ON THE IMPULSE CONTROL OF JUMP DIFFUSIONS
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Abstract. Regularity of the impulse control problem for a non-degenerate n-dimensional jump diffusion with infinite activity and finite variation jumps was recently examined in [4]. Here we extend the analysis to include infinite activity and infinite variation jumps. More specifically, we show that the value function of the impulse control problem has a locally Lipschitz continuous first derivative.

1. Introduction. In this paper we analyze the regularity of the value function in an impulse control problem for an n-dimensional jump diffusion process. We assume that the uncontrolled stochastic process $X$ is governed by the stochastic differential equation:

$$dX_t = ˜b(X_{t^-})dt + \sigma(X_{t^-})dW_t + \int_{\mathbb{R}^l} j(X_{t^-}, z)˜N(dt, dz), \quad X_0 = x. \quad (1.1)$$

Here $W$ is a $d$-dimensional standard Brownian Motion and $N$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^l$, with $W$ and $N$ independent. The Lévy measure $\nu(\cdot) := E[N(1, \cdot)]$ may be unbounded and $\tilde{N}(dt, dz)$ is its compensated Poisson random measure with $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$. Below, we specify the assumptions placed upon $\tilde{b}, \sigma, j$ in order to ensure that the SDE is well-defined. If an admissible control policy $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ is chosen, then $X$ evolves as

$$dX_t = \tilde{b}(X_{t^-})dt + \sigma(X_{t^-})dW_t + \int_{\mathbb{R}^l} j(X_{t^-}, z)\tilde{N}(dt, dz) + \sum_i \delta(t - \tau_i)\xi_i, \quad (1.2)$$

where $\delta$ denotes the Dirac delta function. Given an admissible control $V := (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$, the objective function is

$$J_x[V] := E_x \left( \int_0^\infty e^{-rt} f(X_t)dt + \sum_{i=1}^\infty e^{-r\tau_i} B(\xi_i) \right). \quad (1.3)$$

The goal is to minimize the objective function over all admissible control policies.

$$u(x) = \inf_V J_x[V]. \quad (1.4)$$

Intuitively, we expect from the Dynamic Programming Principle that the value function $u(x)$ satisfies the following quasi-variational inequality

$$\max\{ -Lu + ru - f, u - Mu \} = 0, \quad x \in \mathbb{R}^n, \quad (HJB)$$

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where $\mathcal{M}\varphi(x)$ is the minimal operator such that
\begin{equation}
\mathcal{M}\varphi(x) := \inf_{\xi \in \mathbb{R}^n} (\varphi(x + \xi) + B(\xi)),
\end{equation}
and the partial integro-differential operator $\mathcal{L}$ is defined as
\begin{equation}
\mathcal{L}\varphi(x) := \mathcal{L}_D\varphi(x) + I\varphi(x),
\end{equation}
with
\begin{align}
\mathcal{L}_D\varphi(x) &= \sum_{i,k=1}^n a_{ik}(x) \partial_{x_i x_k}^2 \varphi(x) + \sum_{i=1}^n b_i(x) \partial_{x_i} \varphi(x), \\
I\varphi(x) &= \int_{\mathbb{R}^d} (\varphi(x + j(x,z)) - \varphi(x) - j(x,z) \nabla \varphi(x)) \nu(dz),
\end{align}
where $(a_{ij})_{n \times n} := \frac{1}{2} \sigma(x) \sigma(x)^T$.

Analysis of the impulse control problem finds its roots in the classical works of [2] and [3]. With regard to impulse control, these authors characterized the value function, analyzed optimal policies and discussed regularity of the value function in the non-degenerate diffusion case with bounded data. Subsequent contributions such as [11], [12], [13] focused upon obtaining various characterizations of the value function for impulse control in more general settings such as in degenerate/non-degenerate pure/jump diffusion, bounded/unbounded data settings. The subject of this paper is on the regularity problem for the impulse control of non-degenerate jump diffusions. The regularity in various relevant contexts has been examined by many in the literature, among them [2], [3], [7], [10], [8], [15], [1], [9], [4]. Our focus for this paper is on identifying the regularity of the value function for impulse control under a general jump diffusion setting on the whole space and with unbounded controls. Recently, [9] (resp. [4]) identified the regularity of the value function of impulse control for a pure diffusion (resp. jump diffusion) with unbounded controls. In both of these papers, the authors utilized classical PDE arguments along with recent viscosity results for impulse control [16] to establish $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$. More specifically, in the jump diffusion case, the authors establish regularity for $u$ with integro-differential operators of order $[0,1]$. In what follows, we extend these results to include integro-differential operators of order $[1,2]$. In doing so, we examine whether regularity can be established using the approach taken in [4]. More specifically, this would involve an examination of a Dirichlet problem on a bounded open set with a non-local integro-differential operator. Resources for the regularity of second order elliptic integro-differential problems include [10], [8], [15], [5] and [6] among others. The technical difficulties encountered with examining this local problem with a non-local integro-differential operator of order $[1,2]$ lead us, alternatively, to consider establishing regularity on the whole domain using a weighted space; this follows the analysis in [13].

The paper is organized as follows. Section 2 provides the setup for the problem. Section 3 discusses useful properties of the integro-differential operator. Section 4 presents regularity of the value function in the continuation region. Section 5 presents the main result $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$, i.e., $u$ has a locally Lipschitz first derivative.

### 2. Setup

We adopt the notation used in [4] for function spaces if not explicitly defined and present the following assumptions:

We assume that the drift, volatility and the jump amplitude (in the first variable)
in (1.1) are Lipschitz continuous, i.e., there exists a positive constants \( C_b, C_\sigma > 0 \) and a positive function \( C_j(z) \in L^q(\mathbb{R}^l, \nu) \) for \( q = 1, 2, 4 \) such that for any \( x, y \in \mathbb{R}^n, z \in \mathbb{R}^l, \)

\[
|\tilde{b}(x) - \tilde{b}(y)| \leq C_b |x - y|, \quad |\sigma(x) - \sigma(y)| \leq C_\sigma |x - y|,
\]

\[
|j(x, z) - j(y, z)| \leq C_j(z) |x - y|,
\]

(H1)

where \( \tilde{b} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}, j : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^n \). Further, we assume that coefficients have Lipschitz continuous first derivatives (denoted \( \tilde{b}', \sigma', j' \)), i.e., there exists a positive constant \( C \) such that

\[
|\tilde{b}'(x) - \tilde{b}'(y)|^2 + |\sigma'(x) - \sigma'(y)|^2 + \int_{\mathbb{R}^l} |j'(x, z) - j'(y, z)|^2 \nu(dz) \leq C |x - y|^2.
\]

(H2)

For the jump amplitude \( j \) and the Lévy measure \( \nu \), we assume there exists some positive measurable function \( j_0(z) \) such that

\[
|j(x, z)| \leq j_0(z), \quad \int_{\{j_0(z) \geq 1\}} [j_0(z)]^p \nu(dz) \leq C_0 < \infty,
\]

\[
\int_{\{j_0(z) < 1\}} [j_0(z)]^p \nu(dz) \leq C_0 < \infty, \text{ for any } p \geq \gamma, \gamma \in [1, 2].
\]

(H3)

Assume that \( j(x, z) \) is continuously differentiable in \( x \) for any fixed \( z \) and for any \( x, x' \) and \( 0 \leq \theta \leq 1 \), there exists a constant \( c_0 > 0 \) such that

\[
c_0 |x - x'| \leq |(x - x') + \theta(j(x, z) - j(x', z))| \leq c_0^{-1} |x - x'|.
\]

(H4)

Moreover, the Jacobian of the transformation satisfies

\[
c_1^{-1} \leq \det[I_d + \nabla j(x, z)] \leq C_1,
\]

(2.1)

for any \( x, z \) and some constants \( c_1, C_1 \geq 1 \), where \( I_d \) is the identity matrix in \( \mathbb{R}^n \), \( \nabla j(x, z) \) is the matrix of the first partial derivatives in \( x \), and \( \det[\cdot] \) denotes the matrix determinant. There exists a constant \( M_\gamma > 0 \) such that

\[
|\nabla j(x, z)| \leq M_\gamma [j_0(z)]^{\gamma - 1}, \quad |\nabla \cdot j(x, z)| \leq M_\gamma [j_0(z)]^{\gamma},
\]

(H5)

where \( \nabla \cdot j(x, z) \) denotes the divergence of the function \( x \mapsto j(x, z) \) for any fixed \( z \). The diffusion component of \( X \) satisfies the uniform ellipticity condition, i.e., there exists \( \lambda > 0 \) such that

\[
\sum_{i,j=1}^n \xi_i a_{ij}(x) \xi_j \geq \lambda |\xi|^2; \quad \lambda > 0, \quad x \in \mathbb{R}^n.
\]

(H6)

Suppose the running cost \( f \geq 0 \) is Lipschitz continuous and semi-concave, i.e., there exists a constant \( C_f > 0 \) such that

\[
|f(x) - f(y)| \leq C_f |x - y|, \quad \forall x, y \in \mathbb{R}^n,
\]

(H7)

and for every open ball \( B_r(0) \) (or simply denoted \( B_r \)) of radius \( r > 0 \) centered at \( 0 \), there exists a constant \( C_r > 0 \) such that the function

\[
x \mapsto f(x) - C_r |x|^2 \text{ is concave.}
\]

(H8)
For equivalent characterizations of semi-concavity in this context see Definition 5.3 in Section 5.

The transaction cost function \( B : \mathbb{R}^n \to \mathbb{R} \) is lower semi-continuous and satisfies:

\[
\begin{align*}
\inf_{\xi \in \mathbb{R}^n} B(\xi) &= K > 0, \\
B &\in C(\mathbb{R}^n \setminus \{0\}), \\
|B(\xi)| &\to \infty, \text{ as } |\xi| \to \infty, \\
B(\xi_1) + B(\xi_2) &\geq B(\xi_1 + \xi_2) + K, \forall \xi_1, \xi_2 \in \mathbb{R}^n.
\end{align*}
\] (H9)

Assume the discount rate \( r \) is sufficiently large, i.e.,

\[ r > \kappa \geq \beta \geq 0, \] (H10)

where we define \( \beta \) in (2.5) and \( \kappa \) in (5.22).

The nonlocal integro-differential operator can be written as

\[ I\varphi(x) := \int_{\mathbb{R}^l} \left( \varphi(x + j(x, z)) - \varphi(x) - j(x, z) \cdot \nabla \varphi(x) \mathbb{1}_{\{j_2(z) < 1\}} \right) \nu(dz), \] (2.2)

and the local differential operator has the form

\[ \mathcal{L}_D \varphi(x) := \sum_{i,k=1}^{n} a_{ik}(x) \partial^2_{x_i x_k} \varphi(x) + \sum_{i=1}^{n} b_i(x) \partial_{x_i} \varphi(x), \] (2.3)

where \( b := \tilde{b} - \int_{\mathbb{R}^l} j(x, z) \mathbb{1}_{\{j_2(z) < 1\}} \nu(dz) \).

Given the above assumptions, the following result holds for \( u \) and \( \mathcal{M} u \) respectively.

**Lemma 2.1.** The function \( u(\cdot) \) is Lipschitz continuous with constant \( C_u \). Additionally, \( \mathcal{M} u(\cdot) \) is Lipschitz continuous.

**Proof.** A proof of this lemma can be found in [4] as Lemma 3.3. Within our above setup, we provide a proof that \( u \) is Lipschitz continuous. Given an admissible control \( V \) and two initial states \( x_1, x_2 \), denote by \( X_t^i \) the solution of (1.1). Set \( Y_t = X_t^1 - X_t^2 \) and apply Itô’s formula with \( \varphi(y, t) = |y|^2 e^{-\alpha t} \) to obtain

\[ \begin{align*}
d\varphi(Y_t, t) &= a_t dt + \sum_{k=1}^{d} b_t^k dW_t^k + \int_{\mathbb{R}^l} c(t, z) \tilde{N}(dt, dz), \quad \text{where} \quad (2.4) \\
a_t &:= \partial_t \varphi(Y_t, t) + \sum_{i=1}^{n} \left( \tilde{b}_i(X_t^1) - \tilde{b}_i(X_t^2) \right) \partial_i \varphi(Y_t, t) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \left( \sum_{k=1}^{d} \left[ \sigma_{ik}(X_t^1) - \sigma_{ik}(X_t^2) \right] \left[ \sigma_{jk}(X_t^1) - \sigma_{jk}(X_t^2) \right] \right) \partial^2_{ij} \varphi(Y_t, t) \\
&\quad + \int_{\mathbb{R}^l} \left[ \varphi(Y_t + j(X_t^1, z) - j(X_t^2, z), t) - \varphi(Y_t, t) \\
&\quad - \sum_{i=1}^{n} [j_i(X_t^1, z) - j_i(X_t^2, z)] \partial_i \varphi(Y_t, t) \nu(dz) \right], \\
b_t^k &:= \sum_{i=1}^{n} \left( \sigma_{ik}(X_t^1) - \sigma_{ik}(X_t^2) \right) \partial_i \varphi(Y_t, t), \\
c(t, z) &:= \varphi(Y_t + j(X_t^1, z) - j^*(X_t^2, z), t) - \varphi(Y_t, t),
\end{align*} \]
where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Define

$$
\beta := \sup_{x,x' \in \mathbb{R}^n} \{2\beta_{\beta} + \beta_{\sigma} + \beta_j\},
$$

$$
\beta_{\beta} := \sum_i \frac{\langle x_i - x'_i \rangle \tilde{b}_i(x) - \tilde{b}_i(x')}{|x - x'|^2},
$$

$$
\beta_{\sigma} := \sum_{i,k} \frac{\sigma_{ik}(x) - \sigma_{ik}(x')}{|x - x'|^2},
$$

$$
\beta_j := \int_{\mathbb{R}^l} \left[ |x - x' + j(x, z) - j(x', z)|^2 - |x - x'|^2 
- \sum_i 2(x_i - x'_i)[j_i(x, z) - j_i(x', z)] \right] |x - x'|^{-2} \nu(dz),
$$

where $\beta < \infty$ due to $(H1)$ (see Section 5.2.1 in [14]). Using (2.5) and taking $\alpha > \beta$, we find

$$
\mathbb{E}[\varphi(Y_t, t)] - (x_1 - x_2)^2 \leq (-\alpha + \beta) \int_0^t \mathbb{E}[\varphi(Y_s, s)]ds,
$$

which implies $\mathbb{E}[(X_{t-}^{1} - X_{t-}^{2})^2] \leq e^{\beta t/2} |x_1 - x_2|$ by Gronwall’s and Jensen’s inequality. Using $(H7)$ and $(H10)$, we have $J_{x_1}(V) - J_{x_2}(V) \leq C_u |x_1 - x_2|$ with $C_u = C_f/(r - \beta/2)$. Subsequently,

$$
u(x_1) \leq J_{x_1}(V) \leq J_{x_2}(V) + C_u |x_1 - x_2|.
$$

Taking the infimum over all admissible controls with initial state $x_2$ yields the desired inequality. Now, exchanging the roles of $x_1, x_2$ completes the proof. \(\square\)

3. Integro-Differential Operator. In this section, we discuss some properties of the non-local operator $I$.

**Definition 3.1.** Recall, let $B_r(x)$ denote the open ball of radius $r$ centered at $x$. The outer $\eta$-neighborhood of $\Omega$ is defined as $\Omega^{-} := \{x \in \mathbb{R}^n : x \in B_\eta(y) \text{ for some } y \in \Omega\}$. Similarly, the inner $\eta$-neighborhood of $\Omega$ is defined as $\Omega^{+} := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \eta\}$.

**Lemma 3.1.** ($\varepsilon$-$L^p$-estimates) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose $(H3)$, and $(H4)$ hold. Then, for any given $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ depending on $\varepsilon$, such that for smooth $\varphi$, Lipschitz on $\mathbb{R}^n$ with constant $C_\varphi$, we have for $1 \leq p \leq \infty$,

$$
\|I\varphi\|_{L^p(\mathcal{O})} \leq \varepsilon \|\varphi\|_{W^{2,p}(\mathcal{O}^+)} + C(\varepsilon)C_\varphi.
$$

**Proof.** We proceed as in Lemma 2.2.1 in [6]. Let $\eta \in (0, 1]$ be determined later. Based on (H3), we know

$$
\eta^{-1} \int_{\{\eta \leq j_0(z) < 1\}} j_0(z)\nu(dz) \leq \int_{\{j_0(z) < 1\}} [j_0(z)]^\gamma \nu(dz) \leq C_0,
$$

where $\gamma > 1$. We start with the case where $\eta > 1$.
\[
\int_{\{j_0(z) < \eta\}} [j_0(z)]^2 \nu(dz) \leq \eta^{2-\gamma} r(\eta), \tag{3.3}
\]
where the module of integrability is given by
\[
r(\eta) = \int_{\{j_0(z) < \eta\}} [j_0(z)]^\gamma \nu(dz). \tag{3.4}
\]

Now, we write \( I_\varphi = I_{n,1}^I \varphi + I_{n,2}^I \varphi + I_{n,3}^I \varphi \) with
\[
I_{n,1}^I \varphi = \int_{\{j_0(z) \geq 1\}} \varphi(-j(\cdot, z)) - \varphi(\cdot)\nu(dz),
\]
\[
I_{n,2}^I \varphi = \int_{\{\eta \leq j_0(z) < 1\}} \varphi(-j(\cdot, z)) - \varphi(\cdot) - \nabla \varphi(\cdot) \cdot j(\cdot, z) \nu(dz), \tag{3.5}
\]
\[
I_{n,3}^I \varphi = \int_0^1 (1-\theta) d\theta \int_{\{j_0(z) < \eta\}} j(\cdot, z) \cdot \nabla^2 \varphi(\cdot + \theta j(\cdot, z)) \cdot j(\cdot, z) \nu(dz).
\]

Using Lipschitz continuity, we have \( |I_{n,1}^I \varphi| \leq C_\varphi \int_{\{j_0(z) \geq 1\}} j_0(z) \nu(dz) \leq C_\varphi C_0 \) and \( |I_{n,2}^I \varphi| \leq 2C_\varphi \int_{\{\eta \leq j_0(z) < 1\}} j_0(z) \nu(dz) \leq 2C_\varphi C_0 \eta^{1-\gamma}. \) For the last term, we have
\[
|I_{n,3}^I \varphi| \leq \int_0^1 d\theta \int_{\{j_0(z) < \eta\}} |j_0(z)|^2 |\nabla^2 \varphi(\cdot + \theta j(\cdot, z))| \nu(dz). \tag{3.6}
\]

Using this, we can estimate the \( L^p \) norm as follows
\[
\|I_{n,3}^I \varphi\|_{L^p(O)}^p \leq \int_O dx \int_0^1 d\theta \left( \int_{\{j_0(z) \leq \eta\}} [j_0(z)]^2 |\nabla^2 \varphi(x + \theta j(x, z))| \nu(dz) \right)^p
\]
\[
\leq \int_O dx \int_0^1 d\theta \left( \int_{\{j_0(z) \leq \eta\}} [j_0(z)]^2 \nu(dz) \right)^{\frac{p}{q}} \cdot \left( \int_{\{j_0(z) \leq \eta\}} [j_0(z)]^2 \|\nabla^2 \varphi(x + \theta j(x, z))\|^p \nu(dz) \right)
\]
\[
\leq (\eta^{2-\gamma} r(\eta))^p \|\nabla^2 \varphi\|_{L^p(O \cap \eta)}^p,
\]

Above, we use Fubini’s theorem, Jensen’s inequality, and the Hölder inequality with \( 1/p + 1/q = 1. \) Thus, \( \|I_{n,3}^I \varphi\|_{L^p(O)} \leq \eta^{2-\gamma} r(\eta) \|\nabla^2 \varphi\|_{W^{2,p}(O \cap \eta)}. \) From the above estimates, we find
\[
\|I \varphi\|_{L^p(O)} \leq \eta^{2-\gamma} r(\eta) \|\nabla^2 \varphi\|_{L^p(O \cap \eta)} + C_0 (1 + 2\eta^{1-\gamma}) C_\varphi. \tag{3.7}
\]

Note that the module of integrability satisfies \( r(\eta) \to 0 \) as \( \eta \to 0. \) Now choose \( \eta \) small enough so that \( \eta^{2-\gamma} r(\eta) < \varepsilon \) and \( \eta < \varepsilon. \)

A direct application of Lemma 3.1 is the following local estimate for the integro-differential operator (see e.g. Proposition 2.4 in [15], Theorem 3.1.20 in [6], Proposition 3.5 in [1]). This result differs from the above references in that it uses knowledge of Lipschitz continuity. The estimate represents a direct extension of the classical \( L^p \) interior estimates of Theorem 9.11 in [7].

**Proposition 3.2.** (Local \( L^p \)-estimates) Suppose (H1), (H3), (H4), and (H6). Let \( \mathcal{O}' \subset \mathcal{O} \) be bounded open subsets of \( \mathbb{R}^n \) with dist\((\partial \mathcal{O}', \partial \mathcal{O}) \geq \delta > 0. \) Suppose that \( v \in W^{2,p}_{\text{loc}}(\mathcal{O}) \), \( v \) is Lipschitz on \( \mathbb{R}^n \) with constant \( C_v \), \( 1 < p < \infty. \) Letting
\[
(-\mathcal{L}_D - I + r)v = f \text{ in } \mathcal{O}, \tag{3.8}
\]
define the function \( f \) in \( \mathcal{O} \), there exists a constant \( C \) depending on \( n, p, \delta, \text{diam}(\mathcal{O}) \) and the bounds imposed by (H1) and (H6) such that

\[
\|v\|_{W^{2,p}(\Omega')} \leq C(\|f\|_{L^p(\mathcal{O})} + C_v + \|v\|_{L^\infty(\mathcal{O})}).
\]  

(3.9)

**Proof.** This proof is similar to Proposition 3.5 in [1]. Let \( C \) denote a generic constant throughout this proof. Let \( R \in (0, \text{dist}(\mathcal{O}', \partial \mathcal{O})) \). Consider \( B_R(x_0) \) (or simply \( B_R \)) for \( x_0 \in \mathcal{O}' \). For a constant \( 0 < \delta < 1 \) to be determined later, consider a smooth cut-off function \( \zeta^\delta \) satisfying

\[
\begin{cases}
\zeta^\delta \equiv 1 \text{ on } B_{\frac{3}{4}R}, \\
\zeta^\delta \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{\frac{3}{4}R}, \\
0 \leq \zeta^\delta \leq 1.
\end{cases}
\]

(3.10)

Moreover, \( \zeta^\delta \) can be chosen to satisfy \( |\partial_i \zeta^\delta| \leq \frac{C}{\delta}, \ |\partial^2_{ij} \zeta^\delta| \leq \frac{C}{\delta^2} \) for a constant \( C \). The function \( w := \zeta^\delta v \) satisfies

\[
\begin{cases}
(-L_D + r)w = \zeta^\delta iv(x) + \zeta^\delta f(x) + h(x) & x \in B_{\frac{3}{4}R}, \\
w(x) = 0 & x \in \partial B_{\frac{3}{4}R},
\end{cases}
\]

(3.11)

where \( h(x) := -\sum_{i,j=1}^n a_{ij}(\partial^2_{ij} \zeta^\delta \cdot v + 2\partial_i \zeta^\delta \cdot \partial_j v) - \sum_{i=1}^n b_i \cdot \partial_i \zeta^\delta \cdot v \). For this classical Dirichlet problem, there exists a constant \( C \) independent of \( w \) such that

\[
\|w\|_{W^{2,p}(B_{\frac{3}{4}R})} \leq C \left( \|\zeta^\delta iv\|_{L^p(B_{\frac{3}{4}R})} + \|\zeta^\delta f\|_{L^p(B_{\frac{3}{4}R})} + \|h\|_{L^p(B_{\frac{3}{4}R})} \right). 
\]

(3.12)

We now estimate the terms on the right-hand side of (3.12) individually. For the first term,

\[
\|\zeta^\delta iv\|_{L^p(B_{\frac{3}{4}R})} \leq \|iv\|_{L^p(B_{\frac{3}{4}R})} \leq \frac{\delta}{4} \|v\|_{W^{2,p}(B_{3R})} + C \left( \frac{\delta}{4} \right) C_v,
\]

(3.13)

where the first inequality follows from the choice of \( \zeta^\delta \); the second inequality follows from Lemma 3.1 with \( \varepsilon = \frac{\delta}{4} \). Next, it is clear that \( \|\zeta^\delta f\|_{L^p(B_{\frac{3}{4}R})} \leq \|f\|_{L^p(B_{\frac{3}{4}R})} \).

Now, we will estimate \( \|h\|_{L^p(B_{\frac{3}{4}R})} \). It follows from our choice of \( \zeta^\delta \) that

\[
\left\| \sum_{i,j=1}^n a_{ij} \partial^2_{ij} \zeta^\delta \cdot v \right\|_{L^p(B_{\frac{3}{4}R})} \leq C \cdot \|v\|_{L^\infty(B_{\frac{3}{4}R})} \cdot \|\partial^2_{ij} \zeta^\delta\|_{L^p(B_{\frac{3}{4}R}) \setminus B_{\frac{1}{2}R}}
\]

\[
\leq C \cdot \|v\|_{L^\infty(B_{\frac{3}{4}R})} \cdot \delta^{-\frac{2p}{p-2}}, \quad \text{and,}
\]

\[
\left\| \sum_{i,j=1}^n 2a_{ij} \partial_i \zeta^\delta \cdot \partial_j v \right\|_{L^p(B_{\frac{3}{4}R})} \leq C \cdot C_v \cdot \delta^{-\frac{2p}{p-2}},
\]

\[
\left\| \sum_{i=1}^n b_i \cdot \partial_i \zeta^\delta \cdot v \right\|_{L^p(B_{\frac{3}{4}R})} \leq C \cdot \|v\|_{L^\infty(B_{\frac{3}{4}R})} \cdot \delta^{-\frac{2p}{p-2}}.
\]
Using the above estimates, we obtain
\[
\|v\|_{W^{2,p}(B_{1/2}R)} \leq \|w\|_{W^{2,p}(B_{3/4}R)} \leq C_\delta \frac{\delta}{4} \|v\|_{W^{2,p}(B_{3R})} + C\left(\|v\|_{L^\infty(B_{3/4}R)} + C_v\right) \left(1 + \delta \frac{n-p}{p} + \delta \frac{n-2p}{p}\right) + C\|f\|_{L^p(B_{3/4}R)}.
\]  
(3.14)

Multiplying $\delta^2$ on both sides of the previous inequality produces
\[
\delta^2 \|v\|_{W^{2,p}(B_{1/2}R)} \leq C\delta \left(\frac{\delta}{2}\right)^2 \|v\|_{W^{2,p}(B_{3R})} + K(\delta),
\]  
(3.15)

where $K(\delta) := C \cdot \left(\|v\|_{L^\infty(B_{3/4}R)} + C_v\right) \cdot (\delta^2 + \delta \frac{n-p}{p} + \delta \frac{n-2p}{p}) + \|f\|_{L^p(B_{3/4}R)}$. Denote $F(\tau) := \tau^2 \|v\|_{W^{2,p}(B_{\frac{3}{4}R}(\beta - \tau))}$. The previous inequality yields the following recursive inequality $F(\delta) \leq C\delta F\left(\frac{\delta}{2}\right) + K(\delta)$. Choosing $0 < \delta < 1$ such that $\delta \leq \frac{1}{2^q}$, we obtain $F(\delta) \leq \frac{1}{2} F\left(\frac{\delta}{2}\right) + K(\delta)$. Now iterating the recursive inequality and noting that $K(\delta)$ is an increasing function, we obtain
\[
F(\delta) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} K\left(\frac{\delta}{2^i}\right) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} K(\delta) = 2K(\delta).
\]  
(3.16)

Hence,
\[
\|v\|_{W^{2,p}(B_{1/2}R)} \leq 2 \left(C \cdot \left(\|v\|_{L^\infty(B_{3/4}R)} + C_v\right) \cdot (\delta^2 + \delta \frac{n-p}{p} + \delta \frac{n-2p}{p}) + \|f\|_{L^p(B_{3/4}R)}\right),
\]
\[
\leq C \left(\|f\|_{L^p(B_{3/4}R)} + \|v\|_{L^\infty(B_{3/4}R)} + C_v\right).
\]  
(3.17)

If we cover $\mathcal{O}$ with a finite number of balls of radius $\frac{1}{2}R$, then the estimate of the proposition follows. \[\square\]

**Lemma 3.3.** Assume (H3) holds. Suppose $\varphi$ is Lipschitz on $\mathbb{R}^n$ with constant $C_\varphi$. Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. If $\varphi \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in [\gamma/2, 1]$, then $I\varphi \in C^{0, \frac{2\alpha-\gamma}{\gamma}}(\overline{\Omega})$ and
\[
\|I\varphi\|_{C^{0, \frac{2\alpha-\gamma}{\gamma}}(\overline{\Omega})} \leq C \left(C_\varphi + \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})}\right),
\]  
(3.18)

for a positive constant $C$ dependent upon $\Omega, \alpha, \gamma$.

**Proof.** This proof is similar to the proof of Lemma 3.2 in [1]. Let $C$ denote a generic constant unless specified otherwise. First, we estimate $\sup_{\overline{\Omega}} |I\varphi|$. For any
\[ x \in \overline{\Omega}, \]

\[ |I \varphi(x)| \leq \int_{\{j_0(z) \leq 1\}} |\varphi(x + j(x, z)) - \varphi(x) - \nabla \varphi(x) \cdot j(x, z)| \nu(dz) + \int_{\{j_0(z) > 1\}} |\varphi(x + j(x, z)) - \varphi(x)| \nu(dz) \]

\[ \leq \int_{\{j_0(z) \leq 1\}} \int_0^1 |\nabla \varphi(x + \theta j(x, z)) \cdot j(x, z) - \nabla \varphi(x) \cdot j(x, z)| \, d\theta \, \nu(dz) \]

\[ + C \varphi \int_{\{j_0(z) > 1\}} j_0(z) \nu(dz) \]

\[ \leq \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})} \int_{\{j_0(z) \leq 1\}} [j_0(z)]^{1+\alpha} \nu(dz) + C \varphi \int_{\{j_0(z) > 1\}} j_0(z) \nu(dz) \]

\[ \leq C_0 \left( C \varphi + \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})} \right). \]

(3.19)

Next, we show \( I \varphi \) is Hölder continuous. Let \( x_1, x_2 \in \overline{\Omega} \) and set \( \delta = |x_1 - x_2|^\frac{1}{2} \wedge 1 \).

Consider \( |I \varphi(x_1) - I \varphi(x_2)| \leq I_1 + I_2 + I_3 \) in which

\[ I_1 := \int_{\{j_0(z) \leq \delta\}} (|\varphi(x_1 + j(x_1, z)) - \varphi(x_1) - j(x_1, z) \cdot \nabla \varphi(x_1)| \]

\[ + |\varphi(x_2 + j(x_2, z)) - \varphi(x_2) - j(x_2, z) \cdot \nabla \varphi(x_2)|) \nu(dz), \]

\[ I_2 := \int_{\{\delta < j_0(z) < 1\}} (|\varphi(x_1 + j(x_1, z)) - \varphi(x_2 + j(x_2, z))| \]

\[ + |\varphi(x_1) - \varphi(x_2)| + |j(x_1, z) \cdot \nabla \varphi(x_1) - j(x_2, z) \cdot \nabla \varphi(x_2)|) \nu(dz), \]

\[ I_3 := \int_{\{j_0(z) > 1\}} (|\varphi(x_1 + j(x_1, z)) - \varphi(x_2 + j(x_2, z))| + |\varphi(x_1) - \varphi(x_2)|) \nu(dz). \]

(3.20)

Estimating \( I_1 \), we have

\[ I_1 = \int_{\{j_0(z) \leq \delta\}} |j(x_1, z) \cdot \nabla \varphi(w_1, z) - j(x_1, z) \cdot \nabla \varphi(x_1)| + |j(x_2, z) \cdot \nabla \varphi(w_2, z) - j(x_2, z) \cdot \nabla \varphi(x_2)| \nu(dz) \]

\[ \leq \int_{\{j_0(z) \leq \delta\}} j_0(z) |\nabla \varphi(w_1, z) - \nabla \varphi(x_1)| + j_0(z) |\nabla \varphi(w_2, z) - \nabla \varphi(x_2)| \nu(dz) \]

\[ \leq \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})} \left( \int_{\{j_0(z) \leq \delta\}} j_0(z) |w_1, z - x_1|^\alpha \nu(dz) + \int_{\{j_0(z) \leq \delta\}} j_0(z) |w_2, z - x_2|^\alpha \nu(dz) \right) \]

\[ \leq 2 \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})} \int_{\{j_0(z) \leq \delta\}} [j_0(z)]^{1+\alpha} \nu(dz) \]

\[ \leq 2 \|\varphi\|_{C^{1,\gamma}(\overline{\Omega})} \delta^{2\alpha - \gamma} \int_{\{j_0(z) < 1\}} [j_0(z)]^{\gamma + 1 - \alpha} \nu(dz) \]

\[ \leq 2C_0 \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})} |x_1 - x_2|^{\frac{2\alpha - \gamma}{\alpha - \gamma}}, \]

(3.21)

for some \( w_1, w_2 \) satisfying \( |w_1, z - x_1| \leq |j(x_1, z)| \) and \( |w_2, z - x_2| \leq |j(x_2, z)| \). Es-
We briefly remark about the last two inequalities above. Let \( \text{diam}(\Omega) := \max_{x,y \in \Omega} |x - y| \).

If \( \delta = |x_1 - x_2|^{\frac{1}{2}} \), we have \( |x_1 - x_2| \delta^{-\gamma} = |x_1 - x_2|^{1 - \frac{\gamma}{2}} \leq |x_1 - x_2|^{\frac{2(\alpha - \gamma)}{2}} \) along with \( |x_1 - x_2|^{\alpha} \delta^{-\gamma} = |x_1 - x_2|^{\frac{2\alpha - 2\gamma}{2}} \).

If instead \( \delta = 1 < |x_1 - x_2|^{\frac{1}{2}} \), then we have \( |x_1 - x_2| \delta^{-\gamma} \leq C |x_1 - x_2|^{\frac{2(\alpha - \gamma)}{2}} \) with \( C = (\text{diam}(\Omega))^{\frac{2\alpha - 2\gamma}{2}} \) along with \( |x_1 - x_2|^{\alpha} \leq C |x_1 - x_2|^{\frac{2\alpha - 2\gamma}{2}} \).

Estimating \( I_3 \), we find

\[
I_3 \leq \int_{\{\delta < j_0(z) < 1\}} C_\varphi \left( |x_2 - x_1 + j(x_2, z) - j(x_1, z)| + |x_1 - x_2| \right) \nu(dz)
\]

\[
\leq |x_1 - x_2| \int_{\{\delta < j_0(z) < 1\}} C_\varphi \left( 2 + C_j(z) \right) \nu(dz)
\]

\[
\leq C |x_1 - x_2|^{\frac{2\alpha - 2\gamma}{2}}, \text{ with } C = (\text{diam}(\Omega))^{\frac{2\alpha - 2\gamma}{2}}.
\]

Combining these estimates for \( I_1, I_2, I_3 \), we have

\[
|I_\varphi(x_1) - I_\varphi(x_2)| \leq C |x_1 - x_2|^{\frac{2\alpha - 2\gamma}{2}},
\]

for \( C \) independent of \( x_1, x_2 \). \( \square \)

**4. Regularity in the Continuation Region.** In this section, we establish the regularity of the value function \( u \) in the continuation region \( \mathcal{C} := \{x \in \mathbb{R}^n : u(x) < \mathcal{M}u(x)\} \) through approximation. For \( \epsilon > 0 \), set

\[
\bar{j}^\epsilon(x, z) := j(x, z)1_{\{j_0(z) > \epsilon\}}.
\]

With this definition, for each fixed \( \epsilon > 0 \), it holds that \( \bar{j}^\epsilon \in L^1(\mathbb{R}^l, \nu) \). Indeed,

\[
\int_{\mathbb{R}^l} |\bar{j}^\epsilon(x, z)| \nu(dz) \leq \int_{\{j_0 > \epsilon\}} j_0(z) \nu(dz) + \frac{1}{\epsilon^2} \int_{\{j_0 \leq 1\}} [j_0(z)]^2 \nu(dz) < \infty.
\]

As we shall see, \( \bar{j}^\epsilon \) is a good approximation for \( j \) in \( \mathcal{C} \). We shall consider the limit as \( \epsilon \to 0 \), and the corresponding estimates for \( j \).
Letting \( u_\epsilon \) denote the value function corresponding to a jump function \( j^\epsilon \), we have that \( u_\epsilon \) is Lipschitz continuous for each \( \epsilon > 0 \).

**Lemma 4.1.** For each \( \epsilon > 0 \), the value function \( u_\epsilon \) is Lipschitz continuous in \( \mathbb{R}^n \) with constant \( C_\epsilon \), the Lipschitz constant for \( u_\epsilon \).

**Proof.** The proof proceeds directly as in Lemma 2.1 since \( |j^\epsilon(x,z) - j^\epsilon(y,z)| \leq |j(x,z) - j(y,z)| \). \( \square \)

At this point, the regularity analysis presented in [4] allows us to conclude \( u_\epsilon \in W^{2,p}_{loc}(\mathbb{R}^n) \) for each fixed \( \epsilon > 0 \). The next goal is to show uniform convergence of \( u_\epsilon \) to \( u \). In doing so, we utilize a general estimate obtained for solutions of jump diffusions (see e.g. Chapter 5 in [14]). For this estimate, we define the norm

\[
\|h - h'\|_{0,p} := \sup_{t,x} \left\{ \left( \int_{\mathbb{R}^d} |h(t,x,z) - h'(t,x,z)|^p \nu(dz) \right)^{1/p} \right\},
\]

for \( p \geq 2 \). Additionally, set

\[
A_{0,p}(h - h') := \|h - h'\|_{0,2p} + \|h - h'\|_{0,2}. \tag{4.4}
\]

**Lemma 4.2.** Suppose the assumptions (H1), (H10) hold. Fix \( \epsilon > 0 \). Letting \( X_t \) be a solution to (1.1) using jump function \( j \) with \( X_0 = x_0 \) and \( X_t^\epsilon \) be a solution using jump function \( j^\epsilon \) and \( X_0^\epsilon = x_0 \), we have for \( \alpha > \beta \),

\[
E \left[ \sup_{0 \leq s \leq t} |X_s - X_s^\epsilon|^{2 \epsilon - \alpha s} \right] \leq MA_{0,2}^2(j - j^\epsilon), \tag{4.5}
\]

for every \( t \geq 0 \) and for some constants \( C, M \) which depend only upon \( \alpha > \beta \), the bounds on \( b, \sigma, j \) and the dimensions \( n, d \).

**Proof.** Set \( Y_t = X_t - X_t^\epsilon \) and apply Itô’s formula with \( \varphi(y,t) = |y|^2 e^{-\alpha t} \) to obtain

\[
d\varphi(Y_t, t) = a_t dt + \sum_{k=1}^d b_k^i \, dW^k_t + \int_{\mathbb{R}^d} c(t,z) \tilde{N}(dt,dz), \tag{4.6}
\]

where

\[
a_t := \partial_t \varphi(Y_t, t) + \sum_{i=1}^n \left[ \partial_i \varphi(Y_t, t) + \frac{1}{2} \sum_{i,j=1}^n \left( \sum_{k=1}^d [\sigma_{ik}(X_t) - \sigma_{ik}(X_t^\epsilon)] [\sigma_{jk}(X_t) - \sigma_{jk}(X_t^\epsilon)] \right) \partial_{ij}^2 \varphi(Y_t, t) + \int_{\mathbb{R}^d} \varphi(Y_t + \epsilon(X_t, z) - j^\epsilon(X_t^\epsilon, z), t) - \varphi(Y_t, t) \right. \\
- \sum_{i=1}^n [j_i(X_t, z) - j_i^\epsilon(X_t^\epsilon, z)] \partial_i \varphi(Y_t, t) \nu(dz),
\]

\[
b_k^i := \sum_{i=1}^n \left( \sigma_{ik}(X_t) - \sigma_{ik}(X_t^\epsilon) \right) \partial_i \varphi(Y_t, t),
\]

\[
c(t,z) := \varphi(Y_t + j(X_t, z) - j^\epsilon(X_t^\epsilon, z), t) - \varphi(Y_t, t),
\]

and,

\[
\partial_t \varphi(y,t) = -\alpha \varphi(y,t),
\]

\[
\partial_i \varphi(y,t) = 2y_i |y|^{-2} \varphi(y,t) = 2y_i e^{-\alpha t},
\]

\[
\partial_{ij}^2 \varphi(y,t) = 2\delta_{ij} |x|^{-2} \varphi(y,t),
\]
where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. From above, we know
\[
\varphi(y + \tilde{j}(z, t), t) - \varphi(y, t) - \sum_i \tilde{j}_i(z, t) \partial_i \varphi(y, t) = |\tilde{j}(z, t)|^2 e^{-\alpha t}, \tag{4.7}
\]
with $\tilde{j}(z, t) := j(X_t, z) - j^*(X_t^*, z)$. Using the fact that for each $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that $(a + b)^2 \leq (1 + \varepsilon)a^2 + (1 + C_\varepsilon)b^2$, we find
\[
\int_{\mathbb{R}^l} |j(X_t, z) - j^*(X_t^*, z)|^2 e^{-\alpha t} \nu(dz)
\leq (1 + \varepsilon) \int_{\mathbb{R}^l} |j(X_t, z) - j^*(X_t^*, z)|^2 e^{-\alpha t} \nu(dz)
\leq (1 + \varepsilon) \beta_j |X_t - X_t^*|^2 e^{-\alpha t} + (1 + C_\varepsilon) e^{-\alpha t} \|j - j^*\|_{0.2}^2. \tag{4.8}
\]

With this estimate, we find
\[
a_t \leq \left[ -\alpha + \beta + \varepsilon \beta_j \right] \varphi(Y_t, t) + (1 + C_\varepsilon) e^{-\alpha t} \|j - j^*\|_{0.2}^2.
\]

Using this inequality and taking expectations in (4.6) yields
\[
\mathbb{E} \left[ |X_t - X_t^*|^2 e^{-\alpha t} \right] \leq \frac{1 + C_\varepsilon}{\alpha} \|j - j^*\|_{0.2}^2. \tag{4.9}
\]

Recall, the following stochastic integral inequalities (see e.g. [14]). For any $p > 0$, there is a constant $C_p > 0$ (in particular, $C_1 = 3$, $C_2 = 4$) such that
\[
\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_0^r f(s) dW_s \right|^p \right] \leq C_p \mathbb{E} \left[ \left( \int_0^t |f(s)|^2 ds \right)^{p/2} \right], \tag{4.10}
\]

and for the stochastic Poisson integral, if $0 < p \leq 2$, then
\[
\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_{\mathbb{R}^l \times (0, r)} g(s, z) \tilde{N}(dz, ds) \right|^p \right] \leq C_p \mathbb{E} \left[ \left( \int_0^t ds \int_{\mathbb{R}^l} |g(s, z)|^2 \nu(dz) \right)^{p/2} \right]. \tag{4.11}
\]

Now, coming back to (4.6) to take first the supremum and then the expectation, we deduce after using (4.10), (4.11) with $p = 1$,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s - X_s^*|^2 e^{-\alpha t} \right] \leq 3 \mathbb{E} \left[ \sum_k \left( \int_0^t |h_k|^2 ds \right)^{1/2} + \left( \int_0^t ds \int_{\mathbb{R}^l} |c(s, t)|^2 \nu(dz) \right)^{1/2} \right]. \tag{4.12}
\]

We now estimate the two terms on the right hand side of the above inequality. First, for some $C$ depending on the Lipschitz constant $C_\sigma$ in (H1), we have that $\sum_k |h_k|^2 \leq
\[ C |\varphi(Y_s, s)|^2 \] by the following inequalities

\[
\sum_k |b_k|^2 = \sum_k \left| \sum_i (\sigma_{ik}(X_t) - \sigma_{ik}(X'_t)) \frac{2(X_i(t) - X'_i(t))}{|X_t - X'_t|^2} \varphi(Y_s, s) \right|^2
\leq 4n \frac{|\varphi(Y_s, s)|^2}{|X_t - X'_t|^4} \sum_{k,i} (\sigma_{ik}(X_t) - \sigma_{ik}(X'_t))^2 (X_i(t) - X'_i(t))^2
\leq 2n \frac{|\varphi(Y_s, s)|^2}{|X_t - X'_t|^4} \left( \sum_{k,i} (\sigma_{ik}(X_t) - \sigma_{ik}(X'_t))^4 + \sum (X_i(t) - X'_i(t))^4 \right)
\leq 2n \frac{|\varphi(Y_s, s)|^2}{|X_t - X'_t|^4} \left( C^4_\sigma |X(t) - X'(t)|^4 + d |X(t) - X'(t)|^4 \right)
\leq 2n(C^4_\sigma + d) |\varphi(Y_s, s)|^2.
\]

Using the above, we now have

\[
E \left[ \left( \sum_k \int_0^t |b_k|^2 \, ds \right)^{1/2} \right] \leq C E \left[ \left( \sup_{0 \leq s \leq t} |\varphi(Y_s, s)| \right)^{1/2} \left( \int_0^t |\varphi(Y_s, s)| \, ds \right)^{1/2} \right].
\]

Thus, by means of the inequality \(2ab \leq \varepsilon a^2 + b^2/\varepsilon\) and the Hölder inequality we deduce that

\[
3E \left[ \left( \sum_k \int_0^t |b_k|^2 \, ds \right)^{1/2} \right] \leq \frac{1}{3} E \left[ \sup_{0 \leq s \leq t} |\varphi(Y_s, s)| \right] + C_1 E \left[ \int_0^t |\varphi(Y_s, s)| \, ds \right]. \tag{4.13}
\]

The term corresponding to Poisson integral can be handled using the same technique. Towards this end, note that

\[
|c(s, z)|^2 \leq |j(X_s, z) - j^*(X'_s, z)|^2 \int_0^1 |\nabla \varphi(Y_s + \theta(j(X_s, z) - j^*(X'_s, z), s))|^2 \, d\theta.
\]

\tag{4.14}

Estimating the gradient \(\nabla \varphi\) and using \(y := X_s - X'_s\) to ease notation, we have

\[
|\nabla \varphi(y + \theta \bar{j}, s)|^2 = 4 \varphi(y + \theta \bar{j}, s) e^{-\alpha s} = 4e^{-2\alpha s} |y + \theta \bar{j}|^2 \leq 8e^{-2\alpha s} (|y|^2 + |\bar{j}|^2).
\]

\tag{4.15}

Thus, we know \(|c(s, z)|^2 \leq 8e^{-2\alpha s} |\bar{j}|^2 \left( |y|^2 + |\bar{j}|^2 \right)\). Now, assuming \(C_j(z) \in L^4(\mathbb{R}^l)\), we have for \(p = 2, 4\)

\[
\int_{\mathbb{R}^l} |j(X_s, z) - j^*(X'_s, z)|^p \nu(dz) \leq 2^{p-1} \|j - j^*\|_{0,p}^p + 2^{p-1} \int_{\mathbb{R}^l} |j^*(X_s, z) - j^*(X'_s, s)|^p \nu(dz)
\leq 2^{p-1} \|j - j^*\|_{0,p}^p + 2^{p-1} |X_s - X'_s|^p \int_{\mathbb{R}^l} |C_j(z)|^p \nu(dz)
\leq C \|j - j^*\|_{0,p}^p + C |X_s - X'_s|^p.
\]
Using this estimate and the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $1/p + 1/q = 1$, the following holds
\[
\int_{\mathbb{R}^l} |c(s, z)|^2 \nu(dz) \leq \int_{\mathbb{R}^l} 8e^{-2\alpha s} |\bar{y}|^2 (|y|^2 + |\bar{y}|^2) \nu(dz) \\
\leq 8e^{-2\alpha s} |y|^2 \left( C \|j - j^*\|_{0.2}^2 + C |y|^2 \right) + 8e^{-2\alpha s} \left( C \|j - j^*\|_{0.4}^4 + C |y|^4 \right) \\
\leq C |\varphi(Y_s, s)|^2 + Ce^{-2\alpha s} \left( \|j - j^*\|_{0.4}^4 + \|j - j^*\|_{0.2}^4 \right) \\
\leq C |\varphi(Y_s, s)|^2 + Ce^{-2\alpha s} \Lambda_{0.2}^4 (j - j^*).
\]

(4.16)

Returning back to (4.12) and using $(a + b)^p \leq a^p + b^p$ for $0 < p < 1$, we find
\[
E \left[ \left( \int_0^t ds \int_{\mathbb{R}^l} |c(s, t)|^2 \nu(dz) \right)^{1/2} \right] \leq E \left[ \left( \int_0^t C |\varphi(Y_s, s)|^2 + Ce^{-2\alpha s} \Lambda_{0.2}^4 (j - j^*)ds \right)^{1/2} \right] \\
\leq E \left[ \left( \int_0^t C |\varphi(Y_s, s)|^2 ds \right)^{1/2} \right] + CA_0^2 (j - j^*).
\]

(4.17)

The first term can be handled in the same manner as the Weiner term above to yield an estimate as in (4.13). Now, combining these two estimates, referring back to (4.12), and using (4.9), we conclude
\[
E \left[ \sup_{0 \leq s \leq t} |X_s - X_s^*| e^{-\alpha s} \right] \leq CA_0^2 (j - j^*).
\]

(4.18)

\[\square\]

**Lemma 4.3.** Assume (H1), (H10), (H7). The value function $u_\epsilon$ corresponding to a jump function $j^\epsilon$ converges uniformly on $\mathbb{R}^n$ to $u$, i.e., $u_\epsilon \to u$ on $\mathbb{R}^n$.

**Proof.** Fix $\epsilon > 0$ and let $X_\epsilon$ denote a solution to (1.1) with initial value $X_0 = x$ and let $X_\epsilon^*$ denote a solution to (1.1) with jump function $j^\epsilon$ and initial value $X_\epsilon^* = x$.

From Lemma 4.2 and Jensen’s inequality, we know for $\alpha > \beta$,
\[
E \left[ \sup_{0 \leq s \leq t} |X_s - X_s^*| \right] \leq e^{\alpha t/2} M^{1/2} \Lambda_{0.2} (j - j^*).
\]

(4.19)

Fix a control $V$ and let $J_x^*[V]$ denote the objection function (1.3) under $X^\epsilon$. Using (H7) and (4.19), we find
\[
J_x[V] \leq J_x^*[V] + E \left[ \int_0^\infty e^{-\alpha s} |f(X_s - f(X_s^*))| ds \right] \\
\leq J_x^*[V] + C f \int_0^\infty e^{-\alpha s} E[|X_s - X_s^*]| ds \]
\[
\leq J_x^*[V] + C f M^{1/2} \Lambda_{0.2} (j - j^*) \int_0^\infty e^{-(r - \alpha/2)s} ds.
\]

The final integral in the last inequality converges by (H10). Let $C(\epsilon)$ denote the last term in the last inequality above. Taking infimum over all controls yields
\[
u(x) \leq u_\epsilon^*(x) + C(\epsilon),
\]

(4.21)
where \(C(\epsilon) \downarrow 0\) as \(\epsilon \downarrow 0\). Exchanging the roles of \(X_t\) and \(X_t^\alpha\) yields \(u_\epsilon(x) \leq u(x) + C(\epsilon)\). Since \(C(\epsilon)\) is independent of \(x\), the convergence is uniform. \(\square\)

**Lemma 4.4.** Assume (H1), (H7), (H6), (H10). In the continuation region \(\mathcal{C}\), we have \(u \in W^{2,p}_{\text{loc}}(\mathcal{C})\).

**Proof.** Let \(B \subset \mathcal{C}\) be closed and bounded. Let \(\delta = \inf_{\partial B} \{|\mathcal{M}u(x) - u(x)|\} > 0\). By Lemma 4.3, \(u_\epsilon\) converges uniformly to \(u\) on \(\mathbb{R}^n\) which, in turn, implies \(\mathcal{M}u_\epsilon\) converges uniformly to \(\mathcal{M}u\) on \(\mathbb{R}^n\). Using this information, there exists a \(\epsilon'(\delta) > 0\) such that for all \(\epsilon \in (0, \epsilon'(\delta))\), it holds that \(B \subset \{x \in \mathbb{R}^n : u_\epsilon(x) < \mathcal{M}u_\epsilon(x)\}\). For an open set \(\mathcal{O} \subset B\) and any \(\epsilon \in (0, \epsilon'(\delta))\), the local estimate Lemma 3.2 along with Lemmas 4.1 and 4.3 yield that \(\|u_\epsilon\|_{W^{2,p}(\mathcal{O})} \leq C\) for some constant \(C\) independent of \(\epsilon\). Thus, a weak limit exists and must coincide with the value function \(u\) due to Lemma 4.3. Since \(B\) was arbitrary, the proof is complete. \(\square\)

As in [4], we can now use a “bootstrap” argument to obtain further regularity of \(u\) in \(\mathcal{C}\).

**Proposition 4.5.** Assume (H1), (H3), (H7), (H6), (H10). For any compact subset \(\mathcal{D} \subset \mathcal{C}\) of the continuation region, the value function \(u\) is in \(C^{2,\frac{2n-\alpha}{\alpha}}(\mathcal{D})\) for any \(\alpha \in [\gamma/2, 1]\) and satisfies \((-\mathcal{L}_D - I + r)u - f = 0\) in \(\mathcal{C}\).

**Proof.** First, consider any compact set \(\mathcal{D}\) such that \(\overline{\mathcal{D}}^* \subset \mathcal{C}\). From Lemma 4.4, \(u \in W^{2,p}(\mathcal{D})^1\) for \(p \in (1, \infty)\) from which Sobolev imbedding implies \(u \in C^{1,\alpha}(\overline{\mathcal{D}}^*)\) for any \(\alpha \in (0, 1)\). Using this result and applying Lemma 3.3, we know that \(Iu \in C^{0,\frac{2n-\alpha}{\alpha}}(\mathcal{D})\) for \(\alpha \in [\gamma/2, 1]\). We now have enough regularity to use the Schauder estimates to improve our results. Indeed, for any open ball \(B \subset \mathcal{D} \subset \overline{\mathcal{D}}^* \subset \mathcal{C}\), the solution \(u\) of the following classical Dirichlet problem

\[
\begin{cases}
(-\mathcal{L}_D + r)u(x) = f(x) + Iu(x) & \text{a.e. } x \in B, \\
u(x) = u(x) & x \in \partial B,
\end{cases}
\tag{4.22}
\]

is in \(C^{2,\frac{2n-\alpha}{\alpha}}(B)\) by the Schauder estimates since \(f + Iu(x) \in C^{0,\frac{2n-\alpha}{\alpha}}(\mathcal{D})\). Now, from classical uniqueness results of viscosity solutions as used in Lemma 5.4 in [4] (see also final paragraph in Theorem 5.5 in [4]), we conclude \(u \in C^{2,\frac{2n-\alpha}{\alpha}}(B)\) for any open ball \(B \subset \mathcal{D}\). The choice of a compact set \(\mathcal{D}\) such that \(\overline{\mathcal{D}}^* \subset \mathcal{C}\) was necessary in order to apply Lemma 3.3. However, the outer 1-neighborhood \(\Omega^1\) appears there as a result of our choice of magnitude 1 to separate large and small jumps. If, instead, we take any \(\epsilon \in (0, 1)\) to separate jump behavior, we would reach an analogous conclusion \(u \in C^{2,\frac{2n-\alpha}{\alpha}}(B)\) for any open ball \(B \subset \mathcal{D}\) where \(\overline{\mathcal{D}}^* \subset \mathcal{C}\). Hence, we find \(u \in C^{2,\frac{2n-\alpha}{\alpha}}(\mathcal{C})\) for any compact set \(\mathcal{C} \subset \mathcal{C}\) and satisfies \((-\mathcal{L}_D - I + r)u - f = 0\) in \(\mathcal{C}\). \(\square\)

### 5. Regularity in \(\mathbb{R}^n\)

In this section, we investigate the regularity of the value function \(u\). The authors in [4] examine the regularity of \(u\) under two specific assumptions concerning the Lévy measure: \(\nu\) is finite and \(j(x, \cdot) \in L^1(\nu)\). These two assumptions describe qualities of the Lévy kernel \(M(x, d\eta)\) where

\[
M(x, A) := \nu\{z : j(x, z) \in A\}, \quad A -\text{Borel measurable subset in } \mathbb{R}^n,
\]

which, in turn, determine the order of integro-differential operator \(I\) (see Definition 2.1.2 in [6]). The assumptions taken in [4] concern integro-differential operators of order \(\leq 1\). Such operators map smooth functions to smooth functions. For example, Lemma 5.1 in [4] shows that \(I\) maps Lipschitz functions to Lipschitz functions when \(I\)
has order 0. Additionally, when \( j(x, \cdot) \in L^1(\nu) \), Lemma 3.2 in [4] shows that \( I \) maps a Lipschitz function to a continuous function when \( C_j(\cdot) \) is \( \nu \)-integrable. Since the value function for impulse control \( u \) is Lipschitz continuous, it is known that \( Iu \) is at least a continuous function under either assumption on \( M(x, d\eta) \). As the authors in [4] demonstrate, the continuity of \( Iu \) allows for a regularity analysis as in the pure diffusion case after defining a new running cost function \( f := f + Iu \). Under our assumptions on \( M(x, d\eta) \), it is not known a priori that \( Iu \) is continuous for Lipschitz continuous \( u \) (for a similar discussion see [1]). As such, we cannot define \( f \) as in [4] and must directly deal with the integro-differential operator.

### 5.1. Bounded Domain Approach.

With an integro-differential operator \( I \) of order \( \leq 1 \), the authors in [4] show \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n) \) by studying the regularity of an associated optimal stopping time problem for a pure diffusion on bounded open sets of \( \mathbb{R}^n \) (see Section 6 in [4]). With a general jump case considered here, it is natural to consider the possibility of a similar proof argument involving an optimal stopping time problem for jump diffusions on bounded open sets of \( \mathbb{R}^n \).

Through penalization, regularity of an associated optimal stopping problem in a bounded open set \( \mathcal{O} \) arises from the regularity of a Dirichlet problem. As such, we may first consider the existence, uniqueness and regularity of the a solution the following Dirichlet problem:

\[
\begin{aligned}
(-L_D - I + r)v(x) &= f(x), \quad x \in \mathcal{O}, \\
v(x) &= u(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O}.
\end{aligned}
\tag{5.1}
\]

Notice that the non-local character of \( I \) requires that the solution \( v \) be defined on the support of the Lévy kernel \( M(x, \cdot) \), namely, \( \mathbb{R}^n \). Integro-differential problems as above have been extensively discussed in the literature (see e.g. [6, 8, 10]). Recalling this analysis, when studying (5.1) with an integro-differential operator \( I \) of order \((1,2], W^{2,p}(\mathcal{O}) \) solutions exist if an extra condition is placed upon jumps outside of \( \mathcal{O} \) (see (5.4)). In the absence of this modification, only variational solutions in \( W^{1,p}(\mathcal{O}) \) exist. The lack of dependence upon the fixed bounded open set \( \mathcal{O} \) for \( I \) of order \( \leq 1 \) renders this approach useful for establishing the regularity of \( u \). In fact, such an argument would essentially be the same as the analysis undertaken in both [4] and [12]. The existence of this extra condition upon jumps outside \( \mathcal{O} \) for integro-differential operators of order \( > 1 \) does not disqualify this method from helping to achieve regularity for an optimal stopping problem associated to impulse control. Indeed, the extra jump condition (5.4) might automatically be satisfied depending on the value of \( \gamma \) taken in (H3). To see this, consider the following two-step problem associated to (5.1).

\[
\begin{aligned}
(-L_D + r)z(x) &= 0, \quad x \in \mathcal{O}, \\
z(x) &= u(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O},
\end{aligned}
\tag{5.2}
\]

and

\[
\begin{aligned}
(-L_D - I + r)w(x) &= f(x) + Iz(x), \quad x \in \mathcal{O}, \\
w(x) &= 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}.
\end{aligned}
\tag{5.3}
\]

If solutions exist to each problem, then \( v = z + w \) will solve (5.1). Sufficient conditions to solve (5.2) are well-known and can be found in [7]. For (5.3), there is a unique
solution \( w \in W^{2,p}(\mathcal{O}) \) (see Theorem III.3 in [8] and Theorem 3.1.22 in [6]) if
\[
\sup_{x \in \mathcal{O}} \int_{\mathbb{R}^n} \mathbb{1}_{\mathbb{R}^n \setminus \mathcal{O}}(x + j(x,z)) |j(x,z)|^{1+\alpha} \nu(dz) < \infty, \quad (5.4)
\]
where \( 0 < \alpha < 1/n \) and if \( f + Iz \in L^p(\mathcal{O}) \) for \( n < p < 1/\alpha \). The condition (5.4) is satisfied if \( \gamma \in [1,2] \) in (H3) is taken to satisfy \( 0 < \gamma - 1 < 1/n \). Thus, we might be able to pursue this technique for showing regularity under a restricted set of \( \gamma \) values in \([1,2]\) which depend upon the dimension \( n \). Even if we are content with this restriction, we cannot conclude the existence of a unique solution \( w \in W^{2,p}(\mathcal{O}) \) until \( Iz \in L^p(\mathcal{O}) \) for \( n < p < 1/\alpha \) is justified. Recalling the classical results of Corollary 9.18 in [7], we know that \( z \in W^{2,p}_{\text{loc}}(\mathcal{O}) \cap C^0(\overline{\mathcal{O}}) \) from which Sobolev embedding implies that \( z \in C^{0,1}(K) \) for any compact \( K \subset \mathcal{O} \). Since \( z = u \) on \( \mathbb{R}^n \setminus \mathcal{O} \), we can conclude that \( z \) is Lipschitz continuous on \( \mathbb{R}^n \). However, \( z \) Lipschitz continuous on \( \mathbb{R}^n \) does not guarantee that \( Iz \in L^p(\mathcal{O}) \). Essentially, unless we know more regularity about the solution \( z \) with Lipschitz boundary function \( u \), we are unable to obtain a \( W^{2,p}(\mathcal{O}) \) solution to (5.3). Due to this complication and the additional restriction to \( \gamma \) beyond (H3), we instead pursue an analysis of an integro-differential problem on the whole space rather than on a bounded open set \( \mathcal{O} \).

5.2. The Whole Space Approach. In this section, we establish the following main theorem.

THEOREM 5.1. Let the assumptions of Section 2 hold. The value function of impulse control \( u \) has a weak derivative up to order 2 in \( L^p(\mathcal{O}) \) for \( 1 < p < 2 \) and any bounded open set \( \mathcal{O} \), i.e., \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n) \). Thus, \( u \) has a locally Lipschitz first derivative.

The subsections to follow pursue a proof of the above result. In the first, we present a characterization of the value function \( u \). In the second, we discuss the semi-concavity of \( u \) and \( Muu \) which assists in establishing regularity in the third.

The following function spaces will be useful in order to examine the regularity of the value function \( u \) on \( \mathbb{R}^n \). Let \( B_p(\mathbb{R}^n) \) denote the space of Borel measurable functions \( h \) from \( \mathbb{R}^n \) into \( \mathbb{R}^n \) such that
\[
||h||_p = \sup\{|h(x)| (1 + |x|^2)^{-p/2} : x \in \mathbb{R}^n\} < \infty. \quad (5.5)
\]
Let \( C_p(\mathbb{R}^n) \) denote the subspace of \( B_p(\mathbb{R}^n) \) composed of \( p \)-uniformly continuous functions, i.e., all functions \( h \) which satisfy: for every \( \epsilon > 0 \) there exists a \( \delta = \delta(\epsilon, p) \) such that for any \( x, x' \in \mathbb{R}^n \), we have
\[
|h(x) - h(x')| \leq \epsilon(1 + |x|^p), \quad |x - x'| < \delta. \quad (5.6)
\]
Let \( C_p^+(\mathbb{R}^n) \) denote the class of all positive functions in \( C_p(\mathbb{R}^n) \).

5.2.1. QVI. Recall the integro-differential operator \( A := -\mathcal{L}_D - I + r \), where
\[
\mathcal{L}_D \phi(x) = \sum_{i,k=1}^n a_{ik}(x) \partial_{x,x_k}^2 \phi(x) + \sum_{i=1}^n b_i(x) \partial_{x_i} \phi(x),
\]
\[
I \phi(x) = \int_{\mathbb{R}^l} (\phi(x + j(x,z)) - \phi(x) - j(x,z) \nabla_x \phi(x)) \nu(dz), \quad (5.7)
\]
From [17], for any functions $u, v \in B_p(\mathbb{R}^n)$, we say
\[ Au = v, \text{ in } \mathbb{R}^n \text{ (resp. } \leq \text{)} \] if the process
\[ Y_t = \int_0^t v(X_s)e^{-rs}ds + u(X_t)e^{-rt}, t \geq 0, \tag{5.8} \]
is a martingale (resp. submartingale), for every initial $x \in \mathbb{R}^n$. The following proposition from [13] characterizes the value function for our impulse control problem $u$.

**Proposition 5.2.** Let the assumptions (H1), (H7), (H9), and (H10) hold. Then the quasi-variational inequality
\[ \begin{cases} \hat{u} \in C^+_p(\mathbb{R}^n) \\
A\hat{u} \leq f \text{ in } \mathbb{R}^n, \hat{u} \leq M\hat{u} \text{ in } \mathbb{R}^n, \\
A\hat{u} = f \text{ in } [u < M\hat{u}], \tag{5.9} \end{cases} \]
with $[\hat{u} < M\hat{u}]$ denoting the set of points $x$ such that $\hat{u}(x) < M\hat{u}(x)$ has one and only one solution, which is given explicitly as the optimal cost for impulse control $u$.

This martingale notion of $A$, developed in [17] for example, is equivalent to the general notion of weak solutions. Indeed, since $u$ is Lipschitz continuous, $Au$ has a meaning as a distribution. Through (H4), (H5), we have for every test function $\varphi \in \mathcal{D}'(\mathbb{R}^n)$
\[ \langle Au, \varphi \rangle = -\sum_{i,j} \int_{\mathbb{R}^n} u(X)\partial^2_{x_i,x_j}(a_{ij}(X)\varphi(X))dX \]
\[ -\sum_{i=1}^n \int_{\mathbb{R}^n} u(X)\partial_i(b_i(X)\varphi(X))dX + \int_{\mathbb{R}^n} ru(X)\varphi(X)dX \]
\[ -\int_{\mathbb{R}^n} u(X)dX \times \int_{\{j_0 < 1\}} [\varphi(X - j^*(X,z)) - \varphi(X) + \nabla\varphi(X) \cdot j^*(X,z)m^*(X,z)\nu(dz) \]
\[ -\int_{\mathbb{R}^n} u(X)dX \times \int_{\{j_0 \geq 1\}} [\varphi(X - j^*(X,z)) - \varphi(X)]m^*(X,z)\nu(dz) \]
\[ -\int_{\mathbb{R}^n} u(X)dX \times \int_{\{j_0 < 1\}} [j(x(X,z),z) - j^*(X,z)m^*(X,z)]\nu(dz) \cdot \nabla\varphi(X) \]
\[ -\int_{\mathbb{R}^n} u(X)\varphi(X)dX \]
\[ \times \left( \int_{\{j_0 \geq 1\}} [m^*(X,z) - 1]\nu(dz) + \int_{\{j_0 < 1\}} [m^*(X,z) + \nabla \cdot j(x(X,z),z) - 1]\nu(dz) \right), \tag{5.10} \]
with $j^*(X,z) = j(x(X,z),z)$, $m^*(X,z) = \det(\partial x(X,z)/\partial X)$ and the change of variable $X = x + j(x,z)$ (c.f. Section 2.4 in [6]). Using that fact that $Au \leq f$ in the martingale sense is equivalent to $Au \leq f$ in $\mathcal{D}'(\mathbb{R}^n)$, the hypotheses leading to Theorem 5.2 allow us to conclude that $Au \leq f$ in $\mathcal{D}'(\mathbb{R}^n)$.

With $Au \leq f$ in $\mathcal{D}'(\mathbb{R}^n)$, our next goal is to show that the distribution $Au$ is actually a function with $Au \in B_2(\mathbb{R}^n)$. This property not only describes the behavior of $Au$ at infinity but also would mean $Au \in L^\infty(O)$ for any bounded open set $O$. In turn, an application of Lemma 3.2 would complete the regularity argument by
allowing us to conclude \( u \in W^{2,p}_{loc}(\mathbb{R}^n) \). Below, we show \( A(\mu u) \geq -C(1+|x|^2) \) which combined with \( Au \leq f \) in \( D'(\mathbb{R}^n) \), \( u \leq \mu u \) in \( \mathbb{R}^n \) and \( Au = f \) in \( D'(\{u < \mu u\}) \) implies that \( Au \in B_2(\mathbb{R}^n) \).

### 5.2.2. Semi-concavity of \( u \) and \( \mu u \)

The property \( A(\mu u) \geq -C(1+|x|^2) \) in \( D'(\mathbb{R}^n) \) follows from the semi-concavity property of \( u \) and \( \mu u \).

**Definition 5.3.** A continuous function \( h \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) is called semi-concave on \( \mathbb{R}^n \) if for every ball \( B_r(0) \), \( r > 0 \) there exists a constant \( C_r > 0 \) such that \( x \mapsto h(x) - C_r|x|^2 \) is concave on \( B_r(0) \), i.e., for every \(|x| < r, |y| < r\), we have

\[
\theta h(x) + (1 - \theta)h(y) - h(\theta x + (1 - \theta)y) \leq C_r(1 - \theta)|x - y|^2,
\]

for any \( \theta \in [0,1] \). If \( h \) is continuous, this is equivalent to the condition

\[
h(x + z) - 2h(x) + h(x - z) \leq C_r|z|^2,
\]

for all \( z \) sufficiently small. Equivalently, for any unit vector \( \chi \in \mathbb{R}^n \) and constant \( C > 0 \), we have

\[
\frac{\partial^2 h}{\partial \chi^2} \leq C, \text{ in } D'(\mathbb{R}^n).
\]

As observed in Section 4.2 in [13] and Section 6 in [4], in order to show the semi-concavity of \( \mu u \) on \( \mathbb{R}^n \), it suffices to show the semi-concavity of \( u \). Indeed, for fixed \( x \in \mathbb{R}^n \),

\[
\mu u(x + z) - 2\mu u(x) + \mu u(x - z) \leq u(y + z) - 2u(y) + u(y - z),
\]

where \( y := x + \xi \) and \( \xi \in \mathbb{R}^n \) is the limit of a convergent subsequence of a minimizing sequence \( (\xi_k)_{k=1}^\infty \) such that \( u(x + \xi_k) + B(\xi_k) \to \mu u(x) \). The following lemma which, for instance, appears as Proposition 5.9 in Section 5.1.2 [14] assists in showing \( u \) is semi-concave.

**Lemma 5.4.** Let \( X_t, X'_t, Z_t \) be three solutions of (1.1) for \( t \geq 0 \) with initial values \( x, x', z \). If \( \alpha \geq \kappa \), as defined in (5.22), then for \( \psi_\theta(x, x', z) := \theta^2(1 - \theta)^2|x - x'|^4 + |\theta x + (1 - \theta)x' - z|^2 \) and under the assumptions (H1) and (H2), we have

\[
\mathbb{E} \left[ (\alpha - \kappa) \int_0^t \psi_\theta(X_s, X'_s, Z_s)e^{-\alpha s}ds + \psi_\theta(X_t, X'_t, Z_t)e^{-\alpha t} \right] \leq \psi_\theta(x, x', z), \text{ for } t \geq 0.
\]

Moreover, there exists a constant \( C > 0 \), depending on the bounds of \( \sigma, j \) through (H1), and (H2), such that

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \psi_\theta(X_s, X'_s, Z_s)e^{-\alpha s} \right] \leq C \left( 1 + \frac{1}{\alpha - \kappa} \right) \psi_\theta(x, x', z), \text{ for } t \geq 0.
\]

**Proof.** The proof follows analogously to the proof of Lemma 4.2. Indeed, we consider \( \psi_{\lambda, \theta}(x, x', z) := \lambda + \psi_\theta(x, x', z) \) and apply Itô’s formula to find

\[
d\psi_{\lambda, \theta}(X_t, X'_t, Z_t) = \sigma_t dt + \sum_{k=1}^d b^k_t dW^k_t + \int_R c(t, z)N(dt, dz),
\]

where...
with \( a_t \leq \kappa \psi_{\lambda,\theta}(X_t, X_t', Z_t) \). As in Lemma 4.2, we also have

\[
\sum_{k=1}^{d} |b^k_t|^2 + \int_{\mathbb{R}^l} |c(t, z)|^2 \nu(dz) \leq C |\psi_{\lambda,\theta}(X_t, X_t', Z_t)|^2, \tag{5.18}
\]

for some constant \( C > 0 \). Proceeding as in Lemma 4.2 completes the proof.

We will apply this estimate as follows in Proposition 5.5 below. Let \( Y_t(x) \) denote the solution of (1.1) with initial condition \( Y_0(x) = x \). From Lemma (5.4), we have

\[
\mathbb{E} \left[ (\alpha - \kappa) \int_0^t |\theta Y_s(x) + (1 - \theta)Y'_s(x') - Y_s(\theta x + (1 - \theta)x')|^2 e^{-\alpha s} ds \right. \\
+ |\theta Y_t(x) + (1 - \theta)Y_t(x') - Y_t(\theta x + (1 - \theta)x')|^2 e^{-\alpha t} \right] \leq \theta^2(1 - \theta)^2 |x - x'|^4, \tag{5.19}
\]

and

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\theta Y_s(x) + (1 - \theta)Y'_s(x') - Y_s(\theta x + (1 - \theta)x')|^2 e^{-\alpha s} \right] \leq C \left( 1 + \frac{1}{\alpha - \kappa} \right) \theta^2(1 - \theta)^2 |x - x'|^4. \tag{5.20}
\]

The following proposition asserts the semi-concavity property of \( u \).

**Proposition 5.5.** Suppose (H1), (H2), (H7), (H8), and (H10) hold. Then \( u \) is semi-concave on \( \mathbb{R}^n \).

**Proof.** Fix an admissible control \( V \). The value function \( u(x) \) will be semi-concave if \( J_x[V] \) is semi-concave since the infimum of semi-concave functions is semi-concave. Appealing to Definition 5.3, we show,

\[
\theta J_x[V] + (1 - \theta)J'_{x'}[V] - J_{\theta x + (1 - \theta)x'}[V] \leq C \theta(1 - \theta) |x - x'|^2. \tag{5.21}
\]
Define
\[ \kappa := \sup_{x,x',y,\theta} \{ 2\kappa_b + \kappa_\sigma + \kappa_j \}, \]
with
\[ \kappa_b := \sum_i 2\theta^2 (1 - \theta)^2 |x - x'|^2 (x_i - x'_i)[\tilde{b}_i(x) - \tilde{b}_i(x')] \]
\[ + \sum_i (\theta x_i + (1 - \theta)x'_i - y_i)[\theta \tilde{b}_i(x) + (1 + \theta)\tilde{b}_i(x') - \tilde{b}_i(y)], \]
\[ \kappa_\sigma := \theta^2 (1 - \theta)^2 \left[ \sum_{h,k} 2|x - x'|^2 + 4(x_h - x'_h)^2(\sigma_{hk}(x) - \sigma_{hk}(x'))^2 \right. \]
\[ + \sum_{i \neq j, k} 4(x_i - x'_i)(x_j - x'_j)(\sigma_{ik}(x) - \sigma_{ik}(x'))(\sigma_{jk}(x) - \sigma_{jk}(x')) \]
\[ + \sum_i \theta \sigma_{ik}(x) + (1 - \theta)\sigma_{ik}(x') - \sigma_{ik}(y))^2, \]
\[ \kappa_j := \int_{\mathbb{R}^l} \left[ |x - x' + j(x, z) - j(x', z)|^4 - |x - x'|^4 - \sum_i 4|x - x'|(x_i - x'_i) \times (j_i(x, z) - j_i(x', z)) \right] \nu(dz) \]
\[ + \int_{\mathbb{R}^l} \left[ |\theta x + (1 - \theta)x' - y + (\theta j(x, z) + (1 - \theta)j(x', z) - j(y, z))| \right] \]
\[ - \sum_i 2(\theta x_i + (1 - \theta)x'_i - y_i) \times (\theta j_i(x, z) + (1 - \theta)j_i(x', z) - j_i(y, z)) \right] \nu(dz), \]
\[ (5.22) \]
where \( x, x', z \in \mathbb{R}^n, \theta \in [0, 1] \) and \( \beta \leq \kappa < \infty \) due to (H1), (H2) (see Section 5.2.1 in [14]). We have for \( \alpha \geq \kappa \geq \beta, \)
\[ \theta J_x[V] + (1 - \theta)J_{x'}[V] - J_{\theta x + (1 - \theta)x'}[V] \]
\[ = E \left[ \int_0^\infty [\theta f(Y_t(x)) + (1 - \theta) f(Y_t(x')) - f(Y_t(\theta x + (1 - \theta)x'))] e^{-rt}dt \right] \]
\[ = E \left[ \int_0^\infty [\theta f(Y_t(x)) + (1 - \theta) f(Y_t(x')) - f(\theta Y_t(x) + (1 - \theta)Y_t(x')) \]
\[ + f(\theta Y_t(x) + (1 - \theta)Y_t(x')) - f(Y_t(\theta x + (1 - \theta)x'))] e^{-rt}dt \right] \]
\[ \leq C \theta (1 - \theta) \int_0^\infty e^{-rt}E[|Y_t(x) - Y_t(x')|^2]dt \]
\[ + C_f \int_0^\infty e^{-rt}E[|\theta Y_t(x) + (1 - \theta)Y_t(x') - Y_t(\theta x + (1 - \theta)x')|^2]dt \]
\[ \leq C \theta (1 - \theta) |x - x'|^2 \int_0^\infty e^{-(r-\alpha)t}dt \]
\[ C_f C^{1/2} \left( 1 + \frac{1}{\alpha - \kappa} \right)^{1/2} \theta (1 - \theta) |x - x'|^2 \int_0^\infty e^{-(r-\alpha)t}dt \]
\[ \leq C \theta (1 - \theta) |x - x'|^2. \]

The first inequality follows using semi-concavity and Lipschitz continuity of \( f \). The
second inequality follows using a standard estimate for the difference of solutions for (1.1) (c.f. Theorem 5.6 in [14]) and (5.20). □

5.2.3. \( u \in W^{2,p}_{{\text{loc}}} (\mathbb{R}^n) \). Using the semi-concavity property of \( Mu \) on \( \mathbb{R}^n \), the following mollification argument shows that \( A(Mu) \geq -C(1 + |x|^2) \) in \( \mathcal{D}'(\mathbb{R}^n) \) for some constant \( C > 0 \). Recall \( A := (-\mathcal{D} - I + r) \),

\[
\mathcal{L}_D \phi(x) = \sum_{i,k=1}^n a_{ik}(x) \partial^2_{x_i,x_k} \phi(x) + \sum_{i=1}^n b_i(x) \partial_{x_i} \phi(x),
\]

\[
I \phi(x) = \int_{\mathbb{R}^n} (\phi(x + j(x,z)) - \phi(x) - j(x,z) \nabla \phi(x)) \nu(dz).
\]

Since \( Mu \) is semi-concave on \( \mathbb{R}^n \), we know

\[
Mu(x + \rho \chi) + Mu(x - \rho \chi) - 2Mu(x) \leq K \rho^2, \quad x \in \mathbb{R}^n,
\]

for any \( \rho > 0 \) and unit vector \( \chi \in \mathbb{R}^n \) and non-negative constant \( K \). Below, \( C \) denotes a generic constant independent of \( \varepsilon \). Let \( g = Mu \) and denote \( g^\varepsilon \) its mollification on \( \mathbb{R}^n \). We first show that \( A(g^\varepsilon(x)) \geq -C(1 + |x|^2) \) for \( C \) independent of \( \varepsilon \). We proceed by estimating each term in \( A(g^\varepsilon) \). For \( x \in \mathbb{R}^n, \rho > 0 \) and unit vector \( \chi \in \mathbb{R}^n, \)

\[
\frac{1}{\rho^2} (g^\varepsilon(x + \rho \chi) + g^\varepsilon(x - \rho \chi) - 2g^\varepsilon(x)) = \frac{1}{\rho^2} \int_{B_\rho(0)} (g(x + \rho \chi) + g(x - \rho \chi) - 2g(x)) \eta^\varepsilon(z) dz
\]

\[
\leq K \int_{B_\rho(0)} \eta^\varepsilon(z) dz.
\]

Sending \( \rho \to 0 \), yields \( \chi^T \nabla^2 g^\varepsilon(x) \chi \leq K \). Using this, we have

\[
Tr[\sigma(x)\sigma(x)^T \nabla^2 g^\varepsilon(x)] = \sum_{i=1}^n \sigma_{i}^T(x) \nabla^2 g^\varepsilon(x) \sigma_i(x)
\]

\[
\leq K \sum_{i,j=1}^n |\sigma_{ij}(x)|^2
\]

\[
\leq C(1 + |x|^2).
\]

Using Lipschitz continuity of \( \tilde{b}, g \), we know

\[
|\tilde{b}(x) \cdot \nabla g^\varepsilon(x)| \leq |\tilde{b}(x)| |\nabla g^\varepsilon(x)| \leq C(1 + |x|) n C_{Mu} = C(n)(1 + |x|),
\]

\[
\leq C(n)(2 + |x|^2)
\]

\[
\leq C(1 + |x|^2),
\]

where \( C_{Mu} \) is the Lipschitz constant for \( Mu \), and \( C(n) \) is a constant depending on the dimension \( n \). Next,

\[
|g^\varepsilon(x) - g(x)| \leq \int_{B_\rho(0)} |g(x - z) - g(x)| \eta^\varepsilon(z) dz
\]

\[
\leq C_{Mu} \int_{B_\rho(0)} |z| \eta^\varepsilon(z) dz
\]

\[
\leq \varepsilon C_{Mu}.
\]
Then, for all $\varepsilon \in \left(0, \frac{1}{c_{Mu}}\right)$, we have

$$|g^\varepsilon(x)| \leq |g(x)| + 1 \leq C(1 + |x|) \leq C(1 + |x|^2). \quad (5.30)$$

With regard to the integro term, we have

$$\varepsilon$$

Then, for all $\varepsilon \in \left(0, \frac{1}{c_{Mu}}\right)$, we have

$$\int_{\mathbb{R}^n} [g^\varepsilon(x + j(x, z)) - g^\varepsilon(x) - \sum_{i=1}^n j_i(x, z) \partial_z g^\varepsilon(x)] \nu(dz)$$

$$\leq \int_{\mathbb{R}^n} \left( \int_0^1 (1 - \theta) |j(x, z)^T \cdot \nabla^2 g^\varepsilon(x + \theta j(x, z)) \cdot j(x, z)| \, d\theta \right) \nu(dz)$$

$$\leq \int_{\mathbb{R}^n} \frac{K}{2} |j(x, z)|^2 \nu(dz)$$

$$\leq C(1 + |x|^2),$$

Gathering these estimates, we have for all $\varepsilon \in \left(0, \frac{1}{c_{Mu}}\right)$,

$$A(g^\varepsilon(x)) = -\frac{1}{2} Tr[\sigma(x)\sigma(x)^T \nabla^2 g^\varepsilon(x)] - \bar{b}(x) \cdot \nabla g^\varepsilon(x) + rg^\varepsilon(x)$$

$$- \int_{\mathbb{R}^n} [g^\varepsilon(x + j(x, z)) - g^\varepsilon(x) - \sum_{i=1}^n j_i(x, z) \partial_z g^\varepsilon(x)] \nu(dz)$$

$$\geq -C(1 + |x|^2),$$

where $C$ depends upon the dimension $n$ but is independent of $\varepsilon$. Now, this pointwise estimate implies that $A(g^\varepsilon) \geq -C(1 + |x|^2)$ in $\mathcal{D}'(\mathbb{R}^n)$. Finally, since $g^\varepsilon \to g$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, we know $(Ag^\varepsilon, \varphi) \to (Ag, \varphi)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Thus, $A(Mu) \geq -C(1 + |x|^2)$ in $\mathcal{D}'(\mathbb{R}^n)$.

At this point, we know

$$\begin{cases} 
-C(1 + |x|^2) \leq Au \leq f, \text{ in } \mathcal{D}'(\{u = Mu\}), \\
Au = f, \text{ in } \mathcal{D}'(\{u < Mu\}).
\end{cases} \quad (5.33)$$

Thus, $Au$ exists as a function and satisfies $|Au(x)| \leq C(1 + |x|^2)$, i.e., $Au(x) \in B_2(\mathbb{R}^n)$. Knowing $Au(x) \in B_2(\mathbb{R}^n)$ allows us to apply Lemma 3.2 with $f = Au$ over any bounded open set $\mathcal{O}$. Thus, we have $u \in W_{\text{loc}}^{2,p}(\mathbb{R}^n)$ for $p \in (1, \infty)$ as desired.

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