DIFFERENTIABLE COHOMOLOGY OF GAUGE GROUPS.

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Introduction

There is a well-known theory of differentiable cohomology $H^p_{\text{diff}}(G, V)$ of a Lie group $G$ with coefficients in a topological vector space $V$ on which $G$ acts differentiably. This is developed by Blanc in [Bl]. It is very desirable to have a theory of differentiable cohomology for a (possibly infinite-dimensional) Lie group $G$, with coefficients in an arbitrary abelian Lie group $A$, such that the groups $H^l_{\text{diff}}(G, A)$ have the expected interpretations. For instance, $H^2_{\text{diff}}(G, A)$ should classify the Lie group central extensions of $G$ by $A$. In this paper we introduce such a theory and study various differentiable cohomology classes for finite-dimensional Lie groups and for gauge groups. We are mostly interested in the coefficient group $A = \mathbb{C}^\ast$. In that case, we have the exponential exact sequence relating the differentiable cohomologies with coefficients $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{C}^\ast$. This allows us easily to compute $H^l_{\text{diff}}(G, \mathbb{C}^\ast)$ for a compact Lie group $G$: it is isomorphic to the cohomology $H^{l+1}(G, \mathbb{Z})$. In the case of gauge groups, we construct various differentiable cohomology classes, including the central extension of a loop group as a special case. We also prove reciprocity laws for gauge groups of differentiable manifolds with boundary embedded in a complex manifold, in the spirit of the Segal-Witten reciprocity law for loop groups.

The definition of $H^l_{\text{diff}}(G, A)$ uses simplicial sheaves. We consider the classifying space $BG$ as a simplicial manifold, which in simplicial degree $p$ is equal to $G^p$. Then over each manifold $G^p$ we have the sheaf $A$ of smooth $A$-valued functions. These sheaves organize into a simplicial sheaf $A$ over $BG$. We then define $H^l_{\text{diff}}(G, A)$ to be the degree $l$ hypercohomology of $BG$ with coefficients in this simplicial sheaf.

Our motivation is to construct differentiable analogs of the classes in the cohomology $H^p(G_\delta, \mathbb{C}^\ast)$ of the discrete group $G_\delta$ constructed by Cheeger and Simons [Chee-S] using geodesic simplices. Similar classes have been constructed by Beilinson using his Chern classes in Deligne cohomology. We construct a differentiable cohomology class in $H^p_{\text{diff}}(G, \mathbb{C}^\ast)$ corresponding to a characteristic class in $H^{2p}(BG, \mathbb{Z})$. In fact, we construct a more powerful holomorphic class in the holomorphic group cohomology $H^{2p+1}_{\text{hol}}(G_\mathbb{C}, \mathbb{C}^\ast)$ where $G_\mathbb{C}$ is the complexification of $G$. We conjecture that these classes map to the classes in [Chee-S] under the natural map from the holomorphic cohomology of $G_\mathbb{C}$ to the cohomology of the discrete group $G_\delta$.

In the spirit of secondary characteristic classes, we construct an extension of these differentiable cohomology classes involving differential forms of degree $0, 1, \cdots, p - 1$ on the various stages $G^p$ of the simplicial manifold $BG$. Again the constructions are done holomorphically on $BG_\mathbb{C}$. The precise content of the construction is that it yields a class in the Deligne (hyper)-cohomology of $BG_\mathbb{C}$.

Deligne cohomology is defined using a complex of sheaves, and the corresponding

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secondary characteristic classes are hard to describe explicitly. For the purpose of local computations at the identity of the group, one would need only to consider the de Rham part of these classes. The meaning of this is that Deligne cohomology is approximately defined as a fiber product of integer-valued cohomology and truncated holomorphic de Rham cohomology over complex-valued cohomology. Working locally, the integer-valued cohomology disappears and one is left with truncated de Rham cohomology. Concretely, given a complex Lie group \( G \), such a truncated de Rham class will be represented by a family \( (\omega_1, \cdots, \omega_p) \), where \( \omega_j \) is a holomorphic \((2p - j)\)-form over \( G^j \).

Our starting point is the theory of Beilinson which says that any characteristic class in \( H^{2p}(G, \mathbb{Z}) \) for a complex Lie group leads to a so-called Beilinson characteristic class, which may be viewed as a holomorphic cohomology class with coefficients, not in \( \mathbb{C}^* \) but in a Deligne complex of sheaves. This can be thought of as a holomorphic enrichment of a differentiable cohomology class with \( \mathbb{C}^* \)-coefficients. The advantage of using Deligne cohomology as coefficients is that there are transgression maps defined for Deligne cohomology, and thus for a closed oriented manifold \( X \) of dimension \( k < p \) we can construct a cohomology class in \( H^{2p-1-k}(Map(X, G), \mathbb{C}^*) \) for the gauge group \( Map(X, G) \). This formalism implies easily that these classes satisfy a reciprocity law in a holomorphic context (Theorem 3.2). This hopefully clarifies the meaning of the reciprocity laws in \([Br-ML1]\) \([Br-ML2]\) \([Br-ML4]\). We note that in \([Br-ML1]\) it is incorrectly stated that the Segal-Witten reciprocity law holds true not only for holomorphic gauge groups on a Riemann surface with boundary, but even for gauge groups of smooth maps.

We tackle the question of writing down explicit cocycles for all these cohomology classes. Since this appears at present to be too difficult a goal for the classes on the whole group, we localize the question in a neighborhood of the identity. In this way, the topological part of Deligne cohomology disappears and we are left with a class in truncated de Rham cohomology. We conjecture (Conjecture 3.3) that the de Rham cohomology class given by the Beilinson characteristic class coincides with the class given by the Bott-Shulman-Stasheff differential forms on \( G, G^2, \cdots, G^p \). This conjecture then allows us to find explicit formulas for local group cocycles with coefficients in \( \mathbb{C}^* \) (or with enriched coefficients) in a neighborhood of the identity. The formulas are in the spirit of \([Br-ML3]\).

Interesting phenomena appear when we take the derivative of a differentiable cocycle to get a Lie algebra cocycle. In the case of a loop group, we take \( p = 2 \) and \( k = 1 \) and get exactly the Kac-Moody 2-cocycle for a loop algebra \( Map(S^1, \mathfrak{g}) \). For arbitrary \( p \), if we take \( k = p - 1 \) we obtain a well-known Lie algebra \( p \)-cocycle on \( Map(X, \mathfrak{g}) \) which is due to Tsygan \([Ts]\) and Loday-Quillen \([L-Q]\) for \( \mathfrak{g} = \mathfrak{gl}(n) \) and to Feigin \([Fe]\) in general. This cocycle is the direct higher-dimensional generalization of the Kac-Moody cocycle.

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1. Differentiable group cohomology

For a Lie group $G$ acting smoothly on a complete topological vector space $M$, there is a notion of differentiable cohomology $H^p_{\text{diff}}(G, M)$ introduced and studied by Blanc [Bl]. This is equal to the cohomology of the complex of smooth cochains $C^p(G, M)$.

If instead we consider an abelian Lie group $A$ with a smooth action of $G$, the smooth cochain complex can still be defined, but the corresponding cohomology theory is not fully satisfactory. We develop here a formalism for differentiable group cohomology which is well-adapted to geometric applications.

In this paper, a Lie group $G$ means a paracompact Fréchet manifold $G$ equipped with a group structure such that the product map and the inverse map are smooth, and there is an everywhere defined exponential map $\exp : \mathfrak{g} \to G$, where $\mathfrak{g}$ is the Lie algebra of $G$. Note that this is more restrictive than the definition in [P-S].

The classifying space $BG$ is the simplicial manifold

$$
G \times G \times G \xrightarrow{d_0} G \times G \xrightarrow{d_1} G \xrightarrow{d_2} \ldots \xrightarrow{d_3} G \xrightarrow{d_4} \text{pt}
$$

(1-1)

The face maps are given by the usual formulas.

For a paracompact manifold $X$ and a Lie group $A$, we denote by $A_X$ the sheaf of smooth $A$-valued functions on $X$.

Given an abelian Lie group $A$, we can then consider the simplicial sheaf $A$ on $BG$, which consists in the sheaves $A_{G^p}$ on each $G^p$, together with the transition morphisms

$$d^*_j A_{G^p} \to A_{G^{p+1}}$$

(1-2)

given by pull-back of smooth $A$-valued functions.

**Definition 1.1.** The differentiable cohomology groups $H^j_{\text{diff}}(G, A)$ are the hypercohomology groups $H^j(BG, A)$ of the simplicial sheaf $A$ over $BG$.

This hypercohomology may be computed in two ways.

First of all, the abstract method is to use a resolution $I^\bullet$ of the simplicial sheaf $A$ over $BG$ by a complex of simplicial sheaves $I^q_p$ such that each sheaf $I^q_p$ over $G^p$ is acyclic. Then $H^p_{\text{diff}}(G, A)$ is the cohomology of the double complex $\Gamma(G^p, I^q_p)$.

Another method is to use Čech cohomology. This is somewhat complicated to describe because we need open coverings of each $G^p$, which are related to each other via the face and degeneracy maps. More precisely, we need to pick a family $U^{(p)} = \{U_{j}^{(p)} \}_{j \in J_p}$ of open coverings of $G^p$, with indexing sets $J_p$, where the sets $J_p$ form a simplicial set. This means (1) for each $p$ and for each $j \in J_p$ and for $0 \leq k \leq p$, there is a map of sets $d_k : J_p \to J_{p-1}$ such that

$$d_k U^{(p)}_j \subseteq U^{(p-1)}_{d_k(j)}$$
(2) for each $p$ and each $0 \leq k \leq p$, there is a degeneracy map $s_k : J_{p-1} \to J_p$ such that

$$s_k U_j^{(p-1)} \subseteq U_j^{(p)}.$$  

It is required that these maps $d_k$ and $s_k$ satisfy the standard relations among face and degeneracy maps. Such a family of coverings is called good if each covering $U^{(p)}$ of $G^p$ is good (i.e., all non-empty intersections are contractible). Then for such a good covering, we can form the Čech double complex

$$C^q(U^{(p)}, A) = \oplus_{j_0, \ldots, j_q \in J_p} C^\infty(U^{(p)}_{j_0 \ldots j_q}, A)$$

where the horizontal differential is the alternating sum of the pull-back maps

$$d_1^* : C^\infty(U^{(p)}_{d_{i}(j_0) \ldots d_{i}(j_q)}, A) \to C^\infty(U^{(p-1)}_{j_0 \ldots j_q}, A)$$

The cohomology of this double complex computes the hypercohomology groups $H^p(BG, A)$.

The Čech double complex gives rise as usual to a spectral sequence

$$E_1^{p,q} = H^q(G^p, A_{G^p}) \Rightarrow H^{p+q}_{\text{diff}}(G, A).$$

We note the following

**Lemma 1.2.** Given an exact sequence $0 \to A \to B \to C \to 0$ of abelian Lie groups, such that the projection map $B \to C$ has local smooth sections, there is a corresponding long exact sequence

$$\cdots \to H^p_{\text{diff}}(G, C) \to H^p_{\text{diff}}(G, A) \to H^p_{\text{diff}}(G, B) \to H^p_{\text{diff}}(G, C) \to \cdots.$$  

**Proof.** This follows easily from an exact sequence of Čech double complexes. \[\square\]

In case $A$ is a topological vector space, we can describe differentiable group cohomology with coefficients in $A$ as the cohomology of the complex of differentiable cochains.

**Proposition 1.3.** If $A$ is a topological vector space, $H^p_{\text{diff}}(G, A)$ is the cohomology of the complex $C^p(G, A) = \{ f : G^p \to A, f \text{ smooth } \}$ of smooth $A$-valued cochains.

**Proof.** This follows since each $A_{G^p}$ is an acyclic sheaf. \[\square\]

Therefore in that case our definition of $H^p_{\text{diff}}(G, A)$ coincides with that of Blanc [Bl]. In general there is a difference between the differentiable cohomology $H^p_{\text{diff}}(G, A)$ introduced here and the traditional differentiable cohomology groups $H^p_{\text{class}}(G, A)$, which
are defined as the cohomology of the complex $C^p(G, A)$ of smooth $p$-cochains $G^p \to A$. There always is a map

$$H^p_{\text{class}}(G, A) \to H^p_{\text{diff}}(G, A)$$

which may be defined as follows: let $I^p$ be a resolution of the simplicial sheaf $A$ over $BG$, where each sheaf is acyclic. There is a map of complexes of simplicial sheaves $A^p \to I^p$, which induces a morphism on the complexes of global sections. The classical cohomology $H^p_{\text{class}}(G, A)$ is equal to the cohomology of the complex $\Gamma(G^p, A_G)$, and the differentiable cohomology $H^p_{\text{diff}}(G, A)$ is equal to the cohomology of the double complex $\Gamma(G^p, I^p)$. This gives the map we announced.

On the other hand, we have:

**Lemma 1.4.** If $A$ is a discrete abelian group, then $H^p_{\text{diff}}(G, A)$ is equal to the topological cohomology group $H^p(BG, A)$.

This leads to the expected for $G$ compact.

**Proposition 1.5.** Let $G$ be a compact Lie group. Then we have a canonical isomorphism

$$H^p_{\text{diff}}(G, C^*) \simeq H^{p+1}(BG, \mathbb{Z}(1)).$$

**Proof.** We use the exponential exact sequence of coefficient groups:

$$1 \to \mathbb{Z}(1) \to \mathbb{C} \to \mathbb{C}^* \to 1.$$

together with the vanishing of $H^p_{\text{diff}}(G, \mathbb{C})$ for $p \geq 1$, proved by Blanc [Bl].

We now describe the low degree differentiable cohomology groups.

**Lemma 1.5.** The group $H^1_{\text{diff}}(G, A)$ is the group of smooth homomorphisms $G \to A$.

**Proof.** The spectral sequence (1-5) gives an exact sequence

$$0 \to H^1_{\text{diff}}(G, A) \to H^0(G, A) \xrightarrow{d^0 - d^1 + d^2} H^0(G \times G, A).$$

The group $H^0(G, A)$ is the group of smooth mappings $\phi : G \to A$; then $\phi$ is a group homomorphism if and only if $\phi$ is in the kernel of $d^0 - d^1 + d^2$.

**Proposition 1.6.** The group $H^2_{\text{diff}}(G, A)$ is the group of isomorphism classes of central extensions of Lie groups

$$1 \to A \to \tilde{G} \xrightarrow{\pi} G \to 1$$
such that \( \pi \) is a locally trivial smooth principal \( A \)-fibration.

**Proof.** Given such a central extension, pick a good open covering \( (U_j)_{j \in J} \) of \( G \) over which \( \pi \) has a smooth section \( s_j \). Then we cover \( G \times G \) by the open sets \( U_{joj_1j_2}^{(2)} \) defined as

\[
U_{joj_1j_2}^{(2)} = d_0^*U_{jo} \cap d_1^*U_{j_1} \cap d_2^*U_{j_2}.
\]

So \( U_{joj_1j_2}^{(2)} \) is the set of \((g_0, g_1)\) such that \( g_1 \in U_{jo}, g_0g_1 \in U_{j_1}, g_0 \in U_{j_2} \). This covering is indexed by \( J^3 \). Similarly we define open coverings \( U^{(p)} \) of \( G^p \). This allows us to construct a Čech double complex. Because \((U_j)\) is a good covering, this double complex calculates the differentiable cohomology in degrees \( \leq 2 \). We can construct a degree 2 cocycle in this double complex as follows: First over \( U_{joj_1} \) we have \( s_1 = s_0g_01 \), where \( g_01 \) is a smooth function \( U_{joj_1} \to A \) (the transition cocycle of the covering). Next, over \( U_{joj_1j_2}^{(2)} \) we have the function \( h_{joj_1j_2} : U_{joj_1j_2}^{(2)} \to A \) defined by

\[
s_{j_2}(g_0)s_{j_0}(g_1) = s_{j_1}(g_0g_1)h_{joj_1j_2}.
\]

(1 - 6)

Then \((g_{joj_1}, h_{joj_1j_2})\) is a 2-cocycle in the Čech double complex. It is easy to check that a change in the choices of the sections \( s_i \) will change this 2-cocycle by a coboundary. Conversely, a degree 2 cohomology class is represented by a 2-cocycle \((g_{joj_1}, h_{joj_1j_2})\). The \( g_{joj_1} \) are the transition cocycles for a principal \( A \)-bundle \( \tilde{G} \to G \), equipped with sections \( s_i : U_i \to \tilde{G} \). There is a unique group structure on \( \tilde{G} \) compatible with the \( A \)-action and such that

\[
s_{j_2}(g_0)s_{j_0}(g_1) = s_{j_1}(g_0g_1)h_{joj_1j_2}.
\]

This gives a homomorphism from \( H^2_{\text{diff}}(G, A) \) to the group of isomorphism classes of central extensions. This is inverse to the map previously constructed. \( \blacksquare \)

Therefore it follows that the image of the map \( H^2_{\text{class}}(G, A) \to H^2_{\text{diff}}(G, A) \) is comprised of the classes of central extensions which are trivial as a bundle over \( G \). As is well-known, the universal central extension of a loop group does not have this property, so it cannot be represented by a class in \( H^2_{\text{class}}(G, \mathbb{C}^*) \).

The description of degree 3 differentiable cohomology requires the notion of a gerbe \( \mathcal{C} \) over a manifold \( X \) with band the sheaf \( \underline{A}_X \). Then given a simplicial manifold \( X_\bullet \) we have the notion of a simplicial gerbe over a simplicial manifold \( X_\bullet \), which was introduced in \([\text{Br-ML1}]\). This consists of a gerbe \( \mathcal{C} \) over \( X_1 \) with band \( \underline{A}_{X_2} \) over \( X_2 \);

1. an equivalence \( \phi : d_0^*\mathcal{C} \otimes d_2^*\mathcal{C} \to d_1^*\mathcal{C} \) of gerbes with band \( \underline{A}_{X_3} \) over \( X_3 \);
2. a natural transformation

\[
\psi : d_0^*\phi \otimes d_2^*\phi \to d_1^*\phi \otimes d_3^*\phi
\]

(1 - 7)

between equivalence of gerbes over \( X_3 \).

The natural transformation \( \psi \) must satisfy a cocycle condition.
The structures (1) and (2) become somewhat more concrete in the case of the simplicial manifold $BG$: (1) can be called a multiplicative structure on the gerbe $C$ over $G$. For instance, on the level of the fibers $C_g$, which are connected groupoids in which the automorphism group of any object is identified with $A$, we have an induced equivalence of groupoids

$$\phi_{g_0g_1} : C_{g_0} \otimes C_{g_1} \to C_{g_0g_1}. \quad (1-8)$$

Then one can view $\psi_{g_0g_1g_2}$ as associativity data for these equivalences $\phi_{g_0g_1}$.

A simplicial gerbe over $BG$ will be called a multiplicative gerbe over $G$. Namely, the equivalences $\phi_{g_0g_1}$ are not strictly associative, but only associative up to the natural transformations $\psi_{g_0g_1g_2}$.

We can then state

**Proposition 1.7.** The group $H^3_{\text{diff}}(G, A)$ identifies with the group of equivalence classes of multiplicative gerbes over $G$ with band $A_G$.

**Proof.** This is a special case of Theorem 5.7 in [Br-ML1], which says that for $A$ the simplicial sheaf over a simplicial manifold $X_\bullet$, associated to an abelian Lie group $A$, the hypercohomology $H^p(X_\bullet, A)$ identifies with the group of equivalence classes of simplicial gerbes over $X_\bullet$ with band $\underline{A}$. \(\square\)

We next briefly discuss the case of a non-trivial differentiable $G$-module $A$. This means that there is an action $\mu : G \times A \to A$ of $G$ on $A$, where the mapping $\mu$ is smooth. Then we define a simplicial sheaf $\underline{A}$ on $BG$ as follows: again $A_\bullet$ is the sheaf $A_{G\bullet}$. The face maps $\nu_i : d_i^* \underline{A}_{G\bullet} \to \underline{A}_{G\bullet}$ are defined as follows:

1. for $i > 0$, $\nu_i$ is the pull-back map $d_i^*$ on smooth $A$-valued functions;
2. for $i = 0$, we have

$$\nu_i(f)(g_0, \cdots, g_{p-1}) = g_0 f(g_1, \cdots, g_{p-1}) \quad (1-9)$$

using the action of $g_0 \in G$ on $A$. We then define $H^p_{\text{diff}}(G, A)$ to be the hypercohomology of this simplicial sheaf.

We then have an easy generalization of Lemma 1.5, for an exact sequence of differentiable $G$-modules such that the map $B \to C$ admits a local smooth section. There is also the following generalization of Proposition 1.6:

**Proposition 1.8.** For any differentiable $G$-module $A$, the group $H^2(G, A)$ identifies with the group of isomorphism classes of extensions of Lie groups

$$1 \to A \to \tilde{G} \to G \to 1$$

compatible with the given action of $G$ on $A$, and such that the mapping $\tilde{G} \to G$ has local smooth sections.

We will need crucially the holomorphic version of differentiable cohomology. For this purpose, $G$ will be a complex-analytic Lie group, $A$ will be an abelian complex-analytic
Lie group, $G$ will act on $A$ in such a way that the mapping $G \times A \rightarrow A$ is holomorphic. Then we can form the simplicial sheaf $A^{\text{hol}}_p$ over $BG$ such that each $A^{\text{hol}}_p$ is the sheaf of germs of holomorphic mappings $G^p \rightarrow A$. Then we define the holomorphic cohomology $H^{\text{hol}}_p(G, A)$ to be the hypercohomology of the simplicial sheaf $A^{\text{hol}}_p$ over $BG$.

In this paper we will need more general coefficients for differentiable group cohomology than differentiable $G$-modules. An important type of coefficients is provided by the smooth Deligne complex of sheaves $Z(k)_{\infty}^D$ over any smooth manifold $X$:

$$Z(k)_{\infty}^D = Z(k) \rightarrow E^0_X \rightarrow \cdots \rightarrow E^{k-1}_X,$$

(1 - 10)

where $E^l_X$ is the sheaf of germs of smooth complex-valued $l$-forms over $X$. Then we can organize the complexes of sheaves $Z(k)_{\infty}^D$ over the cartesian powers $G^p$ into a simplicial complex of sheaves, which we will also denote by $Z(k)_{\infty}^D$.

**Definition 1.9.** The smooth Deligne differentiable cohomology groups $H^p_{\text{diff}}(X, Z(k)^{\infty}_D)$ are the hypercohomology groups of the simplicial complex of sheaves $Z(k)_{\infty}^D$ over $BG$.

Finally, for a complex manifold $X$ we have the Deligne complex of sheaves $Z(k)^{\infty}_D$ over $X$:

$$Z(k)_{\infty}^D = Z(k) \rightarrow \Omega^0_X \rightarrow \cdots \rightarrow \Omega^{k-1}_X,$$

(1 - 11)

where $\Omega^k_X$ is the sheaf of holomorphic $k$-forms on $X$. For a complex Lie group $G$, we can then define the holomorphic Deligne cohomology groups $H^{\text{hol}}_p(G, Z(k)_{\infty}^D)$ are the hypercohomology groups of the simplicial complex of sheaves $Z(k)_{\infty}^D$ over $BG$. In other words, $H^{\text{hol}}_p(X, Z(k)_{\infty}^D)$ is the Deligne cohomology of the simplicial complex manifold $BG$.

### 2. Local cohomology and Lie algebra cohomology.

The differentiable cohomology $H^p_{\text{diff}}(G, A)$ introduced in section 1 involves the sheaf cohomology of the sheaf $A^{\infty}_{G^p}$ over $G^p$. In practice classes in $H^p_{\text{diff}}(G, A)$ are described using a Čech bicomplex for a family of open coverings of the $G^p$. If we localize at the origin of the group the situation becomes simpler. We will therefore introduce the notion of local differentiable (or simply local) cohomology $H^p_{\text{loc}}(G, A)$. This was developed in the general context of differentiable groupoids by Weinstein and Xu [W-X]. We let $C^\infty_{\text{loc}}(G^p, A)$ be the group of germs at $(1, \ldots, 1)$ of smooth functions $G^p \rightarrow A$. Then we have a standard complex

$$\cdots \rightarrow C^\infty_{\text{loc}}(G^p, A) \rightarrow \cdots \rightarrow$$

$$(-1)^p d^p \rightarrow C^\infty_{\text{loc}}(G^{p+1}, A) \rightarrow \cdots.$$  

(2 - 1)

We define the local differentiable cohomology groups $H^p_{\text{loc}}(G, A)$ to be the cohomology groups of this complex.

The main result is

**Proposition 2.1.** There is a canonical map $H^p_{\text{diff}}(G, A) \rightarrow H^p_{\text{loc}}(G, A)$.  

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We will describe this map concretely. We introduce a good open covering $U^{(p)}$ and consider a Čech representative

$$(g^0, \cdots, g^p),$$

where $g^j$ is a Čech $(p - j)$-cocycle with coefficients in $A_{G^j}$. We focus on the last term for which we observe

**Lemma 2.2.** If $U$ is an open set in $U^{(p)}$ which contains 1, then the function $g_U^p$ over $U$ satisfies the cocycle condition

$$\sum_{j=0}^{p} (-1)^j d_j^* g_U^j = 0$$

in a neighborhood of $1 \in G^{p+1}$.

Now $U^{(0)}$ is an open covering of the set $\{1\}$, so has at least one (non-empty) open set $U_0 = \{1\}$. Applying an arbitrary composition of degeneracy maps, we obtain for each $p$ a distinguished element $i_p$ of $J_p$. The corresponding open set $V_p = U_i^{(p)}$ contains the identity element, and we have

$$d_0(V_p) = d_1(V_p) = \cdots = d_p(V_p).$$

It then follows from Lemma 2.2 that $g_U^p(g_1, \cdots, g_p)$ defines a local differentiable group cocycle. This defines our map from differentiable to local differentiable group cohomologies.

**Remarks**

(1) The map $H_{diff}^p(G, A) \to H_{loc}^p(G, A)$ can be described from any open set $U \in U^{(p)}$ containing 1 which satisfies

$$d_0(U) = d_1(U) = \cdots = d_p(U). \quad (2-2).$$

Indeed one can show that the cohomology class of the corresponding local cocycle is independent of $U$. If $U$ and $V$ satisfy (2-2), then we have:

$$g_U^p - g_V^p = \sum_{j=0}^{p} (-1)^j g_U^{p-1}_{U'V'} \circ d_j,$$

where $U' = d_0(U), V' = d_0(V)$.

Thus the difference between $g_U^p$ and $g_V^p$ is a coboundary.

(2) There is a more abstract construction of the localization map, using the notion of topos. We associate a very natural “localized topos” $X_x$ to a topological space $X$ and a point $x \in X$. This topos is the 2-direct limit of the categories $Sheaves(U)$ of sheaves on open neighborhoods $U$ of $x$. The functor of global sections is exact, since it coincides with the functor $F \mapsto F_x$, where $F_x$ is the stalk at $x$. There is a simplicial topos $BG_1$ which in degree $p$ is given by the localized topos $(G^p)_1$. We can then describe $H_{loc}^p(G, A)$ as the hypercohomology $H^p(BG_1, A)$ of the simplicial sheaf $A$ over $BG_1$. 










There is a natural map of simplicial topoi \( j : BG_1 \to BG \), and the map \( H^p_{diff}(G, A) \to H^p_{loc}(G, A) \) is given by the inverse image \( j^* \).

There is also a notion of local smooth Deligne cohomology which we will have to use. It will be denoted by \( H^p_{loc}(G, Z(k)_{\infty}^D) \). It is described concretely as the cohomology of the double complex \( K^{pq} \), where

\[
K^{pq} = \begin{cases} 
  \mathbb{Z}(k) & \text{if } q = 0, \\
  \lim_{1 \in U \subset G^p} E^{q-1}(U) & \text{if } 1 \leq q \leq k \\
  0 & \text{if } q \geq k + 1
\end{cases}
\tag{2-3}
\]

A more convenient double complex is the multiplicative version, where we use the multiplicative version

\[
0 \to \mathbb{C}^* \xrightarrow{d \log} E^1_X \xrightarrow{} \cdots \xrightarrow{} E^{k-1}_X
\]

of the smooth Deligne complex \( Z(k)_{\infty}^D \) on a manifold \( X \). This leads to the double complex \( L^{pq} \), where

\[
L^{pq} = \begin{cases} 
  \lim_{1 \in U \subset G^p} C^\infty(U, \mathbb{C}^*) & \text{if } q = 1 \\
  \lim_{1 \in U \subset G^p} E^{q-1}(U) & \text{if } 1 \leq q \leq k \\
  0 & \text{otherwise}
\end{cases}
\tag{2-4}
\]

Next there is natural mapping from the local cohomology \( H^p_{loc}(G, A) \) to the Lie algebra cohomology \( H^p(g, A) \), which as usual is defined as the cohomology of the standard complex

\[
\cdots \to C^p(g, A) \to C^{p+1}(g, A) \to \cdots
\tag{2-5}
\]

where \( C^p(g, A) \) is the space of smooth alternating multilinear maps \( g^p \to A \). The differential \( d \) is given by the standard formula.

**Proposition 2.3.** There is a natural mapping \( \phi : H^p_{loc}(G, A) \to H^p(g, A) \) given on the level of local differentiable cocycles by the formula

\[
\phi(c)(\xi_1, \cdots, \xi_p) = \left[ \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \sum_{\sigma \in S_p} \epsilon(\sigma)c(\exp(t_{\sigma(1)}\xi_{\sigma(1)}), \cdots, \exp(t_{\sigma(p)}\xi_{\sigma(p)})) \right]_{t_i=0}.
\]

Thus we have a diagram of maps

\[
H^p_{diff}(G, A) \to H^p_{loc}(G, A) \to H^p(g, A).
\]

We discuss briefly the notions of continuous and measurable cohomologies. If \( A \) is a continuous \( G \)-module, where \( G \) is a topological group, we define over \( BG \) the simplicial sheaf \( A_{cont} \) which in degree \( p \) is the sheaf of continuous \( A \)-valued functions on \( G^p \).
Then the continuous cohomology groups $H^p_{\text{cont}}(G, A)$ are the hypercohomology groups $H^p(BG, A_{\text{cont}})$.

If $G$ is a locally compact topological group, then we have the notion of a measurable subset of $G$, and therefore if $A$ is a Lie group we can talk of a measurable function $U \to A$, where $U$ is open in $G$ (or more generally in $G^p$). The measurable cohomology $H^p_{\text{meas}}(G, A)$ is then defined as the hypercohomology of the simplicial sheaf of measurable $A$-valued functions on the $G^p$. However a simplification occurs here because of the obvious

**Lemma 2.5.** The sheaf of germs of measurable $A$-valued functions on $G^p$ is flasque.

Therefore measurable cohomology can be computed simply as the cohomology of the complex of measurable $A$-valued cochains, which reduces to the standard definition of measurable group cohomology.

3. **Beilinson Characteristic classes and Bott-Shulman-Stasheff forms.**

Let $X_\bullet, Y_\bullet$ be simplicial complex manifolds, and let $G$ be a complex Lie group. We say that a holomorphic simplicial map $\pi : X_\bullet \to Y_\bullet$ is a holomorphic principal $G$-bundle if we are given a holomorphic left action of $G$ on each $X_p$, compatible with the face and degeneracy maps, so that each $\pi_p : X_p \to Y_p$ is a holomorphic principal $G$-bundle. The Beilinson theory of characteristic classes in Deligne cohomology applies to holomorphic principal $G$-bundles $X_\bullet \to Y_\bullet$. We have

**Theorem 3.1.** [Be] [E] For any holomorphic $G$-bundle $X_\bullet \to Y_\bullet$ and for any $\kappa \in H^{2p}(BG, \mathbb{Z}(p))$, there is a canonical class $\kappa^{\text{Bei}} \in H^k_{\text{hol}}(Y_\bullet, \mathbb{Z}(p)_D)$.

In case $p = 2$, $Y$ is a compact complex manifold, and $G$ is simply-connected, this class has been described quite concretely in [Br-ML1]. There is also a description in terms of holomorphic gerbes over $G$, which is given in [Br-ML1].

The universal case is that of the classifying space $BG$, which is a simplicial algebraic manifold.

Deligne cohomology is closely related to so-called Hodge cohomology. The simplicial de Rham complex $\Omega^\bullet_{Y_\bullet}$ is filtered by the subcomplexes $F^p\Omega^\bullet_{Y_\bullet}$, which consist of the complexes of sheaves

$$F^p\Omega^\bullet_{Y_n} : 0 \to \cdots \to 0 \to \Omega^p_{Y_n} \to \cdots \to \Omega^{\dim(Y_n)}_{Y_n}$$

on $Y_n$. The Hodge cohomology groups are the hypercohomology groups $H^k(Y_\bullet, F^p\Omega^\bullet_{Y_\bullet})$. We then have an exact sequence

$$\cdots \to H^k(Y_\bullet, \mathbb{Z}(p)_D) \to H^k(Y_\bullet, F^p\Omega^\bullet_{Y_\bullet}) \oplus H^k(Y, \mathbb{Z}(p)) \to H^k(Y, \mathbb{C}) \to H^{k+1}(Y_\bullet, \mathbb{Z}(p)_D) \to \cdots$$

([Be], see also [E-V]).

In the case of $BG$, the construction of Bott-Shulman-Stasheff gives a class in $H^{2p}(Y_\bullet, F^p\Omega^\bullet_{Y_\bullet})$ (see [B-S-S]). We recall the construction: we start with the invariant
polynomial $P$ on the Lie algebra $\mathfrak{g}$ corresponding to the characteristic class $\kappa$. This gives the Chern-Weil representative $P(\Omega)$ of the characteristic class $\kappa \in H^{2p}(Y, \mathbb{C})$ with respect to a principal $G$-bundle $X \to Y$ equipped with a connection whose curvature is $\Omega$. Then one can construct secondary characteristic classes attached to $m$ connections $D_1, \ldots, D_m$ on the bundle. For this purpose, one introduces the product $Y \times \Delta_{m-1}$ and the pull-back bundle $X \times \Delta_{m-1} \to Y \times \Delta_{m-1}$. Let $(t_0, \ldots, t_{m-1})$ denote the barycentric coordinates on $\Delta_{m-1}$, which satisfy $\sum t_i = 1$. Then $D = \sum_{j=0}^{m} t_j D_j$ is a connection for the pull-back bundle. Let $R$ denote the curvature of $D$. One then constructs the $2p - m + 1$-form $\kappa_{sec}(X \to Y, D_1, \ldots, D_m)$ on $Y$ as follows:

$$\kappa_{sec}(X \to Y, D_1, \ldots, D_m) = \int_{\Delta_{m-1}} P(R)$$

(3 - 3)

by integrating the $2p$-form $P(R)$ on $Y \times \Delta_{m-1}$ in the $\Delta_{m-1}$-direction.

Now consider the universal $G$-fibration $\pi : EG \to BG$ which in degree $n$ is given by $\pi_n : G^{n+1} \to G^n$:

$$\pi_n(g_0, \ldots, g_n) = (g_0^{-1}g_1, \ldots, g_i^{-1}g_{i+1}, \ldots, g_{n-1}^{-1}g_n)$$

(3 - 4)

Here $G$ acts on $(EG)_p = G^{p+1}$ by

$$g \cdot (g_0, \cdots, g_p) = (gg_0, \cdots, gg_p).$$

There are $n + 1$ sections $\sigma_0, \ldots, \sigma_n : G^n \to G^{n+1}$ which are characterized by the fact that the image of $\sigma_i$ is the manifold of $n + 1$-tuples $(g_0, \ldots, g_n)$ such that $g_i = 1$. Each section $\sigma_i$ induces a flat connection $D_i$ on the bundle $G^{n+1} \to G^n$. Then the Bott-Shulman-Stasheff form $\omega_n$ on $G^n$ is the secondary characteristic class $\kappa_{sec}(G^{n+1} \to G^n, D_0, D_1, \cdots, D_n)$. This is a $2p - n$-form on $G^n$. We have the following results:

**Theorem 3.2.** (Bott-Shulman-Stasheff, see [B-S-S])

1. We have $\omega_n = 0$ for $n > p$.
2. The family $(\omega_1, \cdots, \omega_p)$ is a cycle in the double complex $\Omega^n(G^n)$.
3. For any degeneracy map $s_j : G^{m-1} \to G^m$, the pull-back $s_j^* \omega_m$ vanishes.

In particular, $\omega = \omega_1$ is the bi-invariant closed form on $G$ representing the transgressive class in $H^{2p-1}(G, \mathbb{C})$ corresponding to $\kappa$.

The $p$-form $\omega_p$ on $G^p$ is very interesting. Let $v_i$ be a tangent vector to $G$ and let $(v_i)_j$ be the corresponding tangent vector to $G^p$, which lives in the $j$-th copy of $G$. if $\tau : \{1, \cdots, n\} \to \{1, \cdots, n\}$ is a map, then we can form the expression

$$\omega_p((v_1)_{\tau(1)}, \cdots, (v_p)_{\tau(p)}).$$

(3 - 5)

It is easily seen that $\omega_p(v_1, \cdots, v_p) = 0$ if two of the $(v_i)_{\tau(i)}$’s are tangent vectors to the same of $G$. This follows readily from property (3) in Theorem 3.2. Thus the expression (3-5) is determined by its value in the case where $\tau : \{1, \cdots, n\} \to \{1, \cdots, n\}$ is a permutation. In fact one sees by direct calculation that we have:

$$\omega_p((v_1)_{\tau(1)}, \cdots, (v_p)_{\tau(p)}) = \epsilon(\tau)\omega_p((v_1)_1, \cdots, (v_p)_p)$$

(3 - 6)
and that when we evaluate at the origin in $G^p$, the resulting multilinear map on $\mathfrak{g}$ defined by
\[
\beta(v_1, \cdots, v_p) = \omega_1((v_1)_1, \cdots, (v_p)_p)
\tag{3-7}
\]
is skew-symmetric.

Because of property (1) in Theorem 3.2 we then can view $(\omega_1, \cdots, \omega_p)$ as a cocycle in the simplicial complex of sheaves $n \mapsto F^p\Omega^\bullet(G^n)$. This leads to the natural conjecture:

\textbf{Conjecture 3.3.} The class of $(\omega_1, \cdots, \omega_p)$ in $H^2_p(BG, F^p\Omega^\bullet BG)$ is the image of the Beilinson class $\kappa_{Bei}(EG \to BG)$ under the mapping $H^2_p(BG, \mathbb{Z}(p)_D) \to H^2_p(BG, F^p\Omega^\bullet BG)$.

This conjecture will play a key role in the rest of the paper.

4. Formulas for local group cohomology classes.

In the previous section we introduced an exact sequence relating Deligne cohomology and Hodge cohomology of a simplicial complex manifold. Equivalently, we have the following exact sequence
\[
\cdots \to H^k(Y_\bullet, \mathbb{Z}(p)_D) \to H^k(Y_\bullet, F^p\Omega^\bullet Y_\bullet) \to H^k(Y, \mathbb{C}/\mathbb{Z}(p)) \to H^{k+1}(Y_\bullet, \mathbb{Z}(p)_D) \to \cdots
\tag{4-1}
\]
([Be], see also [E-V]). This shows that the difference between the two cohomologies is entirely due to the singular cohomology of $Y_\bullet$ with $\mathbb{C}/\mathbb{Z}(p)$-coefficients. When we go the local Deligne cohomology $H^p_{loc}(BG, \mathbb{Z}(k)_D)$ we essentially replace $G$ by the limit of all the open neighborhoods of 1, which results in the disappearance of the higher cohomology groups with $\mathbb{C}/\mathbb{Z}(p)$-coefficients. Hence we obtain:

\textbf{Proposition 4.1.} For $p \geq 1$, $H^2_{loc}(BG, \mathbb{Z}(p)_D)$ is isomorphic to the degree $2p$ cohomology of local truncated de Rham double complex $F^p\Omega^\bullet_{loc}(BG)$, with $(r,s)$-component equal to
\[
\lim_{\overset{1 \in \mathfrak{g} \subseteq \mathfrak{g}_s}{\longrightarrow}} \Omega^r(U) \quad \text{if} \quad r \geq p
\]
\[
0 \quad \text{otherwise}.
\]

We have on any complex manifold $Z$ an exact sequence of complexes of sheaves:
\[
0 \to F^p\Omega^\bullet_Z \to \Omega^\bullet_Z \to [\sigma_{<p}\Omega^\bullet_Z] \to 0, \tag{4-2}
\]
where $\sigma_{<p}\Omega^\bullet_Z$ is the truncated complex $\Omega^0_Z \to \cdots \to \Omega^{p-1}_Z$. We have a similar exact sequence of complexes of sheaves over a simplicial complex manifold $Z_\bullet$.

This induces a boundary map
\[
H^l(Z_\bullet, [\sigma_{<p}\Omega^\bullet_Z]) \to H^{l+1}(Z_\bullet, F^p\Omega^\bullet_Z).
\]
Let make this more concrete when each $Z_q$ is Stein, so that the above cohomologies are simply computed by the double complexes of global sections. A class in $H^l(Z_\bullet, [\sigma_{<p}\Omega^\bullet_{Z_\bullet}])$ is then given by a family of $l-q$-forms $\omega_q$ on $Z_l$, for $q \geq l-p+1$, satisfying

$$\sum_{j=0}^q (-1)^q d_j^* \omega_q = (-1)^q d\omega_{q-1}$$

We will write down a formula for a class $\eta$ in $H^{2p+1}(BG, [\sigma_{<p}\Omega^\bullet_{BG}])$ which we conjecture to be the image of the Bott-Shulman-Stasheff class under the boundary map

$$H^{2p}(BG, F^p\Omega^\bullet_{loc}(BG)) \to H^{2p+1}(BG, [\sigma_{<p}\Omega^\bullet_{BG}])$$

This class $\eta$ is constructed as follows. We start by picking a contractible open set $U \subset G$ containing 1. Just as in [Br-ML3], we construct inductively mappings $\sigma_l : U^l \times \Delta_l \to U$ with the following properties:

1. $\sigma_0(pt) = 1$;
2. for all $l$ and for $0 \leq j \leq l$, denoting by $\delta_j : \Delta_{l-1} \hookrightarrow \Delta_l$ the $j$-th face map, we have

$$\sigma_l(g_1, \cdots, g_l; \delta_j(t_0, \cdots, t_{l-1})) = \begin{cases} \sigma_{l-1}(d_j(g_1, \cdots, g_l); t_0, \cdots, t_{l-1}) & \text{if } j \geq 1 \\ g_1 \cdot \sigma_{l-1}(g_2, \cdots, g_l; t_0, \cdots, t_{l-1}) & \text{if } j = 0 \end{cases}. \quad (4 - 3)$$

For fixed $(g_1, \cdots, g_l) \in U^l$, the resulting map $\Delta_l \to U$ will be denoted by $\sigma_{g_1, \cdots, g_l}$. It is a singular simplex in $U$ with vertices $(1, g_1, \cdots, g_l)$.

Then we have mappings

$$f_{m,q} : U^{m+q-1} \times \Delta_q \to G^m \quad (4 - 4)$$

given by

$$f_{m,q}(g_1, \cdots, g_{m+q-1}; t_0, \cdots, t_q) = (g_1, \cdots, g_{m-1}, \sigma_q(g_m, \cdots, g_{m+q-1}; t_0, \cdots, t_q)) \quad (4 - 5)$$

Now, $\omega_m$ is a $(2p-m)$-form on $G^m$; we can pull it back under $f_{m,q}$ and then integrate it over $\Delta_q$ to get a $(2p-m-q)$-form $\int_{p_1} f_{m,q}^* \omega_m$ over $U^{m+q-1}$. This $(2p-m-q)$-form will be denoted by $\beta_{m,q}$.

We are now ready to introduce an $l$-form $\eta_l$ over $U^{2p+1-l}$, defined by

$$\eta_l = \sum_{m+q=2p-l, m \geq 1}^{} \beta_{m,q}. \quad (4 - 6)$$

Note $b_{m,q}$ vanishes unless $m \leq p$.

We note that $\eta_0 = \beta_{1,2p-1}$. Indeed, for $m+q = 2p$ and $m \geq 2$, the function $\beta_{m,q}$ vanishes, as the first component of $f_{m,q}$ is equal to $g_1$ and thus, when we fix $g_1$ to be a constant, the restriction of the differential form $\int_{p_1} f_{m,q}^* \omega_m$ to $\{g_1\} \times U^{m+q-1} \times \Delta^q$
vanishes. Now the function $\beta_{1,2p-1}$ is exactly the function which Cheeger and Simons [C-S] use to define a $(2p - 1)$ group cocycle of the discrete group $G^\delta$ with values in $\mathbb{C}^*$.

Then we can present our conjecture

**Conjecture 4.2.** The family $\eta_l$ of $l$-forms over $U^{2p+1-l}$ is a cocycle in the truncated de Rham double complex $\Omega^r(U^*)$, which represents the image of the Bott-Shulman-Stasheff class under the boundary map

$$H^{2p}(BG, F^p\Omega^\bullet_{loc}(BG)) \rightarrow H^{2p+1}(BG, [\sigma_{<p}\Omega^\bullet_{BG}])$$  \hspace{1cm} (4 - 7)

We illustrate this for $p = 2$. The 1-form $\eta_1$ over $U^2$ is equal to $\eta_1 = \beta_{1,2} + \beta_{2,1}$. These 1-forms can be written down as follows. We introduce the following notations. Let $\xi_1, \cdots, \xi_{m+q-1}$ denote vector fields on $G$. View each $\xi_j$ as a tangent vector on the $j$-th factor of $U^{m+q-1}$. Then $df_{m,q}(\xi_j)$ can be viewed as a section of the pull-back $f_{m,q}^*T^{Gm}$ of the tangent bundle of $G^m$ to $U^{m+q-1} \times \Delta_q$. Restricting to a point $(g_1, \cdots, g_{m+q-1})$ of $U^{m+q-1}$, we may view $df_{m,q}(\xi_j)$ as a section of $\sigma_{g_1,\cdots,g_{m+q-1}}^*T^{Gm}$.

Our formula for $\beta_{m,q}$ (for $m + q = 3$) is then

$$\beta_{m,q}(g_1, g_2)(\xi_j) = \int_{\Delta_q} df_{m,q}(\xi)|\sigma_{g_1, g_2}^*\omega_m.$$  \hspace{1cm} (4 - 8)

For example, $\beta_{1,2}$ involves integrating the 2-form $df_{1,2}(\xi)|\omega_1$ over the 2-simplex, and $\beta_{2,1}$ involves integrating the 1-form $df_{2,1}(\xi)|\omega_2$ over the 1-simplex.

It should be noted that the presence of $\beta_{2,1}$ is necessary to obtain a local holomorphic cocycle.

We now prove Conjecture 4.2 in the case $p = 2$. In the proof we will use 3 types of facts:

1. the (relative) Stokes theorem in the case of a projection $p_1 : X \times \Delta^q \rightarrow X$. For $\beta$ a differential form on $X \times \Delta^q$, this gives

$$d \int_{p_1} \beta = \int_{p_1} d\beta + \sum_{j=0}^q \int_{p'_1} (Id \times \delta_j)^* \beta,$$

where $p'_1 : X \times \Delta^{q-1} \rightarrow X$ is the projection.

2. the fiber square principle which says that for a fiber square

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ Z & \xrightarrow{f} & Y \end{array}$$

$p$ is a smooth proper fibration, and for a differential form $\beta$ on $X$, we have

$$f^* \int_p \beta = \int_q g^* \beta.$$  

3. the cocycle relation

$$\sum_{j=0}^3 (-1)^j d_j^* \omega_2 = 0.$$
(4) the cocycle relation
\[ \sum_{j=0}^{2} (-1)^j d_j^* \omega = d\omega_2. \]

First we replace the class \((\omega, \omega_2)\) in the hypercohomology of \(F^2\Omega^\bullet_{\text{loc}}(BG)\) by a cohomologous cocycle. More precisely we add to it the total coboundary of \(\beta_{1,1} \in \Omega^2_{\text{loc}}(G)\). This coboundary has components \(-d\beta_{1,1}\) and \(\sum_{j=0}^{2} (-1)^j d_j^* \beta_{1,1}\). We compute \(d_j^* \beta_{1,1}\) using the fiber square principle. We find
\[ d_j^* \beta_{1,1} = \int_{p_1} \nu_j^* \omega, \]
where \(\nu_j = f_{1,1} \circ (d_j \times Id) : U^2 \times \Delta^1 \to G\) so that
\[ \nu_0(g, 1, g_2, t) = \gamma_{g_2}(t) \]
\[ \nu_1(g, 1, g_2, t) = \gamma_{g_1g_2}(t) \]
\[ \nu_2(g, 1, g_2, t) = \gamma_{g_1}(t) \]

We want to compare this alternating sum with \(d\eta_1\). We first compute \(d\beta_{2,1}\). Using the Stokes theorem and the cocycle relation we have
\[ d\beta_{2,1} = \int_{p_1} f_{2,1}^* d\omega_2 + \omega_2 = \sum_{j=0}^{2} (-1)^j \int_{p_1} f_{2,1}^* d_j^* \omega + \omega_2 \]
\[ = \sum_{j=0}^{2} (-1)^j \int_{p_1} \alpha_j^* \omega + \omega_2 \]
where \(\alpha_j = d_j f_{2,1} : U^2 \times \Delta^1 \to U\) so that
\[ \alpha_0(g_1, g_2, t) = \gamma_{g_2}(t) \]
\[ \alpha_1(g_1, g_2, t) = g_1 \cdot \gamma_{g_2}(t) \]
\[ \alpha_2(g_1, g_2, t) = g_1 \]

Now using Stokes’ theorem we find
\[ d\beta_{1,2} = \sum_{j=0}^{2} (-1)^j \int_{p_1} \gamma_j^* \omega, \]
where \( \alpha_j = f_{1,2} \circ (Id \times \delta_j) : U^2 \times \Delta^1 \to U \) is given by

\[
\gamma_0(g_1, g_2, t) = g_1 \cdot \gamma_{g_2}(t)
\]

\[
\gamma_1(g_1, g_2, t) = \gamma_{g_1 g_2}(t)
\]

\[
\gamma_2(g_1, g_2, t) = \gamma_{g_1}(t)
\]

We then obtain

\[
\sum_{j=0}^{2} (-1)^j d^*_j \beta_{1,1} = d \beta_{1,2} + d \beta_{2,1} - \omega_2 = d \eta_1 - \omega_2.
\]

We conclude that the Bott-Shulman-Stasheff cocycle \((\omega, \omega_2)\) is cohomologous to \((0, d \eta_1)\). In order to prove the conjecture for \(p = 2\), all we have left to do is to show that \((\eta_0, \eta_1)\) is indeed a cocycle, i.e.,

\[
dh = d^*_0 \eta_1 - d^*_1 \eta_1 + d^*_2 \eta_1 - d^*_3 \eta_1
\]

We first compute \(\sum_{j=0}^{3} (-1)^j d^*_j \beta_{2,1}\). By the fiber square principle we have \(d^*_j \beta_{2,1} = \int_{p_1} \phi_j^* \omega_2\), where \(p_1 : U^2 \times \Delta_1 \to U^2\) is the projection and the map \(\phi_j = f_{2,1} \circ (d_j \times Id) : U^3 \times \Delta_1 \to U^2\) is given by

\[
\phi_0(g_1, g_2, g_3, t) = (g_2, \gamma_{g_3}(t));
\]

\[
\phi_1(g_1, g_2, g_3, t) = (g_1 g_2, \gamma_{g_3}(t));
\]

\[
\phi_2(g_1, g_2, g_3, t) = (g_1, \gamma_{g_2 g_3}(t));
\]

\[
\phi_3(g_1, g_2, g_3, t) = (g_1, \gamma_{g_2}(t)).
\]

Here \(\gamma_g(t) = \sigma_1(g, t)\).

Similarly we have \(d^*_j \beta_{1,2} = \int_{p_1} \psi_j^* \omega\), where \(\psi_j : U^3 \times \Delta_2 \to U\) is given by

\[
\psi_0(g_1, g_2, g_3, y) = \sigma_{g_2, g_3}(y)
\]

\[
\psi_1(g_1, g_2, g_3, y) = \sigma_{g_1 g_2, g_3}(y)
\]

\[
\psi_2(g_1, g_2, g_3, y) = \sigma_{g_1, g_2 g_3}(y)
\]

\[
\psi_3(g_1, g_2, g_3, y) = \sigma_{g_1, g_2}(y),
\]

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where \( y \) denotes a point in \( \Delta_2 \).

Next we find from Stokes’ theorem that

\[
dh = A - d_1^* \beta_{1,2} + d_2^* \beta_{1,2} - d_3^* \beta_{1,2}
\]

where \( A = \int_{p_1} \lambda^* \omega \), for the mapping \( \lambda : U^3 \times \Delta_2 \to U \) defined by

\[
\lambda(g_1, g_2, g_3, y) = g_1 \cdot \sigma_{g_2, g_3}(y).
\]

So we are led to consider the mapping \( f_{2,2} : U^3 \times \Delta_2 \to U^2 \) which satisfies

\[
d_0 \circ f_{2,2} = \psi_0, d_1 \circ f_{2,2} = \lambda, d_2 \circ f_{2,2} = p_1.
\]

Then we get from the cocycle relation (4):

\[
\int_{p_1} d f_{2,2}^* \omega_2 = \sum_{j=0}^2 (-1)^j \int_{p_1} f_{2,2}^* d_j^* \omega = d_0^* \beta_{1,2} - A + \int_{p_1} p_1^* \omega.
\]

This is evaluated by Stokes’ theorem to be equal to \( d_2^* \beta_{2,1} - d_3^* \beta_{2,1} + \int_{p_1} \theta^* \omega_2 \), where \( \theta(g_1, g_2, g_3, t) = (g_1, g_2 \cdot \gamma_{g_3}(t)) \). We now claim that

\[
d_0^* \beta_{2,1} - d_1^* \beta_{2,1} + \int_{p_1} \theta^* \omega_2 = 0.
\]

To prove this, we note that \( \sum_{j=0}^2 (-1)^j \int_{p_1} f_{2,2}^* d_j^* \omega_2 = 0 \) by the cocycle relation. It is easy to see that the term corresponding to \( j = 3 \) vanishes, and that the other three terms are \( d_0^* \beta_{2,1}, d_1^* \beta_{2,1} \) and \( \int_{p_1} \theta^* \omega_2 \).

We finally gather all this information to obtain:

\[
dh = A - d_1^* \beta_{1,2} - d_2^* \beta_{1,2} + d_3^* \beta_{1,2}
\]

\[
= \sum_{j=0}^3 (-1)^j d_j^* \beta_{1,2} + \sum_{j=0}^3 (-1)^j d_j^* \beta_{2,1}
\]

which proves the statement.

Conjecture 4.2 implies that the family \( (\eta_i) \) will represent the class of \( \kappa \) in the local (differentiable) Deligne cohomology \( H^{2p}(G, \mathbb{Z}(p)) \), assuming the validity of Conjecture 3.3. As evidence for Conjecture 3.3, we note that the formula it implies for the local Deligne cohomology class in the case \( p = 2 \) is very similar to the formula in [Br-ML1].
5. Differentiable cohomology classes for gauge groups.

In this section, we investigate the differentiable cohomology classes for gauge groups \( \text{Map}(X,G) \) which result from transgression of a characteristic class \( \kappa \in H^{2p}(BG, \mathbb{Z}(2p)) \). As a motivation, we briefly recall from \([\text{Br-ML1}]\) the case of central extensions of a smooth loop group \( LG = \text{Map}(S^1, G) \). In this case one starts from a characteristic class \( \kappa \in H^4(BG, \mathbb{Z}(2)) \), and by transgression in the diagram of simplicial manifolds

\[
\begin{array}{ccc}
(LG)^n \times S^1 & \xrightarrow{ev} & G^n \\
\downarrow p_1 & & \\
(LG)^n & & 
\end{array}
\]

one obtains a class in \( H^3(B(LG), \mathbb{Z}(2)^{\infty}) \), hence in particular a class in \( H^3(B(LG), \mathbb{Z}(1)^D) \). Since the complex of sheaves \( \mathbb{Z}(1)^D \) on a manifold \( X \) is quasi-isomorphic to \( \underline{\mathbb{C}}^* \) \([1]\), this can be viewed as a class in \( H^2(L(BG), \underline{\mathbb{C}}^*) = H^2_{diff}(LG, \mathbb{C}^*) \). According to Proposition 1.6, this gives a central extension of Lie groups

\[
1 \to \mathbb{C}^* \to \widetilde{LG} \to LG \to 1.
\]

A different approach to the central extensions is given in \([\text{Pi}]\).

In fact it is more rewarding to consider the holomorphic analog of this construction, for a complex Lie group \( G \). The construction is then performed using Beilinson characteristic classes. Then the result is a holomorphic central extension. This is the point of view developed in \([\text{Br-ML1}]\).

In this case, assuming Conjecture 3.4, we can write down at least a local differentiable group cocycle for the central extension, using the 1-form \( \eta_1 = \beta_{1,2} + \beta_{2,1} \) over \( U^2 \). We transgress this 1-form in the evaluation diagram

\[
\begin{array}{ccc}
LU^2 \times I & \xrightarrow{ev} & U^2 \\
\downarrow p_1 & & \\
LU^2 & & 
\end{array}
\]

to obtain a smooth function \( c = \int_{\partial_1} ev^* \eta_1 \) on \( U^2 \).

This function is given by the following formula

\[
c(g_1, g_2) = a(g_1, g_2) + b(g_1, g_2), \tag{5-3}
\]

where

\( a(g_1, g_2) = \int_{S^1 \times \Delta_2} \phi^* \omega \), for the mapping \( \phi : S^1 \times \Delta_2 \to G \) given by

\[
\phi(u; t_0, t_1, t_2) = f_{1,2}(g_1(u), g_2(u); t_0, t_1, t_2) = \sigma_{g_1(u), g_2(u)}(t_0, t_1, t_2); \tag{5-4}
\]

\( b(g_1, g_2) = \int_{S^1 \times \Delta_1} \psi^* \omega_2 \), for the mapping \( \psi : S^1 \times \Delta_1 \to G^2 \) described by

\[
\psi(u; t_0, t_1) = f_{2,1}(g_1(u), g_2(u); t_0, t_1) = (g_1(u), \sigma_{g_2(u)}(t_0, t_1)). \tag{5-5}
\]
It is interesting to compute the corresponding Lie algebra 2-cocycle, obtained as in §2 by differentiating $c$ at $(1, 1) \in U^2$ and skew-symmetrizing. Let $\xi_1, \xi_2$ belong to the Lie algebra $Lg$ of $LG$. We observe that the coefficient of $y_1 y_2$ in $a(exp(y_1 \xi_1), exp(y_2 \xi_2))$ is 0, because the 3-dimensional integral is clearly $O((|y_1| + |y_2|)^3)$.

To evaluate the dominant term of $b(exp(y_1 \xi_1), exp(y_2 \xi_2))$, we may as well assume that

$$f_{2,1}(exp(y_1 \xi_1(u)), exp(y_2 \xi_2(u)); t_0, t_1) = (exp(y_1 \xi_1(u)), exp(t_0 y_2 \xi_2(u)),$$

so that

$$\psi(u; t_0, 1 - t_0) = (exp(y_1 \xi_1(u)), exp(t_0 y_2 \xi_2(u)).$$

The derivative of this mapping is given (up to terms which are $O((|y_1| + |y_2|)^3)$), by

$$\frac{\partial}{\partial u} \mapsto \left( y_1 \frac{d \xi_1(u)}{d u}, t_0 y_2 \frac{d \xi_2(u)}{d u} \right), \quad \frac{\partial}{\partial t_0} \mapsto (0, y_2 \xi_2(u)).$$

Now using the expression

$$\omega_2 = 3k \text{Tr}(g_1^{-1} dg_1 dg_2 g_2^{-1})$$

we find

$$\frac{\partial^2}{\partial y_1 \partial y_2} b(exp(y_1 \xi_1), exp(y_2 \xi_2)) = 3k \int_0^1 \text{Tr}(\frac{d \xi_1}{d u} \xi_2(u))du.$$

As this expression is clearly already skew-symmetric in $\xi_1, \xi_2$ its skew-symmetrization gives the 2-cocycle

$$\alpha(\xi_1, \xi_2) = 6k \int_0^1 \text{Tr}(\frac{d \xi_1}{d u} \xi_2(u))du. \quad (5 - 6)$$

This is the standard Kac-Moody cocycle [Ka] [Mo].

The generalization of these constructions that we will discuss involves any closed oriented $d$-dimensional manifold $X$. Then we have the smooth gauge group $Map(X, G)$ comprised of the smooth maps $X \to G$. For $X$ compact (possibly with boundary), this is a Lie group with Lie algebra $Map(X, g)$, which is a Fréchet vector space.

**Theorem 5.1.** For any characteristic class $\kappa \in H^{2p}(BG, \mathbb{Z}(p))$ and any closed oriented manifold $X$ of dimension $k \leq p$, there results a class in differentiable Deligne cohomology $\kappa(X) \in H^{2p-k}(B Map(X, G), \mathbb{Z}(p - k))$. $\Gamma$

This class is simply obtained using transgression in smooth Deligne cohomology for the evaluation diagram

$$\xymatrix{ Map(X, G)^p \times X \ar[r]^{ev} \ar[d]^{p_1} & X^p \\
Map(X, G)^p \ar[u]}

We then have a geometric reciprocity law. This is phrased in terms of a real $(k + 1)$-submanifold with boundary inside a complex manifold of dimension $k$. The case $k = 1$ gives the Segal-Witten reciprocity law for Riemann surfaces with boundary. We note
that in [Br-ML1] it is incorrectly stated that the reciprocity laws for groups of smooth group-valued maps as opposed to groups of holomorphic maps.

**Theorem 5.2.** Let $M$ be a smooth compact manifold of complex dimension $k$ and let $X \subset M$ be a real $(k+1)$-dimensional submanifold with boundary $\partial X$. Let $G$ be a complex Lie group and let $\mathcal{G}$ be the Lie group consisting of smooth maps $X \to G$ which have a holomorphic extension to some neighborhood of $X$. Then the pull-back of $\kappa(\partial X)$ to $H^{2p-k}(B \mathcal{G}, \mathbb{Z}(p-k)_\mathbb{C})$ is trivial.

In the case of Riemann surfaces with boundary, the reciprocity law is more precise than the above theorem: it says not only that the pull-back of some central extension splits, but even that it has a canonical splitting.

Then we have

**Theorem 5.3.** Under the assumptions of Theorem 5.2, the class $\kappa(\partial X)$

$$H^{2p-k}(B \mathcal{G} \to B Map(\partial X, G), \mathbb{Z}(p-k)_\mathbb{C}).$$

This is the analog in differentiable group cohomology of a theorem proved for group cohomology in [Br-ML2].

We will write down an explicit formula for the class in local Deligne cohomology $H^{2p-k}_{loc}(Map(X, G), \mathbb{Z}(p-k)_\mathbb{C})$ associated to the characteristic class $\kappa$. Recall from section 4 the cocycle $(\eta_1, \eta_2, \cdots, \eta_p)$ in the double complex $\Omega^q(G^p)$ associated to $\kappa$. Here $\eta_l$ is an $l$-form over $\text{Map}(X, G)^{2p-1-l}$. Then we have

**Proposition 5.4.** Assume the validity of Conjecture 3.3. For a smooth closed manifold $X$ of dimension $k$, and for a characteristic class $\kappa \in H^{2p}(G, \mathbb{Z}(p))$, the corresponding class in local Deligne cohomology $H^{2p-k}_{loc}(Map(X, G), \mathbb{Z}(p-k)_\mathbb{C})$ is represented by the family $\int_X \eta_l$ of $(l-k)$-forms over $Map(X, G)^{2p-1-l}$, where $\int_X \eta_l$ is the transgression of $\eta_l$ in the evaluation diagram

$$\begin{array}{ccc}
\text{Map}(X, G)^p \times X & \stackrel{ev}{\to} & X^p \\
\downarrow p_1 \downarrow & & \\
\text{Map}(X, G)^p \\
\end{array}$$

We now obtain a Lie algebra cocycle simply by differentiating at the origin of $G$. Introduce a mapping

$$\phi_{m,q} : Map(X, G)^{2p-k-1} \times X \times \Delta_q \to G^{2p-k-1}$$

by

$$\phi_{m,q}(g_1, \cdots, g_{2p-k-1}; x; (t_0, \cdots, t_q)) = f_{m,q}(g_1(x), \cdots, g_{2p-k-1}(x); (t_0, \cdots, t_q))$$
For given \((g_1, \ldots, g_{2p-k-1}) \in Map(X,G)^{2p-k-1}\), denote by \([\phi_{m,q}]_{i}(g_1, \ldots, g_{2p-k-1})\) the corresponding mapping \(X \times \Delta_q \rightarrow G^{2p-k-1}\).

Then we see easily that the Lie algebra cocycle defined from differentiating the class of Theorem 5.1 is obtained by skew-symmetrizing the differential at the origin of \(Map(X,G)^{2p-k-1}\) of the local functional
\[
(g_1, \ldots, g_{2p-k-1}) \mapsto \sum_{m+q=2p-k} c_{m,q}(g_1, \ldots, g_{2p-m-1}),
\]
where \(m\) ranges over \(1, \ldots, p\). Here we have put
\[
c_{m,q}(g_1, \ldots, g_{2p-k-1}) = \int_{X\Delta_q} [\phi_{m,q}]^*(g_1, \ldots, g_{2p-m-1}) \omega_m.
\]

We can find an explicit formula for the partial derivative
\[
\frac{\partial^{2p-k-1}c_{m,q}(\exp(t_1\xi_1), \ldots, \exp(t_{2p-m-1}\xi_{2p-k-1}))}{\partial t_1 \cdots \partial t_{2p-k-1}}
\]
evaluated at \(t_j = 0\) as follows. First of all, we see easily that this partial derivative will be 0 unless \(m = k\). In that case, introduce the value \((\omega_m)_1\) of \(\omega_m\) at the identity, which is a \(2p-m-1 = 2p-k-1\)-multilinear form on \(g\). Given \(\xi_1, \ldots, \xi_{2p-k-1}\) in \(Map(X, g)\), we can write down a \(k\)-form \(\alpha\) on \(X\), whose expression in terms of local coordinates \((x_1, \ldots, x_k)\) is
\[
\alpha = (k!)^{-1}[\omega_m]_1(\xi_1, \ldots, \xi_{2p-2k-1}, \frac{\partial \xi_{2p-2k}}{\partial x_1}, \ldots, \frac{\partial \xi_{2p-m-1}}{\partial x_k}) \, dx_1 \wedge \cdots \wedge dx_k. \tag{5-9}
\]

Then we have

**Proposition 5.5.** The Lie algebra \((2p-k-1)\)-cocycle obtained by differentiating the class in \(H^{2p-k-1}(Map(X,G), \mathbb{C}^*)\) is equal to the skew-symmetrization of the cochain \(\int_X \alpha\) where \(\alpha\) is as in (5-9).

Consider for instance the case \(k = p - 1\). In that case we have \(m = k = p - 1\) hence \(q = 2p - k + 1 = 1\). We then get the following Lie algebra cocycle:
\[
c(\xi_1, \ldots, \xi_p) = \int_X \omega_p(\xi_1, d\xi_2, \ldots, d\xi_p) \tag{5-10}
\]
This is a direct generalization of the Kac-Moody cocycle which was given by Feigin [Fe].

We refer the reader to Teleman [Te] for a detailed discussion of this and more general Lie algebra homology classes.

For \(g = gl(n)\), the above construction is consonant with the results of Tsygan [Ts] and of Loday-Quillen [L-Q] on the Lie algebra of infinite matrices over a \(\mathbb{C}\)-algebra \(A\). Recall that the Lie algebra homology
\[
H_\bullet(M_\infty(A), \mathbb{C}) = \lim_{\rightarrow n} H_\bullet(M_n(A), \mathbb{C})
\]
is a Hopf algebra, and that Tsygan and Loday-Quillen prove that the primitive part \( \text{Prim} H_\bullet(M_\infty(A), \mathbb{C}) \) of this Hopf algebra identifies with the cyclic homology of \( A \). More precisely, there is a shift of 1 in the degrees so that

\[
\text{Prim}_p H_\bullet(M_\infty(A), \mathbb{C}) \rightarrow HC_{p-1}(A).
\]

This applies to \( A = C^\infty(X) \), viewed as a Fréchet algebra; the cyclic homology of \( C^\infty(X) \) is defined taking this topology onto account. Then the cyclic homology of \( C^\infty(X) \), computed by Connes [Co], Tsygan [Ts], Loday-Quillen [L-Q] is given by

\[
HC_i(C^\infty(X)) = \begin{cases} 
\left[ E^i(X)/dE^{i-1}(X) \right] \oplus H^{i-2}(X) \oplus \cdots \text{ if } i < k \\
\oplus_{n \in \mathbb{Z}} H^{i+2n}(X) \text{ if } i \geq k.
\end{cases}
\]

Dually, the primitive part in degree \( p \) of the Lie algebra cohomology of \( M_\infty(A) \) identifies with the cyclic cohomology group \( HC^{p-1}(A) \), which is computed by Connes:

\[
HC^i(C^\infty(X)) = \begin{cases} 
T_i(X)_{cl} \oplus H_{i-2}(X) \oplus \cdots \text{ if } i < k \\
\oplus_{n \in \mathbb{Z}} H_{i+2n}(X) \text{ if } i \geq k.
\end{cases}
\]

Here \( T_i(X)_{cl} \) denotes the space of closed degree \( i \) currents on \( X \). We can also consider the relative cyclic cohomology \( HC^i(C^\infty(X), \mathbb{C}) \) which is obtained by deleting the factor \( H_0(X) \) from \( HC_i(X) \) which is present for \( i \) even.

So the fundamental class in \( H_k(X) \) yields a class in \( HC^k(C^\infty(X), \mathbb{C}) \) which corresponds to a primitive class in \( H^{k+1}(M_\infty(C^\infty(X))) \). This class is precisely that defined by the Lie algebra cocycle (5-10).

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