New $L_p$ Affine Isoperimetric Inequalities *

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Abstract

We prove new $L_p$ affine isoperimetric inequalities for all $p \in [-\infty, 1)$. We establish, for all $p \neq -n$, a duality formula which shows that $L_p$ affine surface area of a convex body $K$ equals $L_{p_n^2}$ affine surface area of the polar body $K^\circ$.

1 Introduction

An affine isoperimetric inequality relates two functionals associated with convex bodies (or more general sets) where the ratio of the functionals is invariant under non-degenerate linear transformations. These affine isoperimetric inequalities are more powerful than their better known Euclidean relatives.

This article deals with affine isoperimetric inequalities for the $L_p$ affine surface area. $L_p$ affine surface area was introduced by Lutwak in the ground breaking paper [26]. It is now at the core of the rapidly developing $L_p$ Brunn Minkowski theory. Contributions here include new interpretations of $L_p$ affine surface areas [32, 37, 38], the discovery of new ellipsoids [21, 28], the study of solutions of nontrivial ordinary and, respectively, partial differential equations (see e.g. Chen [9], Chou and Wang [10], Stancu [39, 40]), the study of the $L_p$ Christoffel-Minkowski problem by Hu, Ma and Shen [16], a new proof by Fleury, Guédon and Paouris [11] of a result by Klartag [18] on concentration of volume, and characterization theorems by Ludwig and Reitzner [23].

The case $p = 1$ is the classical affine surface area which goes back to Blaschke [6]. Originally a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention too (e.g. [5, 20, 25, 31, 36]). It is fundamental in the theory of valuations (see e.g. [1, 2, 22, 17]), in approximation of convex bodies by polytopes [14, 38, 24] and it is the subject of the affine Plateau problem solved in $\mathbb{R}^3$ by Trudinger and Wang [41, 43].

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The classical affine isoperimetric inequality which gives an upper bound for the affine surface area in terms of volume proved to be the key ingredient in many problems (e.g. [12, 13, 27, 34]). In particular, it was used to show the uniqueness of self-similar solutions of the affine curvature flow and to study its asymptotic behavior by Andrews [3, 4], Sapiro and Tannenbaum [35].

$L_p$ affine isoperimetric inequalities were first established by Lutwak for $p > 1$ in [26]. There has been a growing body of work in this area since from which we quote only Lutwak, Yang and Zhang [29, 30] and Campi and Gronchi [8].

Here we derive new $L_p$ affine isoperimetric inequalities for all $p \in (-\infty, 1)$. We give new interpretations of $L_p$ affine surface areas. We establish, for all $p \neq -n$, a duality formula which shows that $L_p$ affine surface area of a convex body $K$ equals $L_{n+2-p}$ affine surface area of the polar body $K^\circ$. This formula was proved in [15] for $p > 0$.

From now on we will always assume that the centroid of a convex body $K$ in $\mathbb{R}^n$ is at the origin. We write $K \in C^2_+$ if $K$ has $C^2$ boundary with everywhere strictly positive Gaussian curvature. For real $p \neq -n$, we define the $L_p$ affine surface area $as_p(K)$ of $K$ as in [26] ($p > 1$) and [38] ($p < 1$) by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+2}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+2}}} d\mu_K(x) \quad (1.1)$$

and

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n} d\mu_K(x) \quad (1.2)$$

provided the above integrals exist. $N_K(x)$ is the outer unit normal vector at $x$ to $\partial K$, the boundary of $K$. $\kappa_K(x)$ is the Gaussian curvature at $x \in \partial K$ and $\mu_K$ denotes the usual surface area measure on $\partial K$. $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ which induces the Euclidian norm $\| \cdot \|$. In particular, for $p = 0$

$$as_0(K) = \int_{\partial K} \langle x, N_K(x) \rangle d\mu_K(x) = n|K|,$$

where $|K|$ stands for the $n$-dimensional volume of $K$. More generally, for a set $M$, $|M|$ denotes the Hausdorff content of its appropriate dimension. For $p = 1$

$$as_1(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} d\mu_K(x)$$

is the classical affine surface area which is independent of the position of $K$ in space.

If the boundary of $K$ is sufficiently smooth then (1.1) and (1.2) can be written as integrals over the boundary $\partial B_2^n = S^{n-1}$ of the Euclidean unit ball $B_2^n$ in $\mathbb{R}^n$

$$as_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u).$$
σ is the usual surface area measure on $S^{n-1}$. $h_K(u)$ is the support function of direction $u \in S^{n-1}$, and $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $\kappa_K(x)$ at this point $x \in \partial K$ that has $u$ as outer normal. In particular, for $p = \pm\infty$,

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n|K^\circ|$$

(1.3)

where $K^\circ = \{y \in \mathbb{R}^n, \langle x, y \rangle \leq 1, \forall x \in K\}$ is the polar body of $K$.

In Sections 2 and 3 we give new geometric interpretations of the $L_p$ affine surface areas and obtain as a consequence

**Corollary 3.1** Let $K$ be a convex body in $C^2_+$ and let $p \neq -n$ be a real number. Then

$$as_p(K) = as_{\frac{2}{p}}(K^\circ).$$

In Section 4 we prove the following new $L_p$ affine isoperimetric inequalities. For $p \geq 1$ they were proved by Lutwak [26].

**Theorem 4.2** Let $K$ be a convex body with centroid at the origin.

(i) If $p \geq 0$, then

$$\frac{as_p(K)}{as_p(B^n_2)} \leq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}},$$

with equality if and only if $K$ is an ellipsoid. For $p = 0$, equality holds trivially for all $K$.

(ii) If $-n < p < 0$, then

$$\frac{as_p(K)}{as_p(B^n_2)} \geq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}},$$

with equality if and only if $K$ is an ellipsoid.

(iii) If $K$ is in addition in $C^2_+$ and if $p < -n$, then

$$c^{\frac{np}{n+p}} \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}} \leq \frac{as_p(K)}{as_p(B^n_2)}.$$

The constant $c$ in (iii) is the constant from the Inverse Santaló inequality due to Bourgain and Milman [7]. This constant has recently been improved by Kuperberg [19]. We give examples that the above isoperimetric inequalities cannot be improved.
In Theorem 4.1 we show a monotonicity behavior of the quotient \( \left( \frac{a_r(K)}{n|K|} \right)^{\frac{n+1}{n}} \), namely
\[
\left( \frac{a_r(K)}{n|K|} \right) \leq \left( \frac{as_t(K)}{n|K|} \right)^{\frac{r(n+1)}{r(n+2)}}.
\]
and as a consequence obtain

**Corollary 4.1** Let \( K \) be convex body in \( \mathbb{R}^n \) with centroid at the origin.

(i) For all \( p \geq 0 \)
\[
as_p(K) as_p(K^\circ) \leq n^2 |K| |K^\circ|.
\]
(ii) For \( -n < p < 0 \),
\[
as_p(K)as_p(K^\circ) \geq n^2 |K| |K^\circ|.
\]
If \( K \) is in addition in \( C^2_+ \), inequality (ii) holds for all \( p < -n \).

## 2 \( L_{-\frac{n}{n+2}} \) affine surface area of the polar body

It was proved in [32] that for a convex body \( K \in C^2_+ \)
\[
\lim_{\delta \to 0} c_n \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+2}}} = \int_{S^{n-1}} \frac{d\sigma(u)}{P_K(u)^{\frac{n+1}{n+2}} h_K(u)^{n+1}} = \int_{\partial K} \frac{\kappa_K(x)^{\frac{n+2}{n+1}}}{\kappa_{B_\infty^n}^n(x)^{\frac{n+2}{n+1}}} d\mu_{K^\circ}(x)
\]
\[
= as_{-n(n+2)}(K),
\]
where \( c_n = 2 \left( \frac{|B_\infty^n|}{n+1} \right)^{\frac{2}{n+2}} \) and \( K_\delta \) is the convex floating body [36]: The intersection of all halfspaces \( H^+ \) whose defining hyperplanes \( H \) cut off a set of volume \( \delta \) from \( K \).

Assumptions on the boundary of \( K \) are needed in order that (2.4) holds. To see that, consider \( B_\infty^n = \{ x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i| \leq 1 \} \). As \( \kappa_{B_\infty^n}(x) = 0 \) a.e. on \( \partial B_\infty^n \),
\[
\int_{\partial B_\infty^n} \frac{\kappa_{B_\infty^n}(x)^{\frac{n+2}{n+1}}}{\kappa_{B_\infty^n}(x)^{\frac{n+2}{n+1}}} d\mu_{B_\infty^n}(x) = 0.
\]
However
\[
\lim_{\delta \to 0} c_n \frac{|(B_\infty^n)^\circ| - |B_\infty^n|}{\delta^{\frac{2}{n+2}}} = \infty.
\]

Indeed, writing \( K \) for \( B_\infty^n \), we will construct a 0-symmetric convex body \( K_1 \) such that \( K_\delta \subseteq K_1 \subseteq K \). Then \( K^\circ \subseteq K_1^\circ \subseteq K^\circ_\delta \). Therefore, to show (2.5), it is enough to show that
\[
\lim_{\delta \to 0} c_n \frac{|K^\circ_\delta| - |K^\circ|}{\delta^{\frac{2}{n+2}}} = \infty.
\]
Let \( R^+ = \{(x_j)_{j=1}^n : x_j \geq 0, 1 \leq j \leq n\} \). It is enough to consider \( K^+ = R^+ \cap K \) and to construct \((K_1)^+ = K_1 \cap R^+\).

We define \((K_1)^+\) to be the intersection of \( R^+ \) with the half-spaces \( H_i^+, 1 \leq i \leq n+1 \), where \( H_i = \{(x_j)_{j=1}^n : x_i = 1\}, 1 \leq i \leq n \), and \( H_{n+1} = \left\{(x_j)_{j=1}^n : \sum_{j=1}^n x_j = n \right\} \), \( \delta > 0 \) sufficiently small. Notice that the hyperplane \( H_{n+1} \) (orthogonal to the vector \((1, \ldots, 1)\)) cuts off a set of volume exactly \( \delta \) from \( K \) and therefore \( \hat{K}_\delta \subset K_1 \).

Moreover, \( K_1^\circ \) can be written as a convex hull:

\[
K_1^\circ = \text{co} \left( \{ \pm e_i, 1 \leq i \leq n \} \cup \left\{ \frac{1}{s} (\varepsilon_1, \ldots, \varepsilon_n), \varepsilon_j = \pm 1, 1 \leq j \leq n \right\} \right),
\]

where \( s = n - (n!\delta)^{\frac{1}{n}} \). Hence

\[
|K_1^\circ| = \frac{2^n}{n!} \cdot \frac{n}{n - (n!\delta)^{\frac{1}{n}}}
\]

and therefore

\[
\lim_{\delta \to 0} \frac{|K_1^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = \frac{2^n}{n!} \lim_{\delta \to 0} \delta^{-\frac{2}{n+1}} \frac{(n!\delta)^{\frac{1}{n}}}{(n - (n!\delta)^{\frac{1}{n}})} = \infty.
\]

Now we show

**Theorem 2.1** Let \( K \) be a convex body in \( C_+^2 \) such that \( 0 \in \text{int}(K) \). Then

\[
\lim_{\delta \to 0} c_n \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = as_{-\frac{n}{n+2}}(K^\circ).
\]

As a corollary of (2.4) and Theorem 2.1 we get that for a convex body \( K \in C_+^2 \)

\[
as_{-n(n+2)}(K) = as_{-\frac{n}{n+2}}(K^\circ).
\]

(2.6)

This is a special case for \( p = -n(n+2) \) of the formula \( as_p(K) = as_{\frac{n}{n+2}}(K^\circ) \) proved in [15] for \( p > 0 \). We will show in the next section that this formula holds for all \( p < 0, p \neq -n \) for convex bodies with sufficiently smooth boundary. For \( p = 0 \) (and \( K \in C_+^2 \)) the formula holds trivially as \( as_0(K) = n|K| \) and \( as_{\infty}(K^\circ) = n|K| \) (see [38]).

For the proof of Theorem 2.1 we need the following lemmas.

**Lemma 2.1** Let \( K \in C_+^2 \). Then for any \( x \in \partial K^\circ \), we have

\[
\lim_{\delta \to 0} \frac{\langle x, N_{K^\circ}(x) \rangle}{\delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x\|}{\delta} \right)^n - 1 \right] = \frac{\langle x, N_{K^\circ}(x) \rangle^2}{c_n (\kappa_{K^\circ}(x))^{\frac{1}{n+1}}},
\]

where \( x_\delta \in \partial (K_\delta)^\circ \) is in the ray passing through 0 and \( x \).
Proof
Since $K$, and hence also $K_\delta$, are in $C^2_+$ one has that $K^\circ$ and $(K_\delta)^\circ$ are in $C^2_+$. Therefore, for $x \in \partial K^\circ$ there exists a unique $y \in \partial K$, such that, $\langle x, y \rangle = 1$, namely $y = \frac{N_{K^\circ}(x)}{\langle N_{K^\circ}(x), x \rangle}$. $y$ has outer normal vector $N_K(y) = \frac{x}{\|x\|}$ and $\frac{1}{\|x\|} = \langle y, N_K(y) \rangle$.

Similarly, for $x_\delta \in \partial (K_\delta)^\circ$ there exists a unique $y_\delta$ in $\partial K_\delta$ such that $\langle x_\delta, y_\delta \rangle = 1$, namely $y_\delta = \frac{N_{(K_\delta)^\circ}(x_\delta)}{\langle N_{(K_\delta)^\circ}(x_\delta), x_\delta \rangle}$, $y_\delta$ has outer normal vector $N_{K_\delta}(y_\delta) = \frac{x_\delta}{\|x_\delta\|}$ and $\frac{1}{\|x_\delta\|} = \langle y_\delta, N_{K_\delta}(y_\delta) \rangle$.

Let $y' = [0, y] \cap \partial K_\delta$ ([z₁, z₂] denotes the line segment from $z_1$ to $z_2$) and let $y'_\delta \in \partial K$ be such that $y_\delta = [0, y'_\delta] \cap K_\delta$.

We have
\[
\frac{1}{\|x\|} = \frac{\langle y, N_K(y) \rangle}{\langle y_\delta, N_{K_\delta}(y_\delta) \rangle} = \langle y'_\delta, \frac{x}{\|x\|} \rangle \leq \frac{\langle x, N_{K^\circ}(x) \rangle}{\|y'_\delta\|^n} - 1 \leq \frac{\langle x, N_{K^\circ}(x) \rangle}{\|y'_\delta\|^n} - 1.
\]

Hence
\[
\left[\frac{\|x_\delta\|}{\|x\|} - 1\right] = \left[\frac{\langle y, N_K(y) \rangle}{\langle y_\delta, N_{K_\delta}(y_\delta) \rangle} - 1\right] \leq \left[\frac{\langle x, N_{K^\circ}(x) \rangle}{\|y'_\delta\|^n} - 1\right] \leq \left[\frac{\langle x, N_{K^\circ}(x) \rangle}{\|y'_\delta\|^n} - 1\right].
\]

We first consider the lower bound.
\[
\lim_{\delta \to 0} \frac{n}{\delta + \frac{\|y'_\delta\|^n}{\|x_\delta\|^n} - 1} \geq \lim_{\delta \to 0} \frac{n}{\delta + \frac{\|y'_\delta\|^n}{\|x_\delta\|^n} - 1} = \frac{n}{\delta + \frac{\|y'_\delta\|^n}{\|x_\delta\|^n} - 1}.
\]

As $\delta \to 0$, $y'_\delta \to y$. As $K$ is in $C^2_+$, $N_{K_\delta}(y'_\delta) \to N_K(y)$ as $\delta \to 0$. 6
Therefore \( \lim_{\delta \to 0} \langle y'_0, N_{K_\delta}(y'_0) \rangle = \langle y, N_K(y) \rangle \). By Lemma 7 and Lemma 10 of [36],

\[
\lim_{\delta \to 0} \frac{\langle y'_0, N_{K_\delta}(y'_0) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|y'_0\|}{\|x'_\delta\|} \right)^n - 1 \right] = \frac{(\kappa_K(y))^{\frac{1}{n+1}}}{c_n}.
\]

Hence

\[
\lim_{\delta \to 0} \frac{\langle x, N_{K_\delta}(x) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_{\delta}\|}{\|x\|} \right)^n - 1 \right] \geq \frac{\langle x, N_{K_\delta}(x) \rangle (\kappa_K(y))^{\frac{1}{n+1}}}{c_n} = \frac{\langle x, N_{K_\delta}(x) \rangle \kappa_K(y) (\kappa_{K_\delta}(x))^{\frac{1}{n+1}}}{c_n (\kappa_{K_\delta}(x))^{\frac{1}{n+1}}}.\]

The last equation follows from the fact that if \( K \in C^2_+ \), then, for any \( y \in \partial K \), there is a unique point \( x \in \partial K_\delta \) such that \( \langle x, y \rangle = 1 \) and [15]

\[
\langle y, N_K(y) \rangle \langle x, N_{K_\delta}(x) \rangle = (\kappa_K(y) \kappa_{K_\delta}(x))^{\frac{1}{n+1}}. \tag{2.8}
\]

Similarly, one gets for the upper bound

\[
\lim_{\delta \to 0} \frac{\langle x, N_{K_\delta}(x) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_{\delta}\|}{\|x\|} \right)^n - 1 \right] \leq \frac{\langle x, N_{K_\delta}(x) \rangle \kappa_K(y) (\kappa_{K_\delta}(x))^{\frac{1}{n+1}}}{c_n (\kappa_{K_\delta}(x))^{\frac{1}{n+1}}},
\]

hence altogether

\[
\lim_{\delta \to 0} \frac{\langle x, N_{K_\delta}(x) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_{\delta}\|}{\|x\|} \right)^n - 1 \right] = \frac{\langle x, N_{K_\delta}(x) \rangle \kappa_K(y) (\kappa_{K_\delta}(x))^{\frac{1}{n+1}}}{c_n (\kappa_{K_\delta}(x))^{\frac{1}{n+1}}}.\]

\textbf{Lemma 2.2} Let \( K \in C^2_+ \). Then we have

\[
\frac{\langle x, N_{K_\delta}(x) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_{\delta}\|}{\|x\|} \right)^n - 1 \right] \leq c(K, n),
\]

where \( c(K, n) \) is a constant (depending on \( K \) and \( n \) only) and \( x \) and \( x_\delta \) are as in Lemma 2.1.

\textbf{Proof} By (2.7)

\[
\frac{\langle x, N_{K_\delta}(x) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_{\delta}\|}{\|x\|} \right)^n - 1 \right] \leq \frac{\langle x, N_{K_\delta}(x) \rangle \langle y, N_{K_\delta}(y) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|y\|}{\|y\|'} \right)^n - 1 \right] \leq \frac{\langle x, N_{K_\delta}(x) \rangle}{\langle y, N_{K_\delta}(y) \rangle} \left( \frac{\|y\|}{\|y\|'} \right) \left[ \left( \frac{\|y\|}{\|y\|'} \right)^n - 1 \right].
\]
Since $K_\delta$ is increasing to $K$ as $\delta \to 0$, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$, $0 \in \text{int}(K_\delta)$. Therefore there exists $\alpha > 0$ such that $B_2^n(0, \alpha) \subset K_\delta \subset K \subset B_2^n(0, \frac{1}{\alpha})$ for all $\delta < \delta_0$. $B_2^n(0, r)$ is the $n$-dimensional Euclidean ball centered at 0 with radius $r$.

Hence for $\delta < \delta_0$

$$\frac{\langle x, N_{K_\delta}(x) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_\delta\|}{\|x\|} \right)^n - 1 \right] \leq \alpha^{-2(n+1)} \frac{\langle y, N_K(y) \rangle}{n \delta^{\frac{2}{n+1}}} \left[ 1 - \left( \frac{\|y\|}{\|y\|} \right)^n \right] \leq C' r(y)^{-\frac{n+1}{n+4}}$$
due to Lemma 6 in [36]. Here $r(y)$ is the radius of the biggest Euclidean ball contained in $K$ and touching $\partial K$ at $y$.

Since $K$ is $C^2$, by the Blaschke rolling theorem (see [34]) there is $r_0 > 0$ such that $r_0 \leq \min_{y \in \partial K} r(y)$. We put $c(K, n) = C' r_0^{-\frac{n+1}{n+4}}$.

**Proof of Theorem 2.1.**

$$\frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = \frac{1}{n \delta^{\frac{2}{n+1}}} \int_{\partial K^\circ} \langle x, N_{K_\delta}(x) \rangle \left[ \left( \frac{\|x_\delta\|}{\|x\|} \right)^n - 1 \right] d\mu_{K_\circ}(x).$$

Combining Lemma 2.1, Lemma 2.2 and Lebesgue’s convergence theorem, gives Theorem 2.1:

$$\lim_{\delta \to 0} \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = \lim_{\delta \to 0} \frac{1}{n \delta^{\frac{2}{n+1}}} \int_{\partial K^\circ} \langle x, N_{K_\delta}(x) \rangle \left[ \left( \frac{\|x_\delta\|}{\|x\|} \right)^n - 1 \right] d\mu_{K_\circ}(x)$$

$$= \int_{\partial K^\circ} \lim_{\delta \to 0} \frac{1}{n \delta^{\frac{2}{n+1}}} \langle x, N_{K_\delta}(x) \rangle \left[ \left( \frac{\|x_\delta\|}{\|x\|} \right)^n - 1 \right] d\mu_{K_\circ}(x)$$

$$= \int_{\partial K^\circ} \frac{\langle x, N_{K^\circ}(x) \rangle^2}{c_n \left( \kappa_{K^\circ}(x) \right)^{\frac{1}{n+1}}} d\mu_{K_\circ}(x)$$

$$= \frac{1}{c_n} \text{as } K \to K^\circ.$$

**Remark**

The proof of Theorem 2.1 provides a uniform method to evaluate

$$\lim_{t \to 0} \frac{|(K_t)^\circ| - |K^\circ|}{t^{\frac{2}{n+1}}}$$

where $K_t$ is a family convex bodies constructed from the convex body $K$ such that $K_t \subset K$ or- similarly- such that $K \subset K_t$. In particular, we can apply this method to prove the analog statements as in (2.4) and Theorem 2.1 if we take as $K_t$ the illumination body of $K$ [42], or the Santaló body of $K$ [31], or the convolution body of $K$ [33] - and there are many more.
3 \(L_p\) affine surface areas

We now prove that for all \(p \neq -n\) and all \(K \in C^2_+\), \(as_p(K) = as_{\frac{n}{p}}(K^\circ)\). To do so, we use the surface body of a convex body which was introduced in [37, 38]. We also give a new geometric interpretation of \(L_p\) affine surface area for all \(p \neq -n\).

**Definition 3.1** Let \(s \geq 0\) and \(f : \partial K \to \mathbb{R}\) be a nonnegative, integrable function. The surface body \(K_{f,s}\) is the intersection of all the closed half-spaces \(H^+\) whose defining hyperplanes \(H\) cut off a set of \(f_{\mu_K}\)-measure less than or equal to \(s\) from \(\partial K\). More precisely,

\[
K_{f,s} = \bigcap_{\partial K \cap H^+} H^+.
\]

**Theorem 3.1** Let \(K\) be a convex body in \(C^2_+\) and such that 0 is the center of gravity of \(K\). Let \(f : \partial K \to \mathbb{R}\) be an integrable function such that \(f(x) > c\) for all \(x \in \partial K\) and some constant \(c > 0\). Let \(\beta_n = 2\left(\frac{1}{B_2^{n-1}}\right)^{\frac{2}{n-1}}\). Then

\[
\lim_{s \to 0} \beta_n \left|\frac{\left(K_{f,s}\right)^0}{s^{\frac{2}{n-1}}} - \frac{|K^0|}{s^{\frac{n}{n-1}}}\right| = \int_{S^{n-1}} \frac{d\sigma(u)}{h_K(u)^{n+1} f_K(u)^{\frac{1}{n-1}} \left(f(N_K^{-1}(u))\right)^{\frac{2}{n-1}}}
\]

where \(N_K : \partial K \to S^{n-1}, x \to N_K(x) = u\) is the Gauss map.

**Proof**

Let \(u \in S^{n-1}\). Let \(x \in \partial K\) be such that \(N_K(x) = u\) and let \(x_s \in \partial K_{f,s}\) be such that \(N_{K_{f,s}}(x_s) = u\). Let \(H_\Delta = H(x - \Delta u, u)\) be the hyperplane through \(x - \Delta u\) with outer normal vector \(u\). Since \(K\) has everywhere strictly positive Gaussian curvature, by Lemma 21 in [38] almost everywhere on \(\partial K\),

\[
\lim_{\Delta \to 0} \frac{1}{|\partial K \cap H_\Delta^+|} \int_{\partial K \cap H_\Delta^+} |f(x) - f(y)| d\mu_K(y) = 0.
\]

This implies that

\[
\lim_{\Delta \to 0} \frac{1}{|\partial K \cap H_\Delta^+|} \int_{\partial K \cap H_\Delta^+} f(y) d\mu_K(y) = f(x).
\]

(3.9)

Let \(b_s = h_K(u) - h_{K_{f,s}}(u)\). As \(H(x - b_s u, u) = H(x_s, u)\) (the hyperplane through \(x_s\) with outer normal \(u\)) and as \(b_s \to 0\) as \(s \to 0\), (3.9) implies

\[
\lim_{s \to 0} \frac{1}{|\partial K \cap H^-(x_s, u)|} \int_{\partial K \cap H^-(x_s, u)} f(y) d\mu_K(y) = f(x).
\]

(3.10)
Thus we get from (3.12), (3.13) and (3.14) that
\[
s \leq (1 + \varepsilon) f(x) |B_2^{n-1} | \sqrt{f_K(u)} \left( 2b_s \right) ^{\frac{n-1}{2}} ,
\] or, equivalently
\[
\frac{b_s}{S^{\frac{2}{n-1}}} \geq \frac{1 - c_1 \varepsilon}{\beta_n f(N^{-1}_K(u))^\frac{2}{n-1} f_K(u)^{\frac{1}{n-1}}},
\]
where \( c_1 \) is an absolute constant.

Let now \( x'_s \in [0,x] \cap \partial K_f,s \). Then \( \langle x'_s, u \rangle \leq h_{K_f,s}(u) \). Therefore \( b_s = h_K(u) - h_{K_f,s}(u) \leq \langle x - x'_s, u \rangle \).

Hence for \( s \) sufficiently small
\[
\frac{b_s}{S^{\frac{2}{n-1}}} \leq \frac{\langle x - x'_s, u \rangle}{S^{\frac{2}{n-1}}} \leq \frac{\langle x, u \rangle}{S^{\frac{2}{n-1}}} \left( 1 - \frac{\|x'_s\|}{\|x\|} \right) \leq \frac{\langle x, u \rangle}{S^{\frac{2}{n-1}}} \frac{\|x'_s - x\|}{\|x\|} \leq (1 + \varepsilon) \frac{\langle x, N_K(x) \rangle}{n S^{\frac{2}{n-1}}} \left[ 1 - \left( \frac{\|x'_s\|}{\|x\|} \right) ^n \right].
\]
The last inequality follows as \( 1 - \left( \frac{\|x'_s\|}{\|x\|} \right) ^n \geq (1 - \varepsilon) ^n \frac{\|x'_s - x\|}{\|x\|} \) for sufficiently small \( s \). By Lemma 23 in [38]
\[
\lim_{s \to 0} \frac{n S^{\frac{2}{n-1}}} S^{\frac{2}{n-1}} \langle x, N_K(x) \rangle \left[ 1 - \left( \frac{\|x'_s\|}{\|x\|} \right) ^n \right] = \frac{1}{\beta_n f(N^{-1}_K(u))^\frac{2}{n-1} f_K(u)^{\frac{1}{n-1}}}.
\]
Thus we get from (3.12), (3.13) and (3.14) that
\[
\lim_{s \to 0} \frac{b_s}{S^{\frac{2}{n-1}}} = \frac{1}{\beta_n f(N^{-1}_K(u))^\frac{2}{n-1} f_K(u)^{\frac{1}{n-1}}}. \tag{3.15}
\]

As \( (1 - t)^{-n} \geq 1 + nt \) for \( 0 \leq t < 1 \) and by (3.15),
\[
\lim_{s \to 0} \frac{\beta_n}{n S^{\frac{2}{n-1}}} ([h_{K_f,s}(u)]^{-n} - [h_K(u)]^{-n}) = \lim_{s \to 0} \frac{\beta_n}{n S^{\frac{2}{n-1}}} [h_K(u)]^{-n} \left[ \left( 1 + \frac{b_s}{h_K(u)} \right) ^{-n} - 1 \right] \geq \lim_{s \to 0} \frac{\beta_n}{[h_K(u)]^{n+1} f(N^{-1}_K(u))^\frac{2}{n-1} f_K(u)^{\frac{1}{n-1}}} \frac{b_s}{S^{\frac{2}{n-1}}} = \frac{1}{[h_K(u)]^{n+1} f(N^{-1}_K(u))^\frac{2}{n-1} f_K(u)^{\frac{1}{n-1}}}. \tag{3.16}
\]
As \( h_{K,f,s}(u) \geq \langle x'_s, u \rangle \),
\[
\frac{h_{K,f,s}(u)}{h_K(u)} \geq \frac{\langle x'_s, u \rangle}{\|x\|} = \frac{\|x'_s\|}{\|x\|},
\] (3.17)

Since \( K \in C^2_+ \), \( h_{K,f,s}(u) \to h_K(u) \) as \( s \to 0 \). Therefore,
\[
\lim_{s \to 0} \frac{\beta_n}{n s^{n-1}} \left( [h_{K,f,s}(u)]^{-n} - [h_K(u)]^{-n} \right) = \lim_{s \to 0} \frac{\beta_n}{n s^{n-1}} [h_{K,f,s}(u)]^{-n} \left[ 1 - \left( \frac{h_{K,f,s}(u)}{h_K(u)} \right)^n \right] 
\leq \lim_{s \to 0} \frac{\beta_n}{n s^{n-1}} [h_{K,f,s}(u)]^{-n} \left[ 1 - \left( \frac{\|x'_s\|}{\|x\|} \right)^n \right] 
= \lim_{s \to 0} \frac{\beta_n}{h_K(u) [h_{K,f,s}(u)]^n} \lim_{s \to 0} \frac{1}{n s^{n-1}} \langle x, N_K(x) \rangle \left[ 1 - \left( \frac{\|x'_s\|}{\|x\|} \right)^n \right] 
= \frac{1}{[h_K(u)]^{n+1} f(N_K^{-1}(u))^{\frac{2}{n-1}} f_K(u)^{\frac{1}{n-1}}} 
\] (3.18)

where the last equality follows from (3.14).

Altogether, (3.16) and (3.18) give
\[
\lim_{s \to 0} \frac{\beta_n}{n s^{n-1}} \left( [h_{K,f,s}(u)]^{-n} - [h_K(u)]^{-n} \right) = \frac{1}{[h_K(u)]^{n+1} f(N_K^{-1}(u))^{\frac{2}{n-1}} f_K(u)^{\frac{1}{n-1}}}. 
\]

Therefore
\[
\lim_{s \to 0} \frac{\beta_n}{s^{n-1}} \left( |(K_{f,s})^0| - |K^0| \right) = \lim_{s \to 0} \frac{\beta_n}{n s^{n-1}} \int_{S^{n-1}} \left[ \left( \frac{1}{h_{K,f,s}(u)} \right)^n - \left( \frac{1}{h_K(u)} \right)^n \right] d\sigma(u) 
= \int_{S^{n-1}} \lim_{s \to 0} \frac{\beta_n}{n s^{n-1}} \left[ \left( \frac{1}{h_{K,f,s}(u)} \right)^n - \left( \frac{1}{h_K(u)} \right)^n \right] d\sigma(u) 
= \int_{S^{n-1}} \frac{1}{h_K(u)^{n+1} f_K(u)^{\frac{1}{n-1}} (f(N_K^{-1}(u)))^{\frac{2}{n-1}}} d\sigma(u), 
\]

provided we can interchange integration and limit.

We show this next. To do so, we show that for all \( u \in S^{n-1} \) and all sufficiently small \( s > 0 \),
\[
\frac{1}{n s^{n-1}} \left[ \left( \frac{1}{h_{K,f,s}(u)} \right)^n - \left( \frac{1}{h_K(u)} \right)^n \right] \leq g(u) 
\]
with $\int_{S^{n-1}} g(u) \, d\sigma(u) < \infty$. As $0 \in \text{int}(K)$, the interior of $K$, there exists $\alpha > 0$ such that for all $s$ sufficiently small $B_2(0, \alpha) \subset K_{f,s} \subset K \subset B_2^n(0, \frac{1}{\alpha})$. Therefore, $\alpha \leq h_{K_{f,s}}(u) \leq h_K(u) \leq \frac{1}{\alpha}$ and $\alpha \leq \frac{1}{h_K(u)} \leq \frac{1}{h_{K_{f,s}}(u)} \leq \frac{1}{\alpha}$.

With (3.17), we thus get for all $s > 0$,

$$\frac{1}{n \, s^{\frac{n-2}{n}}} \left( (h_{K_{f,s}}(u))^{-n} - (h_K(u))^{-n} \right)$$

$$= \frac{1}{n \, s^{\frac{n-2}{n}}} \left( h_{K_{f,s}}(u) \right)^{-n} \left( 1 - \frac{(h_{K_{f,s}}(u))^n}{(h_K(u))^n} \right)$$

$$\leq \frac{\alpha^{-n}}{n \, s^{\frac{n-2}{n}}} \left[ 1 - \left( \frac{\|x_s\|}{\|x\|} \right)^n \right]$$

$$\leq \alpha^{-(n+1)} \frac{\langle x, u \rangle}{n \, s^{\frac{n-2}{n}}} \left[ 1 - \left( \frac{\|x_s\|}{\|x\|} \right)^n \right].$$

By Lemma 17 in [38] there exists $s_3$ such that for all $s \leq s_3$

$$\frac{\langle x, u \rangle}{s^{\frac{n-2}{n}}} \left[ 1 - \left( \frac{\|x_s\|}{\|x\|} \right)^n \right] \leq \frac{C}{(M_f(x))^{\frac{2}{n-1}} r(x)},$$

where $C$ is an absolute constant and, as in the proof of Lemma 2.2, $r(x)$ is the biggest Euclidean ball contained in $K$ that touches $\partial K$ at $x$. Thus, as $\partial K$ is $C^2_+$, by Blaschke’s rolling theorem (see [34]) there is $r_0$ such that $r(x) \geq r_0$.

$$M_f(x) = \inf_{0 < s} \int_{\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s))} f \, d\mu_K$$

is the minimal function. It was introduced in [38]. By the assumption on $f$, $M_f(x) \geq c$. Thus altogether

$$\frac{1}{n \, s^{\frac{n-2}{n}}} \left( (h_{K_{f,s}}(u))^{-n} - (h_K(u))^{-n} \right) \leq \frac{\alpha^{-(n+1)} \frac{C}{n \, c^{\frac{2}{n-1}}} \, r_0}{r_0} = g(u),$$

which, as a constant, is integrable.

**Theorem 3.2** Let $K$ be a convex body in $C^2_+$ and such that $0$ is the center of gravity of $K$. Let $f : \partial K \to \mathbb{R}$ be an integrable function such that $f(y) > c$ for all $y \in \partial K$ and some constant $c > 0$. Let $\beta_n = 2 \left( \|B_2^{n-1}\|^{\frac{2}{n-1}} \right)$. Then

$$\lim_{s \to 0} \beta_n \frac{|(K_{f,s})^0 - |K^0|}{s^{\frac{2}{n-1}}} = \int_{\partial K^0} \left( \frac{\langle x, N_{K^0}(x) \rangle}{\langle y(x), N_{K^0}(y(x)) \rangle} \right) \left( \frac{f(y(x))^{\frac{1}{n-1}}}{f(y(x))^{\frac{2}{n-1}}} \right) \, d\mu_{K^0}(x)$$

Here $y(x) \in \partial K$ is such that $\langle y(x), x \rangle = 1$. 

12
Proof

We follow the pattern of the proof of Theorem 3.1 integrating now over \( \partial K^\circ \) instead of \( S^{n-1} \).

As a corollary we get the following geometric interpretation of \( L_p \) affine surface area.

**Corollary 3.1** Let \( K \in C^2_+ \) be a convex body. For \( p \in \mathbb{R}, p \neq -n \), let \( f_p : \partial K \to \mathbb{R} \) be defined by \( f_p(y) = \kappa_K(y) \frac{n^2 + p}{2(n+p)} \langle y, N_K(y) \rangle \), \( \frac{(n-1)(n^2+2n+p)}{2(n+p)} \). Then

\[
\text{(i)} \quad \lim_{s \to 0} \beta_n \left| (K_{f_p,s})^\circ \right| = \text{as}_{\frac{n^2}{p}}(K^\circ).
\]

\[
\text{(ii)} \quad \lim_{s \to 0} \beta_n \left| (K_{f_p,s})^\circ \right| = \text{as}_{\frac{p}{n}}(K) = \text{as}_{\frac{n^2}{p}}(K^\circ).
\]

Proof

Notice first that \( f_p(y) \) verifies the conditions of Theorems 3.1 and 3.2.

(i) For \( x \in \partial K^\circ \), let now \( y(x) \) be the (unique) element in \( \partial K \) such that \( \langle x, y(x) \rangle = 1 \). Then, by Theorem 3.2, with \( f(y(x)) = f_p(y(x)) \), and with (2.8)

\[
\lim_{s \to 0} \beta_n \left| (K_{f_p,s})^\circ \right| = \int_{\partial K^\circ} \frac{\langle y(x), N_K(y(x)) \rangle}{\kappa_K(y(x))} \frac{(n+1)}{n+p} d\mu_K(x)
\]

\[
= \int_{\partial K^\circ} \frac{\kappa_{K^\circ}(x) \frac{n}{n+p}}{\frac{n}{n+p}} d\mu_{K^\circ}(x) = \text{as}_{\frac{n^2}{p}}(K^\circ).
\]

(ii) For \( u \in S^{n-1}, \) let now \( y \in \partial K \) be such that \( N_K(y) = u \). Then \( f_p(N_K^{-1}(u)) = \frac{n^2 + p}{2(n+p)} h_K(u) \frac{(n-1)(n^2+2n+p)}{2(n+p)}. \) By Theorem 3.1 with \( f(N_K^{-1}(u)) = f_p(N_K^{-1}(u)) \)

\[
\lim_{s \to 0} \beta_n \left| (K_{f_p,s})^\circ \right| = \int_{S^{n-1}} \frac{f_K(u) \frac{n}{n+p}}{h_K(u) \frac{n-1}{n+p}} h\sigma(u) = \text{as}_p(K).
\]

(iii) follows from (i) and (ii).
4 Inequalities

Theorem 4.1 Let $s \neq -n, r \neq -n, t \neq -n$ be real numbers. Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at the origin and such that $\mu_K \{x \in \partial K : \kappa_K(x) = 0\} = 0$.

(i) If \( \frac{(n+r)(t-s)}{(n+t)(r-s)} > 1 \), then

\[ a_r(K) \leq \left( \frac{a_s(K)}{n|K|} \right)^{\frac{r(1)}{r(t)}} \left( \frac{a_t(K)}{n|K|} \right)^{\frac{r(t)}{(n)}}. \]

(ii) Similarly, again using Hölder's inequality, which now enforces the condition \( \frac{(n+r)t}{(n+t)r} > 1 \),

\[ \left( \frac{a_r(K)}{n|K|} \right)^{\frac{r(1)}{r(t)}} \leq \left( \frac{a_t(K)}{n|K|} \right)^{\frac{r(t)}{(n)}}. \]

Proof

(i) By Hölder’s inequality, which enforces the condition \( \frac{(n+r)(s-t)}{(n+t)(s-r)} > 1 \)

\[ a_r(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x) \rangle^{\frac{n(r-1)}{n+r}}} \, d\mu_K(x) \]

\[ = \int_{\partial K} \left( \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x) \rangle^{\frac{n(r-1)}{n+r}}} \right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left( \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x) \rangle^{\frac{n(r-1)}{n+r}}} \right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} \, d\mu_K(x) \]

\[ \leq \left( a_s(K) \right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left( a_t(K) \right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}. \]

(ii) Similarly, again using Hölder’s inequality, which now enforces the condition \( \frac{(n+r)t}{(n+t)r} > 1 \),

\[ a_r(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x) \rangle^{\frac{n(r-1)}{n+r}}} \, d\mu_K(x) = \int_{\partial K} \left( \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x) \rangle^{\frac{n(r-1)}{n+r}}} \right)^{\frac{r(t)}{(n)}} \, d\mu_K(x) \]

\[ \leq \left( a_t(K) \right)^{\frac{r(t)}{(n)}} \left( n |K| \right)^{\frac{r(t)}{(n+r)}}. \]

Condition \( \frac{(n+r)(t-s)}{(n+t)(r-s)} > 1 \) implies 8 cases: 

$-n < s < r < t$, $s < -n < t < r$, $r < t < -n < s$, $t < r < s < -n$, $s < r < t < -n$, $r < s < -n < t$, $t < -n < s < r$ and $-n < t < r < s$. 

14
Note also that (ii) describes a monotonicity condition for \( \left( \frac{as_r(K)}{|nK|} \right)^{\frac{n+r}{n+t}} \): if \( 0 < r < t \), or \( r < t < -n \), or \( -n < r < t < 0 \) then

\[
\left( \frac{as_r(K)}{|nK|} \right)^{\frac{n+r}{n+t}} \leq \left( \frac{as_t(K)}{|nK|} \right)^{\frac{n+t}{n+t}}.
\]

We now analyze various subcases of Theorem 4.1 (i) and (ii). For \( r = 0 \), if \( \frac{n(s-t)}{s(n+t)} > 1 \)

\[
n|K| \leq (as_t(K))^{\frac{t}{n+t}} \left( as_s(K) \right)^{\frac{s}{n+t}}.
\]

For \( s = 0 \), if \( \frac{t}{r(n+t)} > 1 \),

\[
as_r(K) \leq \left( n|K| \right)^{\frac{t-r}{n+r}} \left( as_t(K) \right)^{\frac{n+t}{t(n+r)}}.
\]

(4.19)

For \( s \rightarrow \infty \), if \( \frac{n+t}{n+s} > 1 \),

\[
as_r(K) \leq \left( as_s(K) \right)^{\frac{s-t}{n+s}} \left( as_s(K) \right)^{\frac{n+t}{n+s}}.
\]

(4.20)

For \( r \rightarrow \infty \), if \( \frac{t-s}{n+t} > 1 \) and if \( K \) is in \( C_+^2 \),

\[
as_{\infty}(K) = n|K^{\circ}| \leq (as_t(K))^{\frac{n+t}{n+t}} \left( as_s(K) \right)^{\frac{n+t}{n+t}}.
\]

(4.21)

As for all convex bodies \( K \), \( as_{\infty}(K) \leq n|K^{\circ}| \) (see [38]), it follows from (4.19) that, for all convex body \( K \) with centroid at origin,

\[
as_r(K) \leq \left( n|K| \right)^{\frac{n}{n+r}} \left( n|K^{\circ}| \right)^{\frac{r}{n+r}}, \quad r > 0
\]

(4.22)

and from (4.20),

\[
n|K| \left( n|K^{\circ}| \right)^{\frac{r}{n}} \leq \left( as_t(K) \right)^{\frac{n+t}{n}}, \quad -n < t < 0.
\]

(4.23)

Similarly, (4.21) implies that, if in addition \( K \) is in \( C_+^2 \),

\[
n|K^{\circ}| \left( n|K| \right)^{\frac{t}{n}} \leq \left( as_t(K) \right)^{\frac{n+t}{t}}, \quad t < -n
\]

(4.24)

(4.22) can also be obtained from Proposition 4.6 of [26] and Theorem 3.2 of [15].

Inequalities (4.22), (4.23) and (4.24) yield the following Corollary which was proved by Lutwak [26] in the case \( p \geq 1 \).
Corollary 4.1 Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at the origin.

(i) For all $p \geq 0$
\[ \text{as}_p(K) \text{ as}_p(K^o) \leq n^2 |K| |K^o|. \]

(ii) For $-n < p < 0$,
\[ \text{as}_p(K) \text{ as}_p(K^o) \geq n^2 |K| |K^o|. \]

If $K$ is in addition in $C^2_+$, inequality (ii) holds for all $p < -n$.

Thus, using Santaló inequality in (i), for $p \geq 0$, \[ \text{as}_p(K) \text{ as}_p(K^o) \leq \text{as}_p(B^n_2)^2, \] and inverse Santaló inequality in (ii), for $-n < p < 0$, \[ \text{as}_p(K) \text{ as}_p(K^o) \geq c^n \text{as}_p(B^n_2)^2. \] $c$ is the constant in the inverse Santaló inequality [7, 19].

**Proof**

(i) follows immediately from (4.22). (ii) follows from (4.23) if $-n < p < 0$ and from (4.24) if $p < -n$.

Lutwak [26] proved for $p \geq 1$
\[ \frac{\text{as}_p(K)}{\text{as}_p(B^n_2)} \leq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}, \]
with equality if and only if $K$ is an ellipsoid. We now generalize these $L_p$-affine isoperimetric inequalities to $p < 1$.

**Theorem 4.2** Let $K$ be a convex body with centroid at the origin.

(i) If $p \geq 0$, then
\[ \frac{\text{as}_p(K)}{\text{as}_p(B^n_2)} \leq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}, \]
with equality if and only if $K$ is an ellipsoid. For $p = 0$, equality holds trivially for all $K$.

(ii) If $-n < p < 0$, then
\[ \frac{\text{as}_p(K)}{\text{as}_p(B^n_2)} \geq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}, \]
with equality if and only if $K$ is an ellipsoid.

(iii) If $K$ is in addition in $C^2_+$ and if $p < -n$, then
\[ \frac{\text{as}_p(K)}{\text{as}_p(B^n_2)} \geq c^{\frac{n-p}{n+p}} \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}. \]

where $c$ is the constant in the inverse Santaló inequality [7, 19].
We cannot expect to get a strictly positive lower bound in Theorem 4.2 (i), even if $K$ is in $C^2_+$. Consider, in $\mathbb{R}^2$, the convex body $K(R, \varepsilon)$ obtained as the intersection of four Euclidean balls with radius $R$ centered at $(\pm(R - 1), 0)$, $(0, \pm(R - 1))$, $R$ arbitrarily large. We then “round” the corners by putting there arcs of Euclidean balls of radius $\varepsilon$, $\varepsilon$ arbitrarily small. To obtain a body in $C^2_+$, we “bridge” between the $R$-arcs and $\varepsilon$-arcs by $C^2_+$-arcs on a set of arbitrarily small measure. Then $\alpha_p(K(R, \varepsilon)) \leq \frac{16}{R^{2+p}} + 4\pi \varepsilon^{2+p}$. A similar construction can be done in higher dimensions.

This example also shows that, likewise, we cannot expect finite upper bounds in Theorem 4.2 (ii) and (iii). If $-2 < p < 0$, then $\alpha_p(K(R, \varepsilon)) \geq \frac{2^{\frac{3}{2} + \frac{1}{p+1}}}{R^{2+p}} R^{\frac{p}{p+1}}$. If $p < -2$, then $-2 < \frac{4}{p} < 0$ and thus

$$\alpha_p(K(R, \varepsilon)) = \alpha_p(K(R, \varepsilon)) \geq R^{\frac{2}{2+p}} 2^{\frac{12+3p}{4+2p}}.$$

Note also that in part (iii) we cannot remove the constant $c_{np}$ in $\alpha_p(K(R, \varepsilon))$. Indeed, if $p \to -\infty$, the inequality becomes $c^n|B^n_2|^2 \leq |K||K^\circ|$. 

**Proof of Theorem 4.2**

(i) The case $p = 0$ is trivial. We prove the case $p > 0$. Combining inequality (4.22), the Blaschke Santaló inequality, and $\alpha_q(B^n_2) = n|B^n_2|^{\frac{q}{n+q}} |B^n_2|^{\frac{q}{n+q}}$, one obtains

$$\frac{\alpha_p(K)}{\alpha_p(B^n_2)} \leq \left( \frac{|K^\circ|}{|B^n_2|} \right)^{\frac{n}{n+p}} \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n}{n+p}} \leq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n}{n+p}}.$$

This proves the inequality. The equality case follows from the equality case for the Blaschke Santaló inequality.

(ii) Combining inequality (4.23) and $(\alpha_p(B^n_2))^{\frac{n+p}{n}} = n|B^n_2|^{\frac{p}{n+q}} |B^n_2|^{\frac{q}{n+q}}$, one gets, for $-n < p < 0$,

$$\left( \frac{\alpha_p(K)}{\alpha_p(B^n_2)} \right)^{\frac{n+p}{n}} \geq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{p}{n}} \left( \frac{|K^\circ|}{|B^n_2|} \right)^{\frac{n-p}{n}} \geq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n}}$$

where the last inequality follows from the Blaschke Santaló inequality. As $\frac{p}{n} < 0$,

$$\left( \frac{|K|}{|K^\circ|} \right)^{\frac{p}{n}} \geq \left( \frac{|B^n_2|}{|B^n_2|} \right)^{\frac{p}{n}}.$$

As $n + p > 0$,

$$\frac{\alpha_p(K)}{\alpha_p(B^n_2)} \geq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n+p}{n+p}}.$$

The equality case follows from the equality case for the Blaschke Santaló inequality.
(iii) Similarly, combining (4.24), \( n|B^n_2| = (as_p(B^n_2))^{\frac{n+p}{n+p-1}} (as(B^n_2))^{\frac{n+1}{n+p-1}} \), and the Inverse Santaló inequality, we get, for \( p < -n \),

\[
\left( \frac{as_p(K)}{as_p(B^n_2)} \right)^{\frac{n+p}{n+p-1}} \geq \left( \frac{|K^o|}{|B^n_2|} \right)^{\frac{n}{n+p}} \geq c^n \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}.
\]

As \( \frac{n+p}{n+p-1} > 0 \),

\[
\frac{as_p(K)}{as_p(B^n_2)} \geq c^{\frac{n+p}{n}} \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}.
\]

The \( L_{-n} \) affine surface area was defined in [32] for convex bodies \( K \) in \( C^n_+ \) and with centroid at the origin by

\[
as_{-n}(K) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}}.
\]

More generally, one could define the \( L_{-n} \) affine surface area for any convex body \( K \) with centroid at the origin by \( as_{-n}(K) = \sup_{x \in \partial K} \langle x, N_K(x) \rangle^{\frac{n+1}{2}} \). But as in most cases then \( as_{-n}(K) = \infty \), it suffices to consider \( K \) in \( C^n_+ \).

A statement similar to Theorem 4.1 holds.

**Proposition 4.1** Let \( K \) be a convex body in \( C^n_+ \) with centroid at the origin. Let \( p \neq -n \) and \( s \neq -n \) be real numbers.

(i) If \( \frac{n(s-p)}{(n+p)(n+s)} \geq 0 \), then

\[
as_p(K) \leq (as_{-n}(K))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_s(K).\]

(ii) If \( \frac{n(s-p)}{(n+p)(n+s)} \leq 0 \), then

\[
as_p(K) \geq (as_{-n}(K))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_s(K).\]

(iii) The \( L_{-n} \) affine isoperimetric inequality holds

\[
\frac{as_{-n}(K)}{as_{-n}(B^n_2)} \geq \frac{|K|}{|B^n_2|}.
\]

**Proof** (i) and (ii)

\[
as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{n}{n+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x)
\]

\[
= \int_{\partial K} \left( \frac{\kappa_K(x)^{\frac{s}{n+s}}}{\langle x, N_K(x) \rangle^{\frac{n(s-1)}{n+s}}} \right) \left( \frac{\langle x, N_K(x) \rangle^{\frac{n+1}{2}}}{\kappa_K(x)^{\frac{1}{2}}} \right)^{\frac{2n(s-p)}{(n+p)(n+s)}} d\mu_K(x)
\]
which is
\[
\leq (a_{s-n}(K))^{\frac{2n(s-p)}{(n+p)(n+s)}} a_s(K), \quad \text{if } \frac{n(s-p)}{(n+p)(n+s)} \geq 0,
\]
and
\[
\geq (a_{s-n}(K))^{\frac{2n(s-p)}{(n+p)(n+s)}} a_s(K), \quad \text{if } \frac{n(s-p)}{(n+p)(n+s)} \leq 0.
\]

(iii) Note that \( \frac{n(s-p)}{(n+p)(n+s)} > 0 \) implies that \( s > p > -n \) or \( p < s < -n \) or \( s < -n < p \). If \( p = 0 \) and \( s \to \infty \), then
\[
a_{s-n}(K) \geq \sqrt{|K|/|K^o|}. \tag{4.25}
\]

This gives the \( L_n \) affine isoperimetric inequality
\[
\frac{a_{s-n}(K)}{a_{s-n}(B^n_2)} = a_{s-n}(K) \geq \sqrt{|K|/|K^o|} \geq \sqrt{|K^2|/|K^o|^2} \geq \sqrt{|K^2|/|B^n_2|^2} = |K|/|B^n_2|.
\]

Analogous to corollary 4.1, an immediate consequence of (4.25) is the following corollary. It can also be proved directly using (2.8).

**Corollary 4.2** Let \( K \) be a convex body in \( C^2_+ \) with centroid at the origin. Then
\[
a_{s-n}(K\cap K^o) a_{s-n}(K) \geq a_{s-n}(B^n_2)^2.
\]

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