One-Loop N-Point Superstring Amplitudes with Manifest d=4 Supersymmetry

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The hybrid formalism for the superstring is used to compute one-loop amplitudes with an arbitrary number of external d=4 supergravity states. These one-loop N-point amplitudes are expressed as Koba-Nielsen-like formulas with manifest d=4 supersymmetry.

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1. Introduction

Although some nonperturbative aspects of superstring theory can now be studied through duality symmetries, there are still many perturbative aspects of superstring theory which are not well understood. For example, perturbative finiteness of superstring amplitudes has only been proven by patching together amplitude computations using the light-cone Green-Schwarz (GS) formalism with amplitude computations using the Ramond-Neveu-Schwarz (RNS) formalism \[1].\] This patching is necessary since the light-cone GS amplitudes have contact term problems whereas the RNS amplitudes are only spacetime supersymmetric after summing over spin structures.

Another aspect of perturbative superstring theory which is not well developed is the computation of scattering amplitudes involving more than four external fermions. After extracting their low-energy contributions, these amplitude computations could be useful for simplifying analogous computations in QCD \[3\]. However, contact term interactions in the light-cone GS formalism and the complicated nature of Ramond vertex operators in the RNS formalism have made these amplitudes difficult to compute. So to study perturbative finiteness and to obtain amplitude expressions for more than four external fermions, it would be nice to have a superstring formalism which does not suffer from the problems of the GS and RNS formalisms.

Over the last seven years, a “hybrid” formalism has been developed which combines the advantages of the GS and RNS formalisms without including their disadvantages. The hybrid formalism is manifestly spacetime supersymmetric and therefore does not require summing over spin structures and can easily handle an arbitrary number of external fermions. Furthermore, in a flat target-space background, the hybrid worldsheet action is quadratic so scattering amplitudes can be computed using free field OPE’s.

In this paper, one-loop scattering amplitudes will be computed using the d=4 version of the hybrid formalism which describes the Type II superstring\[4\] compactified on any six-dimensional manifold which preserves at least d=4 supersymmetry \[1\]. These one-loop amplitudes will be computed for an arbitrary number of external N=2 d=4 supergravity states which are independent of the compactification. The amplitudes will be expressed as

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3 The proof of \[2\] using the twistor-string formalism is incorrect since the unphysical poles come from the chiral boson correlation function were not treated in a BRST-invariant manner.

4 Although only the Type II superstring will be discussed here, all results are easily generalized to one-loop open or heterotic superstring amplitudes.
Koba-Nielsen-like formulas with manifest N=2 d=4 supersymmetry which closely resemble the formulas computed in \[5\] for tree amplitudes. Although these one-loop amplitudes could be computed in principle using the RNS or light-cone GS formalisms, the problems described earlier have up to now prevented such computations. However, it will be shown that when all external states are NS-NS, the amplitudes agree with the RNS result.

There are various generalizations of these one-loop computations which might be possible. One possible generalization would be to compute one-loop amplitudes involving external compactification-dependent states. Such computations would be interesting for anomaly analysis but require a better understanding of correlation functions involving the chiral boson \(\rho\) in the hybrid formalism. A second possible generalization would be to compute one-loop amplitudes using the d=6 \[6\] or d=10 \[7\] pure spinor versions of the hybrid formalism which manifestly preserve more spacetime symmetries. And a third possible generalization would be to use the hybrid formalism to compute multiloop amplitudes which might be useful for studying perturbative finiteness. Up to now, only special “topological” amplitudes \[8\] which involve trivial correlation functions of the chiral boson \(\rho\) have been computed at higher loops using the hybrid formalism \[9\].

Section 2 of this paper will review the d=4 hybrid formalism. Section 3 will use the correlation functions of the hybrid worldsheet variables on a torus to compute explicit d=4 supersymmetric expressions for one-loop N-point amplitudes with external massless d=4 states. And section 4 will prove that these one-loop expressions are gauge-invariant, single-valued, modular invariant, and agree with the RNS one-loop amplitudes when all external states are in the NS-NS sector.

2. Review of d=4 Hybrid Formalism

2.1. Worldsheet action

After embedding the RNS superstring in a \(\hat{c} = 2\) N=2 superstring and performing a field redefinition on the worldsheet variables, the superstring can be described in a d=4 super-Poincaré covariant manner using the d=4 hybrid formalism \[10\]. This formalism can be used to describe either the uncompactified superstring or any compactification of the superstring which preserves at least d=4 supersymmetry. For the Type II superstring, the worldsheet variables in this formalism consist of the N=2 d=4 superspace variables
Note that all worldsheet variables are periodic and that the chiral boson \( \phi \) is fermionized since \( e^2 \).

### 2.2. Worldsheet N=2 superconformal generators

Where \( S \) is the action for the left and right-moving chiral bosons, and \( S_C \) is the action for the compactification-dependent variables. Note that \( z \) versus \( \bar{z} \) is correlated with \( L \) versus \( R \), and not with \( \theta \) versus \( \bar{\theta} \).

As \( y \to z \), the free-field OPE’s of the four-dimensional variables are

\[
x^m(y)x^n(z) \to -\eta^{mn} \log |y - z|^2,
\]

\[
\rho_L(y)\rho_L(z) \to -\log(y - z), \quad \rho_R(y)\rho_R(z) \to -\log(y - z),
\]

\[
p_L(y)\theta_L^\alpha(z) \to \frac{\delta_\alpha}{y - z}, \quad \bar{p}_L(y)\bar{\theta}_L^{\dot{\alpha}}(z) \to \frac{\delta_\dot{\alpha}}{y - z},
\]

\[
p_R(y)\theta_R^\alpha(z) \to \frac{\delta_\alpha}{y - \bar{z}}, \quad \bar{p}_R(y)\bar{\theta}_R^{\dot{\alpha}}(z) \to \frac{\delta_\dot{\alpha}}{y - \bar{z}}.
\]

Note that all worldsheet variables are periodic and that the chiral boson \( \rho \) can not be fermionized since \( e^{\rho_L(y)} e^{-\rho_L(z)} \to (y - z) \) and \( e^{\rho_R(y)} e^{-\rho_R(z)} \to (y - \bar{z}) \). It has the same behavior as the negative-energy field \( \phi \) that appears when bosonizing the RNS ghosts.

### 2.2. Worldsheet N=2 superconformal generators

Using RNS variables, one can construct the twisted \( \hat{c} = 2 \) N=2 superconformal generators \([T, G, \bar{G}, J] = [T_{\text{matter}} + T_{\text{ghost}}, j_{\text{BRST}}, b, bc + \xi \eta]\) where the \( L/R \) index is being suppressed in this subsection. In terms of the d=4 hybrid variables, these generators are mapped under the field redefinition to\(^4\)

\[
T = -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha - \bar{p}_{\dot{\alpha}} \partial \bar{\theta}^{\dot{\alpha}} - \frac{1}{2}(\partial \rho \partial \rho + \partial^2 \rho) + T^C,
\]

\[
G = e^\rho (d)^2 + G^C, \quad \bar{G} = e^{-\rho}(\bar{d})^2 + \bar{G}^C, \quad J = -\partial \rho + J^C,
\]

where

\[
d_\alpha = p_\alpha + \frac{i}{2} \bar{\theta}^\alpha \partial x_{\alpha \dot{\alpha}} - \frac{1}{4} (\bar{\theta})^2 \partial \theta_\alpha + \frac{1}{8} \theta_\alpha \partial (\bar{\theta})^2,
\]
\[
\bar{d}_\alpha = \bar{p}_\alpha + \frac{i}{2} \theta^\alpha \partial x_{\alpha \dot{\alpha}} - \frac{1}{4} (\theta)^2 \bar{\theta}_{\dot{\alpha}} + \frac{1}{8} \bar{\theta}_{\dot{\alpha}} \partial (\theta)^2 ,
\]

\( x_{\alpha \dot{\alpha}} = x_m \sigma^m_{\alpha \dot{\alpha}} \), and \([T^C, G^C, \tilde{G}^C, J^C]\) are the twisted \( \hat{c} = 3 \) N=2 generators of the superconformal field theory used to describe the compactification manifold.

As was shown by Siegel[11], \( d_\alpha \) and \( \bar{d}_\dot{\alpha} \) satisfy the OPE’s

\[
d_\alpha(y)\bar{d}_\dot{\alpha}(z) \rightarrow i \frac{\Pi_{\alpha \dot{\alpha}}}{y - z}, \quad d_\alpha(y)\bar{d}_\beta(z) \rightarrow \text{regular}, \quad \bar{d}_\alpha(y)\bar{d}_\beta(z) \rightarrow \text{regular}, \quad (2.4)
\]

\[
d_\alpha(y)U(z) \rightarrow \frac{D_\alpha U}{y - z}, \quad \bar{d}_\dot{\alpha}(y)U(z) \rightarrow \frac{\bar{D}_{\dot{\alpha}} U}{y - z},
\]

\[
d_\alpha(y)\delta^\beta_{\alpha}(z) \rightarrow \frac{\delta^\beta_{\alpha}}{(y - z)^2}, \quad \bar{d}_{\dot{\alpha}}(y)\delta^\beta_{\dot{\alpha}}(z) \rightarrow \frac{\delta^\beta_{\dot{\alpha}}}{(y - z)^2},
\]

\[
d_\alpha(y)\Pi^m(z) \rightarrow -i \frac{\sigma^m_{\alpha \dot{\alpha}} \bar{\theta}^\alpha}{y - z}, \quad \bar{d}_{\dot{\alpha}}(y)\Pi^m(z) \rightarrow -i \frac{\sigma^m_{\alpha \dot{\alpha}} \theta^\alpha}{y - z}, \quad \Pi^m(z)\Pi^n(z) \rightarrow -\frac{\eta^{mn}}{(y - z)^2}
\]

where

\[
\Pi^m = \partial x^m - \frac{i}{2} \sigma^m_{\alpha \dot{\alpha}} (\theta^\alpha \partial \bar{\theta}^\dot{\alpha} + \bar{\theta}^\dot{\alpha} \partial \theta^\alpha), \quad D_\alpha = \partial_{\theta^\alpha} - \frac{i}{2} \bar{\theta}^\dot{\alpha} \partial \alpha \dot{\alpha}, \quad \bar{D}_{\dot{\alpha}} = \partial_{\bar{\theta}^\dot{\alpha}} - \frac{i}{2} \theta^\alpha \partial \alpha \dot{\alpha}, \quad (2.5)
\]

and \( U(z) = U(x(z), \theta(z), \bar{\theta}(z)) \) is a scalar superfield. The advantage of working with the variables \( d_\alpha, \bar{d}_{\dot{\alpha}} \) and \( \Pi^m \) is that they commute with the spacetime supersymmetry generators

\[
q_\alpha = \int dz [p_\alpha - \frac{i}{2} \bar{\theta}^\alpha \partial x_{\alpha \dot{\alpha}} - \frac{1}{8} (\bar{\theta})^2 \partial \theta_{\alpha}], \quad \bar{q}_{\dot{\alpha}} = \int dz [\bar{p}_{\dot{\alpha}} - \frac{i}{2} \theta^\alpha \partial x_{\alpha \dot{\alpha}} - \frac{1}{8} (\theta)^2 \partial \bar{\theta}_{\dot{\alpha}}].
\]

2.3. Massless compactification-independent vertex operators

Since the hybrid formalism is a critical N=2 superconformal field theory, physical vertex operators can be described by U(1)-neutral N=2 primary fields with respect to the superconformal generators of \((2.3)\). For massless states of the Type II superstring which are independent of the compactification, the unintegrated vertex operator only depends on the zero modes of \([x^m, \theta_\alpha, \bar{\theta}^\dot{\alpha}, \theta^\alpha, \bar{\theta}^\dot{\alpha}]\) and is therefore described by the scalar superfield \( U(x, \theta_L, \bar{\theta}_L, \theta_R, \bar{\theta}_R) \), which is the prepotential for an N=2 d=4 supergravity and tensor multiplet [11]. The NS-NS fields for the graviton, anti-symmetric tensor, and dilaton are in the \( g^\alpha_L \bar{g}^\dot{\alpha}_L g^\beta_R \bar{g}^\dot{\beta}_R \) component of \( U \), i.e.

\[
[D_{L\alpha}, \bar{D}_{L\dot{\alpha}}][D_{R\beta}, \bar{D}_{R\dot{\beta}}]U|_{\theta = \bar{\theta} = 0} = \sigma^m_{\alpha \dot{\alpha}} \sigma^n_{\beta \dot{\beta}} (h_{mn} + b_{mn} + \eta_{mn} \phi), \quad (2.6)
\]
and the R-R field strengths for the U(1) vector and complex scalar are in the \((\theta_L)^2\bar{\theta}_R^\dagger\) \((\theta_R)^2\bar{\theta}_L^\dagger\) and \(\theta_L^\dagger(\bar{\theta}_L)^2\) \((\theta_R)^2\bar{\theta}_R^\dagger\) components of \(U\), i.e.

\[
(D_L)^2\bar{D}_{L\dot{\alpha}}(D_R)^2\bar{D}_{R\dot{\beta}}U|_{\theta=\bar{\theta}=0} = (\sigma_{\alpha m})_{\dot{\alpha}\dot{\beta}}F_{mn}, \quad D_{L\alpha}(\bar{D}_L)^2(D_R)^2\bar{D}_{R\dot{\alpha}}U|_{\theta=\bar{\theta}=0} = \sigma_{\alpha\beta}^m\partial_m y,
\]

where \(D_{L\alpha}\) and \(\bar{D}_{L\dot{\alpha}}\) are defined as in (2.5) with \(\theta_L^\dagger\) and \(\bar{\theta}_L^\dagger\) variables, and \(D_{R\alpha}\) and \(\bar{D}_{R\dot{\alpha}}\) are defined as in (2.3) with \(\theta_R^\dagger\) and \(\bar{\theta}_R^\dagger\) variables. Note that in standard SU(2) notation for N=2 superspace, \(\theta_L^\dagger = \theta_\alpha^+, \bar{\theta}_L^\dagger = \bar{\theta}_{\dot{\alpha}}^+, \theta_R^\dagger = \theta_\alpha^-, \) and \(\bar{\theta}_R^\dagger = \bar{\theta}_{\dot{\alpha}}^−\).

For \(U\) to be an N=2 primary field, it must satisfy the constraints

\[
(D_L)^2U = (\bar{D}_L)^2U = (D_R)^2U = (\bar{D}_R)^2U = \partial_m\theta^mU = 0.
\]

The first four constraints are the N=2 d=4 supersymmetric generalization of the usual polarization conditions, and the last constraint is the equation of motion in this gauge. Using the OPE’s of (2.2), one can compute that the integrated form of the closed superstring vertex operator is

\[
\int d^2z \ V = \int d^2z \ G_L(G_L(G_R(G_R(U)))) = \int d^2z \ |H(z)|^2 U(z, \bar{z}) \quad (2.7)
\]

where

\[
H(z) = d_L^\alpha(z)(\bar{D}_L)^2D_{L\alpha} + \bar{d}_{\dot{\alpha}}(z)(D_L)^2\bar{D}_{L\dot{\alpha}}
\]

\[
+ \partial\theta_L^\alpha(z)D_{L\alpha} - \partial\bar{\theta}_L^\dagger(z)\bar{D}_{L\dot{\alpha}} + \frac{i}{2}\Pi_{Lo\dot{\alpha}}(z)[D_L^\alpha, \bar{D}_{L\dot{\alpha}}],
\]

\(| \cdot |^2\) signifies the left-right product, and all superspace derivatives \(D_{\alpha}\) and \(\bar{D}_{\dot{\alpha}}\) act on the superfield \(U\).

Using the field redefinition to write (2.7) in terms of RNS variables, one can check that the NS-NS fields of (2.6) couple in (2.7) as

\[
\int d^2z \ V = \int d^2z (\partial x^m + i\bar{\psi}_L^m\bar{\psi}_L^p k_p)(\bar{\partial}x^n + i\bar{\psi}_R^n\bar{\psi}_R^q k_q)(h_{mn} + b_{mn} + \eta_{mn}\phi), \quad (2.9)
\]

which is the usual RNS vertex operator. Furthermore, \(\int d^2z \ V\) is invariant up to a surface term under the linearized gauge transformation

\[
\delta U = (D_L)^2\Lambda_L + (\bar{D}_L)^2\bar{\Lambda}_L + (D_R)^2\Lambda_R + (\bar{D}_R)^2\bar{\Lambda}_R, \quad (2.10)
\]

which is the N=2 d=4 supersymmetric generalization of the gauge transformation \(\delta(h_{mn} + b_{mn} + \eta_{mn}\phi) = \partial_m\lambda_{Ln} + \partial_n\lambda_{Rm}\). For example, under \(\delta U = (D_L)^2\Lambda_L\),

\[
\delta V = (-\partial\bar{\theta}_L^\dagger\bar{D}_{L\dot{\alpha}} + \frac{i}{2}\Pi_{Lo\dot{\alpha}}[D_L^\alpha, \bar{D}_{L\dot{\alpha}}])\bar{H}(\bar{z})(D_L)^2\Lambda_L \quad (2.11)
\]

\[
= (-\partial\theta_L^\dagger D_{L\alpha} - \partial\bar{\theta}_L^\dagger\bar{D}_{L\dot{\alpha}} - \Pi_{L\alpha}^m\partial_m)\bar{H}(\bar{z})(D_L)^2\Lambda_L = -\partial_{\bar{z}}(\bar{H}(\bar{z})(D_L)^2\Lambda_L).
\]
3. One-Loop N-point Scattering Amplitude

3.1. Topological prescription

In [9], a “topological” prescription was given for computing scattering amplitudes for any \( \hat{c} = 2 \) \( \mathcal{N}=2 \) superconformal field theory. This prescription uses the twisted version of the \( \mathcal{N}=2 \) superconformal field theory in which the worldsheet \( \mathcal{N}=2 \) superconformal ghosts contribute zero central charge and decouple from the scattering amplitudes. For the \( \hat{c} = 2 \) \( \mathcal{N}=2 \) superconformal field theory representing the self-dual string on \( T^2 \times \mathbb{R}^2 \), this prescription was used by Ooguri and Vafa [12] to explicitly compute the \( g \)-loop partition function dependence on the \( T^2 \) moduli.

For the \( \hat{c} = 2 \) \( \mathcal{N}=2 \) superconformal field theory which represents the superstring using the \( d=4 \) hybrid formalism reviewed in section 2, there is a subtlety in the prescription caused by the negative energy chiral boson \( \rho \). Like the chiral boson \( \phi \) in the RNS formalism, correlation functions of \( \rho \) can have unphysical poles which need to be treated carefully. Fortunately, as in the special multiloop amplitudes computed in [3][4], this subtlety can be ignored here because the chiral boson \( \rho \) will decouple from the other worldsheet variables.

The topological prescription for the one-loop Type II superstring amplitude is

\[
A = \int d^2\tau (\tau_2)^{-2} \int d^2z_1 \cdots \int d^2z_N \left\langle (\int J_L \wedge J_R)^2 V_1(z_1, \bar{z}_1) \cdots V_N(z_N, \bar{z}_N) \right\rangle
\]

(3.1)

where \( \int J_L \wedge J_R = \int d^2w (-\partial \rho_L + J_L^C)(-\bar{\partial} \rho_R + J_R^C) \) is constructed from the \( U(1) \) currents of [4], and \( \langle \rangle \) signifies the two-dimensional correlation function on a torus. As discussed in [3], this prescription reproduces (up to picture-changing subtleties) the standard one-loop RNS prescription in the large Hilbert space. Note that when written in terms of RNS variables, \( \langle \int J_L \wedge J_R \rangle^2 = \langle \int d^2w (b_L c_L + \xi_L \eta_L)(b_R c_R + \xi_R \eta_R) \rangle^2 \), and this term is necessary for providing the \([b_L, c_L, \xi_L, \eta_L] \) and \([b_R, c_R, \xi_R, \eta_R] \) zero modes in the large RNS Hilbert space.

When the external states are \( d=4 \) supergravity states represented by (2.7), the vertex operators are independent of the \( \rho \) variable and the compactification variables. So the two-dimensional correlation function factorizes into a \( d=4 \) contribution coming from the \([x^m, \theta^\alpha, \theta^\dot{\alpha}, p_\alpha, \bar{p}_{\dot{\alpha}}] \) worldsheet variables which depends on the external states, and a \( d=6 \) contribution coming from the \( \rho \) variable and compactification variables which does not depend of the external states. Since the \( d=4 \) worldsheet variables are free fields, one can easily compute their correlation functions on a torus. Although the compactification variables are not necessarily free fields, they only contribute an overall factor through their partition function which is independent of the external momenta and polarizations.
3.2. Partition functions on a torus

Since the d=4 worldsheet variables \([x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha}, p_\alpha, \bar{p}_{\dot{\alpha}}]\) all are periodic free fields, it is straightforward to compute their partition functions. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight. From the four \(x^m\)’s, the partition function is \(|\eta(\tau)|^{-2}\tau_2^{-\frac{3}{2}}\) conformal weight.

And the partition functions for the fermions \(\theta^\alpha_L, \theta^\dot{\alpha}_L, \bar{\theta}^\dot{\alpha}_R, \bar{\theta}^\alpha_R\) and \(\bar{p}_L\alpha, \bar{p}_L\dot{\alpha}, p_R\alpha, p_R\dot{\alpha}\) contribute \(|\eta(\tau)|^{16}\) for correlation functions involving all sixteen fermionic zero modes. Note that these sixteen fermionic zero modes will come from the external vertex operators.

The partition functions of the remaining variables depend on the twisted \(\hat{c} = 3\) \(\mathbb{N}=2\) superconformal field theory which describes the compactification manifold. For example, for the uncompactified superstring, the remaining variables are \([x^j, \bar{x}^\dot{j}, \Gamma^j_L, \bar{\Gamma}^\dot{j}_L, \Gamma^j_R, \bar{\Gamma}^\dot{j}_R, \rho_L, \rho_R]\) where \(j = 1\) to \(3\), \(\Gamma^j_L/R\) are fermions of zero conformal weight and \(\Gamma^j_L/R\) are fermions of higher weight.

The partition function for the six uncompactified \(x^j\)’s provides the factor \(|\eta(\tau)|^{-12}\tau_2^{-3}\). Naively, the partition function for \([\Gamma^j_L, \bar{\Gamma}^\dot{j}_L, \Gamma^j_R, \bar{\Gamma}^\dot{j}_R]\) is zero since \(\langle (\int J_L \wedge J_R)^2 \rangle = \langle (\int (-\partial \rho_L + \Gamma^j_L \bar{\Gamma}^\dot{j}_L)(-\partial \rho_R + \Gamma^j_R \bar{\Gamma}^\dot{j}_R))^2 \rangle\) contributes at most eight of the twelve fermion zero modes. However, the partition function for the \(\rho\) field diverges in the absence of \(e^\rho\) factors, so the expression \(\langle (\int J_L \wedge J_R)^2 \rangle\) needs to be regularized. This can be done by writing

\[
1 = \lim_{y \to z} \Gamma^1_L(y) \Gamma^1_R(\bar{y}) e^{-\rho_L(y)-\rho_R(\bar{y})} \bar{\Gamma}^\dagger_L(z) \bar{\Gamma}^\dagger_R(\bar{\bar{z}}) e^{\rho_L(z)+\rho_R(\bar{\bar{z}})}
\]

and computing \(\langle 1 \ (\int J_L \wedge J_R)^2 \rangle\). With this regularization, the partition function for \([\Gamma^2_L, \Gamma^3_L, \bar{\Gamma}^2_L, \bar{\Gamma}^3_L]\) and \([\Gamma^2_R, \Gamma^3_R, \bar{\Gamma}^2_R, \bar{\Gamma}^3_R]\) gives \(|\eta(\tau)|^8\tau_2^2\) where the eight fermion zero modes come from \((J_L \wedge J_R)^2\) and the \(\tau_2^2\) factor comes from integrating \((J_L \wedge J_R)^2\) twice over the torus.

Finally, the partition function for the \([\Gamma^1_L, \bar{\Gamma}^1_L, \rho_L]\) and \([\Gamma^1_R, \bar{\Gamma}^1_R, \rho_R]\) variables can be computed using the same method as was used in \([13]\) for the \([\xi_L, \eta_L, \phi_L]\) and \([\xi_R, \eta_R, \phi_R]\) variables in the large RNS Hilbert space. It was shown in \([13]\) that functional integration over \([\xi, \eta, \phi]\) variables reproduces the Verlinde-Verlinde prescription of \([14]\) for \([\beta, \gamma]\) correlation functions if one inserts \(\xi_L(v) \xi_R(\bar{v}) (\int \eta_L \wedge \eta_R)^{\eta}\) into the \(g\)-loop correlation function.

For example, the one-loop \([\beta, \gamma]\) correlation function

\[
\langle \delta(\gamma_L(y)) \delta(\gamma_R(\bar{y})) \delta(\beta_L(z)) \delta(\beta_R(\bar{z})) \rangle
\]
is reproduced by the \([\xi, \eta, \phi]\) correlation function
\[
\langle \xi_L(v)\xi_R(\bar{v}) \rangle \left( \int d^2w \, \eta_L(w)\eta_R(\bar{w}) e^{-\phi_L(y)-\phi_R(\bar{y})} e^{\phi_L(z)+\phi_R(\bar{z})} \right). 
\] (3.4)

Since only the zero modes of \(\xi\) and \(\eta\) contribute to (3.4), this correlation function can be written as
\[
\tau_2 \langle \xi_L(y)\xi_R(\bar{y})\eta_L(z)\eta_R(\bar{z}) e^{-\phi_L(y)-\phi_R(\bar{y})} e^{\phi_L(z)+\phi_R(\bar{z})} \rangle 
\] (3.5)

where the \(\tau_2\) factor comes from the \(d^2w\) integration. But
\[
\langle \delta(\gamma_L(y))\delta(\gamma_R(\bar{y})) \rangle = |\eta(\tau)|^{-4} 
\] (3.6)

for odd spin structure, which is the relevant spin structure for the periodic \(\rho\) variable. Therefore, by comparing with (3.5) one finds that \(\langle 1 \rangle = |\eta(\tau)|^{-4}\tau_2^{-1}\) for the \([\Gamma_L^1, \Gamma_L^1, \rho_L]\) and \([\Gamma_R^1, \Gamma_R^1, \rho_R]\) partition functions where 1 is defined in (3.2).

Multiplying together the above partition functions, one finds for the uncompactified superstring that all \(\eta(\tau)\) factors cancel and
\[
A = \int d^2\tau(\tau_2)^{-6} \int d^2z_1... \int d^2z_N \langle \langle V_1(z_1, \bar{z}_1)...V_N(z_N, \bar{z}_N) \rangle \rangle 
\] (3.7)

where \(\langle \langle \rangle \rangle\) signifies the correlation function on a torus divided by the partition function, and the correlation function must include all sixteen zero modes of the \(d = 4\) fermionic variables. Using techniques similar to those used in (3.1) for tree amplitudes, (3.7) will now be explicitly computed.

### 3.3. Koba-Nielsen-like formula for one-loop amplitude

To evaluate (3.7) for external massless compactification-independent states, it is convenient to use (2.7) to write
\[
\langle \langle \prod_{r=1}^{N} V_r(z_r, \bar{z}_r) \rangle \rangle = \langle \langle \prod_{r=1}^{N} |H_r|^2 U_r(z_r, \bar{z}_r) \rangle \rangle 
\]

\[
= \prod_{s=1}^{N} \frac{\partial}{\partial \epsilon_s} \frac{\partial}{\partial \bar{\epsilon}_s}|_{\epsilon_s=\bar{\epsilon}_s=0} \langle \langle \exp\left( \sum_{r=1}^{N} \epsilon_r H_r \right)^2 \prod_{t=1}^{N} U_t(z_t, \bar{z}_t) \rangle \rangle 
\] (3.8)

where \(H_r\) is defined in (2.8) and \(D_{r\alpha}\) and \(\bar{D}_{r\dot{\alpha}}\) are fermionic derivatives which act only on \(U_r(z_r, \bar{z}_r)\). For example, \(D_{2\alpha} \prod_{r=1}^{N} U_r(z_r, \bar{z}_r) = U_1(z_1, \bar{z}_1) \prod_{r=3}^{N} U_r(z_r, \bar{z}_r)\). Furthermore, it will be convenient to introduce the notation
\[
H_r = d_\alpha(z_r)w_\alpha^r + \bar{d}_\dot{\alpha}(z_r)\bar{w}_{\dot{\alpha}}^r + \partial \theta_\alpha(z_r)a_\alpha^r + \bar{\partial} \theta_{\dot{\alpha}}(z_r)a_{\dot{\alpha}}^r + \Pi_m(z_r) b_r^m 
\] (3.9)
where
\[ w^{\dot{\alpha}}_r = -(\bar{D}_r)^2 D^\alpha_r, \quad \bar{w}^{\dot{\alpha}}_r = -(D_r)^2 \bar{D}^\dot{\alpha}_r, \quad a^\alpha_r = -D^\alpha_r, \quad \bar{a}^{\dot{\alpha}}_r = \bar{D}^{\dot{\alpha}}_r, \quad b^m_r = \frac{i}{2} \sigma^m_{\alpha\dot{\alpha}} [D^\alpha_r, \bar{D}^{\dot{\alpha}}_r]. \]

Note that the $L/R$ index has been suppressed in these formulas and that $| \cdot |^2$ signifies the left-right product.

The first step in evaluating (3.8) is to eliminate the $d^\alpha$ variables using the OPE’s of (2.4). After separating off the zero mode $d^\alpha_{(0)}$ of $d^\alpha$, any correlation function involving $d^\alpha(z)$ is determined by the conditions that it is a periodic function of $z$ and has the pole structure determined by the OPE’s of (2.4). So (3.8) is equal to

\[ \prod_{s=1}^N \frac{\partial}{\partial \epsilon_s} \frac{\partial}{\partial \bar{\epsilon}_s} \bigg|_{\epsilon_s = \bar{\epsilon}_s = 0} \langle \exp(M + \sum_r \epsilon_r d^\alpha_{(0)} w^\alpha_r) \rangle^2 : \prod_{r=1}^N U_r(z_r, \bar{z}_r) \rangle \]  

(3.11)

where
\[ M = \sum_r \epsilon_r [\bar{d}^{\dot{\alpha}}(z_r) \bar{w}^{\dot{\alpha}}_r + \partial \bar{\theta}^{\dot{\alpha}}(z_r) \bar{\theta}^{\dot{\alpha}}_r + \Pi_m(z_r) b^m_r] \]

(3.12)

\[ + \sum_{r,s} \epsilon_r w^\alpha_r F(z_s - z_r)(D_{s\alpha} + \epsilon_s (i \Pi_{\alpha\dot{\alpha}}(z_s) \bar{w}^{\dot{\alpha}}_s - i \partial \bar{\theta}^{\dot{\alpha}}(z_s) b_{s\alpha\dot{\alpha}})) \]

\[ - \sum_{r,s} \epsilon_r \epsilon_s \partial F(z_r - z_s) w^\alpha_r a_{s\alpha} + \sum_{r,s,t} \epsilon_r \epsilon_s \epsilon_t F(z_r - z_s) F(z_t - z_s) \partial \bar{\theta}^{\dot{\alpha}}(z_s) \bar{w}^{\dot{\alpha}}_s w_{t\alpha} w^\alpha_r, \]

the function $F(z)$ in (3.12) is the Dirac propagator for odd spin structure
\[ F(z) = \partial \bar{z} \log \Theta_1(z, \tau), \]

and the normal ordering symbol $: \cdot \cdot :$ signifies that $D^\alpha_r$ is always ordered to the left of $[w^\alpha_r, \bar{w}^{\dot{\alpha}}_r, a^\alpha_r, \bar{a}^{\dot{\alpha}}_r, b^m_r]$. The term proportional to $\epsilon_r \epsilon_s \epsilon_t$ in (3.12) comes from the pole of $d^\alpha(z_t)$ with the residue of the pole of $d^\beta(z_r)$ at $z_s$. Note that all $\theta^\alpha(z)$ variables in (3.11) can be set equal to their zero mode $\theta^\alpha_{(0)}$ since there are no more $d^\alpha$ variables to contract with.

Although $F(z) \to F(z) - 2\pi i n$ under $z \to z + m + n \tau$, the correlation function of (3.11) is single-valued on the torus after integrating out the fermionic zero modes $d^\alpha_{(0)}$. To see this, suppose that $z_u \to z_u + m + n \tau$ for some $u$. Then $M \to M - 2\pi i n \delta M$ where
\[ \delta M = (\sum_r \epsilon_r w^\alpha_r) [D_{u\alpha} + \epsilon_u (i \Pi_{\alpha\dot{\alpha}}(z_u) \bar{w}^{\dot{\alpha}}_u - i \partial \bar{\theta}^{\dot{\alpha}}(z_u) b_{u\alpha\dot{\alpha}} + 2 \partial \bar{\theta}^{\dot{\alpha}}(z_u) \bar{w}^{\dot{\alpha}}_u \sum_t \epsilon_t w_{t\alpha} F(z_t - z_u))] \]

Although one could instead have used the free field OPE’s of (2.2) to eliminate the $p^\alpha$ variables of (3.8), this would break manifest $d=4$ supersymmetry.
\[-\epsilon_u w^\alpha_u \sum_s [D_{s\alpha} + \epsilon_s (i\Pi_{\alpha}\delta(z_s)\bar{w}^\alpha_s - i\partial\bar{\theta}^\alpha(z_s)b_{s\alpha\bar{\alpha}} + 2\partial\bar{\theta}^\alpha(z_s)\bar{w}^\alpha_s + \sum_t \epsilon_tw_tF(z_t - z_s))]].\]

Since integrating over \(\delta a^{(0)}_r\) brings down the term \((\sum_r \epsilon_r w^\alpha_r)^2\), the first term in \(\delta M\) does not contribute since it is proportional to \(\sum_r \epsilon_r w^\alpha_r\). One can check that the second term in \(\delta M\) contributes

\[
\langle\langle \langle \delta M \rangle \exp M^2 : \prod_t U_t(z_t, \bar{z}_t) \rangle \rangle = -\langle\langle \langle \epsilon_u w^\alpha_u (\int dy \ d_\alpha) \rangle \exp M^2 : \prod_t U_t(z_t, \bar{z}_t) \rangle \rangle
\]

where the contour integration in \(\int dy \ d_\alpha\) goes around all N external vertex operator locations \(z_r\). Deforming this contour off the back of the torus, one finds that \(\langle\langle \langle \delta M \rangle \exp M^2 \rangle \rangle = 0\), so the correlation function of \(\langle\langle \delta M \rangle \exp M^2 \rangle \rangle \) is single-valued on the torus.

One can similarly use the OPE’s of \((2.4)\) to eliminate \(\bar{\theta}^\alpha\) in \((3.11)\) and write \(\langle\langle \prod_{r=1}^N V_r(z_r, \bar{z}_r) \rangle \rangle\) as

\[
\prod_{s=1}^N \frac{\partial \partial \epsilon_s \partial \epsilon_s}_{|\epsilon_s = \bar{\epsilon}_s = 0} \langle\langle \epsilon_r (d^{(0)}_\alpha w^\alpha_r + d^{(0)}_\bar{\alpha} \bar{w}^\bar{\alpha}_r) + L : \prod_{r=1}^N U_r(z_r, \bar{z}_r) \rangle \rangle
\]

where

\[
L = \sum_r \epsilon_r \Pi_m(z_r)b^m_r + \sum_{r,s} \epsilon_r F(z_r - z_s)(D_{s\alpha}w^\alpha_r + \bar{D}_{s\bar{\alpha}}\bar{w}^\bar{\alpha}_r)
\]

\[
+ \sum_{r,s} \epsilon_r \epsilon_s [F(z_r - z_s) i\Pi_{\alpha\bar{\alpha}}(z_s)\bar{w}^\alpha_s w^\alpha_r + \partial F(z_r - z_s)(a_{s\alpha}w^\alpha_r + \bar{a}_{s\bar{\alpha}}\bar{w}^\bar{\alpha}_r)]
\]

\[
- i \sum_{r,s,t} \epsilon_r \epsilon_s \epsilon_t b_{t\alpha\bar{\alpha}} w^\alpha_s \bar{w}^\bar{\alpha}_r F(z_t - z_s) \partial F(z_r - z_t)
\]

\[
+ \sum_{r,s,t,u} \epsilon_r \epsilon_s \epsilon_t \epsilon_u w_{t\alpha\bar{\alpha}} w^\alpha_s w^\alpha_u \bar{w}^\bar{\alpha}_r F(z_t - z_s) F(z_u - z_r) \partial F(z_t - z_r),
\]

all \(\bar{\theta}^\alpha(z)\) variables are set equal to their zero mode \(\bar{\theta}^{\hat{\alpha}}_{(0)}\), and the normal ordering symbol \(\langle\langle \epsilon_r \epsilon_s \epsilon_t \epsilon_u\rangle\rangle\) now signifies that \(\bar{D}^{\hat{\alpha}}_r\) is always ordered to the left of \(D^\alpha_r\), which is always ordered to the left of \([w^\alpha_r, \bar{w}^\bar{\alpha}_r, a^\alpha_r, \bar{a}^\bar{\alpha}_r, b^m_r]\).

The final step in evaluating \(\langle\langle \prod_{r=1}^N V_r(z_r, \bar{z}_r) \rangle \rangle\) is to perform the correlation function over the \(x^m\) variables. Note that \(\Pi^m\) in \((3.13)\) is equal to \(\partial x^m\) since \(\theta^\alpha\) and \(\bar{\theta}^{\hat{\alpha}}\) have been set equal to their zero modes. The correlation function over \(x^m\) is easily performed using the scalar Green’s function \(x^m(y, \bar{y})x^m(z, \bar{z}) \rightarrow \eta^{mn}G(y - z)\) where

\[
G(y - z) = -\log |\Theta_1(y - z, \tau)|^2 + \frac{2\pi}{\tau_2} |\Im(y - z)|^2.
\]

(3.16)
For example,

$$\langle \langle \exp(\sum_r c^m_r \partial x_m(z_r)) \rangle \rangle^2 \prod_s e^{ik^m_s x_m(z_s)} \rangle \rangle (3.17)$$

$$= \exp\left(\sum_{r,s} \left[-\frac{1}{2} k^m_r k^s_m G(z_r - z_s) + i c^m_r k^s_m \partial G(z_r - z_s) + i \bar{c}^m_r k^s_m \partial \bar{G}(z_r - z_s) \right] \right)$$

$$- \frac{1}{2} c^m_r c^s_m \partial^2 G(z_r - z_s) - \frac{1}{2} \bar{c}^m_r \bar{c}^s_m \partial^2 \bar{G}(z_r - z_s) - c^m_r \bar{c}^s_m \partial \bar{G}(z_r - z_s) \right]$$

$$= \exp\left(-\frac{2\pi}{\tau_2} \left[ I_m \sum_r (c^m_r + ik^m_r z_r) \right] \right) \exp\left(\ln \left[ \exp\left(-\frac{i c^m_r k^s_m F(z_r - z_s) + \frac{1}{2} c^m_r c^s_m \partial F(z_r - z_s) + \frac{1}{2} k^m_r k^s_m \log \Theta_1(z_r - z_s, \tau) \right) \right] \right)$$

where $\tilde{k}^m_r$ in (3.17) is defined by $\tilde{k}^m_r = -k^m_r$ so that $e^{i \tilde{k}^m_r z_m(x)} = e^{ik^m_r x_m(z_r)}$.

So after performing the correlation function over $x_m$, integrating out the zero modes of $[d^\alpha_0(x), d^{\bar{\alpha}}_0(x), \theta^\alpha_0(x), \theta^{\bar{\alpha}}_0(x)]$, and plugging into (3.17), one finally obtains the N=2 d=4 supersymmetric Koba-Nielsen-like formula for the one-loop amplitude

$$A = \int d^2 \tau_2 d^2 \theta L d^2 \theta R d^2 \bar{\theta} R : \exp(C) : \prod_{r,s}^N \left( \frac{\partial}{\partial \epsilon_s} \frac{\partial}{\partial \bar{\epsilon}_s} \right)$$

$$\int d^2 \theta L d^2 \theta R d^2 \bar{\theta} R : \exp\left(-\frac{2\pi}{\tau_2} \left[ I_m(K^m + i \sum_r k^m_r z_r) \right] \right) \prod_{r,s} \left( \frac{\partial}{\partial \epsilon_s} \frac{\partial}{\partial \bar{\epsilon}_s} \right)$$

where

$$C = \sum_{r,s} \epsilon_r F(z_r - z_s)(D_{s\alpha} w^\alpha_r + \bar{D}_{s\bar{\alpha}} \bar{w}^{\bar{\alpha}}_r - ik^s_m b^m_{r})$$

$$+ \sum_{r,s,t} \epsilon_r \epsilon_s \epsilon_t F(z_r - z_s) F(z_s - z_t) k_{t\alpha\bar{\alpha}} \bar{w}^{\bar{\alpha}}_{s\bar{\alpha}} w^\alpha_r + \partial F(z_r - z_s)(a_{s\alpha} w^\alpha_r + \bar{a}_{s\bar{\alpha}} \bar{w}^{\bar{\alpha}}_r + \frac{1}{2} b^m_r b^m_{s})]$$

$$- i \sum_{r,s,t} \epsilon_r \epsilon_s \epsilon_t b_{t\alpha\bar{\alpha}} w^\alpha_r \bar{w}^{\bar{\alpha}}_s F(z_t - z_s) - F(z_r - z_s)) \partial F(z_r - z_t)$$

$$+ \sum_{r,s,t,u} \epsilon_r \epsilon_s \epsilon_t \epsilon_u w_{t\alpha\bar{\alpha}} w_{s\alpha\bar{\alpha}} F(z_u - z_t) - F(z_u - z_t)) F(z_s - z_r) \partial F(z_r - z_t),$$

$$K^m = \sum_r \epsilon_r b^m_r - i \sigma^m_{\alpha\bar{\alpha}} \sum_{r,s} \epsilon_r \epsilon_s w^\alpha_r \bar{w}^{\bar{\alpha}}_s F(z_r - z_s),$$

(3.20)

$\tilde{U}_t(k, \theta, \bar{\theta})$ is the Fourier transform of $U_t(x, \theta, \bar{\theta})$, $\tilde{k}^m_r$ is defined by $\tilde{k}^m_r = -k^m_r$ in $K^m$ and $C$, $F(z) = \partial_z \log \Theta_1(z, \tau)$, $[w^\alpha_r, \bar{w}^{\bar{\alpha}}_r, a^\alpha_r, \bar{a}^{\bar{\alpha}}_r, b^m_r$] is defined in (B.18), and the normal ordering symbol $: :$ signifies that $\bar{D}^{\bar{\alpha}}_r$ is always ordered to the left of $D^\alpha_r$, which is always ordered to the left of $[w^\alpha_r, \bar{w}^{\bar{\alpha}}_r, a^\alpha_r, \bar{a}^{\bar{\alpha}}_r, b^m_r]$. 

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4. Consistency of the One-Loop Formula

Although the derivation of (3.18) was completely straightforward, the consistency of this one-loop formula will now be checked by showing that it is invariant under gauge transformations of the external states, single-valued as a function of the vertex operator locations, modular invariant, and agrees with the RNS one-loop amplitude when all external states are in the NS-NS sector.

4.1. Gauge invariance

To check the invariance of (3.18) under the gauge transformation of (2.10), first consider \( \delta U = (D_L)^2 L_u \) for the external state at \( z_u \). Since \( D_L^2 \delta U = 0 \), one sees from (3.10) that \( w_L^u = \bar{w}_L^\alpha = a_L^\alpha = 0 \), \( \bar{a}_u^\alpha = \bar{D}_L^\alpha \), and \( b^m_u = -ik^m_u \). So from (3.19) and (3.20),

\[
\frac{\partial}{\partial \epsilon} |_{\epsilon_u=0} : \exp C |^2 \exp(-\frac{2\pi}{\tau_2} [\text{Im}(K^m + i \sum r k^m_r z_r)]^2)
\]

\[
= \left[ -i \sum_r F(z_u - z_r) k_r m_b^m + \sum_r \epsilon_r \partial F(z_r - z_u)(\bar{a}_{u\alpha} \bar{w}_r^\alpha + b^m_r b_{um}) - i \sum_{r,s} \epsilon_r \epsilon_s b_{u\alpha\hat{\alpha}} w_{s\alpha} \bar{w}_r^\alpha \right] (F(z_u - z_s) - F(z_r - z_s)) \partial F(z_r - z_u)
\]

\[
+ \frac{2\pi i}{\tau_2} b_{um} \text{Im}(K^m + i \sum_r k^m_r z_r) | \exp C |^2 \exp(-\frac{2\pi}{\tau_2} [\text{Im}(K^m + i \sum r k^m_r z_r)]^2) : \right]
\]

\[
= \left[ -i \sum_r F(z_u - z_r) k_r m b^m + \sum_r \epsilon_r \partial F(z_r - z_u)(\bar{D}_{u\alpha} \bar{w}_r^\alpha - ib^m_r k_{um}) + \sum_{r,s} \epsilon_r \epsilon_s k_{u\alpha \hat{\alpha}} w_{s\alpha} \bar{w}_r^\alpha F(z_r - z_s) \partial F(z_r - z_u) + \frac{2\pi}{\tau_2} k_{um} \text{Im}(K^m + i \sum r k^m_r z_r) \right]
\]

\[
| \exp C |^2 \exp(-\frac{2\pi}{\tau_2} [\text{Im}(K^m + i \sum r k^m_r z_r)]^2) : \right]
\]

where when changing \( \bar{a}_{u\alpha} \) to \( \bar{D}_{u\alpha} \), one needs to be careful with normal ordering since

\[
e^C : \bar{D}_{u\alpha} = : e^C \bar{D}_{u\alpha} : - : [\bar{D}_{u\alpha}, e^C] : \right) = e^C \bar{D}_{u\alpha} : - : k_{u\alpha\hat{\alpha}} \sum_r \epsilon_r F(z_r - z_u) w_{r\alpha} e^C : \right)
\]

when \( \epsilon_u = 0 \). One can easily check that (4.1) is equal to

\[
- \frac{\partial}{\partial z_u} |_{\epsilon_u=0} : \exp C |^2 \exp(-\frac{2\pi}{\tau_2} [\text{Im}(K^m + i \sum r k^m_r z_r)]^2) \cdot \prod_{r,s} | \Theta_1 (z_r - z_s, \tau) |^{k^m_r k_{s\alpha m}}, \right)
\]

so the amplitude of (3.18) is gauge invariant up to a surface term. Similarly, one can show that the gauge transformations \( \delta U_u = (D_L)^2 \Lambda_L + (D_R)^2 \Lambda_R + (\bar{D}_R)^2 \bar{\Lambda}_R \) only change (3.18) by a surface term.
4.2. Single-valued function of vertex operator locations

Under the transformation \( z_u \to z_u + m + n \tau \) for integer \( m \) and \( n \), the function

\[
F(z) = \partial_z \log \Theta_1(z, \tau) \to F(z) - 2\pi i n.
\]

Since the points \( z_u \) and \( z_u + m + n \tau \) are identified on the torus, one needs to check that the integrand of (3.18) is single-valued under this transformation.

Using that \(
\sum_r \epsilon_r w^\alpha_r = \sum_r \epsilon_r \bar{w}^\alpha_r = \sum_r k^m_r = 0
\) from the \([d_{\alpha}, \bar{d}_{\alpha}, x^m] \) zero mode integration, one finds that :

\[
| \exp C |^2 : \text{transforms into}
\]

\[
-ik^m u_m \sum_r \epsilon_r b^m_r + \epsilon_u w^\alpha_u \sum_{s, t} \epsilon_s F(z_s - z_l) k_{t\alpha\bar{\alpha}} \bar{w}^{\bar{\alpha}_s} + k_{u\alpha\bar{\alpha}} \sum_r \epsilon_r \epsilon_s F(z_r - z_s) \bar{w}^{\bar{\alpha}_s} w^\alpha_r) |^2 :
\]

\[
= : | \exp(C + 2\pi i n k^m u_m |^2 : 
\]

where the terms proportional to \( \sum_s D_{s\alpha} \) and \( \sum_{s} \bar{D}_{s\dot{\alpha}} \) are total derivatives which can be ignored after ordering \( \sum_s D_{s\alpha} \) to the left of \( e^C \) using

\[
: e^C \sum_s D_{s\alpha} := \sum_s D_{s\alpha} : e^C : - : \sum_r \epsilon_r F(z_r - z_s) k_{s\alpha\dot{\alpha}} \bar{w}^{\dot{\alpha}}_r e^C : .
\]

Since the term

\[
\exp(- \frac{2\pi}{\tau_2} [Im(i \sum_r k^m_r z_r)^2]) \prod_{r, s} |\Theta_1(z_r - z_s, \tau)|^{k^m_r k^m_s}
\]

is invariant under \( z_u \to z_u + m + n \tau \),

\[
\exp(- \frac{2\pi}{\tau_2} [Im(K^m + i \sum_r k^m_r z_r)^2]) \prod_{r, s} |\Theta_1(z_r - z_s, \tau)|^{k^m_r k^m_s}
\]

transforms into

\[
\exp(-4\pi i n k^m u_m Im K^m) \exp(- \frac{2\pi}{\tau_2} [Im(K^m + i \sum_r k^m_r z_r)^2]) \prod_{r, s} |\Theta_1(z_r - z_s, \tau)|^{k^m_r k^m_s}.
\]

So the product

\[
| \exp C |^2 : \exp(- \frac{2\pi}{\tau_2} [Im(K^m + i \sum_r k^m_r z_r)^2]) \prod_{r, s} |\Theta_1(z_r - z_s, \tau)|^{k^m_r k^m_s}
\]

is invariant under \( z_u \to z_u + m + n \tau \), implying that the integrand of (3.18) is single-valued.
4.3. Modular invariance

Under the modular transformation \( \tau \rightarrow \tau' = -\tau^{-1} \) and \( z \rightarrow z' = \tau^{-1}z \), the amplitude of (3.18) should remain invariant. To check this, it is useful to define \( \epsilon'_r = \tau^{-1}\epsilon_r \). Using

\[
\tau'_2 = |\tau|^{-2}\tau_2, \quad F(z, \tau) = \tau^{-1}F(z', \tau') - 2\pi iz', \quad \partial F(z, \tau) = \tau^{-2}\partial'F(z', \tau') - 2\pi i\tau^{-1},
\]

one finds that

\[
\mathcal{A} = \int d^2\tau'(|\tau'|^{-6}|\tau|^{-8}) \int d^2z'_N \prod_{s=1}^{N} \frac{\partial}{\partial \epsilon'_s} \frac{\partial}{\partial \bar{\epsilon}'_s} |\epsilon'_s = \bar{\epsilon}'_s = 0 |\]

\[
\int d^2\theta L d^2\bar{\theta} L d^2\theta R d^2\bar{\theta} R : |\exp C|^2 : (\tau \sum_r \epsilon'_r w_{rL})^2 (\tau \sum_s \epsilon'_{sL} \bar{w}_{sL})^2 (\bar{\tau} \sum_r \epsilon''_r w_{rR})^2 (\bar{\tau} \sum_s \epsilon''_{sR} \bar{w}_{sR})^2
\]

\[
\exp \left(-\frac{2\pi}{\tau_2} [\text{Im}(K^m + i \sum_r k^m_r z_r)]^2 \right) \prod_{r,s} |\Theta_1(z_r - z_s, \tau)|^{k^m_r k^m_s} \prod_t \tilde{U}_t(k, \theta_L, \bar{\theta}_L, \theta_R, \bar{\theta}_R)
\]

where \( C \) and \( K^m \) are defined in (3.13) and (3.20).

One can write : \( \exp C \) in terms of \( \epsilon'_r, z'_r, \) and \( \tau' \) using

\[
\exp(C(\epsilon'_r, z'_r, \tau')) = \exp(C(\epsilon'_r, z'_r, \tau')) + 2\pi i \sum_r \epsilon'_r z'_r (w_{rL}^\alpha \sum_s D_{s\alpha} + \bar{w}_{sL}^\alpha \sum_r \bar{D}_{s\alpha})
\]

\[
- i \sum_r \epsilon_r b_{rm} \sum_s k^m_{rs} z'_s - \sum_{r,s,t} \epsilon_r \epsilon_s (z'_r F(z_s - z_t) - z'_t F(z_r - z_s) + \bar{w}^\alpha_s w_{rL}^\alpha F(z_s - z_t) - w_{sL}^\alpha \bar{w}^\alpha_r F(z_t - z_u)\right):
\]

\[
= \exp(C(\epsilon'_r, z'_r, \tau')) + 2\pi i \tau [-iK'_m \sum_t k^m_t z'_t - \frac{1}{2} K'_m K^m] \right):
\]

where

\[
K'^m = \tau^{-1} K^m = \sum_r \epsilon'_r b_{rm} - i\sigma'^m_{\alpha\beta} \sum_{r,s} \epsilon'_r \epsilon'_s w_{rL}^\alpha \bar{w}_{sL}^\alpha F(z'_r - z'_s, \tau')
\]

and (4.3) has been used to order \( \sum_{s} D_{s\alpha} \) to the left of : \( \exp C :\)

Since (4.3) is modular invariant,

\[
\exp\left(-\frac{2\pi}{\tau_2} [\text{Im}(K^m + i \sum_r k^m_r z_r)]^2 \right) \prod_{r,s} |\Theta_1(z_r - z_s, \tau)|^{k^m_r k^m_s}
\]
\[ \exp\left[ \frac{\pi}{2} (\tau^2 (\tau_2)^{-1} - (\tau_2')^{-1})(K'_m K'_m + 2iK'_m \sum_s k^m_s z'_s) \right] \]

\[ \exp(-\frac{2\pi}{\tau_2} [Im(K'^m + i \sum_r k^m_r z'_r)]^2) \prod_{r,s} |\Theta_1(z'_r - z'_s, \tau')|^{k^m_r k^m_s} \]

\[ = |\exp[\pi i \tau (K'_m K'_m + 2iK'_m \sum_s k^m_s z'_s)]|^2 \]

\[ \exp(-\frac{2\pi}{\tau'_2} [Im(K'^m + i \sum_r k^m_r z'_r)]^2) \prod_{r,s} |\Theta_1(z'_r - z'_s, \tau')|^{k^m_r k^m_s} . \]

So

\[ : |\exp C(\epsilon_r, z_r, \tau)|^2 : \exp(-\frac{2\pi}{\tau_2} [Im(K'^m + i \sum_r k^m_r z'_r)]^2) \prod_{r,s} |\Theta_1(z'_r - z'_s, \tau')|^{k^m_r k^m_s} \]

\[ =: |\exp C(\epsilon'_r, z'_r, \tau')|^2 : \exp(-\frac{2\pi}{\tau'_2} [Im(K'^m + i \sum_r k^m_r z'_r)]^2) \prod_{r,s} |\Theta_1(z'_r - z'_s, \tau')|^{k^m_r k^m_s} , \]

which implies using (4.8) that \( A \) is modular invariant.

### 4.4. Equivalence with RNS amplitude for external NS-NS states

In this final subsection, the one-loop amplitude of (3.18) will be shown to be equivalent to the RNS prescription when all external \( d=4 \) states are in the NS-NS sector. The method used for evaluating the RNS amplitude will be an \( N \)-point generalization of the four-point one-loop computation in [15].

In the RNS formalism, the one-loop amplitude for \( N \) external massless \( d=4 \) NS-NS states can be written as

\[ A_{RNS} = \int d^2 \tau (\tau_2)^{-2} \prod_{r=1}^{N} \int d^2 z_r \sum_{spin} (\int d^2 w b_{LC\L}(w)b_{RC\R}(\bar{w})) \prod_{r=1}^{N} \left| V^{RNS}_r(z_r, \bar{z}_r) \right| \quad (4.11) \]

where \( V^{RNS}_r \) is defined in (2.9), \( \int d^2 w b_{LC\L}(w)b_{RC\R}(\bar{w}) \) provides the fermionic ghost zero modes, and \( \sum_{spin} \) signifies the two-dimensional correlation function summed over the four possible spin structures on the torus. Since \( V_r \) only involves \( \psi^m \) in four of the ten directions, the odd spin structure does not contribute to (4.11).

The first step to evaluating (4.11) is to compute the partition functions of the various worldsheet fields. For the uncompactified superstring, the \( [\beta, \gamma] \) partition function cancels the \( [\psi^8, \psi^9] \) partition function, the \( x^\mu \) partition function contributes \( |\eta(\tau)|^{-20} \tau_2^{-5} \), and the
\[ b_L, c_L, b_R, c_R \] partition function contributes \(|\eta(\tau)|^4\tau_2\) where the \(\tau_2\) factor comes from the \(\int d^2w\) integration in (4.11).

To compute the partition function for \(\psi^\mu\) for \(\mu = 0\) to 7, it is convenient to Wick-rotate to Euclidean space and then use SO(8) triality to map \(\psi^\mu\) to a Majorana-Weyl SO(8) spinor variable \(\chi^a\) for \(a = 1\) to 8. As in the GS formalism, \(\chi^a\) satisfies periodic boundary conditions on the torus so there is no need to sum over spin structures. As discussed in [15], the only subtlety in this SO(8) triality map comes from the odd spin structure because of the missing zero modes of \(\psi^8\) and \(\psi^9\). Note that the odd spin structure contributes parity-violating amplitudes involving odd powers of the spacetime \(\epsilon\) tensor. However, as in the one-loop four-point amplitude computed in [15], there is no parity violating contribution to one-loop N-point amplitudes when all external states contain \(d=4\) polarizations and momenta. So the above subtlety coming from odd spin structures can be ignored.

Since the partition function for eight left and right-moving periodic fermions contributes \(|\eta(\tau)|^{16}\), the scattering amplitude of (4.11) is

\[
A_{RNS} = \int d^2\tau (\tau_2)^{-6} \prod_{r=1}^{N} d^2z_r \langle \prod_{r=1}^{N} V_{r}^{RNS}(z_r, \bar{z}_r) \rangle
\] (4.12)

where

\[
V_{r}^{RNS} = (\partial x^m + i \chi^a_L (\Gamma^m)_{ab} \chi^b_L k_p) (\bar{\partial} x^n + i \chi^a_R (\Gamma^n)_{cd} \chi^d_R k_q) (h_{mn} + b_{mn} + \eta_{mn}\phi),
\] (4.13)

\((\Gamma^{mn})_{ab}\) are constructed from the SO(8) Pauli matrices, \(\langle \langle \quad \rangle \rangle\) signifies the correlation function divided by the partition function, and the correlation function must include all sixteen fermionic zero modes of \(\chi^a_L\) and \(\chi^a_R\).

To compare (4.12) with the one-loop amplitude of (3.7), divide the Majorana-Weyl SO(8) spinor \(\chi^a\) into the SO(4) \(\times\) SO(4) spinors \(\chi^{\alpha\beta}\) and \(\chi^{\dot{\alpha}\dot{\beta}}\) for \((\alpha, \dot{\alpha}, \beta, \dot{\beta}) = 1\) to 2 where the first SO(4) acts on the unprimed spinor indices and \(\mu = 0\) to 3, while the second SO(4) acts on the primed spinor indices and \(\mu = 4\) to 7. Then by defining

\[
\theta^\alpha = \chi^{\alpha+}, \quad \bar{\theta}^\alpha = \chi^{\dot{\alpha}-}, \quad p^\alpha = \chi^{\alpha-}, \quad \bar{p}^\dot{\alpha} = \chi^{\dot{\alpha}+},
\] (4.14)

one can equate the correlation function and zero modes of \(\chi^a\) with the correlation functions and zero modes of \([\theta^\alpha, \bar{\theta}^\alpha, p_\alpha, \bar{p}_{\dot{\alpha}}]\). Furthermore, one can use

\[
\chi^a (\Gamma^{mn})_{ab} \chi^b = p^\alpha (\sigma^{mn})_{\alpha\beta} \theta^\beta + \bar{p}^\dot{\alpha} (\bar{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}} \bar{\theta}^\dot{\beta}
\]
to show that the hybrid vertex operator of (2.7) for NS-NS states coincides with the RNS vertex operator of (4.13). So the RNS and hybrid one-loop amplitudes for external NS-NS states have been proven to be equivalent.

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References

[1] S. Mandelstam, “The N Loop String Amplitude: Explicit Formulas, Finiteness and Absence of Ambiguities”, Phys. Lett. B277 (1992) 82.
[2] N. Berkovits, “Finiteness and Unitarity of Lorentz-Covariant Green-Schwarz Superstring Amplitudes”, Nucl. Phys. B408 (1993) 43, [hep-th/9303122].
[3] Z. Bern and D. Kosower, “The Computation of Loop Amplitudes in Gauge Theories”, Nucl. Phys. B379 (1992) 452.
[4] N. Berkovits, “Covariant Quantization of the Green-Schwarz Superstring in a Calabi-Yau Background”, Nucl. Phys. B431 (1994) 258, [hep-th/9404162].
   N. Berkovits, “A New Description of the Superstring”, proceedings of VIII J. A. Swieca Escola de Veraõ, [hep-th/9604123].
[5] N. Berkovits, “Super-Poincaré Invariant Koba-Nielsen Formulas for the Superstring”, Phys. Lett. B385 (1996) 109, [hep-th/9604120].
[6] N. Berkovits, “Quantization of the Type II Superstring in a Curved Six-Dimensional Background”, Nucl. Phys. B565 (2000) 333, [hep-th/9908044].
[7] N. Berkovits, “Super-Poincaré Covariant Quantization of the Superstring”, JHEP 0004 (2000) 018, [hep-th/0001035].
[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic Anomalies in Topological Field Theories”, Nucl. Phys. B405 (1993) 279, [hep-th/9302103].
   I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, “Topological Amplitudes in String Theory”, Nucl. Phys. B413 (1994) 162, [hep-th/9307158].
   M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes”, Comm. Math. Phys. 165 (1994) 311, [hep-th/9309140].
[9] N. Berkovits and C. Vafa, “N=4 Topological Strings”, Nucl. Phys. B433 (1995) 123, [hep-th/9407190].
[10] W. Siegel, “Classical Superstring Mechanics”, Nucl. Phys. B263 (1986) 93.
[11] N. Berkovits and W. Siegel, “Superspace Effective Actions for 4D Compactifications of Heterotic and Type II Superstrings”, Nucl. Phys. B462 (1996) 213, [hep-th/9501016].
[12] H. Ooguri and C. Vafa, “All Loop N=2 String Amplitudes”, Nucl. Phys. B451 (1995) 121, [hep-th/9505183].
[13] U. Carow-Watamura, Z.F. Ezawa, K. Harada, A. Tezuka and S. Watamura, “Chiral Bosonization of Superconformal Ghosts on Riemann Surface and Path Integral Measure”, Phys. Lett. B227 (1989) 73.
[14] E. Verlinde and H. Verlinde, “Multiloop Calculations in Covariant Superstring Theory”, Phys. Lett. B192 (1987) 95.
[15] J. Polchinski, “String Theory, Volume 2: Superstring Theory and Beyond”, Cambridge Univ. Press (1998).