ON THE HOMOTOPY FIXED POINT SETS OF SPHERES
ACTIONS ON RATIONAL COMPLEXES

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Abstract: In this paper, we describe the homotopy type of the homotopy fixed point sets of $S^3$-actions on rational spheres and complex projective spaces, and provide some properties of $S^1$-actions on a general rational complex.

1. Introduction

An action of a group $G$ on a space $M$ gives rise to two natural spaces, the fixed point set $M^G$ and the homotopy fixed point set $M^{hG}$. It is crucially important that there is an injection:

$$k : M^G \hookrightarrow M^{hG}.$$ 

Indeed, one version of the generalized Sullivan conjecture asserts that, when $G$ is a finite $p$-group, and $M$ is a $G$-CW-complex, then the $p$-completion of $k$ is a homotopy equivalence. This conjecture was proved in the case when $M$ is a finite complex by Miller [7].

For a finite group $G$, the rational homotopy theory of $M^{hG}$ has been studied by Goyo [5].

In [1, 2], the authors studied the homotopy type of $M^{hG}$ for a compact Lie group $G$ with particular emphasis when $G$ is the circle.

From now on, and unless explicitly stated otherwise, $G$ will denote a compact connected Lie group and by a topological $G$-space we mean a nilpotent $G$-space with the homotopy type of a CW-complex of finite type and $M^G \neq \emptyset$. Then the action of $G$ on $M$ induces an action of $G$ on $M^Q$.

We then start by setting a sufficiently general context in which $M^{hG}$ has the homotopy type of a nilpotent CW-complex. Identifying the homotopy fixed point set with the space $\text{sec}(\xi)$ of sections of the corresponding Borel fibration

$$\xi : M \to M^{hG} \to BG,$$

we have that if $\pi_{>n}(M)$ are torsion groups for a certain $n > 1$, then $M^{hG}$ is a rational nilpotent complex with the homotopy type of a CW-complex [1].

In this paper, we explicitly describe the rational homotopy type of the homotopy fixed point sets of certain $S^3$ actions.

2010 Mathematics Subject Classification. 55R91, 55R45.

* The author was supported in part by the National Natural Science Foundation of China (No. 11171161), Program for New Century Excellent Talents in University (No. NCET-08-0288), and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.
Theorem 1.1.
(1) When $n$ is odd, $S^n_{Q}hS^3$ has the rational homotopy type of products of odd dimensional spheres, precisely, we have

$$S^n_{Q}hS^3 \simeq Q S^a \times S^{a+4} \times \cdots \times S^n,$$

where

$$a = \begin{cases} 
1, & n = 4k + 1, \\
3, & n = 4k + 3.
\end{cases}$$

(2) If $n = 4k$, $S^n_{Q}hS^3$ is either path connected, and of the rational homotopy type of $S^3 \times K_k$, where $K_k$ has the minimal Sullivan model

$$(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{2 \leq r \leq 2k}), d)$$

with $|x_s| = 4s$, $|y_r| = 4r - 1$, $dx_s = 0$ $(1 \leq s \leq k)$, $dy_r = \sum_{s+t=r} x_s x_t$ $(2 \leq r \leq 2k)$, or else, it has 2 components, each of them has the rational homotopy type of

$$S^{4k+3} \times S^{4k+7} \times \cdots \times S^{8k-1},$$

(3) If $n = 4k + 2$, $S^n_{Q}hS^3$ is path connected, and of the rational homotopy type of $S^3 \times S^7 \times T_k$, where $T_k$ has the minimal Sullivan model

$$(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{3 \leq r \leq 2k+1}), d)$$

with $|x_s| = 4s + 2$, $|y_r| = 4r - 1$, $dx_s = 0$ $(1 \leq s \leq k)$, $dy_r = \sum_{s+t=r-1} x_s x_t$

$(3 \leq r \leq 2k + 1)$.

Theorem 1.2.
(1) If $n$ is odd, $\mathbb{C}P^n_{Q}hS^3$ is path connected, and has the rational homotopy type of one of the following spaces:

$$\mathbb{C}P^1 \times S^7 \times S^{11} \times \cdots \times S^{2n+1},$$

$$S^3 \times \mathbb{C}P^3 \times S^{11} \times \cdots \times S^{2n+1},$$

$$S^3 \times S^7 \times \mathbb{C}P^5 \times \cdots \times S^{2n+1},$$

$$\cdots,$$

$$S^3 \times S^7 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n.$$

(2) If $n$ is even, $\mathbb{C}P^n_{Q}hS^3$ is path connected, and has the rational homotopy type of one of the following spaces:

$$\ast \times S^5 \times S^9 \times \cdots \times S^{2n+1},$$

$$S^1 \times \mathbb{C}P^2 \times S^9 \times \cdots \times S^{2n+1},$$

$$S^1 \times S^5 \times \mathbb{C}P^4 \times \cdots \times S^{2n+1},$$

$$\cdots,$$

$$S^1 \times S^5 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n.$$
In [1, Corollary 2], they give a criterion of an elliptic $S^1$-space. We first show that the condition $M$ is a finite complex is necessary by the following example: there is a nilpotent $S^1$-complex $M$ which is not an elliptic space, such that each component of $M^h_{S^1}$ is elliptic. We also observe that an $S^1$-finite nilpotent complex $M$ is elliptic if and only if one of the component of $M^h_{S^1}$ is elliptic, complementing the mentioned result.

Finally, we show that the injection $k$ is generally not a rational homotopy equivalence.

**Theorem 1.3.** For an $S^1$-complex $M$ which is simply connected with

$$\dim \pi_*(M) \otimes \mathbb{Q} < \infty.$$  

Then

$$k : M^S_{S^1} \to M^h_{S^1}.$$  

is a rational homotopy equivalence if and only if $M$ is rational homotopy equivalent to a product of $CP^\infty$.

In the next section we prove Theorem 1.1 and 1.2. In section 3 we prove Theorem 1.3.

2. $S^3$-RATIONAL SPHERES AND COMPLEX PROJECTIVE SPACES

Our results heavily depend on known facts and techniques arising from rational homotopy theory. All of them can be found with all details in [4]. We simply remark a few facts.

We recall that when $M$ is path connected, the Sullivan model of $M$ is a quasi-isomorphism:

$$m : (AV_M, d) \to A_{PL}(M),$$

where $(AV_M, d)$ is a Sullivan algebra.

We also recall that a space $M$ is elliptic if both $H^*(M; \mathbb{Q})$ and $\pi_*(M) \otimes \mathbb{Q}$ are finite dimensional vector spaces over $\mathbb{Q}$.

For a $G$-space $M$, we have the corresponding Borel fibration:

$$\xi : M \to M_{hG} \to BG,$$

where $M_{hG} = (M \times EG)/G$. It is a classical fact that the homotopy fixed point set

$$M^{hG} = \text{map}_G(EG, M)$$

is homotopy equivalent to the section space $\text{Sec}(\xi)$ of this fibration.

Each fixed point gives rise to a trivial section of the product bundle

$$M^G \to BG \times M^G \to BG.$$  

Composing with the injection $M_G \times BG \hookrightarrow EG \times M/G = M_{hG}$ gives a section of the Borel fibration. So we have a natural injection:

$$k : M^G \to M^{hG}.$$

For any $G$-CW complex $M$, there is an equivariant rationalization $m : M \to M_G$, that is, $M_Q$ is also a $G$-CW complex, $m$ is an equivariant map, and $(M_Q)^G \simeq (M_G)^Q$. Moreover, we have
Proposition 2.1. [1 Proposition 12] If $M$ is a Postnikov piece, that is $\pi_{>N}(M) = 0$ for some $N$, then

(i) $M^{hG}$ has the homotopy type of a nilpotent CW-complex of finite type.

(ii) $(M^{hG})_Q \simeq (M_Q)^{hG}$.

Note that if $M_Q$ is a Postnikov piece, then $(M_Q)^{hG}$ makes sense and is a rational space.

Now, we determine the homotopy type of the homotopy fixed point sets of certain $S^3$-actions. That is, we give the proof of Theorem 1.1 and 1.2.

Proof of Theorem 1.1 (1) Case 1: $n$ is odd.

We only prove the case $n = 4k + 3$, the case $n = 4k + 1$ is similar, so we omit it.

As in the proof of [1] Theorem 19, it is not hard to get the model of the corresponding Borel fibration

$$\xi: (A, 0) \rightarrow ((\Lambda e) \otimes A, D) \rightarrow (\Lambda e, 0),$$

in which $(A, 0) = (Ax/x^k, 0)$ and $|x| = 4, |e| = n$. This fibration is trivial, so $\text{Sec}(\xi) \simeq \text{Map}(\mathbb{H}P^k, S^n)$.

By [1] Theorem 9, the model of $S_Q^{nhS^3}$ is $(\Lambda(x_1, x_2, \ldots, x_{n+1}/4), 0)$. It is exactly the model of $S^3 \times S^2 \times \cdots \times S^n$. It follows that $S_Q^{nhS^3} \simeq Q S^n \times S^{a+4} \times \cdots \times S^n$.

(2) Case 2: $n = 4k$.

As $\pi_{>2n}(S^n) \otimes Q = 0$, a model of the Borel fibration is

$$\xi_{2n}: (A, 0) \rightarrow (\Lambda(e, e') \otimes A, D) \rightarrow (\Lambda(e, e'), d),$$

where $A = \Lambda x/x^{2k+1}$, $x, e, e'$ are of degree 4, $n, 2n - 1$ respectively, $De = 0$, $De' = e^2 + \lambda x^k e$, $de' = e^2$.

(i) If $\lambda = 0$, then $\xi_{2n}$ is trivial and

$$S_Q^{nhS^3} \simeq \text{Map}(\mathbb{H}P^{2k}, S^n)_Q.$$

A straightforward computation shows that this mapping space has a model of the form

$$((\Lambda y_1, 0) \otimes (\Lambda(x_s)_{1 \leq s \leq k}, (y_r)_{2 \leq r \leq 2k}), d)$$

with $|x_s| = 4s, |y_r| = 4r - 1, dx_s = 0 (1 \leq s \leq k), dy_r = \sum_{s + t = r} x_s x_t (r > 1)$.

(ii) If $\lambda \neq 0$, then the fibration $\xi_{2n}$ has two non homotopic sections $\sigma, \tau$ which corresponds to the only two possible retractions of its model,

$$\varphi_\sigma, \varphi_\tau: (\Lambda(e, e') \otimes A, D) \rightarrow (A, 0), \quad \varphi_\sigma(e) = 0, \quad \varphi_\tau(e) = \lambda x^k.$$

By the same way in [1], we have that the model of $\text{Sec}_\sigma(\xi_{2n})$ is of the form

$$((\Lambda(x_s)_{1 \leq s \leq k}, (y_r)_{1 \leq r \leq 2k}), \tilde{d})$$

with $|x_s| = 4s, |y_r| = 4r - 1$. The linear part of $\tilde{d}$ is:

$$\tilde{d}(y_r) = \lambda x_r$$

for $1 \leq r \leq k$. Which means the minimal model of $\text{Sec}_\sigma(\xi_{2n})$ is

$$(\Lambda(y_r)_{k+1 \leq r \leq 2k}, 0).$$

Replace $\lambda$ by $-\lambda$, we have that the model of $\text{Sec}_\tau(\xi_{2n})$ is the same.

(3) Case 2: $n = 4k + 2$. 
As $\pi_{2n}(S^n) \otimes \mathbb{Q} = 0$, a model of the Borel fibration is

$$\xi_{2n} : (\Lambda, 0) \to (\Lambda(e, e') \otimes A, D) \to (\Lambda(e, e'), d),$$

where $A = \Lambda x/x^{2k+1}$. $x, e, e'$ are of degree 4, $n, 2n - 1$ respectively, $De = 0$, $De' = e^2$, $de' = e^2$.

So the fibration $\xi_{2n}$ is trivial, we have

$$S^{hG}_n \simeq \text{Map}(\mathbb{H}P^{2k}, S^n)\mathbb{Q}.$$ 

The model of $S^{n,hG}$ is:

$$(\Lambda(y_1, y_2), 0) \otimes (\Lambda(x_s)_{1 \leq s \leq k}, (y_r)_{3 \leq r \leq 2k+1}, d)$$

with $|x_s| = 4s + 2$, $|y_r| = 4r - 1$, $dx_s = 0$ $(1 \leq s \leq k)$, $dy_r = \sum_{s+t=r-1} x_s x_t$ $(3 \leq r \leq 2k + 1)$.

This implies the result. \(\square\)

**Proof of Theorem 1.2** First, we assume $n = 2k + 1$. As $\pi_{2k+4}(\mathbb{C}P^n) = 0$, we just need to use the model of $\xi_{2n+2}$:

$$(A, 0) \to (\Lambda(e, e') \otimes A, D) \to (\Lambda(e, e'), d),$$

in which $A = (\Lambda x)/x^{k+2}$, $|x| = 4$, $|e| = 2$, $|e'| = 4k + 3$, and

$$De = 0, \quad De' = e^{n+1} + \sum_{j=1}^k \lambda_j e^j x^{n+1-2j}, \quad \lambda \in \mathbb{Q}, j = 1, \ldots, n.$$ 

The retraction of this model of fibration is just $\varphi(e) = 0$. So we have $\text{Sec}(\xi_{4k+4})$ is connected, and the model of it is

$$(\Lambda(e, (e')_{1 \leq r \leq k+1}, d)$$

with $|e| = 2$, $|e'_r| = 4r - 1$, $\overline{d}(e'_r) = \lambda_{k+1-r} e^{2r}$ for $1 \leq r \leq k$ and $\overline{d}(e'_{k+1}) = e^{2k+2}$.

If $\lambda_1 \neq 0$ this is a model of

$$S^2 \times S^7 \times \cdots \times S^{4k+3}.$$ 

If $\lambda_1 = \cdots = \lambda_{i-1} = 0$ and $\lambda_i \neq 0$, this is a model of

$$S^3 \times \cdots \times S^{4k-4i+1} \times \mathbb{C}P^{2k+1-2i} \times S^{4k-4i+3} \times \cdots \times S^{4k+3}.$$ 

Finally, if all $\lambda_i = 0$, then it is a model of

$$S^3 \times S^7 \times \cdots \times S^{4k-1} \times \mathbb{C}P^{2k+1}.$$ 

For $n$ even, the proof is similar, so we omit it. \(\square\)

3. The Inclusion $k : M^{S^1} \hookrightarrow M^{hS^1}$

We begin with some interesting observations on $S^1$-actions.

In [2 Example 12], there is an $S^1$-action on $M = K(Z, n) \times K(Z, n + 1)$, such that the model of it’s Borel fibration is

$$\eta_n : (Ax, 0) \to (\Lambda(x \otimes \Lambda(z, y), D) \to (\Lambda(z, y), d),$$

where $|x| = 2$, $|z| = n$, $|y| = n + 1$, $D(z) = 0$, and $D(y) = xz$. For $n = 2k$, there is only one retraction $\sigma$: $\sigma(z) = \sigma(y) = 0$, so Sec$(\eta_{2k})$ is path connected.

By the same method used in [1], a model of Sec$(\eta_{2k})$ is

$$(\Lambda((z_i)_{1 \leq i \leq k}, (y_j)_{1 \leq j \leq k+1}), d),$$
where \(|z_i| = 2i, |y_j| = 2j − 1\) and \(d(y_i) = z_i\). The minimal model of \(\text{Sec}(\eta_{2k})\) is \((\Lambda y_{k+1}, 0)\), so \(\text{Sec}(\eta_{2k}) \simeq_{\mathbb{Q}} S^{2k+1}\) is an elliptic space. However, \(M\) is not an elliptic space.

Next we complement [1] Corollary 2] with the following:

**Proposition 3.1.** For an \(S^1\)-space \(M\) which is a nilpotent finite complex, the following conditions are equivalent:
1) \(M\) is elliptic,
2) Each component of \(M_{Q^hS^1}\) is elliptic,
3) One of the components of \(M_{Q^hS^1}\) is elliptic.

**Proof of Proposition 3.1.**
1) \(\Rightarrow\) 2): [1] Theorem 15].
2) \(\Rightarrow\) 3): Trivial.
3) \(\Rightarrow\) 1): By [2] Theorem 13], \(2\dim \pi_*(\text{Sec}_{\sigma}(\xi) \otimes \mathbb{Q}) \geq \dim \pi_*(M) \otimes \mathbb{Q}\). By \(\text{Sec}_{\sigma}(\xi)\) is elliptic, \(\dim \pi_*(\text{Sec}_{\sigma}(\xi)) \otimes \mathbb{Q}\) is finite, so \(\dim \pi_*(M) \otimes \mathbb{Q}\) is finite. Then \(M\) is elliptic. \(\square\)

**Remark 3.2.** The theorem holds also for \(G = S^3\). The proof is similar.

The rest of the section is devoted to the proof of Theorem 1.3.

Let \(M\) be an \(S^1\)-space and \(M^G \neq \emptyset\). Then the inclusion \(M^{S^1} \hookrightarrow M\) induces a map of Borel fibrations:

\[
\begin{array}{ccc}
M^{S^1} & \longrightarrow & M \\
\downarrow & & \downarrow \\
\mathbb{C}P^\infty \times M^{S^1} & \longrightarrow & M_{hS^1} \\
\eta & \gamma & \xi \\
\downarrow & \downarrow & \downarrow \\
\mathbb{C}P^\infty & \longrightarrow & \mathbb{C}P^\infty.
\end{array}
\]

If there exists \(N\) such that \(\pi_{\geq N}(M_Q) = 0\) and \(\pi_{\geq N}(M_{Q}^{S^1}) = 0\). Then \(k\) is identified with the corresponding

\[M^{S^1} \hookrightarrow \text{Map}((\mathbb{C}P^\infty)^{(N)}, M^{S^1}) \rightarrow \text{Sec}(\xi_N) \cong M^{hS^1},\]

obtained by truncating in the diagram (3.1):

\[
\begin{array}{ccc}
M^{S^1} & \longrightarrow & M \\
\downarrow & \gamma_N & \downarrow \\
F_N & \longrightarrow & E_N \\
\eta_N & \xi_N & \xi_N \\
\downarrow & \downarrow & \downarrow \\
(\mathbb{C}P^\infty)^{(N)} & \longrightarrow & (\mathbb{C}P^\infty)^{(N)}.
\end{array}
\]
Now let
\[(3.2) \quad (A \otimes \Lambda V, D) \twoheadrightarrow (\Lambda V, d) \]
\[
\begin{array}{c}
(A, 0) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
(A, 0) \otimes (\Lambda Z, d) \\
\downarrow \downarrow \\
(A, 0) \otimes (\Lambda Z, d)
\end{array}
\]
be a model of the above diagram where \((A, 0) = (\Lambda x/(\Lambda z)^{>N}, 0)\), \((\Lambda V, d)\) and \((\Lambda Z, d)\) are minimal Sullivan models of \(M\) and \(M^{S_1}\), respectively.

Then we have the following theorem:

**Theorem 3.3.** [1, Theorem 21] The composition
\[
(\Lambda V \otimes A^\#), \tilde{d} \xrightarrow{\phi} (\Lambda(Z \otimes A^\#), \tilde{d}) \xrightarrow{\gamma} (\Lambda Z, d)
\]
is a model of \(k: M^{S_1}_Q \hookrightarrow M^{hsS_1}_Q\). And the morphism are defined by:
\[
\phi(v \otimes \alpha) = \rho^{-1}[\psi(v) \otimes \alpha], \quad v \otimes \alpha \in V \otimes A^#,
\]
\[
\gamma(z \otimes \alpha) = \begin{cases} z & \alpha = 1, \\ 0 & \alpha \neq 1, \end{cases} \quad z \otimes \alpha \in Z \otimes A^#.
\]

Then we give some information about \(\psi\). First, let \((A x \otimes \Lambda V, D)\) be a model of the fibration \(\xi\), we can decompose the differential \(D\) in \(A \otimes \Lambda V\) into
\[
D = \sum_{i \leq 1} D_i, \quad D_i(V) \subset A x \otimes A^i V.
\]

And we have:

**Proposition 3.4.** [2, Lemma 14] The vector space \(V\) can be decomposed into a direct sum \(W \oplus K \oplus S\) where
(1) \(W \oplus K = \ker D_1\),
(2) \(K\) and \(S\) have the same dimension admitting bases \(\{v_i\}_{i \in I}, \{s_i\}_{i \in I}\), and for any \(i \in I\), there exists \(n_i \geq 1\) such that \(D_1(s_i) = x^{n_i} v_i\).

Let \(K = Q(x)\), the field of fractions of \(A x\), we obtain a morphism of (ungraded) differential vector spaces
\[
\overline{\psi}: (K \otimes V, D_1) \rightarrow (K \otimes Z, 0) = (Z_K, 0).
\]
If we assume \(K\) concentrated in degree 0 and consider in \(V\) and \(Z\) the usual \(\mathbb{Z}_2\)-grading given by the parity of the generators, then the Borel localization theorem claim that:

**Theorem 3.5.** [1, Theorem 22] The morphism
\[
\overline{\psi}: (K \otimes V, D_1) \rightarrow (Z_K, 0)
\]
is a quasi-isomorphism.

By Proposition 3.4, we have
Lemma 3.6. (1) $\dim W = \dim Z$.

(2) There are $\{w_j\}_{j \in J}, \{z_j\}_{j \in J}$ which are homogenous basis of $W$ and $Z$ respectively, and non negative integers $\{m_j\}_{j \in J}$ such that

$$\psi(w_j) = x^{m_j}z_j + \Gamma_j, \quad \Gamma_j \in R \otimes \Lambda^{>2}Z, \quad j \in J,$$

and

$$\psi(s_i) \in R \otimes \Lambda^{\geq 2}Z, \quad \psi(v_i) \in R \otimes \Lambda^{>2}Z, \quad s_i \in S, \quad v_i \in K, \quad i \in I.$$

Theorem 3.7. For an $S^1$-complex $M$ which is simply connected with

$$\dim \pi_\ast(M) \otimes \mathbb{Q} < \infty,$$

Then the inclusion

$$k : M^{S^1} \to M^{hS^1}$$

is a rational homotopy equivalence if and only if $M$ is rational homotopy equivalent to a product of $\mathbb{C}P^\infty$.

Proof. By Theorem 3.3, the model of $k$ is:

$$\alpha : (\Lambda(V \otimes A^\#), \tilde{d}) \to (\Lambda(Z \otimes A^\#), \tilde{d}) \to (\Lambda Z, d).$$

By [1] Theorem 24, $\pi_\ast(k) \otimes \mathbb{Q}$ is injective, so we only consider the surjective part.

By [1] Theorem 11, $(\Lambda(V \otimes A^\#), \tilde{d})$ is a model of $M^{hS^1}_Q$. Then we have

$$H^k(V \otimes A^\#, \tilde{d}_1) \cong \text{Hom}(\pi_k(M^{hS^1}_Q), \mathbb{Q}),$$

where $k \geq 1$.

By Proposition 3.3, $V = W \oplus K \oplus S$. An easy computation shows that $(W \otimes A^\#) \oplus S \subset H^\ast(V \otimes A^\#, \tilde{d}_1)$. It is obvious that

$$\alpha(w_j) = 0 \iff m_j \neq 0,$$

$$\alpha(w_j \otimes (x^i)^\#) = 0 \iff m_j \neq i,$$

$$\alpha(s_j) = 0.$$

If there exists $j$ such that $|w_j| \geq 2$ or $S \neq \emptyset$, then $H(\alpha, \tilde{d}_1)$ is not injective, so $k$ is not a rational homotopy equivalence.

If $|w_j| = 2$, for each $j \in J$, and $S = \emptyset$, we have $(\Lambda W, d)$ is a model of a product of $\mathbb{C}P^\infty$. It is easy to show that $k$ is a rational homotopy equivalence. □

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