Series solutions for a static scalar potential in a Salam-Sezgin Supergravitational hybrid braneworld.

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(Dated: November 1, 2021)

The static potential for a massless scalar field shares the essential features of the scalar gravitational mode in a tensorial perturbation analysis about the background solution. Using the fluxbrane construction of [8] we calculate the lowest order of the static potential of a massless scalar field on a thin brane using series solutions to the scalar field’s Klein Gordon equation and we find that it has the same form as Newton’s Law of Gravity. We claim our method will in general provide a quick and useful check that one may use to see if their model will recover Newton’s Law to lowest order on the brane.

Keywords: Six dimension, Braneworld; fluxbrane; Salam-Sezgin

I. INTRODUCTION

It has long been a dream to derive the properties of our four-dimensional world from the symmetries of some higher dimensional spacetime. Randall-Sundrum models use warped metrics in five dimensions to obtain a low-energy effective Newtonian potential for four-dimensional gravity. Two models have been proposed: Randall-Sundrum I [1] and Randall Sundrum II [2]. Randall-Sundrum I contains two branes in a compact spacetime and Randall-Sundrum II contains only one brane embedded in a five-dimensional non-compact Anti de-Sitter spacetime. However, while the model met with a lot of interest, for example it may provide an explanation for the large energy difference between the weak-unification scale and the Planck scale, it has also led to some problems. It can be shown for example that under general assumptions the compact Randall-Sundrum I model must contain a negative tension brane [3]. It has been pointed out that such negative tension branes violate all the standard energy conditions [4]. Embedding black holes in Randall-Sundrum II-type models has also led to some problems, although the tension of the single brane can now be strictly positive. The Einstein equations evaluated across the brane forbid simple black holes in our universe (on the brane) and black strings extending off the brane develop Gregory-Laflamme instabilities at the AdS horizon and are unstable far from the brane [5].

In six (and higher) dimensions however, it is possible to have solutions that contain only positive tension branes. It may even be possible to have stable black hole solutions in six-dimensions [6]. Here we present an example of a six-dimensional model based on a supersymmetric fluxbrane solution of [6] and [7] and give a general argument to show that its low energy limit is four-dimensional Newtonian gravity. The model contains a single brane embedded in a six-dimensional spacetime. Scalar and electromagnetic fields propagate in the bulk, and we impose regular closure of the geometry off the brane.

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The basic idea is to investigate the behavior of gravity by approximating the full tensorial analysis with a scalar field argument. To do this we solve the Klein-Gordon equation on the background spacetime solution. The background solution must obey certain boundary conditions, in particular the Einstein equations across the brane must hold. Solving this equation allows us to derive a static potential for the Newtonian limit which should qualitatively resemble that of the observed universe.

Further details will appear in [8].

II. THE MODEL

We start with the metric (which equates to the bosonic sector of the model considered in [9]),

\[
S = \int_M d^6x \sqrt{-g} \left( \frac{\mathcal{R}}{4\kappa^2} - \frac{1}{4} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-\kappa \phi} F_{ab} F^{ab} - \frac{\Lambda}{2\kappa^2} e^{\kappa \phi} \right).
\] (1)

The metric assumption we use is

\[
ds^2 = \Delta(r) d\theta^2 + \frac{r^2}{\Delta(r)} dr^2 + r^2 g_{ij} dx^i dx^j,
\] (2)

where \(g_{ij}\) will, for simplicity, be taken to be the Minkowski metric. We consequently find solutions of the form [10]

\[
\Delta(r) = \frac{A}{r^2} - \frac{B^2}{r^6} - \frac{1}{8}\Lambda r^2.
\] (3)

For later use we define \(r_\pm\) to be the two zeroes of \(\Delta\) (there are only two solutions for \(0 \leq r \in \mathbb{R}\)). To this background fluxbrane geometry we add a single brane and interpret our four-dimensional universe to be restricted to the brane in the usual manner. Note that in general both the scalar field \(\phi\) and the Maxwell field will exist off the brane. Further details of the construction will appear in [3].

In order to consistently reproduce our four-dimensional universe on the brane, the gravitational interaction on the brane must reduce, at low energies, to the observed Newtonian potential. The basic idea here is to use a massless scalar field to model the behavior of gravity. Therefore we look for solutions to the massless Klein-Gordon equation

\[
\nabla^2 G_\Phi = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu G_\Phi \right) = \frac{\delta(r - r') \delta(\theta - \theta') \delta(x - x')}{\sqrt{-g}},
\] (4)

with boundary conditions

\[
G_\Phi \big|_{r \to r_-} < \infty ,
\]
(5)

\[
\partial_r G_\Phi \big|_{r = r_*} = 0 .
\] (6)

III. THE SOLUTION

To simplify the analysis of (11) we perform a change of variables to put the differential equation into Sturm-Liouville form with the variable \(\rho\).

\[
\rho = e^{\int (r^4 \Delta(r))^{-1} dr}
\] (7)
which becomes
\[ \rho = \frac{r^4 - r^4}{r^4 - r^4} \geq 0, \]  
where \( r_+ \) and \( r_- \) are the two roots of \( \Delta(r) \). We also perform the Fourier decomposition of the Green’s function
\[ G_{\Phi} = \int \frac{d^4k}{(2\pi)^5} e^{i k \mu (x^\mu - x'^\mu)} \sum_{n=-\infty}^{\infty} e^{i n(\phi - \phi')} y_{q,n}(\rho, \rho'). \]
This gives a differential equation of the form
\[ \partial_\rho (\rho \partial_\rho y_{q,n}) - \frac{\pi^2}{\rho} \left( \frac{\rho + \frac{r^4}{r^4}}{\rho + 1} \right)^2 y_{q,n} + \frac{\pi^2}{(\rho + 1)^2} y_{q,n} = \delta(\rho - \rho'), \]
where
\[ q^2 = -k_\mu k^\mu, \]
\[ \tilde{q}^2 = q^2 \Lambda / 8 (r^4 - r^4)^2; \]
\[ \pi^2 = n^2 r^8. \]
An obvious way to investigate the behavior of solutions to this differential equation is to use the method of Frobenius and expand the solution out as a power series. Provided the \( r \) coordinate is close to \( r_- \) the value of \( \rho \) will be small. The two linearly independent solutions are
\[ y_1 = \sum_{k=0}^{\infty} a_k \rho^{k+c_1}, \]
\[ y_2 = \begin{cases} \sum_{k=0}^{\infty} b_k \rho^{k+c_2} & n \neq 0 \\ \log(\rho) y_1 + \sum_{k=0}^{\infty} b_k \rho^k & n = 0 \end{cases}, \]
where \( c_1 \) and \( c_2 \) are the solutions of the indicial equation for \( y_{q,n} \),
\[ c^2 - (nr^4)^2 = 0. \]
If we impose the requirement that the solution be regular as \( \rho \to 0 \), then we must set the coefficient of the second independent solution to 0 and we are just left with the first solution. The Einstein equations at the brane also require a Neumann-type boundary condition for the function \( G_{\Phi} \) at the brane. This, along with the usual braneworld matching conditions that the metric should be continuous at the brane and its first derivative should just be a step function (the stress-energy tensor contains a delta function source due to the brane,) result in the on-brane solution at \( \rho = \rho_* = r_- / r_+ < 1 \),
\[ y_{q,n} = \frac{y_1(\rho_*)}{\rho_* \partial_\rho y_1(\rho_*)}, \]
The \( n = 0 \) mode represents the lowest order of the potential. For \( n = 0 \) we find
\[ y_1(\rho_*) = \sum_{k=0}^{\infty} \left( (-1)^k \sum_{j=0}^{k} e_j q^{2j} \right) \rho_*^k, \]
where $e_{0,0} = 1$, $e_{0,k} = 0$ for all $k > 0$ and $e_{j,k} \geq 0$ for all $j > 0$ and $k > 0$. In the case where this double summation converges absolutely then all rearrangements converge to the same limit and we can write

$$y_1(\rho_*) = \sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} (-1)^k e_{j,k} \rho_*^k \right) q^{2j}. \tag{19}$$

Similarly

$$\rho_* \partial_\rho y_1(\rho_*) = \sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} (-1)^k k e_{j,k} \rho_*^k \right) q^{2j}. \tag{20}$$

We can then write out (17) as

$$y_{q,0}(\rho_*) = \sum_{k=-1}^{\infty} Q_{k,0} q^{2k} \tag{21}$$

and calculate $Q_{i,0}$ ($Q_{i,n}$ is the general term) order by order in $q^2$ using the relationship

$$y_1(\rho_*) = [y_{q,0}(\rho_*)][\rho_* \partial_\rho y_1(\rho_*)]. \tag{22}$$

For example,

$$Q_{-1,0} = -1 + \frac{\rho_*}{\rho_*}. \tag{23}$$

Substituting the series back into the expression for the potential we find

$$V_\Phi = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{x}-\mathbf{x}')} \int_{-\infty}^{\infty} \frac{dk^0}{i(k^0 - i\epsilon)} \left( \frac{Q_{-1,0}}{q^2} \sum_{j=0}^{\infty} Q_{j,0} q^{2j} \right). \tag{24}$$

The term proportional to $q^{-2}$ just gives a contribution of

$$V_\Phi = \frac{Q_{-1,0}}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\mathbf{k} \cdot (\mathbf{x}-\mathbf{x}')}}{k^2} \tag{25}$$

and thus the potential to lowest order becomes

$$V_\Phi = -\frac{1 + \rho_*}{8\pi^2 \rho_*} \frac{1}{|\mathbf{x} - \mathbf{x}'|}, \tag{26}$$

which has the desired $1/x$ dependence of the familiar Newtonian gravitational potential.

### IV. CONCLUSION

The basic model outlined here is capable of reproducing Newton’s law of gravity at low energies, and our method demonstrated here provides a method for quickly checking this. Our method does not in general allow one of easily calculate the correction to the lowest order of the static gravitational potential on the brane. Reproduction of Newton’s law of gravity to lowest order seems to be a generic property of (6D) models constructed in this manner. Several other models have achieved the same result although often without the added feature of regularity in the bulk and with added restrictions on the brane [10]. It is the regular closure of the bulk geometry at the totally geodesic submanifolds (bolts) that gives hope to the idea that perturbations of black holes and gravitational waves will not grow without limit and thus further study should be carried out in this direction.
V. ACKNOWLEDGEMENTS

We thank Dr. David Wiltshire for introducing us to this problem and for his supervision. This research was supported in part by the Marsden Fund administered by the Royal Society of New Zealand.

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