DIMENSION ON DISCRETE SPACES.

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Abstract

In this paper we develop some combinatorial models for continuous spaces. In this spirit we study the approximations of continuous spaces by graphs, molecular spaces and coordinate matrices. We define the dimension on a discrete space by means of axioms, and the axioms are based on an obvious geometrical background. This work presents some discrete models of n-dimensional Euclidean spaces, n-dimensional spheres, a torus and a projective plane. It explains how to construct new discrete spaces and describes in this connection several three-dimensional closed surfaces with some topological singularities. It also analyzes the topology of (3+1)-spacetime. We are also discussing the question by R. Sorkin [19] about how to derive the system of simplicial complexes from a system of open covering of a topological space S.

Introduction.

A number of workers have been unhappy about applications of the continuum picture of space and spacetime. They have believed that the breakdown of the functional integral at the Plank length shows not merely the failure of the classical field equations but also indicates that a differential manifold upon which they are built should be replaced by some finite theory. This was certainly one of the motivations behind Penrose invention [15] of spin networks and recent works by Finkelstain on a novel spacetime microstructure [1]. Isham, Kubyshin and Renteln [2] introduce a quantum theory on the set
of all topologies on a given set, and show that for a finite basic set almost all metrics can be obtained by embedding this set into a vector space and then varying the norm of this space.

One more approach to a combinatorial model of space and spacetime is studied in work [19] by R.Sorkin. He replaces general topological spaces by a finite ones and describes how to associate a finite space with any locally finite covering of a topological space. He also presents some examples of posets derived from simple spaces.

Another way is Regge calculus [18]. Many suggestions for formulating various Regge calculus versions have been made in order to face a number of problems. Regge calculus describes general relativity spacetime by using a simplicial complex. Its fundamental variables are a set of edge lengths and an incidence matrix that describes how they are connected. One approach supposes that the connectivity of a simplicial complex is fixed, but the lengths of edges can be varied. Another approach fixes the edge length and varies the connectivity of a simplicial complex in order to change the metric of a spacetime. The Regge calculus, however, supposes that there is a continuous underlying spacetime and does not account naturally for the appearance of a minimal length in effective theories.

At the same time in mathematics there exist several quickly developed approaches to discrete spaces in the frame of digital topology which can be useful in physics. Digital topology is the study of topological properties of image arrays. It provides the theoretical foundations for image processing operations such as image thinning, border following and object counting.

The paper [12] reviews the fundamental concepts of digital topology, surveys the major theoretical results in this field and contains the bibliography of almost 140 references.

Traditionally a discrete or digital space is considered as a graph whose edges between vertices define the nearness and connectivity in the neighborhood of any vertex.

This approach was used by Rosenfeld [16,17] who proved the first version of the Jordan curve theorem by using a graph theoretical model of a digital plane. However, this model does not utilize a topological basis and requires different nearness for the curve and its background.

An alternative topological approach to the digital topology uses the notion of a connected topology on a totally ordered set $\mathbb{Z}$ of integers [9-11,13]. The digital plane $\mathbb{Z} \times \mathbb{Z}$ or the three-dimensional digital space $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ are the
topological products of two or three such spaces respectively. Using this construction the Jordan curve theorem in two and three dimensions was proven. Another approach to finite topology is offered by Kovalevsky in [14]. He builds the digital space as a structure consisting of elements of different dimensions by using a such well-known in topology element as a cellular complex.

Our approach to discrete spaces is based on using three combinatorial tools: a graph a molecular space and a coordinate matrix [3-8]. In this panel the material to be presented below begins with short description of some geometrical background for the definition of the dimension on a graph. Then we shall show the connection between a graph, a molecular space and a coordinate matrix. We shall define the dimension on discrete spaces which is based on some geometrical ground. We shall analyze the dimensions of different models of two, three and n-dimensional discrete spaces. We present some examples of three-dimensional discrete closed spaces with strange topological features which do not have direct continuous analogies. Then we prove some theorems showing how to construct closed three dimensional spaces with nonstandard topology. Finally we discuss the topological structure of a (3+1) space-time.

Geometrical background for the definition of the dimension on a graph.

We are going to construct now a graph with certain properties which can be thought as a convenient tool for describing the ideas of nearness and continuity by combinatorial methods. This will be done in the first place by picking out in elementary geometry those properties of nearness which seem to be fundamental and taking them as axioms. To get a glimpse of the intuitive geometrical ground of the dimension consider the following example. Let $E^n$ be n-dimensional Euclidean space and $p$ a point in it. The neighbourhood of $p$ is commonly defined to be any set $U$ such that $U$ contains an open solid disk $D^1_n$ of center $p$. The boundary of this disk is the sphere $S^{n-1}_1$.

The definition of neighborhood is formulated in this way so as to be as free as possible from the ideas of size and shape, concepts that play no part in topology.

Using this definition of a neighbourhood of a $p$ in Euclidean space it is easy to see that the family of sets $U$ satisfy the usual topological axioms.
1. \( p \) belongs to any neighbourhood of \( p \).

2. If \( U \) is a neighbourhood of \( p \) and \( U \subset V \), then \( V \) is a neighbourhood of \( p \).

3. If \( U \) and \( V \) are neighbourhood of \( p \), so as is \( U \cap V \).

4. If \( U \) is a neighbourhood of \( p \), then there is a neighbourhood \( V \) of \( p \) such that \( V \subset U \) and \( V \) is a neighbourhood of each of its points.

Taking these properties as axioms in an abstract formulations we can define a topological space \( E \) as a set \( E \) with a family of subsets of \( E \) satisfying the four properties listed earlier. We can also define a subset \( W \) of \( E \) open if for each point \( p \) in \( W \), \( W \) is a neighbourhood of \( p \).

Note that the disk \( D^n_1 \) plays crucial role in this definition.

In the continuous case the sphere \( S^{n-1}_1 \) contain in itself an infinite sequence of disks \( D^n_i \) and spheres \( S^{n-1}_i \) of center \( p \).

\[
D^n_1 \supset D^n_2 \supset \ldots \supset D^n_i \supset \ldots \\
S^{n-1}_1 \supset S^{n-1}_2 \supset \ldots \supset S^{n-1}_i \supset \ldots 
\]

However the situation is different in the discrete case where the sequences of disks and spheres can not be infinite and axiom 4 is not realized. Therefore we have a finite series of the form

\[
D^n_1 \supset D^n_2 \supset \ldots \supset D^n_t \\
S^{n-1}_1 \supset S^{n-1}_2 \supset \ldots \supset S^{n-1}_t 
\]

The smallest disk \( D^n_t \) and the smallest sphere \( S^{n-1}_t \) can not be reduced in the sense that they do not contain disks and spheres others then themselves (Figure 1).

The topological meaning this construction for a graph reveals that the vertex \( p \) is considered as \( n \)-dimensional if its minimal neighbourhood is the sphere \( S^{n-1} \).

The point \( p \) and the nearest sphere \( S^{n-1} \) together form the smallest disk \( D^n \) of center \( p \). Point \( p \) of a discrete space \( G \) is considered as one-dimensional if
its nearest neighbourhood is a zero-dimensional sphere $S^0$. It is well known
that $S^0$ is a set of two disconnected points. In the other words $S^0$ is disjoined
graph of two points. In one-dimensional discrete sphere $S^1$ all points are one-
dimensional. Obviously the minimal number of points required for $S^1$ is four.
For two-dimensional discrete sphere $S^2$ all its points are two-dimensional. It
means that nearest neighbourhood of any point of $S^2$ should be $S^1$ and so
on.

**A molecular space and a coordinate matrix of a graph.**

In order to make this paper self-contained we shall summarize the necessary
results from our previous papers. Let $E^\infty$ be infinite-dimensional Euclidean
space. Take the coordinates of a point $x, x \in E^\infty$, as a sequence of real
numbers

$$x = (x_1, x_2, ..., x_n, ..) = [x_i], \ i \in N.$$  

We define unit cube $K \in E^\infty$ in the following way: each $x, x \in K$, has
coordinates $x_i$ satisfying conditions presented in [3,4,5,8]:

$$n_i \leq x_i \leq n_i + 1, \ i \in N, \ n_i - \text{integer}.$$  

Therefore, $K$ is an infinite-dimensional cube with unit edges. In [3,4,5,8] $K$
is called a kirpich. We will use this name in the present paper.
The position of $K$ in $E^\infty$ is determined by the left vertex coordinates. For
the given kirpich we have

$$K = (n_1, n_2, ..., n_n, ...) = [n_i], \ i \in N.$$  

Two kirpiches are called adjacent, if they have common points. The
distance $d(K_1, K_2)$ between kirpiches $K_1 = [n_i]$ and $K_2 = [m_i]$ is defined by
using sup norm

$$d(K_1, K_2) = \max |n_i - m_i|, \ i \in N.$$  

Obviously, two kirpiches are adjacent if their appropriate coordinates dis-
tinguish not more then 1, or the distance between them equals 1. Any set of
kirpiches in $E^\infty$ is called a molecular space and is denoted by $M$. Clearly,
any molecular space can be represented by its intersection graph $G(M)$. It
was shown in [4,8] that any graph $G$ can be represented by a molecular space $M(G)$, such that $G = G(M(G))$. Clearly, more than one $M(G)$ can be built for the graph $G$. There exists isomorphism between any two $M(G)$.

Let $M$ be a molecular space with a set of kiripches

$$V = (K_1, K_2, \ldots, K_n), \quad K_1 = [k_{1i}], \quad K_2 = [k_{2i}], \ldots, K_n = [k_{ni}]$$

The matrix $[k_{pi}]$ is called the coordinate matrix of the molecular space $M$ and its intersection graph $G(M)$ and is denoted $A(M)$ or $A(G(M))$. This matrix has $n$ rows and infinite columns.

In fact we shall always use a finite-dimensional Euclidean space. The intuitive background for using the infinite-dimensional unit cube is the attempt to create some universal element not depending on the dimension and suitable for describing elements of different dimensions: zero-dimensional points, one-dimensional lines, two-dimensional surfaces and so on.

Let $S$ be a surface in $E^n$. The molecular space $M(S)$ of $S$ is a set of kiripches intersecting $S$.

Figure 2 shows the graph $G$, its molecular space $M$ and its coordinate matrix $A$.

The dimension and the metric of a discrete space.

Our objective now is to define the dimension on graphs. Later on we will use the names discrete space and point for a graph and its vertex when we want to emphasize the notion of the dimension on it.

Since in this paper we only use induced subgraphs, we shall use the word subgraph for an induced subgraph. We shall also use some symbols, notations and names introduced in our previous works.

Definitions

Let $G$, $G_1$ and $v$ be a graph, its subgraph and its point.

- The subgraph $B(G_1)$ containing $G_1$ is called the ball of $G_1$ if any point of $B(G_1)$ is adjacent to at least one point of $G_1$.

- The subgraph $B(G_1)$ without points of $G_1$ is called the rim of $G_1$ and it is denoted $O(G_1)$.

Obviously $B(G_1) - G_1 = O(G_1)$.

- If $G_1$ is a point $v$ then $B(v)$ and $O(v)$ are called the ball and the rim of $v$ respectively.
• The subgraph $B(v_1, v_2, \ldots v_n)$, $B(v_1, v_2, \ldots v_n) = B(v_1) \cap B(v_2) \cap \ldots B(v_n)$, is called the joint ball of the points $v_1, v_2, \ldots v_n$.

• The subgraph $O(v_1, v_2, \ldots v_n)$, $O(v_1, v_2, \ldots v_n) = O(v_1) \cap O(v_2) \cap \ldots O(v_n)$, is called the joint rim of the points $v_1, v_2, \ldots v_n$.

Let $G$ and $G_1$ be a graph and its subgraph with points $(v_1, v_2, \ldots v_n)$ and $(v_1, v_2, \ldots v_p)$ respectively. It is clear that

\[
O(G_1) = O(v_1) \cup O(v_2) \cup \ldots \cup O(v_p) - G_1
\]

\[
B(G_1) = B(v_1) \cup B(v_2) \cup \ldots \cup B(v_p)
\]

• A graph $K_n(v_1, v_2, \ldots v_n)$ of $n$ points is called completely connected or complete if any its two points are adjacent.

• A graph $H_n(v_1, v_2, \ldots v_n)$ of $n$ points is called completely disconnected if any its two points are disjoined.

• The join $G_1 \ast G_2$ of two graphs $G_1$ and $G_2$ is the graph which consists of two graphs $G_1$ and $G_2$ and all edges joining points of $G_1$ with points of $G_2$.

To this end we begin by defining the dimension on a graph in the following way.

**Definition 1** Zero-dimensional normal space $S^0$ is the graph which consists of two non-adjacent points.

**Definition 2** A point $v$ of a graph $G$ is called a normal $n$-dimensional point if its rim $O(v)$ is a normal $(n-1)$-dimensional space.

**Definition 3** For any integer $n$, $n \geq 1$, define a normal $n$-dimensional space to be a connected graph in which any point is $n$-dimensional normal.

According to these definitions one-dimensional normal space is any circle $C_n$, $n \geq 4$.

Further the notation $p(G)$ will be used for the dimension of a graph $G$.

Figure 3 represents normal zero, one and two-dimensional spheres and one, two and three-dimensional disks.

Figure 4 shows normal two-dimensional discrete flat spaces and their molecular spaces. $E_1^2$ is the two-dimensional discrete space in Khalimsky topology [9-13].
Tree-dimensional normal sphere is the graph $S^3$ depicted in Figure 5. It can be verified without difficulties that the complete $(n+1)$ partite graph $K(2, 2, \ldots, 2)$ is the minimal graph describing $S^n$ [3,4,6]. Therefore, the minimal number of elements necessary to describe $S^n$ is $2n+2$. Notice that the same number of points is used by R. Sorkin [19] to describe $S^n$ in the finitary topology approach.

A normal torus $T^2$ and a projective plane $P^2$ are presented in Figure 5. It can be checked directly that the Euler characteristic and the homology groups of all graphs depicted in Figures 3-5 match the Euler characteristic and the homology groups of their continuous counterparts [5,7].

In [3] normal n-dimensional space is called of the type $\Pi_2$. This separation to the different types is caused by the fact the normal molecular spaces and graphs of any type $\Pi_n$, $n \neq 1, 2$ have some unusual properties different from those of direct discrete models of continuous spaces in $E^m$.

Our objective now is to define a generalization of the dimension which includes the above definition. It is natural to consider a point $v$ as zero-dimensional if its neighborhood does not contain any normal space.

**Definition 4** A point $v$ of a graph $G$ is called zero-dimensional, $p(v) = 0$, if $O(v)$ does not contain the normal zero-dimensional sphere $S^0$.

**Definition 5** A connected graph $G$ is called zero-dimensional, $p(G) = 0$, if any of its points is zero-dimensional.

By this definition in a zero-dimensional connected graph any two points are adjacent. Therefore, this graph is a complete graph on any number of vertices. A disconnected zero-dimensional graph is considered as a zero-dimensional sphere $S^0$ if it has exactly two components. It is clear that $S^0$ contains normal zero-dimensional sphere as its subgraph. We will extend this analogy to higher dimensions.

**Definition 6** A graph $G$ is called closed n-dimensional if
1. For any point $v$ \hspace{1cm} $p(v) \leq n$.
2. $G$ is homotopic to some normal n-dimensional space.

**Definition 7** A point $v$ is called n-dimensional, $p(v) = n$, if
1. $O(v)$ contains a closed $(n-1)$-dimensional space.
2. \( O(v) \) does not contain any closed \( n \) or more-dimensional space.

**Definition 8** A graph \( G \) is called \( n \)-dimensional, \( p(G) = n \), if
1. \( G \) contains at least one \( n \)-dimensional point
2. For any point \( v \) \( p(v) \leq n \).

In definition 6 we use homotopy of graphs. Two graphs are called homotopic if each of them can be turned into the other by contractible transformations which consist of contractible gluing and deleting of vertices and edges of a graph. It was shown [5-7] that these transformations do not change the Euler characteristic and the homology groups of graphs.

Let us look at some examples of \( n \)-dimensional (not normal) discrete spaces and their molecular spaces.

Spheres \( S^0 \), \( S^1 \), their molecular spaces and the molecular space \( M(S^2) \) of sphere \( S^2 \) are drawn in Figure 6. \( M(S^2) \) is a hollow space, it does not contain the central unit cube. These spheres are not normal but satisfy definitions 6-7. Any sphere \( S^n \) depicted in Figure 6 has the same Euler characteristic and homology groups as continuous \( S^n \) and can be transformed to the sphere \( S^n \) drawn in Figure 3 by contractible transformations. Flat one, two and three-dimensional spaces and their molecular spaces are shown in Figure 7. It is easy to construct three and more dimensional spaces but it is difficult to draw it. For a flat three-dimensional space the only molecular space is shown. However a \( n \)-dimensional space can be easily described by its coordinate matrix of the form

\[
\begin{pmatrix}
x_{i1} & x_{i2} & \cdots & x_{in} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{p1} & x_{p2} & \cdots & x_{pn} \\
\cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

where \( x_{ik} = 0, \pm 1, \pm 2, \ldots; \quad i = 1, 2, 3, \ldots; \quad k = 1, 2, \ldots, n \).

The standard definition of the distance on a graph can be applied to a discrete space.

**Definition 9** The distance \( d(v_1, v_2) \) between two points \( v_1 \) and \( v_2 \) in a discrete space \( G \) is the length of a shortest path joining them if any; otherwise \( d(v_1, v_2) = \infty \).
Obviously the distance is a metric. The Plank length can be thought as the length of an edge of the graph.

**Mathematical observations.**

Before proceeding to the main result of this paper let us pause to describe some mathematical observations relating to this approach. The following surprising facts were revealed.

- Suppose that $S^1$ is a circle of radius $R$. Let $A$ be a cover of $S^1$ by arcs whose length is small enough compared with $R$. Denote $G(A)$ the intersection graph of this cover. This graph is called the circular arc graph. It appears that:
  1. Dimension of $G(A)$ is equal to one, $p(G(A)) = dim(S^1) = 1$.
  2. $G(A)$ has the same Euler characteristic and homology groups as $S^1$.
  3. $G(A)$ can be reduced to the cycle graph $C_4$ by contractible transformations [5,6,7] ($S^1$ in Figure 3).

- Suppose we have some two-dimensional closed surface, for example, a sphere $S^2$ of radius $R$. Consider any tiling $A$ of $S^2$ by elements $(a_1,a_2,...a_n)$ whose size is small enough relative to the radius $R$. Construct the intersection graph $G(A)(v_1,v_2,...v_n)$ in the following way: Two vertices $v_1$ and $v_2$ are adjacent iff elements $a_1$ and $a_2$ have at least one common point. In most cases it turns out that
  1. Dimension of $G(A)$ is equal to two, $p(G(A)) = dim(S^2) = 2$.
  2. $G(A)$ has the same Euler characteristic and homology groups as $S^2$.
  3. $G(A)$ can be reduced by contractible transformations into the minimal two-dimensional sphere on 6 vertices [5,6,7] ($S^2$ in Figure 3).

- Suppose that $P^k$ is a surface in $E^n$, $n = 2, 3$ (for spheres $n$ can be any number). Divide $E^n$ into a set of cubes with the scale $l_1$ of the cube edge and call the molecular space $M_1(P^k)$ of $P^k$ the family of cubes intersecting $P^k$. Denote $G_1(P^k)$ the intersection graph of $M_1(P^k)$. Change the scale of the cube edge from $l_1$ to $l_2$ and obtain $M_2(P^k)$ and $G_2(P^k)$ by using the same structure. It is revealed that in most of cases
  1. $p(G_1(P^k)) = p(G_2(P^k)) = dim(P^k)$
  2. $G_1(P^k)$ and $G_2(P^k)$ have the same Euler characteristic and the homology groups as $P^k$.
  3. $G_1(P^k)$ and $G_2(P^k)$ can be transformed from one to the other with four kinds of transformations if the divisions are small enough.
These facts allow us to assume that the graph and the molecular space contain topological and perhaps geometrical characteristics of the surface $P^k$. Otherwise, the molecular space $M$ and the graph $G$ are the discrete counterparts of a continuous space $P^k$.

**Singular spaces.**

This section describes a method of obtaining new spaces from given ones. We will see that there exist n-dimensional normal spaces with some peculiar properties. These spaces give rise to new discrete structures that have different topologies in different points.

**Theorem 1** Let $G(p_1, p_2, ... p_r)$ and $H_2(v_1, v_2)$ be an n-dimensional normal space and the completely disconnected space on two points respectively. Then $H_2(v_1, v_2) * G(p_1, p_2, ... p_r)$ is a (n+1)-dimensional normal space.

**Proof.** The proof is by induction.

(i) For $n=0,1$ the theorem is verified directly.

(ii) Assume that theorem is valid for any $n, n \leq k$. Let $G(p_1, p_2, ... p_r)$ be a normal $(k+1)$-dimensional discrete space. Consider

$$W = H_2(v_1, v_2) * G(p_1, p_2, ... p_r)$$

It is necessary to show that $W$ is a $(k+2)$-dimensional discrete normal space. Take any point $p_i$. With respect to the definition of a normal space, $O(p_i)$ in $G$ denoted $O(p_i)|G$ is a k-dimensional normal space. Therefore, according to the assumption $H_2 * O(p_i)$ is $(k+1)$-dimensional normal space. Hence any point $p_i$ in $W$ has the rim which is $(k+1)$-dimensional normal space.

The rims of points $v_1$ and $v_2$ in $W$ are the $(k+1)$-dimensional normal space $G$ by construction.

$$O(p_i)|W = H_2 * (O(p_i)|G), \ i = 1, 2, ... n, \ O(v_k)|W = G, \ k = 1, 2$$

Therefore, the rim of any point of $W$ is a $(k+1)$-dimensional normal space, and, by the definition, $W$ is a normal $(k+2)$-dimensional space. That completes the proof. □
We are now in a position to describe n-dimensional normal spaces with peculiar properties.

- Firstly construct a space without singularities. Let $G$ be n-dimensional sphere $S^n$. It means that the rim of any point of $S^n$ is a normal sphere $S^{n-1}$, and $S^n$ can be turned into the minimal $S^n$ on 2n+2 points by contractible transformations [3,6,7].

Consider $W=H_2(v_1,v_2)*S^n$. If $p \in S^n$ then $O(p)|W = H_2(v_1,v_2)*S_p^{n-1} = S_p^n$. For points $v_1$ and $v_2$ the rim is $S^n$ itself. Therefore, the rim of any point of $W$ is sphere $S^n$, and $W$ is a normal (n+1)-dimensional space. It is easy to show that $W$ can be reduced to the minimal (n+1)-sphere $S^{n+1}$ by contractible transformations and, therefore, $W = S^{n+1}$.

\[ W = H_2 * S^n = S^{n+1} \]

\[ O(p)|W = H_2 * S_p^{n-1} = S_p^n, \quad p \in S^n; \quad O(v_1)|W = O(v_2)|W = S^n \]

- Suppose that $G$ is a discrete two-dimensional torus $T^2$ depicted in Figure 5. For any point $p$ of $T^2$ $O(p) = S_p^1$. Therefore, in $W = H_2(v_1,v_2)*T^2$ the rim of any point $p$ is a two-dimensional sphere $S_p^2$, $O(p)|W = S_p^2$. However, for points $v_1$ and $v_2$ their rims are the torus $T^2$ itself, $O(v_i) = T^2$, $i=1,2$. Notice that the dimension of $T^2$ is equal to 2. Hence $W$ is a normal three-dimensional space in which the rims of points have a different topology. For points $v_1$ and $v_2$ the space has torus neighborhood $T^2$, in all other points the neighborhood is spherical, $S^2$.

\[ W = H_2 * T^2 \]

\[ O(p)|W = H_2 * S_p^1 = S_p^2, \quad p \in T^2; \quad O(v_1)|W = O(v_2)|W = T^2 \]

- Another peculiar three-dimensional space appears when we choose the projective plane $P^2$ (Figure 5) as a basic space $G$.

In three-dimensional normal space $W = H_2(v_1,v_2)*P^2$ the neighbourhoods of $v_1$ and $v_2$ are the projective plane $P^2$, the neighbourhoods of all other points are usual spheres $S^2$.

\[ W = H_2 * P^2 \]

\[ O(p)|W = H_2 * S_p^1 = S_p^2, \quad p \in P^2; \quad O(v_1)|W = O(v_2)|W = P^2 \]

- In general we can create a number of three-dimensional normal spaces with two singularities by taking discrete models of closed two-dimensional oriented or non-oriented surfaces as a basic space.
The dimensional local structure of a physical discrete (3+1) space-time.

Now we are ready to discuss some general features of the physical (3+1) space-time. We will restrict our consideration by local properties of a point \( v \).

**Theorem 2.** (3+1) space-time is four-dimensional non-normal.

**Proof.** We have to prove that in (3+1) space-time the rim of any point is a closed three-dimensional non-normal discrete space.

Suppose that a physical object is in point \( v \) of a three-dimensional discrete space \( R(t) \) at a given moment \( t \) and at either the same or the nearest point \( v_1 \) at the next moment \( t + Dt \). (Figure 8a). In (3+1) space-time \((R,T)\) we have two three-dimensional spaces \( R(t) \) and \( R(t + Dt) \) corresponding to the different moments. Obviously these spaces are joined together in the following way. Point \( v \) on \( R(t) \) should be connected with the ball \( B(v) \) on \( R(t + Dt) \) (Figure 8b). Therefore, in the (3+1) space-time \((R,T)\) (Figure 8c) the rim \( O(v)|(R,T) \) of point \( v \) is as shown in Figure 8d.

(i) If the rim \( O(v)|R(t) \) of \( v \) in \( R(t) \) is a non-normal closed two-dimensional space, then, for the same reasons as in theorem 1, \( O(v) \) in \((R,T)\) is a non-normal closed three-dimensional space, and \((R,T)\) is a non-normal four-dimensional space.

(ii) Suppose that \( R(t) \) is a normal three-dimensional space. Then \( O(v) \) in \( R(t) \) is a normal two-dimensional discrete space. Obviously \( O(v) \) in \((R,T)\) contains the normal three-dimensional space \( H(u_1, u_2) * O(v)|R(t) \) where \( u_1 \) and \( u_2 \) are \( v \) in \( R(t + Dt) \) and \( R(t - Dt) \). By theorem 1 it is a normal three-dimensional space. Take \( v_1 \) in \( R(t + Dt) \), \( v_1 \in R(t + Dt) \), \( v_1 \in O(v)|(R,T) \). It is easy to see that \( v \) in \( R(t + Dt) \) is adjacent to all points of the rim of this \( v_1 \) in \( O(v)|(R,T) \). Hence \( O(p)|(R,T) \) is a non-normal closed three-dimensional space which can be reduced into normal \( H(u_1, u_2) * O(v)|R(t) \) by contractible transformations. Thus \((R,T)\) is a non-normal four-dimensional space-time. \( \square \)
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**Figure Captions.**

Figure 1: Difference between an infinite and finite number of enclosed disks $D^n$ in continuous and discrete spaces respectively.

Figure 2: Graph $G$, its molecular space $M(G)$ and its coordinate matrix $A(G)$.

Figure 3: Zero ($S^0$), one ($S^1_1, S^1_2$) and two ($S^2$) dimensional normal discrete spheres, and one ($v_1$), two ($v_2, v_3$) and three ($v_4$) dimensional points.
Figure 4: Normal discrete two-dimensional planes and their molecular spaces. $E_1^2$ is the two-dimensional plane in Khalimsky topology.

Figure 5: A discrete normal three-dimensional sphere $S^3$, a two-dimensional torus $T^2$, a two-dimensional projective plane $P^2$. The Euler characteristic and the homology groups of these graphs are consistent with the Euler characteristic and the homology groups of their continuous counterparts.

Figure 6: Zero and one-dimensional non-normal spheres $S^0$ and $S^1$ and their molecular spaces $M(S^0)$ and $M(S^1)$. $M(S^2)$ is a molecular space of the two-dimensional non-normal sphere $S^2$. It does not contain the central unit cube.

Figure 7: Non-normal discrete one and two-dimensional flat spaces $E_1^1$ and $E_2^2$ and their molecular spaces $M(E_1^1)$ and $M(E_2^2)$. $M(E_3^3)$ is a molecular space of a non-normal discrete three-dimensional flat space $E_3^3$.

Figure 8: Theorem 2 for (1+1) space-time. (1+1) space-time is not normal because $O(v)$ is not a normal one-dimensional sphere.
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