EVOLUTION OF INTERFACES FOR THE NONLINEAR DOUBLE DEGENERATE PARABOLIC EQUATION OF TURBULENT FILTRATION WITH ABSORPTION

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Abstract. We prove the short-time asymptotic formula for the interfaces and local solutions near the interfaces for the nonlinear double degenerate reaction-diffusion equation of turbulent filtration with strong absorption

\[ u_t = \left( \left| (u^m)_x \right|^{p-1} (u^m)_x \right)_x - bu^\beta, \quad mp > 1, \beta > 0. \]

Full classification is pursued in terms of the nonlinearity parameters \( m, p, \beta \) and asymptotics of the initial function near its support. Numerical analysis using a weighted essentially nonoscillatory (WENO) scheme with interface capturing is implemented, and comparison of numerical and analytical results is presented.

1. Introduction

Consider the Cauchy problem (CP) for the nonlinear double degenerate parabolic equation:

\[ Lu \equiv u_t - \left( \left| (u^m)_x \right|^{p-1} (u^m)_x \right)_x + bu^\beta = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \]

where \( m, p, b, \beta > 0, \) \( mp > 1, \) \( 0 < T \leq +\infty, \) and \( u_0 \) is nonnegative and continuous. Equation (1) arises in turbulent polytropic filtration of a gas in porous media [9, 15, 25]. Under the condition \( mp > 1, \) the PDE (1) possesses finite speed of propagation property. Assume that \( \eta(0) = 0, \) where \( \eta(\cdot) \) be an interface, or free boundary defined as

\[ \eta(t) := \sup \{ x : u(x, t) > 0 \}. \]

Let

\[ u_0(x) \sim C(-x)_{+}^\alpha, \quad \text{as} \quad x \to 0^-, \quad \text{for some} \quad C > 0, \alpha > 0. \]

The goal of this paper is to present full classification of the short time behavior of the interface \( \eta, \) and local solution near \( \eta \) in terms of parameters \( m, p, b, \beta, C, \) and \( \alpha. \)

Our estimations will be global in time in the special case when

\[ u_0(x) = C(-x)_{+}^\alpha, \quad x \in \mathbb{R}, \]

and the minimal solution to the problem (1), (4) is of self-similar form.
Figure 1. Classification of different cases in the \((\alpha, \beta)\)-parameter space for interface development in problem (1)-(4).

The initial development of interfaces and structures of local solutions near the interfaces is very well understood in the case of the reaction-diffusion equations with porous medium \((p = 1\) in (1)) and \(p\)-Laplace \((m = 1\) in (1)) type diffusion terms. Full classification of the evolution of interfaces and the local behaviour of solutions near the interfaces for the reaction-diffusion equations with porous medium type diffusion \((p = 1\) in (1)) was presented in [6] for the case of slow diffusion \((m > 1)\), and in [2] for the fast diffusion case \((0 < m < 1)\). Similar classification for the reaction-diffusion equations with \(p\)-Laplace type diffusion \((m = 1\) in (1)) is presented in a recent paper [5].

The organization of the paper is as follows: In Section 2 we outline the main results. For clarity of the exposure we describe further technical details of the main results in Section 3. In Section 4 we apply nonlinear scaling techniques for some preliminary estimations which are necessary for the proof of main results. In Section 5 we prove the main results. Finally, in Section 6 we confirm our analytical results from Section 2 numerically by implementing a WENO scheme. We provide explicit values of some of constants in Section 7 and numerical graphs in Section 8.

2. MAIN RESULTS

There are four different subcases, as shown in Fig. 1. The main results are the following:

**Theorem 1.** If \(\alpha < \frac{1+p}{mp-\min\{1, \beta\}}\), then the interface initially expands and

\[
\eta(t) \sim \xi_* t^{1/(1+p-\alpha(mp-1))}, \quad \text{as } t \to 0^+,
\]

where

\[
\xi_* = C \frac{mp-1}{1+p-\alpha(mp-1)} \xi'_*,
\]
and $\xi'_*$ is a positive number depending only on $m$, $p$, and $\alpha$. For arbitrary $\rho < \xi_*$, there exists a positive number $f(\rho)$ depending on $C, m, p$, and $\alpha$ such that:

$$(7) \quad u(x, t) \sim t^{\alpha/(1 + p - \alpha(mp - 1))}, \text{ as } t \to 0^+,$$

along the curve $x = \xi_p(t) = \rho \alpha/(1 + p - \alpha(mp - 1))$.

**Theorem 2.** Let $0 < \beta < 1, \alpha = (1 + p)/(mp - \beta)$ and

$$C_* = \left[ \frac{b(mp - \beta)^{1 + p}}{(m(1 + p))^{\rho}(m + \beta)} \right]^{\frac{1}{mp - \beta}}.$$

Then interface expands or shrinks according as $C > C_*$ or $C < C_*$ and

$$(8) \quad \eta(t) \sim \xi_* t^{rac{mp - \beta}{1 + \alpha(1 - \beta)}}, \text{ as } t \to 0^+,$$

where $\xi_* \leq 0$ if $C \leq C_*$, and for arbitrary $\rho < \xi_*$ there exists $f_1(\rho) > 0$ such that:

$$(9) \quad u(x, t) \sim t^{1/(1 - \beta)} f_1(\rho), \text{ for } x = \rho t^{\frac{mp - \beta}{1 + \alpha(1 - \beta)}}, \text{ as } t \to 0^+.$$

**Theorem 3.** If $0 < \beta < 1$, and $\alpha > (1 + p)/(mp - \beta)$, then the interface shrinks and

$$(10) \quad \eta(t) \sim -\ell_* t^{1/(\alpha(1 - \beta))}, \text{ as } t \to 0^+,$$

where $\ell_* = C^{-\alpha/\rho} (b(1 - \beta))^{1/\alpha(1 - \beta)}$. For arbitrary $\ell > \ell_*$ we have:

$$(11) \quad u(x, t) \sim \left[ C^{1 - \beta}(-x)^{\alpha(1 - \beta)} - b(1 - \beta) t \right]^{1/(1 - \beta)}, \text{ as } t \to 0^+,$$

along the curve $x = \eta(t) = -\ell t^{1/(\alpha(1 - \beta))}$.

**Theorem 4.** If $\alpha \geq (1 + p)/(mp - 1)$ and $\beta \geq 1$, then interface initially remains stationary.

3. **Further Details of the Main Results**

Further details of Theorem 2, $f$ is a shape function of the self-similar solution to the problem (11), (14) with $b = 0$:

$$(12) \quad u(x, t) = t^{\frac{\alpha}{1 + p - \alpha(mp - 1)}} f(\xi), \xi = x t^{-\frac{1}{1 + p - \alpha(mp - 1)}}.$$

In fact, $f$ is a solution of the following nonlinear ODE problem in $\mathbb{R}$:

$$(13) \quad \left( |(f^m(\xi))'|^{p - 1} (f^m(\xi))' + (1 + p - \alpha(mp - 1))^{-1} (\xi f'(\xi) - \alpha f(\xi)) \right) = 0,$$

$$f(\xi) \sim C(-\xi)^{\alpha}, \text{ as } \xi \downarrow -\infty, \ f(+\infty) = 0.$$

Moreover, $\exists \xi_* > 0$ such that: $f(\xi) \equiv 0$ for $\xi \geq \xi_*; \ f(\xi) > 0$ for $\xi < \xi_*$. Dependence on $C$ is given by the following relation:

$$f(\rho) = C^{1 + p/(1 + p - \alpha(mp - 1))} f_0(C^{(mp - 1)/(\alpha(mp - 1) - (1 + p)) \rho}),$$

$$f_0(\rho) = w(\rho, 1), \ \xi_*' = \sup \{ \rho : f_0(\rho) > 0 \} > 0,$$
where \( w \) is a minimal solution of the CP (1), (4) with \( b = 0, C = 1 \). Lower and upper estimations for \( f \) are given in (34). We also have that:

\[
(16) \quad \xi^*_s = A_0^{\frac{mp-1}{1+p}} \left[ \frac{(mp)^p (1 + p - a(mp - 1))}{(mp - 1)^p} \right]^{\frac{1}{mp}} \xi^*_r,
\]

where \( A_0 = w(0,1) \) and \( \xi^*_r \) is some number belonging to the segment \([\xi_1, \xi_2]\) (see Section [7]). In the particular case \( a = p(mp - 1)^{-1} \) and \( mp > 1 + p - p(\min[1, \beta]) \), the explicit solution of (1), (4) with \( b = 0 \) is given by (32) and

\[
(17) \quad \xi_1 = \xi_2 = 1, \xi^*_s = (mp)^p (mp - 1)^{-p}, \quad f_0(x) = (\xi^*_s - x)^{p/(mp - 1)}
\]

Further details of Theorem 2. If \( p(m + \beta) = 1 + p \), the solution to (1), (4) is

\[
(18) \quad u(x, t) = C(\xi, t - x)_{\xi-x}^0, \quad \xi_s = b(1 - \beta)C^\beta (C/C_s)^{mp-\beta} - 1.
\]

Let \( p(m + \beta) \neq 1 + p \). If \( C = C_s \), then \( u_0 \) is a stationary solution to (1), (4). If \( C \neq C_s \), then the minimal solution to (1), (4) is of the self similar form:

\[
(19) \quad u(x, t) = t^{\frac{1}{(1 - \beta)}} f_1(\xi), \quad \xi = x t^{-\frac{mp-\beta}{1+mp(1-\beta)}},
\]

\[
(20) \quad \eta(t) = \xi_s t^{\frac{mp-\beta}{(1+mp(1-\beta))}}, \quad 0 < t < +\infty,
\]

where \( f_1(\xi) \) solves the following nonlinear ODE problem:

\[
(21) \quad \left( (f_1')^{p-1} (f_1')' + \frac{mp-\beta}{1+p(1-\beta)} f_1' - \frac{1}{1-\beta} f_1 - b f_1^\beta = 0, \quad \xi \in \mathbb{R},
\]

\[
(22) \quad f_1(\xi) \sim C(-\xi)^{1+p)/(mp-\beta), \quad \text{as} \ \xi \downarrow -\infty, \quad \text{and} \quad f_1(\xi) \to 0, \quad \text{as} \ \xi \uparrow +\infty.
\]

Moreover, \( \exists s \) such that \( f(\xi) \equiv 0 \) for \( \xi \geq \xi_s \): \( f(\xi) > 0 \) for \( \xi < \xi_s \). If \( C > C_s \) then the interface expands, \( f_1(0) = A_1 > 0 \) (see (6)), and:

\[
(23) \quad C_1 t^{\frac{1}{mp}} (\xi_1 - \xi_s)^\mu_+ \leq u \leq C_2 t^{\frac{1}{mp}} (\xi_2 - \xi_s)^\mu_+, \quad 0 < x < +\infty, \quad 0 < t < +\infty,
\]

where

\[
\begin{align*}
\mu &= p(mp - 1)^{-1}, & \text{if} \ p(m + \beta) > 1 + p, \\
\mu &= (1 + p)(mp - \beta)^{-1}, & \text{if} \ p(m + \beta) < 1 + p,
\end{align*}
\]

which implies:

\[
(24) \quad \xi_1 \leq \xi_s \leq \xi_2.
\]

If \( 0 < C < C_s \), then the interface shrinks. If \( p(m + \beta) > 1 + p \) then:

\[
(25) \quad \left( C^{1-\beta} (-x)^{\frac{(1+p)(1-p)}{mp-\beta}} - b(1 - \beta) t \right)^{\frac{1}{mp}} \leq u \leq \left( C^{1-\beta} (-x)^{\frac{(1+p)(1-p)}{mp-\beta}} - b(1 - \beta) (1 - (C/C_s)^{mp-\beta}) t \right)^{\frac{1}{mp}} \quad x \in \mathbb{R}, \quad t \geq 0,
\]

which also implies (24), where we replace \( \xi_1 \) (respectively, \( \xi_2 \)) with respective negative values given in Section [7]. However, if \( p(m + \beta) < 1 + p \) then:

\[
(26) \quad C_s \left( -\xi_3 t^\frac{mp-\beta}{1+mp(1-\beta)} - x \right)^{\frac{1+p}{mp-\beta}} \leq u \leq C_3 \left( -\xi_4 t^\frac{mp-\beta}{1+mp(1-\beta)} - x \right)^{\frac{1+p}{mp-\beta}}, \quad x \in \mathbb{R}, \quad t > 0,
\]
where the left-hand side is valid for $x \geq -\ell_0 t^{\frac{mp-b}{(1+\rho)(mp-1)}}$, while the right-hand side is valid for $x \geq -\ell_1 t^{\frac{mp-b}{(1+\rho)(mp-1)}}$. From (26), (24) follows if we replace $\zeta_1$ and $\zeta_2$ with $-\zeta_3$ and $-\zeta_4$, respectively.

Further Details of Theorem 4

There are four different subcases (see Fig. 1).

(4a) If $\beta = 1, \alpha = (1 + p)/(mp - 1)$, the unique minimal solution to (1), (4) is

$$u_\ell = C(-x)^{1+\rho}(mp-1) e^{-bt} [1 - (C/\ell)^{mp-1} b^{-1} (1 - e^{-b(mp-1)y})]^{1/(1-mp)}$$

where

$$T = +\infty, \quad \text{if } b \geq (C/\ell)^{mp-1},$$

$$T = (b(1-mp))^{-1} \ln[1 - b(C/\ell)^{mp-1}], \quad \text{if } -\infty < b < (C/\ell)^{mp-1},$$

(4b) Let $\beta = 1$ and $\alpha > (1 + p)/(mp - 1)$. Then $\forall \epsilon > 0 \exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that:

$$(C-\epsilon)(-x)^{\rho} e^{-bt} \leq u \leq (C + \epsilon)(-x)^{\rho} e^{-bt}$$

$$[1 - \epsilon b(mp-1)]^{-1} (1 - e^{-b(mp-1)y})^{1/(1-mp)}, \ x > x_\epsilon, \ 0 \leq t \leq \delta_\epsilon.$$ 

(4c) Let $1 < \beta < mp$ and $\alpha \geq (1 + p)/(mp - \beta)$. Then $\forall \epsilon > 0 \exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that:

$$g_{-\epsilon}(x,t) \leq u(x,t) \leq g_\epsilon(x,t), \ x \geq x_\epsilon, \ 0 \leq t \leq \delta_\epsilon$$

where

$$g_\epsilon(x,t) = \begin{cases} (C + \epsilon)^{1-\beta}[\epsilon^{(1-\beta)} + b(\beta - 1)(1 - \epsilon - \kappa_\epsilon t)]^{1/(1-\beta)}, & \text{if } x_\epsilon \leq x < 0, \\ 0, & \text{if } x \geq 0, \end{cases}$$

where $\kappa_\epsilon = 0$, if $\alpha > (1 + p)/(mp - \beta)$; $\kappa_\epsilon = ((C+\epsilon)/C)^{mp-\beta}$, if $\alpha = (1 + p)/(mp - \beta)$.

(4d) Let either $1 < \beta < mp$, $(1 + p)/(mp - 1) \leq \alpha < (1 + p)/(mp - \beta)$, or $\beta \geq mp$, $\alpha \geq (1 + p)/(mp - 1)$. If $\alpha = (1 + p)/(mp - 1)$ then for $\forall \epsilon > 0 \exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that:

$$u_\ell = C(-x)^{\rho} (1 - \gamma \epsilon t)^{1/\ell} \leq u \leq (C + \epsilon)(-x)^{\rho} (1 - \gamma \epsilon t)^{1/\ell}$$

for $x > x_\epsilon, \ 0 \leq t \leq \delta_\epsilon$ (see Section 7 for $\gamma_\epsilon$). However, if $\alpha > (1 + p)/(mp - 1)$, then for arbitrary sufficiently small $\epsilon > 0$, there exist $x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that:

$$(C-\epsilon)(-x)^{\rho} \leq u \leq (C + \epsilon)(-x)^{\rho} (1 - \epsilon t)^{1/(1-mp)}, \ x > x_\epsilon, \ 0 \leq t \leq \delta_\epsilon.$$ 

Results for the case $b = 0$.

(1) If $\alpha = p/(mp - 1)$ the minimal solution to the problem (1), (4) is

$$u(x,t) = C(\xi_\epsilon t - x)^{p/(mp-1)}, \xi_\epsilon = C^{mp-1}(\frac{mp}{mp-1})^\frac{p}{\rho}.$$ 

If $0 < \alpha < (1 + p)/(mp - 1)$, then the minimal solution to (1), (4) has the self-similar form (12) and

$$\eta(t) = \xi_\epsilon t^{\frac{1}{1+p-\alpha mp-1}}, \ 0 \leq t < +\infty,$$

where $\xi_\epsilon$ and $f$ solve (13)-(14). We have the estimation

$$C_4 t^{\frac{\alpha}{1+p-\alpha mp-1}} (\xi_3 - \xi)^{\frac{1}{p}} \leq u \leq C_5 t^{\frac{\alpha}{1+p-\alpha mp-1}} (\xi_4 - \xi)^{\frac{1}{p}}, \ x \geq 0, \ t \geq 0,$$
If \( \alpha = p/(mp - 1) \), then \( \xi_3 = \xi_4 = \xi_5 \) and both lower and upper estimations in (34) coincide with the solution (32).

(2) If \( \alpha = (1 + p)/(mp - 1) \), then interface initially remains stationary. Explicit solution to (1), (4) is

\[
u_C(x, t) = C(-x)^{(1+p)/(mp-1)}[\lambda(t_\ast - t)(1 - mp)]^{1/(1-mp)}, \quad x \in \mathbb{R}, \quad 0 \leq t < t_\ast,
\]

where

\[t_\ast = 1/\lambda(1 - mp), \quad \text{with } \lambda = -C^{mp-1}p(m + 1)(m + p)^p/(mp - 1)^{(1+p)}.
\]

(3) If \( \alpha > (1 + p)/(mp - 1) \), then interface again remain stationary, and for \( \forall \epsilon > 0 \) \( \exists \epsilon_0 < 0 \) and \( \delta_\epsilon > 0 \) such that:

\[
(C - \epsilon)(-x)^\alpha \leq u \leq (C + \epsilon)(-x)^\alpha(1 - \epsilon t)^{1/(1-mp)}, \quad x_\epsilon \leq x, \quad 0 \leq t \leq \delta_\epsilon.
\]

4. Preliminary Results

The prelude of the mathematical theory of the nonlinear degenerate parabolic equations began with the paper [31] [12] [1] [4] [3] [8]. Boundary value problems for (1) have been investigated in [23] [15] [30] [18] [20] [19].

Definition 1 (Strong Solution). ([15] [30]) Let \( u_0 \in L^1(\mathbb{R}) \) and nonnegative. A measurable nonnegative function \( u(x, t) \) defined in \( \mathbb{R} \times (0, T) \) is a strong solution of (1), (2) if

\[
\begin{align*}
u & \in C((0, T); L^1(\mathbb{R}) \cap L^\infty([\delta, T) \times \mathbb{R}), \quad \forall \delta > 0, \\
u_t & \in W^{1,\infty}(\mathbb{R}), \quad \text{a.e. } 0 < t < T, \\
u_t, (|u|^\alpha, |u_t|^\alpha) & \in L^1_{loc}(0, T; L^1(\mathbb{R})),
\end{align*}
\]

and \( u \) satisfies (1), (2) for a.e. \( 0 < t < T, x \in \mathbb{R} \).

Existence, uniqueness and comparison theorems for the strong solution of the CP (1), (2) was proved in [15] for the case \( b = 0 \), and in [30] for \( b > 0 \). In [15] it is proved that the strong solution of (1), \( b = 0 \), is locally Hölder continuous. Local Hölder continuity of the locally bounded weak solutions (accordingly, strong solutions) of the general second order multidimensional nonlinear degenerate parabolic equations with double degenerate diffusion term is proved in [20] [19]. The following is the standard comparison result, which is widely used throughout the paper:

Lemma 1. Let \( g \) be a non-negative and continuous function in \( \overline{Q} \), where:

\[Q = \{(x, t) : \eta_0(t) < x < +\infty, \quad 0 < t < T \leq +\infty\}\]
then we have that:

\[ g = g(x, t) \text{ is in } C^{2,1}_{x,t} \text{ in } Q \text{ outside a finite number of curves: } x = \eta_j(t), \text{ which divide } Q \text{ into a finite number of subdomains: } Q^j, \text{ where } \eta_j \in C[0, T]; \text{ for arbitrary } \delta > 0\]

and finite \( \Delta \in (\delta, T] \) the function \( \eta_j \) is absolutely continuous in \([\delta, \Delta]\). Let \( g \) satisfy the inequality:

\[ Lg = g_t - \left( (g^m)_x \right)_{x} + bg^{\beta} \geq 0, (\leq 0), \]

at the points of \( Q \) where \( g \in C^{2,1}_{x,t} \). Assume also that the function: \( (g^m)_x \) is continuous in \( Q \) and \( g \in L^\infty(\{Q \cap (t \leq T_1)\}) \) for any finite \( T_1 \in (0, T] \). If in addition we have that:

\[ g(\eta_0(t), t) \geq (\leq) u(\eta_0(t), t), \quad g(x, 0) \geq (\leq) u(x, 0), \]

then

\[ g \geq (\leq) u, \text{ in } \overline{Q} \]

Suppose that \( u_0 \in L^1_{loc}(\mathbb{R}) \), and may have unbounded growth as \( |x| \to +\infty \). It is well known that in this case some restriction must be imposed on the growth rate for existence, uniqueness of the solution to the CP \([1], [2]\). For the particular cases of the equation \([1]\) with \( b = 0 \) this question was settled down in \([11], [16]\) for the porous medium equation \((p = 1)\) with slow \((m > 1)\) and fast \((0 < m < 1)\) diffusion; and in \([13], [14]\) for the \( p \)-Laplacian equation \((m = 1)\) with slow \((p > 1)\) and fast \((0 < p < 1)\) diffusion; The case of reaction-diffusion equation \( m > 1, p = 1, b > 0 \) is analyzed in \([22], [24], [7]\). Surprisingly, only a partial result is available for the double-degenerate PDE \([1]\). The sharp sufficient condition for the existence of the solution to the CP for \([1]\), \( b = 0 \) is established in \([18]\). In particular, it follows from \([18]\) that the CP \([1], [4]\) has a solution if and only if \( \alpha \leq (1 + p)/(mp - 1) \). Moreover, solution is global \((T = +\infty)\) if \( \alpha < (1 + p)/(mp - 1) \) and only local in time if \( \alpha = (1 + p)/(mp - 1) \). Uniqueness of the solution is an open problem. For our purposes it is satisfactory to employ the notion of the minimal solution.

**Definition 2** (Minimal Solution). Let \( u_0 \in L^1_{loc}(\mathbb{R}) \) and nonnegative. Nonnegative weak solution of the CP \([1], [2]\) is called a minimal solution if

\[ 0 \leq u(x, t) \leq v(x, t), \]

for any nonnegative weak solution \( v \) of the same problem \([1], [2]\).

Note that the minimal solution is unique by definition. The following standard comparison result is true in the class of minimal solutions:

**Lemma 2.** Let \( u \) and \( v \) be minimal solutions of the CP \([1], [2]\). If

\[ u(x, 0) \geq (\leq) v(x, 0), \quad x \in \mathbb{R}, \]

then

\[ u(x, t) \geq (\leq) v(x, t), \quad (x, t) \in \mathbb{R} \times (0, T). \]

If the function \( u(x, t) \) is a minimal solution to CP \([1], [4]\) with \( b = 0 \), then the function:

\[ \bar{u}(x, t) = \exp(-bt)u(x, (b(1 - mp))^{-1}(\exp(-b(mp - 1))t - 1)), \]
is a minimal solution to (1) with $b \neq 0$ and $\beta = 1$. Hence, from the above mentioned result it follows that the unique minimal solution to CP (1), (4) with $mp > 1, b > 0$, $\beta = 1$, and $\alpha = (1 + p)/(mp - 1)$, is the function $\bar{u}_C(x,t)$ from (27).

In the following lemmas we establish some preliminary estimations of the solution to the CP.

**Lemma 3.** If $b = 0$, $0 < \alpha < (1 + p)/(mp - 1)$, then the minimal solution $u$ of the CP (1), (4) has a self-similar form (12), where the self-similarity function $f$ satisfies (15). If $u_0$ satisfies (3), then the solution to CP (1), (2) satisfies (5)-(7).

**Lemma 4.** If $0 < \alpha < \frac{1+p}{mp-min(1,p)}$, then solution to the CP (1)-(3) satisfies (7).

In the next lemma we analyze special class of finite travelling wave solutions.

By a finite travelling-wave solution with velocity $0 \neq k \in \mathbb{R}$ we mean a solution $u(x,t) = \phi(kt - x)$, where $\phi(y) \geq 0$, $\phi \neq 0$, and $\phi(y) = 0$ for $y \leq y_0$ for some $y_0 \in \mathbb{R}$.

**Lemma 5.** Let $0 < \beta < 1, p(m + \beta) > 1 + p$. PDE (1) admits a finite travelling-wave solution, $u(x,t) = \phi((kt - x))$, with $\phi(y) = 0$ for $y \leq 0$; $\phi(y) > 0$, for $y > 0$, and:

$$\lim_{y \rightarrow +\infty} y^{-\frac{1+p}{mp}} \phi(y) = C_\ast.$$  

**Lemma 6.** If $0 < \beta < 1, \alpha = (1 + p)/(mp - \beta)$, then the minimal solution $u$ to the CP (1), (4) has a self-similar form (19), where the self-similarity function $f$ satisfies (21), (22). If $C > C_\ast$, then $f_1(0) = A_1 > 0$, where $A_1$ depends on $m, p, \beta, C$ and $b$. If $u_0$ satisfies (3) with $\alpha = (1 + p)/(mp - \beta)$ and $C > C_\ast$, then the solution to CP (1), (2) satisfies:

$$u(0,t) \sim A_1 t^{1/(1-\beta)} \text{ as } t \rightarrow 0^+$$

**Lemma 7.** If $0 < \beta < 1, \text{ and } \alpha > (1 + p)/(mp - \beta)$, then the solution $u$ to the CP (1)-(3) satisfies (11).

**Proof of Lemma 3.** Let $u$ be a unique minimal solution of the problem (1), (4). If we consider a function:

$$u_k(x,t) = ku(k^{-1/\alpha} x, k^{(\alpha(mp-1)-(1+p))/\alpha} t), \quad k > 0,$$

it may easily be checked that this satisfies (1), (4). Since $u$ is a minimal solution we have:

$$u(x,t) \leq ku(k^{-1/\alpha} x, k^{(\alpha(mp-1)-(1+p))/\alpha} t), \quad k > 0.$$

By changing the variable in (42)

$$y = k^{-1/\alpha} x, \quad \tau = k^{(\alpha(mp-1)-(1+p))/\alpha} t,$$

we derive (42) with $k$ replaced with $k^{-1}$. Since $k > 0$ is arbitrary, (42) follows with ”=”. If we choose $k = e^{\beta/(1-p-\alpha(mp-1))}$, the latter implies (12) with $f(\xi) = u(\xi,1)$, where $f$ is a nonnegative and continuous solution of (13), (14). By [9], PDE (1) has a finite speed of propagation property, and minimal solution of (1), (4) has an expanding interface. Therefore, the upper bound $\xi_\ast$ of the support of $f$ is positive and finite; $f$ is positive and smooth for $\xi < \xi_\ast$ and $f = 0$ for $\xi \geq \xi_\ast$. Thus, (33) is valid. Proof of (6) and (15) coincide with the proof given in Lemma 3.2 of [6].
Now suppose that $u_0$ satisfies (3). Then for arbitrary sufficiently small $\epsilon > 0$, there exists an $x_\epsilon < 0$ such that:

$$ (C - \epsilon)(-x)^p \leq u_0(x) \leq (C + \epsilon)(-x)^p, \quad x \geq x_\epsilon. $$

(44)

Let $u_\epsilon(x, t)$ (respectively, $u_{-\epsilon}(x, t)$) be a minimal solution to the CP (1), (2) with initial data $(C + \epsilon)(-x)^p$ (respectively, $(C - \epsilon)(-x)^p$). Since the solution to the CP (1), (2) is continuous, there exists a number $\delta = \delta(\epsilon) > 0$ such that:

$$ u_\epsilon(x, t) \geq u(x, t), \quad u_{-\epsilon}(x, t) \leq u(x, t), \quad 0 \leq t \leq \delta. $$

From (44), (45), and by applying the comparison result, (1), it follows that:

$$ u_{-\epsilon} \leq u \leq u_\epsilon, \quad 0 \leq t \leq \delta. $$

We have:

$$ u_{\pm \epsilon}(\xi_\rho(t), t) = t^{\alpha/(1 + p - \alpha(mp - 1))} f(\rho; C \pm \epsilon), \quad \forall \rho < \xi_\epsilon, \quad t \geq 0. $$

(47)

(Furthermore, we denote the right-hand side of (15a) by $f(\rho, C)$). Now taking $x = \xi_\rho(t)$ in (46), after multiplying by $t^{-\alpha/(1 + p - \alpha(mp - 1))}$ and passing to the limit, first as $t \to 0^+$ and then as $\epsilon \to 0^+$, we can easily derive (7). Similarly, from (46), (47) and (33), (5) easily follows. The lemma is proved.

**Proof of Lemma 4** As in the proof of Lemma 3, (44) and (45) follow from (3). From (18) and (2), it follows that the existence, uniqueness, and comparison result for the minimal solution of the CP (1), (2) with $u_0 = (C \pm \epsilon)(-x)^p$ and $T = +\infty$ hold. As before, from (44) and (45), (46) follows. For arbitrary $k > 0$, the function

$$ u_k^{\pm \epsilon}(x, t) = k \rho \pm \epsilon, \quad k > 0, $$

is a minimal solution of the following problem:

$$ u_t - \left( |u|^p, k^{\alpha(mp - 1) - (1 + p)/\alpha} u \right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, $$

(49a)

$$ u(x, 0) = (C \pm \epsilon)(-x)^p, \quad x \in \mathbb{R}. $$

(49b)

Since $\alpha(mp - \beta) - (1 + p) < 0$, it follows that:

$$ \lim_{k \to +\infty} u_k^{\pm \epsilon}(x, t) = v^{\pm \epsilon}(x, t), \quad x \in \mathbb{R}, \quad t \geq 0, $$

where $v^{\pm \epsilon}$ is a minimal solution to CP (1), (2) with $b = 0$, $u_0 = (C \pm \epsilon)(-x)^p$, and $T = +\infty$. Hence, $v^{\pm \epsilon}$ satisfies (47). Taking $x = \xi_\rho(t)$, where $\rho < \xi_\epsilon$ is fixed, from (50) it follows that for arbitrary $t > 0$

$$ \lim_{k \to +\infty} k \rho \pm \epsilon, \quad x \in \mathbb{R}, \quad t \geq 0, $$

(51)

Letting $\tau = k^{\alpha(mp - 1) - (1 + p)/\alpha} t$, then (51) implies:

$$ u^{\pm \epsilon}(\xi_\rho(t), \tau) \sim \tau^{\alpha/(1 + p - \alpha(mp - 1))} f(\rho; C \pm \epsilon), \quad \tau \to 0^+. $$

As before, (7) easily follows from (46) and (52). The lemma is proved. \qed
Proof of Lemma 5. Plugging \( u(x,t) = \phi(kt-x) \) into (1) and choosing \( y_0 = 0 \) we have the following initial value problem for \( \phi \):

\[
\begin{cases}
(\phi^{(m)})^{p-1}(\phi^{(m)})' - k\phi' - b\phi^\beta = 0, & 0 \leq y < +\infty \\
\phi(0) = (\phi^{(m)})'(0) = 0,
\end{cases}
\]

Proof of the existence and uniqueness of the solution to (53), which is monotonically increasing with asymptotic formula (39) is known in particular cases \( p = 1 \) [17] and \( m = 1 \) [26]. The standard proof based on phase plane analysis applies with minor modifications. By introducing new variables:

\[
X = \phi, \quad Y = ((\phi^{(m)})^\rho)^p,
\]

we have the following problem ODE problem in phase plane:

\[
\begin{cases}
dY/dX = k + bmX^{m+\beta-1}Y^{-\frac{1}{p}}, \\
Y(0) = 0.
\end{cases}
\]

Since \( m + \beta > 1 \), similar proof as in [26] implies the existence and uniqueness of the global increasing solution of (55). Next, we employ a scaling argument to prove:

\[
Y(X) \sim \left[ \frac{bm(1+p)}{p(m+\beta)} \right]^\frac{1}{p} X^\frac{p(m+\beta)}{p}, \quad \text{as} \quad X \to +\infty.
\]

Rescaled function:

\[
Y_l(X) = lY(l^\gamma X), \quad \gamma = -\frac{1+p}{p(m+\beta)}, \quad l > 0,
\]

solves the problem:

\[
\frac{dY_l}{dX} = kl^{(p(m+\beta)-(1+p))} + bmX^{m+\beta-1}Y_l^{-\frac{1}{p}}, \quad Y_l(0) = 0.
\]

As in (55), there exists a unique global solution \( Y_l \) of (58). It can be easily shown that the sequences \( \{Y_l\} \) and \( \{dY_l/dX\} \) are bounded in every fixed compact subset \( \mathbb{R}^+ \) uniformly for \( l \in (0,1] \). By choosing the expanding sequence of compact subsets of \( \mathbb{R}^+ \), and by applying Arzela-Ascoli theorem and Cantor’s diagonalization, it follows that there is a sub-sequence \( \{Y_{l'}\} \) which converges as \( l' \downarrow 0 \) in \( \mathbb{R}^+ \), and the convergence is uniform on compact subsets of \( \mathbb{R}^+ \). Since the limit function is a unique solution of the problem (58) with \( l = 0 \), we have

\[
\lim_{l \downarrow 0} Y_l(X) = \left[ \frac{bm(1+p)}{p(m+\beta)} \right]^\frac{1}{p} X^\frac{p(m+\beta)}{p}.
\]

By changing the variable \( Z = l^\gamma X \), from (59), (56) follows.

Let \( Y \) be a solution of the problem (55). Note that the problem:

\[
\frac{d\phi^m}{dy} = Y^\frac{1}{\gamma} (\phi(y)), \quad \phi(0) = 0,
\]

has a unique maximal solution in \( (-\infty, M) \), such that \( \phi \equiv 0 \) for \( y \leq 0 \), and \( \phi > 0 \) for \( 0 < y < M \). Moreover, whether \( M < \infty \) or \( M = +\infty \), we have \( \phi(y) \to +\infty \) as \( y \to M^- \).
It is enough to show that \( \exists \) and \( f \) propagation property, and minimal solution of (1), (4) has a finite interface. There-

Comparison (1) implies:

Since \( C \) choose

As in the proof of Lemma 3, it follows that (63) is true with equality sign. If we

Passing to the limit as \( y \to M^- \), from (61), (56) it follows that \( M = +\infty \), and ac-

Correspondingly

Case 1: \( p(m + \beta) < 1 + p \)

It is enough to show that \( \exists \ t_0 > 0 \) such that \( u(0, t_0) > 0 \). Let \( g(x, t) = C_1(t - x)^{1 + p \over m + p} \), \( C_1 \in (C_*, C). \)

\[
Lg = bC_1^\beta(t - x)^{\beta(1 + \beta) \over m + p} \left[ 1 - \left( {C_1 \over C_*} \right)^{mp - \beta} + C_1^{1 - \beta} {1 + p \over b(mp - \beta)} (t - x)^{-1 + \beta (mp - \beta)} \right]
\]

Since \( C_1 < C \), we can choose \( x_1 < 0 \) and \( \delta > 0 \) such that:

\[
Lg \leq 0, \text{in } Q := \{(x, t) : x_1 \leq x < t, 0 < t \leq \delta\},
\]

\[ g(x, 0) \leq u(x, 0), \ x_1 \leq x; \ g(x_1, t) \leq u(x_1, t), \ 0 \leq t \leq \delta. \]

Comparison (1) implies:

\[ 0 < g(x, t) \leq u(x, t), \forall \ x_1 \leq x < t, \ 0 \leq t \leq \delta. \]

In particular, we have: \( u(0, t_0) > 0, \forall \ 0 < t_0 \leq \delta \), which implies that \( f_1(0) = A_1 > 0. \)

Case 2: \( p(m + \beta) > 1 + p \)

We apply Lemma 5 with the forward traveling wave \( (k > 0) \). By (39) for some \( M > 0 \) we have:

\[ \phi(y) < Cy^{mp \over m + p}, \text{for } y > M. \]

Let us choose:

\[ K = \max\{\phi(y) : 0 \leq y \leq M\}, \ \xi = \max\left\{ M; \left( {K \over C} \right)^{mp \over 1 + p} \right\}, \]
and consider a family of traveling-wave solutions to (1) of the form: \( g(x, t) = \phi((kt - x - \xi)) \). From (64), (65) it follows that:

\[
\phi((-x - \xi)) \leq C(-x)^{1-\frac{\mu}{m-\beta}}, \text{ for any } x \in \mathbb{R}.
\]

From the comparison theorem it follows that \( g \leq u \) for any \( x \in \mathbb{R}, \ t \geq 0 \). By choosing \( t_0 > 0 \) such that \( k > \xi t_0^{-1} > 0 \), we ensure that:

\[
0 < g(0, t_0) = \phi((kt_0 - \xi)) \leq u(0, t_0) = t_0^{1-\frac{1}{\beta}} f_1(0),
\]

which proves that \( f_1(0) > 0 \). To prove the asymptotic formula (40) we proceed as we did in the proof in Lemma 4. As before, (44) - (46) follow from (3), where \( v_{\pm \epsilon} \) is a solution of the problem:

\[
v_t - \left(\left(|v'|_1^p - (vp)_x\right)_x\right) + bv^\beta = 0, \ |x| < |x_\epsilon|, 0 < t \leq \delta,
\]

\[
v(x_\epsilon, t) = \left(C \pm \epsilon \right)(-x_\epsilon)^p, \ v(-x_\epsilon, t) = u(-x_\epsilon, t), 0 \leq t \leq \delta,
\]

\[
v(x, 0) = \left(C \pm \epsilon \right)(-x)^p, |x| \leq |x_\epsilon|.
\]

Rescaled function:

\[
u_k^{x, \epsilon}(x, t) = ku_{x\epsilon}\left(k^{-\frac{1}{\beta}}x, k^{\beta-1}t\right), \ k > 0,
\]

satisfies the Dirichlet problem:

\[
v_t - \left(\left(|v'|_1^p - (vp)_x\right)_x\right) + bv^\beta = 0, \ |x| < k^{\frac{1}{\beta}}|x_\epsilon|, 0 < t \leq k^{1-\beta} \delta,
\]

\[
v(k^{\frac{1}{\beta}}x_\epsilon, t) = k(C \pm \epsilon)(-x_\epsilon)^p, \ v(-k^{\frac{1}{\beta}}x_\epsilon, t) = ku(-x_\epsilon, k^{\beta-1}t), 0 \leq t \leq k^{1-\beta} \delta,
\]

\[
v(x, 0) = \left(C \pm \epsilon \right)(-x)^p, |x| \leq k^{\frac{1}{\beta}}|x_\epsilon|.
\]

As before in the proof of Lemma 4 we have:

\[
limit_{k \to +\infty} \nu_k^{x, \epsilon}(x, t) = v_{x\epsilon}(x, t), \ (x, t) \in P := \{(x, t) : x \in \mathbb{R}, 0 < t \leq t_0\},
\]

thus,

\[
v_{x\epsilon}(x, t) = t^{1-\frac{1}{\beta}} f_1(\rho; C \pm \epsilon), \forall \rho < \zeta, t \geq 0,
\]

Taking \( x = \eta_\rho(t) = \rho t^{\frac{m-\beta}{m-\beta-1}} \) and \( \tau = k^{\beta-1} t \) it follows from (72) that:

\[
u_{x\epsilon}(\eta_\rho(\tau), \tau) \sim t^{1-\frac{1}{\beta}} f_1(\rho; C \pm \epsilon), \ \text{as} \ \tau \to 0^+.
\]

From (46) and (74), since \( \epsilon > 0 \) is arbitrary and \( f_1(0) = A_1 > 0 \), the desired asymptotic formula (40) follows. The lemma is proved.

**Proof of Lemma 7** As before, (44) - (46) follow from (3), where \( v_{x\epsilon} \) is a solution of the problem:

\[
v_t - \left(\left(|v'|_1^p - (vp)_x\right)_x\right) + bv^\beta = 0, \ |x| < |x_\epsilon|, 0 < t \leq \delta,
\]

\[
v(x_\epsilon, t) = \left(C \pm \epsilon \right)(-x_\epsilon)^p, \ v(-x_\epsilon, t) = u(-x_\epsilon, t), 0 \leq t \leq \delta,
\]

\[
v(x, 0) = \left(C \pm \epsilon \right)(-x)^p, |x| \leq |x_\epsilon|.
\]
Rescaled function:

\[ u^\epsilon_k(x,t) = ku_\epsilon(x,t) \left( k^{-\frac{1}{\alpha}} x, k^{\beta-1} t \right), \quad k > 0, \]

satisfies the Dirichlet problem:

\[
\begin{align*}
\tag{78a}
\nu_t &= k^{1-p-0(m+\beta)} \left( (v^m)_x \right)^{p-1} x - b u^\epsilon, \quad \text{in } E^k, \\
\tag{78b}
\nu(k^{\frac{1}{\alpha}} x, t) &= k(C \pm \epsilon)(-x^\alpha), \quad \nu(-k^{\frac{1}{\alpha}} x, t) = ku(-x^\beta, t), \quad 0 \leq t \leq k^{1-\beta} \delta, \\
\tag{78c}
v(x,0) &= (C \pm \epsilon)(-x^\alpha), \quad 0 < t \leq k^{1-\beta} \delta.
\end{align*}
\]

where

\[ E^k := \{ |x| < k^{\frac{1}{\alpha}} |x|, \quad 0 < t \leq k^{1-\beta} \delta \}. \]

The next step consists in proving the convergence of the sequence \( \{u^\epsilon_k\} \) as \( k \to +\infty \). This step is identical with the proof given in the similar Lemma 3.4 from [6]. For any fixed \( t_0 > 0 \), the function \( g(x,t) = (C + 1)(1 + x^\gamma)^{1/\gamma} \) is a uniform upper bound for the sequence \( \{u^\epsilon_k\} \) in \( E_{0 \epsilon}^k = E^k \cap P \), where \( P = \{(x,t) : 0 < t \leq t_0\} \). The sequences \( \{u^\epsilon_k\} \) are uniformly Hölder continuous on an arbitrary compact subset of \( P \) [13][20]. As in the proof of the Lemma 3.4 of [6] it is proved that some subsequences \( \{u^\epsilon_k\} \) converge to solutions of the reaction equation. This imply that

\[ \eta(\epsilon, \tau) \sim \frac{1}{\tau^{\frac{1}{\gamma}}} \left( \left( (C \pm \epsilon)^{1-\beta} \nu^{1-(1-\beta)} - b(1-\beta) \right)^{\frac{1}{\gamma}} \right), \quad \text{as } \tau \to 0^+. \]

From (46) and (79), since \( \epsilon > 0 \) is arbitrary, the desired formula (11) follows. The lemma is proved.

5. Proofs of the Main Results

Proof of Theorem 7 From Lemma 4 and (7), it follows

\[ \lim_{t \to 0} \inf \eta(t) t^{1/(mp-1)} \geq \xi_\ast. \]

For \( \forall \epsilon > 0 \), let \( u_\epsilon \) be a minimal solution of the CP (1), (4) with \( b = 0 \) and with \( C \) replaced by \( C + \epsilon \). The second inequality of (44) and the first inequality of (45) follow from (5). Since \( u_\epsilon \) is a supersolution of (1), from (44), (45), and a comparison principle, the second inequality of (46) follows. By Lemma 3 we have:

\[ \eta(t) \leq (C + \epsilon)^{\frac{mp-1}{mp-0(m+\beta)}} \xi_\ast t^{1/(1+p-\alpha(m+\beta))}, \quad 0 \leq t \leq \delta, \]

and hence:

\[ \lim_{t \to 0} \sup \eta(t) t^{1/(mp-1)} \leq \xi_\ast. \]

Proof of Theorem 2 Assume that \( u_0 \) is defined by (4) and \( p(m+\beta) \neq 1 + p \). The self-similar form (19) follows from Lemma 6. Let \( C > C_\ast \). For a function:

\[ g(x,t) = t^{1/(1-\beta)} f_1(\zeta), \quad \zeta = xt^{-\frac{mp-\beta}{mp(m+\beta)}}. \]

we have

\[ L g = t^{\frac{p}{mp-\beta}} L^0 f_1, \]
where the operator $L^0$ is defined by (21). By choosing

$$f_1(\zeta) = C_0(\xi_0 - \zeta)^{\gamma_0}, \; 0 < \zeta < +\infty,$$

with $C_0$, $\xi_0 > 0$ and $\gamma_0 = (1 + p)/(mp - \beta)$ we have

$$(84) \quad L^0 f_1 = bC_0^2(\xi_0 - \zeta)^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_0/C_*)^{mp-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)}(\xi_0 - \zeta)^{\frac{1-p(\alpha+1)}{mp-\beta}} \right\}.$$ 

For an upper estimation we choose $C_0 = C_2$ and $\xi_0 = \xi_2$ (see the appendix, Section 7). If $p(m+\beta) > 1 + p$, we have

$$L^0 f_1 \geq bC_2^2(\xi_2 - \zeta)^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_2/C_*)^{mp-\beta} + \frac{C_2^{1-\beta}}{b(1-\beta)}(\xi_2 - \zeta)^{\frac{1-p(\alpha+1)}{mp-\beta}} \right\} = 0, \; \text{for } 0 \leq \zeta \leq \xi_2,$$

while if $p(m+\beta) < 1 + p$, we have:

$$L^0 f_1 \geq bC_2^2(\xi_2 - \zeta)^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_2/C_*)^{mp-\beta} \right\} = 0, \; \text{for } 0 \leq \zeta \leq \xi_2.$$

By (83) we have

$$(85a) \quad Lg \geq 0, \; \text{for } 0 < x < \xi_2 t^{\frac{mp-\beta}{(mp+p-\beta)}}, \; 0 < t < +\infty,$$

$$(85b) \quad Lg = 0, \; \text{for } x > \xi_2 t^{\frac{mp-\beta}{(mp+p-\beta)}}, \; 0 < t < +\infty.$$ 

(1) implies that $g$ is a supersolution of (1) in $(x, t): x > 0, t > 0)$. Since

$$(86a) \quad g(x, 0) = u(x, 0) = 0, \; \text{for } 0 \leq x < +\infty,$$

$$(86b) \quad g(0, t) = u(0, t), \; \text{for } 0 \leq x < +\infty,$$

the right-hand side of (23) follows. If $p(m+\beta) < 1 + p$, to prove the lower estimation we choose $C_0 = C_1$, $\xi_0 = \xi_1$, and $\gamma_0 = (1 + p)/(mp - \beta)$. From (84) and (83) we have

$$L^0 f_1 \leq bC_1^2(\xi_1 - \zeta)^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_1/C_*)^{mp-\beta} + \frac{C_1^{1-\beta}}{b(1-\beta)}(\xi_1 - \zeta)^{\frac{1-p(\alpha+1)}{mp-\beta}} \right\} = 0, \; \text{for } 0 \leq \zeta \leq \xi_1,$$

$$(87a) \quad Lg \leq 0, \; \text{for } 0 < x < \xi_1 t^{\frac{mp-\beta}{(mp+p-\beta)}}, \; 0 < t < +\infty,$$

$$(87b) \quad Lg = 0, \; \text{for } x > \xi_1 t^{\frac{mp-\beta}{(mp+p-\beta)}}, \; 0 < t < +\infty.$$ 

As before from (86) and (1), the left-hand side of (23) follows. If $p(m+\beta) > 1 + p$, then to prove the lower estimation we choose $C_0 = C_1$, $\xi_0 = \xi_1$ and $\gamma_0 = p/(mp-1)$. We have

$$L^0 f_1 \leq C_1(1-\beta)^{-1}(\xi_1 - \zeta)^{\frac{1-p-\beta}{mp-1}} \times$$

$$\times \left\{ \xi_1 - C_1^{mp-1} \left( \frac{(1-\beta)p(mp)^p}{(mp-1)^{1+p}} + b(1-\beta)C_1^{\beta-1}(\xi_1)^{\frac{p(\alpha+1)}{mp-1}} \right) \right\} = 0, \; 0 < \zeta < \xi_1,$$

which again implies (87). From (1), the left-hand side of (23) follows.

Let $p(m+\beta) > 1 + p$ and $0 < C < C_*$. For $\gamma \in (0, 1)$ consider a function

$$g(x, t) = \left[ C_1^{1-\beta}(-x)^{\frac{(1+p)(\alpha+1)}{mp-\beta}} - b(1-\beta)(1-\gamma)t \right]^{\frac{1}{p}}, \; x \in \mathbb{R}, \; t > 0.$$
We estimate $Lg$ in

$$M := \{(x,t) : -\infty < x < \mu_x(t), t > 0\}, \mu_x(t) = -\left[ b(1-\beta)(1-\gamma)C^{\beta-1}t \right]^{mp-\beta/(1+\beta)}.$$  

We have $Lg = b\beta S$, where

$$S = \gamma - C^{mp-\beta} \left[ 1 - \left( \frac{-\mu_y(t)}{(-x)_+} \right)^{\frac{1+p}{mp-\beta}} \right]^{1/\beta} \times$$

\begin{equation}
\times \left[ R_1 + R_2 \left[ 1 - \left( \frac{-\mu_y(t)}{(-x)_+} \right)^{\frac{1+p}{mp-\beta}} \right]^{-1} \right].
\end{equation}

(88a)

(88b) $S|_{t=0} = \gamma - (C/C_*)^{mp-\beta}$, $S|_{t=\mu_y(t)} = \gamma$, where $R_1, R_2 > 0$ (see Section 7). Moreover, $S_t \geq 0$ in $M$.

Thus, $\gamma - (C/C_*)^{mp-\beta} \leq S \leq \gamma$ in $M$.

If we take $\gamma = (C/C_*)^{mp-\beta}$ (respectively, $\gamma = 0$), then we have:

(89a) $Lg \geq 0$ (respectively, $Lg \leq 0$) in $M$,

(89b) $Lg = 0$, for $x > \mu_y(t), t > 0$.

From (1), the estimation (25) follows. Let $p(m+\beta) < 1 + p$ and $0 < C < C_*$. First, we establish the following rough estimation:

$$\left[ C^{1-\beta}(-x)_+^{\frac{1+p}{mp-\beta}} - b(1-\beta)(1-(C/C_*)^{mp-\beta})t \right]^{1/\beta} \leq$$

\begin{equation}
\leq u(x,t) \leq C(-x)_+^{\frac{1+p}{mp-\beta}}, \text{ for } x \in \mathbb{R}, 0 \leq t < \infty.
\end{equation}

(90)

To prove the left-hand side we consider the function, $g$, as in the case when $p(m+\beta) > 1 + p$ with $\gamma = (C/C_*)^{mp-\beta}$. As before, we then derive (88a), and since:

$$S_t \leq 0, \text{ in } M,$$

we have $S \leq 0$ in $M$. Hence, (89) is valid with $\leq$ in (89a). As before, from (1), the left-hand side of (90) follows. To prove the right-hand side of (90) it is enough to observe that:

$$Lu_0 = b\beta (1-(C/C_*)^{mp-\beta}) \geq 0, \text{ for } x \in \mathbb{R}, t \geq 0.$$

Having (90), we can now establish a more accurate estimation (26). Consider a function:

$$g(x,t) = C_0 \left( \frac{\mu_y}{\mu_y + x} \right)^{\frac{1+p}{mp-\beta}}, \text{ in } G_\ell,$$

$$G_\ell := \{(x,t) : \zeta(t) = -\ell t^{\frac{mp-\beta}{1+p}}, x < \infty, 0 < t < +\infty\},$$

where, $C_0, \zeta_0 > 0, \ell > \zeta_0$. Calculating $Lg$ in

$$G_\ell^+ := \{(x,t) : \zeta(t) < x < -\zeta_0 t^{\frac{mp-\beta}{1+p}}, 0 < t < +\infty\},$$
we have:

\[ Lg = bg^\varepsilon S, \quad S = 1 - (C_0/C_\varepsilon)^{mp-\beta} - (b(1 - \beta))^{-1}C_0^{1-\beta} \zeta_0 f^{-(1+p)/(mp-\beta)} \times \]

\[ \times \left( -\zeta_0 t^{mp-\beta}(1-p)/(mp-\beta) - x \right) \]

(91)

By choosing \( C_0 = C_\varepsilon \), we have:

\[ Lg \leq 0, \text{ in } G_\varepsilon^+; \quad Lg = 0, \text{ in } G_\varepsilon \setminus G_\varepsilon^+. \]

To obtain a lower estimation we choose \( \zeta_0 = \zeta_3 \), and \( \ell = \ell_0 \) (see Section 7). Using (90), we have:

\[ g |_{x=\zeta(t)} = t^{1/p} C_\varepsilon (\ell_0 - \zeta_3) \]

\[ \leq (b(1 - \beta)\theta t)^{1/p} = t^{1/p} \times \]

(93a)

\[ \left[ C^{1-\beta} \ell_0^{(1+p)/(mp-\beta)} - b(1 - \beta) (1 - (C/\varepsilon)^{mp-\beta}) \right]^{1/p} \leq \eta(\zeta(t), t), \quad t \geq 0, \]

(93b)

\[ g(x, 0) = u(x, 0) = 0, \quad 0 \leq x \leq x_0, \]

(93c)

where \( x_0 > 0 \) is an arbitrary fixed number. By using (92), (93), we can apply (1) in:

\[ G_{\ell_0}^+ := G_{\ell_0} \cap \{ x < x_0 \}. \]

Since \( x_0 > 0 \) is arbitrary, the left inequality in (26) follows. Since

\[ S_\varepsilon \leq 0, \text{ for } \zeta(t) < x < -\zeta_0 t^{mp/(mp+1)}, \quad t > 0, \]

from (91) it follows that

\[ S \geq S |_{x=\zeta(t)} = 1 - (C_0/C_\varepsilon)^{mp-\beta} - (b(1 - \beta))^{-1}C_0^{1-\beta} \zeta_0 (\ell - \zeta_0) \]

\[ \leq \zeta_0 t^{mp-\beta}(1-p)/(mp-\beta). \]

(92)

By choosing \( C_0 = C_\varepsilon, \quad \zeta_0 = \zeta_4, \quad \ell = \ell_1 \) (see Section 7), we have

\[ S |_{x=\zeta(t)} = 0, \]

\[ Lg \leq 0, \text{ in } G_\varepsilon^+, \quad Lg = 0, \text{ in } G_\varepsilon \setminus G_\varepsilon^+, \]

\[ u(\zeta(t), t) \leq t^{\frac{1}{mp}} C_\varepsilon^{\frac{1}{mp}} = t^{\frac{1}{mp}} C_\varepsilon^\frac{1}{mp} = g(\zeta(t), t), \quad t \geq 0, \]

and, for arbitrary \( x_0 > 0 \), (93b) and (93c) are valid. By applying (1) in \( G_{\ell_0}^c \), due to the arbitrariness of \( x_0 > 0 \), we derive the right-hand side of (26). From (23), (25), and (26) it follows that:

\[ \zeta_1 t^{mp/(mp+1)} \leq \eta(t) \leq \zeta_2 t^{mp/(mp+1)}, \quad 0 \leq t < +\infty, \]

where the constants \( \zeta_1 \) and \( \zeta_2 \) are chosen according to relevant estimations for \( u \). From (20) and the respective estimations (23), (25), and (26), the estimation (24) follows. If \( u_0 \) satisfies (3) with \( \alpha = (1 + p)/(mp - \beta) \) and with \( C \neq C_\varepsilon \), then the asymptotic formulae (8) and (9) may be proved as the similar estimations (5) and (7) were in Lemma 3.
Proof of Theorem 3. For all $\varepsilon > 0$ from (3), (44) follows. Consider a function:

$$\eta(t) = \ell t^{1/(\alpha(1-\beta))}, \quad \ell(\varepsilon) = (C + \varepsilon)^{-1/\alpha}[b(1-\beta)(1-\varepsilon)]^{1/(\alpha(1-\beta))},$$

where $\delta_1 > 0$ is chosen such that $\eta(\ell(\varepsilon))(\delta_1) = x_\varepsilon$. We have

$$\eta(t) = \ell t^{1/(\alpha(1-\beta))}, \quad \ell(\varepsilon) = (C + \varepsilon)^{-1/\alpha}[b(1-\beta)(1-\varepsilon)]^{1/(\alpha(1-\beta))},$$

where $\delta_1 > 0$ is chosen such that $\eta(\ell(\varepsilon))(\delta_1) = x_\varepsilon$. We have

$$L_{\varepsilon}g = b g_\varepsilon^\beta(e + S)$$

$$S = -b^{-1}(am)^p(C + \varepsilon)^{mp-\beta} (-x)^{(mp-\beta)-(1+p)} \left[ g(x)^{1-\beta} \right] (C + \varepsilon)^{1/(1-\beta)}$$

$$S_1 = \left\{ \alpha p(m + \beta - 1) + p(\alpha(1-\beta) - 1) \left[ g(x)^{1-\beta} \right] (C + \varepsilon)^{1/(1-\beta)} \right\}.$$
we derive
\[ Lg = bg^\beta S, \quad S = 1 - (b(1 - \beta))^{-1} \xi_5 c_6^{1/\alpha} \left\{ g t^{1/(\beta - 1)} \right\}^{(\alpha(1 - \beta) - 1)/\alpha} \]
\[ \frac{p}{b} (am)^p (am - 1) c_6^{(1+p)/\alpha} g^{mp-\beta-(1+p)/\alpha} . \]
Since
\[ S_x \geq 0, \quad \text{in } G^+_{t, \delta} , \]
\[ S \geq S \big|_{x = \eta(t)} = 1 - (b(1 - \beta))^{-1} \xi_5 c_6^{1-\beta} (\ell - \xi_5)^{\alpha(1 - \beta) - 1} - \]
\[ -t \frac{a(mp-\beta-1+p)}{a(1-\beta)} b^{-1} p(am)^p (am - 1) c_6^{mp-\beta} (\ell - \xi_5)^{\alpha(mp-\beta)-(1+p)} , \]
we have
\[ S \geq \epsilon - t \frac{a(mp-\beta-1+p)}{a(1-\beta)} b^{-1} p(am)^p (am - 1) c_6^{mp-\beta} (\ell - \xi_5)^{\alpha(mp-\beta)-(1+p)} , \quad \text{in } G^+_{t, \delta} . \]
By choosing \( \delta = \delta(\epsilon) > 0 \) sufficiently small we have
\[ Lg \geq b(\epsilon/2) g^\beta , \quad \text{in } G^+_{t, \delta} . \]
By applying (99) and (11) in \( G^+_{t, \delta} = G_{t, \delta} \cap \{ x < x_0 \} \) we have
\[ Lg = 0 , \quad \text{in } G^+_{t, \delta} \setminus \bar{G}^+_{t, \delta} , \]
\[ u(\eta(t), t) \leq t^{1/\beta} \{ c_{1-\beta} (1-\alpha(1-\beta)) - b(1-\beta)(1-\epsilon) \}^{1/\beta} = \]
\[ c_6 (\ell - \xi_5 t^{1/(1-\beta)} = g(\eta(t), t) , \quad 0 \leq t \leq \delta , \]
\[ u(x_0, t) = g(x_0, t) = 0 , \quad 0 \leq t \leq \delta , \]
\[ u(x, 0) = g(x, 0) = 0 , \quad 0 \leq x \leq x_0 . \]
Since \( x_0 > 0 \) is arbitrary, from (101) and (11), it follows that for all \( \ell > \ell_* \) and \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon, \ell) > 0 \) such that:
\[ u(x, t) \leq c_6 \left( -\xi_5 t^{1/(1-\beta)} - x \right)^\alpha , \quad \text{in } \bar{G}^+_{t, \delta} . \]
In view of (11) (which is valid along \( x = \eta(t) \)), \( \delta \) may be chosen so small that:
\[ -\ell t^{1/\alpha(1-\beta)} \leq \eta(t) \leq -\xi_5 t^{1/\alpha(1-\beta)} , \quad 0 \leq t \leq \delta . \]
Since \( \ell > \ell_* \) and \( \epsilon > 0 \) are arbitrary numbers, (10) follows from (103).

Proof of Theorem 4 and all the results described in Section 3 case (4a)-(4d), and in the special case of \( b = 0 \) are almost identical to the similar proofs given in [6].

6. Numerical Solution

In this section, we investigate the numerical solutions to (1) using on a weighted essentially nonoscillatory (WENO) scheme. In Section 6.1 we briefly introduce the WENO discretization for the PDE. Numerical results and comparisons with analysis are presented in Section 6.2 and Section 6.3. All figures can be found in the appendix, Section 8.
6.1. Finite Difference Discretization. WENO methods refer to a family of finite volume or finite difference methods for solutions of hyperbolic conservation laws and other convection dominated problems. The central idea behind the WENO scheme is to use nonlinear combinations of numerical stencils for solution interpolation/reconstruction, with weights adapted to the smoothness of the solution on these stencils. Therefore, interpolation across discontinuous or nonsmooth part of the solution is avoided as much as possible. This yields numerical solutions with high order accuracy in smooth regions, while maintaining non-oscillatory and sharp discontinuity transitions [29]. These features make WENO schemes well suitable for the study of problems with piecewise smooth solutions containing discontinuities or sharp interfaces. A WENO scheme was proposed in [27] to solve nonlinear degenerate parabolic equation of the form \( u_t = (b(u))_{xx} \). In that paper, the second order derivative term is directly approximated using a conservative flux difference formula. Below we describe a finite difference WENO scheme for the solution of the nonlinear double degenerate parabolic equation (1).

As shown in Fig. 2, the numerical solution is defined at full grid node \( x_i = u(x_i) \), where \( x_i = i\Delta x \) and \( \Delta x = x_{i+1} - x_i \) is the uniform grid size. Defining the flux function \( f(u^m) = |(u^m)_x|^{p-1}(u^m)_x \), and introduce an auxiliary function \( h(\xi) \) such that:

\[
(104) \quad f(u^m)(x) = \frac{1}{\Delta x} \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} h(\xi) d\xi.
\]

Then at grid node \( x_i \), we have:

\[
(105) \quad \left([u^m]_{x}^{p-1}(u^m)_{x}\right)(x_i) = f(u^m)(x_i) = \frac{1}{\Delta x}(h(x_i + \frac{1}{2}) - h(x_i - \frac{1}{2})).
\]

Notice that to evaluate the derivative \( f(u^m)_x \) at \( x_i \), we need the values of \( h(x) \) at half grid nodes \( x_{i-\frac{1}{2}} = x_i - \Delta x \) and \( x_{i+\frac{1}{2}} = x_i + \Delta x \). Therefore, if the function \( h(x) \) can be computed to \( r \)th order of accuracy, then the right hand side of equation (105) would be an \( r \)th order approximation to \( f(u^m) \). Overall, the WENO approximation for \( ([u^m]_{x}^{p-1}(u^m))_{x} \) can be summarized as following:

1. With the given values of \( u(x) \) (and thus the values of \( u^m(x) \)) defined at grid \( x_i = i\Delta x \), approximate the derivative \( u^m_x \) at \( x_i \) using a fifth order WENO interpolation scheme. Based on this, compute the pointwise values of the flux function \( f(u^m)(x_i) \). From (104), this value is also the cell average of \( h(x) \) over the interval \( (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \).

2. From the cell average of \( h(x) \), compute the values of \( h(x) \) at the half grid nodes \( x_{i+\frac{1}{2}} \) with a fifth order WENO reconstruction scheme.

Here the WENO interpolation scheme for \( u^m \) is similar to that proposed in [21] for the solution of Hamilton-Jacobi Equations. In [21], derivatives of the solution are computed on left-biased and right-biased numerical stencils to construct monotone Hamiltonians. For the solution of the nonlinear parabolic equation in this paper, we simply calculate the left and right biased derivatives by WENO approximation and take the arithmetic average of the two to be the value of \( u^m_x \). A similar strategy
The computational domain is $[-b, b]$. The formation of the CP for (1) is considered. The initial condition is taken as the IPS solution $u_{\text{IPS}}$. The periodic boundary condition is imposed to simplify the numerical implementation. Without the focus on efficiency of the algorithm, we always choose the time step $\Delta t$ small enough to get a stable solution. The computational domain is $[-5.5, 5.5]$ with total number of 1024 grid points. The initial condition is taken as the IPS solution $u_{\text{IPS}}$ at $t = 0.05$ with $\Gamma = 1.0$. We set the domain large enough so that the interface does not reach the boundary at the end of numerical simulation. The periodic boundary condition is imposed to simplify the numerical implementation. Without the focus on efficiency of the algorithm, we always choose the time step $\Delta t$ small enough to get a stable solution.

6.2. Comparison with the Instantaneous Point Source (IPS) Solution. Solution of the CP for $\frac{x}{2}$, $b = 0$ with initial function being a Dirac’s point mass (or $\delta$-function) is given by (32, 9):

$$u_{\text{IPS}}(x, t) = \frac{1}{\Gamma} \left( \frac{\Gamma}{k(m, p)} \right) \frac{\Gamma}{\sqrt{\pi}} \left( \frac{t}{k(m, p)} \right).$$

with $k(m, p) = \frac{mp-1}{m(1+p)} \left( \frac{1}{p(m+1)} \right)^{\frac{1}{p}}$. Here $\Gamma$ is an integration constant, defined by the conservation of the energy. This solution has a compact support $[-\eta(t), \eta(t)]$ with the interface function given by:

$$\eta(t) = t \left( \frac{1}{k(m, p)} \right) \frac{\Gamma}{\sqrt{\pi}}.$$

For our numerical test, we use parameters $m = 6$, and $p = 2$ and 3 respectively. The computational domain is $[-5.5, 5.5]$ with total number of 1024 grid points. The initial condition is taken as the IPS solution $u_{\text{IPS}}$ at $t = 0.05$ with $\Gamma = 1.0$. We set the domain large enough so that the interface does not reach the boundary at the end of numerical simulation. The periodic boundary condition is imposed to simplify the numerical implementation. Without the focus on efficiency of the algorithm, we always choose the time step $\Delta t$ small enough to get a stable solution.

The comparison between the numerical and analytical solutions is shown in Fig. 3 for time $t = 2.0$. Here the filled circles represent the numerical solution and the solid line is the solution (107). The agreement is excellent. The WENO scheme can successfully capture the sharp transition in the solution without generating any apparent numerical oscillations. To identify the location of the (right) interface $\eta(t)$, we take the first $x_i$ where $u_i < 10^{-10}$ as the interface location. In Fig. 4, the computed values for $\eta(t)$ (circles) are plotted together with that given by the IPS solution (108) (solid curve) at different stages of the simulation. It is clear that the dynamics of the interface is accurately captured by the WENO scheme.

6.3. Comparison with Analytical Results. In this section, we apply the WENO scheme to equation (1) with initial condition given by (4). To compare the numerical solution with the analytical results for the CP, we use numerical initial condition as shown in Fig. 5. Here $u(x, 0)$ is given by condition (4) near the interface (interval $[-1, 0]$ for this case). As the value of $x$ gets smaller, $u(x, 0)$ is smoothly brought to
zero by a hyperbolic tangent function. Notice that since the solution to (1) has a 
finite speed of propagation, it is expected that as long as the time is short enough, 
the numerical solution locally close to the interface should agree with that from the 
analysis. For all the numerical examples shown below, a grid size of $\Delta x = \frac{1}{125}$ is 
used. Since the interfaces never reach beyond the domain boundary at the end of 
the simulation, periodic boundary conditions are applied.

6.3.1. Region I with Expanding Interface. For region 1, we choose $m = 4$, $p = 2$, 
$\beta = 0.5$, $b = 0.5$, $C = 1.0$, and $\alpha = 0.2 < (1 + p)/(mp - \beta) = 0.4$. For these parameters, 
the interface expands and its location is given by $\eta(t) \sim \xi_0 t^{1/(1 + p - \alpha(mp - 1))} = \xi_0 t^{0.625}$ 
for a positive $\xi_0$. In order to compare numerical results with analysis, we need to 
solve the second order nonlinear ODE (13), up to $\xi = \xi_*$ where $f(\xi_*) = 0$. Since 
$\xi_*$ is unknown, we transfer the BVP $(13)$, $(14)$ to a system of IVP, by introducing 
another variable $g(\xi) = (f^m(\xi))' = mf^{m-1}(\xi)f'(\xi)$. We then solve the system with 
some given initial conditions. However, since the boundary condition $(14)$ is given 
at negative infinity in the analysis, it is not clear how to set the initial conditions 
for $f(\xi)$ and $g(\xi)$, respectively.

From the analysis, we know that $f(\xi) t^{\alpha/(1 + p - \alpha(mp - 1))} \sim u(x,t)$ as $t \to 0^+$, along the 
curve $x = \xi^{1/(1 + p - \alpha(mp - 1))}$. Thus one strategy is to use the numerical solution near 
the interface to estimate $f(\xi)$ and its derivative at specific value of $\xi$. Therefore, 
we have the approximation $f(0) \approx u(0,t) t^{-\alpha/(1 + p - \alpha(mp - 1))} = u(0,t) t^{0.125}$ for small time $t$. 
Specifically, we choose to approximate $f(0)$ by

\begin{equation}
(109) \quad f(0) \approx \frac{1}{3} \sum_{i=1}^{3} u(0,t_i) t_i^{0.125},
\end{equation}

where $t_1 = 0.01$, $t_2 = 0.02$ and $t_3 = 0.03$, respectively. Plug in the values of the 
numerical solution at $x = 0$ and $t = t_i$, we get $f(0) \approx 0.752$. To evaluate $f'(0)$, 
we use the approximation $f(\xi_0) \approx u(\Delta x, t) t^{0.125}$ and $f(2\xi_0) \approx u(2\Delta x, t) t^{0.125}$ for small 
value of $t$, where $\xi_0 = \Delta x t^{1/(1 + p - \alpha(mp - 1))} = \Delta x t^{0.625}$. The evaluation of $f(\xi_0)$ 
and $f(2\xi_0)$ is similar to $(109)$ for $f(0)$. Then we fit a quadratic function to interpo- 
late the three points $(0, f(0))$, $(\xi_0, f(\xi_0))$ and $(2\xi_0, f(2\xi_0))$ and use the derivative 
of the quadratic function at 0 to approximate $f'(0)$. Through some simple calculation, 
we get $f'(0) \approx -0.309$. Finally, the values of $f(0)$ and $f'(0)$ are used as 
initial conditions in the ODE solver (third-order TVD Runge-Kutta Discretization) 
to solve for $g(\xi)$ and $f(\xi)$. The numerical solutions for $f(\xi)$ and its derivative are 
plotted in Fig. (6)(a) and (b), respectively. As the value of $\xi$ increases, function $f(\xi)$ 
decreases and the rate of decreasing gets larger. As $\xi \sim \xi_*$, the function becomes 
nonsmooth and the ODE solver fails to yield an accurate solution, even with a very 
small time step. We choose $\xi_*$ to be the value of $\xi$ which gives the smallest $f(\xi)$ 
and get $\xi_* \approx 0.696$. In Fig. (7)(a), we plot the interface location computed by the 
WENO scheme together with the analytical curve $\eta(t) = \xi_* t^{0.625}$. Good agreement 
is achieved for small time intervals.

We can estimate the range for $\xi_*$ based on analytical results given by (6), (16) 
and bound $[\xi_1, \xi_2]$ given in Section 7. In addition to the constants $C$, $m$, $p$ and
\( \alpha \), the solution to the CP (11), (14) \( w(0,1) \) with \( b = 0 \) and \( C = 1 \) is needed. We approximate the value \( w(0,1) \) using the numerical solution from WENO scheme and get \( w(0,1) \approx 0.725 \). Then from (6), (16) and bound \([\xi_1, \xi_2]\) given in Section 7 we compute the bounds for the range of \( \xi \), as \( \xi_1 = 0.678 \) and \( \xi_2 = 0.764 \). It is clear that the value of \( \xi_\ast \) computed before is within the range. In Fig. 7(b), it is shown that without the absorption term, the interface location given by the numerical solution is indeed bounded by the two curves predicted by analysis.

6.3.2. Region 2 the Borderline Case. For Region 2, we first choose \( m = 2.5, \quad p = 0.5, \quad \beta = 0.5, \quad b = 1.0, \quad \alpha = (1 + p)/(mp - \beta) = 2.0 \). Thus \( p(m + \beta) = 1 + p = 1.5 \). With these parameters, we have \( C_\ast = 0.13572 \). With the choice of \( C > C_\ast \), and \( C < C_\ast \), the analytical solution is given by the explicit formula (18). In Fig. 8, the numerical results show excellent agreement with the analytical traveling wave solution.

For the second set of tests, we choose \( m = 2.0, \quad p = 2.0, \quad \beta = 0.2, \quad b = 1.0, \) and \( \alpha = (1 + p)/(mp - \beta) \approx 0.78947 \). With these choices, we have \( p(m + \beta) > 1 + p \) and \( C_\ast \approx 0.75655 \). For this case, the analytical results are given by (19) and (20). Here we use the similar strategy as described in the previous section to numerically estimate \( \zeta \) through the solution of the nonlinear ODE (21). For the choice of \( C = 1.2 > C_\ast \), and \( C = 0.2 < C_\ast \), we get the estimation \( \zeta_\ast \approx 12.3 \) and \( \zeta_\ast \approx -6.7 \), respectively. The comparison between numerical solution and analysis is shown in Fig 9. In the plot, the analytical curves are given by \( \eta(t) = 12.3t^{1.25} \) for Fig. 9(a) and \( \eta(t) = -6.7t^{1.25} \) for Fig. 9(b). The numerical results agree well with the analysis for short time durations.

Finally we choose \( m = 0.5, \quad p = 2.0, \quad \beta = 0.2, \quad b = 1.0, \) and \( \alpha = (1 + p)/(mp - \beta) = 3.75 \). With these choices, we have \( p(m + \beta) < 1 + p \) and \( C_\ast \approx 0.1032 \). For the choice of \( C = 0.4 > C_\ast \), and \( C = 0.05 < C_\ast \), we solve the nonlinear ODE (21) and get the estimation \( \zeta_\ast \approx 0.895 \) and \( \zeta_\ast \approx -0.586 \), respectively. As shown in Fig. 10, the agreement between numerics and analysis is again very good. In the plot, the analytical curves are given by \( \eta(t) = 0.895t^{1.25} \) for Fig. 10(a) and \( \eta(t) = -0.586t^{1.25} \) for Fig. 10(b).

6.3.3. Region 3 with Shrinking Interface. In Region 3, we choose the parameters \( m = 4, p = 2, \beta = 0.5, \quad b = 0.8, \quad C = 0.5, \) and \( \alpha = 0.8 > (1 + p)/(mp - \beta) = 0.4 \). For this choice, the absorption term dominates and the interface shrinks. The analytical solution and interface location are given by (11) and (10), respectively. Comparison between numerical and analytical results is plotted in Fig. 11. It is interesting to note that for the interface location as shown in Fig. 11(a), excellent agreement is obtained between the numerics and analysis during the whole simulation, even though the analysis is valid only for short time period. In Fig. 11(b), the numerical solution \( u(x, t) \) near the interface matches well with that from the analysis.

6.3.4. Region 4 with Waiting Time. In Region 4, we choose \( m = 2, \quad p = 3, \beta = 1.0, \quad b = 0.5, \quad C = 0.5 \). Corresponding to the analysis for Region (4a), we set \( \alpha = (1 + p)/(mp - 1) = 0.8 \). With these parameters, numerical solutions at different time are plotted in Fig. 12(a). It is clear that the interface at \( x = 0 \) remains stationary up to
$t = 1$. In Fig. [12] b, the numerical solution near the interface agrees well with the analytical result given by (27).

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7. **Appendix A**

Here we give explicit values of the constants used in Section 2:

$$\xi_1 = p^{\frac{1}{1+p}} (\alpha (mp - 1))^{\frac{1}{1+p}}, \xi_2 = 1,$$

$\xi_1 = 1, \xi_2 = p^{\frac{1}{1+p}} (\alpha (mp - 1))^{\frac{1}{1+p}},$ if $0 < \alpha \leq p (mp - 1)^{-1};$

$$\xi_1 = A_1^{\frac{mp-1}{1+p}} (1 + b (1 - \beta) A_1^{\beta - 1})^{\frac{1}{1+p}} (p (mp)^{p} (1 - \beta))^{\frac{1}{1+p}} (mp - 1)^{-1},$$

$C_1 = A_1 \xi_1^{\frac{p}{mp-1}},$ if $p (m + \beta) > 1 + p, C > C_*,$

$$\xi_1 = A_1^{\frac{mp-1}{1+p}} (1 + b (1 - \beta) A_1^{\beta - 1})^{\frac{1}{1+p}} ((m (1 + p))^{p} (m + \beta) (1 - \beta))^{\frac{1}{1+p}} (mp - 1)^{-1},$$

$C_1 = A_1 \xi_1^{\frac{p}{mp-1}},$ if $p (m + \beta) < 1 + p, C > C_*,$

$$\xi_2 = A_1^{\frac{mp-1}{1+p}} (1 + b (1 - \beta) A_1^{\beta - 1})^{\frac{1}{1+p}} ((m (1 + p))^{p} (m + \beta) (1 - \beta))^{\frac{1}{1+p}} (mp - 1)^{-1},$$

$C_2 = A_1 \xi_2^{\frac{p}{mp-1}},$ if $p (m + \beta) > 1 + p, C > C_*,$

$$\xi_2 = (A_1 / C_*)^{\frac{mp-\beta}{1+p}}, C_2 = C_*,$$

$\xi_1 = -C^{\frac{mp-\beta}{1+p}} (b (1 - \beta))^{\frac{mp-\beta}{1+p}},$ if $p (m + \beta) > 1 + p, 0 < C < C_*,$

$$\xi_2 = -C^{\frac{mp-\beta}{1+p}} (b (1 - \beta) (1 - (C / C_*))^{mp-\beta})^{\frac{mp-\beta}{1+p}},$$

if $p (m + \beta) < 1 + p, 0 < C < C_*,$

$R_1 = (m (1 + p))^{p} (1 + p - p (m + \beta) (b (mp - \beta) - 1)^{1+p})^{-1},$

$R_2 = (m (1 + p))^{p} (1 + p) p (m + \beta - 1) (b (mp - \beta - 1)^{1+p})^{-1},$

$$\theta_* = \left[1 - \left(\frac{C}{C_*}\right)^{mp-\beta} \left(\left(\frac{C_*}{C}\right)^{\frac{(mp-\beta)(1-\beta)}{1+p}} - 1\right)\right]^{\frac{1}{1+p}},$$

$\xi_0 = C_*^{\frac{mp-\beta}{1+p}} (C_* / C)^{\frac{(mp-\beta)(1-\beta)}{1+p}} (b (1 - \beta) \theta_*^{\frac{mp-\beta}{1+p}}),$

$\xi_3 = C_*^{\frac{mp-\beta}{1+p}} \left(\left(\frac{C_*}{C}\right)^{\frac{(mp-\beta)(1-\beta)}{1+p}} - 1\right) (b (1 - \beta) \theta_*^{\frac{mp-\beta}{1+p}}),$
\[ \ell_1 = C^{\frac{mp-\beta}{1+p}} b (1 - \beta)(\delta, \Gamma)^{-1} \left(1 - \delta, \Gamma - (1 - \delta, \Gamma)^p (C/C_s)^{mp-\beta}\right)^{\frac{mp-\beta}{1+p}}, \]

\[ \zeta_4 = \delta, \Gamma \ell_1, \; \Gamma = 1 - (C/C_s)^{\frac{mp-\beta}{1+p}}, \; C_3 = C(1 - \delta, \Gamma)^{\frac{1+p}{mp-\beta}}, \]

where \(\delta_* \in (0, 1)\) satisfies:

\[ g(\delta_*) = \max_{[0,1]} g(\delta), \; g(\delta) = \delta^{1+p-\beta} \left(1 - \delta \Gamma - (1 - \delta \Gamma)^p (C/C_s)^{mp-\beta}\right). \]

\[ \hat{C} = \left[ \frac{(mp-1)^{1+p}}{p(m+1)(m(1+p))^p} \right]^{\frac{1}{mp-1}}, \; \gamma_\epsilon = \frac{p(m+1)(m(1+p))^p(C+\epsilon)^{mp-1}}{(mp-1)^p} + \epsilon. \]

\[ \xi_3 = A_0^{\frac{mp-1}{1+p}} \left[ \frac{(mp)^p(1+\alpha(mp-1))}{(mp-1)^p} \right]^{\frac{1}{mp-1}} C^{\frac{mp-1}{1+p-\alpha(mp-1)}} \xi_1, \]

\[ \xi_4 = A_0^{\frac{mp-1}{1+p}} \left[ \frac{(mp)^p(1+\alpha(mp-1))}{(mp-1)^p} \right]^{\frac{1}{mp-1}} C^{\frac{mp-1}{1+p-\alpha(mp-1)}} \xi_2, \]

\[ C_4 = C^{(1+p)/(1+\alpha(mp-1))} A_0 \xi_3^{\alpha(1-mp)}, \; C_5 = C^{(1+p)/(1+\alpha(mp-1))} A_0 \xi_4^{\alpha(1-mp)}. \]

\[ \zeta_5 = \ell_*^{\alpha(1-\beta)} (1 - \epsilon) \ell, \]

\[ C_6 = \left[ 1 - (\ell_*/\ell)^{\alpha(1-\beta)} (1 - \epsilon) \right]^{-\alpha} \left[ C^{1-\beta} - \ell^{-\alpha(1-\beta)} b(1 - \beta)(1 - \epsilon) \right]^{1/(1-\beta)}. \]

8. Appendix B

Here we list the figures corresponding to the numerical results as described in Section 6.

Figure 2. Computational grid for the WENO scheme
Figure 3. IPS Solution: numerical and analytical solution at $t = 2.0$

Figure 4. IPS Solution: Interface Location Vs. Time
Figure 5. Initial condition used for computational study

Figure 6. Numerical solution for $f(\zeta)$ and $f'(\zeta)$

Figure 7. Region 1: Interface Location Vs. Time with (left) and without (right) the absorption term
(a) $C = 0.5 > C_s$

(b) $C = 0.06 < C_s$

**Figure 8.** Region 2 with $p(m+\beta) = 1 + p$. Interface Location Vs. Time for different $C$

(a) $C = 1.2 > C_s$

(b) $C = 0.2 < C_s$

**Figure 9.** Region 2 with $p(m+\beta) > 1 + p$. Interface Location Vs. Time for different $C$

(a) $C = 0.4 > C_s$

(b) $C = 0.05 < C_s$

**Figure 10.** Region 2 with $p(m+\beta) < 1 + p$. Interface Location Vs. Time for different $C$
Figure 11. Region 3 Left: Interface Location Vs. Time; Right: Solution at $t = 0.9$

Figure 12. Region 4 Left: Numerical solution at different time; Right: Numerical solution at $t = 0.7$
References

[1] Ugur G. Abdulla. On the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains. *Journal of Mathematical Analysis and Applications*, 260(2):384–403, 2001.

[2] Ugur G. Abdulla. Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption. *Nonlinear Analysis: Theory, Methods, & Applications*, 50(4):541–560, 2002.

[3] Ugur G. Abdulla. Reaction-diffusion in nonsmooth and closed domains. *Boundary Value Problems*, (2):28, 2005.

[4] Ugur G. Abdulla. Well-posedness of the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains. *Transactions of the American Mathematical Society*, 357(1):247–265, 2005.

[5] Ugur G. Abdulla and Roqia Jeli. Evolution of interfaces for the non-linear parabolic $p$-Laplacian type reaction-diffusion equations. *European Journal of Applied Mathematics*, 28(5), 2017.

[6] Ugur G. Abdulla and John R. King. Interface development and local solutions to reaction-diffusion equations. *SIAM Journal on Mathematical Analysis*, 32(2):235–260, 2000.

[7] U. G. Abdullaev. On existence of unbounded solutions of nonlinear heat equations with absorption. Zh. Vychisl. Mat. i Mat. Fiz., 33:232–245, 1993.

[8] S.N. Antontsev, J.I. Diaz, and S. Shmarev. *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*, volume 48. Springer Verlag, 2012.

[9] G. I. Barenblatt. On some unsteady motions of a liquid or a gas in a porous medium. *Prikl. Mat. Mech.*, 16:67–78, 1952.

[10] G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 1996.

[11] P. Benilan, M. G. Crandall, and M. Pierre. Solutions of the porous medium equation under optimal conditions on initial values. *Indiana University Mathematics Journal*, 33:51–87, 1984.

[12] E. DiBenedetto. *Degenerate Parabolic Equations*. Series Universitext. Springer Verlag, 1993.

[13] E. DiBenedetto and M. A. Herrero. On the Cauchy problems and initial traces for a degenerate parabolic equation. *Transactions of the American Mathematical Society*, 314:187–224, 1989.

[14] E. DiBenedetto and M. A. Herrero. Nonnegative solutions of the evolution $p$-Laplacian equations: Initial traces and Cauchy problem when $1 < p < 2$. *Archive for Rational Mechanics Analysis*, 111:225–290, 1990.

[15] J. R. Esteban and J. L. Vazquez. On the equation of turbulent filtration in one-dimensional porous media. *Nonlinear Analysis: Theory, Methods, & Applications*, 10(11):1303–1325, 1986.

[16] M. A. Herrero and M. Pierre. The Cauchy problem for $u_t = \delta u^m$ when $0 < m < 1$. *Transactions of the American Mathematical Society*, 291:145–158, 1985.

[17] M. A. Herrero and J. L. Vazquez. Thermal waves in absorbing media. *Journal of Differential Equations*, 74:218–233, 1988.

[18] K. Ishige. On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation. *SIAM Journal on Mathematical Analysis*, 27(5):1235–1260, 1996.

[19] A. V. Ivanov. Hölder estimates for equations of slow and normal diffusion type. *Journal of Mathematical Sciences*, 85(1):1640–1644, 1997.

[20] A. V. Ivanov. Regularity for doubly nonlinear parabolic equations. *Journal of Mathematical Sciences*, 83(1):22–37, 1997.

[21] G. Jiang and D. Peng. Weighted ENO schemes for Hamilton-Jacobi equations. *SIAM Journal on Scientific Computing*, 21(6):2126–2143, 2000.

[22] A. S. Kalashnikov. The influence of absorption on the propagation of heat in a medium with heat conductivity that depends on the temperature. Zh. Vychisl. Mat. i Mat. Fiz., 16:689–696, 1976.

[23] A. S. Kalashnikov. On a nonlinear equation appearing in the theory of non-stationary filtration. *Trud. Semin. I. G. Pertovski*, 4:137–146, 1978.
[24] S. Kamin, L. A. Peletier, and J. L. Vazquez. A nonlinear diffusion-absorption equation with unbounded initial data. pages 243–263, 1992.
[25] L. S. Leibenson. General problem of the movement of a compressible fluid in porous medium. Izv. Akad. Nauk SSSR, Geography and Geophysics, IX:7–10, 1945.
[26] Z. Li, W. Du, and C. Mu. Travelling-wave solutions and interfaces for non-Newtonian diffusion equations with strong absorption. Journal of Mathematical Research with Applications, 334:451–462, 2013.
[27] Y. Liu, W. Shu, and M. Zhang. High order finite difference WENO schemes for nonlinear degenerate parabolic equations. SIAM Journal on Scientific Computing, 33(2):939–965, 2011.
[28] O. A. Oleinik, A. S. Kalashnikov, and Ch.Y. Lin. Cauchy problem and boundary value problems for an equation of nonstationary filtration. Izv. Akad. Nauk SSSR, Ser. Mat., 22:667–704, 1958.
[29] C. W. Shu. High order weighted essentially nonoscillatory schemes for convection dominated problems. SIAM Review, 54(1):82–126, 2009.
[30] M. Tsutsumi. On solutions of some doubly nonlinear degenerate parabolic equations with absorption. Journal of Mathematical Analysis and Applications, 132(1):187–212, 1988.
[31] J. L. Vazquez. The Porous Medium Equation: Mathematical Theory. Oxford Science Publications. Oxford University Press, 2007.
[32] Ya. B. Zeldovich and A. S. Kompaneets. On the theory of heat propagation for temperature dependent thermal conductivity, in collection commemorating the 70th anniv. of A. F. Ioffe. Izdat. Akad. Nauk SSSR, 1950.

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