A Note on Traces of Singular Moduli

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(Communicated by M. Kurihara)

Abstract. We generalize Osburn’s work ([6]) about a congruence for traces defined in terms of Hauptmoduli associated to certain genus zero groups of higher levels.

1. Introduction

Let $\mathbb{H}$ denote the complex upper half-plane and $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. For an integer $N (\geq 2)$, let $\Gamma_0(N)^*$ be the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions $W_e$ for $e|N$. There are only finitely many $N$ for which the modular curve $\Gamma_0(N)^* \backslash \mathbb{H}^*$ has genus zero ([5]). In particular, if we let $\mathcal{S}$ be the set of such $N$ which are prime, then

$$\mathcal{S} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

For each $p \in \mathcal{S}$, let $j_p^*(\tau)$ be the corresponding Hauptmodul with a Fourier expansion of the form $q^{-1} + O(q)$ where $q := e^{2\pi i \tau}$.

Let $p \in \mathcal{S}$. For an integer $d \geq 1$ such that $-d \equiv \square (\text{mod } 4p)$, let $Q_d$ be the set of all positive definite integral binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

of discriminant $-d = b^2 - 4ac$. To each $Q \in Q_d$, we associate the unique root $\alpha_Q \in \mathbb{H}$ of $Q(x, 1)$. Consider the set

$$Q_d, p := \{[a, b, c] \in Q_d : a \equiv 0 \pmod{p}\},$$

on which $\Gamma_0(p)^*$ acts. We then define the trace $t^{(p)}(d)$ by

$$t^{(p)}(d) := \sum_{Q \in Q_d, p/\Gamma_0(p)^*} \frac{1}{\omega_Q} j_p^*(\alpha_Q) \in \mathbb{Z},$$

where $\omega_Q$ is the number of stabilizers of $Q$ in the transformation group $\pm \Gamma_0(p)^* / \pm 1$ ([4]).

Osburn ([6]) showed the following congruence:

\[ \text{Received November 29, 2010; revised February 22, 2011} \]
\[ \text{2010 Mathematics Subject Classification: } 11F30, 11F33, 11F37 \]
\[ \text{Key words and phrases: } \text{Singular moduli, modular forms, congruences} \]

The first named author was partially supported by the NRF of Korea grant funded by MEST (2010-0000798). *The corresponding author was supported by Hankuk University of Foreign Studies Research Fund of 2012.\]
THEOREM 1.1. Let $p \in \mathcal{S}$. If $d \geq 1$ is an integer such that $-d \equiv \square \pmod{4p}$ and $\ell \neq p$ is an odd prime which splits in $\mathbb{Q}(\sqrt{-d})$, then

$$t^{(p)}(\ell^2d) \equiv 0 \pmod{\ell}.$$

Although this result is true, we think that his proof seems to be unclear. Precisely speaking, let $D \geq 1$ be an integer such that $D \equiv \square \pmod{4p}$. In §3 we shall define

$$A_\ell(D, d) := \text{the coefficient of } q^D \text{ in } f_{d, p}(\tau)|T_{1/2, p}(\ell^2),$$

$$B_\ell(D, d) := \text{the coefficient of } q^d \text{ in } g_{D, p}(\tau)|T_{3/2, p}(\ell^2),$$

where $f_{d, p}(\tau)$ and $g_{D, p}(\tau)$ are certain half integral weight modular forms, and $T_{1/2, p}(\ell^2)$ and $T_{3/2, p}(\ell^2)$ are Hecke operators of weight $1/2$ and $3/2$, respectively. The key step that is not presented in Osburn’s work is the fact $A_\ell(1, d) = -B_\ell(1, d)$ which would be nontrivial at all. In this paper we shall first give a proof of more general statement $A_\ell(D, d) = -B_\ell(D, d)$ (Proposition 3.1), and then further generalize Theorem 1.1 as follows,

$$t^{(p)}(\ell^{2n}d) \equiv 0 \pmod{\ell^n}$$

for all $n \geq 1$ (Theorem 3.3).

2. Preliminaries

Let $k$ and $N \geq 1$ be integers. If $f(\tau)$ is a function on $\mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, then we define the slash operator $[\gamma]_{k+1/2}$ on $f(\tau)$ by

$$f(\tau)[[\gamma]_{k+1/2} := j(\gamma, \tau)^{-2k-1}f(\gamma \tau),$$

where

$$j(\gamma, \tau) := \left( \frac{c}{d} \right) \varepsilon_d^{-1} \sqrt{ct+d} \quad \text{with} \quad \varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

Here, $\left( \frac{c}{d} \right)$ is the Kronecker symbol and $\sqrt{ct+d}$ takes its argument on the interval $(-\pi/2, \pi/2]$.

We denote by $M_{k+1/2}(N)^{\dagger}$ the infinite dimensional vector space of weakly holomorphic modular forms of weight $k+1/2$ on $\Gamma_0(4N)$ which satisfy the Kohnen plus condition. Namely, the space consists of the functions $f(\tau)$ on $\mathcal{H}$ such that

(i) $f(\tau)$ is holomorphic on $\mathcal{H}$ and meromorphic at the cusps,

(ii) $f(\tau)$ is invariant under the action of $[\gamma]_{k+1/2}$ for all $\gamma \in \Gamma_0(4N)$,

(iii) $f(\tau)$ has a Fourier expansion of the form

$$\sum_{n \equiv \square \pmod{4N}} a(n)q^n.$$
Suppose that $\ell$ is a prime with $\ell \nmid N$. The action of the Hecke operator $T_{k+1/2,N}(\ell^2)$ on a form

$$f(\tau) = \sum_{(-1)^{k}n \equiv \square \pmod{4N}} a(n)q^n \in M_{k+1/2}^{+,+}(N)$$

is given by

$$f(\tau)|T_{k+1/2,N}(\ell^2) := \ell_k \sum_{(-1)^{k}n \equiv \square \pmod{4N}} \left( a(\ell^2n) + \frac{(-1)^{k}n}{\ell} \ell^{k-1}a(n) + \ell^{2k-1}a(n/\ell^2) \right)q^n,$$

where

$$\ell_k := \begin{cases} \ell^{1-2k} & \text{if } k \leq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Here, $a(n/\ell^2) := 0$ if $\ell^2 \nmid n$. As is well-known, $f(\tau)|T_{k+1/2,N}(\ell^2)$ belongs to $M_{k+1/2}^{+,+}(N)$. 

**PROPOSITION 2.1.** Let $p \in \mathbb{S}$.

(i) For every integer $D$ ($\geq 1$) such that $D \equiv \square \pmod{4p}$, there is a unique $g_{D,p}$ in $M_{3/2}^{+,+}(p)$ with the Fourier expansion

$$g_{D,p}(\tau) = q^{-D} + \sum_{d \geq 0, \quad -d \equiv \square \pmod{4p}} B(D, d)q^d \quad (B(D, d) \in \mathbb{Z}).$$

(ii) For every integer $d$ ($\geq 0$) such that $-d \equiv \square \pmod{4p}$, there is a unique form

$$f_{d,p}(\tau) = \sum_{D \in \mathbb{Z}} A(D, d)q^D \quad (A(D, d) \in \mathbb{Z})$$

in $M_{1/2}^{+,+}(p)$ with a Fourier expansion of the form $q^{-d} + O(q)$. They form a basis of $M_{1/2}^{+,+}(p)$. 

(iii) For every integer $d$ ($\geq 0$) such that $-d \equiv \square \pmod{4p}$ and every integer $D$ ($\geq 1$) such that $D \equiv \square \pmod{4p}$, we have

$$A(D, d) = -B(D, d).$$

(iv) For every integer $d$ ($\geq 1$) such that $-d \equiv \square \pmod{4p}$, we get

$$t^{(p)}(d) = -B(1, d).$$

**PROOF.** See [1, Theorem 5.6], [3, §2.2] and [4, Lemma 3.4 and Corollary 3.5].
3. Generalization of Theorem 1.1

We first prove the following necessary proposition by adopting Zagier’s argument ([7, Theorem 5]).

**Proposition 3.1.** Let \( p \in \mathfrak{S} \) and \( \ell \neq p \) be a prime. For each integer \( d \geq 0 \) such that \( -d \equiv \square \pmod{4p} \), we define integers \( A_\ell(D, d) \) and \( B_\ell(D, d) \) in the following manner:

\[
A_\ell(D, d) := \text{the coefficient of } q^D \text{ in } f_{d,p}(\tau)|T_1/2,p(\ell^2) \text{ for each integer } D, \\
B_\ell(D, d) := \text{the coefficient of } q^d \text{ in } g_{D,p}(\tau)|T_3/2,p(\ell^2) \text{ for each integer } D \geq 1 \text{ such that } D \equiv \square \pmod{4p}.
\]

Then we have the relation

\[
A_\ell(D, d) = -B_\ell(D, d) \text{ for every integer } D \geq 1 \text{ such that } D \equiv \square \pmod{4p}.
\]

**Proof.** For a pair of rational numbers \( a \) and \( b \), let

\[
\delta_{a,b} := \begin{cases} 
1 & \text{if } a = b \in \mathbb{Z} \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( d \geq 0 \) be a fixed integer such that \( -d \equiv \square \pmod{4p} \). It follows from the defining property of \( f_{d,p}(\tau) \), namely,

\[
A(D, d) = \delta_{D,-d} \text{ if } D \leq 0
\]

that if \( D \leq 0 \), then

\[
A_\ell(D, d) = \ell A(\ell^2 D, d) + \left( \frac{D}{\ell} \right) A(D, d) + A(D/\ell^2, d) \text{ by the definition (2.1)}
= \ell \delta_{\ell^2 D,-d} + \left( \frac{D}{\ell} \right) \delta_{D,-d} + \delta_{D/\ell^2,-d}
= \ell \delta_{D,-d} + \delta_{D,-d} + \delta_{D,-d}. 
\]

Hence the principal part of \( f_{d,p}(\tau)|T_1/2,p(\ell^2) \) at infinity is

\[
\ell q^{-d/\ell^2} + \left( \frac{-d}{\ell} \right) q^{-d} + q^{-d \ell^2},
\]

where the first term should be omitted unless \(-d/\ell^2\) is an integer. Therefore we achieve

\[
f_{d,p}(\tau)|T_1/2,p(\ell^2) = \ell f_{d/\ell^2,p}(\tau) + \left( \frac{-d}{\ell} \right) f_{d,p}(\tau) + f_{d\ell^2,p}(\tau) \text{ by Proposition 2.1(ii). (3.1)}
\]

And, for every integer \( D \geq 1 \) such that \( D \equiv \square \pmod{4p} \) we derive that

\[
A_\ell(D, d) = \ell A(D, d/\ell^2) + \left( \frac{-d}{\ell} \right) A(D, d) + A(D, d \ell^2) \text{ by (3.1)}
\]
On the other hand, we apply Jenkins’ idea ([2]) to develop a formula for the coefficient $B(D, \ell^2 d)$.

**Proposition 3.2.** Let $p \in \mathbb{S}$ and $\ell (\neq p)$ be a prime. If $d (\geq 0)$ and $D (\geq 1)$ are integers such that $-d \equiv \square \pmod{4p}$ and $D \equiv \square \pmod{4p}$, then

$$B(D, \ell^2 d) = \ell^n B(\ell^2 D, d) + \sum_{t=0}^{n-1} \left( \frac{D}{\ell^2} \right)^{n-t-1} \left( \left( \frac{D}{\ell^2} \right) - \left( \frac{-d}{\ell^2} \right) \right) \ell^t B(\ell^2 D, d)$$

for all $n (\geq 1)$.

**Proof.** From the definition (2.1), we have

$$A(\ell D, d) = \ell A(\ell^2 D, d) + \left( \frac{D}{\ell} \right) A(D, d) + A(D/\ell^2, d), \quad (3.2)$$

$$B(\ell D, d) = \ell B(D, d/\ell^2) + \left( \frac{-d}{\ell^2} \right) B(D, d) + B(D, d/\ell^2). \quad (3.3)$$

Combining Proposition 3.1 with (3.2), we get

$$B(\ell D, d) = \ell B(\ell^2 D, d) + \left( \frac{D}{\ell^2} \right) B(D, d) + B(D/\ell^2, d). \quad (3.4)$$

We then derive from (3.3) and (3.4) that

$$B(D, \ell^2 d) = \ell B(\ell^2 D, d)$$

$$+ \left( \frac{D}{\ell^2} \right) B(D, d) + B(D/\ell^2, d) - \ell B(D, d/\ell^2) = \left( \frac{-d}{\ell^2} \right) B(D, d). \quad (3.5)$$

The remaining part of the proof is exactly the same as that of [2] Theorem 1.1. Indeed, one can readily prove the proposition by using induction on $n$ and applying only (3.5). □

Now, we are ready to prove our main theorem which would be a generalization of Osburn’s result.

**Theorem 3.3.** With the same notations as in Theorem 1.1, we have

$$t^{(p)} (\ell^{2n} d) \equiv 0 \pmod{\ell^n}$$
for all \( n (\geq 1) \).

**Proof.** We achieve that
\[
\ell^{(p)}(\ell^{2n}d) = -B(1, \ell^{2n}d) \text{ by Proposition 2.1(iv)}
\]
\[
= -\ell^n B(\ell^{2n}, d) - \sum_{t=0}^{n-1} \left( \frac{1}{\ell} \right)^{n-t-1} \left( B(1/\ell^2, \ell^{2t}d) - \ell^{t+1} B(\ell^{2t}, d/\ell^2) \right)
\]
\[
- \sum_{t=0}^{n-1} \left( \frac{1}{\ell} \right)^{n-t-1} \left( \left( \frac{1}{\ell} \right) - \left( \frac{-d}{\ell} \right) \right) \ell^t B(\ell^{2t}, d) \text{ by Proposition 3.2}
\]
\[
= -\ell^n B(\ell^{2n}, d) \text{ by the facts that } 1/\ell^2 \text{ and } d/\ell^2 \text{ are not integers, and } \left( \frac{-d}{\ell} \right) = 1
\]
\[
\equiv 0 \pmod{\ell^n},
\]
as desired. \( \square \)

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