Generalized Jordan Right Derivations on Prime and Semiprime $\Gamma$-Rings

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Abstract

In this paper, we analyzed the basic properties and related theorems of generalized Jordan right derivations on prime and semiprime $\Gamma$-rings with their mathematical simulation. We mainly focused on the characterizations of 2-torsion free prime and semiprime $\Gamma$-rings by using Jordan Right Derivations. Important propositions and theorems related to generalized Jordan right derivation on prime and semiprime $\Gamma$-ring have been derived here with sufficient calculations. Our main objective is to prove the theorem that if $M$ be a 2-torsion free $\Gamma$-ring having a commutator right non-zero divisor which satisfies the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$ and if $D:M \to M$ be a generalized Jordan right derivation and $d:M \to M$ be its associated Jordan right derivation then $D$ is a generalized right derivation on $M$.

Keywords: $\Gamma$-Rings, Prime $\Gamma$-Rings, Semiprime $\Gamma$-Rings, Generalized Derivation, Generalized right derivation, Generalized Jordan Right Derivation.

I. Introduction

The notion of a $\Gamma$-ring was first introduced as an extensive generalization of the concept of a classical ring. From its first appearance, the extensions and generalizations of various important results in the theory of classical rings to the theory of $\Gamma$-rings have been attracted a wider attentions as an emerging area of research to the modern algebraists to enrich the world of algebra. All over the world, many prominent mathematicians have worked out on this interesting area of research to determine many basic properties of $\Gamma$-rings and have executed more productive and creative results of $\Gamma$-rings in the last few decades.

N. Nobusawa [VIII] introduced $\Gamma$-ring as a generalization of ternary rings. Barnes [IX] generalized the concept of N. Nobusawa $\Gamma$-ring and gave a concrete definition of a $\Gamma$-ring. Barnes, Luh and Kyuno studied the structure of $\Gamma$-rings and obtained various generalizations analogous to corresponding parts in ring theory. Since then some papers have been published on the topic of $\Gamma$-rings. M. Soyturk [VII] investigated the commutativity of prime $\Gamma$-rings with left and right derivations. He obtained some
significant results on the commutativity of prime Γ-rings of characteristic not equal to 2 and 3. Y. Ceven [X] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring \( M \) that makes \( M \) commutative if \( aab\beta c = a\beta bac \), for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \). With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it. Mustafa Asci and SahinCeran [V] studied on a nonzero left derivation \( d \) on a prime Γ-ring \( M \) for which \( M \) is commutative with the conditions \( d(U) \subseteq U \) and \( d^2(U) \subseteq Z \), where \( U \) is an ideal of \( M \) and \( Z \) is the centre of \( M \). Sapanç and A. Nakajima [VI] defined a derivation and a Jordan derivation on Γ-rings and investigated a Jordan derivation on a certain type of completely prime Γ-ring which is a derivation. They proved that every Jordan derivation on a 2-torsion free completely prime Γ-rings is a derivation. They also gave examples of a derivation and a Jordan derivation of Γ-rings. A.C. Paul and Md. Mizanor Rahman [I] worked on Jordan left derivations on semiprime Γ-rings. Md. Mizanor Rahman and A.C. Paul [III, IV] have also worked on derivations and generalized derivations on Lie ideals of completely semiprime Γ-rings.

II. Preliminaries

In this section some definitions have been discussed which are important for representing our main objective in the later sections.

II.i. Γ-ring: Let \( M \) and \( \Gamma \) be two abelian groups. If there is a mapping \( M \times \Gamma \times M \rightarrow M \) such that the conditions
1. \( xay \in M \)
2. \((x + y)az = xaz + yaz, x(\alpha + \beta)y = xay + x\beta y, x\alpha(y + z) = xay + xaz \)
3. \((xay)\beta z = x\alpha(y\beta z) \)
are satisfied for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \), then \( M \) is called a Γ-ring.

Example: Let \( R \) be a ring of characteristic 2 having a unity element 1. Let \( M = M_{1,2}(R) \) and \( \Gamma = \{ \begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix} : n \in Z \} \), then \( M \) is a Γ-ring. If we assume \( N = \{ (x, x) : x \in R \} \subseteq M \), then \( N \) is also a Γ-ring of \( M \).

II.ii. Prime Γ-ring: A Γ-ring \( M \) is said to be a prime Γ-ring if \( x\Gamma M\Gamma y = 0 \) (with \( x, y \in M \)) implies \( x = 0 \) or \( y = 0 \). In similar manner, \( M \) is said to be prime if the zero ideal is prime.

II.iii. Semiprime Γ-ring: A Γ-ring \( M \) is said to be a prime Γ-ring if \( x\Gamma M\Gamma x = 0 \) (with \( x \in M \)) implies \( x = 0 \).

II.iv. Commutative Γ-ring: A Γ-ring \( M \) is said to be a commutative Γ-ring if \( x\Gamma y \Gamma = y\Gamma x \Gamma \) for all \( x, y \in M \) and \( \alpha \in \Gamma \). Again, \( [x, y]_\alpha = x\Gamma y - y\Gamma x \) is called a commutator of \( x \) and \( y \) with respect to \( \alpha \), where \( x, y \in M \) and \( \alpha \in \Gamma \).

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II.v. n-torsion free or characteristic not equal to n: A Γ-ring $M$ is said to be $n$-torsion free or characteristic not equal to $n$, denoted as char. $M \neq n$, if $nx = 0$ implies $x = 0$ for all $x \in M$. $M$ is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$.

II.vi. Derivation: Let $M$ be a Γ-ring and $d: M \to M$ an additive map. Then $d$ is called a derivation if

$$d(xy) = d(x)yx + xad(y) \quad \text{where } x, y \in M, \alpha \in \Gamma.$$  

Example: Let $M$ be a Γ-ring. If we define the map $D: M \to M$ by $D((x, y)) = (d(x), d(y))$ then $D$ is a derivation on $M$. Let $d: M \to M$ defined by $d(A) = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ then $d$ is a derivation.

II.vii. Right Derivation: Let $M$ be a Γ-ring and $d: M \to M$ an additive map. Then $d$ is called a right derivation if

$$d(ab) = d(a)ab + d(b)a \quad \text{where } a, b \in M, \alpha \in \Gamma.$$  

II.viii. Generalized Derivation: An additive mapping $D: M \to M$ is said to be a generalized derivation if there exists a derivation $d: M \to M$ such that $D(xy) = D(x)yx + xad(y)$ for all $x, y \in M, \alpha \in \Gamma$.

Example: Let $M$ be a Γ-ring and $F: M \to M$ be the additive map defined by $F((x, y)) = (f(x), f(y))$. Then $F$ is a generalized derivation on $M$.

II.ix. Jordan Derivation: Let $M$ be a Γ-ring and $d: M \to M$ an additive map. Then $d$ is called a Jordan derivation if

$$d(xa) = d(x)ax + xad(x) \quad \text{where } x \in M, \alpha \in \Gamma.$$  

Example: Let $M$ be a Γ-ring. If we define the map $D: M \to M$ by $D((x, y)) = (d(x), d(y))$ then $D$ is a Jordan derivation on $M$.

II.x. Generalized Jordan Derivation: An additive mapping $D: M \to M$ is said to be a generalized Jordan derivation if there exists a derivation $d: M \to M$ such that $D(xa) = D(x)ax + xad(x)$ for all $x \in M, \alpha \in \Gamma$.

Example: Let $M$ be a Γ-ring and $N = \{(x, x): x \in R\}$ be the subset of $M$. The map $F: N \to N$ defined by $F((x, x)) = (f(x), f(x))$ is a generalized Jordan derivation on $N$.

II.xi. Jordan Right Derivation: Let $M$ be a Γ-ring and $d: M \to M$ an additive map. Then $d$ is called a Jordan right derivation if

$$d(aa) = 2d(a)a \quad \text{where } a \in M, \alpha \in \Gamma.$$
II.xii. Generalized Jordan Right Derivation: An additive mapping $D: M \to M$ is called a generalized Jordan right derivation if there exists a Jordan right derivation $d: M \to M$ such that $D(axx) = D(x)ax + d(x)ax$ for all $x \in M$, $a \in \Gamma$.

III. Main Results

In this section for the sake of clarity, we prefer to split our presentation into two parts. The first part concerns with lemmas and the second part deals with our main objective.

Lemma I. Let $M$ be a 2-torsion free $\Gamma$-ring and $abc = a\beta b\alpha c$ holds for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Let $D: M \to M$ be a generalized Jordan right derivation and $d: M \to M$ be an associated Jordan right derivation. Then the following statements hold for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

\[ D((a + b)(a + b)) = D(a + b)(a + b) + (a + b)\alpha(a + b) \]
\[ = (D(a) + D(b))\alpha(a + b) + (a + b)\alpha(a + b) \]
\[ = D(a)\alpha a + D(b)\alpha b + D(a)\alpha a + D(b)\alpha b + d(a)b\alpha + d(b)b\alpha \]  
(1)

Also,

\[ D((a + b)(a + b)) = D(aaa + (a + b)(a + b)) + bab \]
\[ = D(aaa) + D(a + b)(a + b) + D(bab) \]
\[ = D(a)\alpha a + D(b)\alpha b + D(a + b)(a + b) + D(b)ab + d(b)b\alpha \]  
(2)

In view of (1) and (2), we get

\[ D(a)\alpha a + d(a)\alpha a + D(aab + baa) + D(b)ab + d(b)b\alpha \]
\[ = D(a)\alpha a + D(a)\alpha b + D(b)\alpha a + D(b)ab + d(a)\alpha a + d(a)b\alpha \]
\[ + d(b)a\alpha + d(b)b\alpha \]
\[ \therefore D(aab + baa) = D(a)\alpha b + d(a)ab + D(b)aa + d(b)aa \]

This completes the proof of (i).

(ii) In view of lemma I (i), consider the following

\[ D(a\beta b + b\beta a) = D(a)\beta b + d(a)\beta b + D(b)\beta a + d(b)\beta a \]
Replacing \( b \) by \( aab + ba \) in the above expression, we have

\[
D(ab(aab + ba) + (aab + ba) \beta a) = D(a)\beta(aab + ba) + d(a)\beta(aab + ba) + D(aab + ba) \beta a + d(aab + ba) \beta a
\]

Again,

\[
D(ab(aab + ba) + (aab + ba) \beta a) = D(a)\beta(aab + ba) + d(a)\beta(aab + ba) + \{D(a)ab + d(a)ab + D(b)\alpha a + d(b)\alpha a\} \beta a + 2d(a)ab \beta a + 2d(b)\alpha a \beta a
\]

\[
\Rightarrow 2D(aab \beta a) + D(ab aab + ba \beta a)
\]

\[
= D(a)\beta aab + D(a)\beta baa + d(a)\beta aab + d(a)\beta baa + D(a)ab \beta a + d(a)ab \beta a + D(b)\alpha a \beta a + d(b)\alpha a \beta a + 2d(a)ab \beta a + 2d(b)\alpha a \beta a
\]

\[
\Rightarrow 2D(ab \beta a) + D(ab aab + d(a)ab + D(b)ab + d(b)ab) + d(a)ab + d(a)ab + D(b)ab + d(b)ab + 2d(a)ab + 2d(b)ab
\]

By canceling identical terms and using the given condition \( ab \beta c = a \beta ba \) for all \( a, b, c \in M \) and \( a, \beta \in \Gamma \), we get

\[
2D(aab \beta a) + 2D(a)ab + 2D(a)ab + 2D(b)ab + 2D(b)ab + 2D(a)ab + 2D(b)ab = 2D(a)ab + d(a)ab + 2D(b)ab + d(b)ab
\]

As \( M \) is a 2-torsion free \( \Gamma \)-ring, so

\[
D(ab \beta a) = D(a)\beta baa + d(a)\beta (2baa - aab) + d(b)\beta baa
\]

This completes the proof of (ii).

(iii) Replacing \( a \) by \( a + c \) in (ii), we have

\[
D((a + c)ab \beta (a + c)) = D((a + c)ab \beta (a + c) + d(a + c)\beta (2b(a + c) + (a + c)ab) + d(b)\beta (a + c)\beta (a + c)
\]

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\[ D(aabbc + caba + aabba + cabbc) = (D(a) + D(c))βba(a + c) + (d(a) + d(c))β(2bba + 2bac - aab - cab) + d(b)βaaa + d(b)βbac \]

By applying lemma I(ii) in the above expression, we get

\[ D(aabbc + caba) + D(aαbc + caba) = D(aαbc + caba) = D(aαbc + caba) = D(aαbc + caba) = D(aαbc + caba) = D(aαbc + caba) \]

This completes the proof of (iii).

Lemma II. Let M be a 2-torsion free Γ-ring and the condition aabbc = aβbac holds for all α, β ∈ Γ. Let \( D: M \rightarrow M \) be a generalized Jordan right derivation and \( d: M \rightarrow M \) be its associated Jordan right derivation. Then the following statements hold for all \( a, b, c \in M \) and \( α, β \in Γ \):

(i) \( d(a)aαβ(aab - baa) = d(a)β(aab - baa)aαa \)

(ii) \( d(aαb) - d(a)αb - d(b)αaβ(aab - baa) = 0 \)

Proof: (i) Replacing c by aab in I(iii), we get

\begin{align*}
D(aabβ(aab)) + (aab)aβbαa = & \ D(aαb)βbα(aab) + D(aαb)βbα(aab) - (aab)αb + d(aαb)β(2baα(aab) - (aab)αb) + d(aαb)β(2baα(aab) - (aab)αb) + d(b)β(aa(aab) + (aab)aa) \\
\end{align*}

Again,

\[ D(aabβ(aab) + (aab)aβbαa) = D((aab)β(aab)) + D(aa(bab)βa) \]

By Definition and I(ii), we get

\[ D(aabβ(aab) + (aab)aβbαa) = D(aabβ(aab)) + D(aa(bab)βa) \]
This completes the proof of Equation (4) by comparing right hand side of (3) and (4) and canceling the identical terms and using \(aabb = abab\), we get

\[
\begin{align*}
D(a)\beta(ba(aab) + D(aab)\beta(aaab) = & D(a)\beta(ba(aab) + D(aab)\beta(aaab) \\
& + D(aaab)\beta(ba(aab) + d(a)\beta(aa(aaab) + (aab)) = D(aab)\beta(aaab) + D(aaab)\beta(aaab) + 2d(a)\beta(ba(aab) & + 2d(b)\beta(aaab) \\
\Rightarrow & d(aaab)\beta(ba(aab) - D(aaab)\beta(aaab) = d(a)\beta(ba(aab) - d(a)\beta(aaab) & - d(b)\beta(aaab) + 2d(b)\beta(aaab) \\
\Rightarrow & d(aaab)\beta(ba(aab) = d(a)\beta(ba(aab) & - d(b)\beta(aaab) + 2d(b)\beta(aaab) \\
\end{align*}
\]

Replacing \(b\) by \(a + b\) in (5), we get

\[
\begin{align*}
d(a)\beta((a + b)\alpha(a + b) & - (a + b)\alpha(a + b)) = d(a)\beta((a + b)\alpha(a + b) & - (a + b)\alpha(a + b) \\
& + d(a + b)\beta((a + b)\alpha(a + b) & - (a + b)\alpha(a + b)) \alpha(a + b) \\
\Rightarrow & (d(\alpha) + d(a))\beta(\alpha + b\alpha - a\alpha - a\alpha - a\alpha) = d(a)\beta(\alpha + b\alpha - a\alpha - a\alpha - a\alpha - a\alpha) \\
& + d(a)\beta(\alpha + b\alpha - a\alpha - a\alpha) + d(b)\beta(\alpha + b\alpha - a\alpha - a\alpha) \\
\Rightarrow & d(\alpha)\beta(\alpha + b\alpha - a\alpha) + d(a)\beta(\alpha + b\alpha - a\alpha) = d(a)\beta(\alpha + b\alpha - a\alpha) \\
& + d(a)\beta(\alpha + b\alpha - a\alpha) + d(b)\beta(\alpha + b\alpha - a\alpha) \\
\end{align*}
\]

Equation (5) and (6) becomes

\[
2d(a)\alpha\beta(b\alpha - a\alpha) = 2d(a)\beta(b\alpha - a\alpha)
\]

Since \(M\) is a 2-torsion free \(\Gamma\)-ring, so

\[
\begin{align*}
d(a)\alpha\beta(b\alpha - a\alpha) = & d(a)\beta(b\alpha - a\alpha) \\
\Rightarrow & d(a)\alpha\beta(aab - baa) = d(a)\beta(aab - baa)
\end{align*}
\]

This completes the proof of (i).

(ii) The replacement of \(a + b\) in (i) yields

\[
\begin{align*}
d(a + b)\alpha((a + b)\alpha(b + a) - b\alpha(a + b))) = & d(a + b)\beta((a + b)ab - b\alpha(a + b) \\
& + d(a + b)\beta((a + b)ab - b\alpha(a + b)) \alpha(a + b) \\
\Rightarrow & (d(a) + d(b))\alpha(b + a + b)\beta(aab + bab - baa - bab) = (d(a) + d(b))\beta(aab + bab - baa - bab)\alpha(a + b)
\end{align*}
\]
\[ d(a)\alpha\beta\alpha\beta(ab - baa) + d(b)\alpha\beta\alpha\beta(ab - baa) + d(a)\alpha\beta\alpha\beta(ab - baa) + d(b)\alpha\beta\alpha\beta(ab - baa) = d(a)\beta\alpha\beta(ab - baa) + d(b)\beta\alpha\beta(ab - baa) + d(a)\beta\alpha\beta(ab - baa) + d(b)\beta\alpha\beta(ab - baa) \]

By using (i) and equation (5) in the last relation, we get
\[ d(b)\alpha\beta\alpha\beta(ab - baa) + d(a)\alpha\beta\alpha\beta(ab - baa) = d(b)\beta\alpha\beta(ab - baa) + d(a)\beta\alpha\beta(ab - baa) \]

\[ d(a)\beta\alpha\beta(ab - baa) - d(a)\beta\alpha\beta(ab - baa) - d(b)\alpha\beta\alpha\beta(ab - baa) = 0 \]

\[ (d(a)b - d(a)b)\alpha\beta(ab - baa) = 0 \]

This completes the proof.

**Lemma III.** Let \( M \) be a 2-torsion free \( \Gamma \)-ring which satisfies the condition \( aab\beta\gamma = a\beta\gamma\alpha \) for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \). Let \( D: M \to M \) be a generalized Jordan right derivation and \( d: M \to M \) be its associated Jordan right derivation. Then \( (D(a)b - d(b)\alpha\beta)(a, b)\alpha\beta = 0 \).

**Proof:** Replacing \( c \) by \( aab - baa \) in lemma I (iii), we get

\[ D(aab\beta(aab - baa) + (aab - baa)\alpha\beta\alpha\beta) = D(a)\beta\alpha\beta(aab - baa) + D(aab - baa)\beta\alpha\beta\alpha\beta + d(a)\beta(2ba(aab - baa) - (aab - baa)ab) + d(aab - baa)\beta(2baa - aab) + d(b)\beta(aa(aab - baa) + (aab - baa)\alpha\beta \]

\[ = D(aab\beta(aab - baa) + (aab - baa)\alpha\beta\alpha\beta) = D(a)\beta\alpha\beta(aab - baa) + D(aab - baa)\beta\alpha\beta\alpha\beta + 2d(a)\beta\alpha\beta(aab - baa) - d(a)\beta[a, b]_\alpha ab + 2d([a, b]_\alpha)\beta\alpha\beta - d([a, b]_\alpha)\beta\alpha\beta + d(b)\beta\alpha\beta[a, b]_\alpha + d(b)\beta[a, b]_\alpha \]

Since \( d([a, b]_\alpha)\beta[a, b]_\alpha = 0 \), hence equation (7) becomes

\[ D(aab\beta(aab - baa) + (aab - baa)\alpha\beta\alpha\beta) = D(a)\beta\alpha\beta(aab - baa) + D(aab - baa)\beta\alpha\beta\alpha\beta + 2d(a)\beta\alpha\beta(aab - baa) - d(a)\beta[a, b]_\alpha ab + d([a, b]_\alpha)\beta\alpha\beta - d([a, b]_\alpha)\beta\alpha\beta + d(b)\beta\alpha\beta[a, b]_\alpha + d(b)\beta[a, b]_\alpha \]

\[ = (d(ab) - d(b)a)\alpha\beta(ab - baa) = 0 \]

On the other hand with the condition \( aab\beta\gamma = a\beta\gamma\alpha \), we have

\[ D(aab\beta(aab - baa) + (aab - baa)\alpha\beta\alpha\beta) = D(a)\beta\alpha\beta(aab - baa) + D(aab - baa)\beta\alpha\beta\alpha\beta + 2d(a)\beta\alpha\beta(aab - baa) - d(a)\beta[a, b]_\alpha ab + d([a, b]_\alpha)\beta\alpha\beta - d([a, b]_\alpha)\beta\alpha\beta + d(b)\beta\alpha\beta[a, b]_\alpha + d(b)\beta[a, b]_\alpha \]

(8)
Again, we have
\[ d(aab)\beta[a, b]_\alpha + (aab - baa)\alpha\beta\alpha = D(aab)\beta aab - (aab)\beta baa \]
\[ \Rightarrow D(aab)\beta(aab - baa) + (aab - baa)\alpha\beta\alpha = D(aab)\beta aab - D(baa)\beta baa \]
\[ \Rightarrow D(aab)\beta(aab - baa) + (aab - baa)\alpha\beta\alpha = D(aab)\beta aab + (aab)\beta baa - D(baa)\beta baa \]
Combining (8) and (9) with \( aab\beta c = a\beta b\alpha c \), we obtain
\[ D(a)\beta ba[a, b]_\alpha + D(aab)\beta baa - D(baa)\beta baa + 2d(a)\beta baa[a, b]_\alpha - d(a)\beta[a, b]_\alpha ab + d([a, b]_\alpha)\beta baa + d(b)\beta[aa][b, b]_\alpha + d(b)\beta[a, b]_\alpha aa = D(aab)\beta aab + d(aab)\beta aab - D(baa)\beta baa - d(baa)\beta baa \]
\[ \Rightarrow D(a)\beta b[b, b]_\alpha + D(aab)\beta baa - D(aab)\beta aab + 2d(a)\beta b[a, b]_\alpha - d(a)\beta[a, b]_\alpha ab + d([a, b]_\alpha)\beta baa + d(b)\beta[a, b]_\alpha aa - d(aab)\beta aab + d(baa)\beta baa = 0 \] (10)
Again, we have
\[ d(aab)\beta[a, b]_\alpha = d(a)\beta[a, b]_\alpha ab + d(b)\beta[a, b]_\alpha aa \]
\[ \Rightarrow d(b)\beta[a, b]_\alpha aa = d(aab)\beta[a, b]_\alpha - (a)\beta[a, b]_\alpha ab \]
Hence from (10), we obtain
\[ (D(a)\alpha b - D(aab) + d(b)\alpha a)\beta[a, b]_\alpha + 2d(a)\alpha b\beta[a, b]_\alpha - d(a)\beta[a, b]_\alpha ab + d([a, b]_\alpha)\beta baa + d(aab)\beta aab - d(aab)\beta baa - d(baa)\beta baa = 0 \]
\[ \Rightarrow (D(a)\alpha b - D(aab) + d(b)\alpha a)\beta[a, b]_\alpha + 2d(a)\alpha b\beta[a, b]_\alpha - d(a)\beta[a, b]_\alpha ab + d([a, b]_\alpha)\beta baa + d(aab)\beta baa - d(aab)\beta baa + d(baa)\beta baa = 0 \]
\[ \Rightarrow (D(a)\alpha b - D(aab) + d(b)\alpha a)\beta[a, b]_\alpha + 2d(a)\alpha b\beta[a, b]_\alpha - d(a)\beta[a, b]_\alpha ab = 0 \]
\[ \Rightarrow (D(a)\alpha b - D(aab) + d(b)\alpha a)\beta[a, b]_\alpha + 2d(a)\alpha b\beta[a, b]_\alpha - d(aab)\beta[a, b]_\alpha + 2d(b)\beta[a, b]_\alpha aa = 0 \]
\[ \Rightarrow (D(a)\alpha b - D(aab) + d(b)\alpha a)\beta[a, b]_\alpha + (2d(a)\alpha b - 2d(aab))\beta[a, b]_\alpha + 2d(b)\beta[a, b]_\alpha aa = 0 \]
\[ \Rightarrow (D(a)\alpha b - D(aab) + d(b)\alpha a)\beta[a, b]_\alpha - 2d(b)\alpha a\beta[a, b]_\alpha + 2d(b)\beta[a, b]_\alpha aa = 0 \]
Replacing (11), we obtain
\[ G(x, y)\beta[x, y]_a = 0, \text{ hence } G(x, y) = 0. \]
Replacing \( a \) by \( a + x \) in (11), we get
\[ G(a + x, b)\beta[a + x, b]_a = 0 \]
\[ \Rightarrow G(a, b)\beta[x, b]_a + G(x, b)\beta[a, b]_a = 0 \] (12)
Replacing \( b \) by \( b + y \) in (12), we get
\[ G(a, b + y)\beta[x, b + y]_a + G(x, b + y)\beta[a, b + y]_a = 0 \]
\[ \Rightarrow G(a, b)\beta[x, y]_a + G(a, y)\beta[x, b]_a + G(x, b)\beta[a, y]_a + G(a, y)\beta[x, y]_a = 0 \] (13)
Substituting \( x \) for \( a \) in (13) and using \( G(x, y) = 0 \), we have
\[ 2G(x, b)\beta[x, y]_a = 0 \]
Since $M$ is $2$-torsion free, hence
\[
G(x, b)\beta[x, y]_a = 0
\]
\[\therefore G(x, b) = 0, \text{ for all } b \in M.
\]
Again, putting $y$ for $b$ in (12), we get
\[
G(a, y) = 0, \text{ for all } a \in M
\]
Therefore equation (13) becomes
\[
G(a, b)\beta[x, y]_a = 0, \text{ for all } a, b \in M
\]
Hence
\[
G(a, b) = 0, \text{ for all } a, b \in M
\]
\[\Rightarrow D(aab) - D(a)ab - d(b)aa = 0
\]
\[\Rightarrow D(aab) = D(a)ab + d(b)aa
\]
Thus $D$ is a generalized right derivation on $M$.

**Theorem II.** Let $M$ be a $2$-torsion free $\Gamma$-ring which satisfies the condition $aab\beta c = a\beta bac$ for all $a, b \in M$ and $\alpha, \beta \in \Gamma$. Let $D: M \to M$ be a generalized Jordan right derivation and $d: M \to M$ be its associated Jordan right derivation. Then either $d = 0$ or $M$ is commutative.

**Proof:** For any $x, y \in M$, we have
\[
D(aaa\beta b) = D(aaa)\beta b + d(b)\beta aaa
\]
\[\Rightarrow D(aaa\beta b) = D(a)aaa\beta b + d(a)aaa\beta b + d(b)\beta aaa \quad (14)
\]
Again, we have
\[
D(aaa\beta b) = D(aaa(a\beta b)) = D(a)aaa\beta b + d(a\beta b)aa
\]
\[\Rightarrow D(aaa\beta b) = D(a)aaa\beta b + d(a)\beta baa + d(b)\beta aaa \quad (15)
\]
Combining (14) and (15) with the condition $aab\beta c = a\beta bac$, we have
\[
d(a)\beta aab - d(a)\beta baa = 0
\]
\[\Rightarrow d(a)\beta[a, b]_a = 0, \text{ for all } a, b \in M
\]
Thus for each $a \in M$, the primeness of $M$ implies that either $[a, b]_a = 0$ or $d(a) = 0$ for all $b \in M$. Now, we put $A = \{a \in M | d(a) = 0\}$ and $B = \{a \in M |[a, b]_a = 0 \text{ for all } b \in M\}$. Then clearly $A$ and $B$ are additive subgroups of $M$ whose union is $M$. But a group can not be written as a set theoretic union of two of its proper subgroups and hence we obtain that either $A = M$ or $B = M$. If $A = M$, then $d(a) = 0$ for all $a \in M$. On the other hand, if $B = M$, then $[a, b]_a = 0$ for all $a, b \in M$ and hence $M$ is commutative. This completes the proof of the theorem.

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