A different approach to the MHD equilibrium

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Equations describing plasma equilibria are derived from the total energy of the system. The MHD equilibrium is shown to hold even for systems where the magnetic field may locally vanish. Through conservation of helicity, confinement is shown to be intimately related to electromagnetic tension. For an axisymmetric device, the presence of a magnetic field or current circling the axis is shown to be destabilising. This provides a simple explanation for the poor performance of some devices for which MHD predicts stability.

I. INTRODUCTION

Magnetic confinement of fusion plasmas is a well-studied problem. Most analyses start by considering the motion of particles in a prescribed electromagnetic field and attempt an averaging process. This process leads in one direction to the BBGKY hierarchy and kinetic equations and in another via a fluid approximation to MHD.

In this paper, we instead begin with the symmetries of a system of electromagnetic particles and the corresponding invariants. From these invariants and simple entropy arguments we can arrive at requirements for equilibrium. The result is a more fundamental picture of plasma confinement and a simple explanation of why devices like \( \theta \)-pinches and magnetic mirrors never worked as well as hoped.

II. PLASMA EQUILIBRIUM

The Lagrangian density for a charged fluid moving in an electromagnetic field can be written as

\[
\mathcal{L}(r) = \sum_s \frac{n_s(r)}{2} m_s \langle v^2_s(r) \rangle - q_s n_s(r) \phi(r) + n_s(r) q_s \langle v_s(r) \rangle \cdot A(r)
\]

with \( n \) the number density of particles of charge \( q \) and mass \( m \). \( \phi \) denotes the electric potential and \( A \) the magnetic vector potential, and the subscript \( s \) stands for the fluid species (e.g. electrons or ions).

From the Lagrangian, we can obtain the energy density (total pressure) of the system

\[
\mathcal{E}(r) = p(r) + \rho(r) \phi(r) - j \cdot A;
\]

where we have introduced the definitions

\[
\rho \equiv \sum_s n_s q_s,
\]

\[
j \equiv \sum_s n_s q_s \langle v_s \rangle
\]

and

\[
p \equiv \frac{1}{2} \sum_s m_s n_s \langle v^2_s \rangle.
\]

As required for a physical quantity (and unlike the particle Hamiltonian \( \frac{1}{2} m v^2 + q \phi \)), this is invariant under a gauge transformation preserving the electric and magnetic fields:

\[
\phi \to \phi' = \phi + \frac{\partial \eta}{\partial t}
\]

\[
A \to A' = A + \nabla \eta
\]

with arbitrary scalar field \( \eta \).

Such a system will be stable against perturbations that increase the potential energy of the system; i.e. those that increase the integral of \( \rho \phi \) or decrease the integral of \( j \cdot A \). For an electron, having a velocity antiparallel to the vector potential increases the potential energy. In fact, since the vector potential due to a moving electron is antiparallel to its motion and electron current will be larger than ion current under most conditions, we see that the maximum electron loss current density is bounded.

We can already see the basis of one approach to plasma confinement in an electromagnetic trap:

- Use an outwards-pointing vector potential to confine an excess of electrons against their electrostatic repulsion
- Use the resulting ‘virtual cathode’ to confine ions

This is precisely the scheme advocated by Bussard in his discussion of the Polywell fusion device.

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We can also write the energy density as a sum of the contributions due to matter and the fields as
\[ \mathcal{E}(r) = p(r) + u(r) \]  
(9)
where
\[ u(r) = \frac{1}{2} \left( \epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0} \right). \]  
(10)
Conservation of energy in this second picture can be expressed as
\[ \frac{\partial u}{\partial t} = -\frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mathbf{E} \cdot \mathbf{j}. \]  
(11)
A familiar equilibrium condition is obtained by requiring the energy density to be constant from place to place:
\[ \nabla \left( \frac{1}{2} \left( \epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0} \right) + p \right) = 0. \]  
(12)
Taking gradients,
\[ \nabla \frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E} = \epsilon_0 (\mathbf{E} \cdot \nabla) \mathbf{E} - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \]  
(13)
\[ \nabla \frac{\mathbf{B}^2}{2\mu_0} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \left( \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B}, \]  
(14)
yielding
\[ \epsilon_0 (\mathbf{E} \cdot \nabla) \mathbf{E} - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \left( \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} + \nabla p = 0. \]  
(15)

It is worth stressing here that since this derivation did not involve microdynamics or arguments of adiabatic invariance and since the field energy \( \epsilon_0 E^2 + B^2/\mu_0 \) is conserved even for time-varying fields, the above equilibrium holds even for systems exhibiting a magnetic null, and reduces to the familiar \( \nabla \rho = \mathbf{j} \times \mathbf{B} \) if \( \mathbf{E} \) and \( (\mathbf{B} \cdot \nabla) \mathbf{B} \) are both zero.

If we apply the same argument to Eq. (2) we see that equilibrium can also be expressed as
\[ \nabla \rho + \rho \nabla \phi + \phi \nabla \rho - (\mathbf{j} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{j} - \mathbf{j} \times \mathbf{B} - \mathbf{A} \times (\nabla \times \mathbf{j}) = 0. \]  
(16)

III. TAYLOR STATE

As per Woltjer and Chandrasekhar, and later Taylor, we can consider the question of relaxation of a zero temperature plasma conserving magnetic helicity.

We wish to minimise \( \int_V B^2 \, dV \) subject to constant \( H = \int_V (\mathbf{A} \cdot \mathbf{B}) \, dV \), where \( V \) is the total volume of the system.

As a variational problem, this is
\[ \delta \int_V \left[ \nabla \times \mathbf{A} \right]^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A} \right] \, dV = 0, \]  
(17)
where we have introduced the Lagrange multiplier \( \lambda \).

The solution is simply
\[ \nabla \times \mathbf{B} = \lambda \mathbf{B}. \]  
(18)
Substituting this result into Eq. (13) and taking the limit as all time-derivatives go to zero gives that the equilibrium for small plasma pressures is approximately
\[ \epsilon_0 (\mathbf{E} \cdot \nabla) \mathbf{E} + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla p = 0. \]  
(19)

In this limit it is purely the tension in the electric and magnetic fields that gives rise to a pressure gradient. Since the tension in the magnetic field cannot be simultaneously positive in all three axes, plasma confinement in equilibrium requires either a topological closure or a nontrivial electric tension.

IV. AXISYMMETRIC TRAPS

We may expect drifts in the \( \theta \) (cyclic) direction to decouple from those in the \( r \) and \( z \) directions. For each species there is a unique value of \( \langle v_{\theta s} \rangle \) which minimises the contribution to the energy density;
\[ \langle v_{\theta s} \rangle = \frac{q_s A_\theta}{m_s} \]  
(20)
Substituting Eq. (20) into the definition of \( \mathbf{j} \) and using Eq. (18) gives
\[ j_\theta = \sum_s n_s q_s^2 A_\theta/m_s = \frac{\lambda}{\mu_0} \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right). \]  
(21)
A general axisymmetric magnetic field can be written as
\[ \mathbf{B} = -\frac{\partial A_\theta}{\partial z} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial (r A_\theta)}{\partial r} \hat{\mathbf{z}} + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\theta}. \]  
(22)
Rewriting Eq. (23) using Eq. (21) gives
\[ \mathbf{B} = -\frac{\partial A_\theta}{\partial z} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial (r A_\theta)}{\partial r} \hat{\mathbf{z}} + \frac{\mu_0}{\lambda} \left( \sum_s n_s q_s^2 A_\theta/m_s \right) \hat{\theta}. \]  
(23)
Defining the stream function $\Psi = rA_\theta$, we can express the above as

$$\mathbf{B} = \frac{1}{r} \left( \frac{\partial \Psi}{\partial z} \hat{\mathbf{r}} + \frac{\partial \Psi}{\partial r} \hat{\mathbf{z}} + \frac{\mu_0}{\lambda} \left( \sum_s \frac{n_s q_s^2 \Psi}{m_s} \right) \hat{\mathbf{\theta}} \right). \quad (24)$$

Considering the material derivative

$$\left( \mathbf{B} \cdot \nabla \right) \mathbf{B} = \left( B_r \frac{\partial B_r}{\partial r} + B_z \frac{\partial B_r}{\partial z} - \frac{B_\theta}{r} \right) \hat{\mathbf{r}} + \left( \frac{\partial B_z}{\partial r} + B_z \frac{\partial B_z}{\partial z} \right) \hat{\mathbf{z}}, \quad (25)$$

it is apparent that for a force-free (Taylor equilibrium) plasma (see Eq.19), $B_\theta$ (and equivalently $\Psi$) acts to impede radial confinement and can be suppressed by large $\lambda$, i.e., large currents parallel to the magnetic field.

The destabilising effect of the $A_\theta$ term suggests that we might increase the stability of our plasma traps by ensuring that $A_\theta$ vanishes in the trapping region. This corresponds with the observed good performance of devices such as Multipoles and Polywells.

V. STATISTICAL EQUILIBRIUM

For systems with a significant electric field or more complicated magnetic geometries, analytical descriptions of the equilibrium become either complicated or difficult to obtain. A statistical approach affords some simplicity compared to solving the resultant nonlinear differential equations many times with different starting conditions to obtain averages.

As plasma temperatures are typically much higher than the boiling points of even refractory materials we should not assume that the plasma is in thermal equilibrium with its boundary. Instead, we should assume the converse; that the plasma is isolated, with the number of particles of each plasma species fixed and the total energy equal to $E_0$ (up to rapid fluctuations of order $\sigma$).

We should not expect a Maxwellian energy distribution - that distribution is characteristic only of systems which are in thermal equilibrium with a heat bath. The maximum entropy distribution for an isolated system is a uniform distribution over possible energies.

The macroscopic state of our system is given by $(N_i, N_e, E_0, \mathbf{E}, \mathbf{B}, \sigma)$, with $\mathbf{E}$ and $\mathbf{B}$ the electric and magnetic fields respectively and subscripts $i$ and $e$ here and hereafter denoting ions and electrons. We assume all microstates corresponding to this macroscopic state are equally likely. The limit $\sigma \to 0$ corresponds to the joint microcanonical ensemble obtained by assigning equal probability to all joint electron-ion-field microstates whose energy sums exactly to $E_0$.

Formally we suppose there are fluctuations of order $\sigma$ as giving rise to a distribution of energy density as

$$f(E) = \frac{1}{\sigma \sqrt{2 \pi}} e^{- (E - E_0)^2 / (2\sigma^2)}. \quad (26)$$

Nothing in the above tells us the degree to which the energy $E$ and energy-variance $\sigma$ are due to the ion or electron distributions. A possible parameterisation of the joint distribution is that of a uniform distribution all pairs of independent Gaussians whose convolution gives the above.

Using the notation

$$g(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{- (x-\mu)^2 / 2\sigma^2}$$

we write such a parameterisation as

$$f(E) = \int_{E_{\text{min}}}^{E_0} dE_i \int_0^{\sigma} d\sigma_i g_i(E|E_i, \sigma_i^2) \ast g_e(E|E_0 - E_i, \sigma^2 - \sigma_i^2) \quad (27)$$

over configurations $(\mathbf{X}_e, \mathbf{X}_i)$ with probability

$$P(\mathbf{X}_e, \mathbf{X}_i) = f \left( \int d^3r \mathbf{E} (\mathbf{X}_e, \mathbf{X}_i; \phi(\mathbf{X}_e), \mathbf{A}(\mathbf{X}_e, \mathbf{X}_e)) \right). \quad (30)$$

Given a set of proposal functions which preserve the other invariants of a given system, a set of representative microstates can be found using Metropolis sampling using $f(\mathbf{E}_{\text{candidate}}) / f(\mathbf{E}_{\text{current}})$ as the acceptance ratio. Since, for example, the system is to be regarded as isolated, the total number of electrons and ions must be conserved in generation of proposals. The cost of the sampling process is dominated by the need to solve Poisson equations for the electric and vector magnetic potentials. Code which implements this process is available.
VI. CONCLUSIONS

Examining plasma trapping from the context of energy equilibrium yields new perspective on MHD independent of plasma microdynamics, a suggestion as to the form of equilibrium energy distributions and guidance as to how to construct plasma traps with better performance than the cusp- and mirror-derived devices commonly in use. We also see that the MHD equilibrium holds for systems with a magnetic null such as magnetic cusps. Gauge invariance, an intrinsic feature of the electromagnetic system, leads to the observation that the $\theta$ component of the vector potential in cylindrically symmetric neutral plasmas acts against confinement. A future paper will examine the design of an optimal trap for a force-free plasma.

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