A New Proposal for a Quasi-Electron Trial Wave Function for the FQHE on a Disk.

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Abstract. – In this letter, we propose a new quasi-electron trial wave function for \( N \) interacting electrons in two dimensions moving in a strong magnetic field in a disk geometry. Requiring that the trial wave function exhibits the correct filling factor of a quasi-electron wave function, we obtain \( N + 1 \) angular-momentum eigenfunctions. The expectation values of the energy are calculated and compared with the data of an exact numerical diagonalization.

Trial wave functions have been used very successfully to describe the low-energy properties of interacting electrons moving in two dimensions in a strong magnetic field [1]. The prime example is the Laughlin wave function [2], which describes the ground state of the system at some of the filling factors of the lowest spin-polarized Landau level (i.e. electronic densities), for which one observes the fractional quantum Hall effect (FQHE) [3]. Deviations from such a state to higher or lower filling factors are then described by quasi-electrons or quasi-holes, for which trial wave functions are also proposed [2]. These are the constituents of the hierarchical picture [4,5] which is used to describe FQHE states at the remaining rational filling factors observed which are different from \( 1/m \) (\( m \) odd integer). The quasi-particles have also been proved to be a useful tool in numerical studies of the FQHE system for small particle numbers. Excitation energies have been calculated [5-8] and even some of the quasi-particle interactions have been obtained [9,10]. The situation is rather satisfying for the quasi-holes, but, in the case of the quasi-electrons, questions remain. Some of these we address in the present letter. After introducing our notation, we present a new trial wave function for the quasi-electron state and then give as an illustration the first numerical results for the expectation value of the energy.

We study the disk geometry in which the single-particle basis of the lowest Landau level is given by

\[
\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m \exp \left[-\frac{1}{4} |z|^2\right], \quad z = x - iy,
\]
$m = 0, 1, \ldots$ is the quantum number of the angular momentum, and the magnetic length $l_c = \sqrt{\hbar/eB}$ is set to unity (the angular momentum $m$ is not to be confused with the above odd integer $m$ of the filling factor). For a description of a finite system confined to a disk, only wave functions with $0 \leq m \leq \text{m}_{\text{max}}$ are kept in the basis, and $\text{m}_{\text{max}}$ then determines the area $A$ via $A = 2\pi(\text{m}_{\text{max}} + 1)$. $A$ contains $\text{m}_{\text{max}} + 1$ flux quanta. We consider the case of $N$ spin-polarized electrons in the lowest Landau level. Then, the definition of the filling factor, $\nu$, suited for calculations with finite $N$, is $\nu = (N - 1)/\text{m}_{\text{max}}$ [4]. When we discuss, below, the filling factor of a wave function we shall determine it from the maximum single-particle angular momentum.

The trial wave function for the ground state at filling factor $1/m$ in the disk geometry [2] was proposed by Laughlin as

$$\Psi_m(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^m \exp \left[ -\frac{1}{4} \sum_{j=1}^{N} |z_j|^2 \right].$$

(2)

Here, the maximum single-particle angular momentum is $\text{m}_{\text{max}} = (N - 1)m$, hence $\nu = 1/m$. A quasi-hole is defined as the ground-state wave function of a system the area of which is increased by one additional flux quantum. Its trial wave function is

$$\Psi_m^{(-, z_0)}(z_1, \ldots, z_N) = \prod_{j=1}^{N} (z_j - z_0) \Psi_m .$$

(3)

Similarly, a quasi-electron is the ground state of a system containing one flux quantum less than in the case of $\nu = 1/m$. The original proposal for its trial wave function reads

$$\Psi_m^{(+, z_0)}(z_1, \ldots, z_N) = \prod_{j=1}^{N} \left( \frac{3}{\partial z_j} - z_0 \right) \Psi_m ,$$

(4)

where the derivatives act only on the polynomial part of $\Psi_m$.

While it is not a priori obvious that $\Psi_m$, $\Psi_m^{(-, z_0)}$ and $\Psi_m^{(+, z_0)}$ are good variational wave functions for the ground state at these three different filling factors in the sense that they yield a low-energy expectation value for a given interaction, it is well known that $\Psi_m$ and $\Psi_m^{(-, z_0)}$ are exact eigenfunctions of a Hamiltonian with short-range interaction [11] for arbitrary values of the parameter $z_0$ related to the position of the quasi-hole.

In order to get eigenfunctions of the total angular momentum $M$, suited for the disk geometry, we expand the quasi-electron trial wave function $\Psi_m^{(+, z_0)}$ with respect to $z_0^\star$. This yields $N + 1$ linearly independent components (quasi-electrons) whose angular momenta $M$ vary between $M^* - N$ and $M^*$. $M^* = (N - 1)N/2$ is the total angular momentum of the ground-state wave function (2). Now one clearly sees the problems with the proposal $\Psi_m^{(+, z_0^\star)}$ for the quasi-electron. The component with $M = M^* - 1$ ($\alpha (z_0^\star)^{N-1}$) is zero. That means that one gets $N$, instead of $N + 1$, quasi-electrons. This is different in the case of the quasi-holes. Moreover, all but one of the remaining $N$ quasi-electrons have the wrong filling factor $\nu = 1/m$, because their $m_{\text{max}}$ is $(N - 1)m$. These are the quasi-electrons with $M = M^* - N = 1, \ldots, M^* - 2$, $M^*$. Thus, the whole wave function $\Psi_m^{(+, z_0^\star)}$ exhibits the wrong filling factor $\nu = 1/m$, and is hence not suited for a quasi-electron trial wave function. Only one component ($\alpha (z_0^\star)^{N}$) has $m_{\text{max}} = (N - 1)m - 1$ and thus shows the correct filling factor. All these features of (4) limit the value of a comparison with exact numerical calculations for small systems, cf. [12] and table I below.

There are more proposals for quasi-electron trial wave functions [6,13,14]. While these show the correct filling factor, they, unfortunately, give only one component, i.e.
\( M = M^* - N \) (quasi-electron at the origin). Thus, apparently none of these trial wave functions in the disk geometry corroborates the picture that one has \( N + 1 \) quasi-electrons becoming degenerate in energy in the thermodynamic limit, just as electrons in a Landau level, which is one of the main ingredients in the hierarchical theory.

In order to improve this situation, we propose a different quasi-electron trial wave function

\[
\prod_{j=1}^{N} [(N - 1) m - z_j \partial_{z_j}] \Psi_m,
\tag{5}
\]

which we will motivate in the following. The operator \( z_j \partial_{z_j} \), acting only on the polynomial part, yields the angular momentum of particle \( j \). Thus, the operation in (5) annihilates those parts of \( \Psi_m \) in which the single-particle state with highest angular momentum \( (N - 1) m \) is occupied. Hence, \( m_{\text{max}} = (N - 1) m - 1 \) in (5), and thus the filling factor comes out correctly (it is larger than \( 1/m \) by an amount of order \( 1/N \)). So far, we have only one state, \( \nu \), i.e. the one at \( M = M^* \). In order to get the remaining \( N \) functions, we apply a magnetic translation followed by a projection to the correct filling factor.

The result of the action of a magnetic translation \( T_{r_0} = \exp[-ir_0 t] \) (\( t = p - eA \)) on the basis (1) is

\[
T_{r_0} \Psi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} (z - z_0)^m \exp \left[ - \frac{1}{4} \left( |z|^2 + |z_0|^2 - 2zz_0^* \right) \right],
\]

where \( z_0 \) is given by the translation vector \( r_0 \). The magnetic translation operator for the \( N \)-particle system commutes with the Hamiltonian in the thermodynamic limit. If we now know an energetically favourable wave function (in the best case an eigenfunction of \( H \)), we get more good candidates for low-energy wave functions by applying a magnetic translation followed by a projection to \( \nu \). Starting out from (5), we construct in this way a set of quasi-electrons which should become degenerate in energy in the thermodynamic limit.

First, we demonstrate this translation and projection by applying it to the Laughlin wave function (2). Its polynomial part is unchanged, while additional terms \( z_j z_0^* \) appear in the exponent. We expand in \( z_0^* \), in order to get angular-momentum eigenfunctions. Then, all coefficients of \( (z_0^*)^n \) (\( n = 1, 2, \ldots \)), correspond to a filling factor smaller than \( 1/m \). The exception is the coefficient of \( (z_0^*)^0 \), showing correctly \( \nu = 1/m \). Thus, projecting back to the original filling factor is equivalent to keeping only the coefficient of \( (z_0^*)^0 \) (putting \( z_0^* = 0 \), and it leads back to the function (2). (The term \( \exp[-(1/4)|z_0|^2] \) cancels out in the normalization.) This supports the picture that the Laughlin wave function for the disk geometry is nondegenerate in the thermodynamic limit.

If we apply the same procedure to the component with \( M = M^* + N \) (\( z_0 = 0 \)) of the quasi-hole wave function \( \Psi_m(z_0) \), the magnetic translation followed by a projection to \( m_{\text{max}} = (N - 1) m + 1 \) reproduces the function \( \Psi_m(z_0) \) completely. This shows that the parameter \( z_0 \) in (3) is related to the translation vector of a magnetic translation. Hence, the \((N + 1)\) linearly independent quasi-hole wave functions, resulting from an expansion of (3) in \( M \)-eigenfunctions, are degenerate in energy, already for a finite system. The physical meaning of the magnetic translation followed by the projection is the following: the position of the quasi-hole is shifted against the background of the Laughlin wave function which is kept fixed (1).

In contrast to the case of \( \Psi_m(z_0) \), however, Laughlin's quasi-electron function \( \Psi_m(z) \)

(1) We thank W. Weller for a discussion on this point.
taken at $z_0^* = 0$ is left unchanged by a magnetic translation followed by the appropriate projection. Thus, in contrast to the case of the quasi-hole, this procedure does not lead to a set of energetically degenerate quasi-electron wave functions which, on the other hand, is the fundamental assumption of the hierarchical model of the FQHE.

Applying now a magnetic translation followed by a projection to $m_{\text{max}} = (N - 1) m - 1$ (i.e. $z \rightarrow z - z_0^*$, $\partial_z \rightarrow \partial_z$) to the function (5), we finally arrive at our new proposal for a quasi-electron trial wave function which reads

$$
\Psi^{(+, z_0)}(x_1, \ldots, x_N) = \prod_{j=1}^{N} \left[ \left( (N - 1) m - z_j \partial_j \right) + z_0 \partial_z \right] \Psi_m.
$$

(6)

Expanding the product in powers of $z_0$, we correctly find $(N + 1)$ quasi-electron states which are eigenstates of the total angular momentum with the eigenvalues $M^* - N$, $\ldots$, $M^* - 1, M^*$.

It is interesting to note that the state at angular momentum $M^* - N$ is equal to the corresponding state of the proposal $\Psi^{(+, z_0^*)}$, but that all other states are different. We note a further property of $\Phi_m^{(+, z_0)}$, which makes it appear a very natural choice. The above-mentioned discrepancies with the quasi-electron trial wave function do not occur in the spherical geometry, where the electrons move on the surface of a sphere in a magnetic field produced by a magnetic monopole in the centre [4]. In this geometry, the filling factor of the quasi-electron trial wave function is correct. Also, there are $(N + 1)$ such states. Remarkably, $\Phi_m^{(+, z_0)}$ is just the stereographic projection of the corresponding quasi-electron trial wave function on the sphere [4] onto the plane. In the previous work [15], the stereographic projection has been employed to relate the wave function $\Psi_m$ on the plane to the corresponding wave function on the sphere. The relation between the quasi-electron trial wave function on the sphere and $\Phi_m^{(+, z_0)}$ is a further application of this mapping.

We conclude by comparing the expectation values for the energies of the components of the quasi-electron $\Phi_m^{(+, z_0)}$ with exact finite-size numerical diagonalizations in the disk geometry. We choose a short-range interaction [16], where only the coefficient corresponding to the relative angular-momentum one is kept in the expansion and taken to be equal to its Coulomb value. This interaction is viewed as giving the basic model which incorporates all important features of the problem and from which one could approach a realistic interaction in a perturbative way.

The energy expectation values of (6) were exactly calculated with an algebraic programme (MATHEMATICA). The results are listed in table I for $m = 3$ and $N = 6$ electrons, i.e. $m_{\text{max}} = 14$. The energy expectation values for the components of the quasi-electron trial wave

| $M$ | $\Phi_m^{(+, z_0)}$ | Diagonalization | $\mathcal{V}^{(+, 3\partial)}$ |
|-----|-------------------|----------------|-------------------|
| 39  | 0.2259            | 0.1839         | 0.2259            |
| 40  | 0.2437            | 0.2168         | 0.2451            |
| 41  | 0.3147            | 0.1986         | 0.3422            |
| 42  | 0.3688            | 0.2025         | 0.4546            |
| 43  | 0.4108            | 0.2078         | 0.5678            |
| 44  | 0.4368            | 0.2145         | —                 |
| 45  | 0.0846            | 0.0188         | 0               |

Table I. - The energy expectation values of the components of the quasi-electron wave function in a system with $N = 6$ particles, $m = 3$. Column 1 is the total angular momentum $M$. Column 2 shows the data for our trial wave function $\Phi_m^{(+, z_0)}$; in column 3, the results of an exact diagonalization are displayed. Column 4 shows the data for Laughlin's quasi-electron wave function $\mathcal{V}^{(+, 3\partial)}$. 
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function \( \Phi_m^{(+, z_0)} \) in the second column can be compared with the lowest eigenenergy of the exact numerical diagonalization in the third column of the table for each total angular momentum \( M \). We show the energy expectation values of the components of Laughlin's quasi-electron trial wave function of the fourth column of the table, in spite of the fact that a comparison with them is not justified because of the wrong filling factor. The only exception from this is the case \( M = 39 \) where the two trial wave functions coincide. For \( M = 40, 41, 42, 43, 44, 45 \), the energies of \( \Phi_m^{(+, z_0)} \) are lower than those of the original proposal \( \nu_m^{(+, z_0)} \). Since a particle in \( \nu_m^{(+, z_0)} \) can occupy a state with angular momentum 15 and a particle in \( \Phi_m^{(+, z_0)} \) cannot, and thus particles in \( \nu_m^{(+, z_0)} \) can lower their energy by keeping a larger mutual distance than those in \( \nu_m^{(+, z_0)} \), this is all the more remarkable. As mentioned above, a component of \( \nu_m^{(+, z_0)} \) at \( M = 44 \) does not exist. For \( M = 45 \), the component of \( \nu_m^{(+, z_0)} \) is obviously not an allowed variational wave function, it does not obey the constraint (filling factor). On the other hand, the energy of \( \Phi_m^{(+, z_0)} \) at \( M = 45 \) is a good upper bound for the exact energy.

Viewed as a function of \( M \), the expectation values of the components of the trial wave function \( \Phi_m^{(+, z_0)} \) follow the behaviour of the exact energies with the exception between \( M = 40 \) and \( M = 41 \); the dependence of the variational energies on \( M \), however, is much stronger than that of the exact energies. In any case, we do get an upper bound for the energy from \( \Phi_m^{(+, z_0)} \). While we expect from the above considerations that the energies of the components of \( \Phi_m^{(+, z_0)} \) become degenerate in the thermodynamic limit, we still find strong finite-size corrections at \( N = 6 \).

In this letter, we have proposed a new quasi-electron trial wave function \( \Phi_m^{(+, z_0)} \) for the FQHE in the disk geometry. The request for the correct filling factor is fulfilled. By using the magnetic translation a whole set of quasi-electron wave functions for different total angular momenta has been constructed. On the scale of the bandwidth, the magnitude of the energy is already for \( N = 6 \) quite reasonable compared with the energy resulting from the exact diagonalization. An interesting and, to our knowledge, unique feature of (6) is that the creation operator for the quasi-electron depends explicitly on the state it is applied to. Generalizing our approach to trial wave functions with more quasi-electrons, we expect that the quasi-electron creation operator depends on the number of quasi-particles already present. Work on the consequences for the statistics of these quasi-particles is under way.

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