Decays of near BPS heterotic strings

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Abstract

The decay of highly excited massive string states in compactified heterotic string theories is discussed. We calculate the decay rate and spectrum of states carrying momentum and winding in the compactified direction. The longest lived states in the spectrum are near BPS states whose decay is dominated by a single decay channel of massless radiation which brings the state closer to being BPS.
1. Introduction

In general, highly excited strings are unstable and will decay. The decay process poses interesting questions. What is the lifetime of the unstable string? What is the decay spectrum? Is the decay dominated by massless radiation or the splitting of the string into two massive strings? These questions have been investigated for various string theories in the past.

The decay of highly excited open string states was investigated in [1][2][3][4]. The lifetime of a string state of mass $m_0$ was determined to be $T \sim \frac{1}{g_s^2} \frac{1}{m_0}$. This is consistent with the facts, that an open string has a constant splitting probability per unit length and the size of an excited string grows linearly with the mass.

The decay of closed strings is expected to be qualitatively and quantitatively different. Although a generic closed string state is expected to behave similarly to the open string, the fact, that a closed string can only split (in the absence of D-branes) when two distinct points of the closed string are coincident, leads to interesting new phenomena. In a series of papers [7][8][9][10][11], Iengo and Russo found particular excited string states, whose lifetime grows with mass like $T = \frac{1}{g_s^2} m_0^5$. Therefore, for large masses, these states are very long lived. Furthermore, the decay into two massive strings is exponentially suppressed. The authors explained their results, using a semiclassical argument, relating the long lived quantum string state to a classical solution, representing a rotating ring. The string never self-intersects in this configuration, so that, the dominant decay mode is through the emission of massless radiation.

Supersymmetry provides a completely different mechanism for the stability of massive string states. A BPS state is annihilated by some supersymmetry charges and the supersymmetry algebra implies a relation between the mass and the charge of the state. Such BPS states are generically stable against decay (they can decay into other BPS states at special points of marginal stability). We ask, whether a state being near BPS, in the appropriate sense, would also be long lived.
The specific string theory, we are considering, is the heterotic string compactified on a circle of radius $R$. Due to the difficulty of constructing the vertex operators for general massive string states, we use a very specialized state, involving only level 1/2 and level 1 oscillators (i.e. states on the leading Regge trajectories). The set of states is uniquely parameterized by the mass and the left and right charges. We compute the decay rate and the spectrum for such states.

In particular, the compactification allows us to study massive states which are close to the BPS bound, since a state with many left oscillators, but few right oscillators can be constructed. We show, that these states are, indeed, made long lived by supersymmetry and that, the dominant decay mode is via the emission of massless radiation. In addition to the BPS states, satisfying $m_0^2 = k^2_L$, the heterotic string also has extremal states, which satisfy $m_0^2 = k^2_R$. We contrast the decays of 'near extremal' to the 'near BPS' states.

Let us briefly review several alternative methods that can be used to analyze the decays of massive string states before moving on to the approach used in this paper.

First, the three point function $\langle i \mid V(0) \mid f \rangle$ can be used to directly calculate the amplitude for a specific decay, where the string state $i$ changes to $f$, by the emission of a second string state, represented by the vertex operator $V$. This method is particularly efficient for calculating decays involving massless radiation and extracting the spectrum of the massless radiation.

A variation of this method considers averaging over initial states [12]. The calculation of [13][7][14] starts with the three point function and considers massless emission only, but sums over all final states with a fixed mass and averages over the initial states with a fixed mass. If the masses of the initial string state and the massive decay product are large, the resulting decay amplitude can be evaluated using a saddle point approximation. The resulting massless emission spectrum is that, of a black body, with temperature and greybody factors depending on the particular string which is considered.

A second method [13][16] extracts the mass shift for a massive string state, from the
residue of the double pole in the s-channel of the one loop scattering amplitude of four massless states. The analytic continuation in external momenta is well defined for the four point function and the double pole can be unambiguously identified.

In this paper, we opt to use the optical theorem, which relates the imaginary part of the one loop two point function, i.e. mass shift $\delta m^2_0$ of a string state of mass $m_0$, to the total decay rate $\Gamma$ of this state \cite{17,18,15}.

$$\Gamma = \frac{Im(\delta m^2_0)}{2m_0}$$ \hspace{1cm} (1.1)

The mass shift $\delta m^2_0$ is given by the one loop string amplitude with two vertex operators associated with the initial massive string state inserted.

$$\delta m^2_0 = A_2 = g_s^2 \int d^2\tau \langle V(0) \int d^2z V^\dagger(z, \bar{z}) \rangle$$ \hspace{1cm} (1.2)

where $g_s$ is the string coupling constant. It is possible to extract partial rates for the decay into two string states of a given mass. However, the rates are always inclusive, in the sense that one automatically sums over all decay products and polarizations of a given mass. The final computation is performed by expanding the amplitude in powers of $e^{2\pi i\tau}, e^{i\bar{z}}, e^{-2\pi i\bar{\tau}}, e^{-i\bar{z}}$; picking out the contributing monomials using mass and charge conservation; analytically evaluating the integrals; and numerically evaluating monomial coefficients.

The relation of black holes and excited fundamental strings goes back to \cite{19,20,21,22}. The exact identification of the (small) black hole entropy and BPS string states \cite{23,24,25,26,27} was achieved using $\alpha'$ corrections to the supergravity action \cite{28,29,30}. BPS black holes have zero temperature and are stable, whereas non BPS black holes do decay via Hawking radiation. While it is interesting to pursue the relation of the decay of perturbative string states and Hawking radiation in light of the string/black hole correspondence, we will limit ourselves to the perturbative string in this paper. For some recent work on the relation of absorption in perturbative strings and black holes, see \cite{31}. 

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The organization of this paper is as follows. In section 2, we discuss the spectrum of the heterotic string, the condition for states to be BPS, or near BPS, and construct the vertex operators of the massive string states that we will use in the rest of the paper. In section 3, we calculate the two point function for these states and extract the decay rate. In section 4, we present the results for the decay of near BPS states. In section 5, we address the decays of other string states, which are not close to BPS. We close with a brief discussion of our results in section 6.

2. BPS and near BPS states in heterotic string theory

We consider the compactification of the heterotic $SO(32)$ string on a circle of radius $R$. The $S^1$ direction is denoted by $X^9$ and the non compact spacetime directions are given by $X^\mu$ with $\mu = 0, 1, \cdots, 8$. The leftmoving modes make up the ten dimensional superstring and the rightmoving modes make up the 26 dimensional bosonic strings. The sixteen extra chiral bosons are compactified on a $SO(32)$ lattice. However, we choose not to turn on any Wilson lines along the $S^1$ and to have no oscillators along either the $S^1$ or heterotic lattice directions.

Since the right and left momenta, on the circle, break up into momentum proper, $l$, and winding, $m$, the $U(1)$ charges are of the following form (in the absence of Wilson lines)$^3$,

$$k_L = \frac{l}{R} + \frac{mR}{2}, \quad k_R = \frac{l}{R} - \frac{mR}{2} \quad (2.1)$$

The physical state condition for the left and the right movers

$$L_0 \mid \text{state} \rangle = \left( \frac{1}{2} (k^\mu k_\mu + (\frac{l}{R} + \frac{mR}{2})^2) + \bar{N} \right) \mid \text{state} \rangle = \frac{1}{2} \mid \text{state} \rangle$$

$$L_0 \mid \text{state} \rangle = \left( \frac{1}{2} (k^\mu k_\mu + (\frac{l}{R} - \frac{mR}{2})^2) + \bar{N} \right) \mid \text{state} \rangle = 1 \mid \text{state} \rangle \quad (2.2)$$

lead to the following expressions for the masses

$$m_0^2 = k_L^2 + 2\bar{N} - 1, \quad m_0^2 = k_R^2 + 2(\bar{N} - 1) \quad (2.3)$$

$^1$ In these formulas and in the rest of the paper we have set $\alpha' = 2$. 

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From which the level matching constraint follows

\[ lm + N - \bar{N} + \frac{1}{2} = 0 \]  (2.4)

The supersymmetry algebra of the heterotic string with one compactified direction is

\[ \{Q_a, Q^I_b\} = 2(CT_\mu P_+)^{ab} k^{\mu} + 2(CT^9 P_+)^{ab} k_L \]  (2.5)

Where \( k_L \) is given by (2.1). The supersymmetry algebra (2.5) implies that massive states can preserve eight of the sixteen supersymmetries if they saturate the BPS bound

\[ m^2_0 = k^2_L \]  (2.6)

It follows from (2.3) that a perturbative BPS state has \( N = \frac{1}{2} \). For notational convenience it is useful to introduce a shifted leftmoving excitation number \( K \) defined by

\[ K = N - \frac{1}{2} \]  (2.7)

A BPS state hence has \( K = 0 \) and non-BPS states have \( K > 0 \). It follows from (2.4) that if momentum and winding are nonzero a BPS state can have a large \( \bar{N} \), corresponding to a highly excited string. Since BPS states are exactly stable, in order to discuss the lifetime and decays of states, we have to move away from the BPS bound. In the following we define a measure of a state’s closeness to BPS.

\[ n_1 = \frac{2K}{m^2_0}, \quad n_2 = \frac{2(\bar{N} - 1)}{m^2_0} \]  (2.8)

A state is close to BPS if \( n_1 \rightarrow 0 \) (which implies from (2.3), that there are few left oscillators). For fixed \( n_1 \), the more excited a string is, the larger \( n_2 \) (since from (2.3) this implies the largest allowed number of right oscillators). For example, a state with large \( l \), small \( K \), and zero winding, \( m = 0 \), will have \( n_1 \sim 0 \) and is therefore near BPS. However, in this case, \( k_L = k_R \) and so, \( n_2 \sim 0 \), meaning \( \bar{N} \) is small and the string is not excited.
On the other hand, \( l = m \) with large \( l \) and small \( K \) would yield a state both close to the BPS bound and highly excited. It is this last type of state which most clearly exhibits the anticipated long lifetime.

A state which saturates the BPS bound (2.6) has equal mass and charge. In the circle compactification there is a second \( U(1) \) charge \( k_R \) and hence, there is another limit in which mass is equated with this charge, the so called ‘extremal’ limit

\[ m_0^2 = k_R^2 \]  

The condition for a highly excited extremal state is that \( n_2 \to 0 \) and \( n_1 \to 1 \). This has the same form as the BPS condition with the role of \( n_1 \) and \( n_2 \) exchanged. Note that this bound is, however, unrelated to supersymmetry.

2.1. States and vertex operators

In order to calculate the decay rate and lifetimes one has to choose a particular state and compute the related vertex operator. In the lightcone gauge, the massive spectrum is easy to construct \([32][33]\); solving the covariant physical state conditions for general massive states is quite complicated. We consider special states and polarizations to simplify the calculation. It would be interesting to generalize the calculations in this paper to more generic states.
The left moving part of the state is taken to be

\[ \xi_{\mu_1 \cdots \nu_K} b_{-1}^{\mu} a_{-1}^{\nu_1} a_{-1}^{\nu_2} \cdots a_{-1}^{\nu_K} | 0 \rangle \quad (2.10) \]

Physical state constraints of Virasoro currents and supersymmetric currents \( L_1 | \text{states} \rangle = 0 \) and \( G_{1/2} | \text{state} \rangle = 0 \) necessitate that \( \xi_{\mu_1 \cdots \nu_K} \) is transverse to the momentum in all indices. \( G_{1/2} | \text{state} \rangle = 0 \) also insure symmetry under any 2 index exchange. Finally, \( L_2 | \text{state} \rangle = 0 \) and \( G_{3/2} | \text{state} \rangle = 0 \) guarantee tracelessness in any pair of indices.

\[ \xi_{\nu_1 \nu_2 \cdots \nu_K} = 0, \quad k^\mu \xi_{\mu_1 \cdots \nu_K} = 0 \quad (2.11) \]

The polarization tensor is normalized as \( \xi_{\mu_1 \cdots \nu_K} \xi^{\mu_1 \cdots \nu_K} = 1 \).

The corresponding vertex operator in the \(-1\) picture is given by

\[ V^{(-1)}_L = \frac{1}{\sqrt{K!}} \xi_{\mu_1 \cdots \nu_K} e^{-\phi} \psi^\mu \partial X^{\nu_1} \partial X^{\nu_2} \cdots \partial X^{\nu_K} \quad (2.12) \]

In a one loop calculation the vertex operator in the zero picture instead of (2.12) is needed. The picture changing operator is given by

\[ \Xi(z) = e^{\phi} \partial X_\mu \psi^\mu (z) \quad (2.13) \]

and the zero picture operator is given by

\[ V^{(0)}(w) = \lim_{z \to w} \Xi(z) V^{(-1)}(w) |_{(z-w)^0} \quad (2.14) \]

i.e. one picks the term proportional to \((z - w)^0\) in the OPE of the vertex operator in the \(-1\) picture and the picture changing operator. The normalized vertex operator \( V^{(0)}_L \) is then given by

\[ V^{(0)}_L = \frac{\sqrt{K}}{\sqrt{(K-1)!}} \xi_{\nu_1 \cdots \nu_K} \psi^\nu \partial \psi^{\nu_1} \partial X^{\nu_2} \partial X^{\nu_2} \cdots \partial X^{\nu_K} + \cdots \quad (2.15) \]

where the dots denote terms which vanish in the two point function because of the sum over the spin structures.
The rightmoving part of the state is taken to be
\[
\bar{\xi}_{\rho_1\cdots\rho_N} \bar{\partial}^{\rho_1} \cdots \bar{\partial}^{\rho_N} | k \rangle
\]  
(2.16)
and the associated normalized vertex operator is given by
\[
V_R = \frac{1}{\sqrt{N!}} \bar{\xi}_{\rho_1\cdots\rho_N} \bar{\partial} X^{\rho_1} \cdots \bar{\partial} X^{\rho_N}
\]  
(2.17)
This state satisfies the physical state conditions if the polarization tensor \( \bar{\xi} \) is transverse, symmetric and traceless. The polarization tensor is normalized as \( \bar{\xi}_{\nu_1\cdots\nu_N} \bar{\xi}^{\nu_1\cdots\nu_N} = 1 \).

Note that we, furthermore, simplify the calculation by imposing the conditions that the polarization tensors \( \xi \) and \( \bar{\xi} \) are orthogonal and do not have any leg in the \( S_1 \) direction.
\[
\xi_{\nu_1\cdots\nu_{K+1}} \bar{\xi}^{\nu_1\cdots\mu_2\cdots\mu_S} = 0, \quad \xi_{9\cdots\nu_{K+1}} = 0, \quad \bar{\xi}^{9\cdots\nu_S} = 0
\]  
(2.18)

The complete vertex operator is then given by
\[
V(\xi, \bar{\xi}, k, k_L, k_R) = V_L^{(0)}(z) V_R(\bar{z}) e^{ik_\mu X^\mu + ik_L X_L + ik_R X_R}
\]  
(2.19)

3. Calculation of the two point function

The optical theorem relates the decay rate to the imaginary part of the mass shift at one loop. The mass shift is given by the one loop string amplitude with two vertex operators (2.19) insertions.
\[
\delta m^2 = A_2 = g_s^2 \int d^2 \tau \left\langle V(0) \int d^2 z V^\dagger(z, \bar{z}) \right\rangle
\]  
(3.1)
where the \( z \) integral is over the usual fundamental torus domain, i.e. the parallelogram stretched onto the unit vector along the real axis and the \( \tau = \tau_1 + i\tau_2 \) vector, and the \( \tau \) integral is over the fundamental domain of \( SL(2, Z) \), the conformal transformations of the torus.
In order to calculate $A_2$ we use the vertex operator (2.19) and evaluate the various contractions.

$$A_2 = c' g_s^2 K (\xi \cdot \xi)(\bar{\xi} \cdot \bar{\xi}) \int d^2 \tau \int d^2 z \langle e^{ik_\mu X^\mu}(0) e^{-ik_\mu X^\mu}(z, \bar{z}) \rangle \langle e^{ik_L X_L}(0) e^{-ik_L X_L}(z) \rangle$$

$$\langle e^{ik_R X_R}(0) e^{-ik_R X_R}(\bar{z}) \rangle \langle \partial X(0) \partial X(z) \rangle K^{-1} \langle \bar{\partial} X(0) \bar{\partial} X(\bar{z}) \rangle \hat{N}$$

$$\sum_\nu \epsilon_\nu Z_\nu \langle \psi \partial \psi(0) \psi \partial \psi(z) \rangle_\nu Z_{het}(\bar{\tau})Z,$$

(3.2)

where we used the fact that the vertex operators are normalized and the orthogonality of the polarization tensor to eliminate right with left contractions. The constant $c'$ is an overall normalization which is independent of the initial state. Since we are interested only in the relative comparison of rates and lifetimes we will set $c' g_s^2 = 1$ when evaluating the amplitudes. $Z_\nu$ is the partition function for fermions with spin structure $\nu$; $Z_{het}$ is the partition function for the bosons compactified on the $SO(32)$ lattice; $Z$ is the partition function for bosons along the first ten dimensions and for the ghosts. The factor of $\tau_2$ is from gauge fixing the conformal symmetry of the torus by choosing the first insertion point to be at 0.

The contractions and contributions to the partition functions are evaluated in Appendix A. The fermionic contractions and the sum over spin structures is performed in (A.3), the $SO(32)$ partition function is given in (A.7), the bosonic partition function is given in (A.8) and the bosonic correlators are given in (A.11) and (A.14-15). Putting everything together the result is

$$A_2 = c' g_s^2 K \sum_{n,w} \int \frac{d^2 \tau}{\tau_2^{9/2}} \int d^2 z \sum_{i=1}^4 \frac{\theta_i^{16}(0 \mid \tau)}{\eta^{24}(\tau)} \left( \partial_2^2 \ln \left( \frac{\theta_i^K(\frac{z}{2\pi} \mid \tau)}{\theta_i^L(0 \mid \tau)} \right) + \frac{1}{4\pi \tau_2} \right) K^{-1} \left( \frac{2\pi \theta_i^L(\frac{z}{2\pi} \mid \tau)}{\theta_i^L(0 \mid \tau)} \right)^2 K$$

$$\left( \partial_2^2 \ln \left( \frac{\theta_i^K(\frac{z}{2\pi} \mid \tau)}{\theta_i^K(0 \mid \tau)} \right) + \frac{1}{4\pi \tau_2} \right) \hat{N} \left( \frac{2\pi \theta_i(\frac{z}{2\pi} \mid \tau)}{\theta_i(0 \mid \tau)} \right)^{-2+2\hat{N}} e^{-\frac{n_0^2}{2\pi \tau_2^2} \bar{z}_i^2} q_L^2 q_L^2 q_R^2 e^{-iz p_L k_L e^{i z p_R k_R}}$$

where $z = z_1 + iz_2$. The sum over the integers $n, w$ represents the summation over the discrete loop momenta $p_L = n/R + wR/2$ and $p_R = n/R - wR/2$ in the compactified direction, which also makes them the left and right charges of one of the decay products.
The constant $c'$ is a normalization which is independent of all the parameters. The plan to evaluate the amplitude is as follows. First, we expand the amplitude in infinite series in powers of $q$ and $e^{iz}$ and their complex conjugates. Secondly, we perform an integration over the $\tau_1$ and $z_1$ which yield constraints on the exponents of $q = e^{2\pi i \tau}$ and $e^{iz}$ and their complex conjugates. Next, we cut off the infinite series to obtain finite polynomials, by relating the exponents to the masses of the original state and its decay products, and applying conservation of mass and charge. More precisely, not only is the series cut off, but a finite list of the exact powers of $q$ and $e^{iz}$ and their complex conjugates which contribute to the amplitude is compiled. The integrals for all the monomials, over $\tau_2$ and $z_2$ are, then, performed analytically. Unfortunately, the final computation of coefficients of all the monomials can only be performed, for specified initial states, using Mathematica for the power series expansions of the expressions.

Using the formulae given in appendix B the holomorphic part of (3.3) can be expanded as a power series in $q$ and $e^{iz}$

\[
\left( \frac{\partial_z^2 \ln (\theta_1(\frac{z}{2\pi} | \tau))}{(4\pi \tau_2)} \right)^{K-1} \left( \frac{2\pi \theta_1(\frac{z}{2\pi} | \tau)}{\theta_1'(0 | \tau)} \right)^{2K} = \sum_{r=0}^{K-1} \frac{1}{(4\pi \tau_2)^r} \left( \begin{array}{c} K-1 \\ r \end{array} \right) \sum_{a,b} C_{a,b}^r q^a (e^{iz})^b \quad (3.4)
\]

The antiholomorphic part of the amplitude (3.3) can be expanded as

\[
\left( \frac{\partial_{\bar{z}}^2 \ln (\theta_1(\frac{\bar{z}}{2\pi} | \bar{\tau}))}{(4\pi \tau_2)} \right)^{\bar{N}} \left( \frac{2\pi \theta_1(\frac{\bar{z}}{2\pi} | \bar{\tau})}{\theta_1'(0 | \bar{\tau})} \right)^{2\bar{N}-2} \sum_{i=1}^{4} \frac{\theta_i^{16}(0 | \bar{\tau})}{\eta^{24}(\bar{\tau})} = \sum_{s=0}^{\bar{N}} \frac{1}{(4\pi \tau_2)^s} \left( \begin{array}{c} \bar{N} \\ s \end{array} \right) \sum_{c,d} \bar{C}_{c,d}^s q^c (e^{-i\bar{z}})^d \quad (3.5)
\]

Using (3.4) and (3.5) the two point amplitude $A_2$ becomes

\[
A_2 = c' g_s^2 K \sum_{n,w} \int d^2 \tau \int d^2 z \sum_{r=0}^{K-1} \sum_{s=0}^{\bar{N}} \frac{\tau_2^{-r+s+\frac{1}{2}}}{(4\pi)^{r+s}} \left( \begin{array}{c} K-1 \\ r \end{array} \right) \left( \begin{array}{c} \bar{N} \\ s \end{array} \right) e^{-\frac{m^2}{2\pi^2} \nu_z^2} \times \sum_{a,b} \sum_{c,d} C_{a,b}^r \bar{C}_{c,d}^s q^a + \frac{1}{2} p_L^2 q^c + \frac{1}{2} p_R^2 e^{iz(b-k_L p_L)} e^{-i\bar{z}(d+k_R p_R)} \quad (3.6)
\]
The integration over the modulus $\tau$ in (3.6) is over the fundamental domain of $SL(2, Z)$. However, as explained in the next subsection or in [5], if one is interested in the imaginary part of $A_2$ only (as we are), the fundamental domain can be replaced by a regular strip $-1/2 < \tau_1 < 1/2$ along the entire length.

The integration over $\tau_1$ and $z_1$ then take the form:

$$\int_{-1/2}^{1/2} d\tau_1 e^{2\pi i \tau_1 (nw + a - c)} , \int_0^{2\pi} dz_1 e^{iz_1 (-nm - wl + b - d)}$$ (3.7)

so a non-zero amplitude requires the $a - c + nw = 0$ and $b - d - nm - wl = 0$. After performing the integrations over $\nu_1$ and $\tau_1$ the two point amplitude becomes

$$A_2 = 2\pi c' g^2 K \int d\tau_2 \int d\nu_2 \sum_{n, w} \sum_{r = 0}^{K-1} \sum_{s = 0}^N \left( \frac{K - 1}{r} \right) \left( \frac{\bar{N}}{s} \right) \sum_{a, b} C_{a, b}^{K, r} C_{a + nw, b - nm - wl} \tau_2^{-\left(r+s+\frac{3}{2}\right)} e^{-\nu_2 (2a - 2nl/R^2 - w R^2 / 2 - nm - wl)} e^{-\frac{m_0^2}{2\pi\tau_2} \nu_2^2}$$ (3.8)

3.1. Calculation of decay rate

After a change of variables $2\pi \tau_2 = t$ and $\nu_2 = ty$ the integrals in (3.8) are all of the following form

$$\tilde{a}_2 = \int_0^\infty dt \int_0^1 dy \, t^{\alpha - 1} e^{-t\left(m_0^2 y^2 - y(m_0^2 - m_1^2 + m_2^2)\right)}$$ (3.9)

where $\alpha = -r - s - 5/2$ and is negative.

The choice of variable names $m_1, m_2$ is not accidental. By considering a Schwinger parametrization of a Feynman integral for a loop amplitude, we can convince ourselves that the coefficients of powers of $y$ are associated with the masses of the decay products as above [1], [5], [8]. For completeness, the argument is reproduced in Appendix C.

A comparison of (3.9) with (3.8) gives the following expressions for the masses $m_0, m_1, m_2$

$$m_0^2 = \left( \frac{l}{R} + \frac{m R}{2} \right)^2 + 2K$$
$$m_1^2 = \left( \frac{l - n}{R} + \frac{(m - w) R}{2} \right)^2 + 2(a + b + K)$$
$$m_2^2 = \left( \frac{n}{R} + \frac{w R}{2} \right)^2 + 2a$$ (3.10)
Note that a comparison with the mass-shell condition (2.3) leads to an identification of the momentum \( k_{1,L}, k_{2,L} \) as well as the oscillator level \( K_1, K_2 \) of the two decay products with mass \( m_1 \) and \( m_2 \) respectively.

\[
\begin{align*}
k_{1,L} &= k_L - p_L = \left( \frac{l-n}{R} + \frac{(m-w)R}{2} \right), \quad K_1 = a + b + K \\
k_{2,L} &= p_L = \left( \frac{n}{R} + \frac{wR}{2} \right), \quad K_2 = a
\end{align*}
\] (3.11)

where \( k_L \) is the leftmoving compact momentum of the initial state and \( p_L \) is the compact loop momentum. As we shall see later the power series expansion of the amplitude limits the range of \( b \) to be \(-a - K \leq b \leq -a\).

The optical theorem relates the imaginary part of the loop amplitude to the summed squares of the tree three point amplitudes. The loop amplitude is formally real but the imaginary part comes from the analytic continuation of the amplitude into a region where it is divergent. From (3.9) we can see that the integral is divergent when the polynomial coefficient of \( t \) in the exponent,

\[
P(y) = m_0^2 y^2 - y(m_0^2 - m_1^2 + m_2^2) + m_2^2
\] (3.12)

is negative, which occurs between the roots of this polynomial,

\[
y_{\pm} = \frac{m_0^2 - m_1^2 + m_2^2}{2m_0^2} \pm \frac{1}{m_0} \sqrt{\frac{(m_0^2 - m_1^2 + m_2^2)^2}{4m_0^2} - m_2^2}
\] (3.13)

This restricts the domain of integration in the \( y \) variable. Note that the roots are real and the imaginary part of \( A_2 \) exists if and only if \( m_0 > m_1 + m_2 \).

Let us now turn to the analytic continuation of the \( t \) integral. The two divergences to deal with are a pole at \( t = 0 \) and an essential singularity at \( t = \infty \). We will analytically continue in \( P \) and \( \alpha \). The integral is well defined for \( Re[P] > 0, Re[\alpha] \geq 1 \). We make the substitution \( u = tP \). The \( t \) integral is then:

\[
\frac{1}{P^\alpha} \int_{tP}^\infty \exp^{-u} u^{\alpha-1} du = \frac{1}{P^\alpha} \int_0^\infty \exp^{-u} u^{\alpha-1} du - \frac{1}{P^\alpha} \int_0^{tP} \exp^{-u} u^{\alpha-1} du
\] (3.14)
$P\epsilon$ is the variable lower limit inherited from the $\tau$ domain. In a moment we will see that the exact lower limit does not matter in the calculation, because the second integral is real for all values of $P$ and $\alpha$, and therefore, the complicated $\tau$ domain can be replaced by a strip extending down to 0. (It may worry the reader that this integral domain now includes infinite copies of the fundamental domain and should diverge by modular invariance. However, modular invariance has, already, been broken in our calculation because we do not keep the entire modular invariant expression for the amplitude, but instead pick out only a finite number of monomials.) More explicitly, if we analytically continue the first integral to negative $P$ we obtain:

$$
\frac{1}{(-1)^\alpha (P)^\alpha} \int_0^\infty \exp^{-u} u^{\alpha-1} du = \frac{\pm i}{(-P)^\alpha} \Gamma(\alpha) = \frac{\pm i}{(-P)^\alpha} \frac{\pi}{\Gamma(1-\alpha) \sin(\alpha \pi)} \tag{3.15}
$$

where we used that $\alpha$ is a half-integer (see (3.8)). The sign in (3.15) is chosen to give a positive mass shift. Now the expression can be analytically continued for negative $P$ and negative $\alpha$. For such values the expression is pure imaginary. Now let’s see that there’s no imaginary contribution from the second integral in (3.14). Let $P$ take its negative value and make the variable change: $u \rightarrow -u$ (or equivalently make the $u = tP$ substitution on the $0 - \epsilon$ integral before doing any analytic continuation in $P$)

$$
(-1)^{\alpha-1} \frac{1}{(-P)^\alpha} \int_0^{-P\epsilon} \exp^{u} u^{\alpha-1} du = - \frac{1}{(-P)^\alpha} \int_0^{-P\epsilon} \sum_n \frac{u^n}{n!u^{\alpha-1}} = e^\epsilon \sum_n \frac{(-P\epsilon)^n}{n!(n-\alpha)} \tag{3.16}
$$

This expression is convergent with no poles and it is still so if analytically continued for negative $\alpha$. Since this integral has no imaginary part and the first integral had no dependence on the lower bound, the fundamental domain of $\tau$ can be replaced by a strip

---

2 Note that, were the number of compactified dimensions even, $\alpha$ would be an integer and the analytic continuation would need slight modification.
extending all the way down to zero. All that’s left is the \( y \) integration.

\[
\int_{y^-}^{y^+} (-P)^{-\alpha} dy = \int m_0^{-2\alpha} (y - y^-)^{-\alpha} (y^+ - y)^{-\alpha} m_0^{-2\alpha} (y^+ - y_-)^{1-2\alpha} \frac{\Gamma(1-\alpha)^2}{\Gamma(2-2\alpha)}
\]

(3.17)

where the roots, \( y^+ \) and \( y^- \), were defined in (3.13) and \( \alpha = -r - s - 5/2 \).

Combining with the \( t \) integral we obtain

\[
Im(\tilde{a}_2) = 2^{2r+2s+6} \pi \Gamma(r + s + 7/2) \left( \left( \frac{m_0^2 - m_1^2 + m_2^2}{4m_0^2} - m_2^2 \right)^{r+s+3} \right.
\]

(3.18)

Where the \( \Theta \)-function enforces the on-shell condition \( m_0 > m_1 + m_2 \). Applying (3.18) to the terms appearing in the summation (3.8) gives the imaginary part of the two point amplitude

\[
Im(A_2) = c'g_s^2 K \sum_{r=0}^{K} \sum_{s=0}^{\tilde{N}} \sum_{a,b} \Theta(m_0 - m_1 - m_2) \left( \frac{K - 1}{r} \right) \left( \frac{\tilde{N}}{s} \right) \pi^{\frac{11}{2}} 2^{21/2+r+s}
\]

\[
\times C_{a,b}^r \frac{ \Gamma(7/2 + r + s) }{ \Gamma(7 + 2r + 2s) } \left( \frac{m_0^2 - m_1^2 + m_2^2}{4m_0^2} - m_2^2 \right)^{3+r+s}
\]

(3.19)

The total decay rate is given by

\[
\Gamma = \frac{Im(\delta m_2^2)}{2m_0} = \frac{Im(A_2)}{2m_0}
\]

(3.20)

The lifetime of the unstable string state is the inverse of the total decay rate

\[
T = \frac{1}{\Gamma}
\]

(3.21)

Note that the summation over the loop momentum \( n \) and winding \( w \) and \( a, b \) sums over all particles with mass \( m_1 \) and \( m_2 \) given by (3.10). If the on shell condition is fulfilled and \( C_{a,b}^r \) is nonzero then the the original string state can decay in this channel. The decay rate in this channel is given by the imaginary part (3.19) for a fixed \( n, w, a, b \).
4. Decays of near BPS states

Massive BPS states are generically stable against decays and hence one has to move away from exact BPS states to investigate the lifetime of excited string states. An excited string state represented by the vertex operator (2.19) is characterized by the momentum, \( l \), winding number, \( m \), and the left and right moving excitation level \( K \) and \( \bar{N} \), respectively. The level matching condition (2.4) provides one relation among the four parameters. The measure of how close a state is to BPS is therefore not unique and depends on the definition.

In the following we will consider states which have \( n_1 \) close to 0 and \( n_2 \) close to 1. This can be achieved by choosing \( K \) to be close to zero and choosing \( l, m \) such that \( k_L \) is large and \( k_R \) is small, so that the state has a large \( \bar{N} \). In the following, we also fix the radius of the compactification circle to be \( R = 1 \).

It is expected that the states which are closest to BPS are the longest lived for a given mass. It is therefore interesting to determine the dependence of the lifetime on the mass \( m_0 \) of the initial state. In the following plot, states with \( K = 1 \) and \( p_R = 0, \frac{1}{2}, 1 \) were considered with \( p_L \) varying such that the mass ranges from \( m_0^2 = 98 \) to \( m_0^2 = 308 \).

![Log/Log plot of lifetime T of K = 1 as a function of mass m_0^2. The masses range from m_0^2 = 98 to m_0^2 = 308.](image)

**Fig. 2:** Log/Log plot of lifetime \( T \) of \( K = 1 \) as a function of mass \( m_0^2 \). The masses range from

A \( \chi^2 \)-fit for the mass dependence of the lifetime gives

\[
T = e^{-8.11}(m_0)^{6.06}
\]  

(4.1)
As we shall show below, the exact behavior of the lifetime of a near BPS state in the limit of large masses is $T \sim m_0^6$.

In the following table we list the decay rates and the dominant decay channel of near BPS states with momentum $l = 9$ and winding number $m = 16$ for varying $K$. The lifetimes are calculated using (3.20) setting $g_s^2 c' = 1$ to fix the normalization.

| $K$ | $m_0^2$ | $n_1$ | $n_2$ | lifetime | dominant decay channel | per cent |
|-----|---------|-------|-------|----------|------------------------|---------|
| 1   | 291     | 0.0068| 0.996 | 8705.5   | $K_1 = 0, m_1^2 = 289, m_2^2 = 0$ | 99.9    |
| 2   | 293     | 0.013 | 0.996 | 4376.2   | $K_1 = 1, m_1^2 = 290.7, m_2^2 = 0$ | 98.3    |
| 4   | 297     | 0.0269| 0.996 | 2212.5   | $K_1 = 3, m_1^2 = 294.8, m_2^2 = 0$ | 95.0    |
| 6   | 301     | 0.039 | 0.996 | 1492.1   | $K_1 = 5, m_1^2 = 298.9, m_2^2 = 0$ | 92.0    |

Table 1: Lifetime and dominant decay channel of near BPS states

For near BPS states the lifetime increases when $n_1$ gets closer to zero, which is a measure of closeness to BPS, hence the closer to BPS a state (of comparable mass) is, the longer lived it is. The decay is dominated by a single decay channel, where one of the decay products is a massless graviton and the other is a massive string state where $K$ has been decreased by one (subleading decay channels decrease $K$ by more than one unit but give a small contribution to the total rate). Hence a massive near BPS state will gradually become more BPS by massless radiation.

The mass dependence (4.1) and the dominance of the massless decay channel can be understood in more detail by analyzing (3.19). Since the decay amplitude is symmetric under interchange of $m_1$ and $m_2$, for a massless decay we can pick the second particle to be massless, i.e. $m_2 = 0$. Conservation of energy implies that the first decay product has mass $m_1^2 = k_L^2 + 2(K - \Delta K)$, where $\Delta K$ is a positive integer related to the decrease in non BPS-ness. For the massless decays the last factor in (3.19) then becomes

$$\left( \frac{(m_0^2 - m_1^2 + m_2^2)^2}{4m_0^2} - m_2^2 \right)^{3+r+s} = \left( \frac{\Delta K^2}{m_0^2} \right)^{3+r+s} \quad (4.2)$$

16
The inverse powers of $m_0$ in (4.2) suppress terms with $r,s > 0$, unless these inverse powers are balanced by combinatorial factors coming from the power series expansion. The combinatorial factors on the holomorphic side are small for near BPS decays because they are of order $K$ and $K$ is small, but on the antiholomorphic side combinatorial factors can be large since they can be of order $\tilde{N}$, i.e. of order $m_0^2$ for near BPS states, so we can set $r = 0$ to a good approximation, but different $s$ can contribute equally. Now let’s see what powers of $q, \bar{q}, e^{iz}, e^{-iz}$ we’re after. It follows from $m_2 = 0$ and (3.10) that $a = 0$ as well as $n = w = 0$. This means that in the power series expansion (3.6) we are looking for the terms with $q^0$. In this case the mass of the first decay product is given by $m_1^2 = m_0^2 + 2b$ and it follows from $m_0 > m_2$ that we are looking for a negative power of $(e^{iz})^b$ in (3.4).

From Appendix B we can see that if we keep only terms proportional to $q^0$ and set $r = 0$ we get

$$\left(\sum_{l=1}^{\infty} le^{il\bar{z}}\right)^{K-1}(ie^{-i\bar{z}/2}(1 - e^{i\bar{z}}))^{2K}$$

(4.3)

We can see that only the first negative power of $e^{iz}$ is present so $b = -1$. Comparing with (3.11) one sees, for $a = 0$, $b = \Delta K$, which explains why the dominant decay channel has $\Delta K = 1$. On the antiholomorphic side, from (3.7), we know that $c = 0$ and $d = -1$, so we are looking for $\bar{q}^0 e^{i\bar{z}}$. Keeping only $q^0$ terms from (3.5) we obtain

$$\left(\frac{1}{\bar{q}} + 24\right)(2 + 960\bar{q})(1 + (-6 + 2\tilde{N})\bar{q})(ie^{i\bar{z}/2}(1 - e^{-i\bar{z}} - \bar{q}e^{-2i\bar{z}}))^{2\tilde{N}-2}$$

$$\times \left(\left(\sum_{l=1}^{\infty} le^{-i\bar{z}l}\right)^{\tilde{N}-s} + (\tilde{N} - s)\bar{q}(e^{-i\bar{z}} + e^{i\bar{z}})(\sum_{l=1}^{\infty} le^{-i\bar{z}l})^{\tilde{N}-s-1}\right)$$

(4.4)

Expanding the above expression shows that for small $s$ the coefficient of $\bar{q}^0 e^{i\bar{z}}$ goes like $\tilde{N}^s$ so small $s$ contribute to the computation at the same order of $m_0$. (For large $s$ the combinatorial factors no longer go like $\tilde{N}^s$.) Therefore we see that, in the limit of large initial mass, the decay rate of near BPS states behaves like $\Gamma \sim 1/m_0^6$ in accordance with the numerical result (4.1).
5. Decays of other states

The calculation of the decay rate and lifetimes is not limited to near BPS states. It is an important question, whether the near BPS states are indeed the longest lived states. To this end, we have calculated the lifetimes for all initial states of a given fixed mass, $m_0^2 = 171$. For radius $R = 1$, there are 134 possible initial states. In the following table, we list the ten longest lived states among them.

| $k_L$ | $k_R$ | $K$ | $n_1$ | $n_2$ | lifetime | decay channel |
|------|------|-----|------|------|---------|---------------|
| -13  | -1   | 1   | 0.0117 | 0.9942 | 1750    | massless      |
| -13  | 1    | 1   | 0.0117 | 0.9942 | 1683    | massless      |
| -13  | -3   | 1   | 0.0117 | 0.9474 | 1645    | massless      |
| -13  | 3    | 1   | 0.0117 | 0.9474 | 1197    | massless      |
| -13  | 5    | 1   | 0.0117 | 0.8538 | 375.8   | light         |
| -13  | -5   | 1   | 0.0117 | 0.8538 | 99.75   | light         |
| -13  | 7    | 1   | 0.0117 | 0.7135 | 85.95   | light         |
| -5   | 13   | 73  | 0.8538 | 0.0117 | 62.24   | light         |
| -7   | 13   | 61  | 0.7135 | 0.0117 | 41.95   | light         |
| -11  | -3   | 25  | 0.2924 | 0.9474 | 38.32   | massless      |
| ...  | ...  | ... | ...   | ...   | ...     | ...           |
| 7    | +13  | 61  | 0.7135 | 0.0117 | 0.0152  | light         |
| -13  | +13  | 1   | 0.0117 | 0.0111 |         | massless      |

Table 2: Longest lived states of mass $m_0^2 = 171$

The longest lived states are indeed the near BPS ones with massless radiation as dominant decay channel. A near extremal state appears as the 8th longest lived state, the lifetime is however 30 times shorter than that of the longest lived near BPS state. The dominant 'light' decay channels consist of BPS states with nonzero winding or momentum and a mass close to zero. The states with dominant 'light' decay channels exhibit shorter lifetimes compared to the states with dominant massless decays.

For comparison purposes, we have listed the shortest lived state among the 134 states of mass $m_0^2 = 171$. Note that the state is technically near BPS since $n_1$ is small, however,
it is not highly excited, since \( n_2 \) is small. Hence, it is the highly excited near BPS states which are long lived.

In contrast to the BPS condition, extremality (2.9), is not related to supersymmetry and it is an interesting question whether highly excited extremal states are also long lived. Already from table 2 it is clear that extremal states are shorter lived than BPS states (for a particular mass). We have calculated the lifetimes of near extremal states with \( \bar{N} = 2 \) and masses ranging from \( m_0^2 = 38 \) to \( m_0^2 = 258 \).

![Fig. 3: Log/Log plot of lifetime \( T \) of \( \bar{N} = 2 \) near extremal states as a function of mass \( m_0^2 \).](image)

A fit of the data produces a lifetime growing as \( T \sim m_0^{2.04} \), however, note that, compared to the near BPS states, the lifetimes of the near extremal states are several orders of magnitude shorter.

A general feature of all decays, whether the initial state is near BPS or not, is that all decays are dominated (to at least 99% of the amplitude) by channels where one decay product (say the first one) has oscillator level \( K_1 = 0 \). Note that, in general, the dominant decay channel may be not massless, but the mass of the state will come from momentum and/or winding, not from left oscillators. To understand this, we must again look at the expansion of the holomorphic piece (3.4), but we do not assume \( a = 0 \). Remember that \( a = K_2 \) and \( a + b + K = K_1 \), which imply together with mass conservation, that \( -K \leq a + b < 0 \) and \( a < K \), where \( a \) is the power of \( q \) and \( b \) is the power of \( e^{iz} \). Also, remember that terms with \( r > 0 \) are suppressed. One obvious term one can pick out for
\( r = 0 \) is \((qe^{-iz})K^{-1}e^{-izk}\). (This is obtained by picking up from (B.2) only terms from second sum with \(l, n = 1\) and only terms with \(n = 1\) in (B.3).) This term has \(a + b = -K\), i.e. \(K_1 = 0\), and \(a = K - 1\), as we wanted to argue. Now suppose we try to pick \(a + b\) less negative. Whether you do this by picking \(n > 0\) in one of the \(2K\) factors in (B.3) or by choosing \(n > 1\) in one factor of (B.2), the power of \(q\) becomes too large and we are forced to use \(r > 0\), which produces suppressed decay rates.

6. Discussion

In this paper we have analyzed the decay of highly excited string states for compactified heterotic strings. We found that the longest lived states are highly excited, near BPS states, for which the lifetime grows as \(T \sim 1/g_s^2(m_0)^6\) for large masses. The decay of such states is dominated by massless radiation, which reduces the non-BPS’ness \(K\) by one. The end result of such a decay (over several steps) is a stable BPS state. For large mass \(m_0\), the near BPS states is very long lived. Near extremal states and generic states, which have both left and right excitations, are much shorter lived, although the lifetime of near extremal states grows with mass. It is an interesting question, whether the decay of excited strings can be used for exploring the string/black hole correspondence. To achieve this, one has to consider more generic string states, instead of states on the leading Regge trajectories. Some steps were taken in this direction, for the comparison of absorption of massless scalars by perturbative strings and BPS black holes, in a recent paper [31], and it would be interesting to generalize this work to decays of near BPS states. We plan to address this question in the future.

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Appendix A. Collection of formulae for one loop string amplitudes

The partition functions of the fermionic leftmovers with even spin structure $\nu = 2, 3, 4$ are

$$Z_2 = \frac{\theta_2(0 \mid \tau)^4}{\eta(\tau)^4}, \quad Z_3 = \frac{\theta_3(0 \mid \tau)^4}{\eta(\tau)^4}, \quad Z_4 = \frac{\theta_4(0 \mid \tau)^4}{\eta(\tau)^4} \quad (A.1)$$

The propagator for two world sheet spinors for the three even spin structures, denoted by $\nu = 2, 3, 4$, is given by

$$\langle \psi^\mu(z)\psi^\nu(w) \rangle_\nu = \eta^\mu\nu \frac{1}{2\pi} \frac{\theta_\nu(\frac{z-w}{2\pi} \mid \tau)}{\theta_\nu(0 \mid \tau)} \frac{\theta_1'(0 \mid \tau)}{\theta_1(\frac{z-w}{2\pi} \mid \tau)} \quad (A.2)$$

Correlation functions for the odd spin structure $\nu = 1$ vanish, unless there are ten fermionic zero modes inserted, which is not the case for our amplitude.

After summing over spin structures, the correlators involving fewer than four fermionic fields vanish. Also, fermionic correlators involving four fermions without derivatives vanish.

The only nonzero fermionic correlator in our calculation of the decay rate is

$$\sum_\nu \epsilon_\nu Z_\nu(\tau) \langle \psi^\mu(z)\partial_\nu\psi^\sigma(z)\psi^\rho(w)\partial_\omega\psi^\lambda(w) \rangle_\nu = (\eta^\mu\lambda\eta^{\sigma\rho} + \eta^\mu\rho\eta^{\sigma\lambda})\eta(\tau)^8 \quad (A.3)$$

where $\epsilon_\nu$ are phases implementing the GSO projection via $\epsilon_2 = -1, \epsilon_3 = 1, \epsilon_4 = -1$.

Calculating (A.3) and vanquishing the other fermionic correlators is accomplished by applying the following Riemann identity to derivatives of (A.2) and (A.1).

$$\frac{1}{2} \sum_\nu \epsilon_\nu \prod_{i=1}^4 \theta_\nu(v_i \mid \tau) = - \prod_{i=1}^4 \theta_1(v'_i) \quad (A.4)$$

where

$$v'_1 = \frac{1}{2}(-v_1 + v_2 + v_3 + v_4), \quad v'_2 = \frac{1}{2}(+v_1 - v_2 + v_3 + v_4)$$

$$v'_3 = \frac{1}{2}(v_1 + v_2 - v_3 + v_4), \quad v'_4 = \frac{1}{2}(v_1 + v_2 + v_3 - v_4) \quad (A.5)$$

and

$$\theta_1'(0 \mid \tau) = 2\pi\eta(\tau)^3 \quad (A.6)$$
The partition function for the bosonic rightmovers in the $SO(32)$ lattice directions is
\[ Z_{het} = \frac{1}{2} \sum_{i=1}^{4} \frac{\theta_{i}(0 \mid \bar{\tau})^{16}}{\eta(\bar{\tau})^{16}} \quad (A.7) \]

The partition function (including the zero modes integral) of the nine left and right moving bosons in the uncompactified spacetime directions is
\[ Z_{st} = \int d^{9}p e^{-p^{2} \pi \tau_{2}} \frac{1}{\eta(\tau)^{9} \eta(\bar{\tau})^{9}} = \frac{1}{\tau_{2}^{9/2} \eta(\tau)^{9} \eta(\bar{\tau})^{9}} \quad (A.8) \]

The correlation function for noncompact worldsheet bosons $X^{\mu}, \mu = 0, \cdots, 8$ is given by
\[ \langle X^{\mu}(z, \bar{z})X^{\nu}(w, \bar{w}) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln \left( \frac{\mid E(z, w) \mid^{2} + \frac{1}{8\pi \tau_{2}}(z - w - \bar{z} + \bar{w})^{2}}{\alpha' k_{i}k_{j}} \right) \quad (A.9) \]
where the prime form $E$ is defined as
\[ E(z, w) = 2\pi \theta_{1}(\frac{z - w}{2\pi} \mid \tau) \quad (A.10) \]

The correlators involving $\partial X^{\mu}$ can be derived from (A.9) and read
\[ \langle \partial X^{\mu}(z) \partial X^{\nu}(w) \rangle = \eta^{\mu\nu} \frac{\alpha'}{8\pi \tau_{2}} \partial_{\bar{z}}^{2} \ln \left( \theta_{1}(\frac{z - w}{2\pi} \mid \tau) \right) + \eta^{\mu\nu} \frac{\alpha'}{8\pi \tau_{2}} \quad (A.11) \]

Using (A.9) the following correlation function is also calculated:
\[ \langle \prod_{i=1}^{n} e^{ik_{i}X(z_{i}, \bar{z}_{i})} \rangle = \prod_{i<j} \mid E(z_{i}, z_{j}) \mid^{\alpha' k_{i}k_{j}} \exp \left( \frac{\alpha'}{16\pi \tau_{2}} \sum_{i<j} k_{i}k_{j}(z_{i} - \bar{z}_{i} - z_{j} + \bar{z}_{j})^{2} \right) \quad (A.12) \]

In the 9th direction, compactified along the $S_{1}$, the zero modes integral becomes a sum since the momenta are discrete, so the partition function for the boson along the $S_{1}$ direction is
\[ Z_{S_{1}} = \sum_{p} e^{i\pi \tau p^{2} - i\pi \bar{\tau} p^{2}} \frac{1}{\eta(\tau)\eta(\bar{\tau})} \quad (A.13) \]

For compact bosons we need a 'chiral' version of (A.12)(derived in [34]),
\[ \langle \prod_{i=1}^{n} e^{ikX(z_{i})} \rangle = \exp(ip \sum_{i=1}^{n} k_{i}z_{i}) \prod_{i<j} E(z_{i}, z_{j})^{\frac{\alpha'}{2} k_{i}k_{j}} \quad (A.14) \]
where \( p \) is the loop momentum and \( k_i \) is the left moving incoming momentum.

For the antiholomorphic correlators one gets

\[
\langle \prod e^{ik_i X(\bar{z}_i)} \rangle = \exp(-ip \sum_{i=1}^{n} \bar{k}_i \bar{z}_i) \prod_{i<j} \bar{E}(\bar{z}_i, \bar{z}_j) \alpha' \bar{k}_i \bar{k}_j
\]  

(A.15)

where \( \bar{p}_i, \bar{k}_i \) are the corresponding right moving momenta. Note that, since there is winding, a compact boson has \( k_i \neq \bar{k}_i \). We can check, that if multiply (A.14), (A.15), and \( Z_{S_1} \) and replace the sums over \( p_i \) by integrals, we recover the nonchiral correlator times the partition function of one uncompactified boson i.e. (A.12) multiplied by \( \frac{1}{\tau_2^{1/2} \eta(\tau)\eta(\bar{\tau})} \).

The last piece we need is the ghost partition function

\[
Z_{gh} = \frac{\eta(\tau)^2 \eta(\bar{\tau})^2}{\tau_2}
\]  

(A.16)

Finally, to account for the gauge fixing of the conformal invariance of the torus i.e. fixing the first insertion point to be at 0, we must augment \( Z_{gh} \) with a factor of \( \tau_2 \). All the factors of \( \tau_2 \) combine to yield \( \tau_2^{-9/2} \).

For notational convenience we define a single partition function combining the bosonic and ghost partition function

\[
Z = Z_{st} Z_{S_1} Z_{gh}
\]  

(A.17)

### Appendix B. Expansion of theta functions

In this appendix we gather the formulae for the series expansion of theta functions and related objects in \( q \) and \( e^{iz} \). The basic expansion formulae for the theta functions are

\[
\begin{align*}
\theta_1(v \mid \tau) &= i \sum_n (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} e^{iz(n-\frac{1}{2})} \\
\theta_2(v \mid \tau) &= \sum_n q^{\frac{1}{2}(n-\frac{1}{2})^2} e^{iz(n-\frac{1}{2})} \\
\theta_3(v \mid \tau) &= \sum_n q^{\frac{1}{2}n^2} e^{izn} \\
\theta_4(v \mid \tau) &= \sum_n (-1)^n q^{\frac{1}{2}n^2} e^{izn}
\end{align*}
\]  

(B.1)
From these basic formulas the expansion of the terms appearing in appendix A can be derived

\[ \partial_z^2 \ln \left( \frac{\theta_1}{2\pi} \right) = \sum_{l=1}^{\infty} l (e^{iz})^l + \sum_{l>0, n>0} l q^{n l} (e^{iz})^l + (e^{iz})^{-l} \] (B.2)

and

\[ 2\pi \theta_1 \left( \frac{\theta_1}{2\pi} \right) = ie^{-iz/2} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{4} (n(n-1))} e^{izn} \frac{1}{\prod_{m>0} (1 - q^m)^3} \] (B.3)

and similarly for the antiholomorphic terms.

**Appendix C. Identification of Masses in Amplitude by Comparison with Schwinger Reparametrized Feynman Integral**

Consider the Feynman integral for the one loop amplitude. For simplicity, let’s first consider the loop particles to be scalars.

\[ \int_{-\infty}^{\infty} d^d p \frac{1}{p^2 + m_0^2} \frac{1}{(p - k)^2 + m_1^2} \] (C.1)

where \( p \) is the loop momentum, \( k \) is the incoming momentum, \( d \) is the uncompactified spacetime dimension (which is 9 for us). We preform a Schwinger reparametrization and obtain

\[ -\int_{-\infty}^{\infty} d^d p \int_0^\infty d\gamma \int_0^\infty d\beta e^{-i\gamma (p^2 + m_0^2)} e^{-i\beta ((k-p)^2 + m_1^2)} \] (C.2)

Performing the momentum integration we obtain

\[ -\int_0^\infty d\gamma \int_0^\infty d\beta e^{-i\gamma m_0^2 - i\beta (k^2 + m_1^2)} e^{\frac{k^2 \beta^2}{\gamma + i\beta}} \left( \frac{\pi}{\gamma + i\beta} \right)^\frac{d-1}{2} \] (C.3)

Finally let’s change to variables \( y = \frac{\beta}{\gamma + i\beta}, t = i\gamma + i\beta \), and rotate the \( t \) contour to run along the real axis.

\[ \int_0^\infty dt \int_0^1 dy \left( \frac{\pi}{t} \right)^{\frac{d-1}{2}} e^{-t (m_0^2 y^2 - y (m_0^2 + m_1^2) + m_1^2)} \] (C.4)
where we used $-k^2 = m_0^2$ and that the jacobian for the variable change is $-t$. This integral looks quite similar to (3.9), except the power of $t$ in (3.9) is $-r - s - (d - 1)/2$ whereas here it is just $(d - 1)/2$. The second issue is that there is no reason to assume that the decay products are scalars. Let us, then, consider what happens with some arbitrary polarization. Note that the precise polarizations allowed by string theory are not arbitrary. Furthermore, the states we are working with have even more specialized polarizations. We are not trying to establish an exact correspondence between the string theory calculation and the equivalent field theory calculation, but only to extract the masses of the decay products. To that end, we just note, that the integral must be a linear combination of integrals such as the following

$$
\int_\infty^{-\infty} p^{2l} (p^k)^c \frac{1}{p^2 - m_1^2 (p - k)^2 - m_2^2} d^d p \tag{C.5}
$$

The powers of $p$ must be expanded in components so the expression after Schwinger reparametrization looks something like

$$
-\int_{-\infty}^{\infty} d^d p \int_0^{\infty} d\gamma \int_0^{\infty} d\beta \sum_{j_i=2l+c} \prod_i D_i p_i^{j_i} e^{-i\gamma(m_1^2 - m_2^2)} e^{-i\beta((k - p)^2 - m_2^2)} \tag{C.6}
$$

where $D_i$ are some coefficients, which might depend on $k_i$. The thing to notice here is, that though this expression is much messier than the one for scalars, the exponentials are unchanged, so performing the gaussian integrals we get something like

$$
\sum_\sigma D_\sigma' \int_0^{\infty} d\gamma \int_0^{\infty} d\beta e^{-i\gamma m_1^2 - i\beta(k^2 + m_2^2)} e^{-\frac{k^2 \Delta^2}{i\gamma + i\beta}} \left(\frac{\pi}{i\gamma + i\beta}\right)^{\frac{d}{2} + g_\sigma} \tag{C.7}
$$

The momentum integral must be performed in components so the sum is complicated; $D_\sigma'$ are some messy coefficients, which can be functions of $y = \beta/(\gamma + \beta)$ and $k_i$; $g_\sigma$ are integer-valued. Looking at the gaussian formula, one can be convinced, that the form of the exponentials remains exactly the same for each term as in the scalar case. Therefore, as in the scalar case, we’re justified in associating the masses with the coefficients of the $y$ powers in the exponent of the exponentials.
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