Substructure and the cusp and fold relations

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ABSTRACT
Gravitational lensing of a background source by a foreground galaxy lens occasionally produces four images of the source. The cusp and the fold relations impose conditions on the ratios of magnifications of these four-image lenses. In this theoretical investigation, we explore the sensitivity of these relations to the presence of substructure in the lens. Starting with a smooth lens potential, we add varying amounts of substructure, while keeping the source position fixed, and find that the fold relation is a more robust indicator of substructure than the cusp relation for the images. This robustness is independent of the detailed spatial distribution of the substructure, as well as of the ellipticity of the lensing potential and the presence of external shear.

Key words: gravitational lensing.

1 INTRODUCTION
Gravitational lensing describes the deflection of light rays from a background source due to the presence of a foreground concentration of mass (Refsdal 1964; Blandford & Narayan 1986; Schneider, Ehlers & Falco 1992). The source can be a star, a galaxy or a quasar, while the matter concentration, the lens, can also be a star, an individual galaxy or a cluster of galaxies. The geometry of space–time, defined by a particular cosmological model, is also an essential ingredient in gravitational lensing, as the strength of the deflection depends on the relative positions of the source, the lens and the observer. If the lens is a very massive object, such as a galaxy or a cluster of galaxies, and if it is appropriately aligned with the background source, then it can produce multiple images of a single source. This defines the ‘strong lensing’ regime. The salient feature of gravitational lensing is that this mapping from the source plane to the image plane preserves surface brightness. However, because the intrinsic flux of the source is usually unobservable, we cannot measure the magnifications of the lensed images directly. We can, however, measure their magnification ratios, because they are given by the ratios of the fluxes of the lensed images. Based on these flux ratios, we can then construct a model of the gravitational lens that will most accurately reproduce observed values.

Unfortunately, this modelling has turned out to be very difficult in practice, despite stringent constraints. The primary constraint on lens modelling comes from the fact that all smooth lens models exhibit certain magnification relations (see Section 2 below), which can be expressed in terms of the flux ratios of the lensed images. One of the most important issues in gravitational lensing is the number of lenses whose flux ratios violate these magnification relations.

Some of the lenses with so-called ‘anomalous flux ratios’ are B1422+231, PG1115+080, B0712+472, B2045+265 and SDSS 0924+0219 (Jackson et al. 1998; Fassnacht et al. 1999; Patnaik et al. 1999; Inada et al. 2003; Chiba et al. 2005). These lenses, among others, exhibit flux ratios that cannot be fit with smooth lens models. It is reasonable to assume that these anomalous flux ratios may be due to limitations in our lens models. For example, our lens models may have inaccurate or too few parameters. However, as Mao & Schneider (1998) first demonstrated, for the case of B1422+231 the discrepancy between observed and model-predicted flux ratios cannot be ascribed to an incorrect choice of parametrization, but rather to substructure in the lens.

This suggestion is compelling, because it is in accord with predictions of the granularity of the mass distribution inferred from high-resolution cosmological simulations in a cold dark matter (CDM) dominated universe. These simulations have found copious amounts of substructure on all scales in the universe, including galaxy scales (Moore et al. 1999; Mathis et al. 2002; Diemand, Moore & Stadel 2005). Metcalf & Madau (2001) quantified the effect of CDM substructure on flux ratios and found that $10^{2}–10^{8} \, M_{\odot}$ substructures near the Einstein radius can cause the flux ratios to deviate significantly from their model-predicted values. Soon afterwards, Dalal & Kochanek (2002) introduced a method to measure the abundance of substructure using lensing data, and concluded that substructure comprised $\approx 2$ per cent of the mass interior to the Einstein radius of typical lens galaxies. These deviations of the flux ratios can also be attributed to substructure along the line of sight rather than within the lens as shown by Mao et al. (2004) and Metcalf (2005).

Additionally, substructure induces magnification perturbations that have been shown to depend on image parity (Schechter & Wambsganss 2002). In earlier work, Bradac et al. (2004), examined the effect of substructures of mass $10^{8} \, M_{\odot}$ in a simulated galaxy on the cusp relation and the suppression of saddle points.

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They found that the cusp relation is violated but the presence of a disc even in the absence of substructure also brings about the violation. They concluded that the suppression of saddle points offered a more stringent indication of the presence of substructure in a lens.

Assuming, therefore, that substructures exist and that their effect is important, we would like to develop a robust diagnostic that quantifies their presence. Aided by previous work from Keeton, Gaudi & Petters (2003, 2005) and Macciò & Miranda (2006), we have two diagnostics at our disposal, namely, examining the cusp and the fold relations \( R_{\text{cusp}} \) and \( R_{\text{fold}} \) hereafter. The cusp and fold relations are model-independent predictions for the magnifications of highly magnified images in four-image lens. Deviations from their ideal values can be interpreted to indicate the presence of substructure. In the theoretical investigation presented in this paper, we compare the robustness of these two relations using the publicly available software GRAVLENS by Keeton (2001). We simulate a simple lens model with added substructure, and keep the source position fixed near a cusp caustic so that \( R_{\text{cusp}} \) and \( R_{\text{fold}} \) remain near their ideal values. Doing so ensures that only the added substructure, and not a varying source position, produces deviations in \( R_{\text{cusp}} \) and \( R_{\text{fold}} \). We then investigate how \( R_{\text{cusp}} \) and \( R_{\text{fold}} \) gauge the presence of substructure as a function of its mass and position in the lens.

This paper is organized as follows. In Section 2 we give a brief review of the magnification relations for folds and cusps, and describe the general properties of \( R_{\text{cusp}} \) and \( R_{\text{fold}} \). In Section 3 we use \( R_{\text{cusp}} \) and \( R_{\text{fold}} \) on a singular isothermal ellipsoid (SIE).\(^1\) These two sections also provide an overview of previous work. In Section 4 we add substructure to our lens and investigate the values of \( R_{\text{cusp}} \) and \( R_{\text{fold}} \) as the masses and positions of the substructure are varied. We also examine these relations when the ellipticity of the lens is varied and when external shear is added to the lens. We present our results and discuss its implications in Section 5.

2 THE CUSP AND THE FOLD RELATIONS

We begin with a brief review of the necessary lensing terminology. The lens equation for a gravitational lens system relates the impact parameter of the light ray on the source plane \( L \) to the position of the source on the source plane \( S \), by taking into account the deflection of the ray by the lens mass. It can be written in dimensionless form as \( y = x - \alpha(x) \), where \( y \) is the position of the source, \( x \) is the position of the lensed image on the lens plane and \( \alpha(x) \) is the bending angle vector, which accounts for the deflection of the light ray. The lens equation can also be viewed ‘in reverse’, as a map \( \eta: L \rightarrow S \) tracing the light ray backwards from the lens to the source plane. That is, we can view \( \eta \) as the assignment \( x \mapsto \eta(x) = x - \alpha(x) \). The inverse of the determinant of the Jacobian of this map, \( \det \left[ \text{Jac} \eta \right](x) \), for a lensed image at position \( x \) on the lens plane, gives the magnification of that image, and is conventionally referred to as the amplification matrix \( A(x) \). The magnification is a signed quantity: when it is negative, the image is called a saddle; when it is positive, it is called a minimum or a maximum.\(^2\) More-

\(^{1}\) In this paper, as in GRAVLENS, the ellipticity is defined by \( e = 1 - q \), where \( q \) is the axis ratio (the ratio of the minor axis to the major axis).

\(^{2}\) The terms ‘saddle’, ‘minimum’ and ‘maximum’ are standard in Morse theory. The number of minus signs appearing across the diagonal of the Hessian matrix of a Morse function gives the number of ‘downhill’ directions of that function. In particular, if there are no minus signs, then there are no downhill directions, and hence the function has a minimum at that point. See Petters et al. (2001) for a comprehensive treatment of the use of Morse theory in gravitational lensing.

Figure 1. The three types of image configurations: folds (top right-hand panel), cusps (middle right-hand panel) and crosses (bottom right-hand panel), with the corresponding source position and caustic curve to the left-hand panels. The lens potential used here is an SIE with ellipticity \( e = 0.5 \).

over, those positions \( x \) for which \( \det \left[ \text{Jac} \eta \right](x) = 0 \) formally have an infinite magnification. The collection of all such points on the lens plane defines the critical curve. The corresponding collection of points on the source plane defines the caustic curve. Focusing on the source plane, the smooth portions of the caustic curve are called its folds, while the points where two abutting folds coincide are called its cusps. Typical examples of critical curves, caustic curves, folds and cusps are shown in Fig. 1. For a detailed treatment of these concepts, see Blandford & Narayan (1986), Schneider et al. (1992) and Petters, Levine & Wambsganss (2001).

For folds and cusps, certain local relations between the magnifications of the multiple images are satisfied (Blandford & Narayan 1986; Schneider & Weiss 1992; Schneider et al. 1992; Zakharov 1995; Petters et al. 2001; Gaudi & Petters 2002a,b; Keeton et al. 2003, 2005). For example, when the source lies asymptotically close to a cusp caustic (see Fig. 1), the three closely spaced images (the so-called cusp triplet) should satisfy \( |\mu_A| = |\mu_B| + |\mu_C| \approx 0 \), where \( \mu_i \) is the signed magnification of image \( i \). For four-image lenses, Keeton et al. (2003) used this relation for the cusp triplet to define

\[ R_{\text{cusp}} = \frac{|\mu_A| - |\mu_B| + |\mu_C|}{\mu_A + |\mu_B| + |\mu_C|} = \frac{F_A - F_B + F_C}{F_A + F_B + F_C}, \]

(1)

where \( F_i = F_{\text{uc}} |\mu_i| \) is the flux of image \( i \) if the source has flux \( F_{\text{uc}} \) (we have essentially divided out by \( F_{\text{uc}} \) since it is unobservable, and are left with a dimensionless quantities). \( \mu_B \) is the magnification of the middle image, and there is no need to specify whether it is a minimum or a saddle (for an SIE, it is a saddle if the source lies on the long axis of the caustic, and a minimum if the source lies on the short axis). The ideal cusp relation \( R_{\text{cusp}} \rightarrow 0 \) is satisfied only when the source lies asymptotically close to the cusp caustic. To move beyond the asymptotic regime, Keeton et al. (2003) expanded the lens mapping in a Taylor series about the cusp to get

\[ R_{\text{cusp}} = 0 + A_{\text{cusp}} d^2 + \cdots, \]

(2)

where \( d \) is the maximum separation between the three images and the coefficient \( A_{\text{cusp}} \) is a function that depends on properties of the lens potential at the cusp point. In fact, Keeton et al. (2003) found that the properties of the lens that matter for \( A_{\text{cusp}} \) are the ellipticity, higher order multipole modes and external shear, whereas the radial mass profile is unimportant. Looking at equation (2), we see that
as the source moves a small but finite distance from the cusp, \( R_{\text{cusp}} \) picks up a term to second order in \( d \). Keeton et al. (2003) derived reliable upper bounds on \( R_{\text{cusp}} \), and concluded that these bounds would be violated only if the lens potential has significant structure on scales smaller than the distance between the images.

A similar magnification relation holds when the source lies near a fold caustic (see Fig. 1). In this case, two of the four images lie closely spaced together, straddling the critical curve, constituting the so-called fold image pair. One of these images is a minimum and the other a saddle. When the source lies asymptotically close to the fold caustic, the fold image pair should satisfy |\( \mu_{\text{min}} \) − |\( \mu_{\text{sad}} \) | ≈ 0. For four-image lenses, Keeton et al. (2005) used this relation to define

\[
R_{\text{fold}} \equiv \frac{|\mu_{\text{min}}| - |\mu_{\text{sad}}|}{|\mu_{\text{min}}| + |\mu_{\text{sad}}|} = \frac{F_{\text{min}} - F_{\text{sad}}}{F_{\text{min}} + F_{\text{sad}}},
\]

(3)

Like its predecessor, the ideal fold relation \( R_{\text{fold}} \to 0 \) is satisfied only when the source lies asymptotically close to the fold caustic. To move beyond the asymptotic regime, these authors expanded the lens mapping in a Taylor series about the fold point to get

\[
R_{\text{fold}} = 0 + A_{\text{fold}}d_{1} + \cdots ,
\]

(4)

where \( d_{1} \) is now the distance between the two images in the fold image pair and the coefficient \( A_{\text{fold}} \) is a function that depends on \( R_{\text{fold}} \) once again on those properties of the lens potential listed above. As we approach a cusp in the asymptotic regime, \( A_{\text{fold}} \to \infty \), where the sign depends on whether the cusp is on the long or short axis of the caustic. For a cusp on the long axis, \( A_{\text{fold}} \to +\infty \) because the middle image is a saddle, whereas for a cusp on the short axis, \( A_{\text{fold}} \to -\infty \) because the middle image is a minimum (remember that we are now looking at a cusp triplet, but using \( R_{\text{fold}} \) instead of \( R_{\text{cusp}} \), and that \( R_{\text{fold}} \) is to be evaluated on minimum/saddle pairs of images). This means that \( R_{\text{fold}} \) breaks down asymptotically close to a cusp, which is not too surprising because it is designed to be evaluated only on fold points. We mention this fact because in our calculations below we do indeed evaluate \( R_{\text{fold}} \) on cusp triplets, but we are safe in doing so because our source sits a small but finite distance from the cusp caustic, and is therefore not asymptotically close.

The more interesting feature of \( R_{\text{fold}} \), as Keeton et al. (2005) discovered, is that the validity of the ideal fold relation depends not just on how close the source is to the fold caustic, but also where precisely the source is along the caustic. This is reflected in the coefficient \( A_{\text{fold}} \), which takes on all values, both positive and negative, as one moves along the caustic. For this reason the authors introduced another variable, the distance \( d_{2} \) to the next nearest image (note that, as we are dealing with minimum/saddle image pairs only, neither \( d_{1} \) nor \( d_{2} \) will ever denote the distance between two saddles or two minima). They did so because both the source and the fold caustic are of course unobservable, and therefore the value of \( A_{\text{fold}} \), too, is unknown. Fortunately, however, the position of the source is encoded in the image configuration: not in the separation \( d_{1} \) between the fold image pair, but rather in the distance \( d_{2} \) to the next nearest image. (To give an example: a source near a fold but not near a cusp will have \( d_{1} \ll d_{2} \approx \theta_{E} \), whereas a source near a cusp will have \( d_{1} \sim d_{2} \ll \theta_{E} \), where \( \theta_{E} \) is the angular Einstein radius.\(^3\)) Hence the authors concluded that when deriving an upper bound on \( R_{\text{fold}} \), beyond which one can infer the presence of small-scale structure, \( d_{2} \) must also be taken into account along with \( d_{1} \). The sensitivity of \( R_{\text{fold}} \) with respect to the position of the source along the fold caustic will unfortunately make things difficult for us, and for this reason we will concentrate on cusp points only, not fold points; see Section 4.1 below.

Incidentally, in Fig. 1, and in the majority of our calculations below, we use as our fiducial lens model an SIE with ellipticity \( e = 0.5 \). Although we are restricting our theoretical analysis to an SIE, our results are nevertheless general, for the following reason. Keeton et al. (2005) generated a large ensemble of realistic lens potentials drawn from three different observational samples, with varying values of ellipticity, octupole modes and external shear. For each lens potential in their ensemble, they chose \( \sim 10^{6} \) source positions, solved the lens equation using GRAVLENS, and then computed \( (d_{1}, d_{2}, R_{\text{fold}}) \) for each minimum/saddle image pair. They then plotted \( R_{\text{fold}} \) on the \( (d_{1}, d_{2}) \) plane. This enabled them to extract the probability distribution of \( R_{\text{fold}} \) for fixed values of \( d_{1} \) and \( d_{2} \). Next, using an SIE with \( e = 0.5 \), they chose source positions such that \( d_{2}^{\text{fold}} = 1.54\theta_{E} \) and \( d_{2}^{\text{cusp}} = d_{2}^{\text{min}} = 0.46\theta_{E} \), and calculated \( R_{\text{fold}} \). They compared this value of \( R_{\text{fold}} \) to that of each of the lens potentials in their ensemble, for the same values of \( d_{1} \) and \( d_{2} \). Finally, for each minimum/saddle image pair, they found that the value of \( R_{\text{fold}} \) calculated in the case of the SIE fell well within the probability distribution of their ensemble, and thus concluded that each image pair of the SIE was consistent with lensing by a realistic potential. They repeated this procedure on the 23 known four-image lenses, and concluded that each case in which \( R_{\text{fold}} \) lay outside the probability distribution constituted strong evidence that small-scale structure was present for that particular lens. Given their result, we take as our ‘archetypal smooth lens potential’ an SIE with ellipticity \( e = 0.5 \). However, we also vary the ellipticity and external shear and evaluate their impact on our results.

3 INVESTIGATING \( R_{\text{cusp}} \) AND \( R_{\text{fold}} \)

Prior to adding point-mass substructures to the SIE, we calculate the values for \( R_{\text{cusp}} \) and \( R_{\text{fold}} \) for a smooth lens model, our control case. In what follows, we assume that the source sits a very small but finite distance from the cusp or the fold.

For a source near a fold caustic but not near a cusp (mathematically, this means that we pick a neighbourhood around our source that does not contain a cusp point), the image configuration is given in the top panel of Fig. 1. The two fold pair images (A, B) that straddle the critical curve have roughly equal and opposite magnifications, with ‘A’ a minimum and ‘B’ a saddle. As stated in Section 2, calculating \( R_{\text{fold}} \) for this pair gives

\[
|\mu_{A}| \approx |\mu_{B}| \Rightarrow R_{\text{fold}} \equiv \frac{|\mu_{A} - |\mu_{B}|}{|\mu_{A}| + |\mu_{B}|} \approx 0.
\]

(5)

Of course, we can apply \( R_{\text{fold}} \) to any minimum/saddle image pair. Now, the two fold pair images A and B will have much higher magnifications than the other two images C and D. So, for example, the fold relation for the combination of B (a saddle) and D (a minimum) gives

\[
|\mu_{B}| \gg |\mu_{D}| \Rightarrow R_{\text{fold}} \equiv \frac{|\mu_{B} - |\mu_{D}|}{|\mu_{B}| + |\mu_{D}|} \approx -1.
\]

(6)

\footnote{The Einstein radius defines the radial scale for a lensing configuration. If a star lies exactly behind another one, then due to the symmetry, a ring-like image appears. This ring has an angular radius \( \theta_{E} \) and a linear radius \( R_{E} = d_{E} \theta_{E} \), where \( d_{E} \) is the angular diameter distance from the observer to the lens plane. For distant galaxies acting as lenses, the angular Einstein radius \( \theta_{E} \) is of order 1 arcsec.}

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If, on the other hand, we choose A (a minimum) and C (a saddle), we get $R_{\text{fold}} \approx +1$.

The more interesting case is when a source lies near a cusp point, because we can use both relations $R_{\text{cusp}}$ and $R_{\text{fold}}$; see the middle panels of Fig. 1. In this case the magnifications of the two outer images A and C of the cusp triplet will have the same sign and the same magnitude, while the magnification of the middle image B will have the opposite sign and roughly twice the magnitude, with the sign depending on whether the source lies near a long axis cusp (B is a saddle) or near a short axis cusp (B is a minimum). Of course, as stated in Section 2, for the cusp triplet (A, B, C),

$$|\mu_{A,C}| \approx \frac{1}{2} |\mu_B| \Rightarrow R_{\text{cusp}} = \frac{|\mu_A| - |\mu_B| + |\mu_C|}{|\mu_A| + |\mu_B| + |\mu_C|} \approx 0. \quad (7)$$

We now apply $R_{\text{fold}}$ to the cusp triplet. If the middle image B is a saddle (a long axis cusp), then A and C are minima, so pairing B with either A or C gives

$$|\mu_{A,C}| \approx \frac{1}{2} |\mu_B| \Rightarrow R_{\text{fold}} = \frac{|\mu_{A,C}| - |\mu_B|}{|\mu_{A,C}| + |\mu_B|} \approx -\frac{1}{3}. \quad (8)$$

If, on the other hand, B is a minimum (a short axis cusp), then A and C are saddles, so we get $+1/3$. The fourth image D, not part of the cusp triplet, will have a magnification much less than that of A, B or C. For a long axis cusp, D is a saddle, so the fold relation for the combination of D with either A or C gives

$$|\mu_{A,C}| \gg |\mu_D| \Rightarrow R_{\text{fold}} = \frac{|\mu_{A,C}| - |\mu_B|}{|\mu_{A,C}| + |\mu_B|} \approx +1. \quad (9)$$

If, on the other hand, we have a short axis cusp, then D is a minimum while A and C are saddles, so we get $-1$.

Of all these values, the important ones for our purposes are equations (7) and (8), $R_{\text{cusp}} \approx 0$ and $R_{\text{fold}} \approx -1/3$, for the (long axis) cusp triplet: as we will show in the next section, substructure breaks the cusp triplet symmetry. The breaking of the symmetry causes the two outer images to no longer have identical magnifications (and thus the same value for $R_{\text{fold}}$). They are also no longer equidistant from the middle image.

4 SIMULATING THE EFFECTS OF SUBSTRUCTURE

We investigate the change to both the image configurations and to the values of $R_{\text{cusp}}$ and $R_{\text{fold}}$ when we distribute substructure in the form of point-masses on to our archetype smooth lens potential, an SIE with ellipticity $e = 0.5$. We consider random spatial distributions for 5–10 point-masses within the Einstein radius of our SIE. We examine cases when the spatial distribution of these point-masses is (1) less than, (2) roughly equal to and (3) greater than the distance between the images, but we do not place the substructure near the images: in each case, the distance of each of our substructure point-masses from either a cusp triplet or a fold image pair is greater than the distance $d_i$ of these images to each other. Thus we deliberately choose our spatial distributions such that they fall outside the usual regime of applicability for $R_{\text{cusp}}$ and $R_{\text{fold}}$ (Keeton et al. 2003, 2005; Macciò & Miranda 2006). A typical example of such a spatial distribution is shown in Fig. 2.

Moreover, since our examination involves simulated configurations rather than observational data, we can and choose to keep the source position fixed. For each spatial distribution, we gradually increase the mass of our substructure, starting from 0 (the control case), while keeping the position of the source fixed as close to a fold or cusp as GRAVLENS allows. Doing so ensures that $R_{\text{cusp}}$ and $R_{\text{fold}}$ will always deviate from close to their ideal values. In other words, by minimizing the distance of the source from the caustic, we minimize the effect on $R_{\text{cusp}}$ and $R_{\text{fold}}$ by the position of the source. We may then attribute deviations in $R_{\text{cusp}}$ and $R_{\text{fold}}$ solely to the presence of substructure. Using GRAVLENS to solve the modified lens equation (SIE + point-masses), we then calculate $R_{\text{cusp}}$ and $R_{\text{fold}}$ for these granular lenses. A similar method was pursued by Macciò & Miranda (2006), which we discuss along with our results in Section 5.

4.1 Sources near a fold caustic

When considering sources near a fold caustic, we encounter the following problem. In general, modifying the lens potential by adding substructure changes the shape of the caustic curve, and the nature of the change depends on both the spatial distribution of the substructure, and more importantly, their masses. To give an example: the spatial distribution of five point-masses in the manner shown in Fig. 2, with angular Einstein radii ranging from $2.5 \times 10^{-2} \leq \theta_{\text{pm}} \leq 0.1 \theta_{\text{SIE}}$ forms the caustic curve considerably, so that the position of the source will be shifted relative to the fold caustic. Given the sensitivity of $R_{\text{fold}}$ to where our source lies along the fold (see Section 2), we cannot say with confidence whether a change in the value of $R_{\text{fold}}$ is due to the addition of substructure, or simply because the position of our source has shifted with respect to the fold caustic. For this reason, we ignore sources lying near a fold.

4.2 Sources near a cusp caustic

Fortunately, we do not encounter this problem when we place our source near a cusp, because the cusp is a essentially a small...
wedge. Doing so gives us our most interesting result: when we add substructure on to our SIE, and consider a source near our new long axis cusp, we no longer have the symmetry in our cusp triplet. The cusp triplet tends to be displaced to one side, so that the two outer images are no longer identical. In fact, one of them is closer to the middle image. This is illustrated in Fig. 2. At this point an important issue arises: because the cusp point is more difficult to locate when substructure deforms the caustic curve, one may wonder whether the skewed cusp triplet shown in Fig. 2 is caused by the possibility that we have just missed the cusp point and instead placed our source (inadvertently) next to a fold caustic. If so, then we would be able to reproduce the same skewed cusp triplet in the case of an SIE without substructure, simply by displacing our source off the cusp by a small amount. But in fact this is not the case: a source displaced slightly off the cusp for an SIE without substructure produces a very tight fold configuration (such as in the top panel of Fig. 1), and nothing like the skewed cusp triplet shown in Fig. 2. Thus it is the substructure that breaks the cusp triplet symmetry.

5 RESULTS AND CONCLUSIONS

We now examine the sensitivity of the cusp and the fold relations to the spatial distribution of substructure shown in Fig. 2. In our calculations below, we set the angular Einstein radius of our SIE to \( \theta_{\text{SIE}} = 1.0 \) arcsec. For lens and source redshifts of \( z \) not asymptotically close, so we should not expect to satisfy the ideal cusp relation of Section 3. However, our source still sits a finite distance from the cusp, and Fig. 1). Thus it is the substructure that breaks the cusp triplet symmetry.

5.1 Dependence on substructure mass

First of all, for the control case of the smooth potential and no substructure, we find \( R_{\text{cusp}} = 2.0 \times 10^{-3} \) and \( R_{\text{fold}} = (-0.331, -0.331) \), where \( R_{\text{fold}} = -0.331 \) for both the leftmost/middle image pair and the rightmost/middle image pair. Looking at equation (8) of Section 3, we expect a value of \( R_{\text{fold}} \approx -0.333 \), so a value of \( -0.331 \) is close enough.

We now examine the change to these values as a function of the spatial distribution of substructure shown in Fig. 2, for three particular substructure masses:

\begin{align*}
(1) & \quad M_{\text{pm}} = 0.0 M_{\odot} \quad R_{\text{cusp}} = 2.0 \times 10^{-3} \quad R_{\text{fold}} = (-0.331, -0.331) \\
(2) & \quad M_{\text{pm}} = 1.0 \times 10^{-3} M_{\odot} \quad R_{\text{cusp}} = 5.0 \times 10^{-3} \quad R_{\text{fold}} = (-0.326, -0.332) \\
(3) & \quad M_{\text{pm}} = 5.0 \times 10^{-3} M_{\odot} \quad R_{\text{cusp}} = 6.0 \times 10^{-3} \quad R_{\text{fold}} = (-0.317, -0.340) \\
(4) & \quad M_{\text{pm}} = 1.0 \times 10^{-2} M_{\odot} \quad R_{\text{cusp}} = 6.0 \times 10^{-3} \quad R_{\text{fold}} = (-0.305, -0.352) \\
\end{align*}

The notation \( R_{\text{fold}} = (-0.317, -0.340) \) implies that the leftmost/middle image pair gives \( R_{\text{fold}} = -0.317 \), while the rightmost/middle image pair gives \( R_{\text{fold}} = -0.340 \); likewise for the others. One sees that, as the mass of the substructure increases, the value of \( R_{\text{fold}} \) for the leftmost/middle image pair consistently increases, towards 0. On the other hand, the value of \( R_{\text{fold}} \) for the rightmost/middle image pair consistently decreases, towards -1. These trends make sense, for the following reason. Fig. 2 shows that the leftmost and middle images of the cusp triplet draw closer together. GRAVLENSE indicates that their magnifications also draw closer together. The pair, in other words, begin to resemble a fold image pair, and the resemblance becomes more prominent as the mass of the substructure increases. In view of equation (5) of Section 3, we therefore expect \( R_{\text{fold}} \to 0 \). On the other hand, as it is displaced farther away from the leftmost/middle image pair, the magnification of the rightmost image decreases. In view of equation (6), we therefore expect \( R_{\text{fold}} \to -1 \) for the rightmost/middle image pair of the cusp triplet. The bottom left-hand panel of Fig. 3 confirms this to be the case.

\( R_{\text{cusp}} \), however, is unresponsive to changes in the mass of the substructure. The reason is that the cusp relation considers all three images together. Thus, although the magnifications of the leftmost and rightmost images of the cusp triplet increase and decrease, respectively, \( \Sigma_{\text{pm}} \) remains roughly constant. The top left-hand panel of Fig. 3 confirms this to be the case. The virtue of the fold relation, then, is that it considers only \( \Sigma_{\text{pm}} \) of images. It is consistently modified by the increase in the mass of the substructure, whereas \( R_{\text{cusp}} \) is not.

Our results bear comparison with a similar method pursued by Macciò & Miranda (2006). Using three particular cusp triplet configurations of an SIE with \( e = 0.33 \), they added varying amounts of substructure (using NFW haloes) near the region of the cusp triplet, solved for the modified lens equation and calculated \( R_{\text{cusp}} \). They repeated this procedure \( 10^4 \) times for each of their three cusp triplet configurations, and thus obtained a probability distribution for \( R_{\text{cusp}} \) in the presence of substructure. They concluded that the class of substructures they considered did not consistently modify the value that \( R_{\text{cusp}} \) had in the absence of substructure. This is in agreement with our results using \( R_{\text{cusp}} \) but Fig. 3 shows that the fold relation \( R_{\text{fold}} \), whose value on the cusp triplet Macciò & Miranda (2006) did not consider, is consistently modified by the substructure, and in such a way that it accurately gauges an increase in its mass. Moreover, since our substructure is not localized near the cusp triplet, we have shown that the fold relation continues to be reliable even

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5.2 Sensitivity to external shear

Both $R_{\text{cusp}}$ and $R_{\text{fold}}$ are expected to be sensitive to the presence of external shear. We therefore add external shear with an amplitude of $\gamma \sim 5$ per cent to our SIE. With this modification, we repeat the procedure above and find the following:

(i) $M_{\text{pm}} = 0.0 M_{\text{SIE}}$: $R_{\text{cusp}} = 5.0 \times 10^{-2}$ and $R_{\text{fold}} = (-0.329, -0.329)$;

(ii) $M_{\text{pm}} = 1.0 \times 10^{-3} M_{\text{SIE}}$: $R_{\text{cusp}} = 6.0 \times 10^{-3}$ and $R_{\text{fold}} = (-0.326, -0.330)$;

(iii) $M_{\text{pm}} = 5.0 \times 10^{-3} M_{\text{SIE}}$: $R_{\text{cusp}} = 6.0 \times 10^{-3}$ and $R_{\text{fold}} = (-0.318, -0.338)$;

(iv) $M_{\text{pm}} = 1.0 \times 10^{-2} M_{\text{SIE}}$: $R_{\text{cusp}} = 6.0 \times 10^{-3}$ and $R_{\text{fold}} = (-0.311, -0.346)$.

Once again, $R_{\text{fold}}$ is consistently modified by the change in the mass of the substructure, whereas $R_{\text{cusp}}$ is not. See the right-hand panels of Fig. 3.

5.3 Sensitivity to ellipticity

The ellipticity of our SIE affects the values of $R_{\text{cusp}}$ and $R_{\text{fold}}$. We find that for ellipticities lower than our archetypal value of $e = 0.5$, $R_{\text{cusp}}$ remains unresponsive to the substructure, whereas $R_{\text{fold}}$ is still sensitive to its mass, though not as robustly as for the case with $e = 0.5$. The left-hand panels of Fig. 4 demonstrate this explicitly for an ellipticity $e = 0.25$. For ellipticities higher than our archetypal value, $R_{\text{cusp}}$ displays erratic fluctuations, whereas $R_{\text{fold}}$ remains well correlated with the mass of the substructure. The right-hand panels of Fig. 4 demonstrate this explicitly for an ellipticity $e = 0.75$.

5.4 Concluding remarks

In this theoretical investigation, we attempt to develop a robust diagnostic that quantifies the presence of substructure in lens galaxies. To this end, we simulated a simple lens potential with added substructure, with the aim of seeing how the cusp and the fold relations, calculated while keeping the source position fixed near a cusp, respond to the presence of the substructure as we vary its mass and position on the lens. We took as our lens model an SIE with angular Einstein radius $\theta_{\text{SIE}} = 1.0$ arcsec and mass $M_{\text{SIE}}$ interior to the Einstein radius. We used point-masses as our substructure, and varied their mass from $6.0 \times 10^{-4} \leq M_{\text{pm}} \leq 1.0 \times 10^{-2} M_{\text{SIE}}$, or up to 1 per cent of the mass of our SIE. We then distributed them randomly within the Einstein radius of our SIE.

However, in an effort to move beyond the usual regime of applicability for the cusp and fold relations (Keeton et al. 2003, 2005; Macció & Miranda 2006), we did not restrict ourselves only to structure on scales smaller than the distance between the images. We thus considered cases when their spatial distribution was greater than the distance between the images in a cusp triplet. Moreover, the distance of our substructure point-masses from the cusp triplet was greater than the distance of these images to each other. Finally, we kept the source position fixed throughout, to minimize the effect on our results by the source’s distance to a caustic.

We found that the fold relation was consistently modified when we varied the mass and spatial distribution of our substructure, whereas the cusp relation was not. When we varied the ellipticity of our SIE, we found that for low ellipticities the cusp relation was unresponsive to the substructure, whereas for high ellipticities it became erratic.
The fold relation, on the other hand, remained well correlated with the mass of the substructure throughout. Considering the effect of external shear gave the same result: the fold relation remained well correlated with the substructure, whereas the cusp relation did not. We conclude, therefore, that the fold relation is the more robust diagnostic of substructure.

In order to apply this technique to real data, we will need to use the observed image positions and the distances between them, and then use Monte Carlo methods to constrain the corresponding source positions. While this is beyond the scope of this work, we pursue it in a follow-up paper.

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