho}) \) implies,
\[
|Q'(s)|^2 + |Q(s)|^2 \leq \frac{1}{s_0} |Q(s) - q(s)|^2 + |Q(s)|^2 \leq M(\text{sup}_{(0, \bar{\rho})} |q|, \text{sup}_{(0, \bar{\rho})} |q'|).
\]
Hence \( Q \) satisfies hypothesis of \( b \) as in [14]. Thus from renormalized equation we have,
\[
\left[ \int_{\Omega} Q(\rho)(t, \cdot) \, dx \right]_{t=0}^{t=r} = - \int_0^r \int_{\Omega} q(\rho) \text{div}_x u \, dx \, dt .
\]
Since \([q, u]\) is a renormalized dissipative weak solution, we obtain,
\[
\left[ \int_{\Omega} \left( \frac{1}{2} q|u|^2 + H(\rho) \right)(t, \cdot) \, dx \right]_{t=0}^{t=r} + \int_0^r \int_{\Omega} \mathcal{S}(\nabla_x u) : \nabla_x u \, dx \, dt \leq \int_0^r \int_{\Omega} q(\rho) \text{div}_x u \, dx \, dt .
\]

**Remark 4.4.** Instead of \(|b(\rho)|^2\) and \(|b'(\rho)|^2\) the same calculation can be done for \(|b(\rho)|^\frac{5}{2}\) and \(|b'(\rho)|^\frac{5}{2}\).
4.2. Main Result. Here we state the main theorem,

**Theorem 4.5.** Let \( \Omega \subset \mathbb{R}^d, d = 1, 2, 3, \) be domain with periodic boundary condition. Let the pressure be given by (1.8). Let \([\varrho, \mathbf{u}]\) be finite energy weak solution in the sense of definition (1.1) which satisfies the pressure condition (1.8) for some \( \beta \geq 3. \) Let \([r, \mathbf{U}]\) be a classical solution of the \( (1.1)-(1.3), (1.5), \) i.e. \((r, \mathbf{U}) \in C^1([0, T] \times \Omega) \times C^1([0, T]; C^2(\Omega)) \) solves equation with same initial data as \((\varrho, \mathbf{u})\) and \(0 < r < \bar{\varrho}. \) Then there holds,

\[
(4.10) \quad (\varrho, \mathbf{u}) = (r, \mathbf{U}) \text{ in } (0, T) \times \Omega
\]

The next two sections are devoted to the proof of Theorem (4.5).

5. Relative Energy

We define the relative energy functional:

\[
(5.1) \quad \mathcal{E}(t) = \mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U})(t) := \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + (H(\varrho) - H(r) - H'(r)(\varrho - r))(t, \cdot) \right) \, dx.
\]

We rewrite the entropy functional as

\[
\mathcal{E}(t) = \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) \, dx - \int_\Omega \varrho \cdot \mathbf{U} \, dx
\]

\[
+ \int_\Omega \varrho \left( \frac{1}{2} |\mathbf{u}|^2 - H'(r) \right) \, dx + \int_\Omega (rH'(r) - H(r)) \, dx.
\]

Next we follow the similar lines as in Section 3. We assume \([r, \mathbf{U}] \in C^1([0, T] \times \Omega) \times C^1(0, T; C^2(\Omega)) \) is classical solution of (1.1)-(1.3), (1.5) and pressure law (1.8) with \(0 < r < \bar{\varrho}. \) Hence we obtain,

\[
[\mathcal{E}(t)]_{t=0}^{\tau} + \int_0^\tau \int_\Omega \mathbf{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt
\]

\[
\leq \int_0^\tau \int_\Omega \left( \frac{\varrho}{r} - 1 \right)(\mathbf{U} - \mathbf{u}) \cdot (\nabla_x \mathbf{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)) \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \mathbf{u} \cdot (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \mathbf{U} \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \, div_x \mathbf{U} \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \left( \text{div}_x \mathbf{u} - \text{div}_x \mathbf{U} \right) (q(\varrho) - q(r)) \, dx \, dt.
\]

Next, we state a lemma which indicates the difference in the proof of Theorem (4.5) with Theorem (2.2).

**Lemma 5.1.** Let \( \varrho \geq 0 \) and \( 0 < \alpha_0 \leq r \leq \bar{\varrho} - \alpha_0 < \bar{\varrho}. \) There exists \( \alpha_1 \in (0, \alpha_0) \) and a constant \( c > 0, \) such that

\[
(5.2) \quad H(\varrho) - H(r) - H'(r)(\varrho - r) \geq \begin{cases} \frac{c(\varrho - r)^2}{\bar{\varrho} - \alpha_1}, & \text{if } \alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1, \\ \frac{h(r)}{\bar{\varrho} - \alpha_1}, & \text{if } 0 \leq \varrho \leq \alpha_1, \\ \frac{h(\varrho)}{\bar{\varrho}/2}, & \text{if } \bar{\varrho} - \alpha_1 \leq \varrho < \bar{\varrho}. \end{cases}
\]
5.1. Discussion of Lemma (5.1)

In Lemma (5.1) the constant $c$ depends on $r$ such that $c(r)$ is uniformly bounded on $(\alpha_0, \bar{\alpha}_0)$. From the above two results and hypotheses of 5.1 and 1.8 we can say that for $0 \leq \varrho \leq \bar{\varrho} - \alpha_1$ we have

$$|h(\varrho) - h(r) - h'(r)(\varrho - r)| \leq C(H(\varrho) - H(r) - H'(r)(\varrho - r)).$$

As $\bar{\varrho} - \alpha_1 \leq \varrho < \bar{\varrho}$ we have no control on $h(\varrho) - h(r) - h'(r)(\varrho - r)$ by $H(\varrho) - H(r) - H'(r)(\varrho - r)$, so we need to add one extra term $\int_0^\tau \int_{\Omega} b(\varrho) h(\varrho) \, dx \, dt$ on the left hand side of the equation which takes care of that case where $b$ is a function which satisfies the hypothesis of renormalized equation.

Let the symbol $\Delta_x$ denote the Laplace operator defined on spatially periodic functions with zero mean.

5.2. Relative energy inequality with extra term. Motivated from discussion above we rewrite the expression for relative entropy as

$$[\mathcal{E}(t)]_{t=0}^T + \int_0^T \int_{\Omega} \mathcal{S}(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt + \int_0^T \int_{\Omega} b(\varrho) h(\varrho)$$

$$\leq \int_0^T \mathcal{R}_1(t) \, dt + \int_0^T \mathcal{R}_2(t) \, dt + \mathcal{R}_3(\tau),$$

where, $\mathcal{R}_1(\cdot)$ is given by

$$\mathcal{R}_1(t) = \int_{\Omega} \left( \frac{\varrho}{r} - 1 \right) (U - u) \cdot (\text{div}_x \mathcal{S}(\nabla_x U) - \nabla_x q(r)) \, dx$$

$$+ \int_{\Omega} \varrho (u - U) : ((U - u) \cdot \nabla_x U) \, dx$$

$$+ \int_{\Omega} (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \text{div}_x U \, dx$$

$$+ \int_{\Omega} (\text{div}_x u - \text{div}_x U)(q(\varrho) - q(r)) \, dx$$

$$= \Sigma_{i=1}^4 \mathcal{I}_i.$$
\( \mathcal{R}_2(\cdot) \) is given by

\[
\mathcal{R}_2(t) = \int_{\Omega} h(q) b(q) \, dx - \int_{\Omega} (q(q) - q(r)) b(q) \, dx \\
+ \int_{\Omega} q(r) b(q) \, dx - \int_{\Omega} q(r) b(q) \, dx + \int_{\Omega} q(q) b(q) \, dx \\
- \int_{\Omega} g u \otimes u : \nabla_x (\nabla_x \Delta^{-1}_x (b(q) - \langle b(q) \rangle)) \, dx \\
+ \int_{\Omega} S(\nabla_x u) : \nabla_x (\nabla_x \Delta^{-1}_x (b(q) - \langle b(q) \rangle)) \, dx \\
+ \int_{\Omega} g u \cdot \nabla_x \Delta^{-1}_x \text{div}_x (b(q) u) \, dx \\
+ \int_{\Omega} g u \cdot \nabla_x \Delta^{-1}_x (b'(q) \varrho - b(q)) \text{div}_x u - \langle b'(q) \varrho - b(q) \text{div}_x u \rangle \, dx \\
= \Sigma_{i=5}^{13} I_i,
\]

and \( \mathcal{R}_3(\cdot) \) is given by

\[
\mathcal{R}_3(\tau) = \int_{\Omega} g u \cdot \nabla_x \Delta^{-1}_x (b(q) - \langle b(q) \rangle)(\tau, \cdot) \, dx \\
- \int_{\Omega} g_0 u_0 \cdot \nabla_x \Delta^{-1}_x (b(q_0) - \langle b(q_0) \rangle) \, dx \\
= \Sigma_{i=14}^{15} I_i.
\]

We sum up the above results and state the theorem,

**Theorem 5.4.** Suppose the pressure constraint (4.8) is satisfied. Let \( \{q, u\} \) be a finite energy weak solution in \((0, T) \times \Omega \) in the sense of definition (4.1). Let \((r, U) \in C^1([0, T] \times \Omega) \times C^1(0, T; C^2(\Omega)) \) such that

\[
0 < r < \bar{\varrho}.
\]

Let \( b(s) \in C^1[0, \bar{\varrho}] \) satisfy the condition,

\[
|b'(s)|^2 + |b(s)|^2 \leq C(1 + h(s)) \text{ for some constant } C \text{ and any } s \in [0, \bar{\varrho}].
\]

Then the following relative energy true for a.e. \( \tau \in (0, T), \)

\[
[E(t)]_{t=0}^{\tau} + \int_0^\tau \int_{\Omega} S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt + \int_0^\tau \int_{\Omega} b(q) h(q) \, dt \\
\leq \int_0^\tau \mathcal{R}_1(t) \, dt + \int_0^\tau \mathcal{R}_2(t) \, dt + \mathcal{R}_3(\tau),
\]

with \( \mathcal{R}_1, \mathcal{R}_2 \) and \( \mathcal{R}_3 \) defined as above.

**Remark 5.5.** Condition (5.8) is slightly changed from the similar assumption in definition (4.1). Although \( b \) following (5.8) will be a suitable candidate for \( b \) as prescribed in definition (4.1).

**Proof.** We have to check that all integrals in R.H.S of (5.4) are bounded which is already done in [12].

\[ \square \]
6. Weak-strong uniqueness

We have achieved the derivation of remainder terms. Now we consider a fixed $b$ which satisfies (5.8). Then we want to show that $\mathcal{R}_i(\cdot)$ can be bounded by $\eta(\cdot)\mathcal{E}(\cdot)$ for some positive function $\eta$, for each $i = 1, 2, 3$.

6.1. Choice of $b$ and its properties. Consider $b \in C^\infty[0, \tilde{\rho}]$, $b'(s) \geq 0$ as follows:

$$b(s) = \begin{cases} 
0, & \text{if } s \leq \tilde{\rho} - \alpha_1, \\
-\log(\tilde{\rho} - s), & \text{if } \tilde{\rho} - \alpha_2.
\end{cases}$$

The choice of $\alpha_2$ is in such a way that

$$-\log(\tilde{\rho} - s) \geq 16||\nabla x U||, \text{ if } \tilde{\rho} - \alpha_2 \leq s < \tilde{\rho}.$$

Considering the assumption (4.8) along with (4.7) we have the following results:

$$\Box$$

This along with (5.1) yields the following, for $\gamma \geq 1$:

$$\int_\Omega |b'(\varrho)|^\gamma \, dx \leq C \int_{\varrho \geq \tilde{\rho} - \alpha_1} |b(\varrho)|^\gamma \, dx.$$

Also for any $2 \leq \beta_0 \leq \beta$, we have,

$$\int_\Omega |b'(\varrho)|^{\beta_0 - 1} \, dx \leq C \int_{\varrho \geq \tilde{\rho} - \alpha_1} H(\varrho) \, dx \leq C \int_{\varrho \geq \tilde{\rho} - \alpha_1} (H(\varrho) - H(r) - H'(r)(\varrho - r)) \, dx \leq C \int_\Omega h(\varrho) \, dx.$$

6.2. Estimates for remainder. Now we proceed to estimate the remainder terms. As earlier mentioned, in [12] Feireisl, Lu, Novotný have encountered similar problem with $q \equiv 0$. In a similar way we can compute terms other than $I_1, I_4$ and $I_6$ to $I_9$.

First,

$$I_1 = \int_\Omega (\frac{q}{r} - 1)(U - u) (\nabla_x S(\nabla_x U) - \nabla_x q(r)) \, dx = \sum_{i=1}^3 J_i,$$

with

$$J_1 := \int_{\alpha_1 \leq \varrho \leq \tilde{\rho} - \alpha_1} \int_\Omega (\frac{q}{r} - 1)(U - u) (\nabla_x S(\nabla_x U) - \nabla_x q(r)) \, dx$$

$$J_2 := \int_{\varrho \leq \alpha_1} \int_\Omega (\frac{q}{r} - 1)(U - u) (\nabla_x S(\nabla_x U) - \nabla_x q(r)) \, dx$$

$$J_3 := \int_{\varrho \geq \tilde{\rho} - \alpha_1} \int_\Omega (\frac{q}{r} - 1)(U - u) (\nabla_x S(\nabla_x U) - \nabla_x q(r)) \, dx.$$
Clearly, by Taylor’s formula, Cauchy-Schwarz inequality and Poincaré inequality, we have for any $\sigma > 0$,

$$|J_1| \leq \frac{C}{\sigma} \|(\text{div}_x S(\nabla_x U)(t) - \nabla_x q(r))\|_{L^\infty} \int_{\alpha_1 \leq \vartheta \leq \vartheta - \alpha_1} (q - r)^2 \, dx$$

(6.8)

$$+ \sigma \int_{\Omega} |\nabla_x (U - u)|^2 \, dx.$$

Further using Korn’s inequality and (5.1) we deduce that,

(6.9)

$$|J_1| \leq \frac{C}{\sigma} \|(\text{div}_x S(\nabla_x U)(t) - \nabla_x q(r))\|_{L^\infty} \int_{\alpha_1 \leq \vartheta \leq \vartheta - \alpha_1} H(q) - H(r) - H'(r)(q - r) \, dx$$

$$+ \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx$$

$$\leq \frac{1}{\sigma} \eta(t) \mathcal{E}(t) + \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx.$$

Similarly,

(6.10)

$$|J_2| \leq \frac{C}{\sigma} \|(\text{div}_x S(\nabla_x U)(t) - \nabla_x q(r))\|_{L^\infty} \int_{\vartheta \leq \alpha_1} 1 \, dx$$

$$+ \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx$$

$$\leq \frac{C}{\sigma} \|(\text{div}_x S(\nabla_x U)(t) - \nabla_x q(r))\|_{L^\infty} \int_{\vartheta \leq \alpha_1} h(r) \, dx$$

$$+ \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx$$

$$\leq \frac{1}{\sigma} \eta(t) \mathcal{E}(t) + \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx,$$

and

(6.11)

$$|J_3| \leq \frac{C}{\sigma} \|(\text{div}_x S(\nabla_x U)(t) - \nabla_x q(r))\|_{L^\infty} \int_{\vartheta \geq \vartheta - \alpha_1} 1 \, dx$$

$$+ \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx$$

$$\leq \frac{C}{\sigma} \|(\text{div}_x S(\nabla_x U)(t) - \nabla_x q(r))\|_{L^\infty} \int_{\vartheta \geq \vartheta - \alpha_1} H(q) \, dx$$

$$+ \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx$$

$$\leq \frac{1}{\sigma} \eta(t) \mathcal{E}(t) + \sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx.$$

Combining above estimates, we have

(6.12)  $|I_1| \leq \frac{3}{\sigma} \eta(t) \mathcal{E}(t) + 3\sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx.$
Next for $\mathcal{I}_2$ we have,

\begin{equation}
\int_{\Omega} g(u - U) \cdot ((U - u) \cdot \nabla x) U \, dx \leq \|\nabla_x U(t)\|_{L^\infty} \int_{\Omega} g|u - U|^2 \, dx \leq \eta(t)\mathcal{E}(t).
\end{equation}

Looking at $\mathcal{I}_3 = \Sigma_{i=4}^5 J_i$ we obtain

\begin{align}
J_4 &= \int_{\rho \leq \bar{\rho} - \alpha_1} (-h(\rho) + h(r) + h'(r)(\rho - r)) \nabla_x U \, dx,
J_5 &= \int_{\rho \geq \bar{\rho} - \alpha_1} (-h(\rho) + h(r) + h'(r)(\rho - r)) \nabla_x U \, dx.
\end{align}

By remark of (5.1) we obtain

\begin{equation}
|J_4| \leq \|\nabla_x U\|_{L^\infty} \int_{\rho \leq \bar{\rho} - \alpha_1} H(\rho) - H(r) - H'(r)(\rho - r) \, dx \leq \eta(t)\mathcal{E}(t).
\end{equation}

For $J_5$ we have to estimate carefully. We have $\alpha_2$ from (6.2), then by (5.1) and (6.4) give us,

\begin{equation}
|J_5| \leq \int_{\bar{\rho} - \alpha_1 \leq \rho \leq \bar{\rho} - \alpha_2} (-h(\rho) + h(r) + h'(r)(\rho - r)) \nabla_x U \, dx
+ \int_{\rho \geq \bar{\rho} - \alpha_2} (-h(\rho) + h(r) + h'(r)(\rho - r)) \nabla_x U \, dx
\leq \int_{\bar{\rho} - \alpha_1 \leq \rho \leq \bar{\rho} - \alpha_2} |\nabla_x U| \max_{\bar{\rho} - \alpha_1 \leq \rho \leq \bar{\rho} - \alpha_2} h''(s)(\rho - r)^2 \, dx + \frac{1}{8} \int_{\Omega} h(\rho)b(\rho) \, dx
\leq \eta(t)\mathcal{E}(t) + \frac{1}{8} \int_{\Omega} h(\rho)b(\rho) \, dx.
\end{equation}

\begin{equation}
|\mathcal{I}_3| \leq \frac{1}{8} \int_{\Omega} b(\rho)h(\rho) \, dx + \eta(t)\mathcal{E}(t).
\end{equation}

For $\mathcal{I}_4 = \Sigma_{i=6}^8 J_i$ we write,

\begin{align}
J_6 &= \int_{\alpha_1 \leq \rho \leq \bar{\rho} - \alpha_1} (\nabla_x u - \nabla_x U)(q(\rho) - q(r)) \, dx,
J_7 &= \int_{\rho \leq \alpha_1} (\nabla_x u - \nabla_x U)(q(\rho) - q(r)) \, dx,
J_8 &= \int_{\rho \geq \bar{\rho} - \alpha_1} (\nabla_x u - \nabla_x U)(q(\rho) - q(r)) \, dx.
\end{align}
Now using Cauchy-Schwarz inequality, Korn inequality and lemma (5.1) we have,

\[ |J_6| \leq \frac{C}{\sigma} M \int_{\tilde{a}_1 \leq \tilde{\theta} \leq a_1} (\varphi - r)^2 \, dx + \sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx \]
\[ \leq \frac{C}{\sigma} \int_{\tilde{a}_1 \leq \tilde{\theta} \leq a_1} H(\varphi) - H(r) - H'(\varphi - r) \, dx \]
\[ + \sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx \]
\[ \leq \frac{1}{\sigma} \eta(t)E(t) + \sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx. \]

(6.19)

(6.20)

Since \( q \) is a compactly supported function, using a similar argument we have,

\[ |J_7| + |J_8| \leq \frac{C}{\sigma} \int_{\varphi \leq a_1} 1 \, dx + \frac{C}{\sigma} \int_{\varphi \geq a_1} 1 \, dx \]
\[ + 2\sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx \]
\[ \leq \frac{1}{\sigma} \eta(t)E(t) + 2\sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx. \]

From the above estimates we have,

\[ |I_4| \leq \frac{1}{\sigma} \eta(t)E(t) + 3\sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx. \]

(6.21)

Next we will focus on the terms of \( R_2 \). As a consequence of (6.4) we have

\[ \langle b(\varphi) \rangle = \frac{1}{|\Omega|} \int_{\Omega} b(\varphi) \, dx \leq CE(t) \]

(6.22)

Using above relation in \( I_5 \), we obtain,

\[ |I_5| \leq CE(t) \int_{\Omega} h(\varphi) \, dx \leq \eta(t)E(t). \]

(6.23)

Similarly using that \( q \) has compact support, we have

\[ |I_7| + |I_8| + |I_9| \leq \eta(t)E(t). \]

(6.24)

For \( I_6 \) we rewrite it as

\[ |I_6| \leq C \int_{\Omega} (q(\varphi) - q(r))^2 \, dx + C \int_{\Omega} (b(\varphi))^2 \, dx \leq \Sigma_{i=9}^{11} J_i + \eta(t)E(t), \]

(6.25)

where,

\[ J_9 = \int_{\tilde{a}_1 \leq \tilde{\theta} \leq a_1} (q(\varphi) - q(r))^2 \, dx, \]

(6.26)

\[ J_{10} = \int_{\varphi \leq a_1} (q(\varphi) - q(r))^2 \, dx, \]

\[ J_{11} = \int_{\varphi \geq a_1} (q(\varphi) - q(r))^2 \, dx. \]

It is similar to \( J_6, J_7 \) and \( J_8 \). Thus using a similar argument we have,

\[ |I_6| \leq \frac{1}{\sigma} \eta(t)E(t) + 2\sigma \int_{\Omega} S(\nabla_x(U - u)) : \nabla_x(U - u) \, dx. \]

(6.27)
The estimates below directly follow from [12] with minor modification. The condition \( \beta \geq 3 \) plays a crucial role here. We have,

\[
|I_{10}| + |I_{11}| + |I_{12}| \leq \frac{1}{\sigma} \eta(t) E(t) + 3\sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx. \tag{6.28}
\]

Also

\[
|I_{13}| \leq \frac{1}{8} \int_{\Omega} b(\varrho) h(\varrho) \, dx + \frac{1}{\sigma} \eta(t) E(t) + 2\sigma \int_{\Omega} S(\nabla_x (U - u)) : \nabla_x (U - u) \, dx. \tag{6.29}
\]

Now for initial data \( \varrho_0 = r_0 \in [\alpha_0, \bar{\varrho} - \alpha_0] \), we have \( b(\varrho_0) \equiv 0 \). Hence, \( I_{15} = 0 \). We can write,

\[
|I_{14}| + |I_{15}| \leq \frac{1}{4} \int_{\Omega} e|u - U|^2(\tau, \cdot) \, dx + \frac{1}{2} \int_{g \geq \bar{\varrho} - \alpha_1} \left( H(\varrho) - H(r) - H'(r)(\varrho - r) \right) \, dx.
\]

Combining (6.12), (6.13), (6.17), (6.21), (6.23), (6.24) and (6.27)-(6.30) and choosing \( \sigma \) small we can conclude that

\[
[E(t)]_{t=0}^{T} + \int_0^T \int_{\Omega} S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \, dt \\
+ \int_0^T \int_{\Omega} b(\varrho) h(\varrho) \, dx \, dt \\
\leq \int_0^T \eta(t) E(t) \, dt, 
\]

where \( \eta \in L^1(0, T) \).

6.3. End of the proof.

Proof of Theorem (4.5): Since \( b \geq 0 \), as a consequence of Grönwall’s lemma and hypothesis for same initial data in (4.5) we have \( \dot{E} \equiv 0 \) in \([0, T]\), which ensures our desired weak strong uniqueness result. \qed

7. Concluding remarks

This method cannot be extended to the Euler (inviscid) system as the viscous damping plays a crucial role in the proof.

Acknowledgement

The work was supported by Einstein Stiftung, Berlin.

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