Towards Exact Results in Nodal Antiferromagnetic Planar Liquids

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Abstract

It has been argued in previous works by the authors that nodal excitations in (2+1)-dimensional doped antiferromagnets might exhibit, in the spin-charge separation framework and at specific regions of the parameter space, a supersymmetry between spinons and holons. This supersymmetry has been elevated to a $N = 2$ extended supersymmetry of composite operators of spinon and holons, corresponding to the effective “hadronic” degrees of freedom. In this work we elaborate further on this idea by describing in some detail the dynamics of a specific composite model corresponding to an Abelian Higgs model (SQED). The abelian nature of the gauge group seems to be necessitated both by the composite structure used, but also by electric charge considerations for the various composites. We demonstrate the passage from a pseudogap to an unconventional superconducting phase, which notably is an exact non-perturbative analytic result, due to the underlying $N = 2$ supersymmetric Abelian gauge theory. We believe that these considerations may provide a first step towards a non-perturbative understanding of the phase diagrams of strongly-correlated electron systems.
1 Introduction and Overview of Results

The study of 2 + 1 dimensional gauge theories has been motivated by the search for an understanding of confinement [1]. The degree to which they can be of help in understanding confinement at zero temperature in three space dimensions is unclear because the nature and possible configurations of textures, instantons and other non-perturbative features such as the locality of disorder variables is very different. Their attraction is that their comparative simplicity allows a much more complete understanding of the confinement phenomenon. However another reason for studying such theories is the interest in planar system in high temperature superconductors [2] where the phases are believed not to be of conventional type. In particular string-like structures which are features of confinement in the gauge theories occur as inhomogeneities in the charge and magnetic order [3] in certain parameter regimes of these materials. Both in the study of confinement and the condensed matter analogues, some of the reason for controversy is the room for error in the calculations that can be performed due to the intrinsic limitations of the methods themselves. It is thus important to pursue models with exact solutions. Unlike for 1 + 1 dimensions where infinite-dimensional groups and the factorizability of the S-matrix are at play [4], dynamical supersymmetry is an important ingredient for obtaining exact information about the phases of the theory [5, 6, 7, 8]. In fact, the more extended the supersymmetry the more exact the results. This is related to holomorphicity properties of the theory.

Speaking about supersymmetry in condensed matter systems may at first sight seem absurd. After all, supersymmetry is a space time symmetry, requiring Lorentz invariance, which condensed matter systems do not have, either because they are primarily lattice systems, or, in case one considers effective continuous quantum field theories of such systems, because there are finite fermi surfaces. Excitations near finite fermi surfaces cannot produce relativistic (Lorentz invariant) field theories. However, there are cases where relativistic low-energy effective quantum field theories do appear in condensed matter situations. One famous example is the \( CP^1 \) \( \sigma \)-model describing the low-energy limit of an antiferromagnet. Other examples, in which we shall concentrate in this article, include excitations near nodal points in the fermi surface, i.e. near points where either the fermi surface shrinks to a set of points, as is the case of underdoped cuprates, or a gap function vanishes (as is the case of \( d \)-wave high-temperature superconductors).

It is our view, which is also shared by a large proportion of the community, that such nodes play an important rôle on the rich phase diagram of these materials. It is therefore of outmost importance to look carefully at the physics implied by excitations near such nodes, which, from a continuum effective field theory viewpoint, constitute a relativistic field theorectic system, in which the role of the limiting velocity (‘speed of light’) is played by the fermi velocity of the node.

One can go one step further: upon assuming separation of spin and electric charge degrees of freedom in the fundamental constituents describing the ground state of planar doped antiferromagnets (or more generally strongly correlated electronic systems), one ob-
serves that there is a balance between fermionic and bosonic degrees of freedom. This balance prompts one to seek for possible supersymmetries in certain regions of the parameter space of the underlying condensed matter system.

In [10, 11] it was argued that it is indeed possible to determine regions in the parameter space of condensed matter models, of relevance to the physics of doped antiferromagnets (and hence high-temperature superconductors), where there is such a dynamical supersymmetry between appropriate degrees of freedom. Specifically, one starts from a microscopic lattice system, an appropriately extended $t - j$ model [11] with nodes in its fermi surface. It is an experimental fact that nodes exist in both the underdoped regime of the cuprates (where the fermi surface consists of four nodes), and in the superconducting regime (the high-$T_c$ oxides are $d$-wave superconductors, with nodes in the respective superconducting gap). The continuum low-energy theory of nodal spinons and holon excitations has the form of a relativistic $CP^1 \sigma$-model (magnon-spinons) coupled to Dirac-like fermions (holon degrees of freedom) plus higher contact interactions among the various field modes. In [11] it was shown that there are regions in the microscopic model phase space, where one recovers a supersymmetric theory between the spinon and holon constituents in the presence of strongly-coupled non-dynamical (i.e. no kinetic terms) gauge fields. This is the constituent theory. The non-dynamical gauge fields of their $CP^1 \sigma$-model simply express contact interactions between spinons and holons.

Although $N = 1$ symmetry was demonstrated for specific condensed matter models of doped antiferromagnets [10, 11], however a lack of holomorphicity properties meant that exact non-perturbative information was not available. The situation is much improved for $N = 2$ supersymmetry [6, 8], and such an elevation is indeed necessitated by the electric charge conservation requirement [10]. It is the point of this paper to construct a precise $N=2$ supersymmetric model of composites of spinons and holons, which arguably described the low energy dynamics of nodal liquids, and discuss the associated physics, especially in connection with a passage from the pseudogap to the superconducting phase.

Although at first sight the model of [11] appears to have only a $N = 1$ supersymmetry, however, as argued in [12] it actually has a hidden $N = 2$ supersymmetry [13] due to its low dimensionality (2+1 dimensions), which implies the existence of topologically conserved currents (i.e. currents which are conserved without use of the equations of motion). This is a generic result for such theories [14, 15]. In [13, 12] we demonstrated how the $N = 1$ supersymmetry, in terms of constituent fields can be elevated to a $N = 2$ supersymmetry in terms of suitably constructed composite fields. As shown in [13, 12], once can generate this way dynamical gauge fields, at the composite operator level, obtained after integrating out the non-dynamical gauge fields of the constituent theory. Such gauge structures are made out of appropriate combinations of spinons and holons, up to quartic order in constituent fields. The presence of higher order composites, in contrast to the bilinear composite models of the non supersymmetric case of ref. [16], is necessary here [12], in order to guarantee

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1In (2+1)-dimensions the concept of a boson or fermion is not well defined, and one has the possibility of fractional statistics as well. Formally this can be easily understood by means of bosonization techniques [9]. However, the concept of bosonic (commuting) or fermionic (anticommuting) operator variables can be defined, and it is in this sense that the we use the terminology fermion and boson.
the gauge coupling of the initial $N = 1$ scalar and vector supermultiplets.

It is interesting to notice that the choice of composite operators made in [13, 12] led explicitly to an Abelian gauge field involved in the $N=2$ supersymmetric multiplet, which essentially coupled two $N = 1$ composite supermultiplets, a scalar and a vector. We were unable to find a choice of composite operators that generate the full Non-Abelian $SU(2)$ supersymmetric model of [17], whose breaking $SU(2) \rightarrow U(1)$ would result in the Abelian Higgs composite model discussed explicitly in [13]. The two cases ($N=2$ Super QED and a Broken phase of $N=2$ $SU(2)$ Georgi-Glashow model) lead to very different non-perturbative dynamics [8], for instance as far as confinement properties of the three-dimensional gauge theory are concerned. In the latter case there is no stable monopole phase if supersymmetry is unbroken. We remind the reader that in non supersymmetric theories, Polyakov has shown that a stable monopole-plasma phase in $(2+1)$-dimensions led to a massive photon, and a linear confinement. In the $N=2$ $SU(2)$ case, such a phase does not exist [8], and this latter result is understood to be exact.

It is the purpose of this article to discuss the very different physics of the two above mentioned distinct cases $N = 2$ Supersymmetric QED (SQED) (with compact $U(1)$ group) and $N = 2$ $SU(2) \rightarrow U(1)$ broken phase of a non-Abelian model discussed in [17]. The difference is exemplified by the very different confinement properties. We shall demonstrate, though, that the non compact case also presents interesting physics, namely it allows an exact study of the passage of a pseudogap to a superconducting phase for the nodal liquid. The pseudogap phase of the non compact case will correspond to the Higgs phase of the SQED, while the superconducting phase, which is characterised by unconventional superconductivity of the anomalous type proposed in [18], corresponds to the Coulomb-phase of the SQED, in which the statistical gauge field remains exactly massless.

Our results may be of relevance in attempts towards an analytic understanding at a non-perturbative level of the dynamics of strongly-correlated electron systems, with relevance to high-temperature superconductivity and more generally to antiferromagnetism. Also our model may be viewed as a toy model for an understanding of ideas related to the so-called scaleless limit of gauge theories [19], where the gauge fields appear dynamically from more fundamental interacting constituents in the theory.

However, in the four-dimensional setting of [19], the emergent gauge bosons (photons) appeared as Goldstone bosons of a spontaneous breakdown of Lorentz symmetry, associated with non-zero vacuum expectation values (v.e.v.) of vector fields linearising the four-fermion Thirring interactions of the model. In three space-time dimensions, on the other hand, the photons have only a single degree of freedom and hence they are allowed, in a sense, to get a v.e.v without breaking Lorentz symmetry. It is this fact that allows the extension of such ideas in $(2+1)$-dimensions to incorporate supersymmetry, which is intimately related to Lorentz invariance. From a physical point of view, with relevance to strongly-correlated electrons, we remark that the maintenance of Lorentz symmetry is connected with the fact that we restrict our attention to excitations near nodes of the fermi surface of such systems [11], which exhibit relativistic behaviour, with the rôle of the velocity of ‘light’ played by the fermi velocity at the node.

The structure of the article is as follows: in section 2 we review the construction (in
the continuum) of the N=2 supersymmetric composite model. In section 3 we discuss the electric charge assignment of the composite excitations, which is crucial in determining the physically correct effective action to describe the continuum composite dynamics. In section 4 we apply these considerations to a specific microscopic model of strongly-correlated electrons [11], and discuss the emergence of the correct continuum limit. Moreover, in the same section we discuss exact results concerning the phase structure of this effective continuum composite theory, and demonstrate the passage from an (unconventional [18]) superconducting to a pseudogap (and stripe) phase. We also discuss an exact result concerning the non-fermi liquid behaviour of our composite nodal supersymmetric liquid. Section 5 contains our conclusions and outlook. Technical aspects of our approach, which regard the formalism and the phase structure of N=2 (2+1)-dimensional supersymmetric gauge theories [6, 7], are given in an Appendix.

2 Review of the construction of the (continuum) N=2 Supersymmetric Abelian Composite Model

We will first summarize the results of [13, 12], dealing with the construction of $N=2$ composite supermultiplets, quartic in the constituent fields of holons and spinons.

The $\gamma$–matrices are $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$ and $\gamma^2 = i\sigma^3$, where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. We define for any spinor $\psi$: $\bar{\psi} = \psi^\dagger \gamma^0$.

First we note that, as discussed in [13], the quadratic composites

$$\phi = \bar{\psi}_1 \psi_2$$

$$A_\mu = \bar{\psi}_1 \gamma_\mu \psi_2 - z_1^* \partial_\mu z_2 + z_2 \partial_\mu z_1^*$$

belong respectively to a $N=1$ scalar supermultiplet $(\phi', \psi', f)$ and a $N=1$ vector supermultiplet $(A_\mu, \chi')$ were the composite superpartners read, in terms of the constituent fields

$$\psi' = (f_1^* - \partial z_1^*) \psi_2 + (f_2 - \partial z_2) \psi_1^*$$

$$\chi' = (f_1^* + \partial z_1^*) \psi_2 - (f_2 + \partial z_2) \psi_1^*$$

$$f = 2(f_1^* f_2 - \partial_\mu z_1^* \partial^\mu z_2) - \bar{\psi}_1 \partial \psi_2 - (\bar{\psi}_2 \partial \psi_1)^*.$$  (2)

We note that the transformation of $A_\mu$ has actually the expected form up to a gauge transformation, which implies that it must be a gauge field. The fields defined in Eqs.(1) are complex: they are not the physical degrees of freedom but have the generic composite structure that we want to study. The physical composite degrees of freedom are given in [13]. There are three $N=1$ scalar supermultiplets with bosonic part:
\[ \phi_1 = \overline{\psi}_1 \psi_2 + \overline{\psi}_2 \psi_1 \]
\[ \phi_2 = i(\overline{\psi}_1 \psi_2 - \overline{\psi}_2 \psi_1) \]
\[ \phi_3 = \overline{\psi}_1 \psi_1 - \overline{\psi}_2 \psi_2 \]

(3)

which form an SU(2) triplet, and are parity conserving. The parity violating \( \phi^4 = \overline{\psi}_1 \psi_1 + \overline{\psi}_2 \psi_2 \) is discarded here, given the fact that we shall be interested in phases where \( < \phi_i > \neq 0 \), and as a result of the Vafa-Witten theorem [20] such spontaneous parity violating condensates are excluded in vector-like theories, such as the ones we are interested here.

On the other hand, there are three \( N = 1 \) vector supermultiplets (forming again an SU(2) triplet) with bosonic part:

\[ A^1_\mu = 2 \Re \left( \overline{\psi}_1 \gamma_\mu \psi_2 - z_1^* \partial_\mu z_2 + z_2 \partial_\mu z_1^* \right) \]
\[ A^2_\mu = 2 \Im \left( \overline{\psi}_1 \gamma_\mu \psi_2 - z_1^* \partial_\mu z_2 + z_2 \partial_\mu z_1^* \right) \]
\[ A^3_\mu = \overline{\psi}_1 \gamma_\mu \psi_1 - \overline{\psi}_2 \gamma_\mu \psi_2 - z_1^* \partial_\mu z_1 + z_1 \partial_\mu z_1^* + z_2^* \partial_\mu z_2 - z_2 \partial_\mu z_2^* \]

(4)

and one SU(2) singlet, which is parity violating:

\[ A^4_\mu = \overline{\psi}_1 \gamma_\mu \psi_1 + \overline{\psi}_2 \gamma_\mu \psi_2 - z_1^* \partial_\mu z_1 + z_1 \partial_\mu z_1^* - z_2^* \partial_\mu z_2 + z_2 \partial_\mu z_2^* \]

(5)

which, upon taking into account the \( \sigma \)-model constraint at a constituent level \( \sum_{i=1}^{2} z_i^* z_i = \text{const} \), can be rewritten in terms of spinon \( J^z \) and holon currents \( J^\psi \) as:

\[ A^4_\mu = \sum_{i=1}^{2} \left( \overline{\psi}_i \gamma_\mu \psi_i + 2z_i \partial_\mu z_i^* \right) \equiv J^\psi_\mu + J^z_\mu \]

(6)

Since in the relativistic theory of the nodal liquids we are interested here, the parity violating vector gauge field is not allowed to acquire a v.e.v. \( < A_\mu > = 0 \), so as not to break the Lorentz invariance of the nodal theory, the Vafa-Witten theorem [20] is satisfied in this case. In what follows we shall therefore retain the parity violating composite excitation, unlike the case of the scalar \( \phi^4 \). In fact we shall make supersymmetric spectra by considering only \( A^4_\mu \), something which we shall justify in the next section when we discuss in detail the electric charge assignments of the various excitations. It should be noted that the condition \( < A^4_\mu > = 0 \) implies that, on the ground state of our system, the spinon \( J^z_\mu \) and holon \( J^\psi_\mu \) currents are linked in such a way that only one of them plays a physical macroscopic rôle. This is a welcome feature of the spin-charge separation framework.
From the explicit form of the $A^4_\mu$ vector potential (5) we can make the interesting observation that this composite excitation has a form similar to that one would obtain from a constituent $CP^1$ $\sigma$-model coupled to fermions, after elimination of the gauge degree of freedom:

$$S_{CP} = \frac{1}{\gamma_0} \int d^3x \sum_{i=1}^2 (|\partial_\mu - iA^4_\mu| z_i|^2 + \overline{\psi}_i (\partial\psi - iA^4_\mu) \psi_i) + \ldots$$

(7)

As argued in [21], integrating out in this naive continuum model the gauge field produces composites of $z$ and $\psi_i$ fermions resulting in contact interactions among the fermions. This indicates that what we have identified as a gauge composite field in our case is indeed related to strongly coupled U(1) gauge group fields to be integrated out in the much more complicated, but realistic, Lattice gauge theory [16] of fundamental constituents. In such a theory, whose precise form is unknown, there are many more non-minimal, higher derivative contact interactions among spinons and holons, and it is the postulate of supersymmetry that allows the construction of an effective continuum field theory. In realistic situations one may hope that the supersymmetric theory of composites would lie in the same universality class in the infrared as the more realistic effective continuum theory representing the true low-energy dynamics of the system.

From the three scalar composite supermultiplets, we form one real $N = 1$ scalar supermultiplet $(\rho, \xi, \sigma)$ and one complex $N = 1$ scalar supermultiplet $(\phi, \psi, f)$ where:

$$\rho = \phi^3 = \overline{\psi}_1 \psi_1 - \overline{\psi}_2 \psi_2$$
$$\xi = (f^*_1 - \partial z^*_1) \psi_1 + (f_1 - \partial z_1) \psi_1^* - (1 \rightarrow 2)$$
$$\phi = \phi_1 - i\phi_2 = 2\overline{\psi}_1 \psi_2$$
$$\psi = \psi' = (f^*_1 - \partial z^*_1) \psi_2 + (f_2 - \partial z_2) \psi_1^*$$

(8)

The $N = 1$ composite supersymmetric transformations, found in [13], are then

$$\delta \rho = \tau \xi, \quad \delta \xi = (\partial \rho + \sigma) \varepsilon, \quad \delta \sigma = \tau \partial \xi,$$
$$\delta \phi = \tau \psi, \quad \delta \psi = (\partial \phi + f) \varepsilon, \quad \delta f = \tau \partial \psi,$$
$$\delta A_\mu = \tau \gamma_\mu \chi, \quad \delta \chi = -\frac{1}{2} F^\mu_{\nu} \gamma^\mu \gamma^\nu \varepsilon = (\partial^\nu A_\nu - \partial \cdot A) \varepsilon,$$

(9)

where $F_{\mu\nu}$ is the Abelian field strength of $A_\mu = A^4_\mu$ (quadratic in the constituent fields), and the gaugino $\chi$ is real, given by:

$$\chi = i(f^*_1 + \partial z^*_1) \psi_1 - i(f_1 + \partial z_1) \psi_1^* + (1 \rightarrow 2)$$

(10)

We stress again that above we selected the parity violating vector supermultiplet due to electric charge assignments to be discussed in the next section.
In order to elevate the $N = 1$ supersymmetry to a $N = 2$ (Abelian) supersymmetry, we should couple the complex composite scalar supermultiplet to one of the composite vector supermultiplets, which we denote as $(A_\mu, \chi)$. In [12] we constructed explicitly the covariant derivative of $\phi$ by adding to the complex quadratic scalar field a higher order (quartic) composite which will generate the minimal coupling. We took into account the quartic contribution only and neglected the higher orders: we managed to construct a quartic composite scalar $M$ whose supersymmetric transformation generates the quartic fermion $\Lambda$:

$$\delta M = \bar{\epsilon} \Lambda,$$

$$\delta \Lambda = \left( -i A_\phi + \partial M + F \right) \epsilon,$$  \hspace{1cm} (11)

where $F$ is a quartic auxiliary field. An important point of our construction is that the transformation (11) of the fermion was satisfied in a gauge defined by the complex equation

$$\partial^\rho (A_\nu \Box \phi) = 0,$$  \hspace{1cm} (12)

which implies actually two gauge conditions. This is possible in (2+1) dimensions, since a gauge field has one physical degree of freedom [22]. Making the substitutions:

$$\phi \rightarrow \Phi = \phi + gM,$$

$$\psi \rightarrow \Psi = \psi + g\Lambda,$$

$$f \rightarrow F = f + gF,$$  \hspace{1cm} (13)

where $g$ is a dimensionful constant, we obtained the expected covariant derivatives

$$D_\mu \Phi = (\partial_\mu - igA_\mu)\Phi,$$

$$= \partial_\mu \phi - igA_\mu \phi + g\partial_\mu M + \text{sixth order composite}$$  \hspace{1cm} (14)

in the (supersymmetry) transformation laws of the complex fermion $\Psi$.

In the above formulae, the quartic scalar $M$ that we constructed in order to generate the minimal coupling $A\phi$ is given by

$$-i\partial^\rho \Box M = \epsilon^{\mu\nu\rho} f \partial_\mu A_\nu + \partial^{\rho} \left( 2\partial_\mu \phi A_\mu + \phi \partial_\mu A_\mu + \bar{\psi} \chi \right).$$  \hspace{1cm} (15)

Note that the occurrence of this scalar is specific to 2+1 dimensions, as it is proportional to the topologically conserved current $J^\rho = \epsilon^{\mu\nu\rho} \partial_\mu A_\nu$, which plays a central role in the elevation of an $N = 1$ supersymmetry to an extended $N = 2$ supersymmetry [14].

On the other hand, the fermion $\Lambda$ is found to have the form [12]:

$$\Lambda = i \left( \Lambda^{(1)} + 2\Lambda^{(2)} + \Lambda^{(3)} + \Lambda^{(4)} \right)$$  \hspace{1cm} (16)
where:

\[
\square \Lambda^{(1)} = \partial^\mu A_\mu \psi + \phi \partial \chi \\
\square \Lambda^{(2)} = \partial \phi \chi + A_\mu \partial^\mu \psi, \\
\square \Lambda^{(3)} = (f - \partial \phi) \chi + (\partial A - \partial_\mu A^\mu) \psi, \\
\partial \square \Lambda^{(4)} = (\partial^\nu A_\nu - \partial A) \partial \psi - 2f \partial \chi.
\]

We also constructed the complex gaugino \( \lambda = \xi - i\chi \), where \( \xi \) and \( \chi \) (superpartner of \( A_\mu^A \)) are respectively defined in Eq.(8) and Eq.(10):

\[
\lambda = 2 (\partial z_2^2 \psi_2 - \partial z_1 \psi_1^*) + 2 (f_1^* \psi_1 - f_2 \psi_2^*)
\]

and finally obtained the following transformations

\[
\begin{align*}
\delta \Phi &= \Xi \Psi, \\
\delta \Psi &= (\partial \Phi + F) \varepsilon, \\
\delta \rho &= \Xi \xi, \\
\delta A_\mu &= \Xi \gamma_\mu \chi, \\
\delta \lambda &= \left( \partial \rho + \sigma + \frac{i}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu \right) \varepsilon, \\
\delta \sigma &= \Xi \partial \xi
\end{align*}
\]

which, when \( \varepsilon \) is a complex parameter, constitute a set of \( N = 2 \) supersymmetric transformations for an Abelian Higgs model [23].

It should be remarked that the \( N = 2 \) supersymmetric transformations are closed in our construction up to terms which, in a lagrangian formalism, would correspond to higher derivative operators and hence to irrelevant operators in the infrared (low energies) in a renormalization-group sense [12].

Finally, we note that no new constraints arise if we consider higher order composites: the quartic composite can be seen as the truncation of a series of composite operators which lead to the exact covariant derivatives if they are ressummed. The gauge condition (12) gives rise to a gauge fixing term in the Lagrangian which is also the truncation of a series of gauge fixing terms.

This completes our review of the construction of an Abelian Higgs composite model out of spinon and holon nodal fields. At this point we note that in [23], where the authors consider ‘elementary’ fields and not composites, the superpartners transform under an on-shell set of \( N = 2 \) transformations were the equations of motion of the auxiliary fields are used, in the supersymmetric Abelian Higgs model. In [12] we restricted ourselves only on the algebraic aspects of the composite supersymmetric transformations and did not consider the detailed dynamics of this model, and the associated physical consequences in connection with strongly-correlated electron systems. These topics will be the focus of the following sections.

Notice that, as a result of our specific construction, by supersymmetrizing only one of the vector multiplets, which as we shall see later on is also dictated by electric-charge
assignments, we encounter an Abelian Gauge Field model. In addition to the electric-charge argument, we also mention that, with the present composite construction, we were unable to generalize the composite superalgebra to an \(SU(2)\) supermultiplet (constructed out of the parity conserving composites) since too many constraints on the fields would be present: constraints for the generation of the covariant derivatives and also for the generation of the non-Abelian field strengths. It is not clear yet whether the parity-violating Abelian structure, generated by \(A_\mu = A_\mu^4\) could be embedded in a non-Abelian group. In fact, as we shall argue below, interesting physics already occurs in the case where our Abelian group is non compact. Some comments on the non-Abelian extension will be presented as outlook.

3 Electric charge assignment of Composite excitations

An important ingredient, which was so far has been left out from the above discussion, is the electric charge (not to be confused with the statistical charge quantum number discussed so far) assignment of the composite excitations discussed above. In view of the spin-charge separation that characterizes the ground state of the nodal liquid, this issue turns out to be quite crucial for the associated physics, and in particular the superconducting properties of the model.

At the constituent level, the electric charge assignment of the various excitations is well defined [11]: spinons, \(z_{1,2}\) are electrically neutral, while holons carry electric charge, which in terms of the (two-component) Nambu-Dirac spinors \(\psi_{1,2}\) can be given the following electric charge assignment:

\[
\psi_1 = \begin{pmatrix} \hat{\psi}_1^\dagger \cr -\hat{\psi}_2 \end{pmatrix} \rightarrow \text{electric charge} \ - e
\]

\[
\psi_2 = \begin{pmatrix} \hat{\psi}_2^\dagger \cr \hat{\psi}_1 \end{pmatrix} \rightarrow \text{electric charge} \ + e,
\]

where \(\hat{\psi}_i, i = 1, 2\) are Grassmann numbers, denoting annihilation operators of a hole with non zero electric charge. Notice that the above charge assignment are consistent with the microscopic structure of the nambu-Dirac spinors in terms of this Grassmann fields.

In view of this assignment we are now in position to discuss the electric charge content of the composite operators of the previous section. We commence our discussion from the quadratic parts of these operators.

First, let us consider the matter parts \(\phi, \psi\). From (8) and (20), and recalling the electrical neutrality of \(z_{1,2}\) spinon fields, we do observe that \(\phi\) has electric charge \(+2e\), while \(\psi\) has electric charge \(+e\).

In a similar spirit, the gauge field \(A_\mu = A_\mu^4\) (5), is electrically neutral, and so is \(A_\mu^3\) in (4). On the other hand, we remark that the vector fields \(A_{\mu}^{1,2}\) would not have definite electric charge. This is one of the reasons why we form the supersymmetric composite
theory in terms of the $A_\mu^4$ field. The bosonic partner fields $\rho = \phi^3$ and $\phi^4$ are electrically neutral.

The reason why we do not choose the vector $A_\mu^3$ rests with the requirement of a definite electric charge for the gaugino of the N=2 multiplet $\lambda$ (18). First of all we observe that, as it stands, the gaugino $\lambda$ does not have a definite electric charge, unless the auxiliary fields of the constituent scalar multiplets

$$f_1 = f_2 = 0 .$$

This is a question that depends on the details of the constituent dynamics. For instance, (21) characterises the Wess-Zumino model, and some supersymmetric gauge theories. However, in the case of the supersymmetric constituent model of [11], the dynamics is that of a $CP^1$ $\sigma$-model, which has the well known constraints $\sum_{a=1}^2 Z_a Z_a = 1$ in superfield language. Such constraints imply the on-shell condition $f_1 = f_2 = \bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2 = \phi^4$. In our theory we may discard the parity violating condensate $\phi^4$, and hence impose the constraint that on physical states (21) holds, implying that the physical gaugino excitation $\lambda$ (18) would have definite electric charge $+e$.

Notice that the particular construction of the gaugino by combining the superpartners of the (parity conserving) excitation $\rho = \phi^3$ and the (parity violating) vector field $A_\mu^4 = A_\mu^4$ is the only one that leads to a $\lambda$ field that has definite electric charge. Indeed, if the corresponding gaugino $\lambda$ were constructed as in (18) but from the superpartners of $\rho = \phi^3$ and $A_\mu^4$, it would not have a well defined electric charge. Notice that the choice of $A_\mu^3$ in the construction of the gaugino would necessitate the involvement of the parity violating $\phi^4$ condensate, which could not acquire a Vacuum Expectation Value according to the Vafa-Witten theorem [20]. Hence this case is also excluded, leaving the current scenario as a unique one with the desired physical properties to be discussed in later sections.
We next come to the quartic parts of the composite operators. With the above electric-charge assignments, we do observe that $\Lambda (16),(17)$ has a definite electric charge $+e$ provided that the quadratic composite auxiliary field $f$ vanishes:

$$f = 0$$

(22)

There is a small but important point regarding the the electric charge of the scalar $M (15)$. In our N=1 construction, due to the $\bar{\psi} \chi$ term, the scalar $M$ seems not to have a definite charge. However, as remarked after Eq.(19), the elevation of the supersymmetry to the extended N=2 case, requires the relevant parameter $\varepsilon$ to become complex. This allows an analytic continuation of the real spinor $\chi$ to a complex one, which may be chosen in such a way that it has a (global) electric charge $-e$ like. The correct assignment then of electric charges (associated with the definition of the proper global rotations of spinors) is such that in the expression for $M (15) \chi \rightarrow \chi^*$, while the expression for $\Lambda^{(3)}$ stays as in (17). It turns out that this assignment is consistent with N=2 supersymmetry, and also with the conservation of electric charge in the various terms of the Lagrangian that describes the relevant dynamics, to be discussed in the next section.

With this assignment, all the quartic parts of the scalar have charge $+2e$ in accordance with the quadratic parts, which allows the assignment of charge $+2e$ to the scalar $\Phi$ of the matter multiplet. Notice the different electric charge assignments between the boson and fermions of the matter supermultiplet, which is in accordance with the expected explicit breaking of the supersymmetry upon coupling to real electromagnetism.

A summary of the electric charge assignments of the various composite excitations of the supersymmetric model is given in figure 1.

4 Application to Strongly Correlated Electrons

The Effective Continuum Lagrangian

In this section we shall consider the physical application of the above consideration in the phase diagram of the nodal liquid. First of all we remind the reader that the above theory assumed the existence of nodes in the Fermi surface of the system, and concentrated in linearising around such nodes. In terms of realistic situation, with possible relevance to high temperature superconductors, we are working on a temperature-doping diagram on regimes where such nodes are present. This is the case of low doping (underdoped cuprates), where the fermi surface consists of four nodes (as in figure 3), or in the optimal doping superconducting case, where the $d$-wave superconducting gap (which is an experimental fact) has nodes as well.

Linearising the lattice theory of the constituent excitations (spinons and holons) about such nodes implies the usual lattice doubling, that is the presence in the continuum relativistic theory of four component constituent spinors (holons). There are different ways of obtaining the continuum limit, all of which may not be physically equivalent. Let us
Figure 2: An antiferromagnetic sublattice in the scenario of [18]; sublattice hopping (dashed, or dashed-dotted lines) defines effectively two kinds of holons, with opposite statistical gauge couplings, which effective hop within a single sublattice, without direct intersublattice hopping; mixing of the sublattices is only allowed via gauge field frustration, described in [18]. The holon excitations can be taken to be linear excitations of a single nodal point in the corresponding $d$-wave fermi surface of the high temperature superconductor. However, this construction is not the appropriate model to be considered in the non-Abelian spin-charge separation ansatz (25) of [16, 11], adopted here, where direct intersublattice hopping is allowed. It is only mentioned here for instructive purposes in order to understand which way the continuum limit should be taken in our case.

briefly describe them, which will be instructive in deciding on the appropriate one to be adopted in our case here.

One way of obtaining the continuum limit at a constituent level is outlined in [24, 18], and is appropriate for considering excitations about a single node of the fermi surface. Consider a planar lattice (lying, say, on the spatial $x-y$ plane), where the original microscopic antiferromagnetic model lives on. In the model of [18] direct intersublattice hopping is forbidden, and the holons are confined within each sublattice, thereby defining two species ('colors') of holons. This model is appropriate for the Abelian spin-charge separation ansatz where an electronic degree of freedom in each sublattice is split into $c_{i,\alpha} = z_{i,\alpha} \psi_i^\dagger$, where $\alpha = 1, 2$, and $\psi$ are spinless Grassmann numbers (holons), while $z_\alpha$ (magnons) are boson doublets [18]. It is only $\psi$ that carries a sublattice ‘color’ index.

Each sublattice is divided into two sublattices according to the hopping of holons (see figure 2). Then an arbitrary sublattice point, corresponding to the position vector $\vec{i}$ reads $\vec{i} = n_x \hat{a}_x + n_y \hat{a}_y$, where $n_{x,y}$ are positive integers (including zero), and $\hat{a}_{x,y}$ are orthogonal unit vectors of the sublattice. We divide spinor components on each sublattice into four kinds, according to the parity of the integers $n_x$ and $n_y$ as follows:

$$
\psi_1 = (\text{even, even}) , \quad \psi_2 = (\text{even, odd}) , \quad \psi_3 = (\text{odd, even}) , \quad \psi_4 = (\text{odd, odd}) ,
$$

(23)

where $\psi$ are Grassmann numbers, and one constructs four-component spinor (of each
The linearisation about a single node on the fermi surface, then, is done as in [18] by considering the Fourier components of the above spinors; in this way, as one takes the continuum limit, by superimposing the two sublattices, the Lattice doubling yields four component Dirac spinors, around each node, and the sublattice structure provides two colours of such spinors. The Dirac nature is obtained by assuming a flux $\pi$ statistical gauge background per sublattice plaquette [18]. In two component notation, then, one obtains two ‘colours’ and two flavours of four component Dirac spinors (electrically charged).

The above construction is not appropriate, however, for our considerations here, which are concerned with the non-Abelian spin-charge separation ansatz of [16, 11] defined on a planar antiferromagnetic lattice but allowing intersublattice hopping. The ansatz was introduced for a particle-hole symmetric formulation away from half-filling and reads:

$$\chi^{\alpha\beta} \equiv \begin{pmatrix} \psi_1 & \psi_2 \\ -\psi_2^\dagger & \psi_1^\dagger \end{pmatrix}_i \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix}_i$$

(25)

where the fields $z_{\alpha,i}$ obey canonical bosonic commutation relations, and are associated with the spin degrees of freedom (‘spinons’), whilst the fields $\psi$ are Grassmann variables on the lattice, which obey fermi statistics, and are associated with the electric charge degrees of freedom (‘holons’). The ansatz admits hidden non-Abelian local $SU(2)$ spin symmetries at the constituent level, as discussed in our previous works [16, 11, 25]. The important point to notice is that now the holon degrees of freedom do carry a spin index. As shown in detail in [11], from this ansatz one can construct the two Dirac spinors (20), which play the role of the effective electrically-charged degrees of freedom for the problem, with the electric charge assignment of the previous section.

An important question arises at this point as to how one should take the continuum limit of this non-Abelian spin-charge separation theory. We propose to follow the procedure of [26] appropriate for BCS superconductivity, despite the fact that our proposed mechanism for superconductivity, as we shall see below, will not be the conventional BCS, but that of [18]. In [26] the four component spinors of the continuum theory are obtained by combining nodes along the diagonal as indicated in figure 3. This is dictated by the starting point which is a BCS-like pairing interaction (at finite temperature $T \neq 0$) among BCS-like quasiparticle excitations near nodes at opposite ends of the diagonals of figure 3 [26]:

$$S = T \sum_{\vec{k}, \alpha, \omega_n} \left[ (i\omega_n - \xi_{\vec{k}}) c_{\alpha}^\dagger(\vec{k}, \omega_n) c_{\alpha}(\vec{k}, \omega_n) - \frac{\alpha}{2} \Delta(\vec{k}) c_{\alpha}^\dagger(\vec{k}, \omega_n) c_{-\alpha}^\dagger(-\vec{k}, -\omega_n) + h.c. + \ldots \right]$$

(26)

in a standard notation, with $\alpha = 1, 2$ denoting the spin up and down states of the third component of the electron spin, $\omega_n$ being the fermionic Matsubara frequencies, $\Delta(\vec{k})$ being
Figure 3: The fermi surface of underdoped cuprates consists of just four nodes. The dashed line is the putative fermi surface. The continuum effective theory may be obtained by linearisation about such nodes, which leads, at the constituent level, to two flavours of four-component Dirac spinors for the holon degrees of freedom, each flavour being obtained by combining the nodes along the diagonal as indicated in the figure (which is a standard procedure in deriving the continuum limit of lattice gauge theories in particle physics).

the $d$-wave gap, and the ... indicating other possible short range interactions between quasiparticles. In the approach of [26], where spin-charge separation is not considered explicitly, $c_\alpha, c_\alpha^\dagger, \alpha = 1, 2$ are electron operators, while in our approach here and in [11], $c_\alpha, c_\alpha^\dagger$ should be replaced by the corresponding holon operators/Grassmann numbers $\psi_\alpha, \psi_\alpha^\dagger$, $\alpha = 1, 2$ appearing in (20). It must be noted that this way of combining the four nodes in order to form a continuum theory from a lattice one is also the standard procedure of taking the continuum limit in lattice particle theory models.

If one adopts the starting point (26) for the constituent theory, then there are two ways of constructing the four-component continuum constituent spinors, corresponding to holons in our case, as discussed in [26] ², leading possibly to different physics as far as the

²However, it should be stressed that the associated physics in those works is entirely different from that of our model. In contrast to our case, in [26] the authors do not consider direct spin-charge separation. Moreover their effective spinons are viewed as electrically neutral fermions, while the holons are electrically charged bosons. Of course this by itself would not lead to physical differences, given that there is a physical equivalence between the two formalisms in view of bosonisation [9]. But the important difference is the fact that these workers use pure QED3 to describe frustration of holes. In [18], and in all of our works so far on the subject, and here, the important point is that we use QED coupled with opposite statistical charge to the two species. In [18] we have discussed the difference if one used pure QED as in [26]. In that case the anomalous graph would imply chiral symmetry breaking. It is the opposite-coupling QED model of [18], which implies unconventional superconductivity. In contrast, in [26], as far as the proposed mechanisms for superconductivity are concerned, the authors assume that the superconducting state, apart from being
properties of the insulating phase are concerned (to which we are not directly concerned in this article). One (Herbut in [26]) is to define the four-component spinors in each pair of nodes as:

$$\Psi(\vec{q},\omega_n)_{i=1,2} = \left( c_1^\dagger(\vec{k},\omega_n), c_2(-\vec{k},-\omega_n), c_1^\dagger(\vec{k}-\vec{Q}_i,\omega_n), c_2(-\vec{k}+\vec{Q}_i,-\omega_n) \right) ,$$

(27)

where $\vec{Q}_i = 2\vec{K}_i$ is the wave vector connecting the nodes within the diagonal pair $i = 1, 2$ (c.f. figure 3), and $\vec{q}$ is measured from the (putative) fermi surface $\vec{k} = \vec{K}_i + \vec{q}$, $|\vec{q}| \ll |\vec{K}_i|$ for the pair $i = 1, 2$. In our case, of course, the electron operators should be replaced by the Grassmann number operators $\psi_{1,2}$, defined in (20), as mentioned previously.

The alternative way of forming the four-component continuum spinors, within the context of BCS-like quasiparticle pairing interaction (26) is described in Balents et al. in [26], where the four-component spinors are constructed as:

$$\Psi(\vec{q},\omega_n)_{i=1,2} = \left( c_1^\dagger(\vec{k},\omega_n), c_2(-\vec{k},-\omega_n), c_2^\dagger(\vec{k},\omega_n), -c_1(-\vec{k},-\omega_n) \right) ,$$

(28)

again for each pair of nodes $i = 1, 2$. Again, within our spin-charge separation framework, the electron operators should be understood as representing the Grassmann number fields $\psi_{1,2}$.

The main difference between the two approaches lies in the ability to describe properly the insulating phase, which is of no concern to us here. Formally, the $\gamma$-matrix and the emergent gauge symmetry structure of the second approach is that of spin rotations. It is this approach which we adopted in our formalism in [11], and will be adopted here as well. As can be readily seen from (20),(28), in our case the construction of four-component spinors is obtained by combining appropriately the two component Dirac spinors (20) at the nodes along the diagonals of figure 3. As mentioned already, the crucial physical difference of our analysis from that in [26] is that here the four component spinors are electrically charged (as in (20)) and represent holon and not electron degrees of freedom. The Dirac nature of the corresponding constituent spinor lagrangian, is again obtained upon coupling the constituent holons to a background statistical gauge field with flux $\pi$ per lattice plaquette.

Before proceeding further an important comment is in order. In [11] one could have taken the four component continuum limit spinors around a given node by simply combining the two two-component spinors in (20) into a single four component one, with simply taking the Fourier components of the corresponding Grassmann quantities. In this case one can obtain a perfectly well-defined continuum supersymmetric gauge theory at a constituent level in certain regime of parameters of the microscopic model, as discussed in [11], around each node. The construction will then lead to four replicas of such theory for the four nodes. At a composite level, however, such a construction would not yield the second species of chiral matter multiplets (c.f. $Q_{1,2}$ in a superfield notation below and in Appendix), which one needs in order to construct a parity-invariant effective composite a $d$-wave, exhibit otherwise the standard BCS phenomenology.
theory. This opportunity is provided to us by the above-mentioned construction of the continuum effective theory, adopted here, which combines the pair of nodes along the diagonal of figure 3.

At the constituent level, therefore, we form two four component continuum spinors by combining lattice fermionic composite excitations near nodes lying along the diagonal of figure 3. The important point to notice is that each diagonal pair of nodes yields two two-component complex Dirac spinors in the continuum and thus four constituent fermionic degrees of freedom. For our purposes here, the composite supermultiplets will then be constructed in the continuum limit out of the continuum constituent degrees of freedom available to us. Each pair of nodes then implies one composite supermultiplet at supersymmetric points, since the latter is made out of four constituent fermionic degrees of freedom. We thus finally obtain two composite matter supermultiplets $Q_1, Q_2$ in superfield notation (c.f. Appendix and below). It is important to notice that in our construction we consider the composite gauge field excitation, $A^4_{\mu}$, and its partner under supersymmetry (gaugino), as expressing interactions (frustration) among those pairs of nodes. Note that the complex gaugino of the N=2 gauge supermultiplet keeps track of the correct doubling of degrees of freedom in the continuum limit, thereby providing another physical reason for the physical relevance of the N=2 supersymmetry, in addition to the electric charge assignments discussed previously.

A question arises at this point as to the relative sign of the statistical gauge coupling among the two pairs of the diagonal nodes. As a result of energetic arguments in favour of a parity conserving ground state [20], we observe that at the supersymmetric points parity conservation actually necessitates opposite couplings with the statistical gauge field between the two pairs of nodes of figure 3. This is due to the Yukawa type coupling of the gaugino with the members (fermion and boson) of the matter multiplet, $\Phi^* \overline{\lambda} \Psi$, which would be otherwise parity violating as becomes clear from the relevant Lagrangian terms shown below. Notice that such Yukawa type terms are not present in non supersymmetric theories, and hence the arguments on opposite couplings among the diagonal nodes is an exclusive feature of the existence of supersymmetric points.

The resulting dynamics at the supersymmetric points of the parameter space of the spinon-holon composite system may then be summarized by that of the N=2 supersymmetric Abelian-Higgs model or, N=2 Supersymmetric Quantum Electrodynamics (SQED), whose properties, including the various exact results on its phases, are reviewed in the Appendix. For our purposes in this section we give the on-shell supersymmetric Lagrangian describing the dynamics of the nodal liquid at supersymmetric points:

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3Caution is expressed at this point, though, regarding the lattice composite theory where the Vafa-Witten theorem may actually fail [16] due to the existence of lattice operators that violate parity (proportional to the lattice spacing, and thus corresponding in the continuum limit to operators with higher derivatives than the ones considered here). The issue is whether such lattice parity-violating operators can be relevant in a renormalization group sense. From a naive point of view, since such operators correspond to higher derivative operators in the continuum, they are expected not to affect the infrared universality class of the model. But this expectation may be naive, and a precise renormalization group analysis needs to be performed. Such issues will not concern us here.
Figure 4: The current-current diagram which leads to massless pole and superconductivity, via the anomaly mechanism of ref. [18]. The wavy lines represent the statistical photon (from the N=2 vector multiplet of SQED), and the continuous lines are fermions (from the chiral matter multiplet). Each blob at the two fermion loops indicate an insertion of the fermion current operator with respect to the fermions in the chiral matter multiplet of SQED.

\[ \mathcal{L}_{\text{on-shell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{i}{2} \bar{\lambda} \partial \lambda \]

\[ + \frac{1}{2} D_\mu \Phi_1 (D^\mu \Phi_1)^* + \frac{1}{2} D_\mu \Phi_2 (D^\mu \Phi_2)^* + \frac{i}{2} \bar{\Psi}_1 \partial \Psi_1 + \frac{i}{2} \bar{\Psi}_2 \partial \Psi_2 \]

\[ + g \frac{i}{2} (\Phi_1^* \bar{\lambda}^* \Psi_1 - \Phi_2^* \bar{\lambda}^* \Psi_2 - \text{c.c.}) - \frac{g}{2} \rho (\bar{\Psi}_1 \Psi_1 - \bar{\Psi}_2 \Psi_2) - U(\rho, \Phi_1, \Phi_2), \]

where \((\Phi_1, \Psi_1), (\Phi_2, \Psi_2)\) are the two sets of superpartners (corresponding to chiral matter superfields \(Q_{1,2}\) c.f. Appendix), associated with the two pairs of nodes of figure 3, and

\[ D_\mu(...)_1 = \partial_\mu(...)_1 + ig A_\mu(...)_1, \]

\[ D_\mu(...)_2 = \partial_\mu(...)_2 - ig A_\mu(...)_2, \]

\[ U(\rho, \Phi_1, \Phi_2) = \frac{g^2}{2} \rho^2 (|\Phi_1|^2 + |\Phi_2|^2) + \frac{g^2}{8} (|\Phi_1|^2 - |\Phi_2|^2)^2. \]

From a physical point of view it must be stressed that the gauge field here is not the real electromagnetic field, but the statistical one, associated with spin and hole frustrations. We shall consider the coupling of the external electromagnetic field later on.

The diagram of figure 4 is the Landau criterion for superconductivity, the electric current-current correlator proceeding via the anomaly mechanism of [18]. In view of the SQED lagrangian (46), which describes our on shell continuum effective dynamics of the nodal liquid at supersymmetric points, the statistical photon \(A_\mu\) can couple to the electrically-charged fermions of the chiral multiplet.

We should stress at this point that all the considerations in this section will pertain to the continuum composite theory. On the lattice the composite theory may differ from its
continuum counterpart by various field operators. Our conjecture is that the two theories belong to the same renormalization group universality class, in other words the operators by which they may differ are irrelevant in a renormalization-group sense. This remains to be shown rigorously by performing a detailed renormalization group approach to the continuum limit, which however lies beyond the scope of the present article.

**Coulomb Phase implies Superconductivity**

The Coulomb phase of the $N=2$ SQED is characterized by a non zero $<\rho>$, acting as a mass term for the fermions $\psi$ (fermion mass gap). From the point of view of the constituent theory this is the phase where the original holons have a parity invariant condensate $\phi^3 = \overline{\psi}_1\psi_1 - \overline{\psi}_2\psi_2$ which, for instance, can be generated by the strongly coupled $U(1)$ of the constituent theory [16] that lead to the composite picture. In this massive phase, the fermions lead to a non zero value of the anomalous one-loop graphs of fig. 4, in the way explained in [18]. If the photon is massless this leads to a massless pole in the zero-temperature correlator and hence to superconductivity. The presence of monopole plasma phases destroys this pole, but this monopole phase is absent in the non compact Abelian Higgs model discussed here. Hence we can safely identify the Coulomb phase of the $N=2$ SQED with the superconducting phase of the nodal liquid.

The local order parameter $<\rho> 
eq 0$ is electrically neutral, but as discussed in [18] there is electrical pairing, which manifests itself upon coupling the theory to an external (true) electromagnetic field. The pairing manifests itself through the so-called mixed Chern-Simons term of the effective theory obtained after integrating out the (massive) fermionic degrees of freedom in the parity-invariant theory:

$$S_{\text{eff}} \ni \frac{i e}{\pi} \int d^3 x \epsilon^{\mu\nu\rho} A_\rho \partial_\nu A_\mu^\text{ext}$$  \hspace{1cm} (31)

where $A_\mu$ is the statistical gauge field, and $A_\mu^\text{ext}$ is the externally applied electromagnetic potential, due to the electric charge of the holons.

Consider the case of a compact statistical gauge field, which undergoes large gauge transformations. The situation at any finite (no matter how small) temperature $\beta^{-1}$ (so that the time direction is compact) is then the following: the large gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Omega$, with $\Omega(\beta) = \Omega(0) + 2n\pi$, $n$ being the winding number of the topological class in which the gauge transformation is, and $\Omega$ constant on any two dimensional space, one obtains for a static external field [18] the following change of the mixed Chern-Simons term (31):

$$\delta S_{\text{eff}} = 2i e \int \text{space} \, d^2 x \epsilon_{ij} \partial_i A_j^\text{ext} = 2i e \mathcal{F},$$  \hspace{1cm} (32)

where $\mathcal{F}$ is the magnetic flux. Consider now a superconducting annulus (Corbino disc geometry), as standard for a pairing demonstration in superconductors. Due to the Meissner effect, which has been rigorously demonstrated for this case in [18], the total magnetic flux
$\mathcal{F}$ through a region bounded by a closed path winding once $(n = 1)$ round the origin in the interior of the sample, must then be quantized as [18] (Plank’s constant $\hbar$ and the speed of light $c$ are restored here)

$$\mathcal{F} = n \frac{\hbar c}{2e}, \quad n' = \text{integer}$$

indicating pairing $+2e$, as observed experimentally. Clearly the respective pairs could be formed out of condensates of fermions $\Psi$, or holons alone, which have such a charge $+2e$. In terms of four component spinors a pairing condensate could have been $\langle \Psi \gamma_5 \Psi \rangle$, where $\gamma_5$ is a $4 \times 4$ chirality matrix that belongs to a reducible representation of the Dirac algebra in (2+1)-dimensional theory, being appropriate only for theories with even number of continuous two-component spinors. This is always the case of lattice models due to doubling.

There is an issue as to the validity of having such condensates still compatible with supersymmetry. We think that a way out of this issue is the fact that the pairing mechanism described above pertains strictly only in the presence of an external electromagnetic field, which breaks supersymmetry explicitly. It should also be born in mind that such condensates would break the statistical gauge invariance as well (spontaneously).

We also stress that such a pairing in the superconducting phase occurs in our model only in the compact SQED case, in which, as we shall discuss below, and reviewed in the Appendix, there are monopole (singular) configurations of the statistical gauge field $A_\mu$. This is a unique feature of our model which differentiates it from other approaches in the literature, such as that of [26] etc. The important point with the superconductivity scenario of [18] is that the relevant order parameter $\langle \rho \rangle$ does not break any symmetries of the original lagrangian, and hence, at least to all orders in perturbation theory, the superconductivity should be considered of Kosterlitz-Thouless type, where strong phase fluctuations suppress the coherence of its phase. This is an important fact of the nodal liquid superconducting phase, and this lack of phase coherence is crucial in interpreting (as we shall see later on) the monopole plasma phase of the Coulomb branch in the case of compact SQED as corresponding to a pseudogap phase.

The above scenario for superconductivity occur strictly speaking for zero temperature. At any finite temperature, there is a plasmon thermal mass for the longitudinal component of the gauge field, and moreover the supersymmetry is explicitly broken. However, upon coupling the system to an external magnetic field there is still screening of the magnetic field lines up to a given critical temperature, and hence one can still speak about superconducting properties, as commented upon in [18].

**Higgs Phase and Pseudogap**

In the Higgs phase the gauge field is massive, and hence there is no pole in the graph of fig. 4, and hence no superconductivity or phase coherence. This is a phase, however, which is characterised by the presence of parity conserving condensates of the field $\phi$ which have
electric charge 2e. Since $\langle \rho \rangle = 0$ in this phase the condensates are massless (gapless), however their presence implies pairing of holons (charge +2e). Again, in view of the fact that (in the absence of an external electromagnetic field) the condensates do not break any of the original symmetries of the Lagrangian, probably implies a Kosterlitz-Thouless like non-superconducting pairing.

We therefore propose to identify this phase with a pseudogap phase for the nodal liquid. Since in this phase there are no monopoles of the statistical gauge field there will be no stripes here, so this phase would be a non-striped regime of the pseudogap phase.

**Compact SQED and Stripe Phase**

We now turn to a detailed study of the case where the gauge group is compact, as seem to be necessitated by electric pairing arguments in the superconducting case, mentioned above. In the compact case, as discussed in [8], and reviewed briefly in the appendix, one has monopole (singular) configurations of the gauge field $A_\mu = A_4^I$. From (5) we observe that this can indeed be the case if the constituent theory of spinon and holons, the $CP^1$ supersymmetric $\sigma$-model, lives in one of its non-trivial topological sectors. Singularities in the spinon and/or holon current may indeed exist, and such singularities would necessarily imply a compact gauge field. Compact gauge fields may also be expected in general for lattice gauge theories, which is the case of the microscopic theory under consideration here. As we have discussed previously such compact statistical gauge fields are necessary for electric flux quantization, characteristic of electric pairing in the superconducting phase.

From (5) and the connection of the $A_4^I$ field with that of a $CP^1$ $\sigma$-model like field at a constituent level, as discussed previously, we observe that the monopole configurations (singularities) could come from the $z$ (spinon) sector, in the way discussed in detail in [27]. The only extension in our case is the fact that there are monopole configurations in each species, i.e. in each pair of nodes obtained by the combination of figure 3. These monopole configuration are identical between the nodes by symmetry upon the exchange of nodes $^4$.

The monopoles could form a stable monopole phase, which is true in supersymmetric theories as well, with the exception of the SU(2) $N=2$ model of [17], to a discussion of which we shall come later. The presence of stable monopole phases contributes non-trivially to the generation of a superpotential for the theory $W$, whose form depends on the fugacities of the monopoles in the way explained in the Appendix. The monopoles form a plasma phase [28], and are responsible for resulting in a massive dual of the statistical photon, which will destroy the superconductivity due to the anomalous graphs of figure 4. This feature is also valid in the SQED case, except in the case where SQED is just embedded in a supersymmetric $N = 2$ SU(2) case, in which case there is no stable monopole phase [6, 7, 8]. This is discussed briefly in the Appendix.

$^4$In fact it is this symmetry argument that cancels out any trace of such monopole singularities in the parity-conserving vector composite $A_4^I$ (4), thereby singling out the $A_4^I$ as the appropriate compact gauge field of this framework.
Another way of seeing the incompatibility with the stable monopole phase may be inferred from the absence of any flux quantization as follows [29]: when consider the quantum fluctuations of isolated monopole configurations, as is the case of a stable monopole plasma phase, it is necessary to abandon in a path integral fixed boundary conditions for the large gauge transformations and instead consider free ones. This is essential for the mathematical consistency of the theory, as argued in [30]. Free boundary conditions imply that in a path integral one has to integrate over all possible phases $\Omega$ which appear in the change of the mixed Chern-Simons term (31) arising in the effective theory of the putative superconducting phase after integrating out massive (fermionic (holon) and bosonic (spinon)) degrees of freedom [18] (c.f. discussion in previous subsection). If we start from the classical theory, then as discussed above, this would imply quantization of the electromagnetic flux which would lead to pairing, $\Phi = m\pi/2e$, $m$ an integer. When quantum fluctuations of the monopole are added then one has to consider the path integral over the phases $\Omega$:

$$\int_0^{2\pi} d\Omega e^{im\Omega} = \delta_{m0}$$

which implies that any quantization of the electric flux is washed out, hence no flux quantization. This implies that the superconducting phase would correspond only to a phase where the monopoles are bound in pairs with their antimonopoles [29], which is in agreement with the Kosterlitz-Thouless nature of the superconductivity of the present composite model, as well as that of [18].

The stable monopole phase, however, has other attractive properties which are tempting us to identify with a non-superconducting stripe phase, and which we now proceed to discuss briefly. We shall restrict ourselves below to the effect of supersymmetric monopoles. For a possible connection of monopole and stripe phases in non-supersymmetric composite theories the reader is advised to see the analysis in ref. [25]. The gauge monopole phase is characterised by the existence of domain walls of a given flux, which, as discussed in [8, 6] and reviewed briefly in the Appendix, form groups of, say $p$ of them, separating $p$ different vacua of the theory, emanating from a heavy particle/monopole configuration of positive statistical charge, and/or ending in one with negative charge (see figure 5). For the record we mention that exact results as far as the structure of the domain walls is concerned have been derived only in the case where one assumes a bare mass $m_b$ for the matter superfields, which contributes to a bare superpotential term of the form $m_b Q_1 Q_2$. In such a case, the moduli space of SQED turns out to be qualitatively similar to that of pure compact $U(1)$ gauge theory, given that one may integrate out in a Wilsonian effective action the massive degrees of freedom, to obtain the effective theory of the massless one, whose spectrum is similar to pure $U(1)$ in the Coulomb phase. The superpotential, as mentioned in the Appendix, is protected by non-renormalization theorems [6], and hence the above result turns out to be exact.

In conformal SQED, however, which is characterised by the absence of a mass term for the matter multiplets, such a mass term is generated dynamically in the Coulomb phase of the theory, but such a mass term arises from the kinetic terms of the supermultiplet, and thus does not appear in the superpotential. In that case, one also expects the dynamical
Figure 5: Examples of domain wall structures for $p = 2$ in compact N=2 supersymmetric Abelian gauge theories: (a) a configuration in $p = 2$ SQED with massive matter, with a pair of heavy monopole charges, of statistical gauge charge +1 (dark blob) and -1 (white blob), (b) a configuration in pure U(1) N=2 supersymmetric gauge theory with two pairs of heavy monopole charges, of statistical gauge charge +1 and -1. The domain wall structures confine in their interior excitations of the statistical ‘electric’ field (arrows), and separate the two vacua of the theory labeled by the dual photon v.e.v. $\gamma = 0$ and $\gamma = \pi$. In the actual condensed matter situation, with the statistical gauge field being given by (5), there is a real electric current flow in the domain wall structures, which prompts one to identified them with the stripes observed in the stripe phase of the underdoped cuprates.

appearance of domain walls of similar nature (qualitatively) to that in figure 5, but unfortunately there are no exact results available to this case. In our composite theory a bare mass term for the composite matter multiplets is not in contradiction with the fact that we work here with nodal excitations. A composite mass term corresponds to contact quartic (or higher) interactions among the fundamental constituents, (spinon $z$ and holons $\psi_i$), and as such one may have it in the effective theory. To turn the logic around, even if such terms are not appearing in the microscopic theory, one may add them by hand in order to guarantee the exactness of the result, and then turn them off adiabatically claiming that qualitatively the information about the domain wall phase does not change.

Along each domain wall, separating two different vacua, there is an excitation of the flux of the gauge field. From a point of view of a phase (moduli space) diagram, the Coulomb branch of compact SQED is split into two, in one of which there is a stable monopole plasma phase, as discussed in the appendix, where such stripe phenomena occur. The presence of monopoles are responsible for giving a small but finite mass to the statistical gauge field $A_{\mu}^4$, and hence the superconductivity, in the sense of the satisfaction of the
Landau criterion of figure 4, is destroyed. This phase is therefore also a pseudogap phase, but there are no bosonic pair excitations as in the Higgs phase. There is also pairing, which is obtained by means of the mixed Chern-Simons terms in the effective lagrangian, upon coupling the theory to an external electromagnetic field. In [8] it was argued that the domain walls in the configuration of figure 5(a) attract each other, which may result in a bigger domain wall with twice the flux.

From our composite construction the gauge field (5) carries holon current excitations, and hence the interior of the stripes is characterised by real electric charge flow. In view of this one might be tempted to identify this monopole pseudogap phase with the stripe phase, in analogy with the non supersymmetric case of ref. [25]. The important advantage of the current model is that it is Abelian, and thus the results may provide exact (at least qualitatively) non perturbative information, in contrast to the non-Abelian non-supersymmetric model of [25], where one relies on perturbative arguments of weakly coupled gauge theories (another difference of the current work is that here the gauge fields are induced from fundamental constituents, and one has a detailed knowledge of their microscopic constituent structure in terms of spinon and holons, and hence one does not have to rely on mean-field arguments on their form, based upon phase fluctuations in the spin-charge separation ansatz, which was the case of the various constituent approaches so far).

**N=2 SU(2)-like supersymmetry and superconductivity**

Before closing we would like to make an important comment regarding the absence of the stable monopole plasma phase in N=2 supersymmetric SU(2) non-Abelian gauge model of [17]. As emphasized in [8], and reviewed in the Appendix, the SU(2) case is characterised only by monopoles of charge +1, and antimonopoles of charge -1, unlike SQED where both charges are present. In such non-Abelian theories, therefore, the stripe (domain wall) phase, due to monopole plasma, would be absent. On the other hand, as a result of the existence of still non-trivial configurations with charge +1, one might still consider large gauge temporal transformations in the respective effective action which would lead to pairing as in (33). Thus the existence of such N=2 SU(2)-type supersymmetric points would be equivalent to the onset of superconductivity in the model.

From a microscopic point of view one might think of the possibility of approaching such a N=2 supersymmetric SU(2) point by letting the fugacities $\tilde{h}$ of the -1 charge monopole excitations, appearing in the superpotential of the theory (c.f. Appendix for notation), depend on the doping concentration in the sample as: $\tilde{h} \sim (\delta - \delta_c)^\gamma$, where $\delta$ is the doping concentration, $\gamma > 0$ is some critical exponent, and $\delta_c$ is a critical value marking the onset of such ‘N=2 SU(2) supersymmetry-like’ regimes.

This could be a viable way of entering from a pseudogap (stripe) phase to superconductivity, which was the original way envisaged in [16, 11], and we still think describes (in some sense) the situation encountered in nature. In other words, the appearance of a N=2 SU(2)-like supersymmetric situation (corresponding to the vanishing of the fugacity of the charge +1 monopoles as one varies the doping concentration in the sample) would destabi-
lize the pseudogap stripe/monopole plasma phase in favour of the Kosteritz-Thouless type superconductivity scenario of [18], reviewed above.

Such a possibility should be explored further in terms of microscopic and/or composite models. At present it is not possible to construct explicitly a fully non-Abelian SU(2) gauge model of composite supersymmetric excitations, for reasons explained in the text above. Thus, at least at present, the N=2 “SU(2)-like” points have to be viewed only as points of SQED compact theories with zero fugacities for monopoles of charge -1, characterised by the absence of a stable monopole phase of domain wall type. However, the construction of a fully non-Abelian SU(2) composite theory may be a possibility, and it certainly constitutes an issue worthy of further exploration.

A Non-trivial Infrared Fixed Point and Non-fermi liquid Behaviour

A final comment we would like to make, concerning exact results in N=2 supersymmetric (2+1)-dimensional theories, is associated with the existence of a non-trivial (non Gaussian) infrared fixed point. In N=2 theories [6, 8] such a statement becomes qualitatively exact by the discovery of appropriate dual models, with which SQED lies in the same renormalization group universality class in the infrared. This dual model is the so-called XYZ supersymmetric model, with a cubic (in superfields X,Y,Z) superpotential, mentioned briefly in the appendix.

Such a property implies that the nodal supersymmetric liquid exhibits a non-fermi liquid behaviour, if one defines the deviation from the fermi liquid behaviour as being essentially equivalent to the absence of a Landau (Gaussian) fixed point at low energies. Arguments why this is so are given in [31].

Our N=2 supersymmetric model is similar in this respect to N=1 or N=0 (non supersymmetric) models of three dimensional Abelian gauge theories, for which it has been argued that there is a non-trivial infrared structure [32, 33]. The important difference on the N=2 case is, however, that the result is exact, in contrast to these other cases, where the result was argued on the basis of approximate resummation Schwinger-Dyson techniques, such as large N-flavour number of fermions [32], or the so-called pinched technique using appropriately resummed amputated fermion-gauge-boson vertices [33].

5 Discussion and Outlook

In this article we have discussed a way of obtaining (in the continuum limit) a N=2 supersymmetric effective lagrangian, describing the dynamics of d-wave nodal composite excitations made out of constituent spinons and holons within a spin-charge separation framework in a certain regime of the parameter space of extended t – j models of doped antiferromagnets [11]. We have paid particular attention to discussing how the continuum limit of the constituent microscopic theory is taken, and how, once this is done, the composite supersymmetric structure emerges. Of particular importance was the existence of an even number of pairs of fermi-surface-d-wave nodes, which implies the emergence of an
even number of composite chiral matter supermultiplets $Q_{1,2}$ in the effective (continuum) nodal composite lagrangian, and thus the possibility of parity conserving effective theories. The (statistical) gauge composite supermultiplet, on the other hand, whose existence was necessitated by supersymmetry [13, 11] expresses frustration of (interaction between) those pairs of nodes. The nodal pairs couple with opposite statistical gauge charge for energetic reasons (parity conservation).

Although our composite construction has taken place in the continuum limit of the constituent theory, nevertheless our hope is that this supersymmetric continuum theory belongs to the same universality class as the corresponding composite lattice model. In other words, the two theories may differ only by a number of irrelevant operators in a renormalization group sense. This, however, remains to be demonstrated, and we hope to come back to such issues in future works.

Basing our considerations on this naive continuum limit we have demonstrated the existence of several exact results concerning the phase structure of the composite continuous theory, including a passage from the pseudogap to an unconventional superconducting pairing state in the spirit of [18], as well as the existence of a stripe phase, and the non-fermi liquid behaviour of the relativistic nodal composite liquid. As regards the latter property, it should be mentioned that this was the result of the existence of a non-trivial infrared (low energy) fixed point in the renormalization group analysis of (2+1)-dimensional N=2 supersymmetric gauge theories under consideration [6, 7].

As a possible outlook of the current approach we remark on the possibility of having Galilean supersymmetry for the theory away from the nodes, in the same regime of the parameter space of the microscopic model which leads to the Lorentzian nodal supersymmetry. Works with Galilean supersymmetry do exist in the field theoretic literature in (2+1)-dimensions [34], but at present we are far from translating such field-theoretic results to excitations pertaining to realistic condensed-matter situations, describing the physics away from nodes in the fermi surface of $d$-wave antiferromagnetic superconductors. We intend to embark on a study of such important issues in the near future.

We do believe that the present work, together with our previous works on this topic [11, 13, 12, 25], opens up the way for a possible formal understanding of the phase diagram (at least at zero or very low temperatures) of doped antiferromagnets, and hence high temperature superconductors, in an analytically exact way, at least in a specific regime of the parameters of the relevant microscopic lattice models, where extended N=2 supersymmetries between spinon and holon degrees of freedom in a spin-charge separation framework do exist. In this way one may hope to extrapolate such exact results in a qualitative manner away from such parameter regions, where the supersymmetries are explicitly broken. At present we do not know precisely how to do such an extrapolation, but we are strongly encouraged by the current results so as to continue pursuing this research further.
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Appendix

N=2 Supersymmetric Quantum Electrodynamics (SQED): A review of the formalism

A $N = 2$ supersymmetric theory in 2+1 dimensions can be obtained by dimensional reduction of a $N = 1$ supersymmetric theory in 3+1 dimensions, in which the two two-component Grassmann coordinates of superspace lead to one complex two-component Grassmann coordinate $\theta$. We have then the following properties:

$$\bar{\theta} \theta = (\bar{\theta} \theta)^*$$
$$\bar{\theta} \gamma^\mu \theta = (\bar{\theta} \gamma^\mu \theta)^*. \quad (35)$$

The Abelian vector supermultiplet contains the following degrees of freedom: a gauge field $A_\mu$, a complex gaugino $\lambda$ and a real scalar $\rho$. The (real) superfield $V$ corresponding to the vector supermultiplet contains in addition an auxiliary field $D$, which is essential for the generation of a scalar potential, as will be seen later on. $V$ has the following expansion in the Grassmann variables $\theta$ and $\bar{\theta}$:

$$V = \bar{\theta} \theta \rho + (\bar{\theta} \gamma^\mu \theta) A_\mu + i \bar{\theta} \theta^* (\bar{\theta} \gamma^\lambda \lambda) - i \bar{\theta} \theta (\bar{\theta} \gamma^\lambda \lambda^*) + \frac{1}{2} |\bar{\theta} \theta|^2 D. \quad (36)$$

The gauge kinetic term is given by the square of the linear superfield $\Sigma = \bar{D}D \bar{V}$ [8], where the supercovariant derivative is [35]

$$D^\alpha = \frac{\partial}{\partial \theta^\alpha} + i (\bar{\theta} \phi)^\alpha. \quad (37)$$

We have then for the gauge multiplet kinetic term

$$\mathcal{L}_{\text{gauge}} = \int d^2 \theta d^2 \bar{\theta} \left( \frac{1}{4} \Sigma^2 \right)$$
$$= \frac{1}{4g^2} \left( -F_{\mu\nu} F_{\mu\nu} - 2 \rho \Box \rho + i (\bar{\lambda} \partial \phi \lambda + (\bar{\lambda} \partial \phi \lambda)^*) + 2D^2 \right)$$
$$= -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g^2} \partial_\mu \rho \partial^\mu \rho + \frac{i}{g^2} \bar{\lambda} \partial \phi \lambda + \frac{1}{2g^2} D^2 + \text{surface terms}, \quad (37)$$

where the surface terms do not contribute after integration over space-time.
We describe matter with a chiral superfield \( Q \) which contains a scalar \( \Phi \), its superpartner \( \Psi \) and an auxiliary field \( F \), all complex, and its expansion in \( \theta \) and \( \bar{\theta} \) is

\[
Q = \Phi + \bar{\theta} \Psi + \frac{1}{2} \theta^2 F + \frac{i}{2} (\bar{\theta} \gamma^\mu \theta) \partial_\mu \Phi + \frac{i}{2} \bar{\theta} \gamma^\mu \theta \partial_\mu \Psi - \frac{1}{8} |\theta \bar{\theta}|^2 \Box \Phi.
\]  

(38)

The matter kinetic term including its coupling to the vector multiplet is given by

\[
\mathcal{L}_{\text{kin. matter}} = \int d^2 \theta d^2 \bar{\theta} \, Q e^{2gV} Q^* \tag{39}
\]

\[
= \frac{1}{4} \partial_\mu \Phi \partial^\mu \Phi^* - \frac{1}{8} (\Phi \Box \Phi^* + \Phi^* \Box \Phi) + \frac{i}{4} (\bar{\Psi} \partial \Psi + (\bar{\Psi} \partial \Psi)^*)
\]

\[
+ \frac{i}{2} g (\Phi^* A_\mu \partial^\mu \Phi + \Phi A_\mu \partial^\mu \Phi^*) + \frac{g^2}{2} A_\mu A^\mu \Phi \Phi^* - g \bar{\Psi} A \Psi
\]

\[
+ g \frac{i}{2} (\Phi^* \bar{\lambda} \Psi - \Phi \bar{\lambda} \Psi) - g^2 \rho^2 \Phi \Phi^* - g \rho \bar{\Psi} \bar{\Psi} + \frac{1}{2} FF^* + \frac{g}{2} D \Phi \Phi^*
\]

\[
= \frac{1}{2} D_\mu (D^\mu \phi)^* + \frac{i}{2} \bar{\Psi} D \Psi + \frac{i}{2} g (\Phi^* \bar{\lambda} \Psi - \Phi \bar{\lambda} \Psi^*)
\]

\[
- \frac{g^2}{2} \rho^2 \phi \phi^* - \frac{g}{2} \rho \bar{\Psi} \bar{\Psi} + \frac{1}{2} FF^* + \frac{g}{2} D \phi \phi^* + \text{surface terms},
\]

where \( g \) is the gauge coupling and \( D_\mu = \partial_\mu + igA_\mu \).

To \( \mathcal{L}_{\text{kin. matter}} \) can be added a mass term, or more generally a self-interaction term via the superpotential \( W(Q) \) which contributes to the Lagrangian as follows

\[
\mathcal{L}_{\text{self. matter}} = \int d^2 \theta W(Q) + \text{h.c.} \tag{40}
\]

\[
= \frac{\partial W}{\partial Q} F - \frac{\partial^2 W}{\partial Q^2} \bar{\Psi} \Psi + \text{h.c.},
\]

where the derivatives of the superpotential are to be taken at \( \theta = \bar{\theta} = 0 \). A mass term for the matter can be taken into account with the superpotential \( W(Q) = mQ^2 \).

Finally, for an Abelian theory, a Fayet-Iliopoulos term can be added:

\[
\mathcal{L}_{\text{F.I.}} = \int d^2 \theta d^2 \bar{\theta} \left( -2g\phi_0^2 V \right) = -g\phi_0^2 D. \tag{41}
\]

We consider 2 matter flavors \( Q_1 \) and \( Q_2 \), with opposite couplings so as to avoid the parity anomalies and the most general bare Lagrangian reads, when we disregard the surface terms.
\[ \mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \left( \frac{1}{4} \Sigma^2 + Q_1 e^{2gV} Q_1^* + Q_2 e^{-2gV} Q_2^* - 2g \phi_0^2 V \right) \]

\[ + \int d^2 \theta W(Q_1, Q_2) + \text{h.c.} \]

\[ = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{i}{2} \bar{\lambda} \partial \lambda \]

\[ + \frac{1}{2} D_\mu \Phi_1 (D^\mu \Phi_1)^* + \frac{i}{2} \bar{\Psi}_1 \partial \Psi_1 - \frac{g^2}{2} \rho^2 \Phi_1 \Phi_1^* - \frac{g}{2} \rho \bar{\Psi}_1 \Psi_1 \]

\[ + \frac{1}{2} D_\mu \Phi_2 (D^\mu \Phi_2)^* + \frac{i}{2} \bar{\Psi}_2 \partial \Psi_2 - \frac{g^2}{2} \rho^2 \Phi_2 \Phi_2^* + \frac{g}{2} \rho \bar{\Psi}_2 \Psi_2 \]

\[ + g \frac{i}{2} (\Phi_1 \bar{\Psi}_1 - \Phi_2 \bar{\Psi}_2 - \text{c.c.}) \]

\[ + \frac{1}{2} D^2 + \frac{g}{2} D(|\Phi_1|^2 - |\Phi_2|^2) - \frac{g}{2} \phi_0^2 D + \frac{1}{2} |F_1|^2 + \frac{1}{2} |F_2|^2 \]

\[ + \left( \frac{\partial W}{\partial Q_1} F_1 + \frac{\partial W}{\partial Q_2} F_2 + \frac{\partial^2 W}{\partial Q_k \partial Q_l} \Psi_k \Psi_l + \text{h.c.} \right) \]

where \( k, l = 1, 2 \) and

\[ D_\mu (...)_1 = \partial_\mu (...)_1 + ig A_\mu (...)_1 \]

\[ D_\mu (...)_2 = \partial_\mu (...)_2 - ig A_\mu (...)_2. \]

The scalar potential is obtained when writing the equations of motion of the auxiliary fields, which are

\[ F_1 = -2 \left( \frac{\partial W}{\partial Q_1} \right)^*, \quad F_2 = -2 \left( \frac{\partial W}{\partial Q_2} \right)^* \]

\[ D = -\frac{g}{2} (|\Phi_1|^2 - |\Phi_2|^2 - \phi_0^2), \]

and lead to the following potential

\[ U = \frac{g^2}{2} \rho^2 (|\Phi_1|^2 + |\Phi_2|^2) - \frac{1}{2} |F_1|^2 - \frac{1}{2} |F_2|^2 \]

\[ - \frac{1}{2} D^2 - \frac{g}{2} D(|\Phi_1|^2 - |\Phi_2|^2) + \frac{g^2}{2} \phi_0^2 D \]

\[ - \left( \frac{\partial W}{\partial Q_1} F_1 + \frac{\partial W}{\partial Q_2} F_2 + \text{h.c.} \right) \]

\[ = \frac{g^2}{2} \rho^2 (|\Phi_1|^2 + |\Phi_2|^2) + \frac{g^2}{8} (|\Phi_1|^2 - |\Phi_2|^2 - \phi_0^2)^2 \]

\[ + 2 \left( \left| \frac{\partial W}{\partial Q_1} \right|^2 + \left| \frac{\partial W}{\partial Q_2} \right|^2 + \text{h.c.} \right). \]
The on-shell Lagrangian is finally

\[ \mathcal{L}_{\text{on-shell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{i}{2} \bar{\lambda} \not{\partial} \lambda \]

\[ + \frac{1}{2} D_\mu \Phi_1 (D^\mu \Phi_1)^* + \frac{1}{2} D_\mu \Phi_2 (D^\mu \Phi_2)^* + \frac{i}{2} \bar{\Psi}_1 \not{D} \Psi_1 + \frac{i}{2} \bar{\Psi}_2 \not{D} \Psi_2 \]

\[ + \frac{g}{2} (\Phi_1^* \bar{\lambda}^* \Psi_1 - \Phi_2^* \bar{\lambda}^* \Psi_2 - \text{c.c.}) - \frac{g}{2} \rho (\bar{\Psi}_1 \Psi_1 - \bar{\Psi}_2 \Psi_2) - U(\rho, \Phi_1, \Phi_2), \]

where $U$ is given by Eq.(45). Note that the Higgs self-coupling is $g^2/8$, as found in [23, 15] due to the elevation of a $N = 1$ supersymmetry to $N = 2$.

**Phases of N=2 SQED**

Let us discuss the phases of the theory. The analysis is outlined in [8] and in more details in [6]. Here we review only the basic properties to be used in our physical discussion with respect to the various phases of the nodal supersymmetric liquid. The basic toll for understanding the various phases of the theory is the so-called moduli space of supersymmetric vacua.

To understand the topology of the moduli space in the SQED case we first note that classically N=2 SQED without matter has a phase of vacua labeled by the vacuum expectation value of the scalar field $\rho$ in the N=2 vector multiplet $(\rho, A_\mu, \lambda)$, where $\lambda$ is the photino, $A_\mu$ the photon, and $\rho$ the scalar (in what follows we shall refer to the gauge boson of SQED as a ‘photon’ although from our physical point of view this will be the statistical photon, not to be confused with the carrier of the real electromagnetic interactions).

Quantum mechanically, one can replace the gauge field by its dual, $\varepsilon_{\mu \nu \rho} A^\rho = \partial_\mu \gamma$, where $\gamma$ is a scalar periodic under $\gamma \to \gamma + 2\pi$. One defines then a chiral superfield $T$ whose lowest component is $\rho + i\gamma$. The moduli space is then given by a ‘metric’ space whose coordinates are given by the v.e.v.s of the scalar fields generating a supersymmetric vacuum ($\rho$ and $\gamma$ in our case). As discussed in [5], the precise geometry of this space at a quantum level determines the phase space structure of the supersymmetric theory. In the SQED case, the Coulomb branch of the moduli space is classically the cylinder defined by $< T >$. Due to the periodicity of the dual field $\gamma$, $T$ is a constrained chiral superfield and the actual good single-valued (and unconstrained) superfields for the description of the moduli space are $e^{\pm T}$ such that the Coulomb branch is described by $< e^{\pm T} >$.

In a non compact SQED there is no monopole phase. Things get more complicated in case of compact SQED, which is the case in which the compact Abelian gauge field may also be embedded in a non-Abelian subgroup. We shall stress the important differences between the two cases later on.

At the moment we note that, since a supersymmetric theory has zero vacuum energy, the scalar field vacuum expectation values (v.e.v.) $< \rho >, < \Phi_1 >, < \Phi_2 >$ must satisfy:

\[ U(< \rho >, < \Phi_1 >, < \Phi_2 >) = 0. \]
Figure 6: The quantum moduli space of $N=2$ SQED with conformal matter and zero superpotential. Its topology is an exact result. The phase where $<Q_1 Q_2>\neq 0$ is the Higgs phase, while the two branches characterised by $<M = e^T>\neq 0$ and $\tilde{M} = e^{-T} \neq 0$, with $T$ the dual of the vector superfield, constitute the Coulomb phase. In case there is a bare mass for chiral multiplets, corresponding to a superpotential term $m Q_1 Q_2$, the conformal point at the origin is replaced by a small neck, of finite thickness.

We will assume that there is no bare superpotential $W = 0$. This is consistent with our microscopic condensed-matter applications we are interested in; such a superpotential is generated by non-perturbative effects in the physically relevant case of compact SQED, as we shall discuss later on. Then, for a supersymmetric vacuum to occur we have the following possibilities:

- In the presence of the Fayet-Iliopoulos term, the only possibility is a Higgs phase where $<|\Phi_1|^2> = <|\Phi_2|^2> + \phi_0^2 \neq 0$ and $<\rho> = 0$. In this phase, the matter fields remain massless and the gauge field acquires the mass $<|\Phi_1|^2> + <|\Phi_2|^2>$. The theory is then the supersymmetric Abelian Higgs model [23].

- Without Fayet-Iliopoulos term ($\phi_0 = 0$), which is the case relevant for our condensed-matter application here, there is also, besides the Higgs phase where $<\Phi_1> = <\Phi_2> \neq 0$ and $<\rho> = 0$, a Coulomb phase where $<\Phi_1> = <\Phi_2> = 0$ and $<\rho> \neq 0$. In this phase, the matter fields acquire a mass given by $<\rho>$ and the gauge field remains massless. In the moduli space, the Higgs branch is a two dimensional conical (real) surface and has its summit at the origin $<\Phi_1> = <\Phi_2> = <\rho> = 0$ [6, 8]. The Higgs and Coulomb branches intersect at the origin of the moduli space.

In the Lagrangian (42), only the superpotential $W$ is protected by non-renormalization theorems and thus cannot be generated by perturbative quantum corrections if it is not present at the tree level. However, the Higgs and Coulomb phases do get perturbative quantum corrections. But, as explained in [6], the topology of the Higgs phase is not
changed and only the Coulomb branch changes qualitatively as it is shown in figure 6. Another possibility is to have non-perturbative quantum correction, coming from solitons if the Abelian gauge field is compact. Such corrections can generate a superpotential for the chiral field dual to the vector field, as will be discussed next.

Dual of N=2 SQED

As an important remark, before we proceed onto a discussion of the compact case, we would like to mention that one can find [8] an exact dual, in the sense of identical moduli space and spectrum of gauge invariant operators (i.e., they belong to the same universality class, of the N=2 SQED, which is described by the so-called XYZ supersymmetric model. This model consists of three chiral superfields \( X, Y, Z \) with a superpotential \( \mathcal{W} = eXYZ \), where \( e \) is a (dimensionful) coupling constant. The compact case then, and the associated effects of the monopole/instantons on the dynamics of SQED can be understood by studying various superpotential configurations of the dual model, and this is the approach followed in [8], whose results, as far as SQED monopole phase is concerned, are reviewed below. It is this exact duality that can be used in extracting exact information on the non-trivial infrared fixed point structure of SQED, with consequences on the non-fermi liquid behaviour of the nodal systems, as we discussed in the main text.

Compact SQED: the important physical differences

We now make some comments for compact SQED. This will help the reader understand the important physical differences from the non compact case. In compact SQED \( M = e^T, \ M = e^{-T} \) are chiral operators representing point-like instantons (Dirac magnetic monopoles in three dimensions), with charges 1, -1 respectively. The complex conjugates \( M^\dagger, \ M^\dagger \) have charges -1,1 respectively.

There is logarithmic confinement in SQED due to its low dimensionality, but also there is linear confinement in the phase where there is a monopole plasma. There is a detailed discussion in [8], to which we refer the interested reader. The proof that the monopole plasma phase leads to linear confinement is similar to that of [28].

Formally, the presence of a linearly confined phase is described by adding to the SQED Lagrangian a superpotential term (interactions) for the chiral operators \( M, \ M \). Consider \( p \) a positive integer, and the superpotential

\[
\mathcal{W}(T) = hM^p/2 + \tilde{h}\tilde{M}^p/2
\]

where \( h, (\tilde{h}) \) denote densities (fugacities) of monopoles with charges 1, (-1) respectively. \( \mathcal{W}(T) \) is generated by instanton effects and in general \( h \neq \tilde{h} \). There could be various other terms in the superpotential. Most of the results reviewed below for the monopole phase of SQED are obtained by virtue of the dual model [8].

If \( h = \tilde{h} \) then there is a symmetry \( T \to -T \) in the theory, and in such a case there are equal amounts of monopoles of charge \( \pm 1 \) and antimonopoles of charge \( \mp 1 \), implying vacua symmetric about \( \rho = 0 \). In the general case \( h \neq \tilde{h} \) one may reduce the effect of the charge -1 monopoles \( \tilde{M} \) relative to the charge 1 monopoles \( M \) by a field rescaling

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$M \to A M, \bar{M} \to \bar{M}/A$, where $A = \left(\frac{\tilde{h}}{\rho}\right)^{\frac{1}{p}}$. This is equivalent to shifting the symmetry to $T \to -T + \ln \left(\frac{\tilde{h}}{\rho}\right)^{\frac{2}{p}}$, which essentially acts as a shift of $\rho$.

In the limit $\tilde{h} \to 0$ stable supersymmetric vacua occur therefore only for $<\rho> \to \infty$, and hence in that special case there is no stable supersymmetric vacuum with monopole plasma and linearly confined phase for finite vevs of $\rho$.

This last case, for $p = 2$, is the case of the non abelian SU(2) Georgi-Glashow N=2 supersymmetric model discussed in [17]. Indeed an index theorem [8, 7] guarantees that the SU(2) case has only monopoles of charge +1 and antimonopoles of charge -1. In this case $\tilde{h} = 0$ and, hence, in view of the above discussion there is no monopole plasma phase. For a detailed discussion on SU(2) with chiral matter multiplets see section 3.2 of ref. [7].

For a generic exponent $p$ in (48), which, as discussed in [8], must be integer to ensure vanishing vacuum energy, as required by supersymmetric vacua, there are $p$ such vacua separated by $p$ domain walls (flux strings) which meet at a vortex of SQED (which exist because of the complex scalar fields of the gauge or chiral multiplets). For even $p$ the strings may connect pairs of vortices-antivortices. We remind the reader that the embedding in the SU(2) occurs only for $p = 2$. On each flux string the gauge field dual is excited in the sense of the string carrying non trivial flux.

On the other hand, in the case where $\tilde{h} \to 0$ there is no stable monopole plasma phase and one has the superconducting regime according to the exact masslessness of the statistical photon. One may then have an elevation to the SU(2) case at such points, given that in SU(2) N=2 supersymmetric theories there are only monopoles of charge +1, and antimonopoles of charge -1, unlike SQED where both kinds of charges are present. The absence of a stable monopole phase in SU(2) N=2 supersymmetric gauge theories is confirmed by independent arguments [7], based on the so-called Wilsonian effective action to gauge theories, where massive degrees of freedom are being integrated out.

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The approach of the last author differs in the way the four-component continuum spinors are constructed, which leads to a different physical interpretation of the associated gauge symmetry as compared with the results of Balents et al. From our point of view, we are closer in spirit to the construction of Balents et al., but only as far as the construction of the matter spinors is concerned. Our composite gauge field nature is entirely different from any of these approaches.

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