Perturbative BV-BFV formalism with homotopic renormalization: a case study

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Abstract

We report a rigorous quantization of topological quantum mechanics on $\mathbb{R}_{\geq 0}$ and $I = [0, 1]$ in perturbative BV-BFV formalism. Costello’s homotopic renormalization is extended, and incorporated in our construction. Moreover, BV quantization of the same model studied in previous work [WY22] is derived from the BV-BFV quantization, leading to a comparison between different interpretations of “state” in these two frameworks.

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1 Introduction

Among mathematical frameworks for quantum field theory (QFT), the one developed by Costello [Cos11] is a candidate to systematically study perturbative gauge theories. This framework uses Batalin-Vilkovisky (BV) formalism [BVSI] to describe gauge symmetry, and contains a procedure
which we call “homotopic renormalization” to deal with UV problem. These two ingredients are
combined together in the language of homological algebra. A successful example quantized in this
formalism is BCOV(Bershadsk-Cecotti-Ooguri-Vafa) theory, see [CL12][CL15][CL20]. The corre-
sponding string field theory at all topology can only be quantized by this method so far, which is a
degenerate BV theory coupled with large N gauge field.

However, this framework needs modification when the spacetime manifold has boundary. In order
to define the theory still within BV formalism, we need to impose boundary condition and work on a
“restricted bulk field space” only (see e.g., [Rab21] [WY22] [GRW20] [GW19] [Zen21] for this approach).
In this setting, renormalization is less developed than the closed spacetime case, and we can find
discussions for specific models, for examples, in [Alb16, Rab21].

Alternatively, we could generalize BV formalism itself to study QFT on manifold with boundary.
A potential generalization is proposed by Cattaneo, Mnev and Reshetikhin [CMR14, CMR18], called
“BV-BFV formalism”\(^1\). Its central notion, called “modified quantum master equation (mQME)”,
characterizes that the anomaly to quantize the theory in BV formalism is controlled by certain
boundary data. This formalism also contains formulation for (gauge) field theories on manifold with
corners, hence may help with topics such as functorial QFT and bulk-boundary correspondence.

While BV-BFV formalism has been shown fruitful even only at classical level (see e.g., [CS16,
CS19] [Sch15] [RS21] [MS22]), it has not incorporated a systematic renormalization for quantization.
For topological field theories, a successful example in quantum level can be found in [CMR20]. For
field theories which are not topological, we need to add counter terms to make the contributions of
Feynman graphs finite. Counter terms on manifolds with boundaries in homotopic renormalization
has been discussed in [Rab21]. To quantize field theories which are not topological in BV-BFV
formalism, We hope to adapt Costello’s homotopic renormalization to BV-BFV formalism. As the
first step, we would like to clarify the relation between BV-BFV formalism and the approach within
BV formalism mentioned above.

1.1 Main results

We use AKSZ type [ASZK97] topological quantum mechanics (TQM) as the toy model to study the
above questions.

Based on previous work [WY22], we obtain a rigorous BV-BFV description of TQM on \(\mathbb{R}_{\geq 0}\)
and \(I = [0, 1]\), with homotopic renormalization incorporated. The mQME’s are stated in Definition
3.2.1 and Definition 4.0.2. Their generic solutions are described in Theorem 3.2.1 and Theorem
4.0.2 respectively. Then we derive and sharpen the BV description in [WY22] from these BV-BFV
constructions (see Proposition 5.0.1 and Proposition 5.0.2). We use 1D BF theory to demonstrate
our result in Example 4.0.1, 5.0.1.

This brings the study on TQM to a new stage, and we would like to present the result for TQM
on \(\mathbb{R}_{\geq 0}\) (Theorem 3.2.1 and Proposition 5.0.1) here. Concretely, the TQM is of AKSZ type with
target being a finite dimensional graded symplectic vector space \(V\). A Lagrangian decomposition
\(V = L \oplus L’\) is chosen. We have:

\(^1\)“BFV” for Batalin-Fradkin-Vilkovisky [BF83, FV75].
Given a functional $I_0 = \pi^*(J^0) + \int_{\mathbb{R}_{\geq 0}} I^0$ with certain $I^0 \in \mathcal{O}(V)[[\hbar]], J^0 \in \mathcal{O}(L)[[\hbar]]$, it induces a consistent BV-BFV interactive theory with polarization $V = L \oplus L'$ and a BFV operator $\Omega_{\text{BFV}}^{\text{left}}(H^0, -)$ if and only if

$$I^0 \ast_h I^0 = 0 \quad \text{and} \quad H^0 = -e^{J^0/h} \ast_h I^0 \ast_h e^{-J^0/h},$$

where $\ast_h$ denotes the Moyal product on $\mathcal{O}(V)[[\hbar]]$, and $\Omega_{\text{BFV}}^{\text{left}}(H^0, -)$ denotes the Weyl quantization of $H^0 \in \mathcal{O}(V)[[\hbar]]$ on $\mathcal{O}(L)[[\hbar]]$.

The functional $I_0 = \pi^*(J^0) + \int_{\mathbb{R}_{\geq 0}} I^0$ above induces a consistent BV interactive theory with “boundary condition $L$” if and only if

$$I^0 \ast_h I^0 = 0, \quad \text{and} \quad \Omega_{\text{BFV}}^{\text{right}}(e^{J^0/h}, I^0) = 0,$$

where $\Omega_{\text{BFV}}^{\text{right}}(-, I^0)$ denotes the Weyl quantization of $I^0$ on $\mathcal{O}(L)[[\hbar]]$.

We would like to stress the following aspects of the story, which should persist in more general settings.

### Homotopic renormalization in BV-BFV formalism

Perturbative BV-BFV formalism involves a BV structure, a splitting and a BFV operator which all need to be properly regularized (or, renormalized) in order to rigorously quantize a generic theory. For TQM, we only need to solve this problem for the former two structures.

A renormalized BV structure has been constructed in [Rab21] using homotopic renormalization, endowing the “restricted bulk field space” $\mathcal{E}_L$ with a differential BV algebra $(\mathcal{O}(\mathcal{E}_L), d, \partial_K_t)$ for each renormalization scale $t > 0$, see (3.6). As for the splitting, we propose a notion:

- The **renormalized splitting** (at scale $t$) is the map $\theta_t : L' \to \mathcal{E}$ determined by

$$\theta_t(l') = 2(\mathbb{I}(\alpha)(l' \otimes -) \otimes 1)\bar{P}(0, t)|_{C_1}$$

for $l' \in L'$ (details see Definition 3.1.1).

This is defined according to the renormalized BV structure associated to $\mathcal{E}_L$, as it depends on the “propagator from scale 0 to scale $t$” $\bar{P}(0, t)$. $\theta_t$ flows with $t$ in a way compatible with the homotopic renormalization group flow (3.9) (3.15) of relevant BV structures:

- For $\forall \varepsilon, \Lambda > 0$, the renormalized splittings $\theta_\varepsilon, \theta_\Lambda$ make this diagram commute:

$$\begin{align*}
\mathcal{O}(\mathcal{E})[[\hbar]] & \xrightarrow{\mathbb{I}_{\theta_\varepsilon}} \mathcal{O}(L') \otimes \mathcal{O}(\mathcal{E}_L)[[\hbar]] \\
\downarrow e^{\hbar\partial_{P(\varepsilon, \Lambda)}} & \quad \downarrow e^{-2(\alpha)/h(1 \otimes e^{\hbar\partial_{P(\varepsilon, \Lambda)}})\mathcal{E}(\alpha)/h} \\
\mathcal{O}(\mathcal{E})[[\hbar]] & \xrightarrow{\mathbb{I}_{\theta_\Lambda}} \mathcal{O}(L') \otimes \mathcal{O}(\mathcal{E}_L)[[\hbar]]
\end{align*}$$

where $\mathbb{I}_{\theta_\varepsilon}, \mathbb{I}_{\theta_\Lambda} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(L') \otimes \mathcal{O}(\mathcal{E}_L)$ denote the algebraic isomorphisms induced by $\theta_\varepsilon, \theta_\Lambda$, respectively (details see Theorem 3.1.1).
As a consistency check, the renormalized splitting interpolates between the ill-defined “extension by zero” in the original work [CMR18] and the “bulk to boundary propagator” known to physicists:

\[
\text{“extension by zero”} \quad \theta_t \quad \text{“scale \( \infty \) limit”} \quad \text{“bulk to boundary propagator”}
\]

(the “scale \( \infty \)” here is not taking naive \( t \to +\infty \) in our setting, but corresponds to the “\( \Delta^\infty \)” in the proof of [WY22, Proposition 2.3.1]).

Above should be the content of homotopic renormalization in BV-BFV formalism which applies in general.

In this paper, structures purely on boundary (e.g., the BFV operator) do not need regularization. If the dimension of spacetime is larger than one, the boundary field space will be typically infinite dimensional, which suggests that homotopic renormalization should also involve the BFV operator. We leave this consideration for later study.

**BV-BFV formalism and the approach within BV formalism**

From the BV-BFV description of TQM we read out (5.2), which is the QME written in [WY22, Section 3]. Moreover, we characterize its generic solutions in Proposition 5.0.1 based on discussions for mQME. This suggests that BV-BFV formalism could imply the approach within BV formalism. Besides, if (5.2) is satisfied, the mQME in Definition 3.2.1 can be reinterpreted as a condition that the map \( \mathbb{I}_{(0,t)} \) defined in (5.10) is a cochain map:

\[
\mathbb{I}_{(0,t)} : \left( \mathcal{O}(L)[\hbar], \frac{-1}{\hbar} \mathcal{O}_L^{\text{right}}(-, H^0) \right) \to (\mathcal{O}(E_L)[\hbar], d + \hbar \partial_K + \{I_t|_{E_L}, -\})
\]

see (5.11) and its following discussion. This translation connects the “wave function” interpretation of “state” inherited in BV-BFV formalism and the “structure map” interpretation of “state” from factorization algebra perspective in [CG16, CG21]. It may inspire further comparative studies between BV-BFV formalism and other frameworks.

**Configuration space techniques**

The space \( \mathbb{R}_{\geq 0}[n] \) introduced in Definition 3.1.1 simplifies our analysis of crucial properties of TQM, including the UV finiteness (Proposition 3.2.1) and the BV anomaly (Lemma 3.2.1). Such a configuration space technique originates from [Kon94, AS94, GJ94], and reflects the power of geometric considerations. For QFT’s in 2D, another geometric renormalization method of regularized integral introduced in [LZ21] (see also [GL21]) may applies. Maybe this method can be incorporated in BV-BFV formalism as well.

### 1.2 Organization of the paper

The paper is organized as follows.

In Section 2, we briefly introduce perturbative BV formalism and perturbative BV-BFV formalism, and fix notations on structures such as Moyal product and Weyl quantization for later use.

Section 3 is devoted to TQM on \( \mathbb{R}_{\geq 0} \). In Section 3.1, we formulate homotopic renormalization in BV-BFV formalism for free theory. Rigorous mQME for interactive theory is stated in Section 3.2
followed by a description of its generic solutions. For TQM on interval, parallel results are obtained in Section 4.

In Section 5, we extract the BV description of TQM in [WY22] from the BV-BFV description in Section 3 and Section 4. Then, mQME is reinterpreted from the perspective of factorization algebra developed in [CG16, CG21].

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Convention

- Let $V$ be a $\mathbb{Z}$-graded $k$-vector space. We use $V_m$ to denote its degree $m$ component. Given homogeneous element $a \in V_m$, we let $|a| = m$ be its degree.
  
  - $V[n]$ denotes the degree shifting of $V$ such that $V[n]_m = V_{n+m}$.
  
  - $V^*$ denotes its linear dual such that $V^*_m = \text{Hom}_k(V_{-m}, k)$. Our base field $k$ will mainly be $\mathbb{R}$.
  
  - $\text{Sym}^m(V)$ and $\wedge^m(V)$ denote the $m$-th power graded symmetric product and graded skew-symmetric product respectively. We also denote
    \[
    \text{Sym}(V) := \bigoplus_{m \geq 0} \text{Sym}^m(V), \quad \hat{\text{Sym}}(V) := \prod_{m \geq 0} \text{Sym}^m(V).
    \]
    The latter is a graded symmetric algebra with the former being its subalgebra. We will omit the multiplication mark for this product in expressions (unless confusion occurs).
  
  - We call $\mathcal{O}(V) := \hat{\text{Sym}}(V^*)$ the function ring on $V$.
  
  - $V[[\hbar]], V((\hbar))$ denote formal power series and Laurent series respectively in a variable $\hbar$ valued in $V$.

- We use the Einstein summation convention throughout this work.

- We use $(\pm)_{\text{Kos}}$ to represent the sign factors determined by Koszul sign rule. We always assume this rule in dealing with graded objects.
  
- Example: let $j$ be a homogeneous linear map on $V$, then $j^*$ denotes the induced linear map on $V^*$: for $\forall f \in V^*, a \in V$ being homogeneous,
  \[
  j^* f(a) := (\pm)_{\text{Kos}} f(j(a)) \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|j||f|} \text{ here.}
  \]
Example: let \( f, g, h \in V^* \) be homogeneous elements, then \( f \otimes g \otimes h \in (V^*)^\otimes 3 \) is regarded as an element in \((V^\otimes 3)^*\): for \( \forall a, b, c \in V \) being homogeneous,

\[
(f \otimes g \otimes h)(a \otimes b \otimes c) := (\pm)_{\text{Kos}} f(a)g(b)h(c) \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|h||a|+|h||b|+|a||a|} \text{ here.}
\]

Example: let \((\mathcal{A}, \cdot)\) be a graded algebra, then \([-,-]\) means the graded commutator, i.e., for homogeneous elements \( a, b \),

\[
[a, b] := a \cdot b - (\pm)_{\text{Kos}} b \cdot a \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|a||b|} \text{ here.}
\]

- We fix an embedding of vector spaces \( \text{Sym}^m(V) \hookrightarrow V^\otimes m \) by

\[
a_1a_2\cdots a_m \mapsto \sum_{\sigma \in S_m} (\pm)_{\text{Kos}} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(m)},
\]

where \( S_m \) denotes the symmetric group. Accordingly, any \( f_1f_2\cdots f_m \in \text{Sym}^m(V^*) \) is regarded as an element in \((\text{Sym}^m(V))^*)\: for \( \forall a_1a_2\cdots a_m \in \text{Sym}^m(V)\),

\[
f_1f_2\cdots f_m(a_1a_2\cdots a_m) = m! \sum_{\sigma \in S_m} (\pm)_{\text{Kos}} f_1(a_{\sigma(1)})f_2(a_{\sigma(2)})\cdots f_m(a_{\sigma(m)}).
\]

- We call \((V, d)\) a cochain complex if \( d \) is a degree 1 map on the graded vector space \( V \) such that \( d^2 = 0 \). Such \( d \) is called a differential. A cochain map \( f : (V, d) \rightarrow (W, b) \) is a degree 0 map from \( V \) to \( W \) such that \( bf = fd \).

- Given a manifold \( X \), we denote the space of real smooth forms by

\[
\Omega^\bullet(X) = \bigoplus_k \Omega^k(X)
\]

where \( \Omega^k(X) \) is the subspace of \( k \)-forms, lying at degree \( k \).

- Now that we have mentioned differential forms, definitely we will work with infinite dimensional functional spaces that carry natural topologies. The above notions for \( V \) will be generalized as follows. We refer the reader to [Tre06] or [Cos11, Appendix 2] for further details. Besides, [Rab21, Appendix A] contains specialized discussion for sections of vector bundles with boundary conditions.

- All topological vector spaces we consider will be nuclear and we still use \( \otimes \) to denote the completed projective tensor product. For example, given two manifolds \( X, Y \), we have a canonical isomorphism

\[
C^\infty(X) \otimes C^\infty(Y) = C^\infty(X \times Y).
\]

- In the involved categories, dual space is defined to be the continuous linear dual, equipped with the topology of uniform convergence of bounded subsets. We still use \((-)^*\) to denote taking such duals.
2 Algebraic Preliminaries

In this section we collect basics and fix notations on perturbative BV formalism and perturbative BV-BFV formalism.

2.1 Perturbative BV quantization

In BV formalism, the algebraic prototype of a free QFT is the following:

Definition 2.1.1 A differential Batalin-Vilkovisky (BV) algebra is a triple \((A, Q, \Delta)\) where

- \(A\) is a \(\mathbb{Z}\)-graded commutative associative unital algebra. Assume the base field is \(\mathbb{R}\).
- \(Q : A \to A\) is a derivation of degree 1 such that \(Q^2 = 0\).
- \(\Delta : A \to A\) is a linear operator of degree 1 such that \(\Delta^2 = 0\), and \([Q, \Delta] = Q\Delta + \Delta Q = 0\). We call \(\Delta\) the BV operator.
- \(\Delta\) is a “second-order” operator w.r.t. the product of \(A\). Precisely, define the BV bracket \(\{−,−\} : A \otimes A \to A\) to be the failure of \(\Delta\) being a derivation:
  \[
  \{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|}a\Delta b, \quad \text{for } \forall a, b \in A.
  \]
  Then for \(\forall a \in A\), \(\{a, −\}\) is a derivation of degree \(|a| + 1\): for \(\forall b, c \in A\)
  \[
  \{a, bc\} = \{a, b\}c + (\pm)_{\text{Kos}}b\{a, c\}, \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|b||a| + |b|} \text{ here}.
  \]

Let \(\hbar\) be a formal variable of degree 0. We can extend the above \(Q, \Delta\) to \(\mathbb{R}[[\hbar]]\)-linear operators on \(A[[\hbar]]\). Then, \((A, Q, \Delta)\) being a differential BV algebra implies \(Q + \hbar\Delta\) is a differential on \(A[[\hbar]]\). The cochain complex \((A[[\hbar]], Q + \hbar\Delta)\) models “the observables of the free theory”.

There is a systematic way to twist (i.e., perturb) this complex, sketched in the following.

Definition 2.1.2 Let \((A, Q, \Delta)\) be a differential BV algebra. A degree 0 element \(I \in A[[\hbar]]\) is said to satisfy quantum master equation (QME) if

\[
QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0,
\]

or formally,

\[
(Q + \hbar\Delta)e^{I/\hbar} = 0.
\]

It is direct to check that (2.1) implies this formal conjugation of operators on \(A[[\hbar]]\):

\[
Q + \hbar\Delta + \{I, −\} = e^{-I/\hbar}(Q + \hbar\Delta)e^{I/\hbar},
\]

which implies \((Q + \hbar\Delta + \{I, −\})^2 = 0\). The cochain complex \((A[[\hbar]], Q + \hbar\Delta + \{I, −\})\) models “the observables of the interactive theory”. So, roughly speaking, perturbative BV quantization amounts to find such “interactive action functional” \(I\) that solves the QME.
Remark To ensure \((Q + h\Delta + \{I, -\})^2 = 0\), the QME \((2.1)\) can be relaxed to the condition that \(QI + h\Delta I + \frac{1}{2}\{I, I\}\) is a central element with respect to the BV bracket. We will not take this into consideration here, but it can be crucial in certain cases (see e.g., [GLLI17, Section 3.3]).

The differential BV algebras in this paper are all function rings on cochain complexes, with the BV operators being “contraction with closed rank-2 symmetric tensors”. Precisely, let \((\mathcal{E}, Q)\) be a cochain complex, \(K \in \text{Sym}^2(\mathcal{E})\) satisfying \(QK = 0\), \(|K| = 1\), then

\[
(\mathcal{O}(\mathcal{E}) := \hat{\text{Sym}}(\mathcal{E}^*), Q, \Delta := \partial K)
\]

is a differential BV algebra, where

- we still use \(Q\) to denote the derivation on \(O(\mathcal{E})\) extended from \(Q^*\) on \(\mathcal{E}^*\);
- \(\partial_K(\text{Sym}^{\leq 1}(\mathcal{E}^*)) = 0\), and for \(n \geq 2\), \(\forall f_1, f_2, \ldots, f_n \in \mathcal{E}^*\),

\[
\partial_K(f_1 f_2 \cdots f_n) = \sum_{i<j} (\pm)K_{ij}(f_i f_j(K)) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n.
\]

(2.2)

To really use such a differential BV algebra to describe a free QFT, the choice of \(K\) is not arbitrary. The cochain \((\mathcal{E}, Q)\) models the “field space” of the QFT. If the spacetime manifold is closed, there should be a nondegenerate pairing \(\omega \in \wedge^2(\mathcal{E}^*), |\omega| = -1\) such that \(Q\omega = 0\). \((\mathcal{E}, Q, \omega)\) is called a dg \((-1)\)-symplectic vector space, and \(K\) should be chosen according to this structure. For example, if \(\mathcal{E}\) is finite dimensional, \(\omega\) will induce a vector space isomorphism \(\mathcal{E} \rightarrow \mathcal{E}^*\) by sending \(e \in \mathcal{E}\) to \(\omega(e, -) \in \mathcal{E}^*\). This further induces a vector space isomorphism:

\[
\text{Sym}^2(\mathcal{E}) \rightarrow \wedge^2(\mathcal{E}^*).
\]

Then, \(K \in \text{Sym}^2(\mathcal{E})\) is the preimage of \(\frac{1}{2}\omega \in \wedge^2(\mathcal{E}^*)\) under this map. For more general field space \(\mathcal{E}\), the construction of \(K\) is less straightforward. We will adopt Costello’s homotopic renormalization method [Cos11] to deal with the models appearing in this paper.

### 2.2 Perturbative BV-BFV quantization

As mentioned above, we expect the field space of a perturbative QFT on a closed manifold to be modelled by a dg \((-1)\)-symplectic vector space. If the spacetime has boundary, this picture needs modification.

In [CMR14, CMR18], Cattaneo, Mnev and Reshetikhin proposed a candidate modification. Their proposal is called “BV-BFV formalism” (BFV for Batalin-Fradkin-Vilkovisky [BF83, FV75]), which generalizes BV formalism to encompass the presence of spacetime boundary. The process of using this formalism to perturbatively quantize a QFT is sketched as follows.

**Definition 2.2.1** A free BV-BFV pair \((\mathcal{E}, Q, \omega, \mathcal{E}^0, Q^0, \omega^0, \pi)\) is the following:

- \((\mathcal{E}, Q)\), \((\mathcal{E}^0, Q^0)\) are cochain complexes.
- \(\pi : (\mathcal{E}, Q) \rightarrow (\mathcal{E}^0, Q^0)\) is a cochain map, and it is surjective.
• \( \omega^0 \in \Lambda^2((E^0)^*) , |\omega| = 0 \) is a nondegenerate pairing, such that \( Q^0 \omega^0 = 0 \).

• \( \omega \in \Lambda^2(E^*), |\omega| = -1 \) is a pairing, such that \( Q \omega = \pi^* \omega^0 \).

The physical meaning is that, \( E \) is the “bulk field space”, \( E^0 \) is the “boundary field space”, and \( \pi \) is “restriction to the boundary”. Usually \( \omega \) is nondegenerate, then the condition \( Q \omega = \pi^* \omega^0 \) says that, the failure of \((E,Q,\omega)\) being a dg \((-1)\)-symplectic vector space is controlled by the boundary data \((E^0, Q^0, \omega^0)\).

Remark We refer the reader to [CM20, Section 7] for the global version of this notion, which is not necessary in the scope of this paper.

Definition 2.2.2 Given a free BV-BFV pair \((E,Q,\omega, E^0, Q^0, \omega^0, \pi)\), a polarization is a Lagrangian decomposition of \( E^0 \) compatible with \( Q^0 \), namely,

\[
E^0 = L \oplus L' \quad \text{such that} \quad Q^0 L \subset L, Q^0 L' \subset L',
\]

(2.3)

and \( \omega^0 \in (L^* \otimes (L')^*) \oplus ((L')^* \otimes L^*) \) only pairs \( L \) with \( L' \).

A splitting compatible with the polarization \( E^0 = L \oplus L' \) is a degree 0 linear map

\[
\theta : L' \to E,
\]

(2.4)

such that \( \pi \theta : L' \to E^0 = L \oplus L' \) is \((0,\text{id}_{L'})\).

Given a polarization as (2.3) let \( p_L, p_{L'} \) denote the projections from \( E^0 \) to \( L, L' \), respectively. The restricted (bulk) field space with boundary condition \( L \) is defined as:

\[
E_L := \ker(p_L, \pi) \subset E.
\]

Then, \((E_L, Q)\) is a cochain complex, and \((Q \omega)|_{E_L} = 0 \). Usually, \( \omega \) is nondegenerate on \( E_L \), hence \((E_L, Q, \omega)\) is a dg \((-1)\)-symplectic vector space. Suppose we have chosen a \( Q \)-closed tensor \( K \in \text{Sym}^2(E_L) \) according to \((E_L, Q, \omega)\), then \((\mathcal{O}(E_L)), Q, \partial_K)\) is a differential BV algebra. We could use it to define the theory still within BV formalism (which is the method in [WY22]). However, the spirit of BV-BFV formalism is that, we should work on \( E \) instead of the subspace \( E_L \subset E \) only.

The splitting (2.4) induces an isomorphism \( E \simeq L' \oplus E_L \), which further induces an algebraic isomorphism

\[
I_{\theta} : \mathcal{O}(E) \to \mathcal{O}(L') \otimes \mathcal{O}(E_L).
\]

(2.5)

Then, the consistency condition of a perturbative QFT in BV-BFV formalism is the following:

Definition A degree 0 functional \( I \in \mathcal{O}(E)[[h]] \) is said to satisfy the modified quantum master equation (mQME) if there is a certain \( \mathbb{R}[[h]] \)-linear differential \( \Omega \) on \( \mathcal{O}(L')[[h]] \), called the BFV operator, such that

\[
(1 \otimes (h Q + h^2 \partial_K) + \Omega \otimes 1)I_{\theta}(e^{I/h}) = 0
\]

for proper choice of \( K \in \text{Sym}^2(E_L) \) and \( \theta : L' \to E \).

Here we do not explain the meaning of “proper choice” and constraints on \( \Omega \). Roughly speaking, mQME says that the BV anomaly of the theory defined by \( I \) on the (full) bulk field space \( E \) is cancelled by some boundary data. Precise form of mQME for the concrete models we study will be stated in later sections.
2.3 Moyal product and Weyl quantization

We list several standard algebraic structures here for later convenience.

Let

\[(V = L \oplus L', \omega^\partial)\]  

be a finite dimensional graded symplectic vector space endowed with a Lagrangian decomposition. Namely, \(\omega^\partial \in \wedge^2(V^*)\), \(|\omega| = 0\) is a nondegenerate pairing that only pairs \(L\) with \(L'\). For \(v \in V\), the map \(v \mapsto \omega^\partial(v, -)\) induces a graded vector space isomorphism \(V \simeq V^*\), hence also \(\wedge^2 V \simeq \wedge^2(V^*)\).

Let

\[K^\partial \in \wedge^2(V)\]  

be the image of \(\omega^\partial\) under \(\wedge^2(V^*) \simeq \wedge^2V\). Regarded as an element in \(V^\otimes 2\), we can write

\[K^\partial = K_0^\partial + K_+^\partial, \quad K_0^\partial \in L \otimes L', \quad K_+^\partial \in L' \otimes L, \quad \text{s.t.} \quad \sigma K_0^\partial = -K_+^\partial,\]  

where \(\sigma\) permutes the two factors of \(V^\otimes 2\).

Moyal product

There is a graded associative algebra structure \((\mathcal{O}(V)[[\hbar]], \ast_h)\): for \(J, F \in \mathcal{O}(V)[[\hbar]]\),

\[J \ast_h F := e_{\text{cross}}^{-\hbar \partial \Sigma_0^\partial/2}(J, F),\]  

where for \(\forall G \in V^\otimes 2\) of degree 0, \(\forall j_1, \ldots, j_m, f_1, \ldots, f_n \in V^*\),

\[e_{\text{cross}}^\partial((j_1, j_2 \cdots j_m, f_1 f_2 \cdots f_n)) := \sum_{s=0}^{+\infty} \sum_{1 \leq l_1 < \cdots < l_s \leq m} \sum_{1 \leq r_1, \ldots, r_s \leq n} (\pm)^{K_0^\partial}((j_{l_1} \otimes f_{r_1})(G)) \cdots ((j_{l_s} \otimes f_{r_s})(G)) \times (j_1 \cdots \hat{j}_{l_1} \cdots \hat{j}_{l_s} \cdots j_m f_1 \cdots \hat{f}_{r_1} \cdots \hat{f}_{r_s} \cdots f_n).\]  

\(\ast_h\) is called the Moyal product.

Weyl quantization

Given the Lagrangian decomposition \(V = L \oplus L'\), Weyl quantization defines an action:

\[\Omega^\partial_{L'}(-, -) : \mathcal{O}(V)[[\hbar]] \otimes \mathcal{O}(L')[[\hbar]] \to \mathcal{O}(L')[[\hbar]],\]  

for \(J \in \mathcal{O}(V)[[\hbar]], g \in \mathcal{O}(L')[[\hbar]]\),

\[\Omega^\partial_{L'}(J, g) := p_{L'} \left( e_{\text{cross}}^{-\hbar \partial K_0^\partial} \left( e^{\hbar \partial (K_0^\partial - K_+^\partial) / 4} J, g \right) \right)\]  

where \(p_{L'}\) here denotes the projection \(\mathcal{O}(V) \to \mathcal{O}(L')\) induced by \(V = L \oplus L', \) and \(\partial (K_0^\partial - K_+^\partial) / 4\) acting on \(\mathcal{O}(V)\) is defined in the same way as \((2.2)\). \(e^{\hbar \partial (K_0^\partial - K_+^\partial) / 4}\) should be regarded as the transformation from “symmetric (Weyl) ordering” to “normal ordering”. It is direct to verify that \(\Omega^\partial_{L'}(-, -)\) makes \(\mathcal{O}(L')[[\hbar]]\) a graded left module over \((\mathcal{O}(V)[[\hbar]], \ast_h)\).
Similarly, $\mathcal{O}(L)[[\hbar]]$ has a graded right module structure over $(\mathcal{O}(V)[[\hbar]], \star_{\hbar})$:

$$\Omega_{L}^{\text{right}}(-, -) : \mathcal{O}(L)[[\hbar]] \otimes \mathcal{O}(V)[[\hbar]] \to \mathcal{O}(L)[[\hbar]],$$

for $J \in \mathcal{O}(V)[[\hbar]]$, $f \in \mathcal{O}(L)[[\hbar]]$,

$$\Omega_{L}^{\text{right}}(f, J) := p_{L} \left( e^{-\hbar \partial_{K}^{\alpha}} \left( f, e^{\hbar \partial_{(K^{\alpha} - K_{\alpha})/4}} J \right) \right). \tag{2.12}$$

Moreover, we have a nondegenerate pairing

$$\ll - , - \gg : \mathcal{O}(L)[[\hbar]] \otimes \mathcal{O}(L')[[[\hbar]] \to \mathbb{R}[[\hbar]],$$

for $f \in \mathcal{O}(L)[[\hbar]]$, $g \in \mathcal{O}(L')[[\hbar]]$,

$$\ll f, g \gg := p_{L} \left( e^{-\hbar \partial_{K}^{\alpha}} (f, g) \right). \tag{2.13}$$

It is direct to verify

$$\ll f, p_{L'} \left( e^{-\hbar \partial_{K}^{\alpha}} (J, g) \right) \gg = \ll p_{L} \left( e^{-\hbar \partial_{K}^{\alpha}} (f, J) \right), g \gg \tag{2.14}$$

for $\forall J \in \mathcal{O}(V)[[\hbar]]$. So this pairing is compatible with the module structures.

**Constraints on the BFV operator**

In Section 2.2 we mentioned that there should be a BFV operator $\Omega$ on $\mathcal{O}(L')[[\hbar]]$ appearing in mQME. Here we describe it more concretely for the case $E_{\partial}$ equals to $V$ in (2.6).

It is direct to check that, for an $\mathbb{R}[[\hbar]]$-linear operator $\Omega$ on $\mathcal{O}(L')[[[\hbar]]$, these two statements are equivalent:

- $\Omega$ can be expanded as
  $$\Omega = \sum_{n=0}^{+\infty} \hbar^{n} \Omega_{n},$$
  where $\Omega_{n}$ is a differential operator on $\mathcal{O}(L')$ of order $\leq n$.

- $\exists J \in \mathcal{O}(V)[[\hbar]]$, such that $\Omega = \Omega_{L'}^{\text{left}}(J, -)$.

By compatibility between $\Omega_{L'}^{\text{left}}(-, -)$ and $\star_{\hbar}$, for $J \in \mathcal{O}(V)[[\hbar]]$, these two statements are equivalent:

- $\Omega_{L'}^{\text{left}}(J, -)$ is a differential, i.e., $|\Omega_{L'}^{\text{left}}(J, -)| = 1$, $(\Omega_{L'}^{\text{left}}(J, -))^{2} = 0$.

- $|J| = 1, J \star_{\hbar} J = 0$.

In summary, when the data of boundary field space and polarization is given by (2.6), we demand that the BFV operator on $\mathcal{O}(L')[[[\hbar]]$ should be $\Omega_{L'}^{\text{left}}(J, -)$ for some $J \in \mathcal{O}(V)[[\hbar]]$ satisfying $|J| = 1, J \star_{\hbar} J = 0$. 

11
3 BV-BFV Description of TQM on $\mathbb{R}_{\geq 0}$

We have discussed topological quantum mechanics (TQM) on $\mathbb{R}_{\geq 0}$ in [WY22] with Costello’s homotopic renormalization incorporated, based on [Rab21]. There, we only work on “restricted bulk field space”, so that the theory can be defined within BV formalism. In this section, we will refine that discussion using BV-BFV formalism. Moreover, we will find generic solutions to the mQME for TQM.

3.1 The content of free theory

Field spaces

Given $(V = L \oplus L', \omega^0)$ as (2.6), let

$$(\mathcal{E}^\partial := V, Q^\partial := 0, \omega^\partial)$$

(3.1)

be the boundary field data (now the spacetime is $\mathbb{R}_{\geq 0}$, with boundary being $\{0\}$),

$$(\mathcal{E} := \Omega^*(\mathbb{R}_{\geq 0}) \otimes V, Q := d, \omega := \int \omega^0 \in \wedge^2(\mathcal{E}_c^*))$$

(3.2)

be the bulk field data, where $d$ is the de Rham differential on $\Omega^*(\mathbb{R}_{\geq 0})$. Note that $\omega$ is not a pairing on $\mathcal{E}$ now (since integration might be divergent on $\mathbb{R}_{\geq 0}$). Instead, it is a pairing on the space of compactly supported forms $\mathcal{E}_c := \Omega^*(\mathbb{R}_{\geq 0}) \otimes V \subset \mathcal{E}$. Let

$$\pi : \mathcal{E} \to \mathcal{E}^\partial$$

(3.3)

be the pullback of differential forms induced by $\{0\} \hookrightarrow \mathbb{R}_{\geq 0}$. Then, we can verify $(\mathcal{E}_c, d, \omega, \mathcal{E}^\partial, 0, \omega^\partial, \pi)$ is a free BV-BFV pair as in Definition 2.2.1. The noncompactness of $\mathbb{R}_{\geq 0}$ makes $\mathcal{E}_c$ rather than $\mathcal{E}$ fit into this definition, and it also leads to several other technical subtleties in the interactive theory (see Remark 3.2.1). We will not expand on this aspect of the story in this paper and just refer the reader to [WY22] Section 3 for relevant facts.

Renormalized BV structure with restricted bulk field space

The polarization $\mathcal{E}^\partial = L \oplus L'$ allows us to define

$$\mathcal{E}_L := \{ f \in \Omega^*(\mathbb{R}_{\geq 0}) \otimes V | \pi(f) \in L \},$$

(3.4)

which is the field space we have studied in [WY22]. By constructions in [Rab21], we have a family of differential BV structures on $\mathcal{O}(\mathcal{E}_L)$, concluded in [WY22] Proposition 3.0.2]. Here we briefly describe it as follows.

For $\forall t > 0$, let

$$H_t := \frac{1}{\sqrt{4\pi t}} \left( e^{-\frac{(x-y)^2}{4t}}(dy - dx) - e^{-\frac{(x+y)^2}{4t}}(dy + dx) \right),$$

$$\tilde{H}_t := \frac{1}{\sqrt{4\pi t}} \left( \phi(x - y)e^{-\frac{(x-y)^2}{4t}}(dy - dx) - \phi(x + y)e^{-\frac{(x+y)^2}{4t}}(dy + dx) \right)$$
denote two smooth forms on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$, with $\phi \in C^\infty(\mathbb{R})$ being a compactly supported even function which evaluates to 1 in a neighborhood of $\{0\}$. Define the BV kernel at scale $t$ to be

$$K_t := \frac{1}{2} \left( H_t - \frac{1}{2} (1 \otimes \partial + \partial \otimes 1) (d^{\text{GF}} \otimes 1 + 1 \otimes d^{\text{GF}}) \int_0^t ds (\bar{H}_s - H_s) \right) \otimes K_+^0$$

$$- \frac{1}{2} \sigma \left( H_t - \frac{1}{2} (1 \otimes \partial + \partial \otimes 1) (d^{\text{GF}} \otimes 1 + 1 \otimes d^{\text{GF}}) \int_0^t ds (\bar{H}_s - H_s) \right) \otimes K_-^0$$

(3.5)

where $K_+^0, K_-^0$ are defined in (2.8), $\sigma$ here permutes variables $x$ and $y$, and $d^{\text{GF}}$ is the Hodge dual to $d$ induced by metric $\langle \partial_x, \partial_x \rangle = 1$ on $\mathbb{R}_{>0}$:

$$d^{\text{GF}}(f \, dx) = -\partial_x f \quad \text{for } f \in \Omega^0(\mathbb{R}_{>0}).$$

Then, by direct computation we can verify

$$K_t \in \text{Sym}^2(\mathcal{E}_L), \quad |K_t| = 1, dK_t = 0.$$

So,

$$(\mathcal{O}(\mathcal{E}_L), d, \partial_{K_t})$$

(3.6)

is a differential BV algebra for $\forall t > 0$. Moreover, for $\forall \varepsilon, \lambda > 0$, define the propagator from scale $\varepsilon$ to scale $\lambda$ to be

$$P(\varepsilon, \lambda) := \left( -\frac{1}{4} (d^{\text{GF}} \otimes 1 + 1 \otimes d^{\text{GF}}) \int_\varepsilon^\lambda dt \bar{H}_t \right) \otimes K_+^0$$

$$+ \sigma \left( \frac{1}{4} (d^{\text{GF}} \otimes 1 + 1 \otimes d^{\text{GF}}) \int_\varepsilon^\lambda dt \bar{H}_t \right) \otimes K_-^0.$$

(3.7)

It satisfies

$$P(\varepsilon, \lambda) \in \text{Sym}^2(\mathcal{E}_L), \quad K_\lambda = K_\varepsilon + dP(\varepsilon, \lambda),$$

(3.8)

hence leading to a conjugation of cochain complexes

$$e^{h\partial_{P(\varepsilon, \lambda)}}$$

$$(\mathcal{O}(\mathcal{E}_L)[[h]], d + h\partial_{K_\varepsilon}) \quad \Rightarrow \quad (\mathcal{O}(\mathcal{E}_L)[[h]], d + h\partial_{K_\lambda}).$$

(3.9)

This relation exhibits Costello’s homotopic renormalization (for the free theory) within BV formalism.

For later discussion we further present several properties of the propagator.

**Definition 3.1.1** For $n \geq 1$, define $\mathbb{R}_{>0}[n]$ to be

$$\mathbb{R}_{>0}[n] := \bigsqcup_{\sigma \in S_n} \{ (x_1, \ldots, x_n) | 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \}.$$  

(3.10)

Namely, $\mathbb{R}_{>0}[n]$ is the disjoint union of $n!$ connected components. Note that there are $n!$ different copies of $(0, 0, \ldots, 0)$ in $\mathbb{R}_{>0}[n]$, labelled by each $\sigma \in S_n$.

We can regard $\mathbb{R}_{>0}[n]$ as a “partial compactification” of configuration space of $n$ pairwise different points in $\mathbb{R}_{>0}$. Particularly, we can write

$$\mathbb{R}_{>0}[2] = C_1 \sqcup C_2, \quad \text{where } C_1 := \{ (x, y) | 0 \leq x \leq y \}, C_2 := \{ (x, y) | 0 \leq y \leq x \}.$$  

(3.11)

Let $D_2$ denote the diagonal in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$, then there is a canonical embedding $(\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \setminus D_2 \hookrightarrow \mathbb{R}_{>0}[2]$, where the image of the former contains the interior of the latter. By direct check we obtain:
Proposition 3.1.1 On \((\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus D_2\), the expression (3.7) is well defined for \(\varepsilon = 0, \Lambda > 0\). The resulting \(P(0, \Lambda)\) is a smooth function on \((\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus D_2\), and can be smoothly extended to \(\mathbb{R}_{\geq 0}[2]\). We denote this extension by \(\bar{P}(0, \Lambda)\).

Given any \((x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\), the limit
\[
\lim_{\varepsilon \to 0} P(\varepsilon, \Lambda)(x, y)
\]
exists, and
\[
P(\varepsilon, \Lambda)(0, 0) = 0;
\]
\[
\lim_{\varepsilon \to 0} P(\varepsilon, \Lambda)(x, y) = \bar{P}(0, \Lambda)(x, y) \quad \text{for } x \neq y;
\]
\[
\lim_{\varepsilon \to 0} P(\varepsilon, \Lambda)(x, x) = (\bar{P}(0, \Lambda)|_{C_1}(x, x) + \bar{P}(0, \Lambda)|_{C_2}(x, x))/2 \quad \text{for } x > 0,
\]
where \(C_1, C_2\) are defined in (3.11).

It is straightforward to verify:
\[
\bar{P}(0, \Lambda)|_{C_1}(x, x) - \bar{P}(0, \Lambda)|_{C_2}(x, x) = -K^0/2 \quad \text{for } x \geq 0,
\]
and
\[
\bar{P}(0, \Lambda)|_{C_1}(0, 0) = -K^0/2, \quad \bar{P}(0, \Lambda)|_{C_2}(0, 0) = K^0/2.
\]
We can also restrict the relation \(K^\Lambda = K^\varepsilon + dP(\varepsilon, \Lambda)\) in (3.8) to \((\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus D_2\). Then its \(\varepsilon \to 0\) limit reads
\[
K^\Lambda = d\bar{P}(0, \Lambda)
\]
as a relation between smooth forms. Since \((\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus D_2\) contains the interior of \(\mathbb{R}_{\geq 0}[2]\), (3.14) holds on the entire \(\mathbb{R}_{\geq 0}[2]\), where \(K^\Lambda\) is regarded as a form on \(\mathbb{R}_{\geq 0}[2]\) by pullback of the canonical projection \(\mathbb{R}_{\geq 0}[2] \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\).

Remark The idea of using (compactified) configuration space to analyse computations in QFT follows \[Kon94, AS94, GJ94\], while our Definition 3.1.1 is specialized for TQM on \(\mathbb{R}_{\geq 0}\).

Renormalized splitting

Now we come to BV-BFV formalism and consider structures on \(\mathcal{O}(\mathcal{E})\). Since \((\mathcal{E}_L, d)\) is a subcomplex of \((\mathcal{E}, d)\), \((\mathcal{O}(\mathcal{E}), d, \partial_{K^\varepsilon})\) is also a differential BV algebra, and there is also a conjugation
\[
e^{h\partial_{K^\varepsilon}} \mathcal{O}(\mathcal{E})[h], d + h\partial_{K^\varepsilon}) \quad \Leftrightarrow \quad (\mathcal{O}(\mathcal{E})[h], d + h\partial_{K^\Lambda}).
\]
(3.15)

However, homotopic renormalization in BV-BFV formalism has additional content, involving the splitting and the BFV operator. For the current model, the BFV operator does not need regularization because the boundary field space \(\mathcal{E}^\partial\) is finite dimensional. We focus on the splitting in the following.

For \(\forall J \in \mathcal{O}(V) \simeq \mathcal{O}(L') \otimes \mathcal{O}(L)\), define \(I(J) \in \mathcal{O}(L') \otimes \mathcal{O}(\mathcal{E}_L)\) to be
\[
I(J) := (1 \otimes (\pi|_{\mathcal{E}_L}))^*(J),
\]
(3.16)
where \( \pi|_{\mathcal{E}_L} : \mathcal{E}_L \to L \) is the restriction of (3.3) to \( \mathcal{E}_L \subset \mathcal{E} \). It is direct to see that for any splitting \( \theta \) as in Definition 2.2.2 \( \mathbb{I}_\theta (\pi^*(J)) = \mathbb{I}(J) \) where \( \mathbb{I}_\theta \) is the algebraic isomorphism (2.5).

Let \( \alpha \in (L')^* \otimes L^* \) be determined by

\[
\alpha(l \otimes l) := \omega^0(l', l) \text{ for } l' \in L', l \in L.
\]

(3.17)

\( \alpha \) can be regarded as an element in \( \text{Sym}^2(V^*) \). Then,

\[
\mathbb{I}(\alpha) \in (L')^* \otimes (\mathcal{E}_L)^*, \quad \mathbb{I}(\alpha)(l' \otimes f) = \omega^0(l', \pi(f)).
\]

Recall that for \( \forall H \in \mathcal{O}(V)[[\hbar]] \) there is an operator \( \Omega^\text{ren}_L(H, -) \) on \( \mathcal{O}(L')[[\hbar]] \) defined in (2.11). Then it is direct to see that for \( \forall J \in \mathcal{O}(V)[[\hbar]] \),

\[
e^{-\hbar(\alpha)/\hbar}(\Omega^\text{ren}_L(H, -) \otimes 1) \mathbb{I}(e^{\alpha/\hbar} J) = \mathbb{I}
\left(\frac{-h \partial \kappa}{\hbar} \left( e^{\hbar(\kappa_2 - \kappa_1)/\hbar^4} H, J \right) \right)
\]

(3.18)

Now we state the choice of splitting for our model. By (3.13) we have:

**Proposition 3.1.2** The map \( \theta_t : L' \to \mathcal{E} \) determined by

\[
\theta_t(l')(x) := 2(\alpha(l' \otimes -) \otimes 1) (\bar{P}(0, t)|_{C_1}(0, x)) \quad \text{for } \forall l' \in L', x \geq 0
\]

(3.19)

is a splitting in the sense of Definition 2.2.2. Namely, \( \theta_t(l') \) is a \( V \)-valued 0-form on \( \mathbb{R}_{\geq 0} \), with value at \( x \) being the result of contracting \( 2\alpha(l' \otimes -) \) with the first tensor factor of \( \bar{P}(0, t)|_{C_1}(0, x) \in V \otimes V \).

**Definition 3.1.2** We define the renormalized splitting (at scale \( t \)) for the current TQM on \( \mathbb{R}_{\geq 0} \) to be the map \( \theta_t \) described in (3.19). We can rewrite it as

\[
\theta_t(l') = 2(\mathbb{I}(\alpha)(l' \otimes -) \otimes 1) \bar{P}(0, t)|_{C_1}
\]

which makes sense although \( \bar{P}(0, t)|_{C_1} \) is not in \( \mathcal{E}_L \otimes \mathcal{E} \).

Then, for this \( \theta_t \) we can write \( \mathbb{I}_{\theta_t} \) as

\[
\mathbb{I}_{\theta_t}(f) = e^{-\hbar(\alpha)/\hbar} \left( e^{\hbar(\kappa_2 - \kappa_1)/\hbar^4} \mathbb{I}(\alpha)(l' \otimes -), f \right)_{\mathcal{E}_L} \quad \text{for } f \in \mathcal{O}(E)
\]

(3.20)

where \( e^{\hbar(\kappa_2 - \kappa_1)/\hbar^4} \mathbb{I}(\alpha)(l' \otimes -) \) is defined in the same way as (2.10), and the restriction of functional \( \mathbb{I}(\alpha) \) only acts on factors coming from \( f \). Note that by definition the first factor of \( \mathbb{I}(\alpha) \) does not contract with \( 2\bar{P}(0, t)|_{C_1} \) in (3.20).

By (3.14) we can verify:

\[
(1 \otimes d + \{\mathbb{I}(\alpha), -\}_t) \mathbb{I}_{\theta_t} = \mathbb{I}_{\theta_t} d
\]

(3.21)

as maps from \( \mathcal{O}(E) \) to \( \mathcal{O}(L') \otimes \mathcal{O}(E_L) \), where \( \{\mathbb{I}(\alpha), -\}_t \) is the BV bracket of \( \partial K_t \) on \( \mathcal{O}(E_L) \). We may regard this relation as certain compatibility between \( \theta_t \) and the differential BV algebra structure associated to \( d \) and \( K_t \).

The homotopic renormalization of splitting in our model reads:

\[\text{So, } \pi^*(\alpha) \in \text{Sym}^2(\mathcal{E}^*) \text{ looks like } "q(0)p(0)" \text{ in Darboux coordinates on boundary field space.}\]
Proof Since \( P(\varepsilon, \Lambda)(0, 0) = 0 \), \( (1 \otimes \partial P(\varepsilon, \Lambda))e^{\Omega(\alpha)/\hbar} = 0 \). Then, for \( f \in \mathcal{O}(\mathcal{E})[[\hbar]] \),

\[
e^{-\Omega(\alpha)/\hbar}(1 \otimes e^{\hbar \partial P(\varepsilon, \Lambda)})(e^{\Omega(\alpha)/\hbar} \|_{\mathcal{E}}(f))
\]

\[
e^{-\Omega(\alpha)/\hbar}(1 \otimes e^{\hbar \partial P(\varepsilon, \Lambda)}) \left( \frac{\hbar}{e_{\text{cross}}^{\Omega(\alpha)}} \right) (e^{\Omega(\alpha)/\hbar}, e^{\hbar \partial P(\varepsilon, \Lambda)} f) \mathcal{E}_L
\]

\[
e^{-\Omega(\alpha)/\hbar} \frac{\hbar}{e_{\text{cross}}^{\Omega(\alpha)}} (e^{\Omega(\alpha)/\hbar}, e^{\hbar \partial P(\varepsilon, \Lambda)} f) \mathcal{E}_L
\]

\[
\|_{\mathcal{E}}(f)
\]

where we have used the fact \( \partial \mathcal{P}(\varepsilon, \Lambda) \mid \hbar \mathcal{E} \) is a boundary term, and \( \mathcal{E}_L \) denotes \( \mathcal{E} \) with degree \( \varepsilon > 0 \). Then we define the “scale 0” interaction to be

Now we consider the interactive theory. Let \( I^0, J^0 \in \prod_{n \geq 2} \text{Sym}^n(V^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(V^*)[[\hbar]] \), with degree \( |I^0| = 1, |J^0| = 0 \). Then we define the “scale 0” interaction to be

\[
I_0 \in \mathcal{O}^{>0}(\mathcal{E}_c)[[\hbar]], \quad I_0 := \pi^*(J^0) + \int_{\mathbb{R}_{>0}} I^0
\]

where \( \mathcal{O}^{>0}(\mathcal{E}_c) \) denotes \( \prod_{n > 0} \text{Sym}^n((\mathcal{E}_c)^*) \), with \( \mathcal{E}_c \subset \mathcal{E} \) the space of compactly supported forms. Here, \( \pi^*(J^0) \) is a boundary term, and \( \int_{\mathbb{R}_{>0}} I^0 \) denotes the \( \Omega^\bullet(\mathbb{R}_{>0}) \)-linear extension of \( I^0 \) followed by integration over \( \mathbb{R}_{>0} \).

The interaction at scale \( t > 0 \) should be a functional \( I_t \in \mathcal{O}^{>0}(\mathcal{E}_c)[[\hbar]] \) generated from \( I_0 \) by homotopic renormalization. Precisely, we have:

**Proposition 3.2.1 “TQM is UV finite”**

For \( I_0 \) defined in (3.22), the limit

\[
I_t := \lim_{\varepsilon \to 0} \hbar \log(e^{\hbar \mathcal{P}(\varepsilon, t)} e^{I_0/\hbar})
\]

is a well-defined element in \( \mathcal{O}^{>0}(\mathcal{E}_c)[[\hbar]] \). We call it the “scale \( t \) interaction.”
Remark 3.2.1 Actually, being a “local functional”, $I_0$ has “proper support”. The BV kernel $K_t$ and propagator $P(\varepsilon, \Lambda)$ also have “proper supports”. These facts make
\[
\log(e^{\hbar \partial_P(\varepsilon, t)} e^{I_0/\hbar})
\] (3.24)
well-defined, although $P(\varepsilon, t)$ is not compactly supported. Moreover, (3.24) has “proper support” as well. Later calculations involving $\partial K_t$ and its BV bracket also make sense by the same reasoning. We refer the reader to [WY22, Section 3] and references therein for relevant explanation. Clarification on this point will be omitted in the rest of this paper.

Proof By standard arguments, the real content of the formula (3.24) is a summation over connected Feynman graphs. It is direct to check that at any given $\hbar$-order and symmetric tensor order, only finite many terms contributes\footnote{This uses the fact that $P(\varepsilon, \Lambda)$ is 0-form with $P(\varepsilon, \Lambda)(0, 0) = 0$, so each bulk vertex will have at least one external leg to pick up 1-form, and two boundary vertices cannot be directly connected by a propagator.} hence (3.24) is well defined.

Consider a term contracting with $\varphi_1, \ldots, \varphi_4 \in \mathcal{E}_c$, depicted by the graph below (numeric coefficients are omitted):

From this example, it is clear that each term in (3.24) is always an integration over $\mathbb{R}_{\geq 0}^{\mid V_{\text{bulk}}\mid}$ with the integrand constructed from propagators and input fields, where $\mid V_{\text{bulk}}\mid$ is the number of “bulk vertices” corresponding to $I^0$. By Definition 3.1.1, we have a canonical projection $\mathbb{R}_{\geq 0}[n] \to \mathbb{R}_{\geq 0}^n$. So, each term in (3.24) can be expressed as integration over $\mathbb{R}_{\geq 0}^{\mid V_{\text{bulk}}\mid}$.

By Proposition 3.1.1 while taking the $\varepsilon \to 0$ limit of a certain propagator $P(\varepsilon, t)$ in a graph formula, we can replace this $P(\varepsilon, t)$ with a linear combination of $\bar{P}(0, t)$. (The type of vertices at the endpoints of this propagator will tell us the choice of replacement.) So, the $\varepsilon \to 0$ limit of each term in (3.24) is an integration of smooth form over a manifold with corners, namely the $I_t$ in (3.23) is well defined. □

By definition, for $\varepsilon, \Lambda > 0$,
\[
e^{I_\Lambda/\hbar} = e^{b\partial_P(\varepsilon, \Lambda)} e^{I_\varepsilon/\hbar}.
\] (3.25)

Now we are ready to write down the mQME for our model.

Definition 3.2.1 Given $I_0$ in (3.22), we further require
\[
J^0 \in \mathcal{O}(L)[[\hbar]].
\] (3.26)

Then, we say that $I_t$ in (3.23) satisfies modified quantum master equation (mQME) at scale $t$ if there exists
\[
H^0 \in \mathcal{O}(V)[[\hbar]], \quad |H^0| = 1, H^0 \star_\hbar H^0 = 0,
\]
such that

\[
(1 \otimes (\hbar d + \hbar^2 \partial_{K_t}) + \Omega^{\text{left}}_{L'}(H^0, -) \otimes 1) \mathbb{I}_{\theta_t}(e^{(\pi^{*}(\alpha) + I_t)/\hbar}) = 0,
\]

where \(*_h\) and \(\Omega^{\text{left}}_{L'}(-, -)\) are defined in Section 2.3, \(\alpha\) is defined in (3.17). \(\Omega^{\text{left}}_{L'}(H^0, -)\) is called the BFV operator. In equation (3.27), \(\mathbb{I}_{\theta_t}\) is well defined on \(\mathcal{O}(\mathcal{E}_c)\) because the image of \(\theta_t\) is a subset of \(\mathcal{E}_c\).

If \(I_s\) satisfies mQME at scale \(\varepsilon\), Theorem 3.1.1 together with (3.25) implies \(I_\Lambda\) satisfies mQME at scale \(\Lambda\), with the same BFV operator. We then say that the data \((I^0, J^0, H^0)\) defines a consistent interactive TQM on \(\mathbb{R}_{\geq 0}\) in BV-BFV formalism.

**Remark 3.2.2** The condition (3.26) corresponds to [CMR18, (2.29)], which reflects the fact that the action could be modified by proper boundary terms according to the polarization we choose.

**Generic solutions to mQME**

To solve the mQME, we first study the BV anomaly of the interaction.

**Lemma 3.2.1** Given \(I_t\) defined in (3.23), we have

\[
(hd + \hbar^2 \partial_{K_t}) e^{I_t/\hbar} = \lim_{\varepsilon \to 0} e^{h b_{p(c, t)}} \left( e^{I_0/\hbar} \left( e^{J^0/\hbar} \right) \right) + \frac{\hbar}{2} \int_{\mathbb{R}_{\geq 0}} I^0 \star \hbar I^0
\]

where the limit in RHS is well defined by the same reason for Proposition 3.2.1. It is straightforward to see that both sides are products of \(e^{I_t/\hbar}\) with an element in \(\mathcal{O}(\mathcal{E}_c)[[\hbar]]\).

**Proof** Proposition 3.1.1 says that each \(P(\varepsilon, t)\) in \(e^{I_t/\hbar} := \lim_{\varepsilon \to 0} e^{h b_{p(c, t)}} e^{I_0/\hbar}\) can be replaced by a linear combination of \(\hat{P}(0, t)\). So we use the following simplified notation:

\[
e^{h b_{\hat{P}(0, t)}} e^{I_0/\hbar} := \lim_{\varepsilon \to 0} e^{h b_{p(c, t)}} e^{I_0/\hbar}
\]

where we should regard the terms in LHS as integrations over spaces defined in (3.10).

By (3.14), we have

\[
(hd + \hbar^2 \partial_{K_t}) e^{I_t/\hbar} = (hd + \hbar^2 \partial_{d[P(0, t)]}) e^{h b_{\hat{P}(0, t)}} e^{I_0/\hbar},
\]

which means the contraction with \(K_t\) can be replaced by contraction with \(d\) of linear combination of \(\hat{P}(0, t)\), while the replacement rule is the same as in (3.28).\(^6\) So, let \(\varphi \in \mathcal{E}_c^\otimes\) be some input field

---

\(^6\)For example, if \(K_t\) contracts with two factors of a single bulk vertex, it will contribute as \(K_t|_{D_2}\) on \(\mathbb{R}_{\geq 0}\) where \(D_2\) is the diagonal of \(\mathbb{R}_{\geq 0}^2\). It is direct to verify \(K_t|_{D_2} = dx \partial_x(P(0, t)(x, x)c_1 + P(0, t)(x, x)c_2)/2\).
\((r > 0)\),
\[
(h\dot d + \hbar^2 \partial_{K_t}) e^{I_t/\hbar}(\varphi)
= (h\dot d + \hbar^2 \partial_{d[\bar{P}(0,t)]}) e^{h\partial_{[\bar{P}(0,t)]}} \sum \text{ (certain expansion)} \times \int_{\mathbb{R}^2[n]} (I^0)^n (\pi^*(J^0))^m(\varphi)
= \sum \text{ (certain expansion)} \times \int_{\mathbb{R}^2[n]} (I^0)^n (\pi^*(J^0))^m((\hbar[\bar{P}(0,t)])^s \land d\varphi + d(\hbar[\bar{P}(0,t)])^s \land \varphi)
= e^{h\partial_{[\bar{P}(0,t)]}} \sum \text{ (certain expansion)} \int_{\partial\mathbb{R}^2[n]} (I^0)^n (\pi^*(J^0))^m(\varphi),
\]
where we have used Leibniz rule and Stokes’ theorem for the last equality. More precisely,
\[
(h\dot d + \hbar^2 \partial_{K_t}) e^{I_t/\hbar}(\varphi)
= e^{I_t/\hbar} \sum \text{ (certain coefficient)} \times \int_{\mathbb{R}^2[n]} (I^0)^n (\pi^*(J^0))^m((\hbar[\bar{P}(0,t)])^s \land d\varphi + d(\hbar[\bar{P}(0,t)])^s \land \varphi)
= \sum \text{ (certain expansion)} \times \int_{\mathbb{R}^2[n]} (I^0)^n (\pi^*(J^0))^m(\varphi),
\]
with \(\mathbb{R}^2[n][V_{\text{bulk}}(\Gamma)]\) the set of bulk vertices, boundary vertices and internal edges of \(\Gamma\), respectively.

The boundary of \(\mathbb{R}^2[n][V_{\text{bulk}}(\Gamma)]\) has two kinds of components at codimension one. The first kind corresponds to a certain \(x_i\) going to 0, and the second kind corresponds to coincidence of some \(x_i\) and \(x_j\). So,
\[
(h\dot d + \hbar^2 \partial_{K_t}) e^{I_t/\hbar}
= h e^{I_t/\hbar}(e^{h\partial_{[\bar{P}(0,t)]}}) \sum_{n=1}^{\infty} \hbar^{-n}
\left(I^0(x_1 = 0^+) \int_{x_2 \geq x_1} I^0(x_2) \cdots \int_{x_n \geq x_n-1} I^0(x_n) + \sum_{i=1}^{n-1} \int_{x_i \geq 0} I^0(x_1) \cdots \int_{x_{i+1} \geq x_{i+1}} I^0(x_{i+1}) \int_{x_{i+1} \geq x_{i+1}} I^0(x_{i+1}) \cdots \int_{x_n \geq x_n-1} I^0(x_n) \right)
= e^{h\partial_{[\bar{P}(0,t)]}} \left(e^{I^0/\hbar}(\pi^* e^{I^0/\hbar} e^{\hbar^2 K_0^1} e^{h\partial_{(\kappa_0^1 - \kappa_0^1)^4/4 I^0, e^{I^0/\hbar}}}) + h^{-1} \int_{\mathbb{R}^2[n]} I^0 \ast h I^0 \right).
\]
We have used (3.12) and (3.13) to obtain the final result, while omitted details are left as an exercise.

\begin{remark}
Similar BV anomaly computation for TQM on \(S^1\) using configuration space can be found in [GL17, Section 3.4]. For BF-like theories on manifold with boundary, [CMR18, Section 4.2] contains
\end{remark}
a rough description of such computations for partition functions. For Hamiltonian mechanics with constraints, \cite{?, Appendix B} contains a proof of mQME for partition functions as well as a detailed consideration of the dependence of the solutions on gauge fixing conditions by using configuration space.

We can imagine an $\varepsilon \to 0$ version of Theorem 3.1.1, which should imply:

**Lemma 3.2.2** Given $I_t$ defined in (3.23), we have

\[
\lim_{\varepsilon \to 0} \left( 1 \otimes e^{\hbar \partial_P(e, \varepsilon)} \right) e^\left( \left( \int_{\mathbb{R}^n} I^0 \right) \varepsilon \right) = e^{\frac{\langle \alpha \rangle}{\hbar}} I_t \varepsilon^\frac{\partial_P(e, \varepsilon)}{\hbar}. \]

The verification is left as an exercise.

Now we can write down generic solutions to mQME.

**Theorem 3.2.1** Given

\[
I^0 \in \prod_{n \geq 2} \text{Sym}^n(V^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(V^*)[[\hbar]], \quad |I^0| = 1,
\]

\[
J^0 \in \prod_{n \geq 2} \text{Sym}^n(L^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(L^*)[[\hbar]], \quad |J^0| = 0,
\]

and $H^0 \in \mathcal{O}(V)[[\hbar]]$ such that $|H^0| = 1, H^0 \ast \hbar H^0 = 0$, the mQME (3.27) is satisfied iff

\[
I^0 \ast \hbar I^0 = 0 \quad \text{and} \quad H^0 = -e^{J^0/\hbar} \ast \hbar I^0 \ast \hbar e^{-J^0/\hbar}.
\]

(It is direct to check $e^{J^0/\hbar} \ast \hbar I^0 \ast \hbar e^{-J^0/\hbar}$ is a well-defined element in $\mathcal{O}(V)[[\hbar]]$.)

**Proof** By (3.21) and Lemma 3.2.1

\[
(1 \otimes (\hbar d + \hbar^2 \partial_{K_1})) \ll_I (e^{(\pi^*(\alpha) + I)} \hbar) = e^{\frac{\langle \alpha \rangle}{\hbar}} (1 \otimes (\hbar d + \hbar^2 \partial_{K_1}) + \hbar (\ll_I (\alpha), -) \ll_I (e^{I} \hbar)) = \frac{\langle \alpha \rangle}{\hbar} \ll_I \left( (\hbar d + \hbar^2 \partial_{K_1}) e^{I_\hbar} \right) = e^{\frac{\langle \alpha \rangle}{\hbar} \ll_I} \lim_{\varepsilon \to 0} \left( (\hbar d + \hbar^2 \partial_{K_1}) e^{I_\hbar} \right) \times \left( \pi^* \left( e^{-J^0/\hbar} \ll_{cross} \left( e^{\frac{\hbar \partial_{K_1}^2}{4}} I^0 \right) e^{J^0/\hbar} \right) \right) + \hbar^{-1} \left( \int_{\mathbb{R}^n} I^0 \ast \hbar I^0 \right). \]
By Lemma 3.2.2 and (3.18),
\[(\Omega_{L'}^\text{left}(H^\partial, -) \otimes 1) \Pi_{\theta_i}(e^{(\pi^*(\alpha_i)+I_0)/\hbar})\]
\[= (\Omega_{L'}^\text{left}(H^\partial, -) \otimes 1) \lim_{\varepsilon \to 0} (1 \otimes e^{\hbar \partial_{P(\varepsilon,t)}}) e^{\left(\int_{\mathbb{R}^2} \frac{I^\partial}{\varepsilon_L} + \frac{J^\partial}{\varepsilon_L} + \frac{I(\alpha)}{\hbar}\right)}\]
\[= \lim_{\varepsilon \to 0} (1 \otimes e^{\hbar \partial_{P(\varepsilon,t)}}) e^{\left(\int_{\mathbb{R}^2} \frac{I^\partial}{\varepsilon_L} + \frac{J^\partial}{\varepsilon_L} + \frac{I(\alpha)}{\hbar}\right)}\]
\[= \lim_{\varepsilon \to 0} (1 \otimes e^{\hbar \partial_{P(\varepsilon,t)}}) e^{\left(\int_{\mathbb{R}^2} \frac{I^\partial}{\varepsilon_L} + \frac{J^\partial}{\varepsilon_L} + \frac{I(\alpha)}{\hbar}\right)} \times \Pi_{\theta_i}(e^{\frac{-\hbar \partial_{K^\partial}}{e_{\text{cross}}}} \left(e^{\frac{\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial, e^{J^\partial/\hbar}\right))\]
\[= e^{\frac{I(\alpha)}{\hbar}} \Pi_{\theta_i}(e^{\hbar \partial_{P(\varepsilon,t)}}) \left(e^{\hbar \partial_{P(\varepsilon,t)}} \times \pi^* \left(e^{\frac{-\hbar \partial_{K^\partial}}{e_{\text{cross}}}} \left(e^{\frac{\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial, e^{J^\partial/\hbar}\right)\right)\right),\]
with the last equality by the same reason as for Lemma 3.2.2.

So, the equation (3.27) is equivalent to
\[I^\partial \star_{\hbar} I^\partial = 0,\]
\[-e_{\text{cross}} \frac{-\hbar \partial_{K^\partial}}{e_{\text{cross}}} \left(e^{\frac{\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial, e^{J^\partial/\hbar}\right) = e^{J^\partial/\hbar} e^{\frac{\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial.\] (3.29)

Since we have imposed that $J^\partial \in \mathcal{O}(L)[[\hbar]]$, by the fact that $K^\partial \in L \otimes L'$,
\[e_{\text{cross}} \frac{-\hbar \partial_{K^\partial}}{e_{\text{cross}}} \left(e^{\frac{\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial, e^{J^\partial/\hbar}\right) = e^{J^\partial/\hbar} e^{\frac{\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial.\]

So (3.29) leads to
\[H^\partial = -e_{\text{cross}} \frac{-\hbar \partial_{K^\partial}}{e_{\text{cross}}} \left(e^{\frac{-J^\partial/\hbar}{e_{\text{cross}}}} \left(e^{\frac{-\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial, e^{J^\partial/\hbar}\right)\right)\]
\[= e_{\text{cross}} \frac{-\hbar \partial_{K^\partial}}{e_{\text{cross}}} \left(e^{\frac{-J^\partial/\hbar}{e_{\text{cross}}}} \left(e^{\frac{-\hbar \partial_{(K^\partial_+ - K^\partial_-)/4} H^\partial}{e_{\text{cross}}}} H^\partial, e^{J^\partial/\hbar}\right)\right)\]
\[= -e^{J^\partial/\hbar} \star_{\hbar} I^\partial \star_{\hbar} e^{J^\partial/\hbar}.\]

\[\square\]

**Remark 3.2.3** The condition $I^\partial \star_{\hbar} I^\partial = 0$ is equivalent to $\int_{S^1} I^\partial$ determining a consistent interaction for TQM on $S^1$ in perturbative BV formalism (see [GLL17 Theorem 3.10]).

**Remark 3.2.4** Actually the $J^\partial \neq 0$ solution can be obtained by a “boundary gauge transformation”
from the $J^\partial = 0$ solution: if $I^\partial \ast_h I^\partial = 0$,

$$0 = (1 \otimes (\hbar d + \hbar^2 \partial_{K^t}) - \Omega_{L'}^\partial(I^\partial, -) \otimes 1) \lim_{\epsilon \to 0} (1 \otimes e^{\hbar \partial P(\epsilon, t)} e^{\left(\int_{[0, \epsilon]} I^\partial \right)_{\epsilon L}})^{+1}/h$$

$$= (1 \otimes (\hbar d + \hbar^2 \partial_{K^t}) - \Omega_{L'}^\partial(e^{\hbar J^\partial/\hbar} \ast_h I^\partial \ast_h e^{-J^\partial/\hbar}, -) \otimes 1) \lim_{\epsilon \to 0} (1 \otimes e^{\hbar \partial P(\epsilon, t)} e^{\left(\int_{[0, \epsilon]} I^\partial \right)_{\epsilon L}})^{+1}/h$$

$$= (1 \otimes (\hbar d + \hbar^2 \partial_{K^t}) - \Omega_{L'}^\partial(e^{\hbar J^\partial/\hbar} \ast_h I^\partial \ast_h e^{-J^\partial/\hbar}, -) \otimes 1) \lim_{\epsilon \to 0} (1 \otimes e^{\hbar \partial P(\epsilon, t)} e^{\left(\int_{[0, \epsilon]} I^\partial \right)_{\epsilon L}})^{+1}/h.$$ 

\section{BV-BFV Description of TQM on Interval}

Now, consider TQM on interval $I = [0, 1]$. The spacetime is compact, so we are free of those subtleties in Remark 3.2.1. We will be sketchy since the story is similar to that in Section 3.

Field spaces

We have a free BV-BFV pair with bulk field data

$$\left(\mathcal{E} := \Omega^\bullet(I) \otimes V, d, \omega := \int_I \omega^\partial\right)$$

and boundary field data

$$(\mathcal{E}^\partial := V \oplus V = (\Omega^\bullet\{0\} \otimes V) \oplus (\Omega^\bullet\{1\} \otimes V), 0, \omega^\partial \oplus (-\omega^\partial))$$

($\Omega^\bullet\{0\}$ and $\Omega^\bullet\{1\}$ denote the de Rham complexes of the boundary points), and the restriction map

$$\pi := (\pi_0, \pi_1),$$

where $\pi_0, \pi_1$ are pullbacks of forms induced by $\{0\}, \{1\} \hookrightarrow I$, respectively. Now $(V, \omega^\partial)$ is a finite dimensional graded symplectic vector space endowed with two Lagrangian decompositions

$$V = L_0 \oplus L'_0 = L_1 \oplus L'_1,$$

by which we can decompose $K^\partial$ (the inverse to $\omega^\partial$, see (2.7)) as

$$K^\partial = K^\partial_{0,-} + K^\partial_{0,+} = K^\partial_{1,-} + K^\partial_{1,+}, \quad K^\partial_{i,-} \in L_i \otimes L'_i, \quad K^\partial_{i,+} \in L'_i \otimes L_i$$

for $i = 0, 1$.

The restricted bulk field space with boundary condition $L_0$ at $\{0\}$ and $L_1$ at $\{1\}$ is:

$$\mathcal{E}_{L_0, L_1} := \{f \in \Omega^\bullet(I) \otimes V|\pi_i(f) \in L_i \text{ for } i = 0, 1\}.$$

Then, $(\mathcal{E}_{L_0, L_1}, d, \omega)$ is a dg $(-1)$-symplectic vector space.
Renormalized BV structure with restricted bulk field space

Now we formulate the renormalized BV structure on $\mathcal{O}(E_{L_0,L_1})$. For Lagrangian decomposition $V = L_0 \oplus L'_0$, we denote the propagator defined in (3.7) as:

$$P_{L_0}^{R_{\geq 0}}(\varepsilon, \Lambda) := \left( -\frac{1}{4} (d^{GF} \otimes 1 + 1 \otimes d^{GF}) \int_\varepsilon^\Lambda dt \tilde{H}_t \right) \otimes K_{0,+}^\partial$$

$$+ \sigma \left( \frac{1}{4} (d^{GF} \otimes 1 + 1 \otimes d^{GF}) \int_\varepsilon^\Lambda dt \tilde{H}_t \right) \otimes K_{0,-}^\partial$$

with

$$\tilde{H}_t = \frac{1}{\sqrt{4\pi t}} \left( \phi(x-y)e^{-\frac{(x-y)^2}{4t}} (dy - dx) - \phi(x+y)e^{-\frac{(x+y)^2}{4(t-\varepsilon)}} (dy + dx) \right).$$

Suppose $\phi$ is supported on a subset of the open region $(-0.1, 0.1)$. Then it is direct to see that, on $(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus ([0, 0.1] \times [0, 0.1])$,

$$P_{L_0}^{R_{\geq 0}}(\varepsilon, \Lambda) = \left( -\frac{1}{4} (d^{GF} \otimes 1 + 1 \otimes d^{GF}) \int_\varepsilon^\Lambda dt \frac{1}{\sqrt{4\pi t}} \phi(x-y)e^{-\frac{(x-y)^2}{4t}} (dy - dx) \right) \otimes K^\partial.$$ 

Similarly we can write down a propagator for Lagrangian decomposition $V = L_1 \oplus L'_1$ with the spacetime modified to $\mathbb{R}_{\leq 1} := (-\infty, 1]$:

$$P_{L_1}^{\geq 0}(\varepsilon, \Lambda) := \left( -\frac{1}{4} (d^{GF} \otimes 1 + 1 \otimes d^{GF}) \int_\varepsilon^\Lambda dt \frac{1}{\sqrt{4\pi t}} \phi(x-y)e^{-\frac{(x-y)^2}{4t}} (dy - dx) \right) \otimes K^\partial.$$ 

Then, on $(\mathbb{R}_{\leq 1} \times \mathbb{R}_{\leq 1}) \setminus ([0, 0.1] \times [0, 0.1])$,

$$P_{L_1}^{\leq 1}(\varepsilon, \Lambda) = \left( -\frac{1}{4} (d^{GF} \otimes 1 + 1 \otimes d^{GF}) \int_\varepsilon^\Lambda dt \frac{1}{\sqrt{4\pi t}} \phi(x-y)e^{-\frac{(x-y)^2}{4t}} (dy - dx) \right) \otimes K^\partial.$$ 

So, we can define $P_{L_0,L_1}(\varepsilon, \Lambda)$ on $[0, 1] \times [0, 1]$ by gluing:

$$P_{L_0,L_1}(\varepsilon, \Lambda) := \begin{cases} 
P_{L_0}^{R_{\leq 1}}(\varepsilon, \Lambda) & \text{on } ([0, 1] \setminus [0, 0.1]) \\
\mathbb{I}[2]) \otimes V^\otimes 2 \quad (4.7)
\end{cases}$$

This is the propagator from scale $\varepsilon$ to scale $\Lambda$ of the renormalized free theory on $\mathcal{O}(E_{L_0,L_1})$. Similarly we can define the BV kernel at scale $t$ by gluing, denoted by $K_{L_0,L_1,t}$ (explicit formula omitted). Just as in Proposition 3.1.1 we can also define the extended propagator

$$\overline{P}_{L_0,L_1}(0, t) \in \Omega^0(\mathbb{I}[2]) \otimes V^\otimes 2 \quad (4.7)$$

with $\mathbb{I}[2] := \{(x, y) | 0 \leq x \leq y \leq 1 \} \cup \{(x, y) | 0 \leq y \leq x \leq 1 \}$. Then $K_{L_0,L_1,t} = d\overline{P}_{L_0,L_1}(0, t)$ on $\mathbb{I}[2]$.

In summary, $(\mathcal{O}(E_{L_0,L_1}), d, \partial_{K_{L_0,L_1,t}})$ is a differential BV algebra for $t > 0$, and we have the following conjugation:

$$e^{h\partial_{P_{L_0,L_1}(\varepsilon, \Lambda)}} (\mathcal{O}(E_{L_0,L_1})[[\hbar]], d + \hbar \partial_{K_{L_0,L_1,t}}) \quad (4.8)$$

If we replace $\mathcal{O}(E_{L_0,L_1})$ in (4.8) with $\mathcal{O}(E)$ where $E$ is defined in (4.1), the resulting relation still holds.

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Renormalized splitting

Similar to (3.16), for $\forall J \in \mathcal{O}(V), i = 0, 1$, define $\mathbb{I}_i(J) \in \mathcal{O}(L_i') \otimes \mathcal{O}(\mathcal{E}_{L_0,L_1})$ to be

$$\mathbb{I}_i(J) := (1 \otimes (\pi_i|_{\mathcal{E}_{L_0,L_1}})^*) (J), \quad (4.9)$$

where we regard $J$ as in $\mathcal{O}(L_i') \otimes \mathcal{O}(L_i)$ and $\pi_i|_{\mathcal{E}_{L_0,L_1}} : \mathcal{E}_{L_0,L_1} \to L_i$ is the restriction of $\pi_i$ in (4.3) to $\mathcal{E}_{L_0,L_1} \subset \mathcal{E}$. Now, for any splitting $\theta : L_0' \oplus L_1' \to \mathcal{E}$ as in Definition 2.2.2 and the induced $\mathbb{I}_\theta : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(L_0') \otimes \mathcal{O}(L_1') \otimes \mathcal{O}(\mathcal{E}_{L_0,L_1})$ in (2.5), it is direct to see that $\mathbb{I}_\theta(\pi_i^* (J)) = \mathbb{I}_i(J)$.

For $i = 0, 1$, let $\alpha_i \in (L_i')^* \otimes L_i^*$ be determined by

$$\alpha_i(l' \otimes l) := \omega^0(l', l) \text{ for } l' \in L_i', l \in L_i. \quad (4.10)$$

Regard $\alpha_i$ as an element in $\text{Sym}^2(V^*)$. Then,

$$\mathbb{I}_i(\alpha_i) \in (L_i')^* \otimes (\mathcal{E}_{L_0,L_1})^*, \quad \mathbb{I}_i(\alpha_i)(l' \otimes f) = \omega^0(l', \pi_i(f)).$$

Similar to Definition 3.1.2, we define:

**Definition 4.0.1** Given the extended propagator in (4.7) and $\alpha_0, \alpha_1$ in (4.10), the scale $t$ renormalized splitting $\theta_t : L_0' \oplus L_1' \to \mathcal{E}$ for the current TQM on $I$ is

$$\theta_t (l_i^0, l_i^1) := 2(\mathbb{I}_0(\alpha_0)(l_i^0 \otimes -) \otimes 1)\overline{P_{L_0,L_1}}(0, t)|_{C_1} - 2(\mathbb{I}_1(\alpha_1)(l_i^1 \otimes -) \otimes 1)\overline{P_{L_0,L_1}}(0, t)|_{C_2} \quad (4.11)$$

where $l_i^0 \in L_0', l_i^1 \in L_1', C_1 := \{(x, y)|0 \leq x \leq y \leq 1\}, C_2 := \{(x, y)|0 \leq y \leq x \leq 1\}$ here.

The renormalization of $\theta_t$ reads:

**Theorem 4.0.1** For $\forall \varepsilon, \Lambda > 0$, the renormalized splittings $\theta_\varepsilon, \theta_\Lambda$ in (4.11) and the conjugation map $e^{\hbar \beta_{L_0,L_1}(\varepsilon, \Lambda)}$ in (4.8) make this diagram commute:

$$\begin{align*}
\mathcal{O}(\mathcal{E})[[\hbar]] & \xrightarrow{\mathbb{I}_0^\Lambda} \mathcal{O}(L_0') \otimes \mathcal{O}(L_1') \otimes \mathcal{O}(\mathcal{E}_{L_0,L_1})[[\hbar]] \\
\downarrow e^{\hbar \beta_{L_0,L_1}(\varepsilon, \Lambda)} & \quad \downarrow e^{-(\mathbb{I}_0(\alpha_0) - \mathbb{I}_1(\alpha_1))/\hbar} \left(1 \otimes 1 \otimes e^{\hbar \beta_{L_0,L_1}(\varepsilon, \Lambda)}\right) e^{(\mathbb{I}_0(\alpha_0) - \mathbb{I}_1(\alpha_1))/\hbar}
\end{align*}$$

This completes the description of homotopic renormalization for free TQM on $I$ in BV-BFV formalism.

**The mQME and its generic solutions**

Using the extended propagator (4.7), we can repeat the proof of Proposition 3.2.1 to show that, given

$$\begin{align*}
I^0 & \in \prod_{n \geq 2} \text{Sym}^n(V^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(V^*)[[\hbar]], \quad |I^0| = 1, \\
J_0^0 & \in \prod_{n \geq 2} \text{Sym}^n(L_0^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(L_0^*)[[\hbar]], \quad |J_0^0| = 0, \\
J_1^0 & \in \prod_{n \geq 2} \text{Sym}^n(L_1^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(L_1^*)[[\hbar]], \quad |J_1^0| = 0,
\end{align*}$$

(4.12)
the scale $t$ interaction
\[ I_t := \lim_{\epsilon \to 0} \hbar \log(e^{I_0}e^{I_0/\hbar}) \]  
(4.13)
with $I_0 := \pi_0^*(J_0^0) + \pi_1^*(J_1^0) + \int_{\mathbb{I}_1} I^0$ is a well-defined element in $\mathcal{O}(\mathcal{E})[[\hbar]]$. Let
\[ H_0^0, H_1^0 \in \mathcal{O}(\mathcal{V})[[\hbar]], \quad |H_i^0| = 1, H_i^0 \star_h H_i^0 = 0 \quad \text{for} \quad i = 0, 1. \]  
(4.14)
They induces a BFV operator
\[ \Omega^{\text{left}}_{L_0^0}(H_0^0, -) \otimes 1 + 1 \otimes \Omega^{\text{right}}_{L_1^0}( -, H_1^0) \]  
(4.15)
on $\mathcal{O}(L_0^0) \otimes \mathcal{O}(L_1^0)[[\hbar]]$, where $\Omega^{\text{right}}_{L_1^0}( -, -)$ is defined in (2.12) with the substitution $L \to L_1, L' \to L_1$. Similar to Definition 3.2.1, we define:

**Definition 4.0.2** Given $I^0, J_0^1, J_1^1$ as in (4.13), we say that $I_t$ in (4.13) satisfies scale $t$ modified quantum master equation (mQME) with (4.13) being the BFV operator if
\[ ((\Omega^{\text{left}}_{L_0^0}(H_0^0, -) \otimes 1 + 1 \otimes \Omega^{\text{right}}_{L_1^0}( -, H_1^0)) \otimes 1 + 1 \otimes \left( \hbar d + \hbar^2 \partial_{\pi_0(t_1)} \right)) \int_0 \left( e^{(\pi_0^*(\alpha_0) - \pi_1^*(\alpha_1) + I_t)/\hbar} \right) = 0. \]  
(4.16)

By arguments similar to those in Section 3.2, we have:

**Theorem 4.0.2** Given $I^0, J_0^1, J_1^1$ as in (4.13), $H_0^0, H_1^0$ as in (4.14), the mQME (4.16) is satisfied iff
\[ I^0 \star_h I^0 = 0, \quad H_0^0 = -e^{J_0^0/\hbar} \star_h I^0 \star_h e^{-J_0^0/\hbar}, \quad H_1^0 = e^{-J_1^0/\hbar} \star_h I^0 \star_h e^{J_1^0/\hbar}. \]  
(4.17)

This describes generic solutions to the mQME for TQM on interval.

**Example 4.0.1** “BF theory with B-A boundary condition”

Let $\mathfrak{g}$ be a Lie algebra with basis $\{ t^a \}_{a=1}^\ell, \{ t^a, t^b \} = f^{abc} t^c$.

For $\beta \in \mathfrak{g}^*$, we use $\epsilon \beta$ to denote the element in $(\mathfrak{g}^*)[-1]$ corresponding to $\beta$, where $\epsilon$ is a formal variable of degree 1; similarly for $\alpha \in \mathfrak{g}$ we have $\eta \alpha \in \mathfrak{g}[1]$ where $\eta$ is a formal variable of degree $-1$.

There is a degree 0 symplectic pairing $\omega^0$ on
\[ V := (\mathfrak{g}^*)[-1] \oplus \mathfrak{g}[1], \]
determined by
\[ \omega^0(\epsilon t_a, \eta t_b) := \delta_a^b \]
where $\{ t_a \}_{a=1}^\ell$ is the basis of $\mathfrak{g}^*$ dual to $\{ t^a \}_{a=1}^\ell$. Then, for the graded symplectic vector space $(V, \omega^0)$, the inverse to $\omega^0$ (see (2.7) for definition) is
\[ K^0 = -\eta t^a \otimes \epsilon t_a - \epsilon t_a \otimes \eta t^a. \]
The model we talk about is the so-called “BF theory” on $\mathbb{I}$, with bulk field space $(\mathcal{E} = \Omega(\mathbf{I}) \otimes V, d, \omega = \int_{\mathbb{I}} \omega^0)$ and boundary field space $(\mathcal{E}^0 = V \oplus V, 0, \omega^0 \oplus (-\omega^0))$. We choose the following polarization:
\[ L_0 = (\mathfrak{g}^*)[-1], L_0' = \mathfrak{g}[1]; \quad L_1 = \mathfrak{g}[1], L_1' = (\mathfrak{g}^*)[-1], \]  
(4.18)
which is the so-called “B boundary condition” at \{0\} and “A boundary condition” at \{1\}. Let \( B^a \) be the basis of \((g^*[1])^*\) and \( A_a \) be the basis of \((g[1])^*\) so that

\[
B^a(\epsilon t) = A_b(\eta t^a) = \delta_b^a.
\]

Then, BF theory corresponds to the following choice:

\[
I^\partial = \frac{1}{2} f_{c}^{ab} B^c A_a A_b, \quad J_0^\partial = J_1^\partial = 0, \quad H_0^\partial = -H_1^\partial = -I^\partial.
\]

This data satisfies \((4.17)\) because

\[
I^\partial \star_h I^\partial = -\hbar f_{b}^{ab} f_{c}^{a'b'} B^c A_a A_{a'} = 0
\]

by Jacobi identity. Hence we indeed obtain a consistent interactive TQM in BV-BFV formalism.

\section{Return to the BV Description}

In \cite{WY22} we only work on the restricted bulk field space \( \mathcal{E}_L \), and define the theory within BV formalism. This approach is taken in several other works to deal with QFT’s on manifold with boundary, such as \cite{Rab21, GRW20, GW19, Zen21}. Moreover, factorization algebras in the sense of \cite{CG16, CG21} can be constructed on the spacetime to describe observables of the theory (see e.g., \cite{Rab21}). In this section, we will extract such a BV description from the BV-BFV description for TQM in previous sections. Then, the mQME is translated to a condition arising from factorization algebra data. This translation exhibits the connection between the BV-BFV interpretation and the factorization algebra interpretation of the “state”.

\subsection{QME with restricted bulk field space}

We first consider TQM on \( \mathbb{R}_{\geq 0} \) discussed in Section \( \ref{sec:3} \).

There is a differential BV algebra \( (\mathcal{O}(\mathcal{E}_L), d, \partial_{K_t}) \), where the restricted bulk field space \( \mathcal{E}_L \) is defined in \((3.4)\) and the BV kernel \( K_t \) is defined in \((3.5)\). The complex \((\mathcal{O}(\mathcal{E}_L)[[\hbar]], d + \hbar \partial_{K_t})\) will be perturbed by restriction of interaction \( I_t \) in \((3.23)\) to \( \mathcal{E}_L \), which is

\[
I_t|_{\mathcal{E}_L} = \lim_{\varepsilon \to 0} \hbar \log \left( e^{h\partial_{P(t \epsilon)}(I_0|_{\mathcal{E}_L}/\hbar)} \right)
\]

with

\[
I_0|_{\mathcal{E}_L} = \mathbb{I}(J^\partial) + \left( \int_{\mathbb{R}_{\geq 0}} I^\partial \right)_{\mathcal{E}_L}
\]

and \( J^\partial \in \mathcal{O}(L)[[\hbar]] \) so that \( \mathbb{I}(J^\partial) \in \mathcal{O}(\mathcal{E}_L)[[\hbar]] \) (\( \mathbb{I} \) is defined in \((3.16)\)). Then, BV quantization amounts to consider the following QME:

\[
(\mathcal{L} + \hbar \partial_{K_t}) e^{(I_0|_{\mathcal{E}_L}/\hbar)} = 0.
\]

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By Lemma [3.2.1] it is direct to see
\[(\hbar d + \hbar^2 \partial_{K_t}) e^{(I_t |_{\varepsilon_L})/\hbar}
= \lim_{\varepsilon \to 0} e^{\hbar \partial_P^{(\varepsilon)}} \left( e^{(I_0 |_{\varepsilon_L})/\hbar} \left( \Pi |_{\varepsilon_L} \left( e^{-J^{\theta}/\hbar} \frac{\hbar \partial_{K^a}}{\epsilon_{\text{cross}}} \left( e^{\hbar \partial(K_L^a - K_{-L}^a)/4} \mathcal{I}^\theta, e^{J^{\theta}/\hbar} \right) \right)
+ 2\hbar^{-1} \left( \int_{\mathbb{R}^3 \times \mathcal{I}^\theta} \mathcal{I}^\theta \right) \right) \middle|_{\varepsilon_L} \right) \right) \].

So we have:

**Proposition 5.0.1** The QME \([5.2]\) is equivalent to
\[\Omega_L^\text{right}(e^{J^{\theta}/\hbar}, \mathcal{I}^\theta) = 0, \quad \text{and} \quad \mathcal{I}^\theta \ast_{\hbar} \mathcal{I}^\theta = 0. \quad (5.3)\]

(5.3) ensures that \((\mathcal{O}(E_L)([h]), d + \hbar \partial_{K_t} + \{I_t |_{E_L}, -\})\) is a cochain complex \((\{-, -\}_t\) is the BV bracket of \(\partial_{K_t}\) on \(\mathcal{O}(E_L)\)).

**Remark 5.0.1** (5.3) also ensures that
\[\left( \mathcal{O}(L)([h]), \frac{1}{\hbar} \Omega_L^\text{right}(-, e^{J^{\theta}/\hbar} \ast_{\hbar} \mathcal{I}^\theta \ast_{\hbar} e^{-J^{\theta}/\hbar}) \right) \quad (5.4)\]
is a derived BV algebra in the sense of [Ban20, Definition 2.1] (we also refer to [WY22, Section 5] for a review of this notion). If we use homological perturbation theory to transfer the operator \(\hbar \partial_{K_t} + \{I_t |_{E_L}, -\}_t\) on \(\mathcal{O}(E_L)([h])\) to d-cohomology of \(\mathcal{O}(E_L)([h])\) following [WY22, Section 4.1], (5.4) will be the resulting “effective observable complex” for the current model.

Similarly, if we consider TQM on \(I\) with restricted bulk field space \(E_{L_0, L_1}\) defined in \([4.5]\), the QME would be
\[(d + \hbar \partial_{K_{L_0, L_1}}) e^{(I_1 |_{E_{L_0, L_1}})/\hbar} = 0 \quad (5.5)\]
where \(I_t\) here is defined in \([4.13]\). Repeating the arguments for the previous case, we have:

**Proposition 5.0.2** The QME \([5.5]\) is equivalent to
\[\Omega_{L_0}^\text{right}(e^{J_{\theta_0}/\hbar}, \mathcal{I}^\theta) = 0, \quad \Omega_{L_1}^\text{left}(\mathcal{I}^\theta, e^{J_1^{\theta}/\hbar}) = 0, \quad \text{and} \quad \mathcal{I}^\theta \ast_{\hbar} \mathcal{I}^\theta = 0 \quad (5.6)\]
where \(\Omega_{L_1}^\text{left}(\cdot, \cdot)\) is defined in \([2.11]\) with the substitution \(L \to L'_1, L' \to L_1\).

**Example 5.0.1** We continue discussing BF theory with B-A boundary condition mentioned in Example 4.0.1.

With polarization \([4.18]\) and \((I^{\theta}, J_{\theta_0}^{\theta}, J_{\theta_1}^{\theta})\) in \([4.19]\), we have \(\mathcal{I}^\theta \ast_{\hbar} \mathcal{I}^\theta = 0\), and
\[\Omega_{L_0}^\text{right}(e^{J_{\theta_0}/\hbar}, \mathcal{I}^\theta) = p_{L_0} \left( e^{\hbar \partial_{(K_{L_0}^a - K_{-L_0}^a)/4} \mathcal{I}^\theta} \right) = 0,
\Omega_{L_1}^\text{left}(\mathcal{I}^\theta, e^{J_{\theta_1}^{\theta}/\hbar}) = p_{L_1} \left( e^{\hbar \partial_{(K_{L_1}^a - K_{-L_1}^a)/4} \mathcal{I}^\theta} \right) = -\frac{1}{2} f^{cb}_{c} A_b.\]
\[ (K_{0,-}^0 = K_{1,+}^0 = -\epsilon t_a \otimes \eta t^a, K_{0,+}^0 = K_{1,-}^0 = -\eta t^a \otimes \epsilon t_a \text{ here.}) \]

So, if we work on the restricted bulk field space and study BV quantization, the B boundary condition will be anomaly free\(^7\) but there will be an anomaly associated to the A boundary condition. If the Lie algebra \( \mathfrak{g} \) is unimodular (i.e., \( f^a_{bc} = 0 \)), then A boundary condition is also anomaly free, and we obtain a consistent theory within BV formalism.

The mQME revisited

Now, suppose \( (I^0, J^0) \) satisfies the condition \([5.3]\), hence defining a TQM on \( \mathbb{R}_{\geq 0} \) within BV formalism. The QME solution \( \left. I_t \right|_{\mathcal{E}_L} \) induces an interactive observable complex

\[
(\mathcal{O}(\mathcal{E}_L)[[\hbar]], d + \hbar \partial K_t + \{I_t|_{\mathcal{E}_L}, -\})_t.
\]  

(5.7)

Recall the mQME \( [3.27] \) (with \( H^0 = -e^{J_0^0/\hbar} * h J^0 e^{-J^0_0/\hbar} \)):

\[
(1 \otimes (\hbar d + \hbar^2 \partial K_t) + \Omega^\text{left}_L (H^0, -) \otimes 1) \mathbb{I}_E (e^{(\pi^+(\alpha)+K_t)/\hbar}) = 0,
\]

it can be rewritten as

\[
(1 \otimes (\hbar d + \hbar^2 \partial K_t + \hbar \{I_t|_{\mathcal{E}_L}, -\})_t + \Omega^\text{left}_L (H^0, -) \otimes 1) e^{-(I_t|_{\mathcal{E}_L})/\hbar} \mathbb{II}_E (e^{(\pi^+(\alpha)+I_t)/\hbar}) = 0.
\]  

(5.8)

It is direct to verify that

\[
e^{-(I_t|_{\mathcal{E}_L})/\hbar} \mathbb{II}_E (e^{(\pi^+(\alpha)+I_t)/\hbar}) = e^{-(I_t|_{\mathcal{E}_L})/\hbar} \lim_{\epsilon \to 0} (1 \otimes e^{\hbar \partial P(\epsilon, t)}) e^{I(\alpha)+I_t|_{\mathcal{E}_L}} \hbar
\]

(5.9)

is of the form \( e^{A_t/\hbar} \) with \( A_t \in \mathcal{O}(L') \otimes \mathcal{O}(\mathcal{E}_L)[[\hbar]] \).

Then, we can use the pairing

\[ \ll -,- \gg: \mathcal{O}(L)[[\hbar]] \otimes \mathcal{O}(L')[[\hbar]] \to \mathbb{R}[[\hbar]] \]

defined in \([2.13]\) to dualize the \( \mathcal{O}(L') \) component of \( e^{-(\hbar|_{\mathcal{E}_L})/\hbar} \mathbb{II}_E (e^{(\pi^+(\alpha)+I_t)/\hbar}) \). A consequence of \([3.18]\) is that, the map \( \ll -, e^{\beta(\alpha)/\hbar} \gg \) actually equals to \( \mathbb{I} \) in \([3.16]\), mapping \( \mathcal{O}(L)[[\hbar]] \) to \( \mathcal{O}(\mathcal{E}_L)[[\hbar]] \). So by \([5.9]\),

\[ \mathbb{I}_{(0,t)} := \ll -, e^{-(\hbar|_{\mathcal{E}_L})/\hbar} \mathbb{II}_E (e^{(\pi^+(\alpha)+I_t)/\hbar}) \gg = e^{-(\hbar|_{\mathcal{E}_L})/\hbar} \lim_{\epsilon \to 0} e^{\hbar \partial P(\epsilon, t)} (e^{(\hbar|_{\mathcal{E}_L})/\hbar} (\cdot)),
\]

(5.10)

which also maps \( \mathcal{O}(L)[[\hbar]] \) to \( \mathcal{O}(\mathcal{E}_L)[[\hbar]] \). \( \mathbb{I}_{(0,t)} \) identifies the observable at scale \( t \) corresponding to a given “boundary observable at scale 0” under renormalization group flow. Then, by \([2.14]\), the mQME \([5.8]\) is equivalent to the condition that

\[ \mathbb{I}_{(0,t)} : \left( \mathcal{O}(L)[[\hbar]], \frac{-1}{\hbar} \Omega^\text{light}_L (-, H^0) \right) \to (\mathcal{O}(\mathcal{E}_L)[[\hbar]], d + \hbar \partial K_t + \{I_t|_{\mathcal{E}_L}, -\})_t \]

(5.11)

is a cochain map.

There is a natural way to understand this condition. For the TQM induced by \( (I^0, J^0) \), we can construct a factorization algebra \( \mathcal{F} \) of observables in the sense of \([CG16, CG21]\), as described in

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\(^7\)This validates an expectation in \([Rab21, \text{Remark 5.0.3}]\).

\(^8\)\( A_t \) is the summation over connected Feynman graphs containing vertices corresponding to \( \mathbb{I}(\alpha) \).
This factorization algebra assigns a cochain complex $\mathcal{F}(U)$ to each open subset $U$ of $\mathbb{R}_{\geq 0}$, and assigns a cochain map

$$\mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \otimes \cdots \otimes \mathcal{F}(U_n) \rightarrow \mathcal{F}(U)$$

to non-intersecting $U_1, \ldots, U_n$ which all lies in $U$. Particularly, the “global observables” $\mathcal{F}(\mathbb{R}_{\geq 0})$ (at scale $t$) is just $[5,7]$. $\mathcal{F}(U)$ is the subset of $\mathcal{F}(\mathbb{R}_{\geq 0})$ consisting of functionals supported on $U$.

For nested open subsets $[0, x_1) \subset [0, x_2) \subset \cdots \subset \mathbb{R}_{\geq 0}$, the factorization algebra data leads to the following sequence of embeddings:

$$\mathcal{F}([0, x_1)) \hookrightarrow \mathcal{F}([0, x_2)) \rightarrow \cdots \rightarrow \mathcal{F}(\mathbb{R}_{\geq 0}) = (\mathcal{O}(\mathcal{E}_L)[[\hbar]], d + \hbar \partial_{K_l} + \{ I_l | \mathcal{E}_L, - \} ).$$

We can imagine a limit among all such $\mathcal{F}([0, x))$’s, which should consist of functionals supported on $\{0\}$. [CG21] Section 10.1 contains a rigorous formulation on this kind of limit based on the factorization algebra data, which we do not explain here for brevity. By the content of (5.10), it is convincing that

$$\left( \mathcal{O}(L)[[\hbar]], -\frac{1}{\hbar} \Omega^\text{right}_L(-, H_\hbar^0) \right)$$

will be the expected limit of these $\mathcal{F}([0, x))$’s if we make things precise. Besides, $\Pi_{(0,t)}$ should be the limit of the “local to global maps” $\mathcal{F}([0, x)) \hookrightarrow \mathcal{F}(\mathbb{R}_{\geq 0})$, hence (5.11) follows from axioms of factorization algebra.

As for TQM on interval, the story is similar. Suppose $(I^0, J^0_0, J^0_1)$ satisfies (5.6), we have an interactive observable complex

$$\left( \mathcal{O}(\mathcal{E}_{L_0, L_1})[[\hbar]], d + \hbar \partial_{K_{L_0, L_1}} + \{ I_l | \mathcal{E}_{L_0, L_1}, - \} \right)$$

induced by the solution $I_l | \mathcal{E}_{L_0, L_1}$ to QME (5.5). Similar to (5.10), we define a map

$$\Pi_{(0,t)} := e^{-\left( I_l | \mathcal{E}_{L_0, L_1}, - \right) / \hbar} \lim_{\varepsilon \to 0} e^{\hbar \partial_{K_{L_0, L_1}(\varepsilon, t)}} \left( e^{(I_0 | \mathcal{E}_{L_0, L_1}, -) / \hbar} m_{\mathcal{O}(\mathcal{E}_{L_0, L_1})} (\Pi_0 \otimes \Pi_1 (\)) \right)$$

(5.12)

from $\mathcal{O}(L_0) \otimes \mathcal{O}(L_1)[[\hbar]]$ to $\mathcal{O}(\mathcal{E}_{L_0, L_1})[[\hbar]]$, where $I_0 | \mathcal{E}_{L_0, L_1} = I_0 (J^0_0) + I_1 (J^0_1) + \left( \int_{t_0}^t I^0 \right) \mathcal{E}_{L_0, L_1}$, $\Pi_0, \Pi_1$ are defined in (4.9) and $m_{\mathcal{O}(\mathcal{E}_{L_0, L_1})} : \mathcal{O}(\mathcal{E}_{L_0, L_1}) \otimes \mathcal{O}(\mathcal{E}_{L_0, L_1})$ denotes the symmetric product on $\mathcal{O}(\mathcal{E}_{L_0, L_1})$. Then, the mQME (4.16)

$$((\Omega^\text{left}_{L_0} (H^0_0, -) \otimes 1 + 1 \otimes \Omega^\text{right}_{L_1} (-, H^0_1)) \otimes 1 + 1 \otimes (\hbar d + \hbar^2 \partial_{K_{L_0, L_1}, t})) \Pi_{(\varepsilon, t)} (e^{(\varepsilon_0 (\alpha_0) - \varepsilon_1 (\alpha_1) + I_l) / \hbar}) = 0$$

(with $H^0_0 = -e^{-J^0_0 / \hbar} \ast_k I^0 \ast_k e^{-J^0_0 / \hbar}, H^0_1 = e^{-J^0_1 / \hbar} \ast_k I^0 \ast_k e^{J^0_1 / \hbar}$) is equivalent to the condition that (5.12) is a cochain map:

$$\Pi_{(0,t)} : \left( \mathcal{O}(L_0)[[\hbar]], -\frac{1}{\hbar} \Omega^\text{right}_L (-, H^0_0) \right) \otimes \left( \mathcal{O}(L_1)[[\hbar]], -\frac{1}{\hbar} \Omega^\text{left}_L (H^0_1, -) \right) \rightarrow \left( \mathcal{O}(\mathcal{E}_{L_0, L_1})[[\hbar]], d + \hbar \partial_{K_{L_0, L_1}, t} + \{ I_l | \mathcal{E}_{L_0, L_1}, - \} \right).$$

(5.13)
Just like the previous case, suppose $F$ is the factorization algebra constructed for the current TQM on interval, then the map $\mathbb{I}_{(0,t)}$ should be regarded as the limit among all such “local to global maps”:

$$F([0,x]) \otimes F((x',1]) \to F(I) = \left( \mathcal{O}(\mathcal{E}_{L_0,L_1})[[\hbar]], d + \hbar \partial K_{L_0,L_1,t} + \{ I_t|_{\mathcal{E}_{L_0,L_1}}, - \} \right)$$

associated to $[0,x] \sqcup (x',1] \to [0,1]$. This explains mQME in the framework of [CG16],[CG21].

**Remark 5.0.2** As mentioned in Remark 5.0.1, we can use homological perturbation theory to construct a projection from $\mathcal{O}(\mathcal{E}_L)[[\hbar]]$ (or $\mathcal{O}(\mathcal{E}_{L_0,L_1})[[\hbar]]$) to the effective observable complex. Then, by composing $\mathbb{I}_{(0,t)}$ with this projection, we obtain a map from $\mathcal{O}(L)[[\hbar]]$ (or $\mathcal{O}(L_0) \otimes \mathcal{O}(L_1)[[\hbar]]$) to the effective observable complex. This map is the “state” from the factorization algebra perspective. (Moreover, if we go through the calculations in Remark 5.0.1 for TQM on $\mathbb{R}_{\geq 0}$, this “state” will be the identity map on $\mathcal{O}(L)[[\hbar]]$, reflecting the trivial time evolution of TQM.)

Accordingly, we can take $\mathbb{I}_{t_0}(e^{(\pi^*(\alpha)+L_1)/\hbar})$ in (3.27) (or $\mathbb{I}_{t_0}(e^{(\pi^*_0(\alpha_0)-\pi^*_1(\alpha_1)+L_1)/\hbar})$ in (4.16)) and project its $\mathcal{O}(\mathcal{E}_L)$ (or $\mathcal{O}(\mathcal{E}_{L_0,L_1})$) component to the $d$-cohomology using homological perturbation theory. The outcome is the “state” in perturbative BV-BFV formalism, which mimics physicists’ “wave function” description.

So, our translation of mQME connects these two interpretations of the “state”.

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