Motion camouflage in three dimensions

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Abstract—We formulate and analyze a three-dimensional model of motion camouflage, a stealth strategy observed in nature. A high-gain feedback law for motion camouflage is formulated in which the pursuer and evader trajectories are described using natural Frenet frames (or relatively parallel adapted frames), and the corresponding natural curvatures serve as controls. The biological plausibility of the feedback law is discussed, as is its connection to missile guidance. Simulations illustrating motion camouflage are also presented. This paper builds on recent work on motion camouflage in the planar setting [8].

I. INTRODUCTION

Motion camouflage is a stealth strategy employed by various visual insects and animals to achieve prey capture, mating or territorial combat. In one type of motion camouflage, the predator camouflages itself against a fixed background object so that the prey observes no relative motion between the predator and the fixed object. In the other type of motion camouflage, the predator approaches the prey such that from the point of view of the prey, the predator always appears to be at the same bearing. (In this case, we say that the object against which the predator is camouflaged is the point at infinity.) For background on motion camouflage, see [8] and the references therein. Motion camouflage behavior in insects is described in [13] (based on earlier work in [4] on hoverflies) and in [9] (for dragonflies). Related themes in insect vision and flight control are also found in [14].

The essential features of motion camouflage are not limited to visual insects. Recent work on the neuroethology of insect-capture behavior in echolocating bats reveals a strategy geometrically indistinguishable from motion camouflage, referred to as the "constant absolute target direction" (CATD) strategy [5]. Because the bat under study, *Eptesicus fuscus*, hunts at night, there is no reason to suppose that camouflage (i.e., misleading its prey’s visual system) is the bat’s goal in using the CATD strategy. In this paper, we are concerned with describing in the simplest possible, biologically plausible way how the motion camouflage or CATD strategy can be achieved using feedback control. This is a small first step toward understanding the much more difficult question of why an animal like the bat *Eptesicus fuscus* uses such a strategy.

What sets this work apart is the structured approach used to derive feedback laws for motion control in three dimensions. We model the pursuer (i.e., predator) and evader (i.e., prey) as point particles subject to curvature (steering) control. Although the speeds of the particles may vary, this variation is considered to result primarily from flight conditions the animal experiences - not primarily as a result of explicit speed control for purposes of achieving motion camouflage. Indeed, the feedback law we derive for motion camouflage is well-defined for constant-speed motion. However, for comparing the theoretical feedback law to the experimentally-derived bat trajectory data, it is useful to retain speed variability in the model, since speed variations on the order of 50 percent are observed as the bat maneuvers.

This focus on systematic formulation and analysis of biologically plausible feedback laws for motion camouflage is a distinguishing feature of our work. For example, in [6] motion camouflage trajectories are studied, but without explicitly providing feedback laws which give rise to them. In [1], feedback using neural networks is used to achieve motion camouflage, but our approach has the advantage of giving an explicit form and straightforward physical interpretation for the feedback control law.

In earlier work, motion camouflage in the planar setting was studied, and a feedback law to achieve motion camouflage was derived [8]. The name given to the feedback law was motion camouflage proportional guidance (MCPG). Here, we extend this work by formulating the problem in three dimensions and generalizing the feedback law to the three dimensional setting. The key is to describe the particle trajectories using natural Frenet frames [3] - the same approach demonstrated successfully in the context of formation control for constant-speed particles [7]. This formulation can also be used to describe missile guidance, specifically, pure proportional navigation guidance (PPNG) [12], [10], [11], cleanly in three dimensions.

II. PURSUIT-EVASION MODEL

For concreteness, we consider the problem of motion camouflage in which the predator (which we refer to as the “pursuer”) attempts to intercept the prey (which we refer to as the “evader”) while appearing to the prey as though it is always at the same bearing (i.e., motion camouflaged against the point at infinity). The dynamics of the pursuer are given
Frenet frames. The position of the pursuer is $\mathbf{r}_p$, and its natural Frenet frame is $\{x_p, y_p, z_p\}$, where $x_p$ is the unit tangent vector to its trajectory, and $\{y_p, z_p\}$ span the corresponding normal plane (and similarly for the evader). The pursuer moves with speed $\nu_p$, and the evader with speed $\nu_e$.

by

\[
\begin{align*}
\dot{r}_p &= \nu_p x_p, \\
\dot{x}_p &= \nu_p (y_p u_p + z_p v_p), \\
\dot{y}_p &= -\nu_p x_p u_p, \\
\dot{z}_p &= -\nu_p x_p v_p,
\end{align*}
\]

where $r_p$ is the position of the pursuer, $\nu_p$ is the speed of the pursuer, $x_p$ is the unit tangent vector to the trajectory of the pursuer, $y_p$ and $z_p$ span the normal plane to $x_p$ (completing a right-handed orthonormal basis with $x_p$), and the natural curvatures $u_p$ and $v_p$ are the controls for the pursuer. Similarly, the dynamics of the evader are

\[
\begin{align*}
\dot{r}_e &= \nu_e x_e, \\
\dot{x}_e &= \nu_e (y_e u_e + z_e v_e), \\
\dot{y}_e &= -\nu_e x_e u_e, \\
\dot{z}_e &= -\nu_e x_e v_e,
\end{align*}
\]

where $r_e$ is the position of the evader, $\nu_e$ is the speed of the evader, $x_e$ is the unit tangent vector to the trajectory of the evader, $y_e$ and $z_e$ span the normal plane to $x_e$ (completing a right-handed orthonormal basis with $x_e$), and the natural curvatures $u_e$ and $v_e$ are the controls for the evader. Figure 1 illustrates equations (1) and (2). Note that $\{x_p, y_p, z_p\}$ and $\{x_e, y_e, z_e\}$ are natural Frenet frames (also known as relatively parallel adapted frames) for the trajectories of the pursuer and evader, respectively [3].

We model the pursuer and evader as point particles, and use natural frames and curvature controls to describe their motion, because this is a simple model for which we can derive both physical intuition and concrete control laws. Flying insects and animals (also unmanned aerial vehicles) have limited maneuverability and must maintain sufficient airspeed to stay aloft, so modeling them in this way is physically reasonable, at least for some range of flight conditions.

Note that the forces supplied by the curvature controls are perpendicular to the instantaneous direction of motion, and therefore do not change the speed: these forces are gyroscopic forces. However, in (1) and (2) we do allow for the possibility of speed variations, as well.

A. Characterizing motion camouflage

Motion camouflage with respect to the point at infinity is given by [8]

\[ r_p = r_e + \lambda r_{\infty}, \]

where $r_{\infty}$ is a fixed unit vector and $\lambda$ is a time-dependent scalar (see also Section 5 of [6]).

Let

\[ r = r_p - r_e \]

be the vector from the evader to the pursuer. We refer to $r$ as the “baseline vector,” and $|r|$ as the “baseline length.” We restrict attention to non-collision states, i.e., $r \neq 0$. In that case, the component of the pursuer velocity $\dot{r}_p$ transverse to the base line is

\[ \dot{r}_p - \left( \frac{r}{|r|} \cdot \dot{r}_p \right) \frac{r}{|r|}, \]

and similarly, that of the evader is

\[ \dot{r}_e - \left( \frac{r}{|r|} \cdot \dot{r}_e \right) \frac{r}{|r|}. \]

The relative transverse component is

\[
\begin{align*}
\mathbf{w} &= (\dot{r}_p - \dot{r}_e) - \left( \frac{r}{|r|} \cdot (\dot{r}_p - \dot{r}_e) \right) \frac{r}{|r|} \\
&= \dot{r} - \left( \frac{r}{|r|} \cdot \dot{r} \right) \frac{r}{|r|}. \quad (5)
\end{align*}
\]

Lemma (Infinitesimal characterization of motion camouflage): The pursuit-evasion system (1) and (2) is in a state of motion camouflage without collision on an interval iff $\mathbf{w} = 0$ on that interval.

Proof: ($\Longrightarrow$) Suppose motion camouflage holds. Thus

\[ r(t) = \lambda(t) r_{\infty}, \quad t \in [0, T]. \quad (6) \]

Differentiating, $\dot{r} = \lambda \dot{r}_\infty$. Hence,

\[
\begin{align*}
\mathbf{w} &= \dot{r} - \left( \frac{r}{|r|} \cdot \dot{r} \right) \frac{r}{|r|} \\
&= \lambda \dot{r}_\infty - \left( \frac{\lambda}{|\lambda|} r_\infty \cdot \lambda \dot{r}_\infty \right) \frac{\lambda}{|\lambda|} r_\infty \\
&= 0 \text{ on } [0, T]. \quad (7)
\end{align*}
\]

($\Longleftarrow$) Suppose $\mathbf{w} = 0$ on $[0, T]$. Thus

\[ \dot{r} = \left( \frac{r}{|r|} \cdot \dot{r} \right) \frac{r}{|r|} \triangleq \xi r, \quad (8) \]
B. Measuring departure from motion camouflage

Remark: The above Lemma and its proof are identical to the corresponding Lemma and proof in [8], but with the vectors interpreted as three-dimensional rather than planar vectors. □

Figure 2 illustrates the pursuer and evader in a state of motion camouflage with respect to the point at infinity.

B. Measuring departure from motion camouflage

Consider the ratio

\[
\Gamma(t) = \frac{\frac{d}{dt} |r|}{\frac{d}{dt} |\tilde{r}|},
\]

which compares the rate of change of the baseline length to the absolute rate of change of the baseline vector [8]. If the baseline experiences pure lengthening, then the ratio assumes its maximum value, \(\Gamma(t) = 1\). If the baseline experiences pure shortening, then the ratio assumes its minimum value, \(\Gamma(t) = -1\). If the baseline experiences pure rotation, but remains the same length, then \(\Gamma(t) = 0\). Noting that

\[
\frac{d}{dt} |r| = \frac{r}{|r|} \cdot \tilde{r},
\]

we see that \(\Gamma(t)\) may alternatively be written as

\[
\Gamma(t) = \frac{r}{|r|} \cdot \frac{\tilde{r}}{|\tilde{r}|}.
\]

Thus, \(\Gamma(t)\) is the dot product of two unit vectors: one in the direction of \(r\), and the other in the direction of \(\tilde{r}\).

From

\[
|w|^2 = |\tilde{r}|^2 - 2 \left( \frac{r}{|r|} \cdot \frac{\tilde{r}}{|\tilde{r}|} \right)^2 + \left( \frac{r}{|r|} \cdot \tilde{r} \right)^2
= |\tilde{r}|^2 \left( 1 - \Gamma^2 \right),
\]

it follows that \(1 - \Gamma^2\) is a measure of departure from motion camouflage.

III. FEEDBACK LAW FOR MOTION CAMOUFLAGE

Using the planar setting as a guide, the curvature controls to achieve motion camouflage in three dimensions can be systematically derived. Indeed, this is a major advantage of representing trajectories using natural Frenet frames. However, for ease of exposition, we instead begin by presenting the control law in an intuitively appealing and biologically plausible form, followed by the calculations demonstrating its effectiveness.

A. Feedback law and interpretation

Using the BAC-CAB identity, \(\hat{a} \times (\hat{b} \times \hat{c}) = \hat{b}(\hat{a} \cdot \hat{c}) - \hat{c}(\hat{a} \cdot \hat{b})\), for arbitrary vectors \(\hat{a}, \hat{b}, \hat{c}\), we observe that

\[
w = \hat{r} \left( \frac{r}{|r|} \times \frac{\tilde{r}}{|\tilde{r}|} - \frac{r}{|r|} \right) = \frac{r}{|r|} \times \left( \hat{r} \times \frac{r}{|r|} \right),
\]

and we conclude from (15) that \(|\hat{r} \times \frac{r}{|r|}|\) is a biologically plausible quantity to appear in a feedback law, since it only requires sensing \(w\) and \(r/|r|\).

The quantity \(|\hat{r} \times \frac{r}{|r|}|\) can be interpreted in terms of an angular-velocity-like quantity. From the point of view of the pursuer, consider an extensible rod connecting the pursuer and evader positions. The motion of the evader (relative to the pursuer) contributes to change in the length of this rod, as well as to angular velocity of the rod (viewed from the pursuer - see figure 3). The transverse component of the velocity of the evader (viewed from the pursuer) is simply

\[
\hat{v}_c - \hat{v}_p = \left( \hat{v}_c - \hat{v}_p \right) \cdot \frac{r_e - r_p}{|r_e - r_p|} \frac{r_e - r_p}{|r_e - r_p|} = -\hat{r} \cdot \left( \frac{r}{|r|} \right) \left( \frac{r}{|r|} \right) = -\omega,
\]

which can also be expressed as

\[
-w = \omega \times (-r),
\]

where \(\omega\) is the corresponding angular velocity of the rod. From (14) and (17) we conclude that

\[
\frac{r}{|r|} \times \left( \hat{r} \times \hat{r} \right) = \left( \frac{r}{|r|^2} \times \hat{r} \right) \times r = \omega \times r.
\]
and hence
\[ \omega = \frac{\mathbf{r}}{|\mathbf{r}|} \times \dot{\mathbf{r}}. \]  
Thus, the quantity \((\dot{\mathbf{r}} \times \mathbf{r}/|\mathbf{r}|)\) is simply \(-\omega\) scaled by \(|\mathbf{r}|\).

For convenience in the calculations below, we define
\[ \mathbf{a} = \mathbf{x}_p \times \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|} \right), \]  
and express the feedback law as
\[ u_p = \mu (\mathbf{a} \cdot \mathbf{y}_p), \]  
\[ v_p = \mu (\mathbf{a} \cdot \mathbf{z}_p), \]  
where \(\mu > 0\) is a constant feedback gain. The quantity \(\mu \nu_p^2 \mathbf{a}\) can then be interpreted as the lateral component of the acceleration vector of the pursuer. Consistent with the fact that \(u_p\) and \(v_p\) can only change the direction of the pursuer’s motion and not its speed, we note that \(\mu \nu_p^2 \mathbf{a}\) is transverse to the direction of motion of the pursuer, \(\mathbf{x}_p\); i.e., \(\mathbf{a} \cdot \mathbf{x}_p = 0\).

Using the formula \(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})\) for the scalar triple product, where \(\mathbf{a}, \mathbf{b}, \mathbf{c}\) are arbitrary vectors, we compute
\[ u_p = \mu \left[ \mathbf{x}_p \times \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|} \right) \right] \cdot \mathbf{y}_p \]
\[ = \mu \left[ \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|} \right) \cdot (\mathbf{y}_p \times \mathbf{x}_p) \right] \]
\[ = -\mu \left[ \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|} \right) \cdot \mathbf{z}_p \right], \quad (22) \]
and similarly,
\[ v_p = \mu \left[ \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|} \right) \cdot \mathbf{y}_p \right]. \quad (23) \]

Remark: It is easy to see that in the planar setting, we recover the planar steering law for motion camouflage presented in [8]. If \(\mathbf{x}_p, \mathbf{x}_e,\) and \(\mathbf{r}\) all lie in the same plane, then \(\dot{\mathbf{r}}\) also lies in that plane, and \((22)\) becomes
\[ u_p = -\mu \left[ \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|} \right) \cdot \mathbf{z}_p \right] = -\mu \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}} \right), \quad (24) \]
where the notion \(q^\perp\) represents the vector \(q\) rotated counterclockwise in the plane by \(\pi/2\). Furthermore, without loss of generality, we identify \(\mathbf{y}_p\) with \(\mathbf{x}_p^\perp\), and \(\mathbf{z}_p\) with the unit vector perpendicular to the plane of motion. \(\square\)

B. Behavior of \(\Gamma\) under the feedback law

Differentiating \(\Gamma\) along trajectories of \((1)\) and \((2)\) gives
\[ \dot{\Gamma} = \left( \frac{\dot{\mathbf{r}} \cdot \mathbf{r} + \mathbf{r} \cdot \ddot{\mathbf{r}}}{|\mathbf{r}|^3} \right) - \left( \frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{|\mathbf{r}|^3} \right)^2 - \left( \frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{|\mathbf{r}|^3} \right)^2 \]
\[ = \left\lfloor \frac{1}{|\mathbf{r}|^3} \right. \left[ 1 - \left( \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{|\mathbf{r}|} \right)^2 \right] + \frac{1}{|\mathbf{r}|^3} \left[ \mathbf{r} - \left( \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{|\mathbf{r}|} \right) \cdot \dot{\mathbf{r}} \right] \cdot \ddot{\mathbf{r}}. \]

We also have
\[ \dot{\mathbf{r}} = \nu_p \mathbf{x}_p - \nu_e \mathbf{x}_e + \nu_x \mathbf{y}_p + \nu_y \mathbf{z}_p (29) \]
\[ = \nu_p \mathbf{x}_p - \nu_e \mathbf{x}_e + \nu_y (\mathbf{y}_p u_p + \mathbf{z}_p v_p) - \nu_e^2 (\mathbf{y}_e u_e + \mathbf{z}_e v_e). \]

If we define
\[ \mathbf{b} = \frac{1}{|\mathbf{r}|^3} \left\lfloor \frac{\mathbf{r} \cdot \mathbf{r}}{|\mathbf{r}|} \right. \left. \right\rfloor \left( \mathbf{r} \cdot \ddot{\mathbf{r}} \right), \]  
then
\[ \mathbf{b} \cdot \dot{\mathbf{r}} = \nu_p (\mathbf{b} \cdot \mathbf{x}_p) - \nu_e (\mathbf{b} \cdot \mathbf{x}_e) \]
\[ + \nu_y^2 \left( \mathbf{b} \cdot \mathbf{y}_p \right) u_p + \left( \mathbf{b} \cdot \mathbf{z}_p \right) v_p \]
\[ - \nu_e^2 \left( \mathbf{b} \cdot \mathbf{y}_e \right) u_e + \left( \mathbf{b} \cdot \mathbf{z}_e \right) v_e, \]
and the only term of \(\Gamma\) into which the controls \(u_p\) and \(v_p\) explicitly enter is
\[ \nu_p^2 \left( \mathbf{b} \cdot \mathbf{y}_p \right) u_p + \left( \mathbf{b} \cdot \mathbf{z}_p \right) v_p, \]
(29)

Using \((20)\) and \((21)\),
\[ \nu_p^2 \left( \mathbf{b} \cdot \mathbf{y}_p \right) u_p + \left( \mathbf{b} \cdot \mathbf{z}_p \right) v_p \]
\[ = \mu \nu_p^2 \left[ (\mathbf{b} \cdot \mathbf{y}_p) (\mathbf{a} \cdot \mathbf{y}_p) + (\mathbf{b} \cdot \mathbf{z}_p) (\mathbf{a} \cdot \mathbf{z}_p) \right] \]
\[ = \mu \nu_p^2 \left[ (\mathbf{b} \cdot \mathbf{a}) - (\mathbf{b} \cdot \mathbf{x}_p) (\mathbf{a} \cdot \mathbf{x}_p) \right] \]
\[ = \mu \nu_p^2 (\mathbf{b} \cdot \mathbf{a}), \]
(30)
where we have also used the identity
\[ \mathbf{a} \cdot \mathbf{b} = (\mathbf{a} \cdot \mathbf{x}_p) (\mathbf{b} \cdot \mathbf{x}_p) + (\mathbf{a} \cdot \mathbf{y}_p) (\mathbf{b} \cdot \mathbf{y}_p) + (\mathbf{a} \cdot \mathbf{z}_p) (\mathbf{b} \cdot \mathbf{z}_p), \]
(31)
and \(\mathbf{a} \cdot \mathbf{x}_p = 0\).

Using the BAC-CAB identity, we observe that
\[ \mathbf{b} = \frac{1}{|\mathbf{r}|^3} \left( \frac{\dot{\mathbf{r}} \times \mathbf{r}}{|\mathbf{r}|^3} \right) \]  
\[ = -\frac{1}{|\mathbf{r}|^3} \left[ \dot{\mathbf{r}} \times \left( \frac{\mathbf{r} \times \mathbf{r}}{|\mathbf{r}|} \right) \right], \quad (32) \]
so that
\[ \mathbf{b} \cdot \mathbf{a} = -\frac{1}{|\mathbf{r}|^3} \left[ \dot{\mathbf{r}} \times \left( \frac{\mathbf{r} \times \mathbf{r}}{|\mathbf{r}|} \right) \right] \cdot \mathbf{x}_p \times \left( \frac{\mathbf{r} \times \mathbf{r}}{|\mathbf{r}|} \right). \]
(33)
Using the identity \((\hat{a} \times \hat{b}) \cdot (\hat{c} \times \hat{d}) = (\hat{a} \cdot \hat{c})(\hat{b} \cdot \hat{d}) - (\hat{a} \cdot \hat{d})(\hat{b} \cdot \hat{c})\), for arbitrary vectors \(\hat{a}, \hat{b}, \hat{c}, \hat{d}\), and
\[
\left| \hat{r} \times \frac{r}{|r|} \right|^2 = |r|^2 (1 - \Gamma^2),
\]
we compute
\[
\hat{b} \cdot \hat{a} = -\frac{1}{|r|} \left( \hat{r} \cdot x_p \right) (1 - \Gamma^2) + \frac{1}{|r|^3} \left[ \hat{r} \cdot \left( \hat{r} \times \frac{r}{|r|} \right) \right] \left[ \left( \hat{r} \times \frac{r}{|r|} \right) \cdot x_p \right] = - (1 - \Gamma^2) \left( \frac{\hat{r}}{|r|} \right) \cdot x_p.
\]

**Remark:** For the foregoing calculations to make sense, we require \(|r| > 0\) and \(|\hat{r}| > 0\). The condition \(|r| > 0\) is a non-collision condition, and does not pose any difficulty for us because our analysis of approach to the state of motion camouflage takes place away from the collision state. Later, we will impose hypotheses that also ensure \(|\hat{r}| > 0\) for all time. (Note that in the constant-speed setting, \(0 < \nu_e/\nu_p < 1\) is sufficient to ensure \(|\hat{r}| > 0\) [8].)

**Remark:** Provided \(\nu_p > \nu_e\), we have
\[
\left( \frac{\hat{r}}{|r|} \right) \cdot x_p = \frac{1}{|r|} \left[ |r|^2 \nu_p - \nu_e (x_p \cdot x_e) \right] > 0,
\]
so that
\[
\hat{b} \cdot \hat{a} \leq 0,
\]
and therefore the only term in \(\hat{\Gamma}\) explicitly involving the controls \(u_p\) and \(v_p\) satisfies
\[
|v|^2 \left[ (\hat{b} \cdot y_p)u_p + (\hat{b} \cdot z_p)v_p \right] \leq 0.
\]

To summarize, \(\hat{\Gamma}\) becomes
\[
\hat{\Gamma} = - (1 - \Gamma^2) \left[ \frac{\mu_p^2}{|r|^2} \left( \nu_p - \nu_e (x_p \cdot x_e) \right) - \frac{|\hat{r}|}{|r|} \right]
+ \nu_p (\hat{b} \cdot x_p) - \nu_e (\hat{b} \cdot x_e)
- \nu_e^2 \left[ (\hat{b} \cdot y_e)u_e + (\hat{b} \cdot z_e)v_e \right].
\]

Noting that
\[
|\hat{b}|^2 = \frac{1}{|r|^2} (1 - \Gamma^2),
\]
we see that
\[
|\nu_e^2 \left[ (\hat{b} \cdot y_e)u_e + (\hat{b} \cdot z_e)v_e \right] |
\leq \frac{\nu_e^2}{|r|} \sqrt{1 - \Gamma^2} \max \left( \sqrt{u_e^2 + v_e^2} \right),
\]
where \(\max \left( \sqrt{u_e^2 + v_e^2} \right)\) is an a priori bound on the maximum absolute curvature of the evader trajectory. Similarly,
\[
|\nu_e (\hat{b} \cdot x_p) - \nu_e (\hat{b} \cdot x_e) |
\leq \frac{1}{|r|} \sqrt{1 - \Gamma^2} \left( |\nu_p| + |\nu_e| \right)
\leq \frac{1}{|r|} \sqrt{1 - \Gamma^2} (\alpha_p + \alpha_e),
\]
where \(\alpha_p\) is an upper bound on \(|\nu_p|\), and \(\alpha_e\) is an upper bound on \(|\nu_e|\). From \(\hat{\Gamma}\) we then conclude
\[
\hat{\Gamma} \leq - (1 - \Gamma^2) \left[ \frac{\mu_p^2}{|r|^2} \left( \nu_p - \nu_e (x_p \cdot x_e) \right) - \frac{|\hat{r}|}{|r|} \right]
+ \frac{1}{|r|} \sqrt{1 - \Gamma^2} \left[ \alpha_p + \alpha_e + \nu_e^2 \max \left( \sqrt{u_e^2 + v_e^2} \right) \right].
\]

**C. Bounds and estimates**

Having bounded \(\hat{\Gamma}\) as in \(\hat{\Gamma}\), we proceed in analogy with the planar setting [8]. We hypothesize that a constant \(\nu_{max}\) exists such that
\[
\frac{\nu_e}{\nu_p} \leq \nu_{max} < 1,
\]
for all time. We also assume that constants \(\nu_p^{low}, \nu_p^{high}, \nu_e^{low}, \nu_e^{high}\) exist such that
\[
0 < \nu_p^{low} \leq \nu_p \leq \nu_p^{high} < \infty,
0 < \nu_e^{low} \leq \nu_e \leq \nu_e^{high} < \infty,
\]
for all time, and observe that
\[
0 < \nu_p^{low} (1 - \nu_{max}) \leq |\hat{r}| \leq \nu_p^{high} (1 + \nu_{max}).
\]

We define the constant \(c_1 > 0\) as
\[
c_1 = \left[ \frac{\alpha_p + \alpha_e + (\nu_e^{high})^2 \max \left( \sqrt{u_e^2 + v_e^2} \right)}{\nu_p^{low} (1 - \nu_{max})} \right].
\]
Given \(\mu > 0\) sufficiently large and \(r_o > 0\), we define \(c_0 > 0\) by
\[
c_0 = \left( \frac{(\nu_p^{low})^3 (1 - \nu_{max})}{\nu_p^{high} (1 + \nu_{max})} \right) \mu - \frac{\nu_p^{high} (1 + \nu_{max})}{r_o},
\]
so that
\[
\mu = \left( \frac{(\nu_p^{low})^3 (1 - \nu_{max})}{\nu_p^{high} (1 + \nu_{max})} \right) \left( \frac{\nu_p^{high} (1 + \nu_{max})}{r_o} + c_0 \right),
\]
and hence
\[
\mu \geq \left( \frac{(\nu_p^{low})^3 (1 - \nu_{max})}{\nu_p^{high} (1 + \nu_{max})} \right) \left( \frac{\nu_p^{high} (1 + \nu_{max})}{|r|} + c_0 \right),
\]
\[\forall \|r\| \geq r_o. \text{ Thus, for } \|r\| \geq r_o, \text{ (55) becomes}\]
\[\dot{\Gamma} \leq - (1 - \Gamma^2) \left[ \frac{\nu_p^{\text{high}}(1 + \nu_{\text{max}})}{(\nu_p^{\text{low}})^3(1 - \nu_{\text{max}})} \frac{(\nu_p^{\text{high}}(1 + \nu_{\text{max}})}{\|r\|} + c_0 \right]
\times \left[ \frac{(\nu_p^{\text{low}})^3(1 - \nu_{\text{max}})}{(\nu_p^{\text{high}})^3(1 - \nu_{\text{max}})} - \frac{\nu_p^{\text{high}}(1 + \nu_{\text{max}})}{\|r\|} \right] + \left( \sqrt{1 - \Gamma^2} \right) c_1 \]
\[= - (1 - \Gamma^2) c_0 + \left( \sqrt{1 - \Gamma^2} \right) c_1. \quad (52)\]

Suppose that given \(0 < \epsilon << 1\), we take \(\mu > 0\) sufficiently large so that there exists \(c_0\) satisfying \(c_0 \geq 2c_1/\sqrt{\epsilon}\). Then for \((1 - \Gamma^2) > \epsilon\),
\[\dot{\Gamma} \leq - (1 - \Gamma^2) c_0 + \left( \sqrt{1 - \Gamma^2} \right) c_1 \]
\[= - (1 - \Gamma^2) \left( c_0 - \frac{c_1}{\sqrt{1 - \epsilon}} \right) \]
\[\leq - (1 - \Gamma^2) \left( c_0 - \frac{c_1}{\sqrt{\epsilon}} \right) \]
\[= - (1 - \Gamma^2) c_2, \quad (53)\]
where
\[c_2 = c_0 - \frac{c_1}{\sqrt{\epsilon}} > 0. \quad (54)\]

**Remark:** There are two possibilities for
\[(1 - \Gamma^2) \leq \epsilon. \quad (55)\]

The state we seek to drive the system toward has \(\Gamma \approx -1\); however, (55) can also be satisfied for \(\Gamma \approx 1\). (Recall that \(-1 \leq \Gamma \leq 1\).) There is always a set of initial conditions such that (55) is satisfied with \(\Gamma \approx 1\). We can address this issue as follows: let \(\epsilon_0 > 0\) denote how close to \(-1\) we wish to drive \(\Gamma\), and let \(\Gamma_0 = \Gamma(0)\) denote the initial value of \(\Gamma\). Take
\[\epsilon = \min(\epsilon_0, 1 - \Gamma_0^2), \quad (56)\]
so that (53) with (54) applies from time \(t = 0\). \(\square\)

From (53), we can write
\[\frac{d\Gamma}{1 - \Gamma^2} \leq -c_2 dt, \quad (57)\]
which, integrating both sides, leads to
\[\int_{\Gamma_0}^{\Gamma} \frac{d\tilde{\Gamma}}{1 - \tilde{\Gamma}^2} \leq -c_2 \int_0^t dt = -c_2 t, \quad (58)\]
where \(\Gamma_0 = \Gamma(t = 0)\). Noting that
\[\int_{\Gamma_0}^{\Gamma} \frac{d\tilde{\Gamma}}{1 - \tilde{\Gamma}^2} = \int_0^\Gamma d(\tanh^{-1} \tilde{\Gamma}) = \tanh^{-1} \Gamma - \tanh^{-1} \Gamma_0, \quad (59)\]
we see that for \(\|r\| \geq r_o\), (53) implies
\[\Gamma(t) \leq \tanh(\tanh^{-1} \Gamma_0 - c_2 t), \quad (60)\]
where we have used the fact that \(\tanh^{-1}(\cdot)\) is a monotone increasing function.

Now we consider estimating how long \(\|r\| \geq r_o\), which in turn determines how large \(t\) can become in inequality (60), and hence how close to \(-1\) will \(\Gamma(t)\) be driven. From (11) and (12) we have
\[\frac{d}{dt} \|r\| = \Gamma(t) \|r\|, \quad (61)\]
which from (47) and \(\Gamma(t) \leq 1, \forall t\), implies
\[\frac{d}{dt} \|r\| \geq -|\Gamma(t)| \nu_p^{\text{high}}(1 + \nu_{\text{max}}) \geq -\nu_p^{\text{high}}(1 + \nu_{\text{max}}). \quad (62)\]
From (62), we conclude that
\[|\Gamma(t)| \geq |\Gamma(0)| - \nu_p^{\text{high}}(1 + \nu_{\text{max}}) t, \forall t \geq 0, \quad (63)\]
and, more to the point,
\[|\Gamma(t)| \geq r_o, \forall t \leq \frac{|\Gamma(0)| - r_o}{\nu_p^{\text{high}}(1 + \nu_{\text{max}})}. \quad (64)\]
For (64) to be meaningful for the problem at hand, we assume that \(|\Gamma(0)| > r_o\). Then defining
\[T = \frac{|\Gamma(0)| - r_o}{\nu_p^{\text{high}}(1 + \nu_{\text{max}})} > 0 \quad (65)\]
to be the minimum interval of time over which we can guarantee that \(\Gamma \leq 0\), we conclude that
\[\Gamma(T) \leq \tanh(\tanh^{-1} \Gamma_0 - c_2 T). \quad (66)\]

From (66), we see that by choosing \(c_2\) sufficiently large (which can be accomplished by choosing \(c_0 \geq 2c_1/\sqrt{\epsilon}\) sufficiently large), we can force \(\Gamma(T) \leq -1 + \epsilon\). Noting that
\[\tanh(x) \leq -1 + \epsilon \iff x \leq \frac{1}{2} \ln \left( \frac{\epsilon}{2 - \epsilon} \right), \quad (67)\]
for \(0 < \epsilon << 1\), we see that
\[\Gamma(T) \leq -1 + \epsilon \iff \tanh^{-1} \Gamma_0 - c_2 T \leq \frac{1}{2} \ln \left( \frac{\epsilon}{2 - \epsilon} \right). \quad (68)\]
Thus, if \(c_0 \geq 2c_1/\sqrt{\epsilon}\) is taken to be sufficiently large that
\[c_2 \geq \nu_p^{\text{high}}(1 + \nu_{\text{max}}) \tanh^{-1} \Gamma_0 - \frac{1}{2} \ln \left( \frac{\epsilon}{2 - \epsilon} \right), \quad (69)\]
then we are guaranteed (under the conditions mentioned in the above calculations) to achieve \(\Gamma(t_1) \leq -1 + \epsilon\) at some finite time \(t_1 \leq T\).

**D. Statement of result**

**Definition** [8]: Given the system (1), (2) with \(\Gamma\) defined by (10), we say that “motion camouflage is accessible in finite time” if for any \(\epsilon > 0\) there exists a time \(t_1 > 0\) such that
\[\Gamma(t_1) \leq -1 + \epsilon. \quad (61)\]

**Proposition:** Consider the system (1), (2) with \(\Gamma\) defined by (10) and control law given by (19) - (21), with the following hypotheses:

(A1) \(0 < \nu_p^{\text{low}} \leq \nu_p^{\text{high}} < \infty\), where \(\nu_p^{\text{low}}\) and \(\nu_p^{\text{high}}\) are constants,
(A2) \(0 < \nu_e^{\text{low}} \leq \nu_e \leq \nu_e^{\text{high}} < \infty\), where \(\nu_e^{\text{low}}\) and \(\nu_e^{\text{high}}\) are constants,
(A3) \(\nu_e/\nu_p \leq \nu_{\max} < 1\), where \(\nu_{\max}\) is constant,
(A4) \(\nu_e\) and \(\nu_p\) are piecewise continuous and \(\sqrt{\nu_e^2 + \nu_p^2}\) is bounded,
(A5) \(\nu_e\) and \(\nu_p\) are piecewise continuous, \(|\dot{\nu}_p| < \alpha_p\), and \(|\dot{\nu}_e| < \alpha_e\), where \(\alpha_p\) and \(\alpha_e\) are finite constants,
(A6) \(\Gamma_0 = \Gamma(0) < 1\), and
(A7) \(|r(0)| > 0\).

Motion camouflage is accessible in finite time using high-gain feedback (i.e., by choosing \(\mu > 0\) sufficiently large).

**Proof:** Analogous to the corresponding proof in [8]. Choose \(r_o > 0\) such that \(r_o < |r(0)|\). Choose \(c_2 > 0\) sufficiently large so as to satisfy (69), and choose \(c_0\) accordingly to ensure that (65) holds for \(\Gamma > -1 + \epsilon\). Then defining \(\mu\) according to (66) ensures that \(\Gamma(T) \leq -1 + \epsilon\), where \(T > 0\) is defined by (65). □

**IV. Simulation Results**

Figures 4-7 illustrate the behavior of the three-dimensional motion camouflage system under control law (19) - (21) for the pursuer, and various open-loop curvature controls for the evader. The speeds of the pursuer and evader are constant, and the ratio of speeds is \(\nu_e/\nu_p = .9\). For each simulation, two views of the resulting three-dimensional trajectories are shown: one perpendicular to the \(r_\infty\)-direction (upper plot), and one along the \(r_\infty\)-direction (lower plot). In figure 4 the evader moves in a straight line (i.e., its curvature controls are identically zero). The corresponding motion camouflage trajectory for the pursuer is then also a straight line. The upper plot of figure 4 shows these straight-line trajectories, along with the baselines at equally-spaced intervals of time. Recall that by definition, these baselines are parallel when the system is in a state of motion camouflage. In the lower plot of figure 4 the trajectories of the pursuer and evader overlap, and the baselines are essentially normal to the page.

In figure 5 the curvature controls for the evader are sinusoidal functions of time. Whereas in figure 4 the motion is very nearly planar (with the plane determined by the initial heading of the evader), in figure 5 the motion is seen to be truly three-dimensional. Nevertheless, the baselines are observed to be nearly parallel. In figure 6 the curvature controls for the evader are randomly varying, and similarly to figure 5 the trajectories are truly three-dimensional in character, with the baselines nearly parallel. In figure 7 the curvature controls for the evader are constant and nonzero, so that the trajectory of the evader is circular.

Although there is a brief transient period at the start of each simulation during which \(\Gamma\) is driven close to \(-1\) by the control law, this transient period is such a small fraction of the total simulation time that the transient behavior is not evident in figures 4-7. The effect of the gain \(\mu\) on both the duration of the transient and the ultimate tolerance within which \(\Gamma\) remains near \(-1\) is illustrated for the planar setting in [8]. Since the bounds and estimates for the three-dimensional problem are analogous to the planar problem, similar behavior is expected.

**V. Connection to Missile Guidance**

For the planar setting, the connection between motion camouflage and the pure proportional navigation guidance (PPNG) law has been described in [8]. There is also a three-dimensional version of the PPNG law, which has been studied in [12] and [10]. The PPNG law (by definition) produces an acceleration which is perpendicular to the velocity of the missile and proportional to the angular velocity of the line of sight (LOS) vector. If \(A_M\) denotes the lateral acceleration of the missile, \(V_M\) its velocity, and \(\Omega_L\) the angular velocity of the LOS vector, then the three-dimensional PPNG law is
Thus, the MCPG law uses range information to provide high gain during the initial phase of the engagement, and ramps the gain down to a lower value in the terminal phase ($|r| \approx r_o$). This type of gain control is plausible for echolocating bats (see [5]) which have remarkable ranging ability.

VI. DIRECTIONS FOR FURTHER WORK

In the biological context, one direction being pursued is the interpretation of three-dimensional trajectory data taken from experiments in which a bat, *Eptesicus fuscus*, pursues a flying praying mantis (whose hearing organ is dislocated so that its trajectory is not influenced by the presence of the bat). The hypothesis is that the bat uses an MCPG strategy during the capture phase of its engagement with the mantis is currently being tested using experimental data collected in the Auditory Neuroethology Laboratory at the University of Maryland. This work represents part of a larger program to understand sensory-motor processing and feedback in biological model systems.

Another aspect of motion camouflage currently under study is discovering feedback laws for motion camouflage with respect to a finite point (as opposed to the point at infinity). In finite-point motion camouflage, the pursuer uses a fixed object as camouflage as it approaches the evader, and this strategy also appears to be biologically revelant. Various scenarios for motion camouflage involving teams of pursuers are also of interest, particularly in combination with formation-control laws based on gyroscopic interactions [7]. Some possible scenarios for team motion camouflage appear in [2].

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