Using Simulation to Estimate Reliability Function for Transmuted Kumaraswamy Distribution

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Abstract
In this study, some estimation methods (moments, modification moments, least square, weighted least square, maximum likelihood) were used to compare with each other the estimation of the parameters and reliability function of Transmuted Kumaraswamy (TK) distribution by using the simulation through four experiments (E1, E2, E3, E4) including the real values of the distribution parameters and by taking different sample sizes (n = 10, 25, 50, 100). The comparison was done using the mean square error (MSE) criterion, and the results were set in special tables included research.

Keywords: Transmuted Kumaraswamy Distribution, some estimation methods and mean square error.

1. Introduction
Poondi Kumaraswamy (1980) have been suggested Kumaraswamy Distribution as the probability distribution function is defined on the closed interval [0,1], where as, each of its cumulative (cdf) and probability (pdf) functions is expressed in the following, respectively [3]

\[ G_{Kw}(x; \alpha, \beta) = 1 - (1 - x^{\alpha})^{\beta} \]

\[ g_{Kw}(x; \alpha, \beta) = \alpha \beta x^{\alpha - 1}(1 - x^{\alpha})^{\beta - 1} \]

Where \( \alpha, \beta > 0 \) are the shape parameters and, X is a random variable that follows a (Kw) distribution, such that \( 0 \leq x \leq 1 \).

Muhammad, Robert and Irene (2016) was proposed a generalization of the Kumaraswamy distribution referred to as the Transmuted Kumaraswamy (TKW) distribution which has (cdf) and (pdf) as following, respectively [2]

\[ F_{TKw}(x; \alpha, \beta, \lambda) = \left( 1 - (1 - x^{\alpha})^{\beta} \right) \left( 1 + \lambda (1 - x^{\alpha})^{\beta} \right) \]

\[ f_{TKw}(x; \alpha, \beta, \lambda) = \alpha \beta x^{\alpha - 1}(1 - x^{\alpha})^{\beta - 1}(1 - \lambda + 2\lambda(1 - x^{\alpha})^{\beta}) \]

Where \( |\lambda| \leq 1 \) is transmuted parameter \( x \in (0,1) \).

And from Eq.(1) the quantial \( X_{q} \) of the (TKw) is

\[ X_{q} = \left( 1 - \frac{(1+\lambda)^{-\frac{1}{\beta}} - \sqrt{(1+\lambda)^{2 - 4\lambda F}}}{2\lambda} \right)^{\frac{1}{\alpha}}, 0 < F < 1 \]

The reliability function is given by
\[ R(x) = 1 - \left( \left(1 - (1 - x^\alpha)^\beta \right) \left(1 + \lambda (1 - x^\alpha)^\beta \right) \right) \]

2. Estimation Methods
In this section we introduce some estimation method as follows

1.2 Estimate initial value for estimators
The idea of this proposed method is based on the use of the median for distribution and the sample generated for distribution [6]. From median of (TKw)
Since
\[ x_{med} = \left( \frac{1 - \left(1 + \lambda \right)^{\frac{1}{\alpha}}}{2\lambda} \right)^{\frac{1}{\beta}} \]...
(4)

Let \( k = 1 - \frac{(1+\lambda)^{\frac{1}{\alpha}}}{2\lambda} \)

Eq.(4) became
\[ \ln(x_{med}) = \frac{1}{\alpha} \ln \left( 1 - k^{\frac{1}{\beta}} \right) \]...
(5)

\[ \hat{\alpha}_{MED} = \frac{\ln \left( 1 - k^{\frac{1}{\beta}} \right)}{\ln(x_{med})} \]...
(6)

From Eq.(5)
\[ x_{med}^\alpha = 1 - k^{\frac{1}{\beta}} \]
\[ \frac{1}{\beta} \ln(k) = \ln(1 - x_{med}^\alpha) \]

\[ \hat{\beta}_{MED} = \frac{\ln(k)}{\ln(1 - x_{med}^\alpha)} \]...
(7)

Where \((\alpha, \beta)\) are the real values taken to generate the samples, and we can use \(\hat{\alpha}_{MED}\) and \(\hat{\beta}_{MED}\) in Eq.(6) and Eq.(7) are Initial value \((\alpha_0, \beta_0)\) to another parameter estimator formula, and \(x_{med}\) can be obtained it from the generating sample.

2.2 Moments Method (MOM)
The idea of this method is to equate moments of population with the moments of the sample and to extract the parameter estimates from it. [4]

The first moment for (TKw) is
\[ E(x) = (1 - \lambda)\beta \frac{\Gamma \left( \frac{1}{\alpha} + 1 \right) \Gamma(\beta)}{\Gamma \left( \frac{1}{\alpha} + 1 + \beta \right)} + 2\lambda\beta \frac{\Gamma \left( \frac{1}{\alpha} + 1 \right) \Gamma(2\beta)}{\Gamma \left( \frac{1}{\alpha} + 1 + 2\beta \right)} = \frac{\sum_{i=1}^{n} x_i}{n} \]...
(8)
The second moment for \( TKw \) is

\[
E(x^2) = (1 - \lambda) \frac{\Gamma \left( \frac{2}{\alpha} + 1 \right) \Gamma(\beta) + 2\lambda \beta \frac{\Gamma \left( \frac{2}{\alpha_0} + 1 \right) \Gamma(2\beta_0)}{\Gamma \left( \frac{2}{\alpha_0} + 1 + \beta_0 \right)} - \sum_{i=1}^{n} x_i^2}{n} \tag{10}
\]

Then approximate reliability estimation is given by

\[
\hat{R}_{MOM}(t) = 1 - \left( \left(1 - (1 - x_{MOM})^{\hat{\beta}_{MOM}} \right) \left( 1 + \lambda \left(1 - x_{MOM}^{\hat{\beta}_{MOM}} \right) \right) \right) \tag{12}
\]

### 3.2 Modification Moments Method (MM)

This method is based on equating the expected approximate value of the function, at the first value of the observation, with a formula for the distribution function [7], as follows:

\[
E \left( \bar{F}(x_{(1)}) \right) = \frac{1}{n + 1} \tag{13}
\]

Replacing \( \bar{F}(x_{(1)}) \) by unbiased estimator, the plotting position formula

\[
P_i = \frac{i}{n + 1}, \quad i = 1, 2, ..., n \tag{14}
\]

We get:

\[
E \left( \bar{F}(x_{(1)}) \right) = E \left( \frac{1}{n + 1} \right) = \frac{1}{n + 1} \tag{15}
\]

From the equations (1) and (16) we get:

\[
\left( 1 - (1 - x_{(1)})^{\beta} \right) \left( 1 + \lambda \left(1 - x_{(1)}^{\beta} \right) \right) = \frac{1}{n + 1}
\]
\[ x_{(1)} = \left( 1 - \left( 1 - \left( 1 + \lambda - \frac{(1 + \lambda)^2 - \frac{4\lambda}{n + 1}}{2\lambda} \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\alpha}} \right) \] 

... (16)

Let

\[ h = 1 - \frac{(1 + \lambda) - \sqrt{(1 + \lambda)^2 - \frac{4\lambda}{n + 1}}}{2\lambda} \]

Eq.(16) became

\[ x_{(1)} = \left( 1 - h^\frac{1}{\beta} \right)^{\frac{1}{\alpha}} \] 

... (17)

By taking natural logarithm for Eq.(17) we get

\[ \hat{\alpha}_{MM} = \frac{\ln \left( 1 - h^{\frac{1}{\beta_0}} \right)}{\ln(x_{(1)})} \] 

... (18)

Since

\[ \text{Var} = E(x^2) - (E(x))^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1} \] 

... (19)

By substituting Eq.(8) and Eq. (10) in Eq.(19), we get

\[ \beta \left( 1 - \lambda \right) \frac{\Gamma \left( \frac{2}{\alpha} + 1 \right) \Gamma(\beta)}{\Gamma \left( \frac{2}{\alpha} + 1 + \beta \right)} + 2\lambda \frac{\Gamma \left( \frac{2}{\alpha} + 1 \right) \Gamma(2\beta)}{\Gamma \left( \frac{2}{\alpha} + 1 + 2\beta \right)} \]

\[ = \sum_{i=1}^{n} (x_i - \bar{x})^2 \left( 1 - \lambda \right) \beta \frac{\Gamma \left( \frac{1}{\alpha_0} + 1 \right) \Gamma(\beta_0)}{\Gamma \left( \frac{1}{\alpha_0} + 1 + \beta_0 \right)} + 2\lambda \beta_0 \frac{\Gamma \left( \frac{1}{\alpha_0} + 1 \right) \Gamma(2\beta_0)}{\Gamma \left( \frac{1}{\alpha_0} + 1 + 2\beta_0 \right)} \]

\[ \hat{\beta}_{MM} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 + \sum_{i=1}^{n} (x_i - \bar{x}) \frac{\Gamma \left( \frac{1}{\alpha_0} + 1 \right) \Gamma(\beta_0)}{\Gamma \left( \frac{1}{\alpha_0} + 1 + \beta_0 \right)} + 2\lambda \beta_0 \frac{\Gamma \left( \frac{1}{\alpha_0} + 1 \right) \Gamma(2\beta_0)}{\Gamma \left( \frac{1}{\alpha_0} + 1 + 2\beta_0 \right)}}{\left( 1 - \lambda \right) \frac{\Gamma \left( \frac{2}{\alpha_0} + 1 \right) \Gamma(\beta_0)}{\Gamma \left( \frac{2}{\alpha_0} + 1 + \beta_0 \right)} + 2\lambda \frac{\Gamma \left( \frac{2}{\alpha_0} + 1 \right) \Gamma(2\beta_0)}{\Gamma \left( \frac{2}{\alpha_0} + 1 + 2\beta_0 \right)}} \] 

... (20)

Then approximate reliability estimation is given by
\[ R_M(t) = 1 - \left( \left( 1 - \left( 1 - x_{AM} \right)^{\beta_{MM}} \right) \left( 1 + \lambda \left( 1 - x_{AM} \right)^{\beta_{MM}} \right) \right) \]  \hspace{2cm} \text{(21)}

### 4.2 Least square Method (LS)

The basic idea of this method is to minimize the value of the quantial function after matching it with the linear regression equation, \([1]\) as follows

From Eq.(3) and substitute \((F)\) by \((P_i)\) in Eq.(14), we get

\[ x(i) = \left( 1 - m_i^{1/\alpha} \right)^{1/\alpha}, \quad i = 1, 2, ..., n \]  \hspace{2cm} \text{(22)}

Where

\[ m_i = 1 - \frac{(1 + \lambda) - \sqrt{(1 + \lambda)^2 - 4 \lambda \frac{i}{n + 1}}}{2 \lambda} \]

\[ \ln(x(i)) = \frac{1}{\alpha} \ln \left( 1 - m_i^{1/\alpha} \right) \]

\[ \ln \left( -\ln(x(i)) \right) = -\ln(x(i)) + \ln \left( -\ln \left( 1 - m_i^{1/\alpha} \right) \right) \]  \hspace{2cm} \text{(23)}

Compare the Eq. (23) with the following linear regression equation

\[ Y_i = a + b \sigma_i + \epsilon \]  \hspace{2cm} \text{(24)}

We get

\[ Y_i = \ln \left( -\ln(x(i)) \right), \quad b = 1 \]

\[ a = -\ln(x(i)) \Rightarrow a = \exp(-a) \]  \hspace{2cm} \text{(25)}

\[ \sigma_i = \ln \left( -\ln \left( 1 - m_i^{1/\alpha} \right) \right) \]  \hspace{2cm} \text{(26)}

From Eq. (24), we get

\[ \epsilon = Y_i - a - b \sigma_i \]  \hspace{2cm} \text{(27)}

\[ \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (Y_i - a - b \sigma_i)^2 \]  \hspace{2cm} \text{(28)}

By taking the partial derivative for Eq. (28) with respect to \((a)\) and equating it with zero, we get

\[- \sum_{i=1}^{n} Y_i + na + \sum_{i=1}^{n} \sigma_i = 0 \]

\[ \hat{a}_{LS} = \frac{\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} \sigma_i}{n} \]  \hspace{2cm} \text{(29)}

By substituting values of \((a)\) from Eq.(29) into Eq.(25), we get

\[ \hat{a}_{LS} = \exp(-\hat{a}_{LS}) \]  \hspace{2cm} \text{(30)}

From Eq. (24), we get

\[ \hat{a}_{(LS)} = Y_i - \hat{a}_{LS} \]  \hspace{2cm} \text{(31)}
From Eq.(26) and Eq.(31), we get
\[
1 - m_i = \exp \left( - \exp \left( \hat{\alpha}_{i(LS)} \right) \right)
\]
\[
\hat{\beta}_{LS} = \frac{\sum_{i=1}^{n} \ln(m_i)}{n} \left( \frac{1}{n} \sum_{i=1}^{n} \ln \left( 1 - \exp \left( - \exp \left( \hat{\alpha}_{i(LS)} \right) \right) \right) \right)
\]
\[\cdots (32)\]

Then approximate reliability estimation is given by
\[
\hat{R}_{LS}(t) = 1 - \left( 1 - (1 - x \hat{a}_{LS})^{\hat{\beta}_{LS}} \right) \left( 1 + \lambda(1 - x^{\hat{a}_{LS}})^{\hat{\beta}_{LS}} \right)
\]
\[\cdots (33)\]

5.2 Weighted Least square method (WLS)

This method is based on Eq.(27) by divided it on \( \ell_i \) so we get the following:[5]
\[
\sum_{i=1}^{n} \left( \frac{\epsilon_i}{Y_i} \right)^2 = \sum_{i=1}^{n} \left( 1 - a \frac{1}{Y_i} - b \frac{\overline{Y_i}}{Y_i} \right)^2
\]
\[\cdots (34)\]

Let
\[
\xi_i = \frac{1}{Y_i}, Y_i = \frac{\overline{Y_i}}{Y_i}
\]

Equation (28) becomes as follows
\[
\delta = \sum_{i=1}^{n} \left( \frac{\epsilon_i}{Y_i} \right)^2 = \sum_{i=1}^{n} \left( 1 - a \xi_i - b Y_i \right)^2
\]
\[\cdots (35)\]

By taking the partial derivative for Eq.(35) with respect to \( a \) and equating it with zero, we get
\[
\frac{\partial \delta}{\partial a} = \sum_{i=1}^{n} \left( \xi_i - a \xi_i^2 - b Y_i \xi_i \right) = 0
\]
\[\cdots (36)\]
\[
\hat{a}_{WLS} = \frac{\sum_{i=1}^{n} \xi_i - b \sum_{i=1}^{n} Y_i \xi_i}{\sum_{i=1}^{n} \xi_i^2}
\]
\[\cdots (37)\]

Where, \( b = 1 \) and from Eq.(25) and Eq.(37) we get
\[
\hat{a}_{WLS} = \exp(-\hat{a}_{WLS})
\]
\[\cdots (38)\]

From Eq.(24), we get
\[
\hat{\alpha}_{i(WLS)} = Y_i - \hat{a}_{WLS}
\]
\[\cdots (39)\]

From Eq.(26) and Eq.(39), we get
\[
1 - m_i = \exp \left( - \exp \left( \hat{\alpha}_{i(WLS)} \right) \right)
\]
\[
\hat{\beta}_{LS} = \frac{\sum_{i=1}^{n} \ln(m_i)}{n} \left( \frac{1}{n} \sum_{i=1}^{n} \ln \left( 1 - \exp \left( - \exp \left( \hat{\alpha}_{i(WLS)} \right) \right) \right) \right)
\]
\[\cdots (40)\]

Then approximate reliability estimation is given by
\[ R_{WLS}(t) = 1 - \left( 1 - (1 - x^{\alpha_{WLS}})^{\beta_{WLS}} \right) \left( 1 + \lambda (1 - x^{\alpha_{WLS}})^{\beta_{WLS}} \right) \] ... (41)

6.2 Maximum likelihood method (MLE)

Let \[ x_1, x_2, x_3, ..., x_n \] be a r.v. of size n drawn from pdf of (TKw) distribution. The likelihood function for equation (2) is given by [2]:
\[ L = \alpha^n \beta^n \prod_{i=1}^{n} x_i^{\alpha - 1} \prod_{i=1}^{n} (1 - x_i^{\alpha})^{\beta - 1} \prod_{i=1}^{n} (1 - \lambda + 2\lambda(1 - x_i^{\alpha})^{\beta} ) \] ... (42)

The log likelihood function for Eq.(42) is
\[ LL = n\ln(\alpha) + n\ln(\beta) + (\alpha - 1) \sum_{i=1}^{n} \ln(x_i) + (\beta - 1) \sum_{i=1}^{n} \ln(1 - x_i^{\alpha}) + \sum_{i=1}^{n} \ln(1 - \lambda + 2\lambda(1 - x_i^{\alpha})^{\beta}) \] ... (43)

By taking partial derivative w.r.t. (\(\alpha\)) and equating to zero for Eq. (45), we get:
\[ \frac{\partial LL}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(x_i) - (\beta - 1) \sum_{i=1}^{n} x_i^{\alpha} \ln(x_i) - 2\lambda \beta \sum_{i=1}^{n} x_i^{\alpha} \ln(x_i)(1 - x_i^{\alpha})^{\beta - 1} = 0 \]

\[ \hat{\alpha}_{MLE} = \frac{n}{\beta_0 - 1} \sum_{i=1}^{n} x_i^{\alpha_0} \ln(x_i) + 2\lambda_0 \beta_0 \sum_{i=1}^{n} x_i^{\alpha_0} \ln(x_i)(1 - x_i^{\alpha})^{\beta - 1} - \sum_{i=1}^{n} \ln(x_i) \] ... (44)

By taking partial derivative w.r.t. (\(\beta\)) and equating to zero for Eq. (44), we get:
\[ \frac{\partial LL}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln(1 - x_i^{\alpha}) + 2\lambda \sum_{i=1}^{n} \ln(1 - x_i^{\alpha})(1 - x_i^{\alpha})^{\beta} = 0 \]

\[ \hat{\beta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \ln(1 - x_i^{\alpha_0}) + 2\lambda \sum_{i=1}^{n} \ln(1 - x_i^{\alpha_0})(1 - x_i^{\alpha_0})^{\beta_0} - \sum_{i=1}^{n} \ln(x_i) \] \]

... (45)

Now, by taking partial derivative w.r.t. (\(\beta\)) and equating to zero for Eq. (44), we get:
\[ \frac{\partial LL}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln(1 - x_i^{\alpha}) + 2\lambda \sum_{i=1}^{n} \ln(1 - x_i^{\alpha})(1 - x_i^{\alpha})^{\beta} = 0 \]

\[ \hat{\beta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \ln(1 - x_i^{\alpha_0}) + 2\lambda \sum_{i=1}^{n} \ln(1 - x_i^{\alpha_0})(1 - x_i^{\alpha_0})^{\beta_0} - \sum_{i=1}^{n} \ln(x_i) \] \]

Then reliability estimation is given by
\[ R_{MLE}(t) = 1 - \left( 1 - (1 - x^{\hat{\alpha}_{MLE}})^{\hat{\beta}_{MLE}} \right) \left( 1 + \lambda (1 - x^{\hat{\alpha}_{MLE}})^{\hat{\beta}_{MLE}} \right) \] ... (46)

3. Experiments and Results:

In this item, we will review simulation steps in terms of selecting sample sizes, real values of parameters and life time values that were used to estimate reliability:
1. The sample size n: (n=10, 25, 50, and 100).
2. Several values of the scale, transmuted parameters (\(\theta, \lambda\)) as shown in table (1) below:
3. Choose life time for estimating reliability
In all cases, \( E_1, E_2, E_3, E_4 \), we chosen \( 0 < t < 1 \).
Such that, \( t(i) = 0.1, 0.3, 0.5, 0.7, 0.9 \), \( i = 1, 2, 3, 4, 5 \).
Choose the number of sample replicated (N): \( N = 1000 \).

4. At this stage, random data is generated by \( TKw \) distribution by Eq.(3) and using MATLAB language version R2015a.

5. At this stage finding the value of parameter and reliability estimated according to the equations (9), (11), (12), (18), (20), (21), (30), (32), (33), (38), (40), (41), (44), (45) and (46).

6. Finally comparison between the estimators is done by
\[
MSE(\hat{\theta}) = \frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}
\]
Where \( \hat{\theta} \) is an estimator for parameter(\( \theta \)).

Table (1)
The default value for parameters

| Parameters | \( \alpha \) | \( \beta \) | \( \lambda \) |
|------------|------------|------------|-------------|
| \( E_1 \)  | 0.5        | 1          | 2           |
| \( E_2 \)  | 2          | 3          | 0.5         |
| \( E_3 \)  | -0.9       | -0.5       | 0.5         |
| \( E_4 \)  | 2          | 3          | 2           |

Table (2)
Estimated values for \( R \) and \( (\alpha, \beta) \) using \( E_1; R = 0.25080 \)

| Methods | \( n \) | Mean Estimated Values | MSE |
|---------|-------|-----------------------|-----|
|         |       | \( \hat{\alpha} \) | \( \hat{\beta} \) | \( \hat{R} \) | \( \alpha \) | \( \beta \) | \( R \) |
| MOM     | 10    | 0.63118               | 2.14383 | 0.38410 | 0.10787 | 3.54980 | 0.10877 |
| MM      | 25    | 0.20645               | 2.14171 | 0.11530 | 0.09541 | 1.37975 | 0.05714 |
| LS      |       | 0.50303               | 1.87716 | 0.28013 | 0.00617 | 0.18477 | 0.00707 |
| WLS     |       | 0.50324               | 1.89008 | 0.28119 | 0.00988 | 0.29027 | 0.00633 |
| MLE     |       | 0.72517               | 2.77621 | 0.26565 | 0.70836 | 4.82968 | 0.05361 |
| MOM     | 50    | 0.55152               | 2.35558 | 0.31087 | 0.05374 | 2.90540 | 0.07586 |
| MM      |       | 0.23357               | 2.20964 | 0.10880 | 0.07699 | 0.90492 | 0.05340 |
| LS      |       | 0.50466               | 1.96559 | 0.26345 | 0.00368 | 0.11023 | 0.00294 |
| WLS     |       | 0.50436               | 1.96818 | 0.26424 | 0.00786 | 0.17651 | 0.00304 |
| MLE     |       | 0.68302               | 2.29869 | 0.28128 | 0.50493 | 1.31493 | 0.03931 |
| MOM     | 100   | 0.52404               | 2.25484 | 0.28231 | 0.02711 | 1.62295 | 0.04792 |
| MM      |       | 0.25803               | 2.14183 | 0.11556 | 0.06365 | 0.46770 | 0.04439 |
| LS      |       | 0.50425               | 1.98422 | 0.25790 | 0.00203 | 0.06446 | 0.00145 |
| WLS     |       | 0.50490               | 1.98971 | 0.25823 | 0.00483 | 0.10741 | 0.00153 |
| MLE     |       | 0.58578               | 2.1317 | 0.27108 | 0.12568 | 0.26289 | 0.02182 |
| MOM     | 100   | 0.51063               | 2.13859 | 0.26283 | 0.01398 | 0.66226 | 0.02944 |
| MM      |       | 0.28504               | 2.07455 | 0.12614 | 0.05077 | 0.18106 | 0.03537 |
| LS      |       | 0.50323               | 1.99307 | 0.25507 | 0.00106 | 0.03440 | 0.00070 |
| WLS     |       | 0.50321               | 1.99424 | 0.25542 | 0.00256 | 0.05446 | 0.00076 |
| MLE     |       | 0.53565               | 2.05592 | 0.26195 | 0.02166 | 0.09843 | 0.01035 |
Table (3): Estimated values for \((R)\) and \((\alpha, \beta)\) using \((E_2; R = 0.0.30095)\)

| Methods | n | Mean Estimated Values | MSE |
|---------|---|-----------------------|-----|
|         |   | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(R\) | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(R\) |
| MOM     | 10 | 1.23019   | 2.94090   | 0.40234 | 0.30001 | 4.46806 | 0.08414 |
| MM      | 10 | 0.33589   | 2.95973   | 0.09408 | 0.46468 | 1.52964 | 0.10390 |
| LS      | 25 | 0.98506   | 2.72019   | 0.32665 | 0.01516 | 0.53846 | 0.00596 |
| WLS     | 25 | 0.99391   | 2.75112   | 0.32688 | 0.01470 | 0.61003 | 0.00584 |
| MLE     | 25 | 1.32726   | 3.68928   | 0.30739 | 1.42454 | 2.31805 | 0.04239 |
| MOM     | 50 | 1.09103   | 3.27263   | 0.33738 | 0.13699 | 3.55548 | 0.05706 |
| MM      | 50 | 0.41594   | 3.11172   | 0.10301 | 0.35817 | 0.95639 | 0.08738 |
| LS      | 50 | 0.99798   | 2.88591   | 0.31384 | 0.00945 | 0.29164 | 0.00241 |
| WLS     | 50 | 1.00173   | 2.90324   | 0.31400 | 0.01116 | 0.35701 | 0.00239 |
| MLE     | 50 | 1.30950   | 3.26441   | 0.32989 | 1.07377 | 0.60901 | 0.03616 |
| MOM     | 100| 1.02897   | 3.32140   | 0.30893 | 0.06894 | 2.48274 | 0.03663 |
| MM      | 100| 0.47856   | 3.14085   | 0.11696 | 0.28681 | 0.62751 | 0.07238 |
| LS      | 100| 1.00441   | 2.96347   | 0.30727 | 0.00505 | 0.17803 | 0.00124 |
| WLS     | 100| 1.00504   | 2.96985   | 0.30748 | 0.00677 | 0.20210 | 0.00123 |
| MLE     | 100| 1.22052   | 3.10155   | 0.32963 | 0.61792 | 0.23368 | 0.02397 |

Table (4): Estimated values for \((R)\) and \((\alpha, \beta)\) using \((E_2; R = 0.73149)\)

| Methods | n | Mean Estimated Values | MSE |
|---------|---|-----------------------|-----|
|         |   | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(R\) | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(R\) |
| MOM     | 10 | 2.62987   | 0.66063   | 0.67238 | 5.89628 | 0.26587 | 0.06930 |
| MM      | 10 | 3.31018   | 0.53422   | 0.77663 | 5.30981 | 0.04547 | 0.02425 |
| LS      | 25 | 2.45003   | 0.49485   | 0.75675 | 1.83556 | 0.03089 | 0.00995 |
| WLS     | 25 | 1.87413   | 0.46399   | 0.73642 | 0.30722 | 0.03651 | 0.00616 |
| MLE     | 25 | 2.55338   | 0.52733   | 0.75693 | 1.15971 | 0.01000 | 0.00627 |
| MOM     | 50 | 2.25816   | 0.59613   | 0.69046 | 1.50773 | 0.13352 | 0.04175 |
| MM      | 50 | 2.76333   | 0.52035   | 0.76646 | 1.66519 | 0.02249 | 0.01244 |
| LS      | 50 | 2.25635   | 0.49833   | 0.74647 | 0.77186 | 0.01675 | 0.00486 |
| WLS     | 50 | 1.97727   | 0.47999   | 0.73586 | 0.20498 | 0.01641 | 0.00298 |
| MLE     | 50 | 2.24353   | 0.51340   | 0.74325 | 0.33524 | 0.00431 | 0.00259 |
| MOM     | 100| 2.10191   | 0.45495   | 0.70790 | 0.54840 | 0.04206 | 0.02035 |
| MM      | 100| 2.57964   | 0.51158   | 0.76442 | 0.90279 | 0.00958 | 0.00750 |
| LS      | 100| 2.15505   | 0.49943   | 0.74262 | 0.31291 | 0.00825 | 0.00234 |
| WLS     | 100| 1.99287   | 0.48676   | 0.73545 | 0.13036 | 0.00818 | 0.00157 |
| MLE     | 100| 2.09402   | 0.50656   | 0.73678 | 0.06835 | 0.00198 | 0.00107 |

| Methods | n | Mean Estimated Values | MSE |
|---------|---|-----------------------|-----|
|         |   | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(R\) | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(R\) |
| MOM     | 10 | 2.06161   | 0.52007   | 0.72096 | 0.26492 | 0.01724 | 0.00991 |
| MM      | 10 | 2.48295   | 0.50419   | 0.76201 | 0.59423 | 0.04057 | 0.00507 |
| LS      | 100| 2.07674   | 0.49848   | 0.73688 | 0.14135 | 0.00431 | 0.00116 |
| WLS     | 100| 2.00344   | 0.49273   | 0.73391 | 0.06941 | 0.00423 | 0.00082 |
| MLE     | 100| 2.04900   | 0.50413   | 0.73387 | 0.02741 | 0.00100 | 0.00054 |
Table (5): Estimated values for \((R)\) and \((\alpha, \beta)\) using \((E_2; R = 0.19053)\)

| Methods | n  | Mean Estimated Values | MSE |
|---------|----|-----------------------|-----|
|         |    | \(\bar{\alpha}\)   | \(\bar{\beta}\) | \(\bar{R}\) | \(\hat{\alpha}\) | \(\hat{\beta}\) | \(\hat{R}\) |
| MOM     | 10 | 4.73002               | 0.95482   | 0.74549   | 3.24217 | 0.17228 | 0.02373 |
| MM      |    | 4.12219               | 0.93907   | 0.71244   | 2.98860 | 0.08737 | 0.02207 |
| LS      |    | 4.04383               | 0.88065   | 0.74145   | 0.63512 | 0.07686 | 0.00510 |
| WLS     |    | 3.52027               | 0.79813   | 0.72952   | 0.78301 | 0.11397 | 0.00447 |
| MLE     |    | 4.58162               | 1.08379   | 0.72851   | 2.38505 | 0.05141 | 0.00857 |
| MOM     | 25 | 4.28839               | 1.01591   | 0.7856    | 1.21304 | 0.12701 | 0.01572 |
| MM      |    | 3.82445               | 0.98963   | 0.70230   | 1.22734 | 0.05339 | 0.01219 |
| LS      |    | 4.04705               | 0.94553   | 0.73262   | 0.33747 | 0.04177 | 0.00230 |
| WLS     |    | 3.71543               | 0.88628   | 0.72474   | 0.40309 | 0.06000 | 0.00199 |
| MLE     |    | 4.45369               | 1.03858   | 0.72715   | 1.94843 | 0.02008 | 0.00631 |
| MOM     | 50 | 4.08675               | 1.04230   | 0.71635   | 0.57857 | 0.08662 | 0.00995 |
| MM      |    | 3.79434               | 1.01178   | 0.69943   | 0.88562 | 0.03429 | 0.00835 |
| LS      |    | 4.07646               | 0.97483   | 0.72889   | 0.18801 | 0.02540 | 0.00115 |
| WLS     |    | 3.89447               | 0.94671   | 0.72457   | 0.26834 | 0.03698 | 0.00107 |
| MLE     |    | 4.39109               | 1.01492   | 0.73137   | 1.36028 | 0.00876 | 0.00420 |
| MOM     | 100| 4.02671               | 1.03467   | 0.71125   | 0.32466 | 0.05304 | 0.00616 |
| MM      |    | 3.75491               | 1.01342   | 0.69869   | 0.65732 | 0.02062 | 0.00595 |
| LS      |    | 4.04844               | 0.99349   | 0.72517   | 0.09920 | 0.01524 | 0.00054 |
| WLS     |    | 3.96302               | 0.97768   | 0.72265   | 0.18886 | 0.02237 | 0.00057 |
| MLE     |    | 4.22904               | 1.00952   | 0.72817   | 0.62121 | 0.00434 | 0.00241 |

4. Conclusion

From the results in Tables 2, 3, 4 and 5, we note the following:

- In the first case \((E_1)\), \((LS)\) is the best in estimating \((\alpha, \beta)\) and the reliability function of the distribution, except in the case of \((n = 10)\) \((WLS)\) is the best in estimating the reliability.
- In the second case \((E_2)\), we note that \((LS)\) is the best in estimating \((\alpha, \beta)\) in all cases, except for \((n = 10)(WLS)\) is the best in estimating the \((\alpha)\), but in the reliability estimate, \((WLS)\) is the best except for \(\alpha = 10\) \((WLS)\) is the best.
- In the third case \((E_3)\), in the estimation of \((\alpha)\), \((WLS)\) is best when \((n = 10,25)\), but in \((n = 10,25)\) \((MLE)\) is the best, while in \((\beta)\) estimation, the \((MLE)\) is the best in all cases of \((n)\). In estimating reliability \((MLE)\) is best, except when in the case of \((n = 10)\), \((WLS)\) is the best.
- In the fourth case \((E_4)\), \((LS)\) is the best in estimating \((\alpha)\), whereas in \((\beta)\) estimation, the \((MLE)\) is the best, and in the reliability estimation, \((LS)\) is best when \((n = 10,100)\), but when \((n = 25,50)\), \((WLS)\) is the best.
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