Quantum Kinetic Theory VI: The Growth of a Bose-Einstein Condensate

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A detailed analysis of the growth of a Bose-Einstein condensate is given, based on quantum kinetic theory, in which we take account of the evolution of the occupations of lower trap levels, and of the full Bose-Einstein formula for the occupations of higher trap levels, as well as the Bose stimulated direct transfer of atoms to the condensate level introduced by Gardiner et al. We find good agreement with experiment at higher temperatures, but at lower temperatures the experimentally observed growth rate is somewhat more rapid. We also confirm the picture of the “kinetic” region of evolution, introduced by Kagan et al., for the time up to the initiation of the condensate. The behavior after initiation essentially follows our original growth equation, but with a substantially increased rate coefficient.

Our modelling of growth implicitly gives a model of the spatial shape of the condensate vapor system as the condensate grows, and thus provides an alternative to the present phenomenological fitting procedure, based on the sum of a zero-chemical potential vapor and a Thomas-Fermi shaped condensate. Our method may give substantially different results for condensate numbers and temperatures obtained from phenomenological fits, and indicates the need for more systematic investigation of the growth dynamics of the condensate from a supersaturated vapor.

I. INTRODUCTION

Although the race to produce a Bose-Einstein condensate was preceded by intense debate concerning the likely rate of its formation, the discovery that a Bose-Einstein condensate of alkali atoms could be produced relatively simply [1–3], and that the growth time was of the order of one second, moved most theoretical activity into the investigation of the properties of the condensates so produced. Since the production of the first Bose-Einstein condensate there have been few theoretical investigations into condensate growth, and only one experiment [4] has made any measurements of growth rates. Only the work of the present authors and co-workers, based on quantum kinetic theory, has made quantitative predictions on the growth rate of a Bose-Einstein condensate. This work started when we showed how to introduce the concept of stimulated condensate growth resulting from kinetic processes [5], leading to a very simple formula for the growth rate. The MIT experiment [4] took the form of a verification of the validity of our theoretical prediction. At the same time in [6] we refined the basic concept of Bosonic stimulation to generate a less idealized theoretical picture, and to compare it with experiment. These initial papers were of necessity brief, and developed neither the full theoretical justification on the numerical modeling nor the full range of possible comparison with the available experimental data. In particular, no account was taken of the information available on the spatial distribution of the atoms in the vapor-condensate system as the condensate grows from the vapor.

This paper will therefore give the detailed justifications and a full range of comparison with experimental data. Most particularly, we want to present a theoretically justifiable method of describing the condensate vapor system as it grows. The absence of such a description has led to a phenomenological fitting of vapor profiles to a zero chemical potential [6] Bose-Einstein distributions, which may be an imperfect model whose results could well be misleading.

The theoretical description of condensate growth that we present is largely able to be viewed as a modification of the quantum Boltzmann equation, in which, however, explicit note is taken of the modification of the excitation spectrum by the existence of the condensate, including of course the fact that the lowest single particle excitation energy is the chemical potential \( \mu_C(n_0) \) of the condensate of \( n_0 \) atoms. Equilibrium arises as a result of the equality of the chemical potentials of uncondensed vapor and condensate, a picture which is rather similar to that normally adopted for chemical reactions. The quantum Boltzmann equation itself automatically provides the Bose stimulation, which makes transition rates into the condensate and other highly occupied levels achieve a speed which permits the production of the condensate in a finite time. Without Bose stimulation, the production of a condensate of about 1,000,000 sodium atoms would take 30 hours, rather than the 100ms observed.

At first glance it might appear that a description which appears to be based on the quantum Boltzmann equation would have nothing to say about condensate coherence or the origin of that coherence. This is emphatically not the case—the kinetics of the transfer of the between energy levels in a trap requires the existence of a wavefunction for each energy level. The condensate level has its own wavefunction, and this obeys the Gross-Pitaevskii equation. The coherence arises because this level becomes macroscopically occupied. There is no precise moment when one can say that the condensate initiates. This picture applies in a trap, in which the energy levels about which we have been speaking are rather well separated. The picture of a Bose-Einstein condensate as developed in the middle part of this century as a part of condensed matter theory is of a homogeneous and thus infinitely extended system—a system for which the thermodynamic limit is achieved. Looked at from our viewpoint, this would be achieved by making the trap broader and ultimately
II. MODEL FOR GROWTH OF A CONDENSATE

In this section, the formalism of Quantum Kinetic theory \cite{8}, will be used to form a model of the growth of a trapped Bose-Einstein condensate. The Bose atoms are described by a second-quantized field, in the pseudopotential approximation; that is, we write

\[ H = H_{\text{kin}} + H_I + H_T, \]

where

\[ H_{\text{kin}} = \int d^3x \psi^\dagger(x) \left( -\frac{\hbar^2}{2m} \nabla^2 \right) \psi(x), \]

\[ H_I = \frac{\hbar^2}{2} \int d^3x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x), \]

and the term \( H_T \) arises from a trapping potential as

\[ H_T = \int d^3x V_T(x) \psi^\dagger(x) \psi(x). \]

The pseudopotential method is used—its validity for this kind of system has been justified in QKV\cite{4}—where \( u = 4\pi a \hbar^2 / m \), and \( a \) is the \( s \)-wave scattering length arising from the interatomic potential.

The situation being considered is that of a vapor cloud confined in a trap in which the lower energy levels are not significantly populated, whilst the higher energy levels contain thermalized equilibrium populations, characterized by a temperature \( T \) and chemical potential \( \mu \), unstable against condensate formation.

This situation is likely to arise, to a degree of approximation, if a system, which is initially in equilibrium at a temperature slightly greater than the critical temperature, is cooled very suddenly to a temperature below the critical temperature, by removing the very high energy atoms in a rapid evaporative cooling ‘cut’. The higher energy levels will very quickly come to their equilibrium distributions, since the difference between the distributions before and after the cut are quite small at the higher energies. The lower levels will however be far from equilibrium and evolve to form a condensate. This is possibly one of the easiest scenarios to model, it is also the situation investigated by the only detailed experimental study of the growth of a condensate in a gas of \( ^{23}\text{Na} \) atoms \cite{8}.

A. The System

The system is contained in a three-dimensional harmonic potential, characterized by the frequencies \( \omega_r \), \( \omega_p \) and \( \omega_z \). It will be useful to define the geometrical mean frequency of the trap as \( \omega = \left( \omega_r \omega_p \omega_z \right)^{1/3} \).

1. Effective potentials arising from mean-field effects

In this system the energies and wavefunctions of the lower trap levels are quite strongly affected by the presence of the condensate, and the effect will of course change as the condensate grows. In QKV it was shown that it is reasonable to account for this by introducing mean-field effects, which make the effective potentials depend on the occupations of the bands. The situation is illustrated in Fig.\ref{fig:1}. As the condensate grows, it expels the vapor from the center of the trap, and this expulsion serves to reduce the mean-field of the vapor as experienced by the condensate. The growth will be assumed to be so slow that the condensate- and non-condensate-bands are always in thermal equilibrium—that is, they will have a well defined temperature shared by both of them, but will not have the same chemical potential. Growth therefore occurs as atoms are transferred from the vapor to the condensate, leading eventually to a unique chemical potential for the whole system.

FIG. 1. Representation of the modification of the trapping potential for a) the non-condensate band and b) the condensate band due to mean-field effects.

2. Condensate- and non-condensate bands

In the formalism of quantum kinetic theory, the system is divided into condensate and non-condensate bands. We will treat in this paper the situation in which the non-condensate band is assumed to be in thermal equilibrium with a temperature \( T \) and chemical potential \( \mu \), and to contain the vast majority of the atoms so that it is essentially undepleted by the process of condensate formation.

The picture of growth we will use is that presented in QKV. In that paper, it was shown that a legitimate division into condensate and non-condensate bands can be made in which one
distinguishes between particle-like excitations, to which it is possible to assign a definite number of atoms, and phonon-like excitations, which are collective modes, which normally involve a large average number of atoms, but are not eigenstates of the atom number. In practice, it has been shown [13] that the energy above which all excitations are essentially particle-like is relatively small.

For the purposes of our modeling, however, there are two criteria which must be considered in the definition of the condensate band.

i) The non-condensate band is considered to be time independent, therefore the condensate band must include all levels whose populations change significantly during the condensate growth process. For the non-condensate band, the thermal distribution is given in the bulk by $[e^{(E-\mu)/kT} - 1]^{-1}$. This is only valid for $E > \mu$, and gives very large populations when $E \approx \mu$. The transition rates in and out of levels in this vicinity also become very large, which contradicts the assumption that the distribution of the non-condensate band is time-independent. These lower states are therefore must be treated time dependently, and hence must be included in the condensate band.

ii) The condensate band consists principally of levels whose energy eigenvalues are significantly affected by the presence of a condensate—but levels which are not affected may be included if this is desirable, which must be done if the first criterion is to be met.

Consequently, in this paper we will choose the condensate band to consist of all levels with energy less than the value $E_R$. We will also introduce an energy $E_\Delta < E_R$, which is the energy above which we can consider the energy levels to be unaffected by the condensate, as illustrated in Fig.

3. Grouping of energy levels into bands

The inclusion of all the condensate band energy levels in the model means that simulations of the system require, in principle, the calculation of all the eigenfunctions of the condensate band, and detailed summations over these. In practice the number of energy levels involved is of the order of tens of thousands, which makes an exact description impractical. However, progress can be made by grouping together energy levels in the condensate band into small ‘sub-bands’, with only the ground state (condensate state) being described as a single level. Each sub-band is described by an energy $e_m$ and contains all the energy eigenstates found within the energy range $[e_m - \Delta e_m/2, e_m + \Delta e_m/2]$. The value of $\Delta e_m$ is chosen partially by the requirement that the lowest of these sub-bands contains at least 3 levels. Smaller values of $\Delta e_m$ would lead sub-bands containing only fractions of individual levels, which is obviously unphysical. As the condensate grows, the mean-field effects from the high occupation of the condensate level will cause the energies of the levels in the sub-bands to increase. The values of $e_m$ and $\Delta e_m$ are therefore dependent on the condensate occupation, and the manner in which they are altered will be discussed later.

B. Notation

For clarity, we set out some of our notation.

\[
N : \text{number of atoms in the condensate band }, \quad (5)
\]

\[
n_0 : \text{number of atoms in the condensate }, \quad (6)
\]

\[
n_{0,f} : \text{equilibrium number of atoms in the condensate }, \quad (7)
\]

\[
\mu_C(n_0) : \text{chemical potential of the condensate }, \quad (8)
\]

\[
\mu : \text{chemical potential of the non-condensate band }, \quad (9)
\]

\[
\xi_{n_0}(x) : \text{wavefunction of an n_0 atom condensate }. \quad (10)
\]

In the situations we will consider, the number of condensate atoms $n_0$ will vary from zero to almost $N$, but this will always be substantially less than the number of atoms in the whole system, composed of both condensate and non-condensate bands. Thus, when the condensate is fully grown, the approximation $n_0 \approx N$ will be valid, and will often be used.

C. Density of States for the System

In the absence of any condensate, the density of states $G(E)$ is taken to be that of a non-interacting gas in a harmonic well. That is

\[
G(E) \equiv \frac{dN(E)}{dE} = \frac{(E - \frac{3}{2}\hbar\omega)^2}{2\hbar^3\omega_x\omega_y\omega_z}, \quad (11)
\]

where $N(E)$ is the cumulative number of states with energy less than $E$ and $\bar{\omega} = (\omega_x + \omega_y + \omega_z)/3$. The number of states in the sub-band with average energy $e_m$ is thus $g_m = G(e_m)\Delta e_m$. The energy scale is such that the value of $V_T(r) = m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)/2$ is zero at the origin.

Once the condensate begins to form, the mean field effects need to be taken into account. The mean field repulsion due to the condensate changes the energies of the lower trapped states. The energy of the condensate level is equal to the chemical potential $\mu_C(n_0)$, which increases with $n_0$. In the Thomas-Fermi approximation the rise is proportional to $n_0^{1/5}$. Obviously the condensate level must remain the lowest energy state, and thus the energies of the other states below $E_\Delta$ must also rise in some fashion. The exact nature of the energy change is difficult to calculate, but some reasonable approximations can be proposed which should reproduce the significant behavior caused by the mean field effects.

The most simply calculated estimate of the energy changes is to assume the energies of the sub-bands $e_m$ are evenly distributed between the fixed upper limit of $E_\Delta$ and the lower limit of $\mu_C(n_0) \approx \alpha n_0^{2/5}$. This is illustrated schematically in Fig.

Both the values of $e_m$ and the values of $\Delta e_m$ are now $n_0$ dependent.

The final value of $\Delta e_m$ (after the growth of the condensate) is set to be $\hbar\omega$. This condition also always fulfills the requirements that there are at least about 3 discrete levels contained in
the lowest energy sub-band, and yet ensures that the sub-band has only a relatively small energy range.

The density of states for the condensate band, in the presence of the condensate, is thus taken to be approximately \( G_{n_0}(E) = N'[(E - \mu_C(n_0))^2] \), where \( N' \) is a normalization chosen so that the cumulative number of states at \( E_\Delta \) is the same as for the non-interacting harmonic oscillator potential. This behavior is illustrated in Fig. 1. As the discontinuity at \( E_R \) shows, this model is obviously quite simplistic, and more realistic models will be discussed later.

It should be noted that in the inset of Fig. 1 the number of particles per energy interval \( f(E) \) is shown (the occupation of the condensate level is not shown) for equilibrium conditions. From this inset it can be seen that the vast majority of atoms do indeed reside at energies higher than \( E_R \), and so the assumption that the non-condensate band is undepleted should be valid.

1. Modified Thomas-Fermi Chemical Potential

In the Thomas Fermi approximation the chemical potential of the condensate is given by

\[
\mu_C(n_0) = \left( \frac{15a_0\omega_x\omega_y\omega_z m^1/2\hbar^2}{2^{5/2}n_0} \right)^{2/5},
\]

which vanishes as \( n_0 \to 0 \). However, the ground state of a non-interacting gas in a harmonic well is \( \hbar(\omega_x + \omega_y + \omega_z)/2 \), and so the real chemical potential should approach this value as \( n_0 \to 0 \). In order to interpolate the Thomas-Fermi chemical potential to satisfy this requirement, the following form for the chemical potential will be used

\[
\mu_C(n_0) = \alpha(n_0 + \nu)^{2/5}
\]

where \( \alpha = (15a_0\omega_x\omega_y\omega_z m^{1/2}\hbar^2/4\sqrt{2})^{2/5} \), and \( \nu \) is a constant such that \( \alpha\nu^{2/5} = \hbar(\omega_x + \omega_y + \omega_z)/2 \).

2. Estimate of \( E_\Delta \)

An estimate for the value of \( E_\Delta \), above which the excitation spectrum is well described by that for a non-interacting gas, can be obtained using the number-conserving Bogoliubov spectrum \([5][1]\). The quantity of interest now is the ratio of the corrections to the energy level arising from the presence of the condensate, to the energy level determined by the non-interacting gas model. For the case of the trap used in the \(^{23}\)Na growth experiments at MIT \([4]\), numerical calculations show that the ratio is less than 10% for energies \( \geq 2\mu_C(n_0) \), and the corrections are of the order of only 5% for \( E \sim 2.5\mu_C(n_0) \). Thus a reasonable estimate of \( E_\Delta \) to be used in the simulations is

\[
E_\Delta = 2\mu_C(n_{0,f})
\]

where \( n_{0,f} \) is the equilibrium occupation of the condensate level, and this is the value of \( E_\Delta \) which will be used in this paper.

![FIG. 2. Schematic representation of the simple model to change the energy levels due to the mean field effects of the condensate. Levels are evenly distributed between \( E_\Delta \) and \( \mu_C(n_0) \) which increases with \( n_0 \).](image)

3. Comparison with More Accurate Density of States

As a result of the predominantly single particle nature of the excitation spectrum, the cumulative number of states \( N(E) \) is expected to be quite well described at high energies by the semiclassical approximation

\[
N(E) = \frac{1}{(2\pi\hbar)^3} \int_0^E d\varepsilon \int d\mathbf{r} \int d\mathbf{p} \delta(\varepsilon - E_{sp}(\mathbf{p}, \mathbf{r})),
\]

where \( E_{sp}(\mathbf{p}, \mathbf{r}) = p^2/2m + V_T(\mathbf{r}) + (8\pi\hbar^2a/m)|\varepsilon|^2|n_{sp}(\mathbf{r})| - \mu_C(n_{0}) \) is the semiclassical energy of a single particle in the potential created by the combination of the trapping potential and the mean field repulsion of the condensate. Equation (15) thus represents the summation over all phase space cells which contain a single particle excitation state of energy less than \( E \). In \([16]\), by carrying out the momentum and space
phasized that this semiclassical result applies to an isotropic trap, and the consideration of the anisotropy of realistic traps, which has not yet been accounted for, may have a significant effect on the spectrum.

\[
\frac{N(\tilde{E})}{n_0} = \tilde{E}^2 \sqrt{(1 - x)(x + \eta/\tilde{E})} + \int_0^{\tilde{E}} d\tilde{E} \frac{4}{\pi \zeta(3)} \int_0^1 dx \tilde{E} \sqrt{\frac{x^2 + \tilde{E}^2/\eta^2}{x^2 + \tilde{E}^2/\eta^2}}^{1/2 - x} \tag{16}
\]

where

\[
\eta = \frac{\mu_C(n_0)}{kT_c}, \quad kT_c = \hbar \omega \left( \frac{n_0}{\zeta(3)} \right)^{1/3}. \tag{17}
\]

The chemical potential \(\mu_C(n_0)\) is given by the Thomas-Fermi approximation form (\(\mu(x)\)) and the energy is given in the dimensionless units \(\tilde{E} = E/kT_c\). This semiclassical form for \(N(E)\) was found in \([16, 14]\) to be practically indistinguishable from that found by numerical solutions of the Bogoliubov spectrum over the entire range of energies.

\[
\text{FIG. 4. The cumulative number of states } N(E) \text{ below an energy } E. \text{ The solid line shows the results obtained using the semiclassical approximation for an isotropic trap equation (16), the dotted line shows the case for the non-interacting harmonic well. The dashed line represents the form used in this paper with } E_{\Delta} = 2\mu_C(n_0). \text{ The results were obtained for a condensate of 5 million atoms at a temperature of 900nK.}
\]

In Fig. 4, the semiclassical form of \(N(E)\) obtained from equation (16) is compared to that of the non-interacting harmonic oscillator (given by equation (11)), and the density of states obtained using our model with \(E_{\Delta}\) equal to \(2\mu_C(n_0)\). The figure does not show very good agreement of our model with the semiclassical results at moderate energies, although at low energies the agreement is good. At high enough energies (not shown) the non-interacting potential results become practically indistinguishable from those obtained from the semiclassical method, the energy at which this occurs is about \(5\mu_C(n_0)\) for the results in Fig. 4. It should be emphasized that this semiclassical result applies to an isotropic trap, and the consideration of the anisotropy of realistic traps, which has not yet been accounted for, may have a significant effect on the spectrum.

4. Approximations

Several approximations that have been made in the derivation of this model rely on the condensate band being small relative to the non-condensate band, and if \(E_R\) and \(E_{\Delta}\) are too large then these approximations will not be valid. The approximations concerned are:

i) That the non-condensate band is so large that it is essentially undepleted by the process of condensate growth.

ii) That the scattering processes between condensate band atoms may be taken as being negligible compared to the inter-band scattering processes, as will be assumed later.

It should be noted, and will be shown later, that the major effect on the overall growth due to mean field effects is caused by the changes in energies of the lowest energy levels, and for these levels the model proposed here is in quite good agreement with the semiclassical results. Because of this, as well as for the above reasons, the value of \(E_{\Delta}\) used will generally remain equal to \(2\mu_C(n_0)\). The value of \(E_R\) is chosen somewhat larger. This provides a check that the solutions we find do match smoothly onto the distribution above \(E_R\), which is assumed not to change.

\[
\text{FIG. 5. Illustration of the two types of dynamical processes, growth and scattering. Only the two non-condensate particle growth process is shown (process (a) in the text), and only non-condensate/condensate band scattering is shown (process (c)).}
\]

D. Dynamical Processes

The dynamics which will be considered in order to describe the evolution of the condensate band arise from the following processes:
a) Two particles in the non-condensate band collide, one of the particles leaves with an increased energy, and the remaining particle enters the condensate band, having now an energy less than $E_R$. Of course the reverse process must also be considered—a non-condensate band particle colliding with a condensate band particle and exciting it out of the condensate band.

b) A non-condensate band particle collides with a condensate band particle and exchanges energy such that both particles end up in the condensate band, and the reverse process.

c) A non-condensate band particle collides with a condensate band particle, transferring some energy, but both particles remain in their respective bands.

d) Two particles in the condensate band collide, transferring energy, with the result that both particles remain in the condensate band, but having different energies than before the collision.

Processes (c) and (d) will be termed scattering processes, since they do not change the occupation number of either band. Processes (a) and (b) cause the number of particles in the condensate band to increase, and so will be referred to as growth processes. The distinction between the two types of processes is illustrated in Fig.5. Because the number of atoms in the non-condensate band is much larger than that in the condensate band, the scattering will be dominated by processes of type (c), and those of type (d) will be neglected.

These processes are described by the full quantum kinetic master equation obtained in QKIII, which can be used to determine rate equations for the evolution of the system.

E. Growth Processes

The formalism of [5][8][9][10][11][12][13][14][15][16][17][18] gives rise to rate equations for $N$, the number of particles in the condensate band as a whole, and $n_m$, which represent the number of quasiparticles in the $m$th quasiparticle level. The derivation gives equations in the limit that $N$ is sufficiently large for us to write $n_0 \approx N$. The rate equations take the form:

$$\frac{dn_m}{dt} = \dot{n}_m\big|_{growth} + \dot{n}_m\big|_{scatt}$$

The form of $\dot{n}_m\big|_{growth}$ was given in QKIII, Sec.IV.E.3, and can be written in terms of the transition rates as follows

$$\dot{n}_m = \hat{n}_m^+ + \hat{n}_m^-$$

$$N = 2W^+ \left( 1 - \epsilon(\mu_C(N) - \mu/k_B T) N + 1 \right)$$

where

$$\dot{n}_m^+ = 2W_m^+(N) \left( 1 - \epsilon(\mu_C(N) - \mu + \epsilon_m) / k_B T ) n_m + 1 \right)$$

$$\dot{n}_m^- = 2W_m^-(N) \left( 1 - \epsilon(\mu_C(N) + \mu + \epsilon_m) / k_B T ) n_m + 1 \right)$$

the relationships between forward and backward rates can be shown to be

$$W^+(N) = \epsilon(\mu - \mu_C(N)) / k_B T W^-(N)$$

$$W_m^+(N) = \epsilon(\mu - \mu_C(N) - \epsilon_m) / k_B T W_m^-(N)$$

$$W_m^+(N) = \epsilon(\mu - \mu_C(N) + \epsilon_m) / k_B T W_m^-(N).$$

In these equations, the energies of the quasiparticle excitations $\epsilon_m$ are measured from the ground state energy, which is $\mu_C(N)$.

I. Transitions

There are six processes which are described by these equations:

i) $N \rightarrow N + 1$, with no change in $n$. The transition probability for this process is $W^+(N) = R^+ (\xi_N, \mu_C(N)/\hbar)$.

ii) $N \rightarrow N - 1$, with no change in $n$. The transition probability for this process is $W^-(N) = R^- (\xi_{N-1}, \mu_{N-1}/\hbar)$.

iii) $N \rightarrow N + 1$, with $n_m \rightarrow n_m + 1$. The transition probability for this process is $W_m^+(N) = R^+ (f_m, (\epsilon^m_N + \mu_C(N))/\hbar)$.

iv) $N \rightarrow N - 1$, with $n_m \rightarrow n_m - 1$. The transition probability for this process is $W_m^-(N) = R^- (f_m, (\epsilon^{m^*}_N - \mu_{N-1}) / \hbar)$.

v) $N \rightarrow N + 1$, with $n_m \rightarrow n_m - 1$. The transition probability for this process is $W_m^+(N) = R^+ (g_m, (-\epsilon^m_N + \mu_N)/\hbar)$.

vi) $N \rightarrow N - 1$, with $n_m \rightarrow n_m + 1$. The transition probability for this process is $W_m^+(N) = R^- (g_m, (-\epsilon^{m^{*}}_N + \mu_{N-1}) / \hbar)$.

Here the functions $f_m$ and $g_m$ are amplitudes for the creation and destruction of atoms in quasiparticle states with energies $\epsilon_m$, which are defined in [5][7][11], but whose explicit form will not need to be used here; $\xi_N$ is the condensate wavefunction for $N$ atoms. The functions $R^\pm(y, \omega')$ are defined by

$$R^+(y, \omega') = \frac{\hbar^2}{(2\pi)^3} \int d^3r \int d^3k_1 d^3k_2 d^3k_3 d^3k \delta(\Delta \omega_{123}(r) - \omega') \delta(k_1 + k_2 - k_3 - k) F_1 F_2 (1 + F_3) W_y(r, k)$$

$$R^-(y, \omega') = \frac{\hbar^2}{(2\pi)^3} \int d^3r \int d^3k_1 d^3k_2 d^3k_3 d^3k \delta(\Delta \omega_{123}(r) + \omega') \delta(k_1 + k_2 - k_3 - k) (1 + F_1)(1 + F_2) F_3 W_y(r, k)$$
In these equations the following notation is used
\[ \hbar \omega_{K_i}(r) = \frac{\hbar^2 K_i^2}{2m} + V_T(r), \]
\[ \Delta \omega_{123} = \omega_{K_1} + \omega_{K_2} - \omega_{K_3}, \]
\[ \Delta K = K_1 + K_2 - K_3 - k, \]
\[ u = \frac{4\pi \hbar^2 a}{m}. \]

The function \( F_i = F(K_i, r) \) is the distribution function for the non-condensate particles, and the Wigner function is given by
\[ W_N(r, k) = \frac{1}{(2\pi)^3} \int d^3v \ y^* (r + \frac{v}{2}) y (r - \frac{v}{2}) e^{ik \cdot v}. \]

The \( R^+ \) function is related to collisions between two non-condensate band particles with momenta \( K_1 \) and \( K_2 \), from which the particles leave with momenta \( K_3 \) and \( k \). The particle with momentum \( k \) is now in the condensate band (ie: \( k \) is small) so that the particle with momentum \( K_3 \) remains in the non-condensate band. The functions \( R^\pm (y, \omega') \) therefore represent the rates for collisions which result in a particle entering (+) or leaving (−) the condensate band with an energy \( \hbar \omega' \).

2. Approximate evaluation of transition rates

The rate factors \( W^\pm (N) \) contain integrals over all space of terms containing the product of \( F(K, r) \) terms times the Wigner function corresponding to the ground state wavefunction \( W_{\xi_N}(r, K) \). In practice the ground state wavefunction is very sharply peaked in comparison to the spatial dependence of the \( F(K, r) \) functions which describe the remainder of the cloud of atoms. This means that in the spatial integral for the \( W^\pm (r) \) terms (from equation (24)) the \( F(K, r) \) terms can be approximated by their values at \( r = 0 \). This gives
\[ W^+(N) = \frac{u^2}{(2\pi)^5 \hbar^2} \int d^3K_1 \int d^3K_2 \int d^3K_3 \int d^3k \delta(\Delta K) \delta(\Delta \omega_{123}(0) - \mu_C(N)/\hbar) F(K_1, 0) F(K_2, 0) \]
\[ \times (1 + F(K_3, 0)) |\xi_N(k)|^2 \]
\[ W^-(N + 1) = \frac{u^2}{(2\pi)^5 \hbar^2} \int d^3K_1 \int d^3K_2 \int d^3K_3 \int d^3k \delta(\Delta K) \delta(\Delta \omega_{123}(0) - \mu_C(N)/\hbar) (1 + F(K_1, 0)) \]
\[ \times (1 + F(K_2, 0)) F(K_3, 0) |\xi_N(k)|^2 \]

in which \( \xi_N(k) \) is the momentum-space ground-state wavefunction, obtained from the spatial form by
\[ \xi_N(k) = \frac{1}{(2\pi)^{3/2}} \int d^3r \ e^{-ik \cdot r} \xi_N(r) \]

In \([5]\), progress was made by assuming that the non-condensate band distribution \( F(K, r) \) was given by the classical Maxwell-Boltzmann distribution
\[ F(K, r) \approx \exp \left( -\frac{\hbar^2 K^2}{2m} + V_T(r) - \mu \right). \]

with values of \( T \) and \( \mu \) which ensure the formation of a condensate once the system reaches equilibrium. Furthermore, in calculating the integrals in equations (33) and (34), it was assumed that the range of condensate band energies was negligible compared to that of the non-condensate band. Thus the range of \( k \) was negligible compared to the range of \( K_{1,2,3} \) and the integrals in \( K_{1,2,3} \) were calculated over all energies rather than just over the non-condensate band. The function \( F(K, r) \) was also assumed to be negligible compared to unity. The result obtained was \([5]\).

\[ W^+(N) = \frac{4m(akT)^2}{\pi \hbar^3} \exp \left[ \frac{\mu_C(N)}{kT} - K_1 \left( \frac{\mu_C(N)}{kT} \right) \right]. \]

Here \( K_1(x) \) is a modified Bessel function. In almost all practical situations the term in square brackets in the above equation is approximately equal to unity, and so \( W^+ \) is essentially independent of \( N \). The value of \( W^-(N) \), the rate of transitions out of the condensate band, can be obtained in a similar fashion to that for \( W^+ \). The ratio of forward to backward rates is found to be given by
\[ W^+(N) = e^{(\mu - \mu_C(N))/kT} W^-(N), \]
which stems from the assumption of the thermal undepleted bath, and the definitions of \( W^+ \) and \( W^- \). From this equation and the master equation it can be seen that equilibrium is achieved when \( \mu_C(N) = \mu \) to order \( 1/N \).

3. Simple growth equation

A rate equation for the mean number of atoms in the condensate \( \langle N \rangle \) (written as \( N \) for convenience for the rest of this section) was obtained in \([5][1]\).
\[ \dot{N} = 2W^+(N) \left\{ 1 - e^{(\mu_C(N) - \mu)/k_B T} \right\} N + 1 \]  

This equation is the simple growth equation used for the simulations of condensate growth in [34]. If the system starts with \( N = 0 \), the growth begins slowly (but at a finite rate). Once a significant condensate occupation is attained the term proportional to \( N \) becomes dominant, causing a much faster growth rate. The growth eventually slows as \( \mu_C(N) \) approaches \( \mu \) and the system settles into equilibrium. This gives the curve an S-shape as will be shown in the following section.

By using the Thomas-Fermi chemical potential \( \mu_C(N) \), and the Maxwell-Boltzmann form for \( W^\pm(N) \), the first simulations for the growth of a realistic condensate were presented in [3]. The growth equation is simple to solve numerically, for whatever number of particles is necessary (for example, the growth of a condensate containing 5 million atoms was simulated in [5]).

4. Beyond the simple growth equation

The derivation of the simple growth equation contains a number of approximations and simplifications. The major behavior once \( N \) becomes large should be described quite well by the simple growth equation, but terms which were neglected may have significant effects during the initial stages of growth. Possibly significant factors which should first be considered are

i) The effect of considering all quasiparticle levels (the excited levels in the condensate band).

ii) The effect of scattering processes (as defined in section 1D).

iii) Corrections to the \( W^\pm(N) \) terms to consider the more realistic Bose-Einstein distribution function.

iv) The fluctuations around the mean number.

This paper aims to consider the effects of incorporating the first three of these factors into the growth equation. During the process of BEC formation, the spectrum of eigenvalues makes a transition from the unperturbed spectrum of trap levels to the case where the spectrum is strongly affected by the condensate in the ground state. The Bogoliubov spectrum of a condensed gas is valid in the case where the number of particles in the condensate, \( n_0 \), is so large that it is valid to write \( n_0 \approx N \). Thus, during the initial stages of condensate formation, where this is not true, one must use another formalism. In this paper we will consider the situation in which the interaction between the particles is very weak, as is in practice the case. This means that we will be able to use the unperturbed spectrum for the initial stages of condensation, and only use the Bogoliubov description once enough condensate has formed to make the effective interaction rather stronger.

The basic formalism of [11] can still be carried out in this case, and the modification that is found is rather minor—essentially, we make the substitution \( N \rightarrow n_0 \) in the chemical potential and the \( W^+(N) \), \( W^++(N) \) functions, and set \( W_{m}^{+} \rightarrow 0 \), since this term comes from the mixing of creation and annihilation operators which arises from the Bogoliubov method.

By making these adjustments, and now grouping the levels into sub-bands of energy \( \epsilon_m \) (measured now from zero, rather than from \( \mu_C(N) \) as was the case for \( \epsilon_m \)), with each sub-band containing \( g_m \) levels, the equations of motion for the growth processes are now

\[
\begin{align*}
\dot{n}_m|_{\text{growth}} &= 2W_{m}^{+}(n_0) \left\{ 1 - e^{\frac{\mu_C(n_0) - \mu}{k_B T}} \right\} n_m + g_m, \quad \text{(40)} \\
\dot{n}_0|_{\text{growth}} &= 2W^+(n_0) \left\{ 1 - e^{\frac{\mu_C(n_0) - \mu}{k_B T}} \right\} n_0 + 1. \quad \text{(41)}
\end{align*}
\]

We will make the further—possibly rather drastic—simplification, and neglect entirely the effect of phonon-like quasiparticles, which are known to comprise only a very small fraction of the levels normally occupied at the temperatures considered. Thus the excited states are now taken to be of a purely single-particle nature, and the condensate band is now described by the occupation number of the condensate level (the lowest energy level) \( n_0 \), and by the occupation numbers of each of the excited states \( n_m \). In this case, the equations \( (40) \) \( (41) \) become the same as the equations \( (18) \), \( (25) \), and may therefore be used to represent the full condensate growth process.

5. Evaluation of Transition Probabilities

The value for the transition probability \( W^+(N) \) found in the simple growth equation \( [35] \) was derived by making some rather sweeping assumptions, and as such equation \( (37) \) is really just an order of magnitude estimate. To obtain a more accurate value the full Bose-Einstein distribution must be used for \( F(K, r) \) and the ranges of integration of the non-condensate functions must exclude the condensate band in which \( F(K, r) \) would become very large. We then have from \( [33] \):

\[
W^+(n_0) = \frac{n^2}{(2\pi)^3} \int d^3r \int d^3K_1 \int d^3K_2 \int d^3K_3 \int d^3k \delta(D\Delta K) \delta(D\Delta K - \mu_C(n_0)/\hbar) F(K_1, r) \\
\times F(K_2, r) (1 + F(K_3, r)) V_{\xi_n}(r, k) \]

\( (42) \)
with
\[ F(K, r) = \left[ \exp \left( \frac{\hbar^2 K^2}{2m} + V_T(r) - \mu \right) \right]^{-1} \]
\[ = \sum_{s=1}^{\infty} \exp \left( -s \left[ \frac{\hbar^2 K^2}{2m} + V_T(r) - \mu \right] \right) . \]

Again we have to make the approximations:

i) That the spatial dependence can be neglected, so \[ F(K, r) \rightarrow F(K, 0) \] and \[ W_{\xi N}(r, k) \rightarrow |\xi_N(k)|^2 . \]

ii) That we can neglect the k dependence, except in \[ \xi_N \], so that the only k dependence left is removed by \[ \int d^3k |\xi_N(k)|^2 = 1 . \] The integrals over \( K_1, K_2 \) and \( K_3 \) can then be performed, to give a final form for \( W^+(n_0) \), found by Davis \[ 17 \], of

\[ W^+(n_0) = \frac{1}{2} \left( \frac{k_B T}{\hbar \omega} \right)^2 \left\{ \left[ \log(1 - z) \right]^2 + z^2 \sum_{r=1}^{\infty} [z(n_0)]^r \left[ \Phi(z, 1, r + 1) \right]^2 \right\} , \]

where

\[ z = e^{-\frac{E_{m} - E_{l}}{k_B T}} \quad z(n_0) = e^{-\frac{E_{m}(n_0) - E_{l}}{k_B T}} . \]

The function \( \Phi \) is the Lerch transcendent \[ 18 \], defined by

\[ \Phi(x, s, a) = \sum_{k=0}^{\infty} x^k / (a + k)^s . \]

This form of \( W^+(n_0) \) gives values of about a factor of three greater than the previous form in equation \[ 17 \], depending on the exact parameters of the system, and this gives a correspondingly faster growth than that in \[ 8 \].

The values for \( W_{+m}^+(n_0) \) are more difficult to obtain. The \( W_{+m}^+(n_0) \) terms are the average of the \( W^+ \) terms for all the individual levels in the sub-band. The \( W^+ \) terms are given by similar overlap integrals as used for the \( W^+ \) terms, and for the lower energy levels in the condensate band, the overlap of the wavefunction with the spatial distribution of a non-condensate band particle should be similar to that for the condensate level. Thus it is expected that the \( W^+ \) terms should be of the same order of magnitude as \( W^+(n_0) \). Progress can therefore be made by approximating \( W_{+m}^+(n_0) \approx W^+(n_0) \).

The effect and validity of this approximation will be investigated in the following section.

F. Scattering Processes

Scattering processes in this paper also need to be included in the evolution of \( n_m \). Scattering between two atoms in the non-condensate band does not have to be explicitly considered, since it has been dealt with in making the assumption that the non-condensate band is an equilibrated time-independent thermal bath. Furthermore, the scattering between two condensate band atoms will be neglected since, at any time, the number of atoms in the condensate band is small relative to the number in the non-condensate band. The dominant scattering processes, and the only ones which will be considered, are the scattering of atoms between levels in the condensate band, due to interactions with non-condensate band atoms (see Fig. 5). These stem from terms in the full master equation which involve two condensate field operators \( \phi \). These terms give rise to a master equation of the form, as shown in QKIII equation \( 50d \):

\[ \dot{\rho}_{\text{scatt}} = \sum_{m \neq \pm m} \gamma_{km} \left[ 2X_{km} \rho X_{km}^\dagger - \left[ X_{km}^\dagger X_{km}, \rho \right]_+ \right] + \sum_{m \neq \pm m} \gamma_{km} \left[ N_{km}, \rho \right]_+ \]

\[ + \sum_{m \neq \pm m} \gamma_{km} \left( 2X_{km} \rho X_{km} - \left[ X_{km}^\dagger X_{km}, \rho \right]_+ \right) \]

which is equivalent to the master equation governing the scattering of particles by a heat bath \[ 14 \].

Here the operators are defined by

\[ X_{km} \equiv a_{m}^\dagger a_{k} \]

where \( a_k \) is the destruction operator for an atom in state \( k \) with energy \( \epsilon_k \). As in the previous section, we treat all excitations as being particle-like. The rates of the processes are determined by the factors \( \gamma_{km} \), and the factors \( N_{km} \) are defined by

\[ N_{km} \equiv \frac{1}{\exp \left( \frac{\epsilon_{km} - E_{l}}{k_B T} \right) - 1} . \]

The last line in the master equation \[ 50d \] represents scattering between degenerate energy levels, which will not have any contribution to the time dependence of \( n_m \), once the levels are grouped into sub-bands, and so can be ignored.

The corresponding rate equation for \( n_k = \langle a_k^\dagger a_k \rangle \), the mean occupation of the \( k \)th level, can easily be found from the master equation. When levels are grouped into sub-bands with mean energy \( \epsilon_k \), occupation \( n_k \), and with \( g_k \) levels contained in the sub-bands, it becomes

\[ \dot{n}_k = \sum_{m \neq \pm m} \gamma_{km} \left[ \tilde{N}_{km} n_m (n_k + g_k) - (\tilde{N}_{km} + 1)(n_m + g_m) n_k \right] + \sum_{l \neq \pm l} \gamma_{lk} \left[ (\tilde{N}_{lk} + 1)n_l (n_k + g_k) - \tilde{N}_{lk} (n_l + g_l) n_k \right] . \]

The transition rates \( \gamma_{km} \) now represent averages over all the individual level transition rates which transfer an atom from the \( k \) sub-band to the \( m \) sub-band. The transition probabilities \( \gamma_{km} \) can be found in a similar manner to the \( W^+(n_0) \) terms, from terms of the form \[ 14 \].
\[ R_{km}(N) = \frac{4u^2}{(2\pi)^5} \int d^3r \int d^3K_1 \int d^3K_2 \int d^3k \int d^3k' \delta(K_1 - K_2 - k + k') P(K_1, r) (1 + P(K_2, r)) \times W_k(N, r, k) W_m(N, r, k') \delta(\Delta \omega_{12}(r) - \Omega_m + \Omega_k), \tag{52} \]

where
\[ \Omega_m(N) = \frac{\mu C(N) + \epsilon_m(N)}{\hbar} \tag{53} \]
\[ \epsilon_m(N) = \frac{\epsilon_m(N)}{\hbar} \tag{54} \]
and the rest of the notation is as was used in the previous section.

1. Estimates for \( \gamma_{km} \)

Explicit computation involved in calculating these factors is impractical, and, it will turn out, unnecessary when the scattering is sufficiently strong. We shall instead estimate these \( \gamma_{km} \) rates by using the quantum Boltzmann approach of Holland et al. \[20]\.

By treating the excitation spectrum as given by the eigenstates of the trapping potential, without modification by the presence of the condensate, and by using the ergodic assumption Holland et al. obtained the kinetic equation (equation (12) in \[20]\)
\[ \frac{\partial n_m}{\partial T}_{\text{scatt}} = \sum_q (1 + f_n) f_q \left( \sum_{mp} \delta(n, m; p, q) g_{emf} f_p (1 + f_m) \right) - (1 + f_q) f_n \left( \sum_{mp} \delta(n, m; p, q) g_{emf} f_m (1 + f_p) \right) \tag{56} \]
where the following notation has been used
\[ n_m = f_m g_m \tag{57} \]
\[ \delta(n, m; p, q) \equiv \delta_{e_n + e_m + e_p + e_q}, \tag{58} \]
The terms of this equation can be simplified for the different possible cases.

First Line, case \( e_n > e_q \): In this case, \( e_q = e_{\text{min}} \), and energy conservation is satisfied when
\[ e_p = e_m + \hbar \omega_{nq} \tag{59} \]
where \( \hbar \omega_{nq} = e_n - e_q \). The summation term in the first line of equation (56) then becomes
\[ g_q \sum_{e_m > E_R} (1 + f_m) f_{m + \omega_{nq}} \approx g_q e^{(\mu - \hbar \omega_{nq})/kT} \Gamma(T) \tag{60} \]
where \( \Gamma(T) \equiv \sum_{e_m > E_R} e^{-e_m/kT} \tag{61} \)
The approximation which has been made is that \( (1 + f_m) \approx 1 \), which should be acceptable since state \( m \) is of high energy (ie: in the non-condensate band). The calculation of \( \Gamma(T) \) requires knowledge of the spectrum of energies in the non-condensate band, which is complicated for an anisotropic trap. The form for an isotropic harmonic potential \( \Gamma(T) \) is easily calculated though, and gives
\[ \Gamma(T) = \frac{e^{-E_R/kT}}{1 - e^{-\hbar \omega/kT}} \tag{62} \]
where \( \omega \) is the frequency of the potential. We will therefore make the approximation that \( \Gamma(T) \approx \Gamma(T) \) for the determination of \( \gamma_{km} \), using the geometrical mean frequency of the real trap as the frequency of the isotropic potential. Thus \( \omega = (\omega_1 \omega_2 \omega_3)^{1/3} \) in equation (62).

Remaining terms: Similar reasoning leads to the results
\[ gn \sum_{e_p > E_R} (1 + f_{p + \hbar \omega_{nq}}) f_p \approx gn \sum_{e_p > E_R} e^{(\mu - e_p)/kT} \tag{63} \]
\[ = gn e^{\mu/kT} \Gamma(T) \tag{64} \]
\[ g_q \sum_{e_m > E_R} f_m (1 + f_{m + \hbar \omega_{eq}}) \approx g_q e^{\mu/kT} \Gamma(T, R) \]  
\[ g_n \sum_{e_p > E_R} f_p (1 + f_p) \approx g_n e^{(\mu - \hbar \omega_{eq})/kT} \Gamma(T) \]

2. Total Scattering Equation:

The total kinetic equation governing the scattering processes is now given by

\[ \tilde{n}_{m,\text{scatt}} = \frac{8ma^2 \omega^2}{\pi \hbar} e^{\mu/kB T} \Gamma(T) \times \]
\[ \left\{ \sum_{k < m} \frac{1}{g_m} \left[ n_k (g_m + n_m) e^{-\hbar \omega_{mk}/kB T} - n_m (g_k + n_k) \right] \right. \]
\[ + \sum_{k > m} \frac{1}{g_k} \left[ n_k (g_m + n_m) - n_m (g_k + n_k) e^{-\hbar \omega_{km}/kB T} \right] \].

\[ (67) \]

The notation \( k > m \) is now being used to mean \( e_k > e_m \). This is the rate equation governing scattering processes, it is equivalent to equation (51) if the following transformations are made

\[ \tilde{N}(\omega_{eq}) \rightarrow e^{-\hbar \omega_{eq}/kT} \]
\[ 1 + \tilde{N}(\omega_{eq}) \rightarrow 1 \]
\[ \gamma_{nq} \rightarrow \frac{8ma^2 \omega^2 e^{\mu/kT} \Gamma(T)}{\pi \hbar} \frac{g_n}{n} \quad \text{when } n > q \]
\[ \gamma_{nq} \rightarrow \frac{8ma^2 \omega^2 e^{\mu/kT} \Gamma(T)}{\pi \hbar} \frac{g_q}{n} \quad \text{when } n < q \]

G. Rate equations including scattering and growth

The total rate equation governing the evolution of this system is then given by adding equation (60) to equation (57)

\[ \dot{n}_m = \dot{n}_m |_{\text{growth}} + \dot{n}_m |_{\text{scatt}} \]

and for the condensate level evolution (41) is used in place of equation (40).

It is useful here to review the major approximations that have been made in order to derive these equations. It was assumed that

i) The non-condensate band is very large, so it is essentially undepleted, and it is in equilibrium.

ii) The influence of collective excitations is negligibly small, so that the states in the condensate band are all of single-particle nature.

iii) The density of states in the system is as described in section II C.

iv) The ergodic approximation is valid, and that states in the condensate band which have similar energies may be ‘binned’ together for the purpose of describing their evolution.

v) The fluctuations of the occupation numbers around the mean numbers may be ignored.

vi) The rate constants \( W^{\pm} \) for the growth processes which change the occupations of the excited states in the condensate band are equal to the rate constant for the growth of the condensate level.

vii) The rate constant for the scattering processes in an anisotropic well of geometrical mean frequency \( \omega \) is equal to that in an isotropic well with frequency \( \omega \).

viii) Three body collisional processes may be ignored.

III. NUMERICAL SOLUTIONS OF THE GROWTH

The rate equations derived to describe the growth of a condensate in the previous section are quite straightforward to solve numerically, and the solutions can be obtained in a matter of a few seconds as opposed to other numerical solutions which have been very time consuming. The nature of these solutions will be discussed and comparisons will be made with experimental data published in [4]. The parameters of the model used here should be the same as in the MIT growth experiments, that is:

i) Using a dilute gas of \( ^{23}\text{Na} \) atoms, characterised by an s-wave scattering length of \( a = 2.75\text{nm} \).

ii) With an axially symmetric (‘cigar shaped’) harmonic trapping potential described by the frequencies \( \omega_x = \omega_y = 2\pi \times 82.3\text{Hz} \) and \( \omega_z = 2\pi \times 18\text{Hz} \), giving a geometrical mean frequency of \( \omega = 2\pi \times 50\text{Hz} \).

iii) With a total number of atoms of the order of \( 10^7 \). The numbers found in the condensate level once equilibrium is reached are in the range \( 5 \times 10^5 \) to \( 1 \times 10^7 \), giving a condensate occupation of between 5 and 30% of the total number of atoms [4]. In the majority of cases the thermal bath was therefore depleted by only a small amount and the undepleted model which is used here should be a good approximation.

iv) With temperatures in the range \( 0.5\mu\text{K} \) to \( 1.5\mu\text{K} \).

A. Results

1. Simple Growth Equation Results

Numerical solutions of the simple growth equation

\[ \tilde{n}_0 = 2W^+(n_0) \left\{ 1 - e^{(\mu (n_0) - \mu)/kB T} \right\} n_0 + 1 \].

(73)

are easily obtained [3], and an example of the resulting growth curve is shown in Fig. 3. This curve shows a characteristic S-shape, the slow initial growth occurs as a result of spontaneous \((+1)\) terms in equation (73) and then, once the occupation becomes large enough, the stimulated growth terms...
levels to come to equilibrium. It is therefore unphysical to consider a situation where the low energy levels have reached their equilibrium populations before the condensate level has even started to evolve. For these reasons, it is not consistent to simply use the Bose-Einstein distribution to find $W^+$ in the model considered by the simple growth equation.

In order to develop a consistent description there need to be a number of lower energy levels with time-dependent populations considered, forming the condensate band. Furthermore, the energies of these levels will be increased by the growth of the condensate, due to increased mean field interactions. To describe this situation the model described in the previous section must be implemented. Considering at first only the ‘growth’ processes, the evolution was found in section II E 3 to be given by

\[
\dot{n}_m|_{\text{growth}} = 2W^+_m(n_0) \left\{ 1 - e^{-\frac{\mu_C(n_0)}{kT}} \right\} n_m + g_m \}, \quad (74)
\]

\[
\dot{n}_0|_{\text{growth}} = 2W^+(n_0) \left\{ 1 - e^{-\frac{\mu_C(n_0)}{kT}} \right\} n_0 + 1 \}, \quad (75)
\]

where, for reasons given in the previous section, we approximate $W^{+\,m}_m(n_0)$ by $W^+(n_0)$. This form of evolution is essentially the simple growth equation applied to several energy levels.

It is now possible to perform a more accurate calculation of $W^+$, using the Bose-Einstein distribution to describe the population of levels above $E_R$ (i.e.: in the non-condensate band), and summing only over the non-condensate band levels. The new form of $W^+$ is given by equation (43).

Sample solutions to the coupled differential equations (74) and (75) are shown in Fig.6. The rate of growth of the condensate has increased substantially (generally by at least a factor of 3) over that predicted by the simple growth equation, this is due to the more accurate calculation of $W^+(n_0)$. However, the shape of the growth is still essentially the same as that given by the simple growth equation.

It can be seen from Fig.6 that the lower energy levels also experience very substantial growth in this model. Indeed the occupations of some of these levels can exceed the condensate occupation substantially before relaxing back to their equilibrium values. This is of course not a realistic scenario, and is certainly not one that has been observed in any experiments. Notice that the number of sub-bands used in this figure (and most of the other figures presented) is substantially fewer than would normally be used. Generally the number of sub-bands required is about 20–50 depending on the exact parameters, however these cannot be well distinguished from each other in a graph. Therefore, throughout this paper, most of the depictions of the growth of sub-bands will show only a few of them.
FIG. 7. Typical results of the new growth equations for the growth of a Bose-Einstein Condensate using the same parameters as in Fig.6. The condensate level (bold) reaches the equilibrium population of $5 \times 10^6$ atoms, the other lines represent the evolution of the populations of the other sub-bands in the condensate band. Note the much faster growth than in Fig.6.

Once scattering processes are included (see below), reducing the number of sub-bands used causes the growth to become slower. It also causes the model to become less realistic, since an individual level may then be described by an average energy quite different from its actual energy. If the number of sub-bands is increased, an asymptotic limit to the speed of the growth is reached, however this is also unrealistic since now some sub-bands contain only fractions of individual energy levels. The choice used in practice is such that in their final equilibrium states the sub-bands have widths of $\bar{\hbar}\omega$. This choice is close to the asymptotic limit, and ensures at least three individual levels are contained in the first sub-band.

3. Inclusion of Scattering Processes

If we include scattering processes, as given by (67), the picture is dramatically changed. Solutions for the resulting evolution equations (72) are shown in Fig.8. This figure shows that the scattering has two main effects. Firstly, the initiation of the condensate level growth occurs much more sharply, this gives a substantial change to the shape of the growth, which has now lost much of the S-shape nature that previous solutions had shown. The speed of the growth after the initiation is changed little by the inclusion of the scattering processes, since in this region growth is completely dominated by the growth processes. The second effect is that the populations of the excited states no longer exceed plausible levels.

The reasons behind these changes are interlinked. Without scattering, all of the levels in the condensate band start to grow, and at quite similar rates. The difference in the growth rate between the very lower energy states and that of the condensate level is particularly small. Because of the degeneracy of states in the lower sub-bands a large population can form in them, which can become very much larger than the condensate level population. Once any one sub-band acquires a sufficient population the stimulated term in the growth process begins to dominate and the population increases even further. In the absence of scattering processes, the only way in which the excess population in these states can be transferred to the condensate level, where it will be found in the equilibrium situation, is by a transfer back to the non-condensate band followed by another collision which transfers it directly to the condensate level.

If scattering processes are considered, atoms may now be transferred directly between different levels in the condensate band in a collision. Any excess population in the excited states can then be quickly transferred out of the state before the stimulated growth process becomes too dominant.
With the inclusion of the scattering processes, the effects of two important approximations must be considered. In the derivation of the equations governing the scattering processes it was assumed that the value of \( \Gamma(T) \equiv \sum_{E_n > E_B} e^{-E_n/kT} \) was equivalent to that for an isotropic harmonic oscillator \( \bar{\Gamma}(T) \), which affects the scattering rates.

The second important approximation was made in the solution of the equations governing the growth processes, where the \( W_{m+}^{++}(n_0) \) terms were assumed to be equal to \( W^+(n_0) \) which has a value given by equation (45). This approximation has no effect on the growth rate of the condensate level if the scattering terms are not considered, since then there are no interactions between different levels in the condensate band. The effects of these two approximations are discussed below.

The magnitude of the scattering rate was assumed to be equal to that for an isotropic harmonic oscillator in the derivation of equation (67). In Fig. 9 the growth of the condensate level is shown for several different scattering rates. The growth slows slightly if the rate is decreased. If the rate is increased, the growth becomes faster until it reaches an asymptotic limit at which point it is the rate of the growth processes which determines the speed of growth. The results show that the overall growth changes by only a relatively small amount (and certainly smaller than present experimental uncertainties in growth experiments) provided that the rate is within two orders of magnitude of that for the isotropic trap. Since it seems unlikely that the corrections due to the anisotropy would change the rate by much more than one order of magnitude, the approximation of using the isotropic rate factor seems to be valid. It is interesting to note that a scattering rate of only about 1% of the isotropic case is usually sufficient to prevent the occupations of the sub-bands becoming very large as they do in the absence of any scattering.

The effect of the approximation \( W_{m+}^{++}(n_0) \approx W^+(n_0) \) is shown in Fig. 10 which shows the behaviour of the growth for different values of the \( W_{m+}^{++}(n_0) \) terms. Changing \( W_{m+}^{++}(n_0) \) changes the rate of growth of the excited states, and thus changes the probability of atoms being scattered from the excited states into the condensate level, giving a corresponding change in the overall growth of the condensate. The results show that provided \( W_{m+}^{++}(n_0) \) lies in the range 2\( W^+(n_0) > W_{m+}^{++}(n_0) > W^+(n_0)/2 \) then the growth rate does not change significantly (compared to experimental uncertainties and the change caused by using an accurate scattering rate). However, outside of this range the growth is altered considerably. The expectation is that \( W_{m+}^{++}(n_0) \) will lie in the desired range, since it is an average over quantities similar to \( W^+(n_0) \), and as such should be of the same magnitude.

The ‘standard’ approximations for the rate constants that will be used in the following results will be \( W_{m+}^{++}(n_0) = W^+(n_0) \) and \( \Gamma(T) = \bar{\Gamma}(T) \).

### 6. Initial Conditions

A problem which has to be considered in solving the total rate equations (22) is the determination of the correct initial conditions. Because the non-condensate band is assumed to be a thermal bath of atoms at equilibrium, the initial populations for the explicitly considered levels in this band are found from \( \exp((E-\mu)/kT) \). Obviously this cannot be used to give the initial populations in the condensate band when the model is attempting to describe growth of the condensate.
is not immediately obvious what the appropriate initial conditions should be for the condensate band. In Fig. 11 four different initial conditions are shown:

a) no initial population in the condensate band. This is the most artificial of the four possibilities presented.

b) initial population of zero in the condensate level and excited state occupations given by a linear dependence on energy rising to match that of the non-condensate band at $E_R$.

c) initial populations of condensate band states all equal to the value of $[\exp((E - \mu)/kT) - 1]^{-1}$.

d) initial populations given by a linear extrapolation of the non-condensate populations, meeting $[\exp((E - \mu)/kT) - 1]^{-1}$ at $E_R$ tangentially.

In Fig. 12 the growth curves corresponding to each of the differing initial conditions are shown. The different initial conditions can be seen to have little effect on the shape of the growth, the main effect is really just a small shift in the initiation time. This effect is generally quite small compared to the effects of changing $\Gamma(T)$ and $W_{m+n^+}(n_0)$ as seen in the preceding sections. The fact that the initial conditions can be changed by so much and yet have little effect on the growth curve is due mainly to the inclusion of the scattering terms. The scattering terms very quickly cause the levels in the condensate band to come to a kind of quasi-equilibrium from whatever initial state they are put in. Thus we conclude that an exact knowledge of the initial conditions is not important.

Since the exact initial conditions do not seem to be important, in all further calculations initial condition (c) will be used, as it is about mid-way between the extremes of cases (a) and (d).

The previous results were all obtained by taking the mean field effects due to the condensate into account in the manner described in the previous section. That is to evenly distribute the energies of the sub-bands between the fixed upper limit of $E_\Delta$ and the $n_0$-dependent lower limit $\mu_C(n_0)$. Thus all levels below $E_\Delta$ (fixed at $2\mu_C(n_0)$) where $n_0,f$ is the final occupation of the condensate level) are modified at all times. This artificial model can be improved on, since the number of levels affected by the condensate depends upon its occupation. In other words, when there is only a small condensate present it has a significant effect on only the lower sub-bands, while the upper sub-bands are essentially unmodified. An improved (although still a little artificial) method is the following: all levels below $\gamma\mu_C(n_0)$ (where $\gamma$ is an arbitrary parameter) will be compressed to fit between this upper limit and the lower limit of $\mu_C(n_0)$; the energies of all levels higher than $\gamma\mu_C(n_0)$ will not be altered. This is similar to the previous model, with the alteration that the upper limit of the levels whose energies are changed is no longer fixed, but instead rises with increasing $n_0$. The energy of a level $e_m$ is then given by

$$
e_m = \begin{cases} e_m^0 & \text{for } e_m^0 > \gamma\mu_C(n_0) \\ \mu_C(n_0) + e_m^0 \left(1 - \frac{1}{\gamma}\right) & \text{for } e_m^0 < \gamma\mu_C(n_0) \end{cases}$$

(76)

where $e_m^0$ are the unchanged energy levels, as given by the eigenstates for a non-interacting gas in a harmonic potential. The energy spectrum of the sub-bands given by this new model is shown in Fig. 13, and compared to the previous energy spectrum. In the older model used so far in this paper all energy levels in the condensate band are changed by the growth of the condensate. However, the extent of the condensate band is determined by the final occupation number of the condensate (i.e., by the final value of $\mu_C(n_0)$). In the new model, the levels are modified in a more consistent fashion, the energy levels for any given condensate population are

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**FIG. 11.** Depiction of the four different initial populations trialed, as described in the text.

**FIG. 12.** Growth curves determined using the initial conditions described in the text. The curves, from slowest to fastest growth, were given by initial conditions a-d respectively. Conditions are the same as for Fig. 8.

7. Modifications of the Energies of the Sub-bands
determined by the condensate population at that time, which appears to be a more logical approach.

In section II.C it was shown that a fair estimate for $E_R$ was a value of $2\mu_C(n_{0,f})$. The value for $\gamma$ will normally be taken to be 2, so that at equilibrium all energy levels below $E_R$ will be modified.

![Energy spectrum](image)

**FIG. 13.** The energy spectrum of the sub-bands as a function of $n_0$, modified to account for the mean field interactions with the growing condensate. In (a) the levels are modified by changing all levels in the condensate band, as has been used to obtain the previous results. In (b) these effects are accounted for by the new model, explained in the text. The lowest energy level is the condensate level, whose energy is $\mu_C(n_0)$. The dotted line represents $2\mu_C(n_0)$, the maximum energy at which mean field corrections are assumed to be significant. This figure shows the evolution of the energies for a temperature of 900nK and with a final (equilibrium) condensate occupation of $5 \times 10^3$. The few levels whose energies never change are those levels in the non-condensate band (above $E_R$) which are explicitly considered.

The effect of this new model, compared to the earlier approach, on the overall growth is very small, smaller in fact than any of the other effects discussed in this section, and it is barely discernible. This would seem to indicate that it is the mean-field effects on the energies of the lower few sub-bands which are important when considering the growth of the condensate, since in both the old and new models the lower levels experience quite similar changes. Since the mean-field effects of the condensate on the higher levels seems to have only a very small effect, the precise value of $E_R$ would appear to have little effect so long as it is reasonably high. The value of $2\mu_C(n_{0,f})$ seems to be a good value, since it is high enough that the energy perturbations of higher energies do not have a large effect on the growth, and it is low enough that the majority of atoms are found at higher energies, giving an undepleted thermal bath.

**B. Comparison of Results with Experiment**

At the present point in time there has only been one published experiment which has investigated in any detail the growth of a Bose-Einstein condensate. The experiment was performed at MIT using a trapped gas of $^{23}$Na atoms, and the results were published in 1998 in ref [4].

**1. MIT Experimental Method**

The MIT experiments were performed in the following way: The gas of atoms was confined in an approximately harmonic magnetic trap. It was then cooled using laser cooling and evaporative cooling techniques to a temperature slightly higher than the critical temperature necessary for the formation of BEC. At this point the system is essentially in thermodynamic equilibrium, and then it is suddenly put into a non-equilibrium configuration of lower energy, by means of a rapid evaporative cooling ‘cut’ which removes all atoms in states above a certain energy in a time of about 10ms. The system is then left to relax to equilibrium with no further cooling.

This method attempts to achieve, probably as closely as is realistically possible, an almost thermalised bath in contact with a condensate band, as has been assumed in our theoretical treatment. The cut which removes the higher energy atoms causes the wings of the energy distribution to be truncated. Experience with solutions of the quantum Boltzmann equation and related methods [28,20,10] shows that the effect of this will firstly be felt by the higher energy atoms remaining. The higher levels will therefore be expected to thermalise more quickly, with thermalisation gradually moving through to the lower energies. Thus at some point after the cut the majority of the atoms will be approximately in equilibrium, with the lower energy atoms still in very non-equilibrium states.

**2. MIT Experimental Results and Theoretical Comparisons**

Phase-contrast microscopy produces two dimensional images of the system with an intensity proportional to the column density of the system. From data of this type temperatures, total numbers of atoms, and condensate level occupations were extracted by the MIT group and were presented in [4].

The results found in [4] were that condensate growth took on the order of 100-200ms depending on the exact conditions. It was found that the growth could be well fitted by a solution of the simple growth equation (73), which can be put in the approximate (but in practice very accurate) form

$$\dot{n}_0 = \kappa_1 n_0 [1 - (n_0/n_{0,f})^{2/5}],$$  \hspace{1cm} (77)

where, again, $n_{0,f}$ is the equilibrium condensate population. The solutions to this equation exhibit the S-shaped growth profile of Fig [4].

The conclusion drawn in [4] was that a curve of this shape was evidence for the importance of bosonic stimulation in the growth processes, since a purely relaxational process would be described by solutions of

$$\dot{n}_0 = \kappa_2 (n_{0,f} - n_0).$$  \hspace{1cm} (78)
However, the rate constants $\kappa_1$ found by the MIT group by fitting to the data obtained did not agree to better than an order of magnitude with the predictions of the simple growth equation of

$$\kappa_1 = 2W^+(n_0)\frac{H}{kT}.$$ (79)

The simple growth equation did well to give the correct order of magnitude rates, however most of the predicted rates for higher temperatures seemed to be too small by about factor of three. As the temperature decreased the discrepancy increased, the predicted rates became slower, whilst the experimentally fitted rate constants became larger.

In the growth experiments statistical uncertainties are estimated [29] to be 10% for relative number measurements, 15% for temperatures. Systematic uncertainties are estimated as 20% in absolute number measurements, and 8% for temperature measurements. Condensate occupations of less than $10^5$ atoms could not be discerned against the background of the thermal vapour cloud [4].

As Fig.14 shows, the growth predicted by our model is quite a good fit to the data, and the order of magnitude is certainly predicted well. This is a substantial improvement to the growth predicted by the simple growth equation at the same conditions which gives growth over about 1.5 seconds, as opposed to the experimental results of about 0.15s. At the stated parameters the theoretical fit could still be improved. The dashed line in Fig.14 indicates that by adjusting the scattering rate and values of $W^{++}_m$ (as was discussed in previous sections) a better fit may be obtained.

In contrast, the solutions to the growth model presented in this paper no longer show the S-shape, but are in fact closer in shape to the solutions of equation (78). These curves were found by MIT to describe the data quite poorly if the growth started at time $t = 0$ (the time at which the cut finished). However, if an initiation time was allowed before the growth began, such solutions became quite close fits, although they still did not describe the initially slow growth giving the S-shape to the growth. The results obtained in this paper show that, while the general shape is that of solutions to equation (78), there is also an initiation time present before the growth begins.

As a specific comparison with experiment, Fig.14 shows the comparison between the growth curve predicted here and the experimental data. The data is similar to that in [4], but was not actually published. It was provided by MIT as being the growth of a condensate with an equilibrium population of about $1 \times 10^5$ atoms, at a temperature of 1200nK [22].

The temperature plays a sensitive role in these comparisons, since the rate of growth is quite sensitive to temperature. Furthermore, as will be seen in the following sections, the fitting method may play a significant role. In Fig.15 the theoretical curve is plotted (using the standard values of the scattering rate and $W^{++}_m$) for two lower temperatures. It can be seen that the fit is very good for the 850nK results, and not quite so good, but still quite close, in the case of the 1000nK curve.

This highlights a difficulty in comparing theoretical predictions with experimental measurements of condensate growth. The spatial density distribution of the thermal cloud (from which the temperature is measured) changes only slightly with temperature, whereas the growth rate is quite strongly dependent on the temperature. There are other problems as well, the most prominent of these being the difficulty of experimentally determining the condensate occupation. The spatial distributions of particles in the first few excited states are quite narrow...
functions, and they can overlap the condensate level distribution significantly. It then becomes extremely difficult to distinguish condensate level atoms from low energy excited state atoms in experimental measurement. If the measured ‘condensate occupation’ was in fact the occupation of the lowest 5 levels (for example) this would alter the growth curve in some important regions. The main effect would be that the growth would appear more gradual during the early times, and would not seem to have such a sharp initiation. This is a possible explanation for the S-shape that was found in the MIT data. Clearly what is needed are theoretical predictions for the overall spatial density distributions during the growth, rather than merely occupation numbers for the various states. This will be addressed in the following section.

![Graph](image)

**FIG. 16.** Comparison of theoretical growth (thin lines) with curves fitted by MIT to experimental data in [4] (broad lines). The MIT curve fits are represented as broad lines to indicate that they are fitted curves with unknown (but probably substantial) uncertainties. a) Theory $T = 830\text{nK}$, $n_{0,f} = 7.6 \times 10^6$; Experiment $T = 810 - 890\text{nK}$, $n_{0,f} = 7.5 - 7.85 \times 10^6$; b) Theory $T = 590\text{nK}$, $n_{0,f} = 2.3 \times 10^6$; Experiment $T = 580 - 610\text{nK}$, $n_{0,f} = 2.1 - 2.5 \times 10^6$. The initial populations were treated as free parameters to best match the initiation of the growth with the MIT curves.

The majority of the data contained in [4] was presented in the form of rate constants for fits to experimental data of the type described by equation (77). Fig 16 shows the comparison of our theoretical growth curves as compared to solutions of equation (77) using a selection of the MIT fitted constants $\kappa_1$. The figure shows good overall agreement with the experimental data. The agreement is better at higher temperatures. At the lower temperatures reached in the experiment the agreement is less good.

However, the results of this paper still show an overall decrease in the rate of growth with decreasing temperature. This is the opposite trend to that experimentally observed. This could partly be due to uncertainties in extracting numbers from the experimental data. Another possible explanation is that in order to cool to such low temperatures from just above the critical temperature in 10ms a large proportion of the atoms must be removed. This will give rise to a system in a highly non-equilibrium state, the relaxation from which may be inadequately described by our model.

### IV. SPATIAL DENSITY DISTRIBUTION OF A CONDENSATE SYSTEM

In the MIT experiments into condensate growth [4], the data is collected by phase-contrast microscopy [4-25], the measurement of change in the phase of light after it has passed through the vapour cloud. The result is a two dimensional plot of the column density integrated along the third dimension. As discussed in the previous section, the extraction of the population per energy level from this experimental data is complicated. Therefore, in order to more easily enable comparison with experiment, it is desirable to obtain from our theoretical results predictions of the spatial distribution of the condensate as it grows. When the system is in equilibrium this distribution is well known, but this is not the case during the condensate growth that we are interested in.

#### A. Semiclassical Phase-space Description

In order to convert the results of the model into spatial distributions, the spatial probability distributions for each energy band need to be found. The distribution for the ground state $\rho_{g_0}(r)$ is well described by the Thomas-Fermi approximation for the wave function:

$$\rho_{g_0}(r) = |\psi_{T-F}(r)|^2 = \frac{n}{4\pi a^2} \left| \mu_C(n_0) - V_T(r) \right| \theta \left( \mu_C(n_0) - V_T(r) \right), \quad (80)$$

where $V_T(r)$ is the trapping potential, and $\theta(x)$ is the step function. This is a very good approximation to the shape of the condensate when $n_0$ is large, failing only at the very edge of the condensate where the numerical solution to the Gross-Pitaevskii equation vanishes smoothly.

A description of the spatial distributions for each of the higher energy sub-bands can be found by using a semiclassical phase-space approach. The cumulative number of states below an energy $E$ is given by

$$N(E) = \frac{1}{\hbar^3} \int d^3r \int d^3P \theta \left( E - \frac{P^2}{2m} - V_{\text{eff}}^{NC}(r) \right), \quad (81)$$

where $V_{\text{eff}}^{NC}(r)$ is the effective potential experienced by an atom. This gives a density of states

$$g(E) = \frac{dN(E)}{dE} = \frac{1}{\hbar^3} \int d^3r \int d^3P \delta \left( E - \frac{P^2}{2m} - V_{\text{eff}}^{NC}(r) \right). \quad (82)$$

Using the local density approximation, for an energy sub-band with energy $e_m$ and width $\Delta e_m$, the average spatial distribution atoms in the band (averaged over all the wavefunctions of all the states in the band) may be obtained by removing the $d^3r$ integral and integrating over $d^3P$. This gives
\[
\rho_m(r) = \frac{1}{\hbar^3} \int d^3p \delta \left( \epsilon_m - \frac{p^2}{2m} - V_{\text{eff}}^{\text{NC}}(r) \right) \Delta \epsilon_m \quad (83)
\]
\[
= \left( 4\pi^{1/2} m^{3/2} / \hbar^3 \right) \Delta \epsilon_m \sqrt{\epsilon_m - V_{\text{eff}}^{\text{NC}}(r)}. \quad (84)
\]

![Graph showing density distribution](image)

**FIG. 17.** Sample radial spatial density distribution for condensate having reached thermal equilibrium. a) Solid—total density distribution, dashed—density distribution due to excited states only. b) Density times radius squared. These results are for the trap parameters used in the MIT growth experiments [4,22].

1. Resulting Spatial Density Distribution

We will use the semi-classical distributions for the excited states, and the Thomas-Fermi wavefunction for the condensate. The total radial probability distribution may be calculated by normalising each level, and then summing over all the levels. The distributions are normalised so as to give the appropriate population in each level. For the bands below \( E_{\text{li}} \) these populations are given by the numerical results of our model. For those above \( E_{\text{li}} \) the population is determined by the Bose-Einstein distribution function 
\[
F(E) = \exp\left(\frac{(E - \mu)}{kT}\right) - 1
\]
This gives the spatial density distribution for the whole condensate system in three dimensions.

An example of the resulting spatial density distribution for a system at equilibrium is shown as Fig. [7]. In the first part of this figure the total density is plotted as a function of radius, as well as the density due to the excited states only. The density due to the excited states can be seen to be significantly decreased in the region of the condensate. In Fig. [7] the density multiplied by the radius squared is plotted, this is proportional to the total number of atoms found at any radius (due to the three dimensional nature of the distribution). From this it can be seen that, even though the density in the centre due to the condensate is much larger than elsewhere, the majority of the atoms are still found in the surrounding vapour cloud (as was assumed in the derivation of the model).

When the system is in equilibrium these results can be checked. At equilibrium the non-condensate spatial distribution can be obtained directly from equation (82). The total number of non-condensed atoms, \( M \) is
\[
M = \int_0^\infty g(\epsilon) \frac{1}{e^{(\epsilon - \mu)/kT} - 1} d\epsilon \quad (85)
\]
which corresponds to the local density form of \( \rho(r) \)
\[
\rho(r) = \frac{1}{\hbar^3} \int d\epsilon \int d^3p \delta \left( E - \frac{p^2}{2m} - V_{\text{eff}}^{\text{NC}}(r) \right) \frac{e^{(\epsilon - \mu)/kT} - 1}{\epsilon^{(\epsilon - \mu)/kT} - 1} \quad (86)
\]
\[
= \frac{1}{\hbar^3} \int d^3p \left\{ \exp \left[ \frac{\rho^2}{\epsilon_m} + V_{\text{eff}}^{\text{NC}}(r) - \mu \right] \right\}^{-1} \quad (87)
\]
Carrying out the integral over momentum space gives
\[
\rho(r) = \frac{mkT}{2\pi\hbar^2} G_{3/2} \left( \frac{V_{\text{eff}}^{\text{NC}}(r) - \mu}{kT} \right) \quad (88)
\]
where \( G_{\sigma}(z) \equiv \sum_{n=1}^\infty n^{-\sigma} e^{-nz} \quad (89) \)

A comparison of the non-condensate density obtained using equation (88) and that calculated by summing equation (84) over 1200 levels is shown in Fig. [8]. It can be seen that the agreement between the two methods, for this equilibrium situation, is good. The agreement improves if more energy levels are included in the sum over equation (84), and using about 2000 levels gives very good agreement. Our semiclassical method therefore shows good agreement with the expected distribution at equilibrium, and this is the only case in which we can be certain of the theoretically correct result.

In obtaining the results in this section we have only considered effects of the mean-field repulsion due to the condensate atoms, both on the thermal vapour cloud and on the condensate itself. In order to be truly consistent the model should also include the effects of the mean-field repulsion due to the thermal cloud on the system. This is quite easily achieved mathematically, but it does increase the computational time by a very substantial amount. However, recently Naraschewski and Stamper-Kurn [24] compared the density distributions obtained both with and without considering the mean-field repulsions of the thermal cloud, for an equilibrium condensate system. They found that the overall density distributions for the two cases were practically indistinguishable except for a very small deviation at the edge of the condensate. Furthermore, Holzmann, Krauth and Naraschewski [27] found that the semiclassical density distribution (including the mean-field repulsions by the thermal cloud) gives excellent agreement with exact Quantum Monte Carlo simulations for dilute gas condensates in equilibrium. Thus we expect that the density given by our semiclassical method is a very good description of the realistic system.
with the experimental results, is to perform a column integral in the account. The previous arguments for the exact non-interacting numerically. dimensions only. This is relatively straightforward to achieve trap, whereas the realistic traps used are strongly anisotropic wavefunctions assumed the use of a spherically symmetric arguments the only effect is to change this density profile is fitted, in order to save time and computational resources thefit vapor—can be fitted to the parabolic Thomas-Fermi potential—its behavior in the wings can thus be used to determine the temperature of the vapor.

\[
\bar{\varrho} = \frac{\varrho}{\lambda}
\]

\[
\bar{x} = x \sqrt{\frac{\varrho}{\lambda}}
\]

\[
\bar{y} = y \sqrt{\frac{\varrho}{\lambda}}
\]

2. Description of a Realistic Experimental System

The next step that must be performed, in order to compare with the experimental results, is to perform a column integral along one dimension, yielding a function of two spatial dimensions only. This is relatively straightforward to achieve numerically.

Finally the asymmetry of the real traps must be taken into account. The previous arguments for the exact non-interacting wavefunctions assumed the use of a spherically symmetric trap, whereas the realistic traps used are strongly anisotropic in the \(z\) dimension. However, in all the previous semiclassical arguments the only effect is to change \(V_T\) from \(m\omega^2r^2/2\) to \(m\omega_{xy}^2(x^2 + y^2)/2 + m\omega_z^2z^2/2\) where now

\[
\omega_{xy} = \sqrt{\lambda}\omega
\]

\[
\omega_z = \bar{\omega}/\lambda
\]

\[
\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}
\]

\[
\lambda = (\omega_x/\omega_z)^{2/3}.
\]

Due to the harmonic nature of the trap, one can recover the original form of the potential by scaling the dimensions used. Defining

\[
\bar{z} = z/\lambda
\]

\[
\bar{x} = x \sqrt{\frac{\varrho}{\lambda}}
\]

\[
\bar{y} = y \sqrt{\frac{\varrho}{\lambda}}
\]

the potential now returns to the form \(V_T(\bar{r}) = m\bar{\omega}^2\bar{r}^2/2\), where the scaled radius is given by \(\bar{r} = \sqrt{x^2 + y^2 + z^2}\). Equation (84) can now still be used, and the column density integration performed as for an isotropic potential, and the resulting function of \(\bar{x}\) and \(\bar{z}\) needs only to be rescaled to recover the answer in terms of \(x\) and \(z\). This scaling also affects the numerical column integration, with the effect that the result needs to be scaled overall by a factor of \(\lambda^{-3/2}\) (assuming the integration is in one of the two shorter dimensions and that the result is in the asymmetric (longer) dimension, as is the case for the ‘cigar’ geometry traps of MIT).

B. MIT Fitting Method

In the MIT experiments, the data for the numbers of condensate atoms and temperatures are obtained from fitting to the density profile obtained. The raw data obtained is in the form of two dimensional images and, although these can be fitted, in order to save time and computational resources the fit was mostly performed only to a one dimensional slice through the center of the condensate [22]. This density profile is fitted from a function formed by the combination of a condensate density profile and non-condensate profile. The MIT fitting procedure (presented in [23]) is a phenomenological procedure, only loosely connected with fundamental theory. It is based on two observations:

i) The behavior of the thermal cloud in the wings of the profile is almost independent of the chemical potential—its behavior in the wings can thus be used to determine the temperature of the vapor.

ii) The center of the profile is dominated by the condensate, and this—after subtracting the contribution of the vapor—can be fitted to the parabolic Thomas-Fermi profile, to determine the condensate chemical potential and thus the number in the condensate.

In order to execute the second procedure, some estimate of the vapor contribution has to be given in the region where the condensate is present. This is done by fitting the profile to the sum of a Thomas-Fermi parabola and a zero chemical potential Bose-Einstein distribution. Thus, using our methodology, this would be as follows. Setting the chemical potential to zero and integrating along one dimension gives (for a slice where \(x = 0\))

\[
\rho_{\text{int}}(z, x = 0) = \int_{-\infty}^{\infty} \rho(z, y, x = 0) dy
\]

\[
= \frac{m(kT)^2}{2\pi\hbar^3\omega_y} G_2 \left( \frac{m\omega_y^2z^2}{2kT} \right).
\]
This function can then be used to fit the wings of the distribution to obtain $T$.

The column density function due to the condensate, $\rho_{\text{int, c}}$, is then obtained by integrating the Thomas-Fermi wavefunction given by equation (80) over one dimension

$$\rho_{\text{int, c}}(z, x = 0) = \int_{-y_1(z)}^{y_1(z)} \left[ \frac{\mu_C(n_0)}{u} - \frac{m}{2u} \left( \omega_z^2 + \omega_y^2 \right) \right] dy$$

where $y_1(z) = \sqrt{\frac{2\mu_C(n_0)}{m\omega_y^2} - \frac{\omega_z^2}{\omega_y^2} z^2}$, \(100\)

and where $u = 4\pi\hbar^2 a/m$, giving

$$\rho_{\text{int, c}}(z, x = 0) = \frac{2m\omega_y^2}{3u} \left( \frac{2\mu_C(n_0)}{m\omega_y^2} - \frac{\omega_z^2}{\omega_y^2} z^2 \right)^{3/2}$$ \(101\)

The density profile can therefore be fitted by the sum of equation \(101\) and equation \(98\).

In the measurement procedure used, the phase shift data was not calibrated independently. Instead, the raw data were fitted using semiphenomenological fit functions and an arbitrary scale factor to related the measured phase shift to the column density.

However, the actual procedure used is somewhat more general, since the MIT procedure uses the trap constants as further fitting parameters, and these are allowed to be independently chosen for the condensate and the thermal cloud. This is done since finite resolution effects of the imaging system may have a rather different effect on the very narrow condensate profile from that on the broader vapor profile. In practice, for similar reasons the scale factor relating column density to the phase-contrast signal is allowed to be different for vapor and condensate. The number of condensate atoms $n_0$ can be obtained from fitting directly to equation \(101\), or, by a simpler method used for most cases, determining the extent of this condensate distribution. In most cases, since it is computationally simpler, number of condensate atoms was obtained from the spatial width of the condensate $z_0$. These are related by

$$\rho_{\text{int, c}}(z_0, x = 0) = 0 \quad \text{when} \quad z_0^2 = \frac{2\mu_C(n_0)}{m\omega_z^2}$$

or

$$\left( \frac{15u(m/2)^{3/2}}{8\pi} \right)^{2/5} \frac{2n_0^{2/5}}{m\omega_z^2}$$ \(102\)

thus $n_0 \propto z_0^5$.

In our fits we have assumed the scale factor to be the same for condensate and vapor, since our model is physically based, rather than being semiphenomenological. Hence, in our procedure, there are three free parameters to be fitted: $T$, $n_0$ and a scale factor.

C. Comparison of Fitting Methods

In Fig.19 we show a least squares fit of a zero chemical potential plus a Thomas-Fermi condensate profile for an equilibrium distribution. The data used was obtained from Fig.2 of [4], for which the MIT group obtained values of $T = 800\text{mK}$ with 9 million condensate atoms present in equilibrium after 160ms [22]. Also shown is the contribution to the distribution due to the non-condensate atoms only. Note that the $z$ axis has been scaled by a factor of 3.2 over that in [4] due to a typographical error in the paper [22].

In this fit, as for all the following cases, the density scale factor (necessary to convert from the arbitrary unit scale resulting from the experimental measurements) was determined by the least squares fitting procedure. Since we use the $T$ and $n_0$ determined by the MIT group using their procedure, this scale factor is the only free parameter in this fit.

Because the non-condensate density is taken to be given by a distribution with zero chemical potential, which means in part that the mean field interactions with the condensate are neglected (as now $V_{\text{eff}}(r) = V_T(r)$), this allows the non-condensate distribution to be quite strongly peaked at small $z$, whereas the inclusion of this effect (as shown in Fig.17) causes the distribution in this region to be suppressed. This could lead to an underestimate of the number of condensate atoms, which might be particularly significant at the initial stages of the condensate formation due to the low numbers present. Using the full MIT procedure with up to 5 extra parameters must give a better fit, but this basic problem will still be present, since the qualitative behavior of the vapor in the condensate region will be very similar to this fit.
The temperature determined by our method at 900K is not very different from the 800K determined by the MIT procedure, but the condensate number by our method is at $4 \times 10^6$ less than half of the MIT value of $9 \times 10^6$. Note however that the fit of Fig. 20 does not appear visually unacceptable, leading to the conclusion that the best we can say is that the data do not constrain the condensate number very strongly. The reason for this probably lies in the choice of the least squares fitting procedure, which weights all data equally, and since there is not very much data in the condensate region, the condensate number can be varied significantly without changing the goodness of fit greatly.

In order to appreciate better the differences between the three fits, we have plotted all three fit functions on the same graph in Fig. 22. From this it is clear that the fits using the zero chemical potential Bose-Einstein Fig. 22(a) and the more correct Fig. 22(b) are much more similar to each other than to the best fit, Fig. 22(c), in which the higher temperature makes for a much more pronounced peak in the vapor distribution, upon which a condensate peak is then superposed, giving a larger overall peak than would be expected from the condensate. From this figure, it is clear that the properties of the distribution in the transition region can be very deceptive.

We should also note that the density scale factor does not change significantly from Fig. 20 to Fig. 21. Since this scale factor is a well defined physical quantity which should be determined more or less equally by all data, this is encouraging. By this we mean that, although the scale factor is of no particular interest to the physics of Bose-Einstein condensation, it is nevertheless a well defined physical quantity whose value should be determinable. In contrast the fit of Fig. 19 gives a very different scale factor, which is not unexpected, given that it does not pass through most of the data points.

The same data can be fitted using the distributions from equation (84). Figure 20 shows the same equilibrium data, fitted by using the same parameters but by using the semiclassical distribution—as before, we use the $T$ and $n_0$ determined by the MIT group using their procedure, so that the only free parameter is the scale factor. This spatial distribution can be seen to fit the experimental data much more closely. However this can still be improved using a different choice of $T$ and $n_0$. The best fit was found with parameters of $T = 900nK$ with 4 million condensate atoms and a density scale factor of $5.4 \times 10^{-14}$, and is shown in Fig. 21. As a measure of how good the fit was, the sum of the square of the differences between the data and the fitted curve was calculated. The best curve gave a value of $5.7 \times 10^3$, the fit in Fig. 20 gave a value of $1.23 \times 10^4$ and the fit of Fig. 19 gave $1.76 \times 10^4$. The best fit to the same data used in the previous two figures. The parameters used were $T = 900nK$, $n_0 = 4 \times 10^6$ and the density scale factor was $5.4 \times 10^{-14}$. We should also note that the density scale factor does not change significantly from Fig. 20 to Fig. 21. Since this scale factor is a well defined physical quantity which should be determined more or less equally by all data, this is encouraging. By this we mean that, although the scale factor is of no particular interest to the physics of Bose-Einstein condensation, it is nevertheless a well defined physical quantity whose value should be determinable. In contrast the fit of Fig. 19 gives a very different scale factor, which is not unexpected, given that it does not pass through most of the data points.
FIG. 23. Comparison between experimental data from [4] and theoretical spatial density distributions, calculated using the semiclassical density distributions and the growth model described in this paper. The first frame shows the distribution before the cooling ‘cut’ below the critical temperature was performed. The parameters used were $T = 800\text{nK}$ and $n_{0,f} = 9 \times 10^6$. The (lower) red lines show the theoretical curves using initial condition (c) described in section III A 6. In the second and third frames the (upper) green curves depict the results obtained using initial condition (d) which become essentially indistinguishable from the condition (c) lines after 40ms.
FIG. 24. Further theoretical spatial distributions as the condensate grows. These graphs lead on from those in Fig. 23 and were obtained using the same parameters. The scale has been chosen so as to correspond to that in Fig. 23. The last graph shows the distribution at equilibrium (400ms - solid line) along with the result at 160ms, the latest time for which the experimental data was available. Note the change of scale in the last graph.
D. Comparison of Spatial Distributions with Experiment

With the aid of the spatial distributions calculated using equation (84), comparisons with the MIT growth data are possible. In figure 2 of [4] the density profiles of the system are given for a single condensate growth. The profiles are one-dimensional slices through the centre of the system, in the $z$ (long) axis. The fits to the equilibrium profile were compared in the previous section. From this data, MIT extracted values of $T = 800\,\text{nK}$ and $n_{0,f} = 9 \times 10^6$, and they found that the final equilibrium situation seemed to have been reached by $160\,\text{ms}$ [22]. The theoretical spatial distribution curves are compared to the experimental data in Fig. 23. The figure shows very good agreement with the experimental data during this $160\,\text{ms}$ period. The first frame at $40\,\text{ms}$ shows the least good agreement, however the agreement can be improved by the choice of different initial conditions as is shown in Fig. 23. Using initial condition (d), described in section III A 6, the agreement with experiment again becomes quite respectable. It is interesting to note that the spatial density distribution in this frame depends quite strongly on the initial conditions, which were found to have only a very small effect on the growth curve in section III A 4. This is because the $40\,\text{ms}$ frame is taken at a time very close to the initiation of the fast growth of the condensate population, and the spatial distribution at this time is quite strongly dependent on exactly when the growth starts, since any difference in the initiation time creates a relatively large change in the occupation numbers given that they are still quite low at this time.

The condensate growth curve which corresponds to the spatial distributions in Fig. 23 is shown in Fig. 25. The figure shows that at $160\,\text{ms}$ (the time of the last graph in Fig. 23) the growth of the condensate is theoretically only about halfway to completion. No experimental data for this particular growth event at times greater than $160\,\text{ms}$ was published in [4], although two further frames were taken at some later times and were found by the MIT group to contain the same number of atoms as for the $160\,\text{ms}$ profile [22].

The spatial density distributions for times later than $160\,\text{ms}$ are shown in Fig. 24 for times up until $400\,\text{ms}$ at which time equilibrium has been reached in the theoretical model. The graphs in this figure show that the spatial distribution changes relatively little after $160\,\text{ms}$ compared to its earlier evolution. In particular the last graph in the figure compares the spatial distribution at equilibrium with that at $160\,\text{ms}$, and it shows only a fairly small change. The peak height has definitely become greater, although not by a great deal more when one considers the size of the fluctuations evident in the experimental data of Fig. 23. However, more importantly, the spatial extent of the condensate has not changed significantly after $160\,\text{ms}$. The condensate number was determined in the majority of the MIT fits from the width of the condensate, as is given by equation (103). A well known difficulty [23] in using this method is that, in the Thomas-Fermi approximation, the condensate occupation depends upon the fifth power of the condensate radius. Thus, for example, a $15\%$ error in the measurement of the radius will result in the value for $n_0$ being incorrect by a factor of $2$. When this is considered, along with the fact that the experimental data will show fluctuations, and given the comparison between the equilibrium and $160\,\text{ms}$ distributions, it seems perhaps possible (though in the opinion of the experimenters [29] very unlikely) that there was some evolution at times later than $160\,\text{ms}$ which was not able to be measured experimentally to sufficient accuracy.

In conclusion therefore, more experimental data is really needed in order to make better comparisons, especially needed is data for times at which equilibrium has definitely been reached (at long $t$, although for very long times processes such as trap-loss may have to be considered). It would also be beneficial to fit the full two-dimensional data, rather than the one-dimensional slices shown previously. This would reduce the influence of fluctuations, and would lead to more weight being given to the measurements of the thermal cloud, which would increase the accuracy of any temperatures found.

V. COMPARISON WITH OTHER THEORETICAL TREATMENTS

Most of the other theoretical treatments have attempted to describe the formation of a condensate in a homogeneous, untrapped situation. No real quantitative predictions have yet emerged from any of the work that has been performed on trapped dilute atomic gas BEC, and so comparisons unfortunately will have to be qualitative at best.

A. Quantum Boltzmann Equation Approach

One of the techniques used to describe condensate growth has been the quantum Boltzmann equation, which has been used by Snoke and Wolfe [30], Semikoz and Tkachev [31], and Holland, Williams and Cooper [20], as well as forming the basis of the theory of the kinetic stages in the work of Kagan, Svistunov and Shlyapnikov [22,33]. Although the theory...
In the context described in this paper, it turns out that essentially the same equations may be obtained by modifying the quantum Boltzmann equation approach as follows:

(i) The quantum Boltzmann equation in an ergodic form is used, a form similar to that used in [2]:

\[ \frac{\partial f(n)}{\partial t} = \frac{8m^2\omega^2}{\pi \hbar} \sum_{e_m, e_p, e_q} \delta(\Delta E)g(\epsilon_{\text{min}}) \times \left[ f(e_p)f(e_q)(1 + f(e_m))(1 + f(e_n)) - f(e_m)f(e_n)(1 + f(e_p))(1 + f(e_q)) \right], \]

where \( n_k = g_k f(e_k) \) is the number of particles with energy \( e_k \), \( \epsilon_{\text{min}} = \min(e_m, e_n, e_p, e_q) \) and \( \Delta E = e_m + e_n - e_p - e_q \).

(ii) The energy levels in the condensate band are modified as discussed in section [II A], in order to account for the mean field interactions due to the presence of the condensate.

(iii) The levels in the non-condensate band are summed over and assumed to be time independent. This allows much larger, and realistic sized, systems to be modelled as opposed to the 100-1000 atom systems typically simulated in previous attempts.

(iv) Collisions between two particles which were both initially in the condensate band were neglected, this is a valid approximation if the vast majority of particles are found in the non-condensate band.

Using these modifications, and the rates for the scattering and growth processes found from quantum kinetic theory (see section [II]), the quantum Boltzmann equation will give rise to the set of differential equations ([2]) whose solutions provided the results in this paper.

Of the above references, only the work of Holland, Williams and Cooper conducted any simulations for the growth of a trapped condensate. They found that their simulations of condensate occupation number evolution behaved as \( n_0 = n_0 f(1 - e^{-t/\tau}) \), where \( \tau \) was a fitted parameter. A function of this form can be made to fit the results obtained by our model reasonably well, provided that an initiation time is allowed for, as was anticipated might be necessary in [2].

The same functional form was obtained using the quantum Boltzmann master equation approach in [1] by Jaksch et al.

It should be noted that, although the quantum kinetic description for the growth of the mean occupation numbers turns out to also be described by a modified quantum Boltzmann equation approach, the full quantum kinetic theory treats aspects of condensate dynamics which are not accessible via the quantum Boltzmann equation. Such aspects include: the treatment of fluctuations, phase and phase decoherence, and the inclusion of Bogoliubov-like quasiparticle states.

B. Comparison with Work of Kagan et al.

Major theoretical work into the growth of condensate in recent years has been performed by Kagan, Svidunov, and Shlyapnikov. They divided the growth into three stages, the first of which was a kinetic stage described by Svistunov in [32]. Svidunov predicted the formation of a particle-flux wave in energy space during the initial stages, which transports particles towards lower energy states. The arrival of this wave at the lowest energy state at the critical time would give rise to an energy distribution function of the form \( f(E) \propto E^{-7/6} \). After the critical time this behaviour would be lost due to a particle-flux wave propagating to higher energies. The simulations of Semikoz and Tkachev [31] showed this behaviour to some extent, although they found that the behaviour at the critical time was \( f(E) \propto E^{-1.24} \).

The work by Svidunov in [32] related to the case of homogeneous systems, however he recently reworked his methodology to consider a gas confined by a harmonic trapping potential. In this case he found [34] that the dependence at the critical time now tended towards \( f(E) \propto E^{-5/3} \).

![FIG. 26. The growth of a condensate of 9 million atoms as predicted by the model in this paper. Plotted is \( f(\epsilon_m) \), the population per individual level with energy \( \epsilon_m \), and the energies of the levels as functions of time. The values of \( f(\epsilon_m) \) and \( \epsilon_m \) are plotted on log scales. The lines almost parallel to the time axis are not lines of constant energy, but rather lines of constant level number, whose energies change with time. Not all levels are shown in order to make the behaviour legible. The solid black curve represents the energy of the condensate level \( \log_{10}[\mu N(\eta_0)] \) as a function of time. The bold line labelled C–C represents the critical time, at which the energy distribution has the form \( f(E) \propto E^{-1.61} \). The temperature of the system was 800nK.](image-url)
small up to the point labelled as the critical time, after which the condensate grows enormously. The populations of the excited states approach equilibrium very rapidly after the critical time, much more rapidly than the condensate level does.

As the critical time is approached, the distribution approaches a straight line as shown in figure 26. At the critical time, when this distribution is linear with the logarithm of the energy, the energy dependence was found to be of the form \( f(E) \propto E^{-1.41} \), which is in good agreement with the prediction of Svistunov of \( E^{-5/3} \).

As far as estimates of timescales are concerned, Kagan, Svistunov and Shlyapnikov predicted that the evolution of the system up to initiation would occur on the timescale of the classical collision time \( \tau_0 \), that the initiation of the condensate growth would occur on a much faster timescale, and that the final growth would occur on the timescale necessary for the annihilation of vortices in homogeneous gases, and the decay of fluctuations in the phase of the condensate.

In the treatment of Kagan, Svistunov and Shlyapnikov, the timescale for the first kinetic stage was postulated for a homogeneous gas, however a comparison can still be made. A first estimate of \( \tau_0 = (\sigma \nu \tau)^{-1} \) can be obtained by using the classical value for \( \nu \tau \), the mean thermal velocity in a gas, of \( \nu \tau = \sqrt{2kT/m} \), and the cross section defined by \( \sigma = 8\pi a^2 \). The value for \( \bar{n} \), the mean density, will be taken as the density of the non-condensate particles in the centre of the trap given by

\[
\bar{n} = \left( \frac{mkT}{2\pi \hbar^2} \right)^{3/2} \frac{G_3/2}{(V_T(0)/kT)} \tag{104}
\]

(see equation [8]) where \( G_o(z) = \sum_{q=1}^{\infty} e^{-q^2} / q^a \). Taking \( V_T(0) = 0 \), and using a temperature of 900nK, the collision time is \( \tau_0 = 27\text{ms} \). This temperature corresponds to that of the growth in Fig.3 which shows that the time until initiation is of the order of \( 2\tau_0 \). Thus our treatment does agree with the picture of Kagan et al. in that this stage occurs over the order of a few \( \tau_0 \).

A note about the collision times is needed here. In examining the validity of the ergodic approximation Jaksch et al. found that it was valid only for quantities averaged over about 10 collision times [10]. The above timescale \( \tau_0 \) is not the timescale over which collisions occur in the condensate system in reality. It is rather the classical collision time for a classical gas in equilibrium below the critical temperature but with no condensate present, and is obviously artificial. Once the condensate begins to form the density increases significantly, and the actual mean collision time was found in [10] to be more than two orders of magnitude smaller than \( \tau_0 \). Thus the ergodic assumption should still be valid for our treatment, even given the large value of \( \tau_0 \).

The timescales found by Kagan et al. for the second and third stages of evolution have only been determined for the case of a homogeneous gas, and so accurate comparisons with our model for these stages are not able to be performed. Our model does agree that the initiation stage occurs on a much faster timescale than the first kinetic stage. The presence of vortices has not been considered in our treatment, nor has any consideration been given to the phase fluctuations, and so comparisons cannot be made with the third stage of evolution in the description of Kagan et al.

VI. CONCLUSIONS

In this paper a model of the growth of a condensate, derived from quantum kinetic theory [3,11], is presented. The main improvements over the simpler description in [5] are the more accurate calculation of the growth rate factor \( W^+(n_0) \), the consideration of the time dependence of the lower energy levels, and the inclusion of the scattering of particles between these levels. The modified \( W^+ \) factors have the greatest effect, generally increasing the rate of growth by a factor of about 3, dependent on the exact parameters. The inclusion of the other levels and their scattering also leads to an increase in the rate of growth, mainly by reducing the amount of time taken for the initiation of the growth, that is the time before the stimulated growth processes, due to the Bose statistics of the system, become dominant.

The model describes the evolution of time-dependent energy levels in the lowest states, coupled with a time independent thermal bath of atoms occupying the higher energy levels.

The results give growth curves whose shapes are approximately those given by

\[
n_0 = \begin{cases} 
0 & \text{for } t < t_i \\
n_0 f(1 - e^{-(t-t_i)/\tau}) & \text{for } t > t_i
\end{cases}
\]

(105)

where \( t_i \) is some initiation time. This form agrees with the general form of the results of Holland et al. [20] and Jaksch et al. [10] once an initiation time is allowed. The results also seem to be in qualitative agreement with features of the description proposed by Kagan, Svistunov and Shlyapnikov.

The results are not very sensitive to the exact nature in which the mean-field effects on the lower levels were accounted for, so long as the energies of the very lowest levels were altered in a consistent fashion. The initial conditions used did not have a large effect on the growth curves, however they can be important when the spatial density profiles of the system are calculated for comparison with experiment.

The evolution of the model depends upon approximations made for the rate factors \( \Gamma(T) \) and \( W^+_{m^+}(n_0) \). The results obtained show that the growth is not very different as long as the actual value of \( \Gamma(T) \) falls within about a factor of 10 of the approximation, and as long as the actual values of \( W^+_{m^+}(n_0) \) lies within a factor of 2 of \( W^+(n_0) \).

Overall the rates of growth predicted now agree very well with the growth rates measured from experimental data [3]. However at lower temperatures the trend in growth rates shows some divergence, with the experimental rates becoming quicker whilst the predicted rates become slower. This may be a result of the substantial cooling necessary to achieve these temperatures, giving rise to a highly non-equilibrium system which is inadequately described by our model.
The extraction from experimental data of energy level populations is fraught with difficulties, leading to large uncertainties. It was found to be better to compare theoretical predictions of the evolution of the spatial density distribution of the system as the condensate to those experimentally measured and the results in sect.V appear to give good agreement with the spatial shapes given in the MIT data. They also show that, once a large condensate occupation has been reached, if the condensate occupation changes by a significant amount the corresponding change in the spatial density distribution of the system is relatively small.

With the aid of more raw experimental data to fit, a determination of exactly how accurate the predictions of this model are, conclusions as to the exact initial conditions present, and more accurate approximations of $\Gamma(T)$ and $W_{m}^{n+}(n_{0})$ may be able to be reached.

Further work which could be undertaken within the framework of this model includes:

i) An accurate determination of the rate factors $\Gamma(T)$ and $W_{m}^{n+}(n_{0})$ analytically, or at least finding constraints on their values by comparison with more experimental data.

ii) An inclusion of some Bogoliubov phonon-like quasiparticle nature in the description of the lower energy levels, since all excited levels in this paper were treated as Hartree-Fock particle-like quasiparticles, which will be valid for most of the higher levels but not for the lower energy excitations.

iii) A consideration of the fluctuations in occupation numbers. These may be significant in determining the initiation time, which is when the occupation of the condensate level becomes large enough for the stimulated growth processes to take over.

iv) As the model stands, the non-condensate band ‘bath’ of atoms is treated as being time independent. A major extension of the model would be to include the dynamics of the non-condensate band in the evolution. Extension to include a time-dependent bath will be treated elsewhere [55].

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