Proton computed tomography reconstruction using a backprojection-then-filtering approach

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Abstract

A novel approach to proton CT reconstruction using backprojection-then-filtering (BPF) is proposed. A list-mode algorithm is formulated accommodating non-linear proton paths. The analytical form is derived for the deblurring kernel necessary for the filtering step. Further, a finite matrix correction is derived to correct for the limited size of the backprojection matrix. High quantitative accuracy in relative stopping power is demonstrated (≤0.1%) using Monte Carlo simulations. This accuracy makes the algorithm a promising candidate for future proton CT systems in proton therapy applications. For the purposes of reconstruction, each proton path in the object-of-interest was estimated based on a cubic spline fit to the proton entry and exit vectors. The superior spatial-resolution of the BPF method over the standard filtering-then-backprojection approach is demonstrated. As the BPF algorithm requires only one backprojection and filtering operation on a scan data set, it also offers computational advantages over an iterative reconstruction approach.

Keywords: proton computed tomography, tomographic reconstruction, proton therapy

(Some figures may appear in colour only in the online journal)
1. Introduction

As protons penetrate into a material, they typically lose energy in a quasi-continuous fashion, due to ionization and excitation of atomic electrons. Elastic scattering from atomic nuclei also occurs, resulting in cumulative small-angle scattering (Multiple Coulomb Scattering). Inelastic nuclear interactions are more catastrophic, but occur more rarely. The utility of imaging with protons instead of x-rays for proton therapy is an old idea that has recently received renewed interest. Cormack, over fifty years ago, considered performing Computed Tomography (CT) with protons to reconstruct stopping-power and obtain increased accuracy in treatment planning (Cormack 1963). He foresaw a fundamental difficulty: the statistical variations in the path and energy (or ‘straggling’) of protons after having passed through a given thickness of tissue. This difficulty did not prove insurmountable and the principle was subsequently demonstrated experimentally (Cormack and Koehler 1976, Hanson et al 1978).

Hanson proposed the incorporation of non-linear approximations to proton trajectories within the reconstruction (Hanson 1979) and a literature has built up on this topic (Schneider and Pedroni 1994, Williams 2004, Schulte et al 2008, Wang et al 2010). Motivated by the potential application in proton therapy planning, a number of research groups have contributed to developments in proton radiography in recent years (e.g. Schneider et al 2004, Civinini et al 2010, Sadrozinski et al 2013, Poludniowski et al 2014). There is also a small but growing body of research addressing the fundamental question of the CT reconstruction itself (Penfold et al 2009, Penfold et al 2010, Cirrone et al 2011, Rit et al 2013).

The canonical way to do CT reconstruction has for a long time been by the Filtered Backprojection (FBP) approach (Kak and Slaney 2001). This consists of:

(a) Uniformly sampling ray-projections through a 2D slice of a body with a detector;
(b) Convolving this array of data with a 1D kernel (or equivalently filtering in the spatial-frequency domain);
(c) Backprojecting the convolved ray-projections through a 2D reconstruction matrix;
(d) Repetition over a set of angles of orientation.

A complete volume (or voxel set) is built up out of a series of 2D slices. The elegance and economy of the approach have led to its dominance and it is still the primary method for reconstruction in hospital CT systems, although iterative/statistical methods of reconstruction are gaining ground (Beister et al 2012).

In proton CT reconstruction, incorporating non-linear estimates of each proton’s trajectory through the imaged object offers advantages, particularly in term of spatial resolution (Li et al 2006). The list-mode nature of the data and the non-parallel and non-linear proton paths, however, puts into question the suitability of the FBP method for such a reconstruction. Indeed a number of authors have instead chosen iterative strategies and accepted the greatly increased computational burden that accompanies the power of that approach (see e.g. Penfold et al (2010)). However, if FBP is pursued, a choice of how to bin the data must be made so that it can be filtered. Some authors have made the decision to apply strict cuts to eliminate large deviations from straight paths, at the cost of increasing imaging dose for the same number of useful protons (Cirrone et al 2011). The approach of performing a voxelwise rebinning of data has also been suggested and produced encouraging results (Rit et al 2013), but would necessitate substantially increased complexity and computation.

Yet the FBP approach is not the only analytic approach to tomographic reconstruction. An alternative approach of Backprojection-then-Filtering (BPF) is proposed here and it is argued that this method more naturally fits the nature of the problem. The quantitative accuracy of the
BPF approach is demonstrated after the application of suitable corrections for the finite size of the backprojection matrix.

2. Method

2.1. Backprojection-then-filtering

Both FBP and BPF are analytic techniques in that they solve the problem of going from the radon transform of an object (the measurements) to the reconstructed object (the 2D image slice) using each datum only once. The approach of Backprojection Filtering (BPF) has long been known (Suzuki and Yamaguchi 1988) but rarely used. The approach consists of:

(a) Sampling ray-projections through a 2D slice (not necessarily uniformly);
(b) Backprojecting the ray-projections through a 2D reconstruction matrix;
(c) Convolving (or filtering) this 2D matrix of data with a 2D kernel;
(d) Repetition over a set of angles of orientation.

The reordering of the filtering and backprojection operations between FBP/BPF approaches may appear trivial but has profound consequences. FBP has generally been preferred over BPF for a number of reasons (Zeng and Gullberg 1994, Zeng and Gullberg 1995). Firstly, the filtering operation in BPF is in 2D and therefore requires more computation and computer memory. Secondly, analytic results for BPF convolution kernels are more difficult to obtain. Thirdly, the BPF approach has had issues with quantitative accuracy. With modern computing power, the somewhat increased computational demand of using 2D Fourier transforms in BPF is not a substantial issue. The second and third points are important but novel contributions to dealing with these are described here. The question remains, why should we ever use BPF at all in preference to FBP? We suggest the following advantages for the proton CT problem: BPF naturally deals with list-mode data (one particle at a time) without the need for binning and it can naturally accommodate non-linear ray-projection paths. Because the filtering operation occurs after the backprojection of each proton path onto a regular matrix in image space, the binning requirement of FBP is completely side-stepped.

2.2. Mathematical formulation of BPF

Suppose a 2D slice of a body has some property described by the function, \( f(x, y) \), which is zero outside some region, \( \Omega_f \). In the case of proton CT the appropriate property is proton relative stopping-power (RSP). A line integral through the body, at an angle, \( \phi \) and displacement, \( t \) (see figure 1), can be written as,

\[
p(t, \phi) = \int \text{d}x \text{d}y \ f(x, y) \ \delta(x \cos \phi + y \sin \phi - t) .
\]  

(1)

This line integral is a ray-projection through the body slice and is related to water-equivalent-path-length (WEPL). The transform from the space \( (x, y) \) to \( (\phi, t) \) is the Radon transform.

The backprojection operation is defined as,

\[
b(x, y) = \int \text{d}t \text{d}\phi \ p(t, \phi) \ \delta(x \cos \phi + y \sin \phi - t)
\]  

(2)

\[
= \int_0^\pi \text{d}\phi \ p(t, \phi) \ \left| x \cos \phi + y \sin \phi = t \right.
\]  

(3)
This is equivalent to tracing back along the ray-projection path and depositing the value of the line-integral at each location that it passes through. The resulting backprojection function, \( b(x, y) \), resembles a blurred-out representation of the density function, \( f(x, y) \). In fact they are related by (Zeng and Gullberg 1994),

\[
 b(x, y) = f(x, y) * \frac{1}{r},
\]

where \(*\) denotes a 2D convolution and \( r = \|(x, y)\| \). Note that even though we required \( f(x, y) \) to be zero outside the region \( \Omega_f \), the function \( b(x, y) \) is non-zero everywhere.

The convolution relation can be expressed as a filtering operation in 2D Fourier space. Forward and inverse Fourier transforms will be indicated by \( \mathcal{F} \{ \ldots \} \) and \( \mathcal{F}^{-1} \{ \ldots \} \), respectively.

Let the Fourier conjugate variables to \((x, y)\) be \((u, v)\) and \( \rho = \|(u, v)\| \). Then we can write:

\[
 f(x, y) = \mathcal{F}^{-1} \left\{ \frac{\mathcal{F} \{ b \}}{\mathcal{F} \{ r^{-1} \}} \right\} = \mathcal{F}^{-1} \left\{ \mathcal{F} \{ b \} * k \right\} = \mathcal{F}^{-1} \{ b(x, y) * k(x, y) \},
\]

where \( k(x, y) \) is a spatial kernel. The following result (valid for 2D Fourier transforms) has also been used: \( \mathcal{F} \{ r^{-1} \} = \rho^{-1} \).

2.3. The backprojection operation

For practical implementation in an algorithm, a discretized formula for the backprojected image, \( b \), needs to be used. The 2D matrix for \( b \) can be calculated from acquired measurements in a number of ways. Imagine that projections are acquired at a set of discrete rotation angles, \( \phi_l = \frac{\pi l}{L} \) where \( l \in \{1, 2, ..., L\} \). At each angle, \( K \) ray-projections are sampled (not necessarily uniformly) over the field encompassing the object with each ray designated an index, \( k \in \{1, 2, ..., K\} \). This provides a discretized radon transform matrix, \( p[k, l] \). Now consider a discrete 2D backprojection matrix, \( b[i, j] \), with corresponding pixel centres \( x_i = (2i - N - 1)\tau/2 \) and \( y_j = (2j - N - 1)\tau/2 \). Here, \( \tau \) is the matrix pitch and \( i \in \{1, 2, ..., N\} \) and

\[
 x_i = (2i - N - 1)\tau/2 \quad \text{and} \quad y_j = (2j - N - 1)\tau/2.
\]
$j \in \{1, 2, ..., N\}$. The backprojection matrix can be expressed in terms of the discrete radon transform as follows:

$$b[i,j] = \frac{\lambda}{L} \sum_{j=1}^{L} \sum_{k=1}^{K} \frac{\lambda_k[i,j]}{\sum_{k=1}^{K} \lambda_k[i,j]} p[k,l].$$  \hspace{1cm} (6)

where $\lambda_k[i,j]$ is the path length of the $k$th ray of the $l$th projection through the pixel $[i,j]$.

### 2.4. The filtering operation

Two observations should be made at this point. Firstly, by selecting a sampling pitch for backprojection the convolution kernel becomes band-limited in spatial-frequency. Secondly, a finite matrix size must be chosen for the kernel. To acquire quantitatively accurate results, the band-limit due to sampling pitch must be imposed on the filter in the frequency domain and then the resulting spatial kernel sampled on a finite matrix in the spatial domain. If this procedure is not followed an almost constant shift in all the reconstructed values results (Zeng and Gullberg 1995).

The band-limit on spatial-frequencies is imposed by modifying the ramp (or ‘rho’) filter representation of the kernel in frequency space:

$$K(\rho) = \rho \quad \text{if} \quad \rho < (2\pi)^{-1}$$

$$= 0 \quad \text{otherwise}$$  \hspace{1cm} (7)

(8)

This is exactly as is done in the classic Filtered Backprojection (FBP) algorithm with one important difference: the relationship between the convolution kernel and filter involves a 2D not a 1D Fourier transform. This renders the analytic evaluation more difficult as remarked by Zeng and Gullberg (1995). In fact, we were unable to find the analytic form of the 2D spatial kernel, $k(r)$, in the literature. However, it can be obtained as follows:

$$k(r) = \mathcal{F}^{-1}\{K(\rho)\}$$

$$= \frac{1}{\pi} \int_{0}^{1/2\pi} \rho \log(\rho) \left[ J_0(2\pi\rho) - \Phi\left(\frac{\pi\rho}{\tau}\right)\right].$$

(10)

(11)

where

$$\Phi(x) = \frac{\pi x}{2} \left[ J_1(x) H_0(x) - J_0(x) H_1(x) \right].$$

(12)

and $J_n$ and $H_n$ are $n$th order Bessel and Struve functions, respectively. The integral over $\Phi$ was performed using an integral definition of the Bessel function (Jeffrey and Zwillinger 2000). The integral over $\rho$ was performed using a known integral identity (Rosenheinrich 2013). The fact that the analytic expression for the kernel can only be expressed in terms of special functions is inconvenient, but not problematic, since fast library routines for calculating these
functions are available. Note, for implementation purposes, that despite the divergent pre-
factor of $r^{-3}$ in the above equation, the kernel remains finite at zero:

$$k(r)_{r=0} = \frac{\pi}{12r^4}. \quad (14)$$

The finite size of the backprojection matrix and kernel must also be considered to find the
appropriate filter in discrete Fourier space. The correct filter to use is a discrete Fourier trans-
form (DFT) of the finite band-limited kernel we have just derived. The filtering operation is
then performed by the following:

$$f[i,j] = \text{DFT}^{-1} \{ \text{DFT} \{ b[i,j] \} \ \text{DFT} \{ k[i,j] \} \}, \quad (15)$$

with appropriate zero-padding (where $\text{DFT}^{-1}$ is the inverse transform). An additional smooth-
ing filter e.g. Shepp–Logan or Gaussian may be applied by sampling the appropriate function
in the frequency domain and multiplying the kernel filter by it in equation (15). In this case,
since any well-designed smoothing filter should leave the low-spatial frequency components
unchanged, it is not critical to find the band-limited kernel of the smoothing filter and then
apply a DFT.

2.5. Finite matrix correction

The mathematical formulation outlined above described the BPF process and how to imple-
ment it. However, if you follow the suggested recipe you still may not reconstruct quantitatively
accurate images of proton stopping power at the one percent level. In practice, a finite region,
$\Omega_b$, must be chosen for the backprojection. This is a problem, because the blurred-out function
$b(x, y)$ is non-zero everywhere and the data truncation leads to quantitative inaccuracy. The
inaccuracy resembles a constant shift in reconstructed values (Suzuki and Yamaguchi 1988),
similar to that arising from a mishandling of the filter (as described in the previous subsection).

Previously, it seems, the only attempt to deal with this has been to make the discretized
backprojection matrix ($N \times N$) finite but much larger than the desired image matrix ($M \times M$).
This is expensive in computer memory and computation time, as well as only asymptotically
improving accuracy.

As observed, the effect of missing data due to a finite backprojection matrix, closely resem-
bles a constant offset in the reconstructed function, $f(x, y)$. After all, the effect of distant data
on the reconstruction region can only manifest in the low-spatial frequencies. A constant-shift
correction can be arrived at in a number of ways. Consider the following. We can write the
backprojection function as a convolution (see equation (4)),

$$b(x, y) = f(x, y) * \frac{1}{r} = \int_{\Omega_b} dx'dy' \frac{f(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}}. \quad (16)$$

The distant values of the backprojection matrix, if far enough away from the central region,
can then be approximated as:

$$b_{r \gg r', (x, y)} \approx \frac{1}{r} \int_{\Omega_b} dx'dy' f(x', y'). \quad (17)$$

The contribution to $f(x, y)$ of the missing data, is therefore approximately (see equation (5)):

$$\Delta (x, y) = \int_{\Omega_b} dx'dy' b_{r \gg r', (x', y')} k(x - x', y - y') \quad (18)$$
\(\Delta = \left( \int_{\in \Omega_f} dx'dy' f(x', y') \right) \left( \int_{\notin \Omega_b} dx'dy' \frac{k(x - x', y - y')}{r'} \right).\)

It should be understood that the integral inside the first brackets is over all points in the reconstruction region (\(\in \Omega_f\)) while that inside the second brackets is over all points outside the backprojection region (\(\notin \Omega_b\)). To the central pixel, \([i = N/2, j = N/2]\), the correction becomes:

\[
\Delta = \left( \sum_{\text{pixels} \in \Omega_f} f[i,j] \right) \left( \sum_{\text{pixels} \notin \Omega_b} k \left[ N/2 - i, N/2 - j \right] / \tau \sqrt{i^2 + j^2} \right). \tag{19}
\]

This quantity need not be calculated at the central pixel. However, at the centre, the a pixel’s minimum distance to the point at which we apply the approximation of equation (17), is at a maximum. The second sum over all pixels outside the backprojection matrix is in principle an infinite sum, but can be evaluated inexpensively to the desired accuracy. In our calculations the infinite sum was truncated by imposing: \(|i - N/2|, |j - N/2| \leq 4N\). This correction, \(\Delta\), calculated for one pixel, should then be applied to all pixels in the image as a constant offset:

\[
f_{\Delta} [i,j] = f[i,j] + \Delta. \tag{20}
\]

2.6. Relaxation of the 2D conditions

It should be noted that the above formulation for reconstruction assumes linear in-plane ray-projections (in this case proton paths). As such, each slice of the 3D volume should factorize into a separate 2D problems with distinct data. This is the essence of the classic tomography problem. In reality, due to beam angular dispersion and straggling, protons will not typically remain in the same axial plane as they progress through the patient. However, both the angular dispersion and straggling of protons is typically small enough to be considered a perturbation. Representative root-mean-square figures for beam divergence before and after the patient might be 5 mrad and 50 mrad respectively (Bruzzi et al 2007), although these values would depend on the beam nozzle design, proton energy and size of patient. We therefore propose the following adaptation to non-linear out-of-plane paths. Firstly, the backprojection is performed as expressed in equation (6), except that ray paths are backprojected along non-linear trajectories, as illustrated in figure 2. The estimated path of a proton through the 3D reconstruction volume is used for backprojection, however, regardless of whether it passes through multiple axial slices. Secondly, the filtering operation, as described in equation (15), is performed for each axial slice as if the protons had progressed linearly, parallel and in-plane.

2.7. Simulations

To demonstrate the proposed algorithm, a simulation of a proton CT scan acquisition was performed using the FLUKA Monte Carlo code (Ferrari et al 2005) and FLAIR interface (Vlachoudis 2009). The test object was a variant of the 3D Shepp–Logan head phantom (Kak and Slaney 2001) consisting of twelve regions constructed from ten ellipsoids. The phantom was scaled to give maximum thicknesses in the x, y and z directions of 13.8, 18.4 and 18.0 cm, respectively. All the regions were modelled as water, but assigned densities between 1.00 and 2.00 g·cm\(^{-3}\), with the background density set as 1.02 g·cm\(^{-3}\). Additionally, a tungsten bar (diameter: 0.5 mm; length 20 mm) was inserted into the phantom to allow quantification of
the modulation-transfer-function (MTF). The centre of the bar was located at: \((x, y, z) = (0, 20, -10)\) mm.

The source was modelled as a 20 cm diameter parallel and mono-energetic beam of 200 MeV protons (initial kinetic energy). The entry and exit positions, directions and energies of each proton were scored at planes parallel to the beam axis, located at 12 cm from the isocentre (see figure 2). The scoring was implemented using a user-modified version of the \textit{mgdraw} routine supplied with FLUKA\(^4\). A total of 180 projections were simulated equally spaced over \(\pi\) radians, each projection consisting of \(10^7\) primary protons. The dose to the centre of the phantom was recorded using the USRBIN scoring card. Only protons which emerged from the phantom and reached the exit plane were considered usable. In addition, during post-processing of the FLUKA output, cuts were applied in angular and lateral deflection at the exit plane using \(0.5 \times 0.5\) mm\(^2\) scoring zones. This was implemented to eliminate events likely to involve inelastic nuclear interactions and to limit the blurring effect of lateral straggling. Sequentially, a \(3\sigma\) then a \(2\sigma\) cut were applied for these purposes in each lateral direction. After particle loss and rejection, approximately 83% of the initial protons remained for reconstruction. The water-equivalent-path (WEPL) for each proton was then calculated by subtracting the residual range of the proton at the exit plane from its initial range. This was performed using the exit and entry energies at those planes and interpolations of the NIST tabulations of proton CSDA range (Berger et al 2005).

2.8. Implementation of proposed BPF algorithm

The BPF algorithm proposed above was implemented in Fortran 95/2003 and compiled with the gfortran compiler. The OpenMP library of gfortran was used to allow multi-threaded execution. The 2D Fourier Transforms were implemented using the external FFTW3 library\(^5\). The Struve functions were calculated using the external \textit{special_functions} Fortran 90 library, derived from Zhang and Jin\(^6\). Backprojection was performed on a \(N \times N \times L\) matrix of voxels.

\(^4\) See the FLUKA online manual at: www.fluka.org
\(^5\) See: www.fftw.org/
\(^6\) See: http://people.sc.fsu.edu/~jburkardt/f_src/special_functions/special_functions.html
Reconstructed stopping-powers were calculated on a $M \times M \times L$ matrix. Unless otherwise indicated, 1 mm$^3$ voxels were used for reconstruction ($M = L = 200$) and a 2D Gaussian smoothing filter was applied ($\sigma = 1 / \sqrt{2}$ pixels). The backprojection matrix was oversized by a low factor of either two ($N = 2M$) or four ($N = 4M$).

Reconstructions for the determination of modulation-transfer-functions (MTFs) were performed at higher resolution ($N = 4M = 8L = 1600$, $0.5 \times 0.5 \times 1.0$ mm$^3$ voxels, no Gaussian filter applied). The MTF determination followed that recommended for the Catphan phantom\(^7\). Note that no correction for the finite size of the tungsten bar was applied.

The backprojection operation was ray-driven on a history by history basis, applying equation (6). Inside the phantom, for each proton, $n$ knot points were calculated equally spaced between the phantom entry and exit points. These points sat on a cubic spline trajectory estimated from the proton’s entry and exit vectors (Williams 2004). We note that the use of a Most-Likely-Path formalism would be a theoretical improvement over the use of cubic splines, at some computational cost (Wang \textit{et al} 2010). Inside the phantom, backprojection was linear between any two knot points (see figure 2). Outside the phantom, the protons were projected back from the surface with trajectories parallel to the beam axis (at an angle $\phi$ in the $x$-$y$ plane). This stops the backprojection rays ‘spreading-out’ as they diverge from the phantom which would exacerbate the departures from the ideal case discussed in section 2.6. Backprojecting away from the phantom into empty space along the actual entry and exit trajectories was found to lead to a small quantitative error in reconstructed stopping powers of up to 1%.

### 2.9. Implementation of a standard FBP algorithm

To compare the proposed BPF algorithm to a more conventional approach, a version of a FBP algorithm was implemented. This is termed standard FBP here (SFBP), because we take the most natural implementation. For each of the 180 projections, the WEPLs of the protons were binned to a 2D matrix at the exit plane, enabling filtering prior to backprojection. The pixel size for this binning was chosen as 0.5 mm. The projections were then filtered with the appropriate 1D ramp filter and backprojected assuming rays parallel to the beam axis. In this formulation, the use of an approximation to curved proton trajectories was not possible. For the same event-rejection cuts applied to the exit trajectories, the FBP algorithm would therefore be expected to result in poorer spatial resolution compared to the BPF approach. Reconstruction was again performed on a matrix of 1 mm$^3$ voxels. For comparison to the BPF reconstructions, the same 2D Gaussian smoothing filter was applied ($\sigma = 1 / \sqrt{2}$ pixels) as a post-processing step in image-space. We note that more complex binning approaches have been suggested for FBP in proton CT (Rit \textit{et al} 2013), but the optimal binning is still an open question and not the subject of this study.

### 3. Results

A central axial slice reconstruction ($x$-$y$ plane) is shown in figure 3(a) for the BPF algorithm (9-knot path, 1.0 mm voxels, Gaussian filter). The two orthogonal planes ($y$-$z$ and $x$-$z$) are shown in figures 3(b) and (c). Interior elliptical regions of relative stopping-power either 1.00 or 1.03 are present and easily distinguished from the background value of 1.02. The tungsten bar is also present in figures 3(a) and (b) and resolvable. The scan ($180 \times 10^7$ protons) produced an absorbed dose of 5.1 mGy at the centre of the phantom.

\(^7\) See the Catphan 500 and 600 manual (The Phantom Laboratory): \texttt{www.phantomlab.com/products/catphan.php}
A non-central axial slice ($z = -2.0$ cm) is shown in figure 4 for: (a) SFBP, (b) BPF (0-knot), (c) BPF (2-knot) and (d) BPF (9-knot) algorithms. The filtered backprojection image is severely blurred. The crudest backprojection-then-filtering approach (0-knot) is a noticeable improvement over SFBP, although some artefacts are apparent: a dark shading inside the outer high-density annulus can be seen. This error is caused by the lack of knot points paced on the surface of the phantom. Increasing the number of knots to two (entry and exit) improves the spatial resolution further and eliminates the artefact. The resolution improvement of a 9-knot spline path over 2-knot is more subtle and not readily apparent in these images (resolution is addressed further below). The standard deviations of relative stopping-power within the region-of-interest (ROI) labelled 1 in figure 4(d), were 0.0066 and 0.0050, respectively, for the SFBP and BPF (9-knot) algorithms. The SFBP image therefore also manifests higher noise, as well as suffering poorer resolution.

Regardless of general image-quality, the usefulness of the BPF algorithm will be compromised if it cannot provide quantitatively accurate reconstructions. Table 1 provides the reconstructed relative stopping-power in the three ROIs depicted in figure 3(d) with and without the finite matrix correction and for two different sizes of backprojection matrix. Two notable observations can be made. Firstly, as the size of the backprojection matrix is increased with respect to the reconstruction matrix, the uncorrected BPF algorithm becomes more accurate. This is expected. Secondly, by applying the finite matrix correction derived in this paper, an accuracy within 0.1% can be obtained even for a modestly oversized back-projection matrix ($N = 2M$). It should be noted that the reconstruction time was dominated by the backprojection step and that this increases approximately linearly with backprojection matrix size. Rates of $1.06 \times 10^8$ and $5.2 \times 10^7$ protons per hour per thread were obtained, with $N = 400$ and $N = 800$, respectively.

The resolution properties of the algorithm can be assessed from figure 5. A region around the tungsten bar is shown in the figure for the BPF algorithm (0.5 × 0.5 × 1.0 mm$^3$ voxels, no Gaussian filter) for: (a) 0-knots, (b) 2-knots, (c) 3-knots and (d) 5-knots. An improvement in the resolution as the number of knots is increased is clear (although negligible after 5-knots and therefore not shown). The corresponding MTFs are shown in figure 6. Little improvement is seen for greater than a 3-knot proton path, although this is likely to depend on the location of the point-of-interest within a phantom and the initial energy of the proton beam.
4. Discussion

A new approach to proton CT reconstruction has been introduced. It has been argued that this method of backprojection-then-filtering naturally suits the nature of the problem: it is a list mode algorithm naturally accommodating non-linear paths. With the appropriate treatment of the 2D kernel and the application of a finite matrix correction derived here, a high quantitative

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Table 1. Reconstructed RSP values averaged in three ROIs in comparison to true relative stopping-power. Results are given with \( f_\Delta \) and without \( f \) finite-matrix correction and are shown for two different backprojection matrix sizes \( (N = 400 \text{ and } 800) \). In each case the reconstruction volume was \( 200^3 \) voxels \( (M = L = 200) \).

| ROI | True RSP | \( N = 400 \) | \( N = 800 \) |
|-----|----------|-------------|-------------|
|     |          | \( f \)     | \( f_\Delta \) | \( f \)     | \( f_\Delta \) |
| 1   | 1.020    | 1.057       | 1.021       | 1.027       | 1.019       |
| 2   | 1.000    | 1.035       | 0.999       | 1.008       | 1.000       |
| 3   | 1.030    | 1.066       | 1.030       | 1.039       | 1.030       |

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Figure 4. A transverse slice \( (\zeta = 2.0 \text{ cm}, 1.0 \text{ mm voxels}, \text{Gaussian filter}) \) reconstructed by (a) SFBP, (b) BPF (9-knot), (c) BPF (2-knot) and (d) BPF (9-knot) methods.
accuracy (≈0.1%) has been demonstrated. This accuracy was obtained by simulating water composition at a variety of densities and should be verified for a mixture of realistic tissue compositions in a subsequent study.

The superiority of the BPF method over the standard FBP, in terms of spatial resolution and noise, has been illustrated. As the spatial resolution and noise would be expected to vary within the reconstruction volume, a more detailed future investigation would be of interest. A piece-wise linear approximation to a cubic spline proton path was applied for reconstruction. Results suggest that five interpolation knots are likely to be sufficient for practical purposes and would provide sufficient spatial resolution for radiotherapy planning purposes. The benefit of using a Most-Likely-Path over a cubic spline approach is also a subject for future study.

The algorithm as presented in this work has been shown to be accurate for protons with small angular dispersions with respect to the beam axis. This reflects the situation at many proton therapy facilities with passively scattered beams, but does not reflect all imaging geometries. The re-weighting of projections appropriate for fan-beam is already long known (Zeng and Gullberg 1994), although rebinning of protons to parallel projection sets is also possible in that case. For cone-beam geometries, however, it may be necessary to introduce a projection

**Figure 5.** A zoomed-in region of a transverse slice around the tungsten rod, reconstructed with (a) 0-knot, (b) 2-knot, (c) 3-knot and (d) 5-knot proton paths.

**Figure 6.** Modulation-transfer-functions calculated from the simulated data based on 0, 2, 3 and 5-knot proton path reconstructions.
weighting, analogous to the pre-convolution weighting in the Feldkamp algorithm (Kak and Slaney 2001).

It is worth commenting on the computational speed of the algorithm. Although the individual backprojection operations are likely to take longer in BPF compared to iterative reconstruction algorithms (due to the over-sized matrix), iterative approaches require multiple steps of both forward and backprojection. The proposed method will still therefore be substantially less computationally demanding than a typical iterative implementation. The BPF algorithm is also highly parallelizable, lending itself to multi-threaded execution.

The use of a Monte Carlo simulation for the illustrative reconstructions here has allowed an appreciation of the fundamental limits of image-quality, deriving from proton interaction mechanisms. It should be noted, however, that perfect trajectory and energy information has been assumed in the entry and exit planes. This is not a realistic representation of the experimental situation where there may be substantial uncertainties on these quantities. Therefore the image-quality presented here should be interpreted as a bound on that obtainable with the same number of protons per projection, event rejection cuts, reconstruction parameters and proton path approximation. A more thorough model of a realistic system would be of interest, but as it would be system-specific, it is beyond the scope of the current work.

5. Conclusion

A new and quantitatively accurate reconstruction method for proton CT using backprojection-then-filtering has been demonstrated. The method is straightforward to implement and possesses superior characteristics to the standard filtering-then-backprojection approach.

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References

Beister M, Kolditz D and Kalender W A 2012 Iterative reconstruction methods in x-ray CT Phys. Med. Biol. 28 94–108
Berger M J, Coursey J S, Zucker M A and Chang J 2005 ESTAR, PSTAR and ASTAR: Computer Programs for Calculating Stopping-Power and Range Tables for Electrons, Protons and Helium Ions (version 1.2.3) (Gaithersburg, MD: National Institute of Standards and Technology) (http://physics.nist.gov/Star [2014-05-03])
Bruzzi M et al 2007 Prototype tracking studies for proton CT IEEE Trans. Nucl. Sci. 54 140–5
Cirrone G A P et al 2011 Monte Carlo evaluation of the filtered back projection method for image reconstruction in proton computed tomography Nucl. Instrum. Methods A 658 78–83
Civinini C et al 2010 Towards a proton imaging system Nucl. Instrum. Methods A 623 588590
Cormack A M 1963 Representation of a function by its line integrals, with some radiological applications J. Appl. Phys. 34 2722
Cormack A M and Koehler A M 1976 Quantitative proton tomography: preliminary experiments Phys. Med. Biol. 21 560–9
Ferrari A, Sala P R, Fasso A and Ranft J 2005 FLUKA: a multi-particle transport code (Geneva: CERN) CERN Yellow Report CERN-2005-10, INFN/TC05/11, SLAC-R-773

Hanson K M 1979 Proton computed tomography IEEE Trans. Nucl. Sci. NS-26 1635–40

Hanson K M, Bradbury J N, Cannon T M, Hutson R L, Laubacher D B, Macek R, Paciotti M A and Taylor C A 1978 The application of protons to computed tomography IEEE Trans. Nucl. Sci. NS-25 657–60

Jeffrey A and Zwillinger D 2000 Gradshteyn and Ryzhik’s Table of Integrals, Series and Products 6th edn (New York: Academic)

Kak A C and Slaney M 2001 Principles of Computed Tomographic Imaging (Philadelphia, PA: SIAM)

Li T, Liang Z, Singanallur J V, Satogata T J, Williams D C and Schulte R W 2006 Reconstruction for proton computed tomography by tracing proton trajectories: a Monte Carlo study Med. Phys. 33 699–706

Penfold S N, Rosenfeld A B, Schulte R W and Schubert K E 2009 A more accurate reconstruction system matrix for quantitative proton computed tomography Med. Phys. 36 4511–8

Penfold S N, Schulte R W, Censor Y and Rosenfeld A B 2010 Total variation superiorization schemes in proton computed tomography image reconstruction Med. Phys. 37 5887–95

Poludniowski G, Allinson N M, Anaxagoras T, Esposito M, Green S, Manolopoulos S, Nieto-Camero J, Parker D J, Price T and Evans P M 2014 Proton-counting radiography for proton therapy: a proof of principle using CMOS APS technology Phys. Med. Biol. 59 2569–81

Rit S, Dedes G, Freud N, Sarrut D and Letang J M 2013 Filtered backprojection proton CT reconstruction along most likely paths Med. Phys. 40 031103

Rosenheinrich W 2013 Tables of some indefinite integrals of bessel functions (www.fh-jena.de/rsh/Forschung/Stoer/besint.pdf)

Sadrozinski H F-W, Johnson R P, Macafee S, Plumb A, Steinberg D, Zatserklyaniy A, Bashkirov V A, Hurley R F and Schulte R W 2013 Development of a head scanner for proton CT Nucl. Instrum. Methods A 699 205–10

Schneider U, Besserer J, Pemler P, Delleart M, Moosburger M, Pedroni E and Kaser-Hotz B 2004 First proton radiography of an animal patient Med. Phys. 31 1046–51

Schneider U and Pedroni E 1994 Multiple Coulomb scattering and spatial resolution in proton radiography Med. Phys. 21 1657–63

Schulte R R, Penfold S N, Tafas J T and Schubert K E 2008 A maximum likelihood proton path formalism for application in proton computed tomography Med. Phys. 35 4849–56

Suzuki S and Yamaguchi S 1988 Comparison between an image reconstruction method of filtering backprojection and the filtered backprojection method Appl. Opt. 27 2867–70

Vlachoudis V 2009 FLAIR: a powerful user friendly graphical interface for FLUKA Int. Conf. on Mathematics, Computational Methods and Reactor Physics (New York, May 2009) (La Grange Park, IL: American Nuclear Society)

Wang D, Mackie T R and Tome W A 2010 On the use of a proton path probability map for proton computed tomography reconstruction Med. Phys. 37 4138–45

Williams D C 2004 The most likely path of an energetic charged particle through a uniform medium Phys. Med. Biol. 49 2899–911

Zeng G L and Gullberg G T 1994 A backprojection filtering algorithm for a spatially varying focal length collimator IEEE Trans. Med. Imaging 13 549–56

Zeng G L and Gullberg G T 1995 Can the backprojection filtering algorithm be as accurate as the filtered backprojection algorithm? IEEE Conf. Record Nuclear Science Symp. and Medical Imaging Conf. (Norfolk, VA, 30 October–5 November 1994) 3 1232–6