ALMOST CONTACT 5–FOLDS ARE CONTACT

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Abstract. The existence of a contact structure is proved in any homotopy class of almost contact structures on a closed 5–dimensional manifold.

1. Introduction

Let \((M^{2n+1}, \xi)\) be a cooriented contact manifold with associated contact form \(\alpha\), i.e. \(\xi = \ker \alpha\). This structure determines a symplectic distribution \((\xi, d\alpha|_\xi) \subset TM\). Any change of the associated contact form \(\alpha\) does not change the conformal symplectic class of \(d\alpha\) restricted to \(\xi\). This allows us to choose a compatible almost complex structure \(J \in \text{End}(\xi)\). Thus given a cooriented contact structure we obtain in a natural way a reduction of the structure group \(\text{Gl}(2n+1, \mathbb{R})\) of the tangent bundle \(TM\) to the group \(U(n) \times \{1\}\), which is unique up to homotopy. A manifold \(M\) is said to be an almost contact manifold if the structure group of its tangent bundle can be reduced to \(U(n) \times \{1\}\). In particular, cooriented contact manifolds are almost contact manifolds and such a reduction of the structure group of the tangent bundle of a manifold \(M\) is a necessary condition for the existence of a cooriented contact structure on \(M\). It is unknown whether this condition is in general sufficient.

Nevertheless there are cases in which the existence of an almost contact structure is sufficient for the manifold to admit a contact structure. For example, if the manifold \(M\) is open then one can apply Gromov’s \(h\)–principle techniques to conclude that the condition is sufficient. See the result 10.3.2 in [EM]. The scenario is quite different for closed almost contact manifolds. Using results of Lutz [Lu] and Martinet [Ma] one can show that every cooriented tangent 2–plane field on a closed oriented 3–manifold is homotopic to a contact structure. A good account of this result from a modern perspective is given in [Ge]. For manifolds of higher dimensions there are various results establishing the sufficiency of the condition. Important instances of these are the construction of contact structures on certain principal \(S^1\)–bundles over closed symplectic manifolds due to Boothby and Wang [BW], the existence of a contact structure on the product of a contact manifold with a surface of genus greater than zero following Bourgeois [Bo] and the existence of contact structures on simply connected 5–dimensional closed orientable manifolds obtained by Geiges [Ge1] and its higher dimensional analogue [Ge2].

The work in this article was presented in the Spring 2012 AIM Workshop on higher dimensional contact geometry. In its course, J. Etnyre commented on a possible alternative approach in the framework of Giroux’s program using an open book decomposition. The argument has been subsequently written and it is the content of the article [Et].

Let us turn our attention to 5–manifolds since the main goal of this article is to show that any orientable almost contact 5–manifold is contact. In this case H. Geiges has been studying
existence results in other situations apart from the simply connected one. In [GT1] a positive result is also given for spin closed manifolds with \( \pi_1 = \mathbb{Z}_2 \), and spin closed manifolds with finite fundamental group of odd order are studied in [GT2]. On the other hand there is also a construction of contact structures on an orientable 5–manifold occurring as a product of two lower dimensional manifolds by Geiges and Stipsicz [GS]. While Geiges used the topological classification of simply connected manifolds for his results in [Ge1], one of the ingredients in [GS] is a decomposition result of a 4–manifold into two Stein manifolds with common contact boundary [AM], [Bk].

Being an almost contact manifold is a purely topological condition. In fact, the reduction of the structure group can be studied via obstruction theory. For example, in the 5–dimensional situation a manifold \( M \) is almost contact if and only if the third integral Steifel–Whitney class \( W_3(M) \) vanishes. Actually, using this hypothesis and the classification of simply connected manifolds due to D. Barden [Ba], H. Geiges deduces that any manifold with \( W_3(M) = 0 \) can be obtained by Legendrian surgery from certain model contact manifolds. Though this approach is elegant, it seems quite difficult to extend these ideas to produce contact structures on any almost contact 5–manifold. We therefore propose a different approach: the existence of an almost contact pencil structure on the given almost contact manifold is the required topological property to produce a contact structure. The tools appearing in our proof use techniques from three different sources:

(i) The approximately holomorphic techniques developed by Donaldson in the symplectic setting [Do, Do1] and adapted in [IMP] to the contact setting to produce the so–called quasi contact pencil.

(ii) The generalization of the notion of overtwistedness to higher dimensions done by Niederkrüger and Gromov [Ni] and the generalized Lutz twist based on that defined in [EP].

(iii) Eliashberg’s classification of overtwisted 3–dimensional manifolds [El] to produce overtwisted contact structures on the fibres of the pencil.

Let us state the main result.

**Theorem 1.1.** Let \( M \) be a closed oriented 5–dimensional manifold. There exists a contact structure in every homotopy class of almost contact structures.

In particular closed oriented almost contact 5–manifolds are contact. It is important to emphasize that using the techniques developed in this article, it is not possible to conclude anything about the number of distinct contact distributions that may occur in a given homotopy class of almost contact distributions. The result states that there is at least one. From the construction it will be clear that the contact structure contains a PS–structure [Ni, NP] and therefore it is non–fillable. The PS–structure is used as a substitute of the overtwisted disk to introduce flexibility as Eliashberg does in the 3–dimensional case.

**Remark 1.2.** The distributions are supposed to be coorientable along the course of the article. Section 10 contains the corresponding results for non–coorientable distributions.

The proof of Theorem 1.1 consists of a constructive argument in which we obtain the contact condition step by step. These steps correspond to the sections of the paper as follows:

- To begin with, we show that any almost contact 5–manifold \((M, \xi)\) admits an almost contact fibration over \( S^2 \) with singularities of some standard type. It is defined on the complement of a link. The construction of this almost contact fibration – in fact,
an almost contact pencil – is the content of Sections 2 and 3.

- In Section 4, we produce a first deformation of the almost contact structure $\xi$ to obtain a contact structure in some neighborhood of the singularities of the fibration and a neighborhood of the link.

- One of the types of singularities has the structure of a base locus of a pencil occurring in algebraic or symplectic geometry. In order to provide a Lefschetz type fibration we blow–up the base locus. This requires the notion of a contact blow–up. For the purposes of the article, it will be enough to define the contact blow–up of a contact 5–manifold along a transverse $S^1$ and the corresponding notion of contact blow–down. This is the content of Section 5.

- Away from the critical points the distribution splits as $\xi = \xi_v \oplus \mathcal{H}$, where $\xi_v$ is the restriction of the distribution to the fibres and $\mathcal{H}$ is the symplectic orthogonal. Section 6 deals with a deformation of $\xi_v$ to produce a contact structure in the fibres. It strongly uses the construction of overtwisted contact manifolds due to Eliashberg [El].

- In Section 7 we begin to deform the horizontal direction $\mathcal{H}$. This is done in two steps. Given a suitable cell decomposition of the base $S^2$, we deform $\mathcal{H}$ in the pre–image of a neighborhood of the 1–skeleton. Section 7 contains this first step.

- The contact condition still has to be achieved in the pre–image of the 2–cells. This is the second step. The contact structure used in order to fill the pre–image of the 2–cells is carefully constructed in Section 8.

- In Section 9 we gather the results in the previous sections and construct the contact structure. Theorem 1.1 is concluded.

- In Section 10 we deal with the case of non–coorientable distributions. We introduce the suitable definitions and explain the non–coorientable version of Theorem 1.1.

It is reasonable to guess after a careful reading of this article the ingredients needed to adapt the proofs in order to work in the 7–dimensional case. We have begun to understand this 7–dimensional setup and it will be the goal of a forthcoming article. E. Giroux has work in progress in which he tries to prove the existence result by using an open book decomposition of the manifold $\mathcal{G}$.

Acknowledgements. The authors are grateful to Y. Eliashberg, J. Etnyre, E. Giroux and H. Geiges for valuable conversations. The proof of Theorem 9.3 was outlined to us by Y. Eliashberg. The original work lacked the construction of the homotopy in the case that 2–torsion existed in $H^2(M, \mathbb{Z})$. This case was proven after a useful discussion with J. Etnyre at the AIM Workshop.

2. Preliminaries.

2.1. Quasi–contact structures. Let $M$ be an almost contact manifold. There exists a choice of a symplectic distribution $(\xi, \omega) \subset TM$ for such a manifold. Namely, we can find a 2–form $\eta$ on $\xi$ with the property that $\eta$ is non–degenerate and compatible with the almost complex structure $J$ defined on $\xi$. By extending $\eta$ to a form on $M$ we can find a 2–form $\omega$ on $M$ such that $(\xi, \omega|_\xi)$ becomes a symplectic vector bundle. This form $\omega$ is not necessarily closed. The triple $(M, \xi, \omega)$ is also said to be an almost contact manifold. In other words, by an almost contact structure is meant to be a triple $(\xi, J, \omega)$ for some $\omega$ as discussed. The
choice of almost complex structure $J$ is homotopically unique and it might be omitted. This article only concerns codimension–1 distributions on $M$ which are almost complex. An almost contact manifold is subsequently described by a triple $(M, \xi, \omega)$.

In order to construct a contact structure out of an almost contact one, the first step is to provide a better 2–form on $M$. That is, we replace $\omega$ by a closed 2–form.

**Definition 2.1.** A manifold $M^{2n+1}$ admits a quasi–contact structure if there exists a pair $(\xi, \omega)$ such that $\xi$ is a codimension 1–distribution and $\omega$ is a closed 2–form on $M$ which is non–degenerate when restricted to $\xi$.

Notice that a quasi–contact pair $(\xi, \omega)$ admits a compatible almost contact structure, i.e. there exist a $J$ which makes $(\xi, J, \omega)$ into an almost contact structure. These manifolds have also been called 2–calibrated \cite{IM} in the literature. The following lemma justifies the appearance of the previous definition:

**Lemma 2.2.** Every almost contact manifold $(M, \xi_0, \omega_0)$ admits a quasi–contact structure $(\xi_1, \omega_1)$ homotopic to $(\xi_0, \omega_0)$ through symplectic distributions and the class $[\omega_1]$ can be fixed to be any prescribed cohomology class $a \in H^2(M, \mathbb{R})$.

**Proof.** Let $j : M \to M \times \mathbb{R}$ be the inclusion as the zero section. We know that we can find a (not–necessarily closed) 2–form $\widetilde{\omega}_0$, such that $\omega_0 = j^*\widetilde{\omega}_0$. Fix a Riemannian metric $g$ over $M$ such that $\xi_0$ and $\ker \omega_0$ are $g$–orthogonal.

Apply Gromov’s classification result of open symplectic manifolds to produce a 1–parametric family $\{\widetilde{\omega}_t\}_{t \in \mathbb{R}}$ of symplectic forms such that for $t = 1$ the form is closed. See \cite{EM}, Corollary 10.2.2. Let $\pi : M \times \mathbb{R} \to M$ be the projection and choose the cohomology class defined by $\widetilde{\omega}_1$ to be $\pi^*a$. Consider the family of 2–forms $\omega_t = j^*\widetilde{\omega}_t$ on $M$. Since $\widetilde{\omega}_t$ is non–degenerate on $M \times \mathbb{R}$ for each $t$, the form $\omega_t$ has 1–dimensional kernel $\ker \omega_t$. Define $\xi_t = (\ker \omega_t)^{g}$. Then $(\xi_t, \omega_t)$ provides the required family. \hfill $\Box$

This is the farthest one can reach by the standard $h$–principle argument in order to find contact structures on a closed manifold. One can start with the almost contact bundle $\xi = \ker \alpha$ and find a 2–form $d\beta$ that makes it symplectic by Lemma 2.2, but there is in general no way to relate $\alpha$ and $\beta$. This is the aim of the article.

2.2. **Obstruction theory.** The content of Theorem 1.1 has two parts. The statement implies the existence of a contact structure in an almost contact manifold. This is a result in itself, regardless of the homotopy type of the resulting almost contact distribution. The construction we provide in this article also concludes that the obtained contact distribution lies in the same homotopy class of almost contact distributions as the original almost contact structure. This is achieved via the study of an obstruction class. Let us review some well–known facts.

Let $M$ be a smooth oriented 5–manifold and $\pi : TM \to M$ its tangent bundle. The projection $\pi$ is considered to be an $SO(5)$–principal frame bundle. An almost contact structure is a reduction of the structure group $G = SO(5)$ to a subgroup $H \cong U(2) \times \{1\} \cong U(2)$. The isomorphism classes of almost contact structures are parametrized by the homotopy classes of such reductions. A reduction of the structure group $G$ to a subgroup $H$ is tantamount to a section of a $G/H$–bundle over $M$. Hence the classification of almost contact structures on $M$ is reduced to the study of homotopy classes of sections of a $SO(5)/U(2)$–bundle over $M$.

**Lemma 2.3.** $SO(5)/U(2) \cong \mathbb{C}P^3$.

**Proof.** First, note that $SO(4)/U(2) \cong S^2$ is given by the choice of a unitary vector in real 3–space. The inclusion $S^2 \to SO(5)/U(2)$ has homotopy cofibre $S^4$ and thus we obtain a fibration $SO(5)/U(2) \to S^4$. Second, there exists an $S^1$–action on the total space of the Hopf
fibration $S^3 \to S^7 \to S^4$ that restricts to the Hopf action on the fibres $S^3$. Hence, the orbit space $\mathbb{C}P^3$ fibres over $S^4$ with fibres $S^3$. The classifying maps of both fibrations coincide as elements in $\pi_3(SO(3))$. In particular, we obtain the diffeomorphism $\mathbb{C}P^3 \cong SO(5)/U(2)$. □

The homotopy groups $\pi_i(\mathbb{C}P^3) = 0$ for $1 \leq i \leq 6$, $i \neq 2$, hence the existence of sections of a fibre bundle with typical fibre $\mathbb{C}P^3$ over the 5–manifold $M$ is controlled by the primary obstruction class $d = W_3(M) \in H^3(M, \pi_2(\mathbb{C}P^3)) \cong H^3(M, \mathbb{Z})$. The hypothesis of Theorem 1.1 is $d = 0$.

Let $s_\xi$ and $s_{\xi'}$ be two sections of this $\mathbb{C}P^3$–bundle. The obstruction class dictating the existence (or the lack thereof) of a homotopy between them lies is the primary obstruction $d(\xi, \xi') \in H^2(M, \mathbb{Z})$. The obstruction theory argument can be made relative to a submanifold $A \subset M$. This implies the following

**Lemma 2.4.** Let $s_\xi$ and $s_{\xi'}$ be two sections of a $\mathbb{C}P^3$–bundle over $M$ that are homotopic over a 2–skeleton of the pair $(M, A)$. Then $s_\xi$ and $s_{\xi'}$ are also homotopic over $(M, A)$.

Let $(M, \xi)$ be an almost contact structure, the construction of the contact structure $\xi'$ obtained in Theorem 1.1 does not modify the homotopy class of the given section, i.e. $s_\xi \sim s_{\xi'}$. This is readily seen at the stages preceding Section 8. In Section 8 we provide a detailed account on the modification of the obstruction class $d(\xi, \xi')$ in the 2–skeleton of certain pieces of $M$ where $\xi'$ has been constructed. This is enough to conclude that $d(\xi, \xi') = 0$ once $\xi'$ is extended to $M$ in Section 9.

2.3. Homotopy of vector bundles. The argument constructing the homotopy between the initial almost contact structure and the resulting contact distribution in Theorem 1.1 uses the following lemma. It is implicitly used in several parts of Sections 4 to 9.

Let $(V, \omega)$ be an oriented vector space of dimension $\dim \mathbb{R} V = 4$. Consider an splitting $V = V_0 \oplus V_1$ with $V_0, V_1$ two oriented 2–dimensional vector subspaces. Since $Sp(2, \mathbb{R})/SO(2)$ is contractible, the space of symplectic structures on $V$ such that $V_0$ and $V_1$ are symplectic orthogonal subspaces is contractible. This essentially implies the following

**Lemma 2.5.** Let $M$ be an almost contact 5–manifold, $A$ an open submanifold of $M$, and $(\xi_0, \omega_0), (\xi_1, \omega_1)$ two almost contact structures on $M$ such that there exists a homotopy $\{\xi_t\}$ of oriented distributions on $(M, A)$ connecting $\xi_0$ and $\xi_1$. Suppose that there exist $L_0$ and $L_1$ two rank–2 symplectic subbundles of $\xi_0$ and $\xi_1$ and a homotopy $\{L_t\} \subset \{\xi_t\}$ of oriented distributions connecting $L_0$ and $L_1$ on $(M, A)$. Then $\{\xi_t\}$ is a homotopy of symplectic distributions connecting $\xi_0$ and $\xi_1$ on $(M, A)$.

**Proof.** Consider $J_0$ and $J_1$ two compatible complex structures on the symplectic distributions $\xi_0$ and $\xi_1$ respectively. These define two fibrewise scalar–product structures

$$g_0 = \omega_0(\cdot, J_0\cdot) \quad \text{and} \quad g_1 = \omega_1(\cdot, J_1\cdot)$$

on $\xi_0$ and $\xi_1$. The space of fibrewise scalar–product structures has contractible fiber, namely $GL^+(4, \mathbb{R})/SO(4)$, and thus is contractible. Hence, there exists a homotopy $\{g_t\}$ of fibrewise scalar–products connecting $g_0$ and $g_1$. The scalar–product $g_t$ provides an orthogonal decomposition $\xi_t = L_t \oplus L_t^{\perp}$. The homotopy of oriented bundles $\{L_t\}$ induces a homotopy of oriented bundles $\{L_t^{\perp}\}$ respecting the symplectic splitting given by $\omega_0$ and $\omega_1$ on $\xi_0$ and $\xi_1$. □

2.4. Notation. Let $\mathbb{R}^{2n}$ be Euclidean space, $B^{2n}(r) = \{ p \in \mathbb{R}^{2n} : \|p\| \leq r \}$ denotes the ball of radius $r$ centered at the origin. The 2–dimensional balls are also referred to as disks and denoted by $D^2(r)$. In case the radius is omitted $B^{2n}$ and $D^2$ denote the ball and disk of radius 1 respectively.
3. Quasi–Contact Pencils.

Approximately holomorphic techniques have been extremely useful in symplectic geometry. Their main application in contact geometry – due to E. Giroux – is to establish the existence of a compatible open book for a contact manifold in higher dimensions. An open book decomposition is a way of trivializing a contact manifold by fibering it over $S^1$. Such objects have also been studied in the almost contact case, see [MMP].

There exists a construction [Pr1] in the contact case analogous to the Lefschetz pencil decomposition introduced by Donaldson over a symplectic manifold [Do1]. It is called a contact pencil and it allows us to express a contact manifold as a singular fibration over $S^1$. It has been extended in [IV2] to the quasi–contact setting. Theorem 3.5 and Corollary 3.6 in this Section show the existence of a quasi–contact pencil with suitable properties. Let us begin with the appropriate definitions.

**Definition 3.1.** An almost contact submanifold of an almost contact manifold $(M, \xi, \omega)$ is an embedded submanifold $j : j : S \to M$ such that the induced pair $(j^*\xi, j^*\omega)$ is an almost contact structure on $S$.

A quasi–contact submanifold of a quasi–contact manifold is defined analogously. In particular this implies in both cases that the submanifold $S$ is transverse to the distribution $\xi$.

A chart $\phi : (U, p) \to V \subset (\mathbb{C}^n \times \mathbb{R}, 0)$ of an atlas of $M$ is compatible with the almost contact structure $(\xi, \omega)$ at a point $p \in U \subset M$ if the push–forward at $p$ of $\xi_p$ is $\mathbb{C}^n \times \{0\}$ and the 2–form $\phi^*\omega(p)$ is a positive $(1, 1)$–form.

**Definition 3.2.** An almost contact pencil on a closed almost contact manifold $(M^{2n+1}, \xi, \omega)$ is a triple $(f, B, C)$ consisting of a codimension–4 almost contact submanifold $B$, called the base locus, a finite set $C$ of smooth transverse curves and a map $f : M \setminus B \to \mathbb{CP}^1$ conforming the following conditions:

1. The set $f(C)$ contains locally smooth curves with transverse self–intersections and the map $f$ is a submersion on the complement of $C$.
2. Each $b \in B$ has a compatible chart to $(\mathbb{C}^n \times \mathbb{R}, 0)$ under which $B$ is locally cut out by $\{z_1 = z_2 = 0\}$ and $f$ corresponds to the projectivization of the first two coordinates, i.e. locally $f(z_1, \ldots, z_n, t) = \frac{z_2}{z_1}$.
3. At a critical point $p \in \gamma \subset M$ there exists a compatible chart $\phi_p$ such that
   \[
   (f \circ \phi_p^{-1})(z_1, \ldots, z_n, s) = f(p) + z_1^2 + \ldots + z_n^2 + g(s)
   \]
   where $g : (\mathbb{R}, 0) \to (\mathbb{C}, 0)$ is a submersion at the origin.
4. The fibres $f^{-1}(p)$, for any $p \in \mathbb{CP}^1$, are almost contact submanifolds at the regular points.

The local models have to be provided through compatible charts in the sense above. That is to say the distribution in $\xi_p$ is mapped to the horizontal distribution $\mathbb{C}^n \times \{0\}$ and $\omega$ is a positive $(1, 1)$–form when restricted to the horizontal distribution with respect to its canonical almost complex structure.

**Remark 3.3.** Quasi–contact pencils for quasi–contact manifolds and contact pencils for contact manifolds are defined by replacing the expression almost contact by the suitable one in each case.

The generic fibres of $f$ are open almost contact submanifolds and the closures of the fibres at the base locus are smooth. This is because the local model (2) in the Definition 3.2 is a parametrized elliptic singularity and the fibres come in complex lines $(z_2 = \text{const} \cdot z_1)$ joining at the origin. We refer to the compactified fibres so constructed as the fibres of the pencil.
Notice that the set of critical values $\Delta = f(C)$ are no longer points, as in the symplectic case, but immersed curves. This is because of Condition (3) in the Definition 3.2. In particular, the usual isotopy argument between two fibres does not apply unless their images are in the same connected component of $\mathbb{CP}^1 \setminus \Delta$. This has been studied in the contact and quasi–contact cases. The set $C$ is a positive link and therefore $\Delta$ is also oriented. There is a partial order in the complementary of $\Delta$: a connected component $P_0$ is less or equal than a connected component $P_1$ if $P_0$ and $P_1$ can be connected by an oriented path $\gamma \subset \mathbb{CP}^1$ intersecting $\Delta$ only with positive crossings. The proposition that follows has only been proved for the contact and quasi–contact cases. An analogous statement probably remains true in the almost contact setting. It is provided to offer some geometric insight about contact and quasi–contact pencils, it is not used in the rest of the article.

**Proposition 3.4** (Proposition 6.1 of [Pr1]). Let $M$ be a quasi–contact manifold equipped with a quasi–contact pencil $(f, B, C)$. Then if two regular values of $f$, $P_0$ and $P_1$, are separated by a unique curve of $\Delta$ then the two corresponding fibres $F_0 = f^{-1}(P_0)$ and $F_1 = f^{-1}(P_1)$ are related by an index $n - 1$ surgery.

Suppose that the manifold and the pencil are contact, then the surgery is a Legendrian one and it attaches a Legendrian sphere to $F_0$ if $P_0$ is smaller than $P_1$. See Figure 2

In the contact case it implies that the crossing of a singular curve in the fibration amounts to a directed Weinstein cobordism. In the quasi–contact case no such orientation appears. For instance, the case in which the quasi–contact distribution is a foliation –in dimension 3 this is a taut foliation– becomes absolutely symmetric and there is no difference in crossing one way or the other.

The main existence result ([IM2] [Pr1]) can be stated as
Figure 2. The drawn orientations make $F_1 = f^{-1}(P_1)$ a Legendrian surgery of $F_0 = f^{-1}(P_0)$.

**Theorem 3.5.** Let $(M, \xi, \omega)$ be a quasi–contact manifold. Given an integral cohomology class $a \in H^2(M, \mathbb{Z})$, there exists a quasi–contact pencil $(f, B, C)$ such that the fibres are Poincaré dual to the class $a + k[\omega]$, for some $k \in \mathbb{N}$.

The proof of this result does not work in the almost contact setting. In order to construct the pencil, the approximately holomorphic techniques are essential and for them to work we need the closedness of the 2–form $\omega$. In general, a quasi–contact pencil may have empty base locus. Nevertheless a pencil obtained through approximately holomorphic sections on a higher dimensional manifold does not.

In case the form $\omega$ of the quasi–contact structure is exact – then called an exact quasi-contact structure – we obtain the following

**Corollary 3.6.** Let $(M, \xi, \omega)$ be an exact quasi–contact closed manifold. Then it admits a quasi–contact pencil such that any smooth fibre $F$ satisfies $c_1(\xi_F) = 0$. Further, the base locus $B$ is non–empty if $\dim M$ is greater than 3.

**Proof.** We use Theorem 3.5 to construct a pencil such that the cohomology class of the Poincaré dual to the smooth fibres equals the first Chern class $c_1(\xi)$ of the complex bundle $\xi$. This follows from the fact that for any $a \in H^2(M, \mathbb{Z})$ there is a quasi–contact pencil with any smooth fibre Poincaré dual to the the class $a + k[\omega]$. Since $\omega$ is exact we get the pencil with the property mentioned earlier by taking $a = c_1(\xi)$.

Since the distribution $\xi$ is transverse to the fibre $F$, we have that

$$\xi|_F = (\xi|_F \cap TF) \oplus \nu(F).$$

Recall that $c_1(\nu(F)) = e(\nu(F)) = PD([F])$ to obtain

$$c_1(\xi|_F) = c_1(\xi|_F \cap TF) + c_1(\nu(F)) = c_1(\xi_F) + a|_F,$$

and so

$$a|_F = c_1(\xi_F) + a|_F.$$
This shows that the first Chern class of the quasi-contact structure in any smooth fibre is zero.

As for the non-emptiness of the set $B$, just recall that by the general theory developed in [IM2, IMP], the submanifold satisfies a Lefschetz hyperplane theorem. It implies that whenever the dimension of $M$ is greater than 3, the morphism

$$H_0(B) \to H_0(M)$$

is surjective. Hence we conclude that $B$ is not the empty set. □

The triviality of the Chern class of the quasi-contact structures on the fibres and the non-emptiness of $B$ are used in the construction of the contact structure.

4. Base locus and Critical loops.

Let $(M, \xi, \omega)$ be an exact quasi-contact 5-manifold and $(f, B, C)$ a quasi-contact pencil on it. Assume that $B \neq \emptyset$ and $c_1(\xi_F) = 0$ for a regular fibre $F$ of $f$. Such a pencil is provided in Corollary 3.6. A fair amount of control on the almost-contact structure can be achieved in the neighborhood of the base locus and the critical loops.

**Definition 4.1.** A submanifold $i : S \to M$ of an almost contact manifold $(M, \xi, \omega)$ is said to be contact if it is an almost contact submanifold and there is a choice of adapted form $\alpha$ for $\xi$ in a neighborhood of $S$, i.e. $\xi = \ker \alpha$, such that $i^*(d\alpha) = i^*\omega$.

Note that a transverse loop is a contact submanifold. An additional property in our almost contact pencil can then be required.

**Definition 4.2.** An almost contact pencil $(f, B, C)$ on $(M, \xi, \omega)$ is called good if there exist neighborhoods of $B$ and $C$ that are contact submanifolds of $(M, \xi, \omega)$.

The following lemma provides a perturbation achieving a suitable almost contact pencil.

**Lemma 4.3.** Let $(M, \xi, \omega)$ be a quasi-contact closed 5-dimensional manifold and let $(f, B, C)$ be a quasi-contact pencil. There exists a $C^0$-small perturbation $\{(\xi_t, \omega)\}$ of almost contact structures such that:

(i) $(\xi_t, \omega)$ is an almost contact structure $\forall t \in [0, 1]$, and $(\xi_0, \omega) = (\xi, \omega)$.

(ii) There exist neighborhoods of $B$ and $C$ that are contact submanifolds of $(\xi_1, \omega)$.

(iii) $(f, B, C)$ is an almost contact pencil for $(M, \xi_1, \omega)$.

(iv) $c_1((\xi_1)_F) = 0$ for a regular fibre $F$ of $f$.

Fix an associated contact form $\alpha$, i.e. $\xi = \ker \alpha$. The proof of the lemma is an exercise. Indeed, in a neighborhood of the link the difference between $\omega$ and $d\alpha$ is exact and this primitive allows us to perturb the defining form until we achieve the contact condition $\omega = d\alpha'$, $\xi' = \ker \alpha'$.

In conclusion, we obtain the following

**Proposition 4.4.** Let $(M, \xi, \omega)$ be an exact quasi-contact closed 5-dimensional manifold. Then there exists an almost contact perturbation $(\xi', \omega)$ of $(\xi, \omega)$ such that $(M, \xi', \omega)$ admits a good almost contact pencil $(f, B, C)$ with $B \neq \emptyset$ and any smooth fibre $F$ satisfying $c_1(\xi'_F) = 0$.

5. Contact blow-up.

Let $(f, B, C)$ be an almost contact pencil on $(M, \xi, \omega)$. The map $f$ does not define a smooth fibration on $M$ for two reasons: it is not defined on $B$ and there exist critical fibres. The former failure can be avoided if we change the domain manifold $M$, i.e. $f$ can be defined on a suitable closed manifold $\tilde{M}$ obtained from $M$ by a specific surgery procedure.

**Definition 5.1.** An almost contact Lefschetz fibration is an almost contact pencil $(f, B, C)$ with $B = \emptyset$. A contact Lefschetz fibration is a contact pencil $(f, B, C)$ with $B = \emptyset$. 
An almost contact Lefschetz fibration can be obtained out of an almost contact Lefschetz pencil by performing a surgery along the base locus. In particular, each connected component of the link \( B \) is replaced by a standard 3–sphere (\( S^3, \xi_{std} \)). This leads to the notion of contact blow–up. The aim of this section is to produce such a fibration for a good almost–contact pencil on a 5–dimensional manifold.

**Theorem 5.2.** Let \((M, \xi, \omega)\) be an almost contact 5–manifold and \((f, B, C)\) a good almost contact pencil with \( B \neq \emptyset \) and \( c_1(\xi_F) = 0 \) for the regular fibres \( F \) of \( f \). Let \((\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})\) be an almost contact blow–up of \((M, \xi, \omega)\) along \( B \). Then there exists a good almost contact Lefschetz fibration \((\widetilde{f}, \widetilde{C})\) for \((\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})\) with \( c_1(\widetilde{\xi}_F) = 0 \). The Lefschetz fibration \((\widetilde{f}, \widetilde{C})\) restricts to \((f, C)\) away from the exceptional divisors.

The necessary definitions and discussions for this statement are given in this Section. Further details on the contact blow–up procedure appear in [CPP]. Theorem 5.2 is a consequence of Theorem 5.3 adapted to an almost contact pencil via Lemma 5.4 and Proposition 5.6. The compatibility of \((\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})\) with \((\widetilde{f}, \widetilde{C})\) is detailed in subsection 5.3.

### 5.1. Contact blow–up surgery.

Throughout this subsection \( \dim(M) \) is not relevant. Let us describe the precise properties of the blown–up manifold \( \widetilde{M} \) and show its existence. Suppose that \((M, \xi)\) is contact.

**Theorem 5.3.** Let \((M^{2n+1}, \xi)\) be a contact manifold. Let \( S \subset M \) be a smooth transverse loop in \( M \). There exists a manifold \( \widetilde{M} \) conforming the following conditions:

- There exists a contact structure \( \widetilde{\xi} \) on \( \widetilde{M} \).
- There exists a codimension–2 contact submanifold \( E \cong (S^{2n-1}, \xi_{st}) \) of \((\widetilde{M}, \widetilde{\xi})\) with trivial normal bundle.
- The manifolds \((M \setminus S, \xi)\) and \((\widetilde{M} \setminus E, \widetilde{\xi})\) are contactomorphic.

A pair \((\widetilde{M}, \widetilde{\xi})\) with the above properties is said to be a contact blow–up of \((M, \xi)\) along \( S \). The submanifold \( E \) is called the exceptional divisor.

**Proof.** By Gray’s stability there is a neighborhood \( U \) of \( S \) and a contactomorphism \( \Phi : S^1 \times B^{2n}(\varepsilon) \rightarrow U, \quad (\theta, r, \sigma) \mapsto \Phi(\theta, r, \sigma), \quad \Phi^*(\xi|_S) = \ker(d\theta - r^2\alpha_{std}), \) where \( \alpha_{std} \) is the standard contact form on \( S^{2n-1} \) induced by restriction of the standard Liouville form on \( \mathbb{R}^{2n} \). Choose an integer \( k \) satisfying that

\[ \frac{2}{\sqrt{1+4k}} < \varepsilon. \]

Then the map

\[ \psi_k : S^1 \times B^{2n}(\varepsilon) \rightarrow S^1 \times B^{2n}(\varepsilon), \quad (\theta, r, w_1, \ldots, w_n) \mapsto (\theta, \frac{r}{\sqrt{1+k^2}}, e^{ik\theta} w_1, \ldots, e^{ik\theta} w_n), \]

is a contact embedding. Hence we obtain a contact embedding from a radius–2 neighborhood of the loop into the manifold, namely

\[ \Phi_k = (\Phi \circ \psi_k) : S^1 \times B^{2n}(2) \rightarrow U \subset M. \]

Consider the diffeomorphism

\[ \phi_1 : S^1 \times (3/2, 2) \times S^{2n-1} \rightarrow S^1 \times (3/2, 2) \times S^{2n-1}, \quad (\theta, r, w_1, \ldots, w_n) \mapsto (\theta, r, e^{i\theta} w_1, \ldots, e^{i\theta} w_n). \]

If \( V = \Phi_k(S^1 \times B^{2n}(3/2)) \), then \( \rho = \Phi_k \circ \phi_1 : S^1 \times (3/2, 2) \times S^{2n-1} \rightarrow U \setminus V \subset M \) satisfies

\[ \rho^*\xi = \ker \left\{ \alpha_{std} + \frac{r^2 - 1}{r^2} d\theta \right\}. \]
Note that the function
\[ h : (3/2, 2) \rightarrow \mathbb{R} \]
\[ r \mapsto h(r) = \frac{r^2 - 1}{r^2} \]
satisfies \( h(r) > 5/9 \). Therefore it is possible to extend it to a smooth function \( \tilde{h} : [0, 2) \rightarrow \mathbb{R} \) satisfying the following conditions (see Figure 3):
- \( \tilde{h}(r) = r^2 \), for \( r \in [0, 1/2] \),
- \( \tilde{h}(r) = h(r) \), for \( r > 3/2 \),
- \( \tilde{h}(r)' > 0 \) for \( r \in [1/2, 3/2] \).

![Figure 3. The function \( \tilde{h} \).](image)

Therefore \( \tilde{\eta} = \alpha_{std} + \tilde{h}(r)d\theta \) is a contact form over \( S^1 \times [0, 2) \times S^{2n-1} \cong B^2(2) \times S^{2n-1} \). We can glue the manifold \((M \setminus V, \xi) \) and \((B^2(2) \times S^{2n-1}, \ker \tilde{\eta})\) with the gluing map \( \rho \) to define a contact manifold \((\tilde{M}, \tilde{\xi})\). This manifold readily satisfies the statement of the theorem. □

5.2. Almost contact blow–up. Let \((f, B, C)\) be a good almost contact pencil on \((M, \xi, \omega)\). In the contact case, the surgery performed in the proof of Theorem 5.3 is local on \( B \). The construction readily applies to a good almost contact pencil since \((M, \xi)\) is a contact structure on a neighborhood of \( B \).

Concerning the almost contact structure, the condition on the parameter \( k \) in the proof of Theorem 5.3 is unnecessary. The size of a neighborhood of a contact submanifold for the almost contact structure can be enlarged. In precise terms:

**Lemma 5.4.** Let \((M, \xi, \omega)\) be an almost contact manifold and \((S, \xi = \ker \alpha)\) be a contact submanifold with trivial normal bundle \( \nu_S \cong S \times \mathbb{R}^{2q} \). Fix a radius \( R \in \mathbb{R} \). Then there exists an almost contact homotopy \((M, \xi_t, \omega)\) such that \((M, \xi_0, \omega) = (M, \xi, \omega)\) and it conforms the following conditions:
- The homotopy is supported in an annulus around \( S \), i.e. given a smooth fiberwise metric on \( \nu_S \) there exist \( \rho_1, \rho_2 \in \mathbb{R}^+ \) with \( \rho_1 < \rho_2 \) such that
  \[ \xi|_{\mathbb{D}(\nu_S, \rho_2) \setminus \mathbb{D}(\nu_S, \rho_1)} = \xi|_{\mathbb{D}(\nu_S, \rho_2) \setminus \mathbb{D}(\nu_S, \rho_1)}, \]
where \( \mathbb{D}(\nu_S, r) \) is the disk bundle of radius \( r \).
- There exists a neighborhood \( U \) of \( S \) and a diffeomorphism \( \varphi \) such that
\[
\varphi : S \times B^{2q}(R) \to U, \quad \varphi^*\xi_1 = \ker(\alpha - r^2\alpha_{std}), \quad \varphi^*\omega = d\alpha - 2rd \wedge d\alpha_{std}.
\]
The 1–form \( \alpha_{std} \) is the standard contact form on \( \partial B^{2q}(R) \).

Proof. This is a statement about a neighborhood \( S \times B^{2q}(\varepsilon) \). Suppose that \( R > \varepsilon \). In this area \((\xi, \omega)\) is a contact structure described as the kernel of the 1–form \( \eta_0 = \alpha - r^2\alpha_{std} \). Consider a function \( H \in C^\infty([0,\varepsilon], \mathbb{R}^+) \) such that:
\begin{enumerate}
\item \( H(r) = r^2 \) for \( r \in [0,\varepsilon/4] \cup [3\varepsilon/4, \varepsilon] \),
\item \( H'(r) > 0 \) for \( r \in (0,\varepsilon/2) \),
\item \( H(\varepsilon/2) = R^2 \).
\end{enumerate}
There exists a homotopy \( \{H_t\} \) of functions in \( C^\infty([0,\varepsilon], \mathbb{R}^+) \) with \( H_0(r) = r^2, H_1(r) = H(r) \) and any \( H_t \) satisfying properties a and b above. The homotopy of 1–forms \( \eta_t = \alpha - H_t(\alpha_{std}) \) defines a homotopy of almost contact distributions. The diffeomorphism
\[
\Psi : S \times B^{2q}(R) \to S \times B^{2q}(\varepsilon/2) \quad (s,r,\theta) \mapsto (s,\sqrt{H(r)},\theta)
\]
satisfies \( \Psi^*\eta_0 = \eta_1 \) as required. \( \square \)

The lemma does not hold for a contact structure since the contact condition is violated in the homotopy. Thus the restriction on the parameter \( k \) in the contact case.

The surgery in the almost contact case can be performed with any given integer \( k \in \mathbb{Z} \). Indeed, the construction follows the proof of Theorem 5.3. Consider a transverse embedded loop \( S^1 \subset (M,\xi,\omega) \) and use the map \( \psi_k \) to obtain a standard contact neighborhood of a certain size. Enlarge this neighborhood using Lemma 5.4 and apply \( \phi_1 \).

5.3. Compatibility with an almost contact pencil. Let \((f,B,C)\) be a good almost contact pencil on a 5–dimensional contact manifold \((M,\xi)\). There are several choices in the previous construction. To begin with, the map \( \Phi : S^1 \times B^{2n}(\varepsilon) \to U \). This amounts to a choice of framing of the trivial normal bundle along \( S^1 \). Since \( S^1 \subset B \) we can use the adapted charts in Definition 3.2 and require that \( \Phi \) satisfies that the map
\[
f \circ \Phi : S^1 \times (B^4(\varepsilon) \setminus \{0\}) \to \mathbb{CP}^1
\]
is precisely \((f \circ \Phi)(\theta, w_1, w_2) = [w_1 : w_2]\). Therefore the compactified fibres are of the form \( S^1 \times L \), for any complex line \( L \subset \mathbb{C}^2 \).

Let us focus on the compactification of fibres in the blow–up, i.e. the extension of \( \tilde{f} \) from \( M \setminus B \) to \( \tilde{M} \). We first restrict ourselves to the transition region \( S^1 \times (3/2, 2) \times S^3 \subset S^1 \times \mathbb{C}^2 \).

The gluing map is \( \rho = \Phi \circ \psi_k \circ \phi_1 \). In order to understand the fibres of the blown–up pencil we just need to describe the map \( \tilde{f} = f \circ \Phi \circ \psi_k \circ \phi_1 \). The reader can readily verify that \( \tilde{f}(\theta,r,w_1,w_2) = (f \circ \Phi)(\theta, rw_1, rw_2) = [w_1 : w_2] \), since \( \psi_k \circ \phi_1 \) acts as complex scalar multiplication in the transition area.

Notice that the domain of definition of \( \tilde{f} \) is \( S^1 \times (3/2, 2) \times S^3 \), and it is invariant with respect to the coordinates \((\theta,r) \in S^1 \times (3/2, 2) \). Hence, the map \( \tilde{f} \) extends trivially to the model \((B^2(2) \times S^3, \ker \eta)\). In particular, the extension of \( \tilde{f} \) restricted to the exceptional divisor \( \{0\} \times S^3 \) consists of the Hopf fibration.

The fibres of the blown–up fibration are thus almost contact submanifolds. Indeed, the fibres of \( \tilde{f} \) restricted to \((B^2(2) \times S^3, \ker \eta)\) are diffeomorphic to \( B^2(2) \times S^1 \), the \( S^1 \) factor being a transverse Hopf fibre. These submanifolds are certainly contact.
5.4. **Blown–up contact Lefschetz Fibration.** The fibres \( \widetilde{F} \) of the Lefschetz fibration \((\tilde{f}, C)\) differ from the fibres \( F \) of \((f, B, C)\). Let us provide a precise description of \( \widetilde{F} \) and show that the blow–up procedure can be perfomed to obtain \( c_1(\tilde{\xi}_F) = 0 \).

The trivialization of a neighborhood of a connected component \( \gamma \subset B \) of the base locus provided in Definition 3.2 induces a natural framing \( \nu_\mathcal{S} \cong S^1 \times \mathbb{C}^2 \), i.e. \( \langle (1, 0), (i, 0), (0, 1), (0, i) \rangle \). It restricts to a framing of \( S^1 \times \mathbb{C} \) in each of the two fibres corresponding to the two complex axes of \( \mathbb{C}^2 \). Hence it induces framings in any complex line \( S^1 \times \mathbb{C} \subset S^1 \times \mathbb{C}^2 \). for the complex line \( \{ (z, w) \in \mathbb{C}^2 : z - \alpha w = 0 \} \), we use \( \langle (\alpha, 1), i(\alpha, 1) \rangle \). Denote by \( \mathbb{F}_p(0) \) such framing of \( B \subset \tilde{f}^{-1}(p) \). Let \( \mathbb{F}_p(n) \) be the \( n \)-twist of \( \mathbb{F}_p(0) \) and \( k_\gamma \) be the parameter used in the construction of Theorem 5.3 when performing a blow–up along \( \gamma \).

**Lemma 5.5.** Let \((f, B, C)\) be a contact pencil for the 5–manifold \((M, \xi)\). Its contact blow–up \((\widetilde{M}, \tilde{\xi})\) has a contact fibration \((\tilde{f}, C)\) that coincides with \((f, B, C)\) away from \( B = \gamma_1 \cup \ldots \cup \gamma_s \). Its fibre over \( p \in \mathbb{CP}^1 \) is contactomorphic to a transverse contact \((0, 1)\)–surgery performed on \( \tilde{f}^{-1}(p) \) along \( \gamma_i \) with framing \( \mathbb{F}_p(-k_i - 1) \), for some \( k_i \in \mathbb{Z} \). The restriction of the map \( f \) to each of the exceptional divisors is given by the Hopf fibration.

**Proof.** The map \( \psi_k \) in Theorem 5.3 modifies the initial framing from \( \mathbb{F}_p \) to \( \mathbb{F}_p(-k_i) \), \( k_i \) being the corresponding squeezing parameter \( k \) in the surgery along \( \gamma_i \). Using the map \( \phi_1 \) substracts another twist and sends the meridian to the longitude of the added solid torus. It is thus a \((p, q) = (0, 1)\)–Dehn surgery with respect to \( \mathbb{F}_p(-k_i - 1) \). \( \square \)

Note that the coefficients \( k_i \) can be arbitrarily chosen. The constructive argument will use the fact that \( c_1(\tilde{\xi}_F) = 0 \) for any fibre \( \widetilde{F} \) of the blown–up pencil. This has been achieved for the initial fibres of the pencil. The blown–up procedure changes the almost contact manifold \((F, \xi)\) to \((\widetilde{F}, \tilde{\xi})\) and we cannot directly assume that \( c_1(\tilde{\xi}_F) = 0 \). This can be fixed using the following:

**Proposition 5.6.** There is a choice of \((k_1, \ldots, k_s) \in \mathbb{Z}^s \), such that the first Chern class of the almost contact structure \((\widetilde{M}, \tilde{\xi}, \tilde{\omega})\) in any regular fibre \( \widetilde{F} \) is zero.

**Proof.** Consider a connected component \( \gamma \subset B \). The complex rank–2 distribution \( \xi \) yields a line bundle \( \bigwedge^2 \xi \). The almost contact pencil is obtained from two generic sections of this line bundle. The determinant bundle of \( \xi \) is needed because of Corollary 3.6. Equivalently, from a section

\[
s = (s_0, s_1) : M \rightarrow \mathbb{C}^2 \otimes \bigwedge^2 \xi.
\]

A point \( p \in M \) maps to \([s_0(p) : s_1(p)] \in \mathbb{CP}^1 \). This is well–defined if \( p \) is not contained in the base locus \( B = \{ p : s_0(p) = s_1(p) = 0 \} \). The construction is detailed in [Pr3].

Let \( \widetilde{F} \) be a regular fibre of \( \tilde{f} \). Along this blown–up fibre \( \widetilde{F} \) the distribution \( \tilde{\xi} \) satisfies

\[
c_1(\tilde{\xi})|_{\widetilde{F}} = c_1(\tilde{\xi}_F) + c_1(\nu_\mathcal{F}).
\]

The submanifold \( \widetilde{F} \subset \widetilde{M} \) is a fibre of a locally trivial fibration, hence \( c_1(\nu_\mathcal{F}) = 0 \). The first Chern class measures the obstruction to the existence of a trivialization along the 1–skeleton. For a transverse loop this is equivalent to a framing. Hence, it is enough to justify that the section \( (s_0, s_1) \) can be lifted to a non-vanishing section \((\tilde{s}_0, \tilde{s}_1)\) from the blown–up manifold to the bundle \( \mathbb{C}^2 \otimes \bigwedge^2 \tilde{\xi} \). Then \( c_1(\tilde{\xi})|_{\widetilde{F}} = 0 \) and \( c_1(\tilde{\xi}_F) = 0 \).

A surgery procedure takes place, thus we should ensure that the relevant vector fields used in the blow–up process extend without adding zeroes. Let \( X_r, X_i, X_j, X_k \) be the canonical
quaternionic basis generating $\mathbb{T} \mathbb{C}^2 = \mathbb{C}^2 \cong \mathbb{H}^1$. The blow–up provides an interpolation between the following two bundles,

$$\xi = \ker \{ d\theta - r^2 \alpha_{std} \} = \langle X_r, r^2 \partial_r + X_i, X_j, X_k \rangle \simeq H = \ker d\theta = \langle X_r, X_i, X_j, X_k \rangle \text{ for } r \text{ large},$$

and

$$\ker \{ r^2 d\theta + \alpha_{std} \} = \langle X_r, r^2 X_i - \partial_r, X_j, X_k \rangle \text{ for } r \text{ small}.$$

Let $\varphi(r) : (0, \infty) \to [0, \pi/2]$ be a function constant equal to 0 for small $r$ and constant equal to $\pi/2$ for large $r$. The distribution being fixed for large radii, we interpolate through $\sin \varphi \cdot X_i - \cos \varphi \cdot \partial_r$ between $X_i$ and $-\partial_r \simeq r^2 X_i - \partial_r$, for $r$ small. Suppose the original sections $(s_0, s_1)$ restrict to an $m$–twisted frame, i.e. their trivialization is $e^{m \psi \theta}$ times the fixed trivialization. Then the choice $k = m$ allows us to extend the field $\partial_r$ in the framing to the interior of the dual sphere without zeroes.

Consider an increasing smooth function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $g(r) = \varepsilon/2$ for $r < \varepsilon/2$ and $g(r) = r$ for $r > \varepsilon$ and use it as a cut–off function to extend the sections to the whole blown–up manifold. This is the required section $\tilde{s} = (\tilde{s}_0, \tilde{s}_1)$ extending the previous section $s = (s_0, s_1)$ away from the surgery area and restricting to the Hopf section at $r = 0$. Thus the sections can be extended to the blown–up manifold in a non–vanishing manner and $c_1(\tilde{\xi}(\tilde{s})) = 0$. \qed

The previous results are of a local nature and therefore they apply, accordingly modified, to good almost contact pencils. In that case, we obtain an almost contact Lefschetz fibration that is contact in neighborhoods of the exceptional divisors and the critical loops, i.e. a good almost contact Lefschetz fibration, and the first Chern class of the distribution along the regular fibres vanishes. This concludes the proof of Theorem 5.2

5.5. **Contact blow–down in 5 dimensions.** In the subsequent sections the argument uses a fibration obtained by blowing–up the base locus of a good almost contact pencil. In order to recover the initial manifold we require the inverse procedure. This is used in Section 9.

**Lemma 5.7.** Let $(M, \xi)$ be a contact manifold and $E \cong (\mathbb{S}^3, \xi_{st})$ be a codimension–2 contact submanifold with trivial normal bundle. Then the submanifold $E$ can be replaced by a knot $K$ such that the resulting manifold $\tilde{M}$ admits a contact structure $(\tilde{M}, \tilde{\xi})$, $K$ is transverse and the complements $(M \setminus E, \xi) \cong (\tilde{M} \setminus K, \tilde{\xi})$ are contactomorphic.

**Proof.** The blow–up procedure can be reversed smoothly. Use $\phi_1^{-1}$ to glue instead of $\phi_1$. In this case any choice of $k$ can be made. \qed

Note that the diffeomorphism type of the blown–up manifold depends on the choice of $k \in \mathbb{Z}$. Even choices of $k$ not being diffeomorphic to odd ones.

The argument developed in this article to prove Theorem 1.1 requires a smooth fibration, hence the reason for performing a blow–up. There is an alternative approach not involving a blow–up that leads to a quite complicated version of the local models used in Sections 6, 7 and 8. These models are essential to describe the deformation of the almost contact structure. The simpler, the better. In particular, the description in Section 8 would be rather technical if the modified model was used.

The use of a blow–up to obtain an almost contact Lefschetz fibration eases the technicalities. In order to prove Theorem 1.1 we perform both blow–up and blow–down processes with the same parameter $k = (k_1, \ldots, k_s)$. This preserves the homotopy type of the almost contact structure.

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1Due to the choice of contact form, we should be strictly working with $\mathbb{C}$ instead of $\mathbb{C}$. 
6. Vertical Deformation.

In Section 3 we endowed our initial 5–dimensional closed orientable almost contact manifold \((M, \xi, \omega)\) with an almost contact pencil \((f, B, C)\) such that \(B \neq 0\) and \(c_1(\xi_F) = 0\) for the fibres \(F\) of \(f\). In Proposition 4.4 we have obtained a contact structure in a neighborhood of the base locus \(B\) and the critical curves \(C\). According to Theorem 5.2 there exists an almost contact Lefschetz fibration \((\tilde{f}, C)\) in an almost contact blow–up \((\tilde{M}, \tilde{\xi}, \tilde{\omega})\). In order to obtain a contact structure in the manifold \((M, \xi, \omega)\) we use the splitting induced by the existence of the fibration \((\tilde{f}, C)\) on \((\tilde{M}, \tilde{\xi}, \tilde{\omega})\). Henceforth we shall consider an almost contact manifold with an almost contact Lefschetz fibration. These will be respectively denoted \((M, \xi, \omega)\) and \((f, C)\) even though in our situation they refer to the blown–up initial manifold and the blown–up pencil. This should not lead to confusion. The initial manifold is recovered in Section 9.

Let \((M, \xi, \omega)\) be a 5–dimensional closed orientable almost contact manifold and \((f, C)\) an almost contact Lefschetz fibration on it.

Definition 6.1. An almost contact structure \((M, \xi, \omega)\) is called vertical with respect to an almost contact fibration \((f, C)\) if the fibres of \(f\) are contact submanifolds for \((\xi, \omega)\) away from the critical points.

A piece of terminology. If an almost contact Lefschetz fibration \((f, C)\) is obtained as a blown–up good almost contact pencil it contains a set \(E = \{E_j\}\) of exceptional contact divisors. The data of an almost contact Lefschetz fibration \((f, C, E)\) with a non–empty set of exceptional divisors will be referred to as an almost contact exceptional fibration, and shortened to ace fibration. An ace fibration is good if the curves of \(C\) and the divisors in \(E\) are contact submanifolds of \((M, \xi, \omega)\).

The main result of this section reads:

Theorem 6.2. Let \((M, \xi, \omega)\) be an almost contact manifold and \((f, C, E)\) an associated good ace fibration. Then there exists a homotopic deformation of the almost contact structure relative to \(C\) and \(E\) such that the almost contact structure becomes vertical for \((f, C)\).

The proof of the theorem relies on the existence of an overtwisted disk in each fibre, such structure allows more flexibility in handling families of distributions. Hence, it will be essential for the argument to apply that the fibres of the ace fibration are 3–dimensional manifolds. In order to obtain a vertical fibration we need Eliashberg’s classification result of overtwisted contact structures \([El]\). The required details are provided.

6.1. 3–dimensional Overtwisted Structures. Our setup provides a fibration with a distribution on each fibre. Given such an almost contact fibration \(f : M \to \mathbb{C}P^1\), let \(F_z\) denote the fibre over \(z \in \mathbb{C}P^1\) and \((\xi_z, \omega_z)\) the induced almost contact structure on \(F_z\). Then the family \((F_z, \xi_z)\) can locally be viewed as a 2–parametric family of 2–distributions on a fixed fibre.

In the proof of Theorem 6.2 we use a relative version of the following:

Theorem 6.3 (Theorem 3.1.1 in \([El]\)). Let \(M\) be a compact closed 3–manifold and let \(G\) be a closed subset such that \(M \setminus G\) is connected. Let \(K\) be a compact space and \(L\) a closed subspace of \(K\). Let \(\{\xi_t\}_{t \in K}\) be a family of co-oriented 2–plane distributions on \(M\) which are contact everywhere for \(t \in L\) and are contact near \(G\) for \(t \in K\). Suppose there exists an embedded 2–disk \(D \subset M \setminus G\) such that \(\xi_t\) is contact near \(D\) and \((D, \xi_t)\) is equivalent to the standard overtwisted disk for all \(t \in K\). Then there exists a family \(\{\xi_t\}_{t \in K}\) of contact structures of \(M\) such that \(\xi_t\) coincides with \(\xi_t\) near \(G\) for \(t \in K\) and coincides with \(\xi_t\) everywhere for \(t \in L\).
Moreover $\xi_t'$ can be connected with $\xi_t$ by a homotopy through families of distributions that is fixed in $(G \times K) \cup (M \times L)$.

In order to allow the case of a 3–manifold with non–empty boundary we also need:

**Corollary 6.4.** Let $M$ be a compact 3–manifold with boundary $\partial M$ and let $G$ be a closed subset of $M$ such that $M \setminus G$ is connected and $\partial M \subset G$. Let $K$ be a compact space and $L$ a closed subspace of $K$. Let $\{\xi_t\}_{t \in K}$ be a family of cooriented 2–plane distributions on $M$ which are contact everywhere for $t \in L$ and are contact near $G$ for $t \in K$. Suppose there exists an embedded 2–disk $D \subset M \setminus G$ such that $\xi_t$ is contact near $D$ and $(D, \xi_t)$ is equivalent to the standard overtwisted disk for all $t \in K$. Then there exists a family $\{\tilde{\xi}_t\}_{t \in K}$ of contact structures of $M$ such that $\tilde{\xi}_t'$ coincides with $\xi_t$ near $G$ for $t \in K$ and coincides with $\xi_t$ everywhere for $t \in L$. Moreover $\tilde{\xi}_t'$ can be connected with $\xi_t$ by a homotopy through families of distributions that is fixed in $(G \times K) \cup (M \times L)$.

**Outline.** The proof for the closed case uses a suitable triangulation $P$ of the 3–manifold having a subtriangulation $Q$ containing $G$, for which the distributions are already contact structures. Then Eliashberg’s argument is of local nature, working with neighborhoods of the $0$, $1$, $2$ and $3$–skeleton of $P \setminus Q$ and assuring that no changes are made in a neighborhood of $Q$. Hence, the method is still valid since $P$ and $Q$ do exist in the case of a manifold with boundary, and only $Q$ contains the boundary. □

We locally treat an almost contact fibration as a 2–parametric family of distributions over a fixed fibre, thus we may use a disk as a parameter space and the central fibre as the fixed manifold. It will be useful to be able to obtain a continuous family of distributions such that the distributions in a neighborhood of the central fibre become contact structures while the distributions near the boundary are fixed. Such a family is provided in the following

**Corollary 6.5.** Under the same hypothesis and notation of Corollary 6.4, assume that $K$ is a topological ball, so $S = \partial K$ is a sphere. Let $$\tilde{K} = K \cup_0 (S \times [0,1]) \quad \text{and} \quad \tilde{L} = L \cup_0 (\partial L \times [0,1])$$
where we identify $\partial K = S \cong S \times \{0\}$ and $\partial L \cong \partial L \times \{0\}$. Let the family of distributions be defined for $\{\xi_t\}_{t \in K}$, the distributions being contact near $G$ and $D$. Then there exists a deformation $\{\tilde{\xi}_t\}_{t \in K}$ such that:

- Satisfies the properties obtained in Corollary 6.4 when the family of parameters is restricted to $K$. In particular, for any $t \in K$ the distributions $\xi_t$ and $\tilde{\xi}_t$ coincide on an open neighborhood of $G \cup D$.

- The distributions $\xi_t$ and $\tilde{\xi}_t$ coincide over $M$ for $t \in \partial \tilde{K} \cong S \times \{1\}$. Further, $\tilde{\xi}_t$ can be connected with $\xi_t$ by a homotopy through families of distributions fixed in $(G \times \tilde{K}) \cup M \times (\tilde{L} \cup \partial \tilde{K})$.

**Proof.** Corollary 6.4 provides the family $\tilde{\xi}_t$ for $t \in K$, it is left to extend the parameter over $\tilde{K}$. Since both families must coincide at $S \times \{1\}$, we use the structures $\xi_t$ in the annulus $S \times [1/2, 1]$ and start deforming to $\tilde{\xi}_t$ at $S \times \{1/2\}$ until we reach $\partial \tilde{K} \cong S \times \{0\}$, where it glues with $\xi_t$.

In more precise terms, let us denote the family $\xi_t$ restricted to $S \times [0,1]$ by $\{\eta^s_{\xi_t}\}_{s \in [0,1]}$, i.e. $\xi_t = \xi_{(p,s)} = \eta^s_{\xi_t}$. Let $\{\xi^s_t\}_{s \in S}$ be the deformation of $\xi_{(t,0)} = \xi^0_t = \xi^1_t$ to $\xi_{(t,1)} = \xi^1_t$, restricted to $S \times \{0\} \cong \partial K$. Define the continuous deformation $$\left\{\eta^s_{\xi^s_t}\right\}_{s \in S} = \left\{\begin{array}{ll} \eta^{1-2s}_t, & s \in [0, 1/2], \\ \xi^{2s-1}_t, & s \in [1/2, 1]\end{array}\right.$$

Finally, the following deformation extends $\{\xi^s_t\}$ to the domain $S \times [0,1]$ and satisfies all the
required properties:
\[
\{(\vec{\xi})_{t,(r)}^{s} \}_{(t,r) \in \mathcal{S} \times [0,1]} = \{\nu_{t}^{q}\}, \quad q = \frac{1}{2} (1 + s) (1 - r).
\]
See Figure 4 for a visual realization.

Remark 6.6. Notice that the assumption that \(K\) is a topological ball is not necessary. However, since we will apply Corollary 6.5 when \(K\) is a topological ball, we consider more appropriate to state it in this particular case.

We need at least one overtwisted disk over each fibre in order to apply Corollary 6.4. The family should behave continuously. Let us provide such a family of disks.

6.2. Families of overtwisted disks. There are two basics issues to be treated: the location of the disks and their overtwistedness. The second is simply guaranteed since once a disk with a contact neighborhood is placed in each fibre we can produce overtwisted disks using Lutz twists. In order to decide the location of the disks in each fibre we need to find a section of the good ace fibration.

Let \((f, C, E)\) be a good ace fibration. Denote by \(U(C), U(E_{i})\) open neighborhoods of the critical curves \(C\) and the exceptional spheres \(E_{i}\). Consider \(U(f) = U(C) \cup U(E_{i})\) the union of these open neighborhoods, so in the complementary of \(U(f)\) the \(f\) becomes a submersion. Instead of finding a global section mapping away from \(U(f)\), we shall construct two disjoint sections that will provide at least one overtwisted disk in each fibre. The global situation we achieve is described as follows:

Proposition 6.7. Let \((f, C, E)\) be a good ace fibration for \((M, \xi, \omega)\). There exists a deformation \((F_{z}, \vec{\xi}_{z})_{z \in \mathbb{CP}^{1}}\) of the family \((F_{z}, \xi_{z})_{z \in \mathbb{CP}^{1}}\) fixed at the intersection of the set \(U(f)\) with each \(F_{z}\) satisfying:

(i) There exist two open disks \(B_{0}, B_{\infty} \subset \mathbb{CP}^{1}\), containing 0 and \(\infty\) respectively such that the intersection \(B_{0} \cap B_{\infty}\) is an open annulus and the curves \(\partial B_{i}\) are disjoint from the set of curves \(f(C)\).
(ii) There exists two disjoint families of embedded 2–disks \( D_z^i \subset F_z \), with \( z \in B_i \), for \( i = 0, 1 \), not intersecting \( U(f) \). Further, the structure \( \xi_z \) is contact in a neighborhood of such families and \( (D_z^1, \xi_z) \) are equivalent to standard overtwisted disks.

Remark 6.8. The complement of \( B_0 \cap B_\infty \) consists of two disjoint disks. The diameter of those disks can be chosen as small as desired. This will be needed in the subsequent sections.

The fact that \( \tilde{\xi}_z \) equals \( \xi_z \) in the intersection of the set \( U(f) \) with \( F_z \) ensures that no deformation is performed near the critical curves nor the exceptional spheres. This is mainly a global statement, involving the whole of the fibres. In order to prove the result we study the local model of a tubular neighborhood of an exceptional divisor of the good ace fibration \((f, C, E)\).

A good ace fibration \((f, C, E)\) is obtained by blowing–up a certain good almost contact Lefschetz pencil along its base locus \( B \). Let \( K \) be a knot belonging to this base locus \( B \). After the blow–up procedure it is replaced by an exceptional contact divisor \( E \cong (S^3, \xi_{st}) \).

As explained in Section 5, the restriction of the fibration \( f \) to \( E \) is the Hopf fibration. Since the distribution \( \xi \) is locally a contact structure the tubular neighborhood theorem provides a chart
\[
\Psi : U \longrightarrow S^3 \times B^2, \quad \Psi^*\xi_{st} = \xi
\]
where \( \xi_{st} = \ker\{\alpha_{S^3} + r^2d\theta\} \). The fibres of the induced map \( f_U \) defined as
\[
\begin{array}{ccc}
S^3 \times B^2 & \xrightarrow{\Psi^{-1}} & U \\
\downarrow f_U & & \downarrow f \\
\mathbb{CP}^1 & \to & \mathbb{CP}^1
\end{array}
\]
correspond to tubular neighborhoods of the Hopf fibres. That is to say, they are submanifolds \((S^1 \times B^2, \xi_v = \ker(d\beta + r^2d\theta))\) for each \( z \in \mathbb{CP}^1 \). Note that the variable \( \beta \in S^1 \) parametrizing each Hopf fibre is not global since the fibration is not trivial. The differential \( d\beta \) is globally well–defined since it is dual to the vector field generating the associated \( S^1 \) action. Using an almost contact connection\(^2\) the standard contact structure in \( S^3 \times B^2 \) can be expressed as the direct sum of distributions
\[
\xi_{st} = \xi_v \oplus H,
\]
where \( \xi_v \) is the standard contact structure in \( S^1 \times B^2 \), the vertical direction, and \( H \) is the contact\(^3\) connection associated to the fibration of \( S^3 \times B^2 \) over \( \mathbb{CP}^1 \).

Topologically, the 4–distribution \( \xi_{st} \) is expressed as a direct sum of two distributions of 2–planes. Since the 2–form \( \omega \) providing the almost contact structure is given and so is \( \xi \), we may interpret \((S^3 \times B^2, \xi_v)\) as a non–trivial family of contact structures parametrized by the base, \( \{\xi_q = \xi_v\} \), \( q \in \mathbb{CP}^1 \). So far we understand the topology and contact structure of the local model of the ace fibration along an exceptional divisor. Indeed, in a neighborhood of the exceptional divisor \( E \) it is precisely a piece of the blown–up fibration and the knots are the intersection of the fibres of the almost contact pencil with the exceptional sphere \( E \).

The local model described above allows us to prove the local version of Proposition 6.7.

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\(^2\)If \((f, C)\) is an almost contact fibration for \((M, \xi, \omega)\), an almost contact connection \( H_\xi \) is defined at a point \( p \in M \setminus C \) as the distribution \( \omega_p \)-symplectic orthogonal to the distribution of the vertical fibre \( \xi_p \cap \ker df_p \) inside \( \xi_p \).

\(^3\)An almost contact connection is called a contact connection if the fibration and \( \xi \) are, with the compatibility condition: \( \exists \alpha \) a contact form such that \( \omega = d\alpha \) and \( \xi = \ker \alpha \).
Lemma 6.9. There exists a deformation of the contact structures \( (\mathbb{S}^3 \times B^2, \xi_u) \) relative to a neighborhood of the boundary such that the deformed structures are contact and with respect to them:

(i) There exist two continuous families of overtwisted disks \( \{D^0_q\}_{q \in B_0} \) and \( \{D^\infty_q\}_{q \in B_\infty} \), i.e., on each point \( q \in B_0 \setminus \partial B_0 \) (resp. \( q \in B_\infty \setminus \partial B_\infty \)) there is an overtwisted disk \( D^0_q \) (resp. \( D^\infty_q \)).

(ii) The disks \( D^0_q \) and \( D^\infty_q \) are disjoint for \( q \in B_0 \cap B_\infty \).

Both \( B_0 \setminus \partial B_0, B_\infty \setminus \partial B_\infty \) can be thought as neighborhoods of the upper and lower semi-spheres.

Proof. Let \( h : \mathbb{S}^3 \to \mathbb{CP}^1 \) be the Hopf fibration, extend the fibration to \( h : \mathbb{S}^3 \times B^2 \to \mathbb{CP}^1 \) by projection onto the first factor. Let \( B_0, B_\infty \) be two disks containing \( 0, \infty \in \mathbb{CP}^1 \), e.g. the complements of a tubular neighborhood of \( \infty \) and \( 0 \). As explained before, the idea is to use the exceptional divisor to create a couple of sections. On the one hand, the exceptional divisor has a contact structure and we would rather not perturb around a small neighborhood of it, and on the other hand the exceptional divisor is not \( \mathbb{CP}^1 \) but \( \mathbb{S}^3 \). We use two copies of the exceptional divisor away from \( \mathbb{S}^3 \times \{0\} \subset \mathbb{S}^3 \times B^2 \) and we trivialize the base \( \mathbb{CP}^1 \) with the two disks \( B_0, B_\infty \).

Let \( q_0 = (1/2, 0), q_\infty = (0, 1/2) \in B_0 \) be two fixed points and denote the two corresponding \( 3 \)-spheres
\[
\mathbb{S}^3_0 = \mathbb{S}^3 \times \{q_0\}, \quad \mathbb{S}^3_\infty = \mathbb{S}^3 \times \{q_\infty\}.
\]
The fibre of the restriction of the fibration \( (\mathbb{S}^3 \times B^2, \xi_u) \to \mathbb{CP}^1 \) to the submanifold \( \mathbb{S}^3_0 \) (resp. \( \mathbb{S}^3_\infty \)) is a transverse knot \( K^p_0 \) (resp. \( K^p_\infty \)). We will now insert two families of overtwisted disks.

Applying a full Lutz twist in a small neighborhood of each of those knots \( K^p_0 \in h^{-1}(p) \) produces a \( 3 \)-dimensional Lutz twist on each fibre, see [Lu, Ge]. This family of overtwisted disks is parametrized as \( \{D^0_t\}_{t \in \mathbb{S}^3_0} \), thus we obtain a \( \mathbb{S}^3 \)-family of overtwisted disks at each fibre. Perform the same procedure for the family of knots \( K^p_\infty \in h^{-1}(p) \) to obtain another family of disks \( \{D^\infty_t\}_{t \in \mathbb{S}^3_\infty} \). The two families of disks can indeed be disjoint by letting the radius in which we perform Lutz twists be small enough. This construction provides the deformed family of contact structures, coinciding with the previous distribution near the boundary. Let us remark that the support of the pair of Lutz twists also does not intersect the exceptional divisor. See Figure 5.

We need the base to be the parameter space, instead of a \( 3 \)-sphere: restricted to \( B_0 \) or \( B_\infty \) the Hopf fibration becomes trivial and therefore there exist two sections \( s_0 : B_0 \to \mathbb{S}^3_0 \) and \( s_\infty : B_\infty \to \mathbb{S}^3_\infty \). The required families are defined as
\[
\{D^0_q\} = \{D^0_{s_0(q)}\}; q \in B_0,
\]
\[
\{D^\infty_q\} = \{D^\infty_{s_\infty(q)}\}; q \in B_\infty.
\]
Note that the two families of overtwisted disks are disjoint since the two families of Lutz twists were. Further, there exists a small neighborhood of the exceptional divisor \( \mathbb{S}^3 \times \{0\} \) where no deformation is performed.

The global construction can be simply achieved:

Proof of Proposition 6.7. Apply Lemma 6.9 to a neighborhood of one exceptional sphere \( E_0 \in E = \{E_0, E_1, \ldots, E_s\} \). The families of overtwisted disks do not meet \( C \) or any \( E_j \). Indeed, the two families are arbitrarily close to \( E_0 \) and the exceptional divisors are pairwise disjoint and none of them intersect the critical curves \( C \). Thus, maybe after shrinking the neighborhood \( U(E_0) \) in the construction, the families are located away from \( U(f) \).
Thus we obtain the families of overtwisted disks required to apply Theorem 6.3. The vertical contact condition is ensured progressively above the 0–cells, the 1–cells and the 2–cells.

6.3. Adapted families. Let \((f, C)\) be an almost contact fibration. A finite set of oriented immersed connected curves \(T\) in \(\mathbb{CP}^1\) will be called an adapted family for \((f, C)\) if it satisfies the following properties:

- The image of the set of critical values \(f(C)\) is part of \(T\). Let \(C_i\) denote the image of each of the components of \(C\).
- Given any element \(c \in T\), there exists another element of \(c' \in T\) having a non–empty intersection\(^4\) with \(c\). Any two elements of \(T\) intersect transversally.

Let \(|T| \subset \mathbb{CP}^1\) be the underlying set of points of the elements of \(T\). The elements of an adapted family \(T\) that are not of the form \(C_i\) are denoted by \(F_j\) and referred to as fake components.

Let \(N \in \mathbb{N}\) be fixed. The insertion of fake curves proves the existence of an adapted family with \(\text{diam}_{g_0}(\mathbb{CP}^1 \setminus |T|) \leq 1/N\), \(g_0\) the standard round metric.

There is a cell decomposition of \(\mathbb{CP}^1\) associated to an adapted family, the 1–skeleton being \(|T|\). See Figure 6. We shall first deform in a neighborhood of each vertex relative to the boundary, proceed with a neighborhood of the 1–cells and finally obtain the vertical contact condition in the 2–cells. To be precise in the description of the procedure, we introduce some notation.

Let \(L_j \in T\) be a curve, \(U(L_j)\) be a tubular neighborhood and denote \(\partial U(L_j) = L_j^0 \cup L_j^1\). Suppose that \(\bigcup_{j \in J} |L_j^i|\) is isotopic to \(|T|\) for both \(i = 0, 1\); this can be achieved by taking a small enough neighborhood of each \(L_j\). See Figure 7. We use \(V(L_j)\) to denote a slightly larger tubular neighborhood satisfying this same condition. Fix an intersection point \(p\) of

\(^4\)In case \(c\) has a self–intersection, then \(c' = c\) is allowed.
two elements $L_j, L_k \in T$. Denote by $A_p$ the connected component of the intersection of $U(L_j) \cap U(L_k)$ containing $p$. Similarly, let $V A_p$ be the connected component of the intersection of $V(L_j) \cap V(L_k)$ that contains $p$, and denote $V A_p = V A_p \setminus A_p$.

The open connected components of $U(T) \setminus \bigcup A_p$ are homeomorphic to rectangles $B_i$. Here the index $p$ over the intersection points is assumed, as well as a suitable indexing for $i$. The third class of pieces constitute the interior of the complementary in $\mathbb{C}P^1$ of the open set formed by the union of the sets $A_p$ and $B_i$; its connected components are denoted $C_i$. Thus, neighborhoods of the 0–cells, 1–cells and 2–cells are labeled $A_p$, $B_i$ and $C_i$ respectively. See Figure 7.

Finally, we define the sets $BB_i$. Let $B_i$ connect a couple of open sets of the form $A_p$. There exists a curve $L_{B_i}$ contained in $B_i$ which is a part of a curve $L_i \in T$; so $L_{B_i}$ is a 1–cell in the decomposition associated to the adapted family $T$. Let $L^0_{B_i}$ and $L^\infty_{B_i}$ denote the two boundary components of $B_i$ which are part of the curves $L^0_i$ and $L^1_i$ defined above. Then we declare $BB^0_i$ (resp. $BB^1_i$) to be the connected component of $V(L_i) \setminus B_i$ containing the boundary curve $L^0_i$ (resp. $L^1_i$). Their union $BB^0_i \cup BB^1_i$ will be denoted $BB_i$. See figures 8 and 9.

6.4. The vertical construction. We prove Theorem 6.2. The following lemma is a simple exercise in differential topology (particular case of Ehresmann’s fibration theorem). It will be used in the proof of Theorem 6.2 and its idea also appears in Lemma 7.4, cf. Section 7. We include it for completeness.

**Lemma 6.10.** Let $f : E \to B^2$ be a locally trivial smooth fibration over the disk with compact fibres. Assume that $E$ has a smooth closed boundary $\partial E$. Suppose also that there is a collar neighborhood $N$ of $\partial E$ and a closed submanifold $S$ such that restricting $f$ to $S, N$ or

---

\[\text{Figure 6. Part of an adapted family } T. \text{ The associated subdivision consists of certain 2–cells with their boundaries being a union of parts of various elements in the family } T.\]
Figure 7. In dark gray the sets $A_p$, in light gray the sets $B_i$, associated to the subdivision of the figure.

Figure 8. Example of two components $V\mathcal{A}_p$ and $V\mathcal{A}_q$ in light gray, containing $\mathcal{A}_p$ and $\mathcal{A}_q$, in dark gray.

\[ \partial E \text{ induces locally trivial fibrations. Let } S_0, N_0 \text{ and } E_0 \text{ be their fibres over } 0 \in B^2. \]

Then there exists a diffeomorphism $g : E \rightarrow E_0 \times B^2$ making the following diagram commute

\[ \begin{array}{ccc}
E & \xrightarrow{g} & E_0 \times B^2 \\
\pi & & \pi_0 \\
B^2 & \xrightarrow{} & B^2
\end{array} \]

such that $g(N) = N_0 \times B^2$ and $g(S) = S_0 \times B^2$.

Proof. Let $g$ be Riemannian metric in $E$ such that $(TE_z)^{\perp g} \subset TS$ as well as $(TE_z)^{\perp g} \subset T(\partial E)$, for the points $z$ where the condition can be satisfied. Let $X = \partial_r$ be the radial vector field in $B^2 \setminus \{0\}$ and construct the connection associated to the Riemannian fibration:

\[ H_\pi(e) = (T_e F_\pi(e))^{\perp g}. \]
The condition imposed on the Riemannian metric implies that $\partial E$ and $S$ are tangent to the horizontal connection $H_\pi$. Let $\tilde{X}$ be a lift of $X$ through $H_\pi$ and $\phi_t(e)$ the flow of this vector field. Define

$$E \xrightarrow{\gamma} E_0 \times B^2$$
$$e \mapsto \left(\phi_{(-||\pi(e)||)}(e), \pi(e)\right).$$

This map satisfies the required properties. \hfill \Box

**Proof of Theorem 6.2.** Let $(f, C, E)$ be a good ace fibration. Note that a horizontal connection $H$ is defined away from $U(C)$ and provides the splitting specified in (2). Fix an exceptional divisor $E_0$. Use it to apply Proposition 6.7 to the family of distributions to ensure the existence of at least one overtwisted disk in each fibre. In particular we obtain $B_0$ and $B_\infty$.

Let $T$ be an adapted family to the almost contact fibration such that $\partial B_0$ and $\partial B_\infty$ are both contained in two different 2–cells $C_0$ and $C_\infty$. In order to establish Theorem 6.2 we need to perform a deformation which is fixed in a neighborhood of $U(C)$ and leaves the distribution $H$ unchanged, i.e. it should be a strictly vertical deformation.

**Deformation at the 0–cells:** Let $p$ be a vertex with neighborhood $A_p$. The fibration trivializes over $\mathcal{V}A_p$ and let $(F_z, \xi_z)$ be the family of fibres and distributions. In case $\mathcal{V}A_p$ is small enough the manifolds with boundary $F_z = F_z \setminus (F_z \cap U(C))$ are all diffeomorphic. Let $N_z$ be a collar neighborhood of $\partial F_z$ in which the distribution is contact. Given an exceptional divisor $E_j \in E$ denote by $U(E_j)_z$ the intersection of $U(E_j)$ with the fibre $F_z$. Applying the trivializing diffeomorphism provided in Lemma 6.10 we may assume $F_z \times \mathcal{V}A_p \cong \mathcal{F}$, $U(E_j)_z \times \mathcal{V}A_p \cong U(E_j)$ and $N_z \times \mathcal{V}A_p \cong N$.

We thus have: a manifold with boundary $\mathcal{F}$ with a family of distributions $\xi_z$ parametrized by the topological disk $K' = \mathcal{V}A_p$ containing $K = A_p$. Also a good set $G$ of submanifold that are already contact for any parameter in $K'$, $G$ consists of the union of $N, U(E_j)$ and a neighborhood of one of the two overtwisted disk. \hfill \footnote{These disks are trivialized along with $N$ using Lemma 6.10} Let us say a neighborhood of $D^\infty$. A neighborhood of this set will not be perturbed. The remaining disk $D^0$ is contactomorphic to the standard overtwisted disk for each element of the family of distributions. This set–up
satisfies the hypothesis of Corollary 6.5 with $L' = \emptyset$. Since we are able to obtain a deformation relative to the boundary we may perform the deformation at each neighborhood of the 0–cells and extend trivially to the complement of $\mathcal{V}A_p$ in $\mathbb{C}P^1$.

*Deformation at the 1–cells:* Almost the same strategy applied to the 0–cells applies, although we should not undo the deformation in a neighborhood of the 0–cells. Corollary 6.5 allows us to perform deformations relative to a subfamily, so in this case $L'$ will be non–empty. See Figure 10. The reader is invited to precise the remaining details.

**Figure 10.** The set $L' \subset \mathcal{V}\mathcal{B}_i$ is already a contact distribution.

*Deformation at the 2–cells:* In this situation Theorem 6.3 also applies after a suitable trivialization of the smooth fibration provided by Lemma 6.10. The set $L$ is a small tubular neighborhood of the boundary of the 2–cells. Except at $C_0$ and $C_\infty$, we may use any of the two families of overtwisted disks to apply the result. Let it be $\mathcal{D}_2^0$. In the remaining family the distributions are contact and so we include the disks in the set $G$, that also contains $N$ and $U(E_j)$. At $C_0$ we use the family $\mathcal{D}_2^0$, since it is the only one well defined over the whole set. Proceed analogously at $C_\infty$. Note that this argument is possible because the deformation is relative to the boundary. Then Theorem 6.3 applies to the 2–cells and we extend trivially the deformation. We obtain a vertical contact distribution $(F_z, \tilde{\xi}_z)$ away from $U(C)$.

To conclude the statement, consider the direct sum $\tilde{\xi}_z \oplus H$ to include the critical set, which has not been deformed. This is the required vertical contact structure. Notice that this construction preserves the almost contact class of the distribution since it is performed homotopically only in the vertical direction. □

7. Horizontal Deformation I

Consider the initial almost contact distribution $(M, \xi)$ and a good ace fibration $(f, C, E)$ with associated adapted family $T$. Theorem 6.2 deforms $\xi$ to a vertical contact structure with respect to $(f, C, E)$. To obtain a honest contact structure the distribution has to be
suitably changed in the horizontal direction. As in the previous section, this is achieved in
three stages. The content of this Section consists of the first two of these: deformation in
the pre–image of a neighborhood of the 1–cells of the adapted family $T$. The main result of
this Section is the following theorem.

**Theorem 7.1.** Let $(M, \xi, \omega)$ be a vertical contact structure with respect to a good ace fibration $(f, C, E)$ and $T$ an adapted family. Then there exists a deformation $(\xi', \omega')$ of $(\xi, \omega)$ relative to $C$ and $E$ such that $(f, C, E)$ is a good ace fibration for $(\xi', \omega')$ and $\xi'$ is a contact structure in the pre–image of a neighborhood of $|T|$.

The resulting distribution $(\xi', \omega')$ is still vertical. In fact, the vertical distribution is fixed
along the deformation. In this sense the deformation in the statement is horizontal. The
blown–up fibration $(f, C)$ will not be deformed to prove this fact, just the almost contact
structure.

Theorem 7.1 follows from Proposition 7.3 and Proposition 7.7. To prove the statement we
trivialize the vertical contact fibration over a neighborhood of the 0–cells. Then the
deformation is performed using an explicit local model. Then we proceed with the pre–image
of a neighborhood of the 1–cells. The same local model is used to deform.

7.1. Local model. The following lemma is used to prove Proposition 7.3 and Proposition 7.7. It is a version of results in Section 2.3 of [El] concerning deformations of a family of
distributions near the 1 and 2–skeleta of a 3–manifold. The connectedness condition is stated
there as the vanishing of a relative fundamental group.

**Lemma 7.2.** Let $(F, \xi(s,t))$ be a family of contact structures over a compact 3–manifold $F$ parametrized by $(s, t) \in [-\varepsilon, \varepsilon] \times [0, 1]$. Let $\alpha(s,t)$ be the associated contact forms. Consider the projection

$$F \times [-\varepsilon, \varepsilon] \times [0, 1] \to F \times [0, 1],$$

and the distribution $\xi$ on $F \times [-\varepsilon, \varepsilon] \times [0, 1]$ defined globally by the kernel of the form

$$\alpha_H(p, s, t) = \alpha(s,t) + H(p, s, t)dt, \quad H \in C^\infty(F \times [-\varepsilon, \varepsilon] \times [0, 1]).$$

Suppose that $|H(p, s, t)| \leq c \cdot |s|$ and $\xi$ is constant along the $s$–lines, i.e. $\partial_s \alpha(s,t) = 0$. Assume
that the 1–form $\alpha_H$ is a contact form in a compact set $G$ such that $\pi_s^{-1}(p, s) \cap G$ is connected and contains one of the endpoints of the interval $\pi_s^{-1}(p, s)$.

Then, there is a small perturbation $\tilde{H}$ of $H$ relative to $G$ such that $\alpha_{\tilde{H}}$ defines a contact
structure. In precise terms, $|\tilde{H} - H| \leq 3\varepsilon c$ and $\tilde{H}|G = H|_G$.

**Proof.** Let us compute the contact condition on $\alpha = \alpha_H$.

$$d\alpha = d\alpha(s,t) + dH \wedge dt = (d\alpha)^n + (d\alpha(s,t))^{n-1} \wedge dH \wedge dt.$$

Therefore, the contact condition is described as

$$(d\alpha)^n \wedge \alpha = (d\alpha(s,t))^n \wedge Hdt + (d\alpha(s,t))^{n-1} \wedge \alpha(s,t) \wedge dH \wedge dt > 0$$

Since $\partial_s \alpha(s,t) = 0$ the first term of the right hand side of the equation is zero and

$$(d\alpha)^n \wedge \alpha = (d\alpha(s,t))^{n-1} \wedge \alpha(s,t) \wedge (\partial_s H \cdot ds \wedge dt).$$

Thus, the 1–form $\alpha$ is a contact form if and only if $\partial_s H > 0$.

Given $p \in F, t \in [0, 1]$, $\pi_s^{-1}(p, t)$ is a 4–parametric family of 1–dimensional manifolds. The
connectedness of $\pi_s^{-1}(p, t) \cap G$ and the compactness of $G$ assure that it is possible to perturb
$H$ to an $\tilde{H}$ relative to $G$ and satisfying the contact condition. Indeed, the connectedness
condition allows us to perturb at least one end of a curve in $F \times [-\varepsilon, \varepsilon] \times [0, 1]$ and obtain a function $\tilde{H}$ with $\partial_s \tilde{H} > 0$. □
7.2. Deformation along intersection points. In this subsection we obtain a contact structure in a neighborhood of the fibres over a neighborhood of the intersection points in \( f(C) \). The precise statement reads as follows:

**Proposition 7.3.** Let \((M, \xi, \omega)\) be a vertical contact structure with respect to a good ace fibration \((f, C, E)\) and \(T\) an adapted family. Then there exists a deformation \((\xi', \omega')\) of \((\xi, \omega)\) relative to \(C\) and \(E\) such that \((f, C, E)\) is a good ace fibration for \((\xi', \omega')\) and \(\xi'\) is a contact structure in the pre–image of a neighborhood of the \(0\)–cells of \([T]\).

In order to apply 7.2 we consider a trivialization relative to \(C\) and \(E\) and with a particular condition on the parallel transport. This is rather technical. The following lemmata exploit the ideas of Lemma 6.10. We introduce the required notation.

Let \(z\) be a point of intersection of the adapted family \(T\) and \((\phi, U)\) a sufficiently small chart centered at \(z\) with polar coordinates \((r, \theta)\). Recall the definition of the open set \(U(f) = U(C) \cup U(E)\), i.e. \(U(f)\) is a small open neighborhood of the critical set union the exceptional divisors. Let \(N = f^{-1}(U) \setminus U(f)\). We shall also denote by \(f\) the possible restrictions of the homonymous fibering map.

Denote by \(F \cong F \setminus (U(f) \cap F)\) the fibre of \(f\) over \(z\). The map \(f : N \to U\) is a smooth fibration provided that the neighborhoods are small enough, and thus the fibres are diffeomorphic to \(F\). Restricting \(f\) to the boundary \(\partial N\) we also obtain a smooth fibration whose fibre is \(\partial F\). The collar neighborhood theorem provides a neighborhood \(U_{\partial N}\) of the boundary \(\partial N\) such that \(f : U_{\partial N} \subset N \to U\) is a smooth fibration and the fibre \(U_{F}\) is diffeomorphic to \(\partial F \times [0, \varepsilon]\). Let \(\psi : U_{F} \to \partial F \times [0, \varepsilon]\) be such a diffeomorphism. Note that the almost contact structure \((\xi, \omega)\) is contact on \(U_{\partial N}\), because \((f, C, E)\) is a good ace fibration.

The trivialization is carried out using the flow of the radial vector field lifted using a boundary preserving connection. Let us work with a slight modification of the manifold \(F\). Concretely, let \(L \subset U_{L}\) be such that

\[\psi(L) = \partial F \times \{\varepsilon/2\}, \quad \psi(U_{L}) = \partial F \times [\varepsilon/2, 3\varepsilon/4].\]

Then \(F \setminus \psi^{-1}(\partial F \times [0, \varepsilon/2])\) is a manifold with boundary \(L\) and collar neighborhood \(U_{L} \subset U_{F}\). To ease notation we still call this shortened manifold \(F\). In this situation, we are able to perform a trivialization respecting the vertical contact condition:

**Lemma 7.4.** Let \((F, \Xi)\) be a contact manifold with boundary and \(f : O \to B^{2}\) a fibre bundle with typical fibre \(F\). Consider a vertical almost contact structure \((O, \xi)\) restricting to \((F, \partial F, \Xi)\) in the fibre. Suppose that \((O, \xi)\) is a contact structure in a collar neighborhood \(C_{F}\) of the boundary \(\partial(f^{-1}(B^{2})) \setminus \partial(f^{-1}(\partial B^{2}))\). Then there exists \(\varepsilon \in \mathbb{R}^{+}\), a flow \(\varphi_{t}\) on \(F\) and a fibre–preserving diffeomorphism

\[\tau : \tau^{-1}(F \times B^{2}(\varepsilon)) \to f^{-1}(B^{2}(\varepsilon)) \subset O; \quad p \mapsto \tau(p) = (\varphi_{-||f(p)||}(p), f(p)),\]

such that \(\tau_{*}(\xi)\) is still a vertical contact structure with respect to the product fibration over \(B^{2}(\varepsilon)\), and the family \(\{\xi_{z} = \tau_{*}(\xi)|_{f^{-1}(z)}\}_{z \in B^{2}(\varepsilon)}\) is constant when restricted to \(\tau(C_{F})\).

**Proof.** The idea is contained in the proof of Lemma 6.10. We use the almost contact connection to trivialize the fibration \(f : O \to B^{2}\). Near the boundary the distribution \(\xi\) is a contact structure, therefore the parallel transport is by contactomorphisms. Let \(\varphi_{t}\) be the flow of the lift of the radial vector field. The diffeomorphism \(\varphi_{t}(p)\) is defined for \(p \in O \setminus \partial O\) and a finite time depending on \(p\). A uniform time \(\varepsilon\) is obtained by shortening the manifold and restricting to a compact neighborhood near the boundary as before the
statement of the Lemma. Define
\[ \tau : \tau^{-1}(F \times B^2(\varepsilon)) \longrightarrow F \times B^2(\varepsilon), \quad \tau(p) = (\varphi(-||f(p)||)(p), f(p)). \]

The splitting of \( \tau_\ast \xi \) must now be performed using \( \tau_\ast \omega \). The connection has been chosen in order that the vertical factor of the distribution \( \tau_\ast \xi \) is a contact structure once restricted to the fibres \( F \). Since the family of distributions was already a contact distribution close to the boundary, the parallel transport is performed along contactomorphisms and the family is indeed constant in the image of the collar neighborhood \( C_F \).

**Remark 7.5.** Suppose that the almost contact connection preserves the collar neighborhood \( C_F \). Then the radius \( \varepsilon \in \mathbb{R}^+ \) in Lemma 7.4 can be assumed to be \( \varepsilon = 1 \). This follows from the argument since in this case the flow \( \varphi_t \) is defined for \( t \in [0, 1] \).

Lemma 7.4 applies to the situation described above once we have used the chart \((U, \phi)\). Indeed, in this situation we use \( \mathcal{O} = N, F = \mathcal{F} \) and the fibration onto \( B^2 \) is given by the map \( \phi \circ f |_N : N \rightarrow B^2 \). Hence the fibration provided by the good ace fibration \((f, C, E)\) restricted to the pre–image of a small neighborhood of \( z \) inside the manifold can be trivialized preserving the vertical contact condition.

In order to be able to apply Lemma 7.2 we prove the existence of a deformation such that at least in one direction the parallel transport along the deformed almost contact connection is a contactomorphism. This allows us to trivialize with the almost contact connection and obtain a vertical contact distribution constant along that direction. Thus conforming the hypotheses of Lemma 7.2. This is the content of the subsequent lemma.

Let \( I^2 \subset B^2 \) denote a small closed rectangle with coordinates \((s, t) \in [0, 1] \times [0, 1] \). The appearance of both cartesian and polar coordinates can be avoided, however it is more natural to prove the previous lemma in polar coordinates and describe the following local model in cartesian coordinates.

**Lemma 7.6.** Let \((F, \Xi)\) be a contact manifold with boundary. Consider a vertical almost contact structure \((F \times B^2, \xi)\) restricting to \((F, \Xi)\). There exists a horizontal deformation of \( \xi \), supported in the pre–image of a small disk \( D^2 \) containing \( I^2 \), such that the parallel transport along the lift of \( \partial_s \) through the deformed almost contact connection is a contactomorphism.

**Proof.** The almost contact distribution \( \xi \) splits as
\[ \xi = \xi_v \oplus \xi_h \]
Let us consider the vertical part \( \xi_v \) restricted to the preimage of \( I^2 \). To ease notation, denote it by \( \xi_{(s, t)} \). Perform Moser’s trick to each 1–parametric family \( \xi_{(s, t)} = \ker \alpha_{(s, t)} \) where the \( t \)–coordinate is fixed. We obtain an \( s \)–family of diffeomorphisms
\[ m^t_s : F \longrightarrow F \text{ such that } (m^t_s)_\ast \xi_{(s, t)} = \xi_{(0, t)}. \]
Note that these diffeomorphisms are supported away from any region where the family is constant. Moser’s argument can be made parametric and the family \( m^t_s \) can be chosen to behave smoothly with respect to the \( t \)–coordinate. We use this family of diffeomorphisms in the fibration to define a diffeomorphism
\[ J : F \times I^2 \longrightarrow F \times I^2, \quad (p, s, t) \longmapsto (m^t_s(p), s, t). \]
Choose a vertical distribution at each fibre constant equal to the distribution \( \xi_{(0, t)} \). The condition \((m^t_s)_\ast \xi_{(s, t)} = \xi_{(0, t)}\) implies that \( J_\ast (\xi_v) = \xi_{(0, t)} \).

We construct the appropriate distribution. Let \( \tau_0 = \xi_h \) and \( \tau_1 \) be the 2–distribution given by \( TB^2 \) inside \( TF \oplus TB^2 \). They are isotopic through horizontal distributions. Let \( \tau_l \) be such
an isotopy and \( \chi : D^2 \to [0,1] \) be a smooth decreasing function such that
\[
\chi(s,t) = 1, \text{ for } (s,t) \in I^2; \quad \chi(s,t) = 0, \text{ for } (s,t) \in \partial D^2.
\]
The required distribution in a small rectangle of the trivialization is
\[
\tilde{\eta}(p,s,t) = \tau(s,t),
\]
and the parallel transport induced by the lift of the vector field \( \partial_\partial \) consists of the contactomorphisms on \( I^2 \) obtained through Moser’s argument. □

Lemma 7.6 allows us, up to a fibre preserving diffeomorphism, to consider the contact structure near the central fibre of \( f : N \to U \) to be given as the pull–back of a contact structure \( \alpha_{(s,t)} + H(p,s,t)ds, \) satisfying \( \partial_s \alpha_{(s,t)} = 0. \)

Let us gather these results and conclude Proposition 7.3.

**Proof of Proposition 7.3** Both Lemma 7.4 and Lemma 7.6 provide a vertical almost contact structure \( (\tilde{\xi}, \tilde{\omega}) \) that coincides with the original almost contact distribution \( (\xi, \omega) \) away from a neighborhood of the fibres over the intersection points and in a neighborhood of the boundary \( \partial N. \) This almost contact structure is also homotopic to the original distribution on \( M \) and parallel transport through a lift of at least one direction to the almost contact connection of \( f : N \to U \) consists of contactomorphisms.

In conclusion, there exists \( \varepsilon > 0, \) a neighborhood \( V(z) \) of \( z \) and a trivializing map \( \psi \) such that the following diagram commutes:
\[
\begin{array}{ccc}
f^{-1}(V(z)) & \xrightarrow{\psi} & F \times B^2(\varepsilon) \\
\downarrow f & & \downarrow \\
V(z) & \xrightarrow{\phi} & B^2(\varepsilon)
\end{array}
\]
and the fibre of \( f \) is a contact manifold \( F \) with boundary. Since the map \( \psi \) is defined as a flow, a smaller neighborhood might be required but such neighborhood exists because of compactness. Lemma 7.6 has been used, hence \( \psi_\varepsilon \xi \) satisfies the hypothesis of Lemma 7.2 for some small rectangle \( I^2 \subset B^2(\varepsilon/2). \) Note that the set \( G \) can be taken to be a suitable neighborhood of the boundary \( \partial F \times B^2(\varepsilon). \) Applying Lemma 7.2 to the domain \( F \times I^2 \) and interpolating back to the almost contact structure \( (\psi_\varepsilon \xi, \psi_\varepsilon) \) in \( F \times ((B^2(\varepsilon) \setminus B^2(\varepsilon/2)) \) we obtain an almost contact structure whose pull–back through \( \psi \) has the required properties once we extend it to \( M \) by \( \xi. \) □

### 7.3. Deformation along curves

Once we have achieved the contact condition in a neighborhood of the fibres over the 0–skeleton, we proceed with a neighborhood of the fibres over the 1–skeleton. Let \( T \) denote a neighborhood of \( T. \)

**Proposition 7.7.** Let \( (M, \xi, \omega) \) be a vertical contact structure with respect to a good ace fibration \( (f,C,E) \) and \( T \) an adapted family. Suppose that \( (M, \xi) \) is a contact structure on a neighborhood \( \mathcal{O} \) of the fibres over the 0–cells of \( T. \) Then there exists a deformation \( (\xi', \omega') \) of \( (\xi, \omega) \) relative to \( C, E \) and \( \mathcal{O} \) over the 0–cells of \( T \) such that \( (f,C,E) \) is a good ace fibration for \( (\xi', \omega') \) and \( \xi' \) is a contact structure in the pre–image of \( T. \)

Let \( S \) be a small neighborhood of the set of fibres over \( \mathbb{T} \setminus \mathcal{O}. \) See Figure [11] The argument applied over \( \mathcal{O} \) in the previous subsection works analogously when applied to \( S. \) Thus, no detailed proof is given. The only subtlety lies in the appropriate choice of the compact set...
Let \( z, w \in \mathbb{CP}^1 \) with corresponding neighborhood \( \mathcal{O}_z, \mathcal{O}_w \); we focus on a line segment \( S \subset |T| \) joining these two points. Let \( (\phi, U) \) be a local chart around \( S \setminus (\mathcal{O}_z \cup \mathcal{O}_w) \) with cartesian coordinates \((s, t)\) such that

\[
\phi(U) = [-\varepsilon, \varepsilon] \times [0, 1], \quad \phi(S) = \{0\} \times [0, 1].
\]

The existence of a suitable trivialization is obtained with the same arguments used in Lemma 7.6. In precise terms, we use the following pair of lemmata.

**Lemma 7.8.** Let \((F, \Xi)\) be a contact manifold with boundary and \( f : \mathcal{O} \longrightarrow [-1, 1] \times [0, 1] \) a fibre bundle with typical fibre \( F \). Consider a vertical almost contact structure \((\mathcal{O}, \xi)\) restricting to \((F, \partial F, \Xi)\) in the fibre. Suppose that \((\mathcal{O}, \xi)\) is a contact structure in a collar neighborhood \( \mathcal{C}_F \) of the boundary \( \partial(f^{-1}([-1, 1] \times [0, 1])) \setminus \partial(f^{-1}(\partial([-1, 1] \times [0, 1]))). \) Then there exists \( \varepsilon \in \mathbb{R}^+ \), a flow \( \varphi_t \) on \( F \) and a fibre–preserving diffeomorphism \( \tau : \tau^{-1}(F \times [-\varepsilon, \varepsilon] \times [0, 1]) \longrightarrow F \times [-\varepsilon, \varepsilon] \times [0, 1] \), \( \tau^{-1}(F \times [-\varepsilon, \varepsilon] \times [0, 1]) \subset \mathcal{O} \)

\[
p \mapsto \tau(p) = (\varphi_{-\|f(p)\|}(p), f(p)),
\]

such that \( \tau_\ast(\xi) \) is still a vertical contact structure with respect to the product fibration over \([-\varepsilon, \varepsilon] \times [0, 1] \), and the family \( \{\xi(s,t) = \tau_\ast(\xi)|_{f^{-1}(s,t)}\}_{(s,t) \in (-\varepsilon, \varepsilon) \times [0, 1]} \) is constant when restricted to \( \tau(\mathcal{C}_F) \). \( \square \)

This is proven as Lemma 7.4 using the vector field \( \partial_s \) instead of the radial vector field \( \partial_r \).

**Lemma 7.9.** There exist an arbitrarily small neighborhood \( S \) of \( S \) and a horizontal deformation of the vertical almost contact structure \((\xi, \omega)\) supported in the pre–image of \( S \), relative to the pre–images of \( S \cap \mathcal{O}_z \) and \( S \cap \mathcal{O}_w \), and conforming the following properties:

- The distribution is deformed relative to \( U(f) \), where it was already a contact distribution.
- There exists a trivialization \((\phi, U)\) such that the parallel transport of the associated almost contact connection along the vector field \( \phi^\ast \partial_s \) consists of contactomorphisms.
This is proven with the same methods used in subsection 7.2.

**Proof of Proposition 7.7.** Trivialize the vertical contact fibration using Lemma 7.9. Choose the coordinates in the trivialization in such a way that the curves which provide the lift of $\phi^*\partial_s$ have at most one of the ends in the fibres over the 0–skeleton. See Figure 12. This allows us to choose a compact set $G$ containing the fibres over the two endpoints plus a neighborhood of the boundary of all the fibers such that the intersection of $G$ with any such arc is connected. There might be the need to progressively shrink the neighborhoods of the fibres over the 0–skeleton. Apply Lemma 7.2 to produce a contact structure in a neighborhood of the fibres over 1–skeleton without perturbing the existing contact structure in a small neighborhood of fibres over the endpoints.

\[ \square \]

8. **Fibrations over the 2–disk.**

Let $(F, \xi_v)$ be a contact 3–manifold, $\xi_v = \ker \alpha_v$ and $D^2$ a 2–disk. In this Section we study contact structures on the product manifold $F \times D^2$. Consider the coordinates $(p, r, \theta) \in F \times D^2$. The existence part of Theorem 1.1 can be essentially reduced to the existence of a contact structure on $F \times D^2$ restricting to a prescribed contact structure on a neighborhood of the boundary $F \times \partial D^2$. This is explained in Section 9.

Fix an $\varepsilon \in (0, 1)$ and consider $H \in C^\infty(F \times D^2(1))$ to be a smooth function such that $\partial_r H > 0$ for $r \in (1 - \varepsilon, 1]$. Then the 1–form

\[ \alpha = \alpha_v + H(p, r, \theta)d\theta \]

defines a distribution $\xi = \ker \alpha$. It can be endowed with the symplectic form

\[ \omega = d\alpha_v + (1 - \tau(r)) \cdot rdr \wedge d\theta + \tau(r)dH \wedge d\theta, \]

where $\tau : [0, 1] \rightarrow [0, 1]$ is an strictly increasing smooth function such that

$\tau(x) = 0$ for $x \in [0, 1 - \varepsilon]$ and $\tau(x) = 1$ for $x \in [1 - \varepsilon/2, 1]$.

Then $(\xi, \omega)$ is an almost contact structure on $F \times D^2(1)$ which is a contact structure on the neighborhood $F \times (1 - \varepsilon/2, 1] \times S^1$ of the boundary $F \times \partial D^2(1)$.

The main result in this Section is the following:
Theorem 8.1. Let \((F,\xi_v)\) be a contact 3–manifold with \(c_1(\xi_v) = 0\), \(\xi_v = \ker \alpha_v\) and \(L\) a transverse link. Given \(\varepsilon \in (0, 1)\), consider a function \(H \in C^\infty(F \times \mathbb{D}^2(1))\) such that \(\partial_r H > 0\) in \(r \in (1 - \varepsilon, 1)\), and the almost contact structure

\[(\xi, \omega) = (\ker(\alpha_v + H(p, r, \theta)d\theta), d\alpha_v + (1 - \tau(r)) \cdot rdr \wedge d\theta + \tau(r) dH \wedge d\theta),\]

where \(\tau\) is the function described above.

Then there exists a 1–parametric family of almost contact structures \(\{(\xi_t, \omega_t)\}\), constant along the boundary \(F \times \partial \mathbb{D}^2(1)\) and with \((\xi_0, \omega_0) = (\xi, \omega)\) such that:

a. \((\xi_1, \omega_1) = (\ker \alpha, d\alpha)\) is a contact structure for some contact form \(\alpha\) on \(F \times \mathbb{D}^2(1)\).

b. The submanifold \(L \times \mathbb{D}^2(1)\) is a contact submanifold of \((F \times \mathbb{D}^2(1), \xi_1)\) and the induced contact structure is a full Lutz twist along \(L \times \{0\}\).

In coordinates \((z, r, \theta) \in L \times \mathbb{D}^2(1)\), the contact structure obtained by a full Lutz twist of a neighborhood of \(L\) along \(L\) is described as

\[\xi_{L \times \mathbb{D}^2(1)} = \ker(\cos(2\pi r)dz + r \sin(2\pi r)d\theta).\]

This theorem is used to conclude Theorem 1.1 in Section 9. In brief, it is used to deform the almost contact structure over the 2–cells of the decomposition associated to an adapted family \(T\) of a vertical good ace fibration \((f, C, E)\). In this description of the fibration over the 2–cells, the part corresponding to the exceptional divisors is the submanifold \(L \times \mathbb{D}^2(1)\). Although the deformation in the statement is not relative to a neighborhood of them, the resulting contact structure is described in the part b. of Theorem 8.1.

Example. Suppose that the function \(H \in C^\infty(F \times \mathbb{D}^2(1))\) also satisfies

\[H(p, 1, \theta) > 0, \text{ for all } (p, \theta) \in F \times \mathbb{S}^1.\]

The contact condition for the initial form \(\alpha_v + H(p, r, \theta)d\theta\) is \(\partial_r H > 0\). Consider a smooth family \(\{H_t\}_{t \in [0, 1]}\) of functions in \(F \times \mathbb{D}^2(1)\) such that

\[H_0 = H, \quad H_1(p, 0, \theta) = 0, \quad \partial_r H_1 > 0 \text{ for } r \in (0, 1) \text{ and } H_t(p, 1, \theta) = H_0(p, 1, \theta).\]

Suppose that \(H_1\) vanishes quadratically at the origin. Then \(\alpha_t = \alpha_v + H_t(p, r, \theta)d\theta\) is a family of almost contact distributions constant along the boundary \(F \times \partial \mathbb{D}^2(1)\) such that \(\ker \alpha_1\) is a contact structure. The corresponding symplectic structures on \(\ker \alpha_t\) is readily constructed as in the previous discussion, and an interpolation to the symplectic form \(\alpha_v + dH_1 \wedge d\theta\) is required to obtain the almost contact structure \((\ker \alpha, d\alpha)\). This contact structure does conform the first property in Theorem 8.1, but not necessarily the second one. The construction in the proof differs from that in this example and satisfies both properties.

Theorem 8.1 also covers harder cases, such as almost contact distributions defined by functions \(H(p, r, \theta)\) with positive and negative values along \(F \times \partial \mathbb{D}^2(1)\).

8.1. The model. In this subsection we describe the model used to obtain the contact structure in the statement of Theorem 8.1.

Consider the manifold \(F \times \mathbb{S}^2\). The submanifolds

\[i_0 : F_0 = F \times \{(1, 0, 0)\} \to F \times \mathbb{S}^2 \quad \text{and} \quad i_\infty : F_\infty = F \times \{(-1, 0, 0)\} \to F \times \mathbb{S}^2\]

are referred to as the fibres at zero and infinity. A construction made relative to \(F_\infty\) should be thought as construction on \(F \times \mathbb{D}^2(1)\), relative to the boundary. In the manifold \(\mathbb{S}^1 \times \mathbb{S}^2\) there exists a unique tight contact structure and a unique overtwisted contact structure isotopic to it. The latter is obtained by performing a full Lutz twist in the former along \(\mathbb{S}^1 \times \{0\}\). This is said to be the standard overtwisted structure on \(\mathbb{S}^1 \times \mathbb{S}^2\).
The basic geometric construction used to prove Theorem 8.1 is the content of the following result. A minor enhancement of the Proposition is also required, it is explained in Corollary 8.3.

**Proposition 8.2.** Let \((F,\xi_v)\) be a contact 3–manifold with \(c_1(\xi_v) = 0\), \(\xi_v = \ker \alpha_v\) and \(L\) a transverse link. Consider the manifold \(F \times S^2\), \(\omega_{S^2}\) the standard area form on \(S^2\) and the almost contact structure 

\[(\xi, \omega) = (\ker \alpha_v, d\alpha_v + \omega_{S^2}).\]

Then there exists a contact structure \(\xi_f = \ker \alpha_f\) on \(F \times S^2\) conforming the properties:

a. The contact form \(\alpha_f\) restricts to the initial contact form at the fibres \(F_0\) and \(F_{\infty}\):

\[i^*_0 \alpha_f = \alpha_v\text{ and }i^*_\infty \alpha_f = \alpha_v\]

b. Consider the inclusion \(i_L : L \times S^2 = \bigsqcup(S^1 \times S^2) \longrightarrow F \times S^2\). Then the contact form \(i_L^* \alpha_f\) is the standard overtwisted form on each \(S^1 \times S^2\).

c. The almost contact structures \((\xi, \omega)\) and \((\ker \alpha_f, d\alpha_f)\) are homotopic relative to \(F_{\infty}\).

**Proof.** This is a rather long proof. It is divided according to the construction and the verification of each of the three properties.

**Construction.** Since \(c_1(\xi_v) = 0\), there exist a global framing \(\{X_1, X_2 \in \Gamma(\xi_v)\}\) of the contact distribution \(\xi_v\). Denote by \(X_0\) the Reeb vector field associated to the contact form \(\alpha_0 = \alpha_v\). Therefore \(\{X_0, X_1, X_2\}\) is a global framing of \(TF\). Let \(\{\alpha_0, \alpha_1, \alpha_2\}\) be the dual framing. Denote the standard embedding of the 2–sphere as \(e = (e_0, e_1, e_2) : S^2 \longrightarrow \mathbb{R}^3\). It is a computation to verify that

\[
\lambda = e_0 \cdot \alpha_0 + e_1 \cdot \alpha_1 + e_2 \cdot \alpha_2
\]

is a contact form on \(F \times S^2\). The important properties are that \(\{\alpha_0, \alpha_1, \alpha_2\}\) is a framing and the map \(e\) is a star–shaped embedding.

In spherical coordinates \((t, \theta) \in [0, 1] \times [0, 1]\) the embedding can be described as

\[
\begin{align*}
e_0(t, \theta) &= \cos(\pi t), \\
e_1(t, \theta) &= \sin(\pi t) \cos(2\pi \theta), \\
e_2(t, \theta) &= \sin(\pi t) \sin(2\pi \theta).
\end{align*}
\]

Note that \(F_{\infty} = F \times (-1, 0, 0)\) and \(F_0 = F \times (1, 0, 0)\) are contactomorphic contact submanifolds of \((F \times S^2, \ker \lambda)\) with trivial normal bundle. Consider two copies of \(F \times S^2\), we can perform a contact fibered sum along their \(F_{\infty}\) fibres, see [Ge]. This operation is done in order to obtain two fibres with the contact form \(\alpha_0\). Those coming from the two zero fibres \(F_0\) in the two copies of \(F \times S^2\). Let us provide an explicit equation for the contact form in this connected sum.

A tentative modification of \(\lambda\) is obtained by considering the following map

\[
\begin{align*}
\kappa_0(t, \theta) &= \cos(2\pi t), \\
\kappa_1(t, \theta) &= \sin(2\pi t) \cos(2\pi \theta), \\
\kappa_2(t, \theta) &= |\sin(2\pi t)| \sin(2\pi \theta),
\end{align*}
\]

and the 1–form \(\kappa_0 \cdot \alpha_0 + \kappa_1 \cdot \alpha_1 + \kappa_2 \cdot \alpha_2\). Due to the appearance of the absolute value this form is just continuous. Observe though that in the smooth area it is a contact form. Let us perturb it to a smooth 1–form.

Define a smooth map \(t : [0, 1] \longrightarrow [0, 1]\) such that:

\[
t(0) = 0, \ t(1/2) = 1/2, \ t(1) = 1, \ t'(v) > 0 \text{ for } v \in [0, 1/2) \cup (1/2, 1] \text{ and } t^{(k)}(1/2) = 0 \forall k \in \mathbb{N}.
\]
This allows us to reparametrize the sphere with coordinates \((v, \theta) \in [0, 1] \times [0, 1]\). The following map is denoted by \((e_0, e_1, e_2)\) in order to ease notation. This should not lead to confusion since the map formerly referred to as \((e_0, e_1, e_2)\) is not to be considered again. Consider the smooth map
\[
e_0(v, \theta) = \cos(2\pi t(v)),
\]
\[
e_1(v, \theta) = \sin(2\pi t(v)) \cos(2\pi \theta),
\]
\[
e_2(v, \theta) = |\sin(2\pi t(v))| \sin(2\pi \theta).
\]

It is indeed smooth because \(t^{(k)}(1/2) = 0\). This almost provides the desired 1–form for the fibre connected sum. Define the smooth function \(h(v) = v(1 - v)\sin(2\pi v)\) and the 1–form \(\eta = c \cdot h(v)d\theta\), where \(c\) is a small positive constant.

**Assertion.** There exists a choice of \(c \in \mathbb{R}^+\) such that the 1–form defined as
\[
\alpha_f = e_0\alpha_0 + e_1\alpha_1 + e_2\alpha_2 - \eta
\]
is a contact form over the fibre connected sum of two copies of \(F \times S^2\) along the fibres \(F_\infty\).

This concludes the construction of the contact form in the manifold \(F \times S^2\) obtained in the Theorem. The contact form \(\alpha_f\) also conforms property a. in the statement of the Theorem.

**Proof of Assertion.** Consider the following volume form \(\nu = \sin(\pi v)dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2\) on \(F \times S^2\) and compute the exterior differential
\[
d\alpha_f = de_0 \wedge \alpha_0 + de_1 \wedge \alpha_1 + e_1 \wedge \alpha_2 + e_0d\alpha_0 + e_1d\alpha_1 + e_2d\alpha_2 - d\eta.
\]
The contact condition states that \(\alpha_f \wedge (d\alpha_f)^2\) is a positive of \(\nu\). Let us express it as
\[
\alpha_f \wedge (d\alpha_f)^2 = \eta_1 + \eta_2 + \eta_3,
\]
where \(\eta_1, \eta_2, \eta_3\) are the following 5–forms:
\[
\eta_1 = \begin{vmatrix}
e_0 & e_1 & e_2 \\
\partial_t e_0 & \partial_t e_1 & \partial_t e_2 \\
\partial_\theta e_0 & \partial_\theta e_1 & \partial_\theta e_2 \\
\end{vmatrix} t'(v)^2 dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2 = 4\pi^2 |\sin(2\pi t(v))| (t'(v))^2 dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2,
\]
\[
\eta_2 = -e_0^2 \cdot h'(v) \cdot \alpha_0 \wedge d\alpha_0 \wedge dv \wedge d\theta,
\]
\[
\eta_3 = -\sum_{i+j \geq 1} (e_i \cdot e_j \cdot h'(v)) \cdot \alpha_i \wedge d\alpha_j \wedge dv \wedge d\theta + \sum_{i,j} (e_i \cdot h(v)) \cdot de_j \wedge d\alpha_i \wedge \alpha_j \wedge d\theta.
\]
The indices belong to \(i, j \in \{0, 1, 2\}\). Evaluating at \(v = 1/2\) we obtain:
\[
\eta_2(p, 1/2, \theta) = \pi \alpha_0 \wedge \alpha_1 \wedge \alpha_2,
\]
\[
\eta_1(p, 1/2, \theta) = \pi \alpha_0 \wedge \alpha_1 \wedge \alpha_2,
\]
\[
\eta_3(p, 1/2, \theta) = 0.
\]

Therefore, there is a small constant \(\delta > 0\) such that the 5–form \(\eta_2 + \eta_3\) is a positive volume form in the region \(F \times [1/2 - \delta, 1/2 + \delta] \times [0, 1]\). The function \(t(v)\) is strictly increasing except at \(v = 1/2\). Hence, there exists a constant \(B > 0\) such that \(t'(v) > B\) for any \(v \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1]\).
Let us write \( \eta_1(p,v,\theta) = g_1(p,v,\theta)\nu \) and \( \eta_2 + \eta_3 = g_2(p,v,\theta)\nu \). There exist constants \( C, M \in \mathbb{R}^+ \) such that \( g_1 > C > 0 \) for \( v \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1] \), and \( |g_2| \leq M \).

Choose the initial constant \( c \in \mathbb{R}^+ \) to satisfy \( cM \leq C \). Then we obtain the following bound for \( v \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1] \):

\[
\alpha_f \wedge (d\alpha_f)^2 = \eta_1 + c\eta_2 + c\eta_3 = (g_1 + cg_2)\nu > C - cM \geq 0.
\]

Hence the form \( \alpha_f \) is a contact form in this region. The following bound holds in the remaining region \( v \in [1/2 - \delta, 1/2 + \delta] \):

\[
\alpha_f \wedge (d\alpha_f)^2 = \eta_1 + c\eta_2 + c\eta_3 = (g_1 + cg_2)\nu > cg_2 \geq 0.
\]

Thus \( \alpha_f \) is a contact form in the fibre connected sum \( F \times S^2 \).

**Property b.** Choose the contact form \( \alpha_v \) associated to \( \xi_v \) such that its Reeb vector field \( X_0 \) is tangent to the link \( L \). Restricting the contact form \( \alpha_f \) in the equation (3) to the submanifold we obtain

\[
i_L^*(\alpha_T) = \cos(2\pi t(v))dz - cv(1-v)\sin(2\pi v)d\theta,
\]

where \((z, v, \theta) \in S^1 \times S^2\). This is an equation of the standard overtwisted contact structure on each \( S^1 \times S^2 \). Indeed, consider \( a(v) = \cos(2\pi t(v)) \) and \( b(v) = v(1-v)\sin(2\pi v) \). Then the curve parametrized by \((a(v), b(v))\) rotates once around the origin and the tangent vector field \((a'(t), b'(t))\) is transverse to the radial direction, i.e. \( \partial_r \), on \((0, 1)\).

**Property c.** Let \( f_F : F \to [0, 1] \) be a Morse function on the 3–manifold \( F \) with a single minimum \( q \in F \). Then

\[
f(p, v, \theta) = f_F(p) - (1 + f_F(p))v^2 : F \times S^2 \to [-1, 1]
\]

is a smooth Morse function on \( F \times S^2 \) whose critical points belong to the central fibre \( F_0 \). Let us use the associated cell decomposition relative to the level \( f^{-1}([-\infty, -1]) = F_\infty \). It is generated by the descending manifolds associated to each critical point. It has a unique 2–cell \( \sigma_q^2 = \{q\} \times (S^2 \setminus \{\infty\}) \), corresponding to the critical point \((q, 0, 0)\).

Due to Lemma 2.4, a pair of almost contact distributions homotopic over the disk \( \sigma_q^2 \) relative to its boundary are homotopic on the 5–manifold \( F \times S^2 \). To conclude Property c. we verify that such relative homotopy exists along \( \sigma_q^2 \). The almost contact distribution \( \xi \) in the statement of the Proposition can be written as \( \xi = \ker \alpha_v \oplus TS^2 \). Its symplectic structure is induced by the symplectic structure on each of the factors. Note that both \( \ker \alpha_v \) and \( TS^2 \) are \( \text{rk}_{\mathbb{R}} = 2 \) symplectic bundles. This is tantamount to \( \text{rk}_{\mathbb{R}} = 2 \) oriented bundles.

Consider a trajectory \( \gamma \) of the Reeb flow through \( q \)

\[
\gamma : (-\varepsilon, \varepsilon) \to F, \quad \gamma(0) = q.
\]

The submanifold \((V, \xi_{ot}) = (\gamma \times S^2, \xi_f|_{\gamma \times S^2})\) is a contact submanifold of the contact manifold \((F \times S^2, \ker \alpha_f)\). A contact form is given by the equation (4). As suggested by the notation, the contact form \( \alpha_{ot} = \alpha_f|_V \) defines the standard overtwisted structure on \((-\varepsilon, \varepsilon) \times (S^2 \setminus \{\infty\})\).

Hence the two subbundles of \( TV \)

\[
\xi_{ot} \to \sigma_q^2, \quad TS^2 \to \sigma_q^2
\]

are homotopic as oriented subbundles relative to the boundary of the disk. Thus relative homotopic as symplectic bundles. This provides a homotopy in the 2–dimensional horizontal part. Let us deal with the vertical bundle.
The initial vertical subbundle is $\xi_v = \ker \alpha_v$, it does satisfy the splitting

$$\xi_v|_{\sigma_q^2} \oplus TV|_{\sigma_q^2} = T(F \times S^2)|_{\sigma_q^2}.$$ 

The resulting vertical subbundle in the distribution $\xi_f$ can be constructed as the symplectic orthogonal subbundle $\nu_{ot}$ of $\xi_{ot}$. This yields the decomposition

$$\nu_{ot}|_{\sigma_q^2} \oplus TV|_{\sigma_q^2} = T(F \times S^2)|_{\sigma_q^2}.$$ 

The space of rank–2 oriented vector bundles transverse to the rank–3 vector bundle $TV$ is contractible. Hence $\nu_{ot}|_{\sigma_q^2}$ is homotopic to $\xi_v|_{\sigma_q^2}$ as rank–2 symplectic distributions.

On the unique 2–cell $\sigma^2_q$ both splittings $\xi = \xi_v \oplus TS^2$ and $\xi_f = \nu_{ot} \oplus \xi_{ot}$ hold. Since the subbundles are pairwise homotopic as symplectic distributions, $\xi$ and $\xi_f$ are also homotopic as symplectic distributions.

In the proof of Property c. of Proposition 8.2 we have only used the 2–skeleton to verify the statement. Obstruction theory ensures that this is enough. There is an alternative geometric approach to produce the homotopy. Indeed, the Reeb trajectories of $\alpha_v$ produce a foliation $\mathcal{L}$ on $F$. This induces a foliation $\mathcal{L} \times \mathbb{D}^2$ with 3–dimensional contact leaves. The argument in the proof of Property c. can be made parametric to construct an explicit almost contact homotopy.

The norm of the function $H$ in the statement of Theorem 8.1 does translate into a geometric feature. This is the size of a certain neighborhood. This is explained in the subsequent subsection. Let us enhance the conclusion of Proposition 8.2 in order to obtain an arbitrarily large contact neighborhood of a fibre.

**Property d.** Let $R \in \mathbb{R}^+$ be given. There exists a neighborhood $U_\infty$ of the fibre $F_\infty$ and a trivializing diffeomorphism $\psi: F \times \mathbb{D}^2(R) \to U_\infty$ such that

- $\psi(F \times \{0\}) = F_\infty$,
- $\psi^* \alpha_f = \alpha_v + r^2 \, d\theta$.

This property could have been included in the statement of Proposition 8.2. It is stated apart to ease the comprehension.

**Corollary 8.3.** There exists a contact manifold $(F \times S^2, \xi_f = \ker \alpha_f)$ conforming a. to d.

**Proof.** The contact structure $(F \times S^2, \xi_f = \ker \alpha_f)$ obtained in Proposition 8.2 does satisfy properties a.– c. Let us modify it in order to satisfy Property d. The contact neighborhood theorem provides a neighborhood $U_\infty$ of the fiber $F_\infty$ and a contactomorphism $\psi_\varepsilon: F \times \mathbb{D}^2(\varepsilon) \to U_\infty$, for some $\varepsilon \in \mathbb{R}^+$. In case $R \leq \varepsilon$ the statement follows.

Suppose $R \geq \varepsilon$. Let $k \in \mathbb{N}$ be an integer and consider the ramified covering

$$\phi_k: F \times S^2 \to F \times \mathbb{CP}^1 \quad \mapsto \quad (p, z) \mapsto (p, z^k).$$

The branch locus consists of the fibres $F_0$ and $F_\infty$. Both fibres are contact submanifolds in $(F \times S^2, \ker \alpha_f)$ and we can lift the contact form to a contact form $\alpha_f^k = \phi_k^* \alpha_f$ in the domain of the covering map. Lifting the formula (3), we obtain

$$\alpha_f^k = \cos(2\pi t(v))\alpha_0 + \sin(2\pi t(v))\cos(2\pi k\theta)\alpha_1 + |\sin(2\pi t(v))|\sin(2\pi k\theta)\alpha_2 + k\eta$$

Hence properties a.– c. are still satisfied by the contact structure $\ker \alpha_f^k$. Regarding Property d, observe that $\psi^* \alpha_f^k = \alpha_v + k r^2 \, d\theta$. Consider the scaling diffeomorphism $g_k: F \times \mathbb{D}^2(\sqrt{k} \cdot \varepsilon) \to F \times \mathbb{D}^2(\varepsilon)$

$$g_k: (p, r, \theta) \mapsto (p, r/\sqrt{k}, \theta).$$
Then the trivializing diffeomorphism $\psi_k \circ g_k$ satisfies $(\psi_k \circ g_k)^* \alpha_f^k = \alpha_v + r^2 d\theta$. Choose $k \in \mathbb{N}$ such that $\sqrt{k} \cdot \varepsilon \geq R$ to conclude the statement. \hfill $\square$

To ease notation, we can refer to the contact structures resulting either of Proposition 8.2 or Corollary 8.3 as $\xi_f$. Since the latter has better properties than the former, $\xi_f$ refers to that in Corollary 8.3.

8.2. The proof. In this subsection we conclude the proof of 8.1. The essential geometric ideas have been introduced in Proposition 8.2. The necessary details to conclude are provided.

Let us introduce a definition. It is given in order to stress the relevance of the size in a neighborhood.

**Definition 8.4.** Let $(F, \xi_v = \ker \alpha_v)$ be a contact manifold. For $A \in \mathbb{R}^+$, the manifold $F \times [-A, A] \times \mathbb{S}^1$ with the contact structure $\alpha_A = \alpha_v + td\theta$ is called the $A$–standard contact band associated to $(F, \ker \alpha_v)$.

The role of this definition is elucidated in the following lemma.

**Lemma 8.5.** Let $(F, \xi_F)$ be a contact manifold, $\xi_F = \ker \alpha_F$. Consider a contact manifold $(F \times [0, 1] \times \mathbb{S}^1, \xi)$ with contact form $\alpha_F + H d\theta$, $H \in C^\infty(F \times [0, 1] \times \mathbb{S}^1)$.

Suppose that $|H| < A$, for some $A \in \mathbb{R}^+$. Then, there exists a strict contact embedding of $(F \times [0, 1] \times \mathbb{S}^1, \alpha)$ in the $A$–standard contact band associated to $(F, \alpha_F)$.

**Proof.** The embedding is defined as

$$
\Psi_A : F \times [0, 1] \times \mathbb{S}^1 \to F \times [-A, A] \times \mathbb{S}^1,
(p, t, \theta) \mapsto (p, H(p, t, \theta), \theta).
$$

The remaining ingredient for the proof of Theorem 8.1 is the subsequent lemma.

Let $l \in \mathbb{R}^+$ be a constant. Consider a smooth function $\kappa_l : [0, 2l + 1] \to [0, l]$ with

$$
\kappa_l(r) = 0 \text{ for } r \in [0, l], \quad \kappa_l(r) = r - l - 1 \text{ for } r \in [2l, 2l + 1].
$$

Consider $(r, \theta) \in \mathbb{D}_l^2$ to be polar coordinates for the 2–disk $\mathbb{D}_l^2$ of radius $2l + 1$.

**Lemma 8.6.** Let $(F, \xi_v)$ be a contact 3–manifold with $c_1(\xi_v) = 0$, $\xi_v = \ker \alpha_v$, $C \in \mathbb{R}^+$ and $L$ a transverse link. Consider the standard area $\omega_\mathbb{D}$ on the 2–disk $\mathbb{D}_l^2$ and the almost contact structure on $F \times \mathbb{D}_l^2$ described as

$$(\xi, \omega) = (\ker(\alpha_v + \kappa_l(r)d\theta), d\alpha_v + \omega_\mathbb{D}).$$

Then there exists a contact structure $\xi_1 = \ker \alpha_1$ on $F \times \mathbb{D}_l^2$ such that:

A. The region $F \times [1, 2l + 1]$ is an $l$–standard contact band for $(F, \ker \alpha_v)$:

$$\alpha_1|_{F \times [1, 2l + 1]} = \alpha_v + (r - l - 1)d\theta.$$  

B. Consider the inclusion $i_L : L \times \mathbb{D}_l^2 = \bigsqcup (\mathbb{S}^1 \times \mathbb{D}_l^2) \to F \times \mathbb{D}_l^2$. Then the contact form $i_L^*\alpha_1$ is the standard overtwisted form on each $\mathbb{S}^1 \times \mathbb{D}_l^2$.

C. $(\xi, \omega)$ and $(\xi_1, d\alpha_1)$ are homotopic relative to the boundary $F \times \partial \mathbb{D}_l^2$.

**Proof.** Consider Property d with radius $R = \sqrt{l}$. Let $(F \times \mathbb{S}^2, \xi_f = \ker \alpha_f)$ be the contact manifold obtained in Corollary 8.3. Then there exists a contact neighborhood $U_\infty$ of the fibre $F_\infty$ and a trivializing diffeomorphism

$$
\psi : F \times [0, \sqrt{l}] \times \mathbb{S}^1 \to U_\infty \text{ such that } \psi^*\alpha_f = \alpha_v + r^2 d\theta.
$$
Note that \( \psi \) also identifies \( \psi : F \times (0, \sqrt{l}] \times S^1 \to U_\infty \setminus F_\infty \).

Define the following map
\[
m : F \times [-l, 0) \times S^1 \to F \times (0, \sqrt{l}] \times S^1, \quad m(p, x, \theta) = (p, \sqrt{-x}, -\theta).
\]
It satisfies \((\psi \circ m)^* \alpha_f = \alpha_v + r d\theta\). This form extends to the region \( F \times [-l, l] \times S^1 \) with the same expression.

Then the manifold \( F \times \mathbb{D}^2 \) is obtained by gluing the regions \( F \times [0, \sqrt{l}] \times S^1 \) and \( F \times [-l, l] \times S^1 \) via the contactomorphism \( \psi \circ m \). The construction implies that Property A holds. Properties B and C follow from Properties b and c in Corollary 8.3 since the manifold \((F \times S^2) \setminus F_\infty\) satisfies them.

**Proof of Theorem 8.1.** The function \( H \) is \( C^0 \)-bounded on the compact manifold \( F \times \mathbb{D}^2(1) \). Let \( l \in \mathbb{R}^+ \) be an upper bound, \( \|H\|_{C^0} < l \). Consider a smooth function \( h \in C^\infty(F \times [0, 1] \times S^1) \) such that
\[
\begin{align*}
- h(p, r, \theta) &= 0 \text{ for } r \in [0, 1 - 2\varepsilon], \\
- h(p, r, \theta) &= r - l - (1 - \varepsilon) \text{ for } r \in [1 - \varepsilon, 1 - 3\varepsilon/4], \\
- \partial_r h &> 0 \text{ for } r \in [1 - 3\varepsilon/4, 1 - \varepsilon/2], \\
- h(p, r, \theta) &= H(p, r, \theta) \text{ for } r \in [1 - \varepsilon/2, 1].
\end{align*}
\]

The almost contact structure \((\xi, \omega)\) is homotopic relative to the boundary to the almost contact structure defined by
\[
(\xi_h, \omega_h) = (\ker(\alpha_v + h(p, r, \theta)), d\alpha_v + (1 - \tau(r)) \cdot rdr \wedge d\theta + \tau(r) dh \wedge d\theta).
\]
The homotopy is provided by a relative homotopy between the functions \( h(p, r, \theta) \) and \( H(p, r, \theta) \) and Lemma 2.5. Hence the departing the almost contact structure can be considered to be \((\xi_h, \omega_h)\).

The neighborhood \( F \times (1 - \varepsilon, 1] \times S^1 \) of the boundary \( F \times \partial \mathbb{D}^2(1) \) is a contact manifold. By Lemma 8.5, it embeds in an \( l \)-standard contact band \( F \times [-l, l] \times S^1 \). Denote such an embedding by \( \phi \).

Consider the almost contact manifold \((F \times \mathbb{D}^2, \ker \alpha_1)\) in the statement of Lemma 8.6 Property A implies the existence of a contactomorphism \( \iota \) embedding the \( l \)-standard contact band in a neighborhood of size \( 2l \) of the boundary of \( F \times \mathbb{D}^2 \).

The composition \( \iota \circ \phi \) embeds a neighborhood of the boundary \( F \times \{1 - \varepsilon\} \times S^1 \) via

\[
(\iota \circ \phi)(p, r, \theta) = (p, r + \varepsilon, \theta).
\]

The required contact structure in the statement of Theorem 8.1 is obtained by extending the contact structure induced by \((F \times \mathbb{D}^2, \ker \alpha_1)\) to the area \( F \times [0, 1 - \varepsilon] \times S^1 \). This is achieved with a coordinate transformation
\[
F \times [0, 1] \times S^1 \to F \times [0, 1 - \varepsilon] \times S^1, \quad (p, r, \theta) \mapsto (p, c(r), \theta),
\]
where \( c : [0, 1] \to [0, 1 - \varepsilon] \) is a smooth function such that
\[
\begin{align*}
- c(t) &= t \text{ near } t = 0, \\
- c(t) &= t - \varepsilon \text{ near } t = 1, \\
- c'(t) &> 0 \text{ for } t \in (0, 1].
\end{align*}
\]
Property B in Lemma 8.6 implies Property b in the Theorem. Property C ensures that the obtained almost contact structure is homotopic to the initial almost contact structure relative to the boundary.
9. Horizontal Deformation II

9.1. Contact Structure in the fibration.

**Theorem 9.1.** Let \((M, \xi, \omega)\) be an almost contact structure and \((f, C, E)\) a good ace fibration adapted to it. There exists a contact distribution \(\xi'\) homotopic to \(\xi\). The restriction of \(\xi'\) to the exceptional 3–spheres in \(E\) induces the unique overtwisted contact structure homotopic to the standard contact structure \(\xi_{\text{std}}\).

A neighborhood of the intersection of an exceptional 3–sphere with a fibre of \(f\) is diffeomorphic to \(S^1 \times D^2 \times D^2\). Let \((z, r, \theta, \rho, \phi)\) be coordinates for such a neighborhood, the triple \((z, \rho, \phi)\) belong to the fibre. It can be considered as a trivial fibration over the first pair of factors

\[
\pi : S^1 \times D^2 \times D^2 \longrightarrow S^1 \times D^2, \quad (z, r, \theta, \rho, \phi) \longmapsto (z, r, \theta).
\]

There also exists a contact structure given by the contact form \(\alpha = dz + r^2 d\theta + \rho d\phi\) on the neighborhood. This induces a contact connection \(A_\pi\) for the fibration \(\pi\). Let \(\delta \in \mathbb{R}^+\) and suppose the horizontal 2–disk \((\rho, \phi) \in D^2(\delta)\) is of radius \(\delta\).

**Lemma 9.2.** Consider the contact manifold \((S^1 \times D^2 \times D^2(\delta), \ker(dz + r^2 d\theta + \rho d\phi)), \pi\) the projection onto the first pair of factors and \(A_\pi\) the associated contact connection. The flow of the lift of \(\partial_\rho\) to \(A_\pi\) preserves the submanifold \(\{(z, r, \theta, \rho, \phi) \in X : \rho = \delta/2\}\).

**Proof.** The vector field \(\partial_\rho\) belongs to the contact distribution. The vertical directions are generated by \(\partial_\rho, \partial_\phi\) and the symplectic form pairs them via \(\rho \cdot d\rho \wedge d\phi\). Hence \(\partial_\rho\) is itself the lift to \(A_\pi\). The statement follows. \(\square\)

**Proof of Theorem 9.1** Apply Theorem 6.2 to the almost contact structure and the given good ace fibration. Chosen an adapted family \(T\) for \((f, C, E)\) and use Theorem 7.1 to obtain a distribution which is a contact structure away from a disjoint union of pre–images of the balls \(\{B_1, \ldots, B_a\} \subset \mathbb{CP}^1\). Let us still refer to this distribution as \(\xi\). The distribution \(\xi\) is a contact structure in the fibres close to the boundary of \(\{B_i\}\), maybe after enlarging the balls if necessary. The restriction of \(f\) to the preimages of each \(B \in \{B_i\}\) is a smooth fibration since the critical values of \(f\) lie in the complement of the set \(B_1 \cup \ldots \cup B_a\). In order to conclude the statement of the Theorem we produce a deformation over each ball \(B \in \{B_i\}\) supported away from the boundary and resulting in a contact structure. The deformation can then be extended to a global deformation.

The attentive reader is probably able to conclude the proof of the statement since the results in Section 8 are the essential ingredients. Nevertheless, let us precise the necessary details regarding the trivializations. Choose the ball \(B_1 \in \{B_1, \ldots, B_a\}\) and a trivializing diffeomorphism \(\varphi : B \longrightarrow B^2(1)\). Consider \(U = f^{-1}(B)\) and the map

\[
g = \varphi \circ f : U \longrightarrow B^2(1).
\]

For \(\varepsilon > 0\) a small constant, we may assume that \(g^{-1}(B^2(1) \setminus B^2(1 - \varepsilon))\) is an open set where the distribution \(\xi\) is a contact structure. The interior must be deformed, the boundary has already been deformed in the previous sections.

The only boundary contribution to this fibration is given by the neighbourhood of the exceptional divisors. Consider an exceptional divisor \(E\). According to the local model used in the contact blow–up, there exists a neighbourhood \(\mathcal{E}\) of \(E\) and a contactomorphism

\[
\varphi_E : (S^3 \times D^2(\delta), \alpha_{\text{std}} + \rho^2 d\phi) \longrightarrow \mathcal{E}.
\]
The composition $f \circ \varphi_E : S^3 \times \mathbb{D}^2(\delta) \to S^2$ restricts to the Hopf fibration at $S^3 \times \{0\}$. Restricting to the region $f^{-1}(U) \cap E$ we obtain a fibration

$$\varphi \circ f \circ \varphi_E : S^3 \times B^2(1) \times \mathbb{D}^2(\delta) \to B^2(1)$$

over the 2–ball. Lemma 9.2 implies that the contact parallel transport along the neighborhood of the boundary is tangent to it. Therefore we can apply Lemma 7.4 and Remark 7.5 to obtain a radius–1 trivialization of the fibration $g$ respecting the vertical contact condition and preserving the boundary. In precise terms, we obtain a trivializing map

$$\tau : U \to F \times B^2(1), \text{ such that } \tau_*\xi = \ker(\alpha(\tau, \theta) + \tilde{G}d\theta + Hd\theta)$$

using the parallel transport along the almost contact connection. The corresponding trivialization for the symplectic structure is also obtained. The trivialization is performed with the radial direction and hence $\tilde{G} = 0$. However, the fact that the contact form $\alpha(\tau, \theta)$ depends on the point $(\tau, \theta) \in \mathbb{D}^2$ beclouds the vanishing of $\tilde{G}$. Theorem 8.1 only applies to contact fibrations with constant vertical contact structure. Let us achieve this.

Consider a fixed angle $\theta \in S^1$ and the radial family of contact structures

$$\xi(\tau, \theta) = \ker(\alpha(\tau, \theta)), \text{ for } \tau \in [0, 1].$$

In a neighborhood of the boundary $\partial F$ this family has a constant contact structure. Hence Gray’s stability theorem applies, relative to $\partial F$, to produce a family $m(\tau, \theta)$ of diffeomorphisms such that

$$m(\tau, \theta) : F \to F, \quad (m(\tau, \theta))^*\xi_0 = \xi(\tau, \theta).$$

This radial family depends smoothly in the parameter $\theta \in S^1$. Consider the map

$$\mathcal{M} : F \times B^2(1) \to F \times B^2(1), \quad \mathcal{M}(p, r, \theta) = (m(\tau, \theta)(p, r, \theta), r, \theta).$$

It trivializes the distribution $\xi$ as

$$(\mathcal{M} \circ \tau)_*\xi = \ker\{\alpha_0 + G(p, r, \theta)d\theta + H(p, r, \theta)d\theta\}.$$ 

In a neighborhood $U$ of the boundary $\{r = 1\}$ the distribution $\xi$ is a contact structure and the almost contact connection is a honest contact connection in $U$. Thus the flow obtained using Gray’s stability coincides with the radial parallel transport by contactomorphisms in $U$. In particular $G$ vanishes on $U$ because the lift of the radial direction is contained in the contact distribution.

The appearance of the function $G$ in the trivialization does not ease the attainment of a contact distribution. Perform a homotopy of almost contact structures relative to the boundary, by using Lemma 2.5, to obtain an almost contact structure $\tilde{\xi}$ such that

$$(\mathcal{M} \circ \tau)_*\tilde{\xi} = \ker(\alpha_0 + H d\theta).$$

Let us denote $\xi = \tilde{\xi}$. This setup satisfies the hypotheses of Theorem 8.1. It applies producing a homotopy $\xi_t$ of almost contact structures over $U$ relative to its boundary such that $\xi_0 = \xi$ and $\xi_1$ is a contact structure. The exceptional divisors remain contact submanifolds and as contact submanifolds of $\xi_1$, their induced contact structure is the standard contact structure $\xi_{std}$ with a full Lutz twist performed. The construction is made relative to the pre–image of a neighborhood of the boundary of the ball $B$. The argument successively applies to the elements of $B = \{B_1, \ldots, B_a\}$. This concludes the statement.
9.2. Interpolation at the exceptional divisors. Let \((M, \xi, \omega)\) be an almost contact manifold. The argument for proving Theorem 1.1 begins with a good almost contact pencil \((f, C, E)\). It is then blown–up to obtain a good acf fibration. The results in Section 6, 7 and 8 confer acf fibrations. These exist not on the manifold \((M, \xi, \omega)\) but in an almost contact blow–up. A contact structure has been obtained in the almost contact blow–up such that a neighborhood of the exceptional spheres has remained contact. It is left to perform an appropriate contact blow–down and obtain a contact structure in the initial manifold \(M\).

The exceptional spheres in \((\widetilde{M}, \widetilde{\xi})\) have the standard tight contact structure \((S^3, \xi_{\text{std}})\) at the beginning of the argument. In the deformation performed in Section 6 the exceptional spheres become overtwisted. Hence the contact blow–down procedure cannot be performed directly. This has a simple solution. We deform the contact distribution on a neighborhood of the exceptional spheres to the standard one. This is the content of the following

**Theorem 9.3.** Let \((S^3 \times B^2(4), \xi_0)\) have the contact form

\[
\eta = \alpha_{\text{ot}} + \delta \cdot r^2 d\theta,
\]

where \(\delta \in \mathbb{R}^+\) is a constant and \(\alpha_{\text{ot}}\) is any contact form associated to an overtwisted contact structure homotopic to the standard contact structure on \(S^3\).

Then there exists a deformation \(\eta_1\) of \(\eta_0\) supported in \(S^3 \times B^2(3)\) such that the \(\eta_1\) is a contact structure and \(S^3 \times \{0\}\) inherits the standard contact structure.

This result is a consequence of Lemma 3.2 in [EP]. Let us give an alternative argument, pointed out to us by Y. Eliashberg.

**Proof of Theorem 9.3.** Let us begin with the standard contact 3–sphere \((S^3, \xi_{\text{std}})\). Performing a Lutz twist along a given transverse trivial knot \(K\) produces an overtwisted contact structure \(\xi_{\text{ot}}^1\) in \(S^3\) homotopic to \(\xi_{\text{std}}\) as almost contact distribution. The contact structure \(\xi_{\text{ot}}^1\) is isotopic to the contact structure \(\xi_{\text{ot}}^2 = \ker \alpha_{\text{ot}}\). Consider both a trivial Legendrian knot \(L \subset (S^3, \xi_{\text{std}})\) whose positive transverse push–off is \(K\), and its Legendrian push–off \(L'\) with two additional zig–zags. According to [DGS] a Lutz twist along \(K\) is tantamount to a contact (+1)–surgery along \(L\) and \(L'\). Hence, given \((S^3, \xi_{\text{ot}}^1)\) there exists a \((-1)\)–surgery on \((S^3, \xi_{\text{ot}}^1)\) producing \((S^3, \xi_{\text{std}})\). Such surgery provides a Liouville cobordism \((W, \lambda)\) from \((S^3, \xi_{\text{ot}}^1)\) to \((S^3, \xi_{\text{std}})\).

The cobordism obtained by a \((-1)\)–surgery along \(L\) and \(L'\) is smoothly trivial, see [DGS]. Consider \(\theta \in S^1\) and \(\eta_1 = \lambda + \mu \cdot d\theta\), for a constant \(\mu \in \mathbb{R}^+\). Then the contactization \((W \times S^1, \eta_1)\) of the exact symplectic manifold \((W, \lambda) \cong (S^3 \times [0, 1], \lambda)\) is diffeomorphic to \(S^3 \times S^1\). We have obtained a contact structure on the 3–sphere times the annulus such that the inner boundary \(S^3 \times \{0\}\) has fibres \((S^3, \xi_{\text{std}})\), and \((S^3, \xi_{\text{ot}}^1)\) are the fibres of the outer boundary \(S^3 \times \{1\}\). The inner part is a convex boundary and it can be filled with the contact manifold

\[(S^3 \times \mathbb{D}^2, \ker(\alpha_{\text{std}} + \delta \cdot r^2 d\theta))\]

in order to obtain a contact structure on \(S^3 \times \mathbb{D}^2\) with \((S^3, \xi_{\text{std}})\) as central fibre. For a choice of \(\mu\) small enough, there exists a small constant \(\delta \in \mathbb{R}^+\) such that in a neighborhood \(S^3 \times (1 - \varepsilon, 1) \times S^1\) of the outer boundary the contact structure can be expressed as

\[
\eta_1 = \alpha_{\text{ot}}^1 + \delta \cdot r^2 d\theta.
\]

The contact forms \(\alpha_{\text{ot}}^1\) and \(\alpha_{\text{ot}}^2 = \alpha_{\text{ot}}\) are isotopic via a family of contact forms \(\{\alpha_{\text{ot}}^r\}, \ r \in [1, 2]\). On the manifold \(S^3 \times [1, 4] \times S^1\) consider the 1–form

\[
\eta_2 = \alpha_{\text{ot}} + \delta \cdot r^2 d\theta \quad \text{for} \ r \in [1, 2], \quad \text{and} \quad \eta_2^2 = \alpha_{\text{ot}}^2 + \delta \cdot r^2 d\theta \quad \text{for} \ r \in [2, 4].
\]
where $\tilde{\alpha}_{oder}(p,r,\theta) = \alpha^*_o(p)$. The form $\eta^2$ is a contact form because the form $r^2d\theta$ does not depend on the point $p \in S^3$. The gluing of the contact forms $\eta^1$ and $\eta^2$ is the required contact structure $\xi_1$ on $S^3 \times B^2(4)$. \qed

Notice that this deformation gives a homotopy of almost contact structures.

9.3. **Proof of Theorem 1.1.** Let $(M, \xi, \omega)$ be an almost contact structure. Proposition 4.4 allows us to construct a good almost contact pencil. Then Theorem 5.3 provides a good $\alpha$-fibration on an almost contact blow-up $(\tilde{M}, \tilde{\xi}, \tilde{\omega})$. We apply Theorem 9.1 to this almost contact manifold. The construction provides the standard overtwisted structure on the exceptional spheres since a sequence of full Lutz twists are performed. Let us use Theorem 9.3 to deform the contact structure to be standard near the exceptional spheres. Lemma 5.7 allows us to blow-down along the exceptional divisors to obtain a contact structure over the initial manifold $M$. It is only required to use the same parameter $k$ in the choice of framing in the blow-up construction and use the same $k$ in the blow-down process. This concludes the proof of the existence of a contact structure $\xi'$ in the manifold $M$.

Let us prove that $\xi$ and $\xi'$ are homotopic as almost contact distributions. This has been proven except in the blow-down process. That is, suppose that two distributions $\xi$ and $\xi'$ are homotopic in $\tilde{M}$ and they coincide along the exceptional divisors $(S^3, \xi_{std})$. Then the two resulting distributions are also homotopic after performing a blow-down. Indeed, the blow-down distributions corresponding to $\xi$ and $\xi'$ are homotopic over $M \setminus B$. Let us consider a cell decomposition of the manifold $M$ such that $B$ contains only 4 and 5-cells. Such decomposition exists due to genericity of transversality. Then Lemma 2.4 implies that the blow-down distributions are also homotopic over $M$. In geometric terms, the Poincaré dual of the obstruction class is not modified by the blow-down process. \qed

9.4. **Uniqueness.** The uniqueness of a contact structure in every homotopy class of almost contact structures does not hold in a 5-fold. There are examples in the literature, for instance [NK] proves that every fillable contact structure has a non-fillable contact structure in the same almost contact homotopy class.

The construction described in this article requires a fair amount of choices. Though, the dependence of the contact structure with respect to them may be understood. The three main ingredients are the stabilization procedure of almost contact pencils, in the same spirit than Giroux’s stabilization for a contact open book decomposition [Gi]; the addition of fake curves in the triangulation increasing the amount of holes filled with the local model and the canonicity of the contact blow-up procedure.

10. **Non-coorientable case**

10.1. **Definitions.** Let $M$ be a $(2n + 1)$-dimensional closed manifold, not necessarily orientable. In order to state the Theorem 1.1 in the non-coorientable setting, we need to give a definition of a non-coorientable almost contact structure. This is a distribution with a suitable reduction of the structure group along with a property requiring a relation between the normal bundle and the distribution. First we introduce the Lie group $\mathfrak{A}(n)$ defined as

$$\mathfrak{A}(n) = \{ A \in O(2n) : \ AJ = \pm JA \},$$

where $J = \begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix}$.

Notice the following properties:

1. The group $\mathfrak{A}(n)$ has two connected components. It is homeomorphic to $U(n) \times \mathbb{Z}_2$. 


2. Its group structure is isomorphic to a semidirect product $U(n) \rtimes_{\rho} \mathbb{Z}_2$. More precisely, let $I = \begin{pmatrix} I_{d_n} & 0 \\ 0 & -I_{d_n} \end{pmatrix}$, then the action

$$\rho : \mathbb{Z}_2 \to \text{Aut}(U(n)), \quad a \mapsto (U \mapsto I^a U I^0)$$

induces the semidirect product structure in the usual way.

3. There is a natural group morphism $s : \mathfrak{A}(n) \to \mathbb{Z}_2$ defined as

$$s(A) = \text{tr}(JAJ^{-1}A^{-1})/(2n),$$

i.e. under the previous isomorphism, $s$ is the projection onto the second factor of $U(n) \rtimes_{\rho} \mathbb{Z}_2$.

Let us deduce some topological implications of the existence of a contact structure. Let $\xi \subset TM$ be a possibly non–coorientable contact structure on $M$ with a fixed set $\{U_i\}$ of trivializing contractible charts. Choose $\alpha_i$ as a local equation for $\xi|_{U_i}$, then

$$\alpha_i = a_{ij}\alpha_j, \quad \text{with } a_{ij} : U_i \cap U_j \to \{\pm 1\}.$$

This implies that $\{a_{ij}\}$ are the transition function of the normal line bundle $TM/\xi$. Further, $(d\alpha_i)_{\xi} = a_{ij}(d\alpha_j)_{\xi}$. In particular, we may choose a family of compatible complex structures $\{J_i\}$ for the bundle $\xi$ satisfying $J_i = a_{ij}J_j$.

First, note that there is a group injection

$$\mathfrak{A}(n) \to O(2n + 1), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & s(A) \end{pmatrix}$$

and thus the structure group of $M$ reduces to $\mathfrak{A}(n)$. And second, a $\mathfrak{A}(n)$–bundle $E$ induces via the morphism $s$ a real line bundle $s(E)$. This construction applied to $\xi$ gives the line bundle $TM/\xi$ in the case above. These two properties will be the ones required in the following:

**Definition 10.1.** An almost contact structure on a manifold $M$ is a codimension 1 distribution $\xi \subset TM$ such that the structure group of $\xi$ reduces to $\mathfrak{A}(n)$ and $s(\xi) \cong TM/\xi$.

Observe that the definition for a cooriented almost contact distribution coincides with the one previously given. There are some immediate topological consequences of the existence of such a $\xi$. Indeed:

(i) If $n$ is an even integer, then $\mathfrak{A}(n) \subset SO(2n)$. Thus the distribution $\xi$ is oriented.

(ii) If $n$ is an even integer, there is an isomorphism

$$TM/\xi \cong \text{det}(TM).$$

Hence, any almost contact structure in an orientable 5–dimensional manifold is cooriented. Conversely, any non–orientable 5–fold can only admit non–coorientable almost contact structures.

(iii) If $n$ is an odd integer, then $s = \text{det}$ as morphisms from $\mathfrak{A}(n)$ to $\mathbb{Z}_2$. Therefore $M$ is orientable since

$$\text{det}(TM) \cong \text{det}(\xi \oplus (TM/\xi)) \cong \text{det}(\xi) \otimes s(\xi) \cong \text{det}(\xi)^2 \cong \mathbb{R}$$

Let $M^{2n+1}$ be a non–orientable manifold with $n$ an even integer. Then there exists a canonical $2 : 1$ cover

$$\pi_2 : M_2 \to M$$

satisfying the following properties:

1. $M_2$ is an orientable manifold.

2. Any almost contact structure $\xi$ on $M$ lifts to an almost contact structure $\pi_2^*\xi$ on $M_2$.

Moreover, such a distribution is cooriented because of equation (7).
10.2. **Statement of the main result.** Let us state the equivalent of Theorem 1.1 in the non–coorientable setting:

**Theorem 10.2.** Let $M$ be a non–orientable closed 5–dimensional manifold. Let $\xi$ be an almost contact structure. Then there exists a contact structure $\xi'$ homotopic to $\xi$.

**Proof.** Let $\pi_2 : (M_2, \pi_2^*\xi) \rightarrow (M, \xi)$ be an orientable double cover. The constructions developed in this article can be performed in a $\mathbb{Z}_2$–invariant manner. Let us discuss it:

(i) An almost contact pencil $(f, B, C)$ can be made $\mathbb{Z}_2$–invariant. To be precise, the loci $B$ and $C$ are $\mathbb{Z}_2$–invariant subsets and $f$ is a $\mathbb{Z}_2$–invariant as a map. In particular the action preserves the fibres. This is because the approximately holomorphic techniques can be developed in that setting. See [IMP] for the details of the construction in the $\mathbb{Z}_2$–invariant setting.

(ii) The deformations performed in Section 4 can easily be done in a $\mathbb{Z}_2$–invariant way. Also, the contact blow–up along a $\mathbb{Z}_2$–invariant loop can be built to preserve that symmetry.

(iii) Subsection 6.2 is also prepared for the $\mathbb{Z}_2$–invariant setting. Instead of having a single pair of overtwisted disks, we require two pairs of overtwisted disks. Each pair in the image of the other through the $\mathbb{Z}_2$–action.

(iv) Eliashberg’s construction is not $\mathbb{Z}_2$–invariant. Therefore we proceed by quotienting the whole manifold by the $\mathbb{Z}_2$–action, we then obtain an almost contact pencil over the quotient. The fibres are oriented since they are 3–dimensional almost contact manifolds. The induced almost contact distribution on them is non–coorientable. However, there is no hypothesis on the coorientability in the results of [El]. Once the procedure described in Section 6 is applied, we consider the orienting double cover.

(v) Section 7 is trivially adapted to the $\mathbb{Z}_2$–invariant setting if a serious increase of notation is allowed.

(vi) Filling the 2–cells as in Section 8 and 9. We need to produce a $\mathbb{Z}_2$–invariant standard model over $M \times S^2$, with $(\hat{M}, \alpha_0)$ a contact manifold with a $\mathbb{Z}_2$–invariant action. The only required ingredient is to ensuring that the framing $\{\alpha_0, \alpha_1, \alpha_2\}$ is chosen $\mathbb{Z}_2$–invariant. The rest of the proof works through up to notation details.

(vii) Blowing–down is still a $\mathbb{Z}_2$–invariant procedure if the previous choices have been done $\mathbb{Z}_2$–invariantly. Therefore, we obtain a $\mathbb{Z}_2$–invariant contact structure $\xi'_2$ on $M_2$. Its quotient produces a contact structure on $M$.

This proves the existence part of the statement. The statement concerning the homotopy follows since the homotopies can be easily made $\mathbb{Z}_2$–invariant. This is left to the careful reader. □

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