NONPOSITIVELY CURVED METRIC IN THE POSITIVE CONE OF A FINITE VON NEUMANN ALGEBRA

E. ANDRUCHOW AND G. LAROTONDA

Dedicated to Angel Rafael Larotonda (1939–2005),
beloved father, dear friend and profound mathematician

ABSTRACT

In this paper we study the metric geometry of the space Σ of positive invertible elements of a von Neumann algebra \( A \) with a finite, normal and faithful tracial state \( \tau \). The trace induces an incomplete Riemannian metric \( \langle x, y \rangle_a = \tau(ya^{-1}xa^{-1}) \), and, though the techniques involved are quite different, the situation here resembles in many relevant aspects that of the \( n \times n \) matrices when they are regarded as a symmetric space. For instance, we prove that geodesics are the shortest paths for the metric induced, and that the geodesic distance is a convex function; we give an intrinsic (algebraic) characterization of the geodesically convex submanifolds \( M \) of \( \Sigma \); and under a suitable hypothesis we prove a factorization theorem for elements in the algebra that resembles the Iwasawa decomposition for matrices. This factorization is obtained via a nonlinear orthogonal projection \( \Pi_M : \Sigma \to M \), a map which turns out to be contractive for the geodesic distance.

1. Introduction

Let \( A \) be a von Neumann algebra with a finite (normal, faithful) trace \( \tau \). Denote by \( A_h \) the set of selfadjoint elements of \( A \), by \( G_A \) the group of invertible elements, and by \( \Sigma \) the set

\[ \Sigma = e^{A_h} = \{ a \in G_A : a \geq 0 \}; \]

\( \Sigma \) is an open subset of \( A_h \) in the norm topology. Therefore if one regards it as a manifold, its tangent spaces identify with \( A_h \). We endow these tangent spaces with the (incomplete) Hilbert–Riemann metric

\[ \langle x, y \rangle_a = \tau(xa^{-1}ya^{-1}), \quad a \in \Sigma, x, y \in A_h. \]

(1.1)

Note that \( \|x\|^2_a = \langle x, x \rangle_a := \tau(xa^{-1}xa^{-1}) \), and also that this metric is invariant for the action \( I_g : x \mapsto gxg^* \), where \( g \in G_A \).

As in classical differential geometry, one obtains a metric ‘dist’ for \( \Sigma \) by considering

\[ \text{dist}(a, b) = \inf \{ \text{Length}(\gamma) : \gamma \text{ is a smooth curve joining } a \text{ and } b \}, \]

(1.2)

where smooth means differentiable in the norm induced topology and the length of a curve \( \gamma(t), t \in [0, 1] \), is measured using the inner product above (1.1):

\[ \text{Length}(\gamma) = \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}^{1/2} dt. \]

Received 9 May 2005; revised 12 September 2005.

2000 Mathematics Subject Classification 53C22, 58B20 (primary), 46L45 (secondary).
The purpose of this paper is the geometric study of the resulting metric space, and, particularly, of its convex subsets.

If $\mathcal{A}$ is finite dimensional, that is, a sum of matrix spaces, this metric is well known: it is the nonpositively curved Riemannian metric on the set of positive definite matrices, which is a universal model space for (finite dimensional) nonpositively curved manifolds of noncompact type (see [4] and [5]).

If $\mathcal{A}$ is of type II$_1$, the trace inner product is not complete, so that $\Sigma$, with the inner products $\langle \cdot, \cdot \rangle_a$, is not a Hilbert–Riemann manifold, properly speaking. For instance, the exponential map

\[ \exp : \mathcal{A}_h \to \Sigma, \quad \exp(x) = e^x, \]

which is a global diffeomorphism in the norm topology, is continuous but non-differentiable in the 2-norm $\| \cdot \|_2$ induced by $\tau$ (namely $\|x\|_2 = \tau(x^*x)^{1/2}$). The set $\Sigma$ itself is not a differentiable manifold with this norm.

However, the metric space $(\Sigma, \text{dist})$ behaves in many senses like in the finite dimensional setting. Let us mention a few issues:

1. the isometric action of the group $G_A$ via $g \mapsto I_g$, where $I_g(x) = gxg^*$;
2. minimality of geodesics (that is, solutions of Euler’s equation are minimizing for the distance introduced above in (1.2), see Theorem 3.1);
3. convexity of the map $t \mapsto \text{dist}(\gamma(t), \delta(t))$ which gives distance among geodesics (Corollary 3.4);
4. algebraic structure of (geodesically) convex subsets (Theorem 4.4);
5. normal projections to convex submanifolds and their minimality (Lemma 5.3 and Theorem 5.4);
6. existence and uniqueness of a factorization for invertible elements by means of convex submanifolds (Corollary 5.7).

2. Main inequalities

The following inequality will be useful; its proof for $n \times n$ real matrices can be found in the inspiring paper of Mostow [5]. It is called by Bhatia [1] the \textit{exponential metric increasing property}. Bhatia proves it for matrices (and for more general norms). However, his proof for the 2-norm is valid almost verbatim in the infinite dimensional context for an arbitrary (finite, faithful) tracial state. We transcribe it. We use the fact that selfadjoint elements in a von Neumann algebra can be approximated by selfadjoint elements with finite spectrum.

\[ \text{Lem} . \text{a} 2.1. \] \textit{Let $\tau$ be a tracial faithful state in $\mathcal{A}$, and $x, y$ selfadjoint elements of $\mathcal{A}$. If $\exp$ denotes the usual exponential map, $\exp(x) = e^x$, then}

\[ \|y\|_2 \leq \|e^{-x} d \exp_x(y)\|_2. \]  \hfill (2.1)

\textit{Proof.} First we must establish the formula

\[ d \exp_x(y) = \int_0^1 e^{tx} ye^{(1-t)x} \, dt. \]

Note that

\[ d \exp_x(y) = \frac{d}{dt} e^{x+ty} \bigg|_{t=0}. \]
Then 
\[ d \exp_x (y) = y + \frac{1}{2} (yx + xy) + \frac{1}{6} (yx^2 + xyx + x^2 y) + \ldots. \]

On the other hand,
\[ e^{tx} ye^{(1-t)x} = y + txy + (1-t)yx + \frac{1}{2} (1-t)^2 yx^2 + t(1-t)xy + \frac{1}{2} t^2 x^2 y + \ldots. \]

Integrating this series (which is absolutely convergent) term by term proves the equality. Denote \( a = e^x \). Let us show now that, if \( b \) is positive in \( A \), then
\[ \| a^{1/2} ba^{1/2} \|_2 \leq \| \int_0^1 a^t ba^{1-t} \, dt \|_2. \]

Assume first that \( a \) has finite spectrum: \( a = \sum_{i=1}^n \alpha_i p_i \) with \( \alpha_i > 0 \) and \( \sum_{i=1}^n p_i = 1 \). Then
\[ a^{1/2} ba^{1/2} = \sum_{i,j=1}^n \alpha_i^{1/2} \alpha_j^{1/2} p_i b p_j. \]

Therefore
\[ \| a^{1/2} ba^{1/2} \|_2^2 = \sum_{i,j=1}^n \alpha_i \alpha_j \tau(p_i b p_j p_i). \]

Analogously
\[ \int_0^1 a^t ba^{1-t} \, dt = \sum_{i,j=1}^n \int_0^1 \alpha_i^{1/2} \alpha_j^{1/2} \, dt p_i b p_j \]
and
\[ \left\| \int_0^1 a^t ba^{1-t} \, dt \right\|_2^2 = \sum_{i,j=1}^n \int_0^1 \alpha_i^{2} \alpha_j^{2(1-t)} \, dt \tau(p_i b p_j b p_i) \]
\[ = \sum_{i,j=1}^n \frac{\alpha_i^{2} - \alpha_j^{2}}{2 \ln \alpha_i - 2 \ln \alpha_j} \tau(p_i b p_j b p_i). \]

Note that \( p_i b p_j b p_i \) is positive. Also one has the elementary inequality
\[ \sqrt{s t} \leq \frac{s - t}{\ln s - \ln t} \]
for \( s, t > 0 \). Then
\[ \alpha_i \alpha_j p_i b p_j b p_i \leq \frac{\alpha_i^{2} - \alpha_j^{2}}{2 \ln \alpha_i - 2 \ln \alpha_j} p_i b p_j b p_i. \]

Taking traces and adding yields (2.2) in this case. In the general case, the inequality follows by approximating (in norm) the element \( a \) with positive elements with finite spectrum.

As in [1], put \( b = e^{-x/2} ye^{-x/2} \) in (2.2):
\[ \| y \|_2 \leq \| \int_0^1 e^{tx} (e^{-x/2} ye^{-x/2}) e^{(1-t)x} \, dt \|_2 \leq \| e^{-x/2} \int_0^1 e^{tx} ye^{(1-t)x} \, dt e^{-x/2} \|_2 \]
\[ = \| e^{-x/2} (d \exp_x (y)) e^{-x/2} \|_2. \]

If \( a \) is positive and invertible and \( b \) is selfadjoint, by the Cauchy–Schwarz inequality for \( \tau \), one has
\[ \| a^{-1/2} ba^{-1/2} \|_2^2 = \tau(a^{-1} ba^{-1} b) \leq \tau(a^{-1} b^2 a^{-1})^{1/2} \tau(b a^{-2} b) \tau(b a^{-2} b)^{1/2} = \| a^{-1} b \|_2^2. \]
Using this inequality for \(a = e^x\) and \(b = d\exp_x(y)\) one obtains
\[
\|y\|_2 \leq \|e^{-x/2}(d\exp_x(y))e^{-x/2}\|_2 \leq \|e^{-x}(d\exp_x(y))\|_2. 
\]
\[
\square
\]

**Corollary 2.2.** For any \(x \in A_h\), the map \(T_x : y \mapsto e^{-x/2}d\exp_x(y)e^{-x/2}\) is bounded, symmetric for the 2-inner product (when restricted to \(A_h\)) and invertible. The inverse is contractive, that is, \(\|T_x^{-1}(z)\|_2 \leq \|z\|_2\).

**Proof.** The map is clearly bounded and invertible; the bound for the inverse follows from the proof of the previous lemma. To prove that it is symmetric, note that
\[
\langle T_x(y), z \rangle_2 = \tau(zT_x(y)) = \tau(e^{-x/2} \sum_{n \geq 0} \frac{1}{n!} \sum_{p+q=n-1} x^p y^q e^{-x/2} z)
\]
\[
= \sum_{n \geq 0} \frac{1}{n!} \sum_{p+q=n-1} \tau(e^{-x/2} x^p y^q e^{-x/2} z)
\]
\[
= \sum_{n \geq 0} \frac{1}{n!} \sum_{p+q=n-1} \tau(x^p y e^{-x/2} e^{-x/2} x^q z)
\]
\[
= \sum_{n \geq 0} \frac{1}{n!} \sum_{p+q=n-1} \tau(e^{-x/2} e^{-x/2} x^q z x^p e^{-x/2} y)
\]
\[
= \tau(T_x(z)y) = \langle y, T_x(z) \rangle_2. \quad \square
\]

### 3. Geodesic distance

For \(X, Y\) smooth vector fields in \(\Sigma\) and \(p \in \Sigma\), we introduce the expression
\[
(\nabla_X Y)_p = \{X(Y)\}_p - \frac{1}{2}(X_p p^{-1} Y_p + Y_p p^{-1} X_p), \tag{3.1}
\]
where \(X(Y)\) denotes derivation of the vector field \(Y\) in the direction of \(X\) (performed in the linear space \(A_h\)). Note that \(\nabla\) is clearly symmetric and verifies all the formal identities of a connection. The compatibility condition between the connection and the metric
\[
\frac{d}{dt} \langle X, Y \rangle_\gamma = \langle \nabla_\dot{\gamma} X, Y \rangle_\gamma + \langle X, \nabla_\dot{\gamma} Y \rangle_\gamma
\]
is fulfilled for any smooth curve \(\gamma \subset \Sigma\) and \(X, Y\) tangent vector fields along \(\gamma\). This identity is straightforward from the definitions for both terms and the cyclicity of the trace. This says that \(\nabla\) is the ‘Levi-Civita’ connection of the metric introduced.

Euler’s equation \(\nabla_\dot{\gamma} \dot{\gamma} = 0\) reads \(\dot{\gamma} = \dot{\gamma} \gamma^{-1} \dot{\gamma}\), and it is easy to see that the (unique) solution of this equation with \(\gamma(0) = p, \dot{\gamma}(1) = q\) is given by the curve
\[
\delta_{pq}(t) = p^{1/2} (p^{-1/2} q p^{-1/2})^t p^{1/2}. \tag{3.2}
\]
Note that \(\delta_{pq} \subset \Sigma\) because \(aba\) is positive and invertible whenever \(a, b\) are positive and invertible.

We will prove that the shortest path joining \(p\) to \(q\) is given by the formula above (Theorem 3.1); these curves look formally equal to the geodesics between positive definite matrices (regarded as a symmetric space).

We will use \(\text{Exp}_p\) to denote the exponential map of \(\Sigma\). Note that
\[
\text{Exp}_p(v) = p^{1/2} e^{p^{-1/2} v p^{-1/2}} p^{1/2}
\]
Rearranging the exponential series we get a simpler expression,
\[ \text{Exp}_p(v) = pe^{p^{-1}v} = e^{vp^{-1}p}. \]
A straightforward computation also shows that for \( p, q \in \Sigma \) we have
\[ \text{Exp}_p^{-1}(q) = p^{1/2} \ln(p^{-1/2}qp^{-1/2})p^{1/2}. \]
As mentioned in the Introduction, we measure curves in \( \Sigma \) using the norms in the tangent space, namely
\[ \text{Length}(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{\alpha(t)} dt. \]
We have \( \|\tilde{\delta}_{p,q}(t)\|_{\delta_{p,q}(t)} \equiv \|\ln(p^{-1/2}qp^{-1/2})\|_2 \), so, for the geodesics introduced in equation (3.2), we have \( L(\delta_{p,q}) = \|\ln(p^{-1/2}qp^{-1/2})\|_2 \).

**Theorem 3.1.** Let \( a, b \in \Sigma \). Then the geodesic \( \delta_{a,b} \) is the shortest curve joining \( a \) and \( b \) in \( \Sigma \), if the length of curves is measured with the metric defined above.

**Proof.** Let \( \gamma \) be a smooth curve in \( \Sigma \) with \( \gamma(0) = a \) and \( \gamma(1) = b \). We must compare the length of \( \gamma \) with the length of \( \delta_{a,b} \). Since the invertible group acts isometrically for the metric, it preserves the lengths of curves. Thus we may act with \( a^{-1/2} \), and suppose that both curves start at 1, or equivalently \( a = 1 \). Therefore \( \delta_{1,b}(t) = \delta(t) = e^{tx} \), with \( x = \ln b \). The length of \( \delta \) is therefore \( \tau(x^2)^{1/2} = \|x\|_2 \).

The proof follows easily from the inequality proved above. Indeed, since \( \gamma \) is a smooth curve in \( \Sigma \), it is of the form \( \gamma(t) = e^{\alpha(t)} \), with \( \alpha = \ln \gamma \). Then \( \alpha \) is a smooth curve of selfadjoints with \( \alpha(0) = 0 \) and \( \alpha(1) = x \). Moreover,
\[ \tau(\gamma^{-1} \dot{\gamma} \gamma^{-1} \dot{\gamma})^{1/2} = \|e^{-\alpha} \dot{\alpha}\|_2 = \|e^{-\alpha} d\exp_\alpha(\dot{\alpha})\|_2. \]

By the inequality in the above lemma, this is not smaller than \( \|\dot{\alpha}\|_2 \). Then
\[ \int_0^1 \tau(\gamma^{-1} \dot{\gamma} \gamma^{-1} \dot{\gamma})^{1/2} dt \geq \int_0^1 \|\dot{\alpha}\|_2 dt \geq \left\| \int_0^1 \dot{\alpha} dt \right\|_2 = \|x\|_2 = \tau(x^2)^{1/2}. \]

**Remark 3.2.** The geodesic distance induced by the metric is given by
\[ \text{dist}(a, b) = \tau(\ln(a^{-1/2}ba^{-1/2})^2)^{1/2}. \]

The curvature tensor \([2] \) is given by
\[ R_a(x, y)z = -\tfrac{1}{4} a[[a^{-1}x, a^{-1}y], a^{-1}], \]
where \([ , , ]\) is the usual commutator, that is, \([x, y] = xy - yx \).

Let \( J(t) \) be a Jacobi field along a geodesic \( \delta \) of \( \Sigma \). That is, \( J \) is a solution of the differential equation
\[ \frac{D^2 J}{dt^2} + R_{\delta}(J, \dot{\delta}) \dot{\delta} = 0. \tag{3.3} \]

Next we show that the norm of a Jacobi field is convex. If \( x, y \in A_b \) are regarded as tangent vectors of \( \Sigma \) at the point \( a \), then the following condition (which is a nonpositive sectional curvature condition) holds:
\[ \langle R_a(x, y)y, x \rangle_a = \tau(R_a(x, y)yx^{-1}xa^{-1}) \leq 0. \]
The proof of this fact is straightforward. Then
\[
\frac{d^2}{dt^2} \langle J, J \rangle_\gamma = 2 \left\{ \left\langle \frac{\partial^2 J}{\partial t^2}, J \right\rangle_\gamma + \left\langle \frac{\partial J}{\partial t}, \frac{\partial J}{\partial t} \right\rangle_\gamma \right\} = 2 \left\{ -\langle R_\gamma (J, \gamma) \gamma, J \rangle_\gamma + \left\langle \frac{\partial J}{\partial t}, \frac{\partial J}{\partial t} \right\rangle_\gamma \right\} \geq 0.
\]

In other words, the smooth function \( t \mapsto \langle J, J \rangle_\gamma \) is convex. We shall need convexity of the norm of the Jacobi field (and not of the square of the norm just noted).

**Proposition 3.3.** Let \( \gamma \) be a geodesic of \( \Sigma \) and \( J \) a Jacobi field along \( \gamma \). The real map \( t \mapsto \langle J, J \rangle_\gamma^{1/2} \) is convex.

**Proof.** Clearly, it suffices to prove this assertion for a field \( J \) which does not vanish. As in Theorem 1 of [3], by the invariance of the connection and the metric under the action of \( G_A \), it suffices to consider the case of a geodesic \( \gamma(t) = e^{tx} \) starting at \( 1 \in \Sigma \). For the field \( K(t) = e^{-tx/2} J(t) e^{-tx/2} \) the Jacobi equation translates into
\[
4\ddot{K} = Kx^2 + x^2 \dot{K} - 2xKx. \tag{3.4}
\]

Moreover
\[
\langle J, J \rangle_\gamma^{1/2} = \tau(\gamma^{-1} J \gamma^{-1} J)^{1/2} = \tau(K^2)^{1/2} = \|K\|_2.
\]

Let us prove therefore that the map \( t \mapsto f(t) = \|K(t)\|_2 \) is convex, for any (non-vanishing) solution \( K \) of (3.4). Note that \( f(t) \) is smooth, and \( \dot{f} = \tau(K^2)^{-1/2} \tau(K \dot{K}) \).

Then
\[
\dot{f} = -\tau(K^2)^{-3/2} \tau(K \dot{K})^2 + \tau(K^2)^{-1/2} \{ \tau(\dot{K}^2) + \tau(K \ddot{K}) \}.
\]

Let us multiply this expression by \( \tau(K^2)^{3/2} \) to obtain
\[
-\tau(K \dot{K})^2 + \tau(K^2) \tau(\dot{K}^2) + \tau(K^2) \tau(K \ddot{K}).
\]

The first two terms add up to a nonnegative number. Indeed, one has \( \tau(K \dot{K})^2 \leq \tau(K^2) \tau(\dot{K}^2) \) by the Cauchy–Schwarz inequality for the trace \( \tau \). Let us examine the third term \( \tau(K^2) \tau(K \ddot{K}) \). It suffices to show that \( \tau(K \ddot{K}) \) is nonnegative. Using (3.4),
\[
\tau(K \ddot{K}) = \frac{1}{4} \{ \tau(K^2 x^2) + \tau(Kx^2 K) - 2 \tau(KxKx) \} = \frac{1}{2} \{ \tau(K^2 x^2) - \tau(KxKx) \}.
\]

This number is positive, again by the Cauchy–Schwarz inequality:
\[
\tau(KxKx) = \tau((xK)^* Kx) \leq \tau((xK)^* xK)^{1/2} \tau((Kx)^* Kx)^{1/2} = \tau(K^2 x^2). \tag{\*}
\]

**Corollary 3.4.** If \( \gamma \) and \( \delta \) are geodesics, the map \( f(t) = \text{dist}(\gamma(t), \delta(t)) \) is a convex function of \( t \).

**Proof.** As in Theorem 2 of [3], the distance between \( \gamma(t) \) and \( \delta(t) \) is given by the geodesic \( \alpha_t(s) \) obtained moving the \( s \) variable in a geodesic square \( h(s, t) \) with vertices \( \gamma(t_0), \delta(t_0), \gamma(t_1) \) and \( \delta(t_1) \). Taking the partial derivative along the \( s \) direction gives a Jacobi field \( J(s, t) \) along the geodesic \( \beta_s(t) = h(s, t) \) and it also
gives the speed of $\alpha_t$. Hence
\[
f(t) = \int_0^1 \left\| \frac{\partial \alpha_t(s)}{\partial s} \right\|_{\alpha_t(s)} ds = \int_0^1 \| J(s, t) \|_{h(s, t)} ds.
\]
This equation says that $f(t)$ can be written as the limit of a convex combination of convex functions $u_i(t) = \| J(s_i, t) \|_{h(s_i, t)}$, so $f$ must be convex itself.

**Lemma 3.5.** The following inequality holds for any $x, y \in A_h$:
\[
\text{dist}(ex, ey) = \| \ln(\frac{e^{tx/2}e^{-ty/2}e^{tx/2}}{2e^{-ty/2}e^{tx/2}}) \|_2 \geq \| x - y \|_2. \tag{3.5}
\]

**Proof.** Take $\gamma(t) = e^{tx}$, $\delta(t) = e^{ty}$ and $f$ as in the previous corollary. Note that $f(0) = 0$, hence $f(t)/t \leq f(1)$ for any $0 < t \leq 1$; hence $\lim_{t \to 0^+} f(t)/t \leq f(1)$.

Now note that
\[
f(t) = \frac{1}{t} \| \ln(\frac{e^{tx/2}e^{-ty/2}e^{tx/2}}{2e^{-ty/2}e^{tx/2}}) \|_2 = \tau \left( \frac{1}{t} \ln(\frac{e^{tx/2}e^{-ty/2}e^{tx/2}}{2}) \right)^{1/2}.
\]

Since
\[
\lim_{t \to 0^+} \frac{1}{t} \ln(\frac{e^{tx/2}e^{-ty/2}e^{tx/2}}{2}) = \frac{d}{dt} \ln(\frac{e^{tx/2}e^{-ty/2}e^{tx/2}}{2})|_{t=0} = x - y,
\]
and the logarithm of $\beta(t) = e^{tx/2}e^{-ty/2}e^{tx/2}$ can be approximated uniformly by polynomials $p_n(\beta) = \sum_k \alpha_{n,k} \beta^k$ for $t$ close enough to zero (note that $\beta(0) = 1$), and
\[
\frac{d}{dt} \beta|_{t=0} = x - y,
\]
we have the desired inequality.

**Lemma 3.6.** The sum of the inner angles of any geodesic triangle in the manifold $\Sigma$ is less than or equal to $\pi$.

**Proof.** Squaring both sides of inequality (3.5) leads (by the invariance of the metric for the action of $G_A$) to
\[
l_i^2 \geq l_{i+1}^2 + l_{i-1}^2 - 2l_{i+1}l_{i-1} \cos(\alpha_i),
\]
where $l_i$ are the sides of any geodesic triangle and $\alpha_i$ is the angle opposite to $l_i$.

These inequalities say that we can construct an Euclidean comparison triangle in the affine plane with sides $l_i$; they also say that the angle $\beta_i$ (opposite to $l_i$ for this flat triangle) is bigger than $\alpha_i$. Adding the three angles we have $\alpha_1 + \alpha_2 + \alpha_3 \leq \beta_1 + \beta_2 + \beta_3 = \pi$.

4. Convex sets

We are interested in the convex subsets of $\Sigma$, that is, subsets $M \subset \Sigma$ such that the ambient geodesic joining two points in $M$ stays in $M$ for any value of $t$. Note that the simplest of such objects are the geodesics.
It is not hard to see that, when two elements \( a, b \in \Sigma \) commute, the geodesic triangle spanned by \( a, b \) and 1 is convex, hence there is a flat surface containing \( a, b \) and 1; indeed, the triangle in \( \Sigma \) is the image of the plane triangle with vertices 0, \( e_1, e_2 \) under the map \( T : \mathbb{R}^2 \to \Sigma \) given by

\[
T(x, y) = e^{x \ln(a) - y \ln(b)}.
\]

In particular the geodesic joining \( a \) and \( b \) is the image of the segment

\[
\{(x, y) : x, y \geq 0, x + y = 1\}.
\]

This is not true in the general case (though the length of the segment is a lower bound for the length of that geodesic, as Lemma 3.5 shows).

**Definition 4.1.** An exponential set \( M \subset \Sigma \) is the exponential of a (closed, selfadjoint) subspace through the origin. In other words, \( M = e^H \) with \( H \) a closed subspace of \( A_h \).

**Lemma 4.2.** If \( M \) is a convex exponential set in \( \Sigma \), the geodesic symmetry \( \sigma_p : q \mapsto pq^{-1}p \) maps \( M \) into \( M \) for any \( p \in M \).

**Proof.** The map \( \sigma_p \) maps any geodesic through \( p \), \( \gamma(t) = p^{1/2}e^{tp^{-1/2}vp^{-1/2}p^{1/2}} \), onto \( \gamma(-t) \); now it is clear that it is an isometry of \( \Sigma \) and it maps \( M \) into \( M \).

Note that, if \( M \) is convex and \( a \in M \), then \( a^\alpha = e^{\alpha \ln a} \) is in \( M \) for any real \( \alpha \). This observation together with the previous lemma leads to the following characterization of convexity.

**Proposition 4.3.** If \( M \) is a convex, exponential set in \( \Sigma \), then

\[
aba \in M \quad \text{whenever} \quad a, b \in M.
\]

**Proof.** Note that \( aba = a^{3/2}(a^{1/2}b^{-1}a^{1/2})^{-1}a^{3/2} = \sigma_{a^{3/2}} \circ \sigma_{a^{1/2}}(b) \).

The converse of this last statement is also true (this can be easily seen iterating the property above in order to construct \( \gamma(t) \) for given \( p, q \in M \).

Let us see how this property looks in the tangent \( H \) (recall that \( M = e^H \)).

This result is related to the results of Porta and Recht [7].

**Theorem 4.4.** If \( H \) is a closed subspace of \( A_h \) (in the norm topology of \( A \)), then \( M = e^H \) is a geodesically convex subset of \( \Sigma \) if and only if \([x, [x, y]] \in H \) for any \( x, y \in H \).

**Proof.** We use property (4.1) above to identify convex sets; the proof follows the guidelines of [5] for matrices, and we translate it here.

We first assume that \( H \) has the double bracket property. Set \( D_x : A \to A \), \( D_x = L_x - R_x \), the difference between left and right multiplication by \( x \) in \( A \).

Let us consider the completion of \( A \) with respect to the trace inner product, namely \( \mathcal{H} = L^2(A, \tau) \). Clearly \( \mathcal{H}_\mathbb{R} = L^2_\mathbb{R}(A_h, \tau) \) contains the completion of \( \mathcal{H} \) as a proper, closed subspace. Put \( \mathcal{H}_1 = L^2_\mathbb{R}(H, \tau) \). Since \( \tau \) is a normal trace,
the involution $*$ extends to a bounded antilinear operator $J$ of $\mathcal{H}_R$, and the map $D_x$ extends uniquely to a bounded linear operator of $\mathcal{H}_R$ (which we still call $D_x$).

First we establish the identity

$$T_x(y) = g(D_x/2)(y),$$  \hspace{1cm} (4.2)

where $T_x$ is the extension of the map from Corollary 2.2, and

$$g(z) := \frac{\sinh(z)}{z} = \sum_{n \geq 0} \frac{z^{2n}}{(2n + 1)!}$$

is an entire function. Note that $g(z) = (2z)^{-1}(e^z - e^{-z})$. To prove (4.2), we take the derivative with respect to $t$ in the identity $X(t)e^{X(t)} = e^{X(t)}X(t)$, where $X(t) = x + ty$; after rearranging the terms we come up with

$$\frac{1}{2}(e^{D_x/2} - e^{-D_x/2})y = (D_x \circ T_x)(y).$$

Note that if $D_x$ were invertible, we would be done; this is not necessarily the case. However, $D_x^2 = D_x \circ D_x$ is selfadjoint when restricted to $A$, and since $T_x$ (more precisely, its extension) is also selfadjoint (Corollary 2.2), the operator $T = T_x - g(D_x/2)$ is selfadjoint on $A$, and hence on $\mathcal{H}$ (note that $g(z)$ involves only even powers of $z$). The equation above says that we have proved that $(D_x \circ T)(y) = 0$ for any $y \in A$; in other words $T$ maps $\mathcal{H}$ into $\{x\}' = \{\xi \in A_b: \xi x = x\xi\}$. A straightforward computation shows that $Tb = 0$ for any $b \in \{x\}'$, which proves that $T = 0$, that is, equation (4.2) holds.

Now, for $x, y \in H$ consider the curve $e^{\alpha(t)} = e^{tx} ye^{tx}$. Clearly $\alpha(0) = y \in H \subset \mathcal{H}_1$; we will prove that $\alpha$ obeys a differential equation in $\mathcal{H}_2$ which has a flow that maps $\mathcal{H}_1$ into $\mathcal{H}_1$, and by the uniqueness of the solution of such an equation we will have $e^{tx} ye^{tx} = e^{\alpha(1)} \in e^H = M$.

Differentiating at $t = t_0$ the equation yields

$$xe^{\alpha(t_0)} + e^{\alpha(t_0)}x = d\exp_{\alpha(t_0)}(\dot{\alpha}(t_0)) = e^{\alpha(t_0)/2}T_{\alpha(t_0)}(\dot{\alpha}(t_0))e^{\alpha(t_0)/2} = e^{\alpha(t_0)/2}g(D_{\alpha(t_0)/2})(\dot{\alpha}(t_0))e^{\alpha(t_0)/2}.$$ 

Note that $g(z)$ is invertible whenever $z$ is a bounded linear operator, and also that the power series for $z \coth(z/2)$ involves only even powers of $z$. On the other hand, $D_x^2 = D_{x/2}$ and $e^{z}x e^{-z} = e^{D_x}x$, hence

$$\dot{\alpha} = g^{-1}(D_{\alpha/2}) \circ (e^{-\alpha/2}xe^{\alpha/2} + e^{\alpha/2}xe^{-\alpha/2}) = g^{-1}(D_{\alpha/2}) \circ (e^{D_{\alpha/2}} + e^{-D_{\alpha/2}})(x) = D_{\alpha} \coth(D_{\alpha/2})(x) = \sum_{n} c_n D_{\alpha}^{2n} x = \sum_{n} c_n D_{\alpha}^2 \circ \ldots \circ D_{\alpha}^2(x) = F(\alpha).$$

Since $D_x^2(x) = [z, [z, x]]$, $F(z) = \sum_n c_n D_x^{2n}(x)$ can be regarded as a map from $\mathcal{H}_1$ to $\mathcal{H}_1$, and since it is clearly an analytic map of $\mathcal{H}$ into $\mathcal{H}$, it fulfils a Lipschitz condition. Now the unique solution must be $\alpha(t) = \ln(e^{tx} ye^{tx})$. This proves that $M$ is convex whenever $H$ has the double bracket property.

To prove the other implication, assume that $M = e^H$ is convex and $H$ is closed in the norm topology of $A_h$. Clearly the path $\alpha$ stays in $H$ for any value of $t$ (here $e^{\alpha(t)} = e^{tx} ye^{tx}$), and the same is true for $\dot{\alpha}$. Now since $\dot{\alpha}(t) = D_{\alpha(t)} \coth(D_{\alpha(t)/2})x$, ...
we have
\[
\lim_{t \to 0} \frac{\dot{\alpha}(t) - \dot{\alpha}(0)}{t^2} = \lim_{t \to 0} \frac{(1 + \frac{1}{12} t^2 D^2 \alpha(t)) x - x}{t^2} + tO(t) = \frac{1}{12} D^2_g(x),
\]
which proves that $D^2_g(x) = [y, [y, x]]$ belongs to $H$ whenever $x$ and $y$ are in $H$. \qed

5. Projections

From now on assume $M = e^H$ is a convex exponential set in $\Sigma$. As before, we identify the derivatives of all the geodesics at $p \in M$ with the tangent space of $M$ at $p$, in order to define the angles between curves and sets in a natural way: note that, in this way, $T_1 M = H$ and $T_p M = p^{1/2} H p^{1/2}$ (which can be thought of as the parallel transport along the geodesic joining 1 and $p$ in $M$).

In particular, $T_1 \Sigma$ is naturally identified with $A_h$ and the same is true for $T_p \Sigma$, for any $p \in \Sigma$, since $p^{1/2} A_h p^{1/2} = A_h$ (this is clear also from the fact that $\Sigma$ is open in $A_h$).

**Lemma 5.1.** Let $r \in \Sigma$. There is at most one point $p = \Pi_M(r)$ in $M$ such that the geodesic joining $r$ and $p$ is orthogonal to $M$ at $p$.

**Proof.** Assume that there are two points $p$ and $p'$ in $M$ and two vectors $v$ and $v'$ orthogonal to $M$ at $p$ and $p'$ respectively such that $\gamma_1(1) = \text{Exp}_p(v) = \gamma_2(1) = \text{Exp}_{p'}(v') = r$, and consider the geodesic triangle with sides the given geodesics and the unique geodesic in $M$ joining $p$ and $p'$. Since the angles at $p$ and $p'$ are right angles, and the sum of the inner angles of any such geodesic triangle is less than or equal to $\pi$, it must be that the angle at $r$ is zero: since geodesics are unique (given an initial velocity and an initial position $r$), it must be that $\gamma_1 = \gamma_2$, hence $p = p'$ and $v = v'$.

Set $NM$ as the normal bundle of $M$, that is, $NM = \{(p, v) : p \in M, v \in (T_p M)^\perp\}$.

Consider the map $E : NM \to \Sigma$ given by $(p, v) \mapsto \text{Exp}_p(v)$; since $E$ is analytic and with the right identifications has differential (at $(p, 0)$) the identity map, $E(NM)$ contains an open neighbourhood of $M$ in $\Sigma$ (with the norm topology).

**Lemma 5.2.** The map $\Pi_M : E(NM) \to M$ that assigns the endpoint of the minimizing geodesic is contractive for the geodesic metric.

**Proof.** If $r, s$ are two points in $E(NM)$, we will prove that this projection is contractive. Assume $\Pi_M(r) = p$, $\Pi_M(s) = q \in M$, $v \in T_p \Sigma$ is orthogonal to $M$ at $p$ and $w$ is orthogonal to $M$ at $q$. Let us consider the distance function
\[
f(t) = \text{dist}^2(\text{Exp}_p(tv), \text{Exp}_q(tw)) = \text{dist}^2(\gamma_1(t), \gamma_2(t)),
\]
where $\gamma_1(t)$ is the only geodesic with initial speed $v$ starting at $p$ and $\gamma_2(t)$ is the only geodesic with initial speed $w$ starting at $q$. Namely,
\[
\gamma_1(t) = p^{1/2} e^{tp^{1/2}v} p^{1/2} \quad \text{and} \quad \gamma_2(t) = q^{1/2} e^{tq^{-1/2}w} q^{1/2}.
\]
Since \( v \in (T_p M)^1 \) and \( w \in (T_q M)^1 \), we have
\[
\langle v, p^{1/2} x p^{1/2} \rangle_p = \tau(x p^{-1/2} v p^{-1/2}) = 0 \quad \text{for any } x \in H = T_1 M,
\]
\[
\langle w, q^{1/2} y q^{1/2} \rangle_q = \tau(y q^{-1/2} w q^{-1/2}) = 0 \quad \text{for any } y \in H = T_1 M.
\] (5.1)

Now we use the formula \( \text{dist}(e^A, e^B) = \|\ln(e^{A/2} e^{-B} e^{A/2})\|_2 \) for \( A = \ln(\gamma_1(t)) \) and \( B = \ln(\gamma_2(t)) \), to write
\[
f(t) = \|\ln(\gamma_1^{1/2} \gamma_2^{-1} \gamma_1^{1/2})\|_2^2 = \tau(\ln^2(\gamma_1^{1/2} \gamma_2^{-1} \gamma_1^{1/2})).
\]

Assume that \( C \) is a simple, positively oriented curve in \( \mathbb{C} \), around the spectrum of \( \alpha_0 = p^{1/2} q^{-1} p^{1/2} \). Then we can use the Cauchy formula to calculate \( \ln^2(a) \) for any element \( a \in A \) such that \( \sigma(a) \subseteq \text{int}(C) \), namely
\[
\ln^2(a) = \frac{1}{2\pi i} \int_C \ln^2(z - a)^{-1} \, dz.
\] (5.2)

Putting \( \alpha(t) = \gamma_1^{1/2}(t) \gamma_2^{-1}(t) \gamma_1^{1/2}(t) \), this formula holds true for \( \alpha_0 = \alpha(0) \) and for \( \alpha(t) \) for \( t \) sufficiently small. Note that
\[
f(t) = \tau(\gamma^{-1/2}(t) \gamma^{1/2}(t) \ln^2(\alpha(t))) = \tau(\gamma^{-1/2}(t) \ln^2(\alpha(t)) \gamma^{1/2}(t)).
\]

If \( x \) is invertible in \( \mathcal{A} \), \( x g(a) x^{-1} = g(xax^{-1}) \) for any element \( a \in \mathcal{A} \) and any analytic function \( g \) in a neighbourhood of \( \sigma(a) \). Then
\[
f(t) = \tau(\ln^2[\gamma_1(t) \gamma_2^{-1}(t)]) = \frac{1}{2\pi i} \int_C \ln^2(z) \tau[(z - \gamma_1(t) \gamma_2^{-1}(t))^{-1}] \, dz.
\]

Now we compute \( f'(0) \); note first that \( \gamma_1(0) \gamma_2^{-1}(0) = pq^{-1} \) and also that
\[
\frac{d}{dt}_{t=0} \gamma_1(t) \gamma_2^{-1}(t) = -vq^{-1} + pq^{-1} wq^{-1}.
\]

Using the properties of the trace we get
\[
\frac{d}{dt}_{t=0} f(t) = -\frac{1}{2\pi i} \int_C \ln^2(z) \tau[(z - pq^{-1})^{-2} (-vq^{-1} + pq^{-1} wq^{-1})] \, dz
\]
\[
= \tau \left[ \left( -\frac{1}{2\pi i} \int_C \ln^2(z - pq^{-1})^{-2} \, dz \right) (-vq^{-1} + pq^{-1} wq^{-1}) \right].
\]

If we integrate by parts the first factor inside the trace, we obtain (note that \( (d/dz) \ln^2(z) = 2 \ln(z) z^{-1} = 2z^{-1} \ln(z) \) and \( C \) is a closed curve)
\[
f(0) = \tau \left[ \left( \frac{1}{2\pi i} \int_C 2 \ln(z) z^{-1} (z - pq^{-1})^{-1} \, dz \right) (-vq^{-1} + pq^{-1} wq^{-1}) \right]
\]
\[
= -\tau \left[ \left( \frac{1}{2\pi i} \int_C 2 \ln(z) z^{-1} (z - pq^{-1})^{-1} \, dz \right) vq^{-1} \right]
\]
\[
+ \tau \left[ \left( \frac{1}{2\pi i} \int_C 2 \ln(z) z^{-1} (z - pq^{-1})^{-1} \, dz \right) pq^{-1} wq^{-1} \right].
\]

Therefore,
\[
f(0) = -\frac{1}{2\pi i} \int_C 2 \ln(z) z^{-1} \tau[q^{-1}(z - pq^{-1})^{-1} v] \, dz
\]
\[
+ \frac{1}{2\pi i} \int_C 2 \ln(z) z^{-1} \tau[q^{-1/2}(z - pq^{-1})^{-1} pq^{-1} wq^{-1/2}] \, dz.
\]

Using the elementary identities
\[
p^{1/2}(z - p^{1/2} q^{-1} p^{1/2})^{-1} p^{-1/2} = (z - pq^{-1})^{-1} = q^{1/2}(z - q^{-1/2} pq^{-1/2})^{-1} q^{-1/2},
\]
one arrives at the expression
\[
\hat{f}(0) = -\frac{1}{2\pi i} \int_{C} 2 \ln(z)z^{-1}\tau[q^{-1}p^{-1/2}(z - p^{-1/2}q^{-1/2})^{-1}p^{-1/2}v] dz
\]
\[
+ \frac{1}{2\pi i} \int_{C} 2 \ln(z)z^{-1}\tau[(z - q^{-1/2}pq^{-1/2})^{-1}q^{-1/2}pq^{-1/2}q^{-1/2}wq^{-1/2}] dz
\]
\[
= -2\tau[q^{-1}p^{-1/2}p^{-1/2}pq^{-1/2} \ln(p^{-1/2}q^{-1/2})p^{-1/2}v]
\]
\[
+ 2\tau[\ln(q^{-1/2}pq^{-1/2})q^{-1/2}p^{-1/2}q^{-1/2}pq^{-1/2}q^{-1/2}wq^{-1/2}]
\]
\[
= -2\tau[\ln(p^{-1/2}q^{-1/2})p^{-1/2}vp^{-1/2}] + 2\tau[\ln(q^{-1/2}pq^{-1/2})q^{-1/2}wq^{-1/2}]
\]
\[
= 0 + 0 = 0,
\]
which holds by the orthogonality relations (5.1), putting \( x = \ln(p^{-1/2}q^{-1/2}) \) (recall that \( M \) is convex) and \( y = \ln(q^{-1/2}pq^{-1/2}) \).

Since \( f(t) \) is a convex function, \( f \) has a global minimum at \( t = 0 \), which proves that \( \text{dist}(\Pi_{M}(r), \Pi_{M}(s)) = \text{dist}(p, q) \leq \text{dist}(r, s) \).

**Lemma 5.3.** Let \( r \in \Sigma \) and let \( M \) be a convex exponential set. Then there exists \( p \in M \) such that \( \text{dist}(r, M) = \text{dist}(r, p) \) if and only if there is a geodesic through \( r \) orthogonal to \( M \).

**Proof.** Assume first that there is a point \( p \in M \) such that the geodesic \( \gamma \) through \( p \) and \( r \) is orthogonal to \( M \) at \( p \). Now for any point \( q \in M \), take a geodesic \( \beta \) joining \( q \) and \( r \), and a geodesic \( \delta \) joining \( p \) to \( q \). Consider the geodesic triangle with sides \( \gamma, \beta, \delta \); the angle opposite to \( \beta \) is a right angle, so (see Lemma 3.6)

\[
\text{Length}(\beta)^2 \geq \text{Length}(\delta)^2 + \text{Length}(\gamma)^2 \geq \text{Length}(\gamma)^2.
\]

This proves that \( \text{dist}(p, r) \leq \text{dist}(q, r) \) for any \( q \in M \).

Assume now that \( p \in M \) has the minimizing property and consider, for any point \( q \in M \), the geodesic \( \gamma_{p,q}(s) \) joining \( p \) to \( q \) (note that it is inside \( M \) for any \( s \) by virtue of the convexity). Now consider the family of geodesics

\[
\gamma_s(t) = \gamma_{r,\gamma_{p,q}}(t) = r^{1/2}(r^{-1/2}p^{-1/2}(p^{-1/2}q^{-1/2})^s p^{1/2}r^{-1/2})^{1/2},
\]

that is, the family of geodesics joining \( r \) to \( \gamma_{p,q}(s) \).

Put \( g(s) = \text{Length}(\gamma_s)^2 = \text{dist}(r, \gamma_{p,q}(s)) \). This function has a minimum at \( s = 0 \), hence (since it is \( C^\infty \)) it must be that \( \dot{g}(0) = 0 \). As in the proof of Lemma 5.2, we have

\[
g(s) = \tau(\ln^2(r\gamma_{p,q}(s)^{-1})) = \frac{1}{2\pi i} \int_{C} \ln^2(z)\tau([z - r\gamma_{p,q}(s)^{-1}]^{-1}) dz.
\]

Taking the derivative at \( s = 0 \) and integrating by parts we obtain

\[
0 = \dot{g}(0) = -2\tau(\ln(rp^{-1})\ln(qp^{-1})p^{-1}) = 2\tau(\ln(qp^{-1})p^{-1}\ln(pr^{-1})).
\]

On the other hand, the angle subtended by \( \gamma_{p,q} \) and \( \gamma_{r,p} \) at \( p \) is

\[
\langle \dot{\gamma}_{r,p}(1), \dot{\gamma}_{p,q}(0) \rangle_p = \tau(\ln(qp^{-1})p^{-1}\ln(pr^{-1})).
\]

This proves that \( \gamma_{r,p} \) is orthogonal to any geodesic at \( p \) contained in \( M \), and, by definition, it is orthogonal to \( M \).

The following is related to the main result in Porta and Recht [6].
Theorem 5.4. If $M = e^H$ is a convex exponential set, and there is a closed, orthogonal supplement $S$ for $H$ (namely $A_h = H \oplus_{\perp r} S$), then for any point $r \in \Sigma$ there is a geodesic $\gamma_r$ through $r$ orthogonal to $M$. If $q_r$ is the point of intersection between $\gamma_r$ and $M$, then $q_r$ is the point of $M$ closest to $r$. This projection $\Sigma \to M$ (which maps $r \mapsto q_r$) is contractive for the geodesic distance.

Proof. Exactly as in [6], there is an equality of sets $E(NM) = \Sigma$, where $NM$ stands for the normal bundle of $M$, that is, the pairs $(p, v)$ with $p \in M$ and $v \perp_p M$. The authors prove in that paper that the set $E(NM)$ is both open and closed in $\Sigma$; to prove both assertions the key observation is that the submanifold $M$ has an open tubular neighbourhood contained in $E(NM)$, obtained by differentiating at $(p, 0)$ the map $(p, v) \mapsto \exp_p(v)$.

The typical examples for this situation arise when $H = B_h$ for a subalgebra $B$ of $A$. In this case, by a result of Takesaki [8], there is a conditional expectation $E : A \to A$ with rank $B$, compatible with $\tau$ (that is, $\tau(E(x)) = \tau(x)$ for any $x \in A$).

Corollary 5.5. If $H$ is a closed subspace in $A_h$ (supplemented as in the previous theorem) such that $[x, [x, y]] \in H$ whenever $x, y \in H$, then for any $z \in A_h$ we can factor

$$e^z = e^y e^w e^y$$

for unique $y \in H$ and $w \in A_h$ such that $\tau(wx) = 0$ for any $x \in H$. Moreover, $e^{2y}$ minimizes the (geodesic) distance between $M = e^H$ and $e^z$, and is unique with that property.

Corollary 5.6. Fix $D$ a maximal abelian subalgebra of $A$. Then for any $x \in A_h$ there are unique $d \in D_h$ and $v \in A_h$ such that $\tau(vz) = 0$ for any $z \in D$ and $e^x = d e^v d$.

Corollary 5.7. If $H$ is a closed, supplemented subspace in $A_h$ such that $[x, [x, y]] \in H$ whenever $x, y \in H$, then for any $g \in G_A$ we can factor

$$g = e^x e^y u$$

for unique $x \in H, y \in H^\perp$ and $u$ in the unitary group of $A$.

Proof. Note that $gg^* \in \Sigma$, hence $gg^* = e^x e^{2y} e^x$ where $x, y$ are as required. Now take $u = e^{-y} e^{-x} g$; a straightforward computation shows that $uu^* = u^* u = 1$. Uniqueness follows from the uniqueness of $x, y$.

References

1. R. Bhatia, ‘On the exponential metric increasing property’, Linear Algebra Appl. 375 (2003) 211–220.
2. G. Corach, H. Porta and L. Recht, ‘The geometry of the space of selfadjoint invertible elements in a C*-algebra’, Integral Equations Operator Theory 19 (1993) 333–359.
3. G. Corach, H. Porta and L. Recht, ‘Convexity of the geodesic distance on spaces of positive operators’, Illinois J. Math. 38 (1994) 87–94.
4. P. Eberlein, ‘Structure of manifolds of nonpositive curvature’, Global differential geometry and global analysis, Lecture Notes in Mathematics 1156 (Springer, Berlin, 1985) 86–153.
5. G. D. Mostow, ‘Some new decomposition theorems for semi-simple groups’, Mem. Amer. Math. Soc. 14 (1955) 31–54.
6. H. Porta and L. Recht, ‘Conditional expectations and operator decompositions’, *Ann. Global Anal. Geom.* 12 (1994) 335–339.
7. H. Porta and L. Recht, ‘Exponential sets and their geometric motions’, *J. Geom. Anal.* (2) 6 (1996) 277–285.
8. M. Takesaki, ‘Conditional expectations in a von Neumann algebra’, *J. Funct. Anal.* 9 (1972) 306–321.

E. Andruchow and G. Larotonda
Instituto de Ciencias
Universidad Nacional de General Sarmiento
J. M. Gutierrez 1150
(1613) Los Polvorines
Buenos Aires
Argentina

eanduch@ungs.edu.ar
glaroton@ungs.edu.ar