Correlation and fluctuation in a random average process on an infinite line with a driven tracer

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Abstract. We study the effect of a single biased tracer particle in a bath of other particles performing the random average process (RAP) on an infinite line. We focus on the long time behavior of the mean and the fluctuations of the positions of the particles and also the correlations among them. In the long time limit these quantities have well defined scaling forms and grow with time as $\sqrt{t}$. A differential equation for the scaling function associated with the correlation function is obtained and solved perturbatively around the solution for a symmetric tracer. Interestingly, when the tracer is totally asymmetric, further progress is enabled by the fact that the particles behind the tracer do not affect the motion of the particles in front of it, which leads in particular to an exact expression for the variance of the position of the tracer. Finally, the variance and correlations of the gaps between successive particles are also studied. Numerical simulations support our analytical results.

Keywords: correlation functions, exact results, stochastic particle dynamics (theory)
1. Introduction

The motion of non-overtaking particles in narrow channels is known as single-file diffusion. In such one-dimensional geometry the motion of any particle is hemmed by its neighbors. As a result the particles cannot bypass each other and keep their initial order the same over time. Study of such restricted motion of particles has been started by Harris [1] and Jepsen [2]. They showed that when the particles evolve according to Hamiltonian dynamics the mean squared displacement (MSD) of a tagged particle (also called tracer particle (TP)) grows diffusively, whereas for Brownian particles the MSD of a TP grows subdiffusively. Recently, several experiments have been able to
observe TP diffusion by passive microrheology in zeolites, transport of colloidal particles or charged spheres in narrow circular channels [3–9]. Such experimental evidence has generated a great revival of interest in tagged particle diffusion. Many different results regarding tagged particle diffusion have been reported for various systems with differently organized dynamics [10–25]. For example, in addition to the mean position and the MSD of the TP, probability distribution functions (PDFs) associated with particle displacements have been studied as well [15, 19–21, 23, 25–27].

One of the simplest systems where tagged particle diffusion has been studied in detail is the simple exclusion process (SEP). This process is usually defined on a one-dimensional lattice, where each lattice site is occupied by one hardcore particle or it is empty. In every small time interval $dt$, each particle moves to the neighboring site on the right with probability $\alpha dt$ and that on the left with probability $\beta dt$ if the target site is empty. The hardcore interaction among the particles plays a dramatic role in the long time asymptotic growth of MSD of a tagged particle. In the absence of bias ($\alpha = \beta = 1/2$), the mean squared fluctuation of the displacement of a TP grows subdiffusively as $\sim A_0 \sqrt{t}$ for large $t$, where the prefactor $A_0$ is given explicitly in terms of particle density $\rho_0$ as $A_0 = \frac{1-\rho_0}{\rho_0} \sqrt{\frac{2}{\pi}}$ [1, 11, 13]. On the other hand, when there is non-zero bias $\alpha \neq \beta$, the MSD grows diffusively as $\sim (\alpha - \beta)(1-\rho_0)t$ for large $t$ [28, 29].

Similar results have also been proved for another interesting interacting and widely studied many particle system called the random average process (RAP), first introduced by Ferrari and Fontes [30]. In RAP particles move on a one-dimensional continuous line, in contrast to SEP where hardcore particles move on a lattice. Each particle moves to the right (left) by a random fraction of the space available until the next nearest particle on the right (left) with some rate $\alpha$ ($\beta$). Thus the jumps in either direction are a random fraction $\eta$ of the gap to the nearest particle in that direction, where the random number $\eta \in [0,1)$ is chosen from some distribution $R(\eta)$. As a result the particles in RAP also never overtake each other, keeping their initial order unchanged over time as in other single-file motions. The RAP appears in a variety of problems such as force propagation in granular media [31, 32], the porous medium equation [33], models of mass transport [32, 34], models of voting systems [35], models of wealth distribution [36] and the generalized Hammersley process [37]. In the unbiased ($\alpha = \beta$) case the MSD of a TP in RAP also grows subdiffusively as $\sim A \sqrt{t}$, whereas it diffuses as $\sim Dt$ in the globally uniform bias ($\alpha \neq \beta$) case for long time [38]. The constants $A$ and $D$ in the prefactors are computed exactly in terms of particle density $\rho_0$ and the moments of the jump distribution $R(\eta)$ [38].

In single-file motion, the movements of individual particles become in general strongly correlated because any large progressive displacement of a given particle in one direction also necessarily requires large displacements of more and more other particles in the same direction. In the context of RAP, such correlation between positions of two tagged particles has been computed explicitly in terms of their label separation $r$ and time $t$. If $x_i(t)$ represents the position of the $i$th particle, then for large $t$ the correlation function $c_{i,j}(t) = \langle [x_i(t) - \langle x_i(t)\rangle][x_j(t) - \langle x_j(t)\rangle] \rangle$ is given by the following scaling form [38]:

$$c_{i,j}(t) = \rho_0^{-2} \frac{\mu_2}{\sqrt{2\pi (\mu_1 - \mu_2)}} \sqrt{2\mu_1(\alpha + \beta)t} \ g \left( \frac{i - j}{\sqrt{2\mu_1(\alpha + \beta)t}} \right),$$

with,

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\[ g(u) = e^{-\frac{u^2}{2}} - \sqrt{\frac{\pi}{2}} u \text{erfc} \left( \frac{|u|}{\sqrt{2}} \right) \]  

where \( \mu_1 \) and \( \mu_2 \) are the first and second moments, respectively, of the jump distribution \( R(\eta) \) and the superscript ‘gb’ indicates ‘global bias’. Note that the scaling function \( g(u) \) is independent of the parameters \( \alpha \) and \( \beta \), i.e. independent of the global bias. Naturally a question arises: what happens if the system is locally biased instead of globally biased? More precisely, if a single particle in RAP moves asymmetrically while all others are moving symmetrically, how does the MSD grow with time? How correlated are the positions of two particles? In this paper we address these questions.

Motion of a single driven tracer particle (DTP) in the pool of other non-driven interacting (hardcore interaction with the TP and among others) particles has been studied in various contexts. In experimental studies, single driven tracers in quiescent media have been used to probe rheological properties of complex media such as DNA [39], polymers [40], granular media [41, 42] or colloidal crystals [43]. Some practical examples of a biased tracer are a charged impurity being driven by applied electric field or a colloidal particle being pulled by optical tweezers in the presence of other colloid particles performing random motion. On the theoretical side, situations have been considered where the surrounding medium is a symmetric simple exclusion process (SSEP) and the tagged particle is a hardcore tracer driven in a preferred direction. In this context the effect of the biased tracer has been quantified in terms of both the tracer motion and the perturbation of the density profile [44–50]. Contrary to what happens in higher dimensions [48–50], the velocity of the tracer moving in a 1D SSEP vanishes [44–47]. It has been shown theoretically that the perturbation of the density field of the bath particles generically consists in a denser region at the front and a depleted region at the back as expected intuitively. The amplitude of the difference between the density profiles in the biased and the unbiased cases decays to zero exponentially as one goes far from the driven tracer on both sides in 1D. However in higher dimensions such decay is dependent on the direction along which one moves away from the driven tracer [44–50]. Non-homogeneous density profiles in the presence of driven tracer have also been evidenced in numerical simulations, for example in colloidal crystals [51], where it has been shown that the defects created by the DTP remain localized near the DTP and affect the frictional drag force.

In the next section we define the model and present the summary of our results.

2. Model definition and summary of the results

We consider an infinite number of particles occupying an infinite line with density \( \rho_0 \). Without any loss of generality, we label the DTP as the zeroth particle and then label other particles according to their positional order with respect to the TP from \(-\infty \) to \( \infty \) (see figure 1). Let us denote the positions of the particles at time \( t \) by
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$x_i(t) \in \mathbb{R}$ for $i \in \mathbb{Z}$. Initially, i.e. at $t = 0$, the particles are arranged according to the following fixed configuration:

$$x_i(0) = \rho_0^{-1} i, \quad i = \ldots, -2, -1, 0, 1, 2, \ldots$$  \hspace{1cm} (3)

Hence all the averages $\langle \ldots \rangle$ in this paper are taken over stochastic evolution. The dynamics of the particles are given as follows. In an infinitesimal time interval $t$ to $t + dt$, any particle (say the $i$th) other than the DTP jumps from $x_i(t)$, either to the right or to the left with probability $\frac{1}{2t}$, and with probability $(1 - \frac{1}{2t})$ it stays at $x_i(t)$. The DTP jumps from $x_0(t)$ to the right with probability $pt$, to the left with probability $qt$ and does not jump with probability $(1 - (p + q)dt)$. The amount of jump, either to the right or to the left, made by any particle is a random fraction of the space available between the particle and its neighboring particle to the right or to the left. For example, the $i$th particle jumps by $\eta_i^r [x_{i+1}(t) - x_i(t)]$ to the right and by $\eta_i^l [x_{i-1}(t) - x_i(t)]$ to the left. The random variables $\eta_i^r, \eta_i^l$ are independently chosen from the interval $[0, 1)$ and each is distributed according to the same distribution $R(\eta)$, with moments

$$\mu_k = \int_{\eta=0}^{1} \eta^k R(\eta) d\eta. \hspace{1cm} (4)$$

The time evolution of the positions $x_i(t)$ can be written as

$$x_i(t + dt) = x_i(t) + \sigma_i^r \eta_i [x_{i+1}(t) - x_i(t)] + \sigma_i^l \eta_i [x_{i-1}(t) - x_i(t)] \hspace{1cm} (5)$$

where the $\eta$ variables are independent and identically distributed according to $R(\eta)$. For $i \neq 0$, $\sigma_i^r$ and $\sigma_i^l$ are 1 with probability $\frac{dt}{2}$ and 0 otherwise. The random variable $\sigma_0^r$ is 1 with probability $pt$ and 0 with probability $1 - pt$. Similarly, $\sigma_0^l$ is 1 with probability $qt$ and 0 with probability $1 - qt$. Clearly, we see that all the particles are symmetrically moving except the zeroth particle, which moves asymmetrically. In this paper we are mainly interested in the effect of this asymmetric motion of TP on the fluctuations and the correlations among the positions of other particles.

We first look at the effect of the biased tracer on the time dependence of the average position $y_i(t) = \langle x_i(t) \rangle$, average gap $h_i(t) = \langle g_i(t) \rangle = \langle x_{i+1}(t) - x_i(t) \rangle$ and average particle density $\rho(w, t) = \langle \sum_{i=-\infty}^{\infty} \delta[w - x_i(t)] \rangle$ profile in section 3. In the long time limit we find that these three quantities have the following scaling forms:

\hspace{1cm} Figure 1. Schematic diagram of the RAP with a DTP (zeroth particle colored in red) on an infinite line. The variable $x_i$ represents the position of the $i$th particle and $g_i = x_{i+1} - x_i$ represents the gap between the $(i+1)$th and $i$th particles. The DTP hops to the left with rate $p$ and to the right with rate $q$, whereas all other particles ($i \neq 0$) hop to the left or to the right with the same rate $1/2$. 

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respectively, are related by

time, \( t \), the scaling functions
also represents the contribution at order smaller than and \( x \) and the pair gap correlation function \( \rho \). In the

\[ y(t) = \rho_0^{-1} \sqrt{2\mu_i t} \mathcal{Y}\left( \frac{i}{\sqrt{2\mu_i t}} \right) + o(\sqrt{t}), \]

\[ h_i(t) = \rho_0^{-1} \mathcal{H}\left( \frac{i}{\sqrt{2\mu_i t}} \right) + o(\sqrt{t}), \]

\[ \rho(w, t) = \Omega\left( \frac{w}{\sqrt{2\mu_i t}} \right) + o(\sqrt{t}), \]

where the index variables \( i \) and the space variables \( w \) are rescaled appropriately by time \( t \). Explicit forms of these scaling functions are given in (31), (37) and (41)–(42) respectively. Here \( o(\ell) \) represents the contribution at order smaller than \( \ell \). Note that in the above three equations the density \( \rho_0 \) appears as an overall factor. This is because the dynamics is invariant under a rescaling of the position variables \( x_i \rightarrow a x_i \). Consequently, we expect that \( \rho_0 \) will appear only as an overall factor in different average quantities, e.g. in mean positions, in correlation functions etc. Since the gap variables \( g_i(t) \) are equal to \( x_{i+1}(t) - x_i(t) \), the scaling functions \( \mathcal{Y}(x) \) and \( \mathcal{H}(x) \) associated to \( \langle x_i(t) \rangle \) and \( \langle g_i(t) \rangle \) respectively, are related by \( \mathcal{H}(x) = \partial_x \mathcal{Y}(x) \). We compute these scaling functions \( \mathcal{Y}(x) \), \( \mathcal{H}(x) \) and also \( \Omega(\ell) \) exactly in section 3 and compare them with numerical measurements. For obvious reasons, we perform our numerical simulations on a ring of size \( L \) with large number \( N \) of particles. In all our simulations we consider \( L = 1 \), \( N = 200 \) (unless otherwise specified) and uniform jump distribution \( R(\eta) = 1 \), whose moments are \( \mu_k = \frac{1}{k+1} \). In the simulation, we observe that the late time growth of the average position \( y_0(t) \) of the DTP changes from \( \sim B_{\text{line}} \sqrt{t} \) to linear growth \( \sim B_{\text{ring}} t \) as \( t \) is increased. We compute the constants \( B_{\text{line}} \) and \( B_{\text{ring}} \) theoretically and compare them with numerical measurements. In particular, we find that the crossover between the line and the ring geometries can be captured through a nice crossover function, which is given explicitly in (29) and plotted in figure 3.

Next in section 4 we study the pair position correlation function \( c_{i,j}(t) = \langle x_i(t)x_j(t) \rangle - y_i(t)y_j(t) \) and the pair gap correlation function \( d_{i,j}(t) = \langle g_i(t)g_j(t) \rangle - h_i(t)h_j(t) \). In the case where all particles hop symmetrically to the right and to the left with a rate, say, \( 1/2 \), many results concerning the two-point correlations of the positions have been derived by Rajesh and Majumdar [38]. In fact, in [38] a more general situation has been considered where all the particles are identical in the sense that all of them have the same hopping rate \( \alpha \) to the right and the same hopping rate \( \beta \) to the left. For this case, the two-point correlation function \( c_{i,j}(t) \) has been computed. This in translationally invariant case the correlation function \( c_{i,j}(t) \) depends only on the label separation (or the initial separation) \( r = |i - j| \) between the two particles and on time \( t \). Moreover, in the large \( t \) limit it was found that the correlation function has a scaling form in terms of the rescaled variable \( \frac{r}{\sqrt{t}} \), see (1). On the other hand, in the model considered in the present paper the system is not translationally invariant, as one particle (zeroth particle) is driven and others are symmetrically moving. As a result, in our case the correlation function \( c_{i,j}(t) \) depends on the indices \( i \) and \( j \) individually. However, as we will see later, \( c_{i,j}(t) \) in our case also has a scaling form:
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\[ c_{i,j}(t) = \rho_0^2 \sqrt{2\mu_1 t} C \left( \frac{i}{\sqrt{2\mu_1 t}}, \frac{j}{\sqrt{2\mu_1 t}} \right) + o(\sqrt{t}), \]  

(7)
in terms of the rescaled variables \( x = \frac{i}{\sqrt{2\mu_1 t}} \) and \( y = \frac{j}{\sqrt{2\mu_1 t}} \), where \( \rho_0 \) is the particle density and \( \mu_1 \) is the first moment of the jump distribution \( \eta \). At the beginning of section 4 we numerically verify that in the large \( t \) limit \( c_{i,j}(t) \) indeed has the scaling form (7). Next inserting the form (7) in the discrete evolution equation for \( c_{i,j}(t) \) and taking the long time limit, we obtain a differential equation for \( C_{xy} \), in section 4.1.

In section 4.1.1 we present a perturbative solution for \( C_{xy} \), where we start with the following expansion:

\[ C(x, y) = C_0(x, y) + \frac{\epsilon}{2} C_1(x, y) + \frac{\epsilon^2}{4} C_2(x, y) + \ldots \]  

(8)
in powers of the drive strength \( \epsilon = p - q \). As a result we obtain individual equations for each \( C_i(x, y) \) with sources depending on lower order functions. One can in principle solve for each \( C_i(x, y) \) separately. In this paper we compute \( C_0(x, y) \) and \( C_1(x, y) \) explicitly and compare them with numerical measurements. Some details of the computation of \( C_i(x, y) \) have been left in appendix A.

The \( q = 0 \) case is a special case, as for this case the boundary conditions associated with the differential equation for \( C(x, y) \) become simpler. This allows us to use the image method to solve \( C(x, y) \) exactly in the first quadrant \( x \geq 0 \) and \( y \geq 0 \). In particular, we compute the variance \( \sigma^2_0(t) \) of the position of the DTP exactly as a function of time \( t \). In section 4.1.3 we prove that

\[ \sigma^2_0(t) = \langle x_0^2(t) \rangle - \langle x_0(t) \rangle^2 = \rho_0^{-2} \frac{\mu_2}{\mu_1 - \mu_2} \sqrt{\frac{2}{\pi}} \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \sqrt{2\mu_1 t} + o(\sqrt{t}). \]  

(9)

Figure 2. Average position \( y_0(t) \) of the DTP as a function of time. Simulations are done on a ring of size \( L = 1 \) with \( N = 50 \) particles. The hopping rates of the DTP for this plot are \( p = 0.5 \) and \( q = 0 \). The red dashed line corresponds to (21) (shifted along the \( y \)-axis) and the orange solid line corresponds to (26). The jump distribution is uniform, i.e. \( R(\eta) = 1 \).

\[ \text{Simulations, line theory, ring} \]
The two-point gap correlation function $d_{i,j}(t)$ is studied in section 4.2. Similar to the position correlation function $c_{i,j}(t)$, the gap correlation $d_{i,j}(t)$ also supports the scaling form under the same rescaling of indices: $x = \frac{i}{2\mu_1 t}$ and $y = \frac{j}{2\mu_1 t}$. In particular, we find that the diagonal $d_{i,i}(t)$ and the non-diagonal $d_{i,j}(t)$ gap correlations have different scaling forms in the large $t$ limit:

$$d_{i,j} = \frac{\rho_0^{-2}}{2\mu_1 t} D \left( \frac{i}{\sqrt{2\mu_1 t}}, \frac{j}{\sqrt{2\mu_1 t}} \right) + O(t^{-1}), \quad i \neq j,$$

$$d_{i,i} = \rho_0^{-2} V \left( \frac{i}{\sqrt{2\mu_1 t}} \right) + \rho_0^{-2} \left[ V_1 \left( \frac{i}{\sqrt{2\mu_1 t}} \right) + D \left( \frac{i}{\sqrt{2\mu_1 t}}, \frac{i}{\sqrt{2\mu_1 t}} \right) \right] + O(t^{-1}).$$

In section 4.2 we compute the scaling functions $V(x)$ and $V_1(x)$ exactly. The scaling function $D(x,y)$ associated with the off-diagonal correlation function $d_{i,j}(t)$ can be obtained from $C(x,y)$, as they are related via $D(x,y) = \partial_x \partial_y C(x,y)$. Finally in section 5, we conclude the paper.

### 3. Average position and particle density profile

When there is no biased TP, the average positions of the particles remain the same as their initial positions i.e. $y_i(t) = x_i(0)$. However, in the presence of a biased TP this will naturally not hold. In this section, we compute its effect on the average position $y_i(t) = \langle x_i(t) \rangle$, mean gap $h_i(t) = \langle g_i(t) \rangle = \langle x_{i+1}(t) - x_i(t) \rangle$ and mean particle density $\rho(w, t) = \langle \sum_{i=-\infty}^{\infty} \delta[w - x_i(t)] \rangle$ profile.

#### 3.1. Average positions: $y_i(t) = \langle x_i(t) \rangle$

Since the TP (zeroth particle) is driven in our model, it will induce an average motion of other particles in the direction of the drive. As a result their average positions $y_i(t)$ at time $t$ will grow from their initial positions $x_i(0) = \rho_0^{-1} i$. When $p \neq q$, we would intuitively expect that the particles will acquire a velocity and hence their average positions will grow linearly with time $\sim t$. However, as we will shortly see, their positions grow as $\sim \sqrt{t}$ in the long time limit, i.e. the velocity vanishes. On the other hand, if one looks at the motion of the particles on a finite ring then in the large $t$ limit the particles’ velocity does not go to zero and their average positions grow linearly with time. To observe this crossover in the asymptotic growth of mean positions $y_i(t)$, we start with the motion of $N$ particles on a ring of size $L$, although originally our model is defined on an infinite line (see (5)). In the end we take the two limits $t \to \infty$ and $N = \rho_0 L \to \infty$ keeping the density $\rho_0 = N/L$ fixed. We find that $t \to \infty$ and $N \to \infty$ do not commute. When $t \to \infty$ before $N \to \infty$ we find $y_i(t) \sim t$, whereas the opposite sequence of limits, i.e. first $N \to \infty$ then $t \to \infty$, yields $y_i(t) \sim \sqrt{t}$. We here emphasize that the model on the ring is considered only in this section 3.1 just to observe this crossover. In all other sections, we work with the original model (5) defined on an infinite line.
Similar to (5), one can write the dynamics of the particles on a ring [52], from which the evolution equation for the average positions \( y_i(t) = \langle x_i(t) \rangle \) can be easily computed. It is however convenient to work with the displacement variables \( z_i(t) = y(t) - y(0) \). One writes the evolution equations for the \( z_i(t) \) as

\[
\begin{align*}
\dot{z}_0 &= \mu_1 p(z_1 - z_0) + \mu_1 q(z_N - z_0) + \mu_1 (p - q) \rho_0^{-1}, \\
\dot{z}_i &= \frac{\mu_1}{2} (z_{i+1} - 2z_i + z_{i-1}), \quad i = 1, \ldots, N - 2 \quad \text{where } z_i(0) = 0 \\
\dot{z}_{N-1} &= \frac{\mu_1}{2} (z_0 - 2z_{N-1} + z_{N-2})\
\end{align*}
\]

and \( \dot{z} = dz/dt \). We solve equations (12) by taking joint Fourier–Laplace transforms. Rescaling time by \( \tau = \mu_1 t/2 \), we define the Laplace transform

\[
\hat{z}_i(s) = \int_0^\infty e^{-\tau s} z_i(\tau) d\tau
\]

and the joint Fourier–Laplace transform

\[
\hat{z}_i(s) = \sum_{k=0}^{N-1} e^{\frac{2\pi j ki}{N}} \hat{z}_i(s),
\]

where \( j^2 = -1 \). The inverse Fourier transform is given by

\[
\tilde{z}_i(s) = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{2\pi j ki}{N}} \hat{z}_i(s).
\]

After performing joint Fourier–Laplace transformation on both sides of (12) we obtain

\[
\hat{z}_i(s) = \frac{2(p - q) \rho_0^{-1}}{s \lambda_i(s)} + \frac{U(s)}{\lambda_i(s)}, \quad \text{where,}
\]

\[
\lambda_i(s) = s + 4 \sin^2 \left( \frac{\pi k}{N} \right),
\]

and \( U(s) = 2(p - 1)(\tilde{z}_i(s) - \tilde{z}_0(s)) + (2q - 1)(\tilde{z}_{N-1}(s) - \tilde{z}_0(s)) \). Determining \( U(s) \) self-consistently and performing inverse Fourier transformation we obtain

\[
\tilde{z}_i(s) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{e^{2\pi j ki}}{\lambda_i(s)} \left( \frac{2(p - q) \rho_0^{-1}}{s} \right),
\]

Now taking the inverse Laplace transform of \( \tilde{z}_i(s) \) one can in principle find \( z_i(t) \) for any \( t \). However we are interested in the long time limit, which is equivalent to studying the \( s \to 0 \) limit of (18). Here two cases arise depending on whether we take the thermodynamic limit \( (N \to \infty \text{ keeping } \rho_0 = N/L \text{ fixed}) \) before \( s \to 0 \) or after. Let us focus on the average displacement of the DTP \( (i = 0) \), separately for these two cases.

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(a) If \( N \) is kept finite and \( s \to 0 \), the sum at the numerator is expected to be dominated by the \( k = 0 \) term, as \( \lambda_0(s) = s \). The sum at the denominator converges to a finite value in the \( s \to 0 \) limit:

\[
\frac{1}{N} \sum_{k=0}^{N-1} \frac{e^{2\pi i k}}{(e^{2\pi/N} - 1) + (2q - 1)(e^{-2\pi/N} - 1)} = -(p + q - 1)N - \frac{1}{N}.
\]

Hence in the \( s \to 0 \) limit

\[
\tilde{z}_0(s) = \frac{2(p - q)\rho_0^{-1}}{(N - 1)(p + q) + 1} s^{-2} + O(1/s^{\frac{5}{2}}),
\]

which after inverse Laplace transformation and restoring \( t = 2\tau/\mu_1 \) gives the following linear asymptotic growth:

\[
y_0(t) = z_0(t) = \frac{(p - q)\rho_0^{-1} \mu_1}{(N - 1)(p + q) + 1} t + O(\sqrt{t}), \quad t \to \infty.
\]

(b) Let us now look at the limits in the opposite order. We first take the thermodynamic limit and then we take the \( s \to 0 \) limit. If \( N \) is sent to infinity first, the sums in (18) become integrals. As a result the numerator of (18) becomes

\[
1 - \int_{x=0}^{1} \frac{(2p - 1)(e^{2\pi j x} - 1) + (2q - 1)(e^{-2\pi j x} - 1)}{s + 4 \sin^2(\pi x)} dx = 2(p + q) \int_{x=0}^{1} \frac{e^{2\pi j x} - 1}{s + 4 \sin^2(\pi x)} dx + O(s^{1/2})
\]

\[
= p + q + O(s^{1/2}).
\]

Inserting the asymptotic forms from (22) and (23) in (18) we obtain \( \tilde{z}_i(s) \) on the infinite line for \( s \to 0 \):

\[
\tilde{z}_i(s) = \frac{p - q}{p + q} \rho_0^{-1} \frac{1}{s^{\frac{3}{2}}} e^{-s^{\frac{1}{2}}} + O(s^{-1}),
\]

which after inverse Laplace transformation and restoring \( t = 2\tau/\mu_1 \) gives

\[
y_i(t) = x_i(0) + z_0(t) \simeq \rho_0^{-1} \sqrt{2\mu_1} \left[ \frac{i}{\sqrt{2\mu_1}} + \frac{p - q}{p + q} \sqrt{\frac{p}{\mu_1}} \left( e^{\frac{p}{\mu_1}} - \sqrt{\frac{p}{\mu_1}} \right) \text{erfc} \left( \frac{|i|}{\sqrt{2\mu_1}} \right) \right]
\]

where \( \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-a^2} da \). Hence the average displacement of the DTP \( (i = 0) \) grows for large \( t \) as

\[
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\]

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\[ y_0(t) = z_0(t) \sim \frac{p - q}{p + q} \rho_0^{-1} \sqrt{\frac{2\mu_1}{\pi}} \sqrt{t}. \]  

(26)

Similar late time growth \( \sim \sqrt{t} \) for the average of the tracer position has been computed in the context of SSEP with a single driven tracer [44, 45]. Comparing the large \( t \) behaviors of \( y_0(t) \) in equations (21) and (26), we see that the limits \( N \to \infty \) and \( t \to \infty \) do not commute. When time \( t \) is longer than the typical time required for an elementary transition to occur but shorter than the time \( t \sim O(\rho_0^2 L^2) \) required for the particles to feel the finiteness of the ring, their positions grow as \( \sim \sqrt{t} \). In this time scale the ring effectively acts as an infinite line. On the other hand, when \( t \gg O(\rho_0^2 L^2) \) the finiteness of the ring comes into play and then their positions grow as \( \sim t \). We verify this behavior numerically in figure 2. In fact this can entirely be described in terms of a nice crossover function which captures both limits (a) and (b) discussed above. We next derive this crossover function.

3.2. Finite size crossover

From the limiting cases (21) and (26) we expect the crossover to be described by a function of the scaling variable \( \phi = \frac{\tau}{N^2} = \frac{\mu_1 t}{2N^2} \) (or equivalently \( \sigma = sN^2 \) in the Laplace space) so that \( \phi \to \infty \) and \( \phi \to 0 \) capture, respectively, the above two limits (a) and (b). To obtain the crossover function we start from the exact expression (18) and compute \( \tilde{z}_0(s) \) for the TP

\[ \tilde{z}_0 \left( \frac{\sigma}{N^2} \right) \sim \frac{2(p - q)\rho_0^{-1} N^2}{p + q} \frac{1}{\sigma} \sum_{k=0}^{N-1} \frac{1}{\sigma N^2 + 4\sin^2\left(\frac{k\pi}{N}\right)} \]

\[ \sim \frac{2(p - q)\rho_0^{-1} N}{p + q} \frac{N^2}{\sigma} \left[ N^2 + 2 \sum_{k=1}^{N/2} \frac{1}{\sigma N + 4\sin^2\left(\frac{k\pi}{N}\right)} + 2 \sum_{k=1}^{N/2} \left( \frac{1}{\sigma N + 4\sin\left(\frac{k\pi}{N}\right)} + \frac{1}{\sigma N^2 + 4\sin^2\left(\frac{k\pi}{N}\right)} \right) \right] \]

\[ \sim \frac{2(p - q)\rho_0^{-1} N^2}{p + q} \frac{N^2}{\sigma} \left[ N^2 + 2N^2 \sum_{k=1}^{\infty} \frac{1}{\sigma + 4\pi^2 k^2} + \frac{N}{2\pi} \int_{x=0}^{\pi/2} \frac{x^2 - \sin^2(x)}{x^2 \sin^2(x)} \, dx \right] \]

\[ \sim \frac{2(p - q)\rho_0^{-1} N^2}{p + q} \frac{1}{2\sigma^{3/2}} \cosh \left( \frac{\sqrt{\sigma^2}}{2} \right). \]  

(27)

In the first line we took the limit of the denominator of (18), which does not introduce any complication. The sum on the numerator can then be taken care of by adding and subtracting the divergent part, as done on the second line. The first sum of the second line can now be evaluated exactly when \( N \to \infty \), while the second sum converges to an integral which turns out to be subdominant. In the end the whole expression indeed converges to a function of the scaling variable \( \sigma \). The inverse Laplace transform is most easily performed on the crossover function expressed as a sum, i.e. on the two first terms on the third line of equation (27). In real space we obtain

\[ z_0(t) \sim \frac{2(p - q)\rho_0^{-1}}{p + q} \sqrt{2\mu_1 t} \Phi \left( \frac{\mu_1 t}{2N^2} \right). \]  

(28)

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Figure 3. Numerical verification of the theoretical crossover function $\Phi(\phi)$ given in (29) for $L = 1$, $N = 200$, $p = 1$ and $q = 0$.

with the crossover function

$$\Phi(\phi) = \sqrt{\phi} + \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{1 - e^{-4\pi^2k^2\phi}}{k^2}.$$  \hspace{1cm} (29)

We can check that the asymptotic behaviors $\Phi(\phi) \sim \phi \rightarrow \infty \sqrt{\phi}$ and $\Phi(\phi) \sim \phi \rightarrow 0 \pi^{-1/2}$ respectively give (21) and (26) in the limiting cases. The prediction (28) is in very good agreement with the numerics, as shown in figure 3.

3.3. Large $t$ scaling limit of $y_i(t)$ on an infinite line

Although in the previous section we mainly looked at the growth of $y_0(t)$, as a by-product we have also found the average position $y_i(t)$ of the $i$th particle in (25) for case (b), where we take the thermodynamic limit before the $t \rightarrow \infty$ limit. Looking at (25) we find that, for large $t$, $y_i(t)$ has the following scaling form:

$$y_i(t) = \rho_0^{-1} \sqrt{2 \mu_i t} \mathcal{Y}\left(\frac{i}{\sqrt{2 \mu_i t}}\right) + \text{O}(1),$$  \hspace{1cm} (30)

where the scaling function is given by

$$\mathcal{Y}(x) = x + \frac{p - q}{p + q} \left[\frac{e^{-x^2}}{\sqrt{\pi}} - |x| \text{erfc}(|x|)\right].$$  \hspace{1cm} (31)

This scaling function can also be computed in a different way as follows. The evolution equation for $y_i(t)$ on a line can be obtained either directly from the dynamics (5) or from the equations on a ring by cutting the ring at $\frac{N}{2}$ and then sending $N$ to infinity. Both methods give

$$\dot{y}_i = \frac{\mu_1}{2} (y_{i+1} - 2y_i + y_{i-1}) + \delta_i q \frac{\mu_1}{2} ((2p - 1)(y_1 - y_0) + (2q - 1)(y_{i-1} - y_0)),$$  \hspace{1cm} (32)
where we recall that $\mu_1 = \int_{\eta=0}^{1} \eta R(\eta) d\eta$ is the average of $\eta$. For long times, we look for solutions of this equation in the scaling form (30). Putting this form in (32) and taking the $t \to \infty$ limit we find that the function $\mathcal{Y}$ satisfies

$$\mathcal{Y}''(x) + 2x\mathcal{Y}'(x) - 2\mathcal{Y}(x) = \delta(x) [(2q - 1)\mathcal{Y}'(0^-) - (2p - 1)\mathcal{Y}'(0^+)].$$

(33)

The delta source at the origin implies that $\mathcal{Y}$ is continuous but its first derivative is discontinuous. By integrating both sides of (33) over an infinitesimal segment from $0^-$ to $0^+$, one finds

$$p\mathcal{Y}'(0^+) = q\mathcal{Y}'(0^-).$$

(34)

Two other boundary conditions are obtained by requiring that particles far enough from the DTP are not perturbed. As a result average positions of the particles far from the DTP are equal to their initial positions. This implies that

$$\mathcal{Y}(x) \sim x \quad \text{when} \quad |x| \to \infty.$$  

(35)

With the boundary conditions (34) and (35), one can easily solve (33) to find $\mathcal{Y}(x)$ as given in (31). We observe nice agreement between this theoretical prediction and numerical measurements in figure 4(a).

### 3.4. Average gap profile

From the average position profile $y_i(t)$ in (30), the average gap profile $h_i(t) = \langle g_i(t) \rangle = y_{i+1}(t) - y_i(t)$ can be easily computed. Similar to $y_i(t)$, the average gap profile $h_i(t)$ also has a scaling form,

$$h_i(t) = \rho_0^{-1} \mathcal{H} \left( \frac{i}{\sqrt{2\mu_1 t}} \right) + O(t^{-1/2}),$$

(36)

in the large $t$ limit. The scaling function $\mathcal{H}(x)$ is obtained by taking the derivative of $\mathcal{Y}(x)$ in (30). We find

$$\mathcal{H}(x) = \mathcal{Y}'(x) = 1 - \frac{p - q}{p + q} \operatorname{Sign}(x) \operatorname{erfc}(|x|),$$

(37)

where $\operatorname{Sign}(x) = x/|x|$. In figure 4(b) we compare this theoretical prediction with numerical measurements and the nice agreement between the two verifies our result.

### 3.5. Particle density profile in the frame of the DTP

From the knowledge of the average position profile $y_i(t)$ and the average gap profile $h_i(t)$, one can now find the mean particle density profile. In the frame of the DTP, the mean particle density at some space point $w$ at time $t$ is defined as

$$\rho(w, t) = \left\{ \sum_{i=-\infty}^{\infty} \delta[w - (x_i(t) - x_0(t))] \right\},$$

(38)
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where the angular average is taken over stochastic evolution. Since both \( y_i(t) \) and \( h_i(t) \) have scaling forms under the transformation

\[
  u = \frac{1}{\sqrt{2\mu_1 t}},
\]

(39)

for large \( t \), we can expect that \( \rho(w, t) \) also has a scaling behavior \( \rho(w, t) \simeq \rho_0 \Omega \left( \frac{w}{\sqrt{2\mu_1 t}} \right) \) for large \( t \). One can observe this scaling behavior in numerical simulations. The question now is what the expression of \( \Omega(\xi) \) is. To find this, let us start with the discrete picture. In terms of the average gaps \( h_i(t) \), the average position of the \( i \)th particle with respect to the DTP is given by

\[
  \bar{y}_i(t) = y_i(t) - y_0(t) = \sum_{i=0}^{i-1} h_i(t),
\]

(40)

which in the large \( t \) limit becomes

\[
  \xi(u) = \rho_0^{-1} \int_0^u \mathcal{H}(a) \, da,
\]

(41)

where \( u \) is given in (39) and \( \xi(u) = \frac{\bar{y}}{\sqrt{2\mu_1 t}} \). On the other hand, the average density near the \( i \)th particle at time \( t \) can approximately be given by \( \rho(\bar{y}, t) \simeq \frac{2}{h_{i-1}(t) + h_i(t)} \), which in the large \( t \) limit gives

\[
  \rho(\xi(u)) = \frac{\rho_0}{\mathcal{H}(u)} + O\left( \frac{1}{\sqrt{t}} \right).
\]

(42)

Figure 4. (a) Scaled average position profile \( \mathcal{Y}(x) \) as a function of \( x = \frac{i}{\sqrt{2\mu_1 t}} \) for \( t = 700 \). The magenta solid line corresponds to the theoretical expression (31). (b) Scaled average gap profile \( \mathcal{H}(x) \) as a function of \( x = \frac{i}{\sqrt{2\mu_1 t}} \) for \( t = 700 \). The black solid line corresponds to (37). The hopping rates of the DTP for this plot are \( p = 0.75 \) and \( q = 0.25 \). The jump distribution is uniform, i.e. \( R(\eta) = 1 \).
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Equations (41) and (42) together constitute the density profile in parametric form. We compare the theoretical expression (41)–(42) of $\rho$ as a function of $\xi$ with numerical measurements in figure 5 and find quite good agreement. We observe that the average density profile is modulated because of the biased motion of the TP; the system is denser in front of the biased TP and sparser behind it. This density modulation $\rho - \rho_0$ decays very fast as one moves away from the tracer on both sides, as $\sim \xi^{-2}$ when $|\xi| \to \infty$.

A similar phenomenon is observed when a biased tracer is present in a one-dimensional simple exclusion process [44, 45]. As in our case, the velocity of the tracer decays as $\sim 1/\sqrt{t}$ [44]. The length scale over which one observes the effect of the tracer also scales as $\sqrt{t}$. Moreover, we note that the decay of the density perturbation at large distances from the DTP is exactly the same as in the RAP case, i.e. $\sim \xi^{-2}$ for $|\xi| \to \infty$ (see equations (26) and (31) of [45]). In contrast, the phenomenon is different for driven tracers in SSEP of higher dimensions: where the velocity of the tracer is finite, the density around the tracer reaches a stationary profile without going to a scaling limit and the decay of the density modulation is exponential everywhere except at the back of the driven tracer, where it becomes algebraic [48–50].

4. Correlations

In this section we study the two-point connected correlation function of the positions,

$$c_{i,j}(t) = \langle x_i(t)x_j(t) \rangle - y_i(t)y_j(t),$$

and of the gaps,

$$d_{i,j}(t) = \langle g_i(t)g_j(t) \rangle - h_i(t)h_j(t).$$

Figure 5. Numerical verification of the particle density profile (42) seen from the frame of the biased TP. Parameters associated with this plot are $p = 0.75$, $q = 0.25$ and $\rho_0 = 200$. The jump distribution is uniform, i.e. $R(\eta) = 1$.

Equations (41) and (42) together constitute the density profile in parametric form. We compare the theoretical expression (41)–(42) of $\rho$ as a function of $\xi$ with numerical measurements in figure 5 and find quite good agreement. We observe that the average density profile is modulated because of the biased motion of the TP; the system is denser in front of the biased TP and sparser behind it. This density modulation $\rho - \rho_0$ decays very fast as one moves away from the tracer on both sides, as $\sim \xi^{-2}$ when $|\xi| \to \infty$.

A similar phenomenon is observed when a biased tracer is present in a one-dimensional simple exclusion process [44, 45]. As in our case, the velocity of the tracer decays as $\sim 1/\sqrt{t}$ [44]. The length scale over which one observes the effect of the tracer also scales as $\sqrt{t}$. Moreover, we note that the decay of the density perturbation at large distances from the DTP is exactly the same as in the RAP case, i.e. $\sim \xi^{-2}$ for $|\xi| \to \infty$ (see equations (26) and (31) of [45]). In contrast, the phenomenon is different for driven tracers in SSEP of higher dimensions: where the velocity of the tracer is finite, the density around the tracer reaches a stationary profile without going to a scaling limit and the decay of the density modulation is exponential everywhere except at the back of the driven tracer, where it becomes algebraic [48–50].
In the previous section we have seen that both the mean position profile \( y_i(t) \) and the mean gap profile \( h_i(t) \) have scaling forms when index \( i \) is scaled by \( \mu t^{1/2} \). When \( p = q = 1/2 \), i.e. when all the particles are moving symmetrically, the position correlation function \( c_{i,j}(t) \) has been computed by Rajesh and Majumdar [38]. Looking at their result \((1)\) with \( \alpha = \beta = 1/2 \), we find that \( c_{i,j}(t) \) has a scaling form as a function of the scaling variable \( u = (i - j)/2\mu t \). On the basis of these facts we expect that both the correlation functions \( c_{i,j}(t) \) and \( d_{i,j}(t) \) have well defined scaling limits under the transformations \( i \rightarrow x = i/\sqrt{2\mu t} \) and \( j \rightarrow y = j/\sqrt{2\mu t} \) for large \( t \). To support this hypothesis, let us first present our numerical results.

We have numerically measured the pair position correlations and pair gap correlations defined, respectively, in \((43)\) and \((44)\) as a function of \( j \) for different fixed values of \( i \) and \( t \). In figure 6 we plot \( \rho_0^2 c_{i,j}(t) \) as a function of \( y = j/\sqrt{2\mu t} \) for \( x = 0 \) and 0.98, and for three different values of \( t = 200, 500 \) and 700, and we observe a clear and excellent data collapse. This verifies our hypothesis and implies the following scaling form of \( c_{i,j}(t) \) for large \( t \):

\[
c_{i,j}(t) = \rho_0^2 2\mu t \left( \frac{i}{\sqrt{2\mu t}}, \frac{j}{\sqrt{2\mu t}} \right) + O(1).
\]

Similar to \( c_{i,j}(t) \) the gap correlation function \( d_{i,j}(t) \) also has a scaling form (numerically verified but not presented here),
\[ d_{i,j}(t) = \frac{\rho^{-2}_0}{\sqrt{2\mu_1 t}} D\left( \frac{i}{\sqrt{2\mu_1 t}}, \frac{j}{\sqrt{2\mu_1 t}} \right) + O(t^{-1}), \quad i \neq j, \]  

where the scaling function \( D(x, y) \) is related to \( C(x, y) \) as
\[ D(x, y) = \partial_x \partial_y C(x, y). \]  

However this scaling form (46) is valid only for off-diagonal gap correlation functions. For diagonal gap correlations we in fact observe numerically (see figure 11) that \( d_{i,i}(t) \) is of order unity not of order \( 1/\sqrt{t} \). Hence, for the \( i = j \) line, we consider the following scaling form for \( d_{i,i}(t) \):
\[ d_{i,i} = \rho^{-2}_0 V\left( \frac{i}{\sqrt{2\mu_1 t}} \right) + \rho^{-2}_0 V\left( \frac{i}{\sqrt{2\mu_1 t}} \right) + D\left( \frac{i}{\sqrt{2\mu_1 t}}, \frac{i}{\sqrt{2\mu_1 t}} \right) \]  

Equations (46) and (48) are supported by numerical evidence. Our next aim is to compute these scaling functions \( C(x, y) \), \( D(x, y) \), \( V(x) \) and \( V_i(x) \) analytically.

4.1. Computation of \( C(x, y) \)

To compute \( C(x, y) \) we start with the discrete evolution equation for \( c_{i,j}(t) \), which can be obtained from the dynamics of the positions (5). It reads as
\[ \dot{c}_{i,j} = \mu_1 (c_{i+1,j} + c_{i-1,j} + c_{i,j+1} + c_{i,j-1} - 4c_{i,j}) + \delta_{i,j} (c_{i+1,i} - 2c_{i+1,i} + 2c_{i+1,i} + c_{i+1,i} + c_{i-1,i} + c_{i-1,i} + (y_{i+1} - y_i)^2 + (y_{i-1} - y_i)^2) + \delta_{i,0} (2p - 1)(c_{i,j} - c_{0,j}) + (2q - 1)(c_{j-1,j} - c_{0,j})) + \delta_{i,0} (2p - 1)(c_{i+1,0} - c_{i,0}) + (2q - 1)(c_{i+1,0} - c_{i,0}) + (2p - 1)(c_{i,j} - c_{i-1,j} + c_{i,j} + (y_{i+1} - y_i)^2 + (y_{i-1} - y_i)^2) + \delta_{i,0} (2p - 1)(c_{i,j} - c_{i-1,j} + c_{i,j} + (y_{i+1} - y_i)^2), \]  

where \( y_i(t) = \langle x_i(t) \rangle \). We are interested in finding the solution of this equation in the form (45) for large \( t \). For this, one can follow the Fourier–Laplace transform method as used in solving (12), to show that \( c_{i,j}(t) \) indeed has the scaling form (45) for large \( t \). However, performing such analysis involves two coupled integral equations arising from the self-consistency conditions and this makes it hard to solve. Instead, assuming that \( c_{i,j}(t) \) has the scaling form (45) for large \( t \), we insert this scaling form in the discrete equations (49) and take the large \( t \) limit to obtain the following differential equation for \( C(x, y) \):
\[ \left[ \partial_x^2 + \partial_y^2 + 2x \partial_x + 2y \partial_y - 2 \right] C(x, y) = \delta(x - y) \frac{2\mu_2}{\mu_1} \left[ \partial_x C|_{x=0^-} - \partial_x C|_{x=0^-} - \partial_y C|_{y=0^-} \right] + \delta(x) [(2q - 1) \partial_x C|_{x=0^-} - (2p - 1) \partial_x C|_{x=0^-}] + \delta(y) [(2q - 1) \partial_y C|_{y=0^-} - (2p - 1) \partial_y C|_{y=0^-}], \]

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in the leading order. Here $\mathcal{H}(x)$ is the average gap profile. We solve this equation for $C(x, y)$ and verify the solution with numerical measurements.

At first glance, equation (50) seems complicated because of the self-consistent terms on the right-hand side (RHS). However we can simplify it further. We start with the $\delta(x)$ and $\delta(y)$ terms on the RHS. From numerical measurements we have seen that $C(x, y)$ is continuous across $x = 0$ but its derivative is possibly discontinuous. Integrating both sides of (50) from $x = 0^{-}$ to $x = 0^{+}$, we find that the derivative should satisfy

$$q \partial_x C(0^-, y) = p \partial_y C(0^+, y)$$

for all $y$. A symmetric argument can be applied to the $y = 0$ line too. Since the equation is of the second order, we expect that the knowledge of two matching conditions at each non-analyticity is enough to determine the solution.

In summary, the $\delta(x)$ and $\delta(y)$ terms on the RHS of (50) can equivalently be replaced by imposing the boundary conditions

$$C(0^-, y) = C(0^+, y), \quad q \partial_x C(0^-, y) = p \partial_y C(0^+, y),$$

$$C(x, 0^-) = C(x, 0^+), \quad q \partial_y C(x, 0^-) = p \partial_y C(x, 0^+).$$

Let us now simplify the $\delta(x - y)$ term; for this we follow the same procedure as for the $\delta(x)$ term, namely integrate both sides of (50) across the $x = y$ line. We therefore make the coordinate transformation $u = x - y$ and $v = x + y$. In terms of the transformed variables $u$ and $v$, equation (50) reads

$$[\partial_u^2 + \partial_v^2 + u \partial_u + v \partial_v - 1]C(u, v) = -\delta(u) \frac{\mu_2}{\mu_1} \left[ 2 \partial_u C(0^-, v) + \mathcal{H}\left(\frac{v}{2}\right)^2 \right],$$

where the $\delta$-source terms for $x = 0$ and $y = 0$ are replaced by the boundary conditions (51). We integrate again across the $u = 0$ line from below to above. As evidenced from numerical measurements of $c_{ij}(t)$ in figure 6, $C$ is continuous at $u = 0$. We therefore obtain an equation for the discontinuity of the first derivative across the $u = 0$ line,

$$\mu_1 (\partial_u C(0^+, v) - \partial_u C(0^-, v)) = -2 \mu_2 \partial_u C(0^-, v) - \mu_2 \mathcal{H}\left(\frac{v}{2}\right)^2.$$

Using the symmetry of $C$ under reflection with respect to the diagonal, we obtain

$$\partial_u C(0^-, v) = -\partial_u C(0^+, v) = \frac{\mu_2}{2(\mu_1 - \mu_2)} \mathcal{H}\left(\frac{v}{2}\right)^2,$$

which is verified numerically in figure 7. Now, inserting the result (54) in the RHS of (50) and transforming back to the original $(x, y)$ coordinates, we obtain

$$[\partial_x^2 + \partial_y^2 + 2x \partial_x + 2y \partial_y - 2]C(x, y) = -\delta(x - y) \frac{2\mu_2}{\mu_1 - \mu_2} \mathcal{H}(x)^2,$$

with boundary conditions (51). Other boundary conditions come from the fact that for large $x$ and $y$, i.e. when both the particles are far from the driven tracer, the correlation among their positions should be equal to the correlation function of the non-driven system, given by equation (1) with $\alpha = \beta = 1/2$. This means that

$$C(x, y) \simeq C_{\text{nl}}(x, y) = \frac{\mu_2}{\sqrt{2\pi(\mu_1 - \mu_2)}} g(x - y), \quad \text{for } |x| \to \infty, \ |y| \to \infty,$$
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\begin{equation}
\pi = -\frac{1}{u}
\end{equation}

where, \( g(u) = e^{-\frac{u^2}{2}} - \sqrt{\frac{\pi}{2}} |u| \text{erfc}\left(\frac{|u|}{\sqrt{2}}\right) \), (57)

and the subscript ‘ub’ denotes the unbiased case. We now have to solve the differential equation (55) with boundary conditions (51) and (56). Computing the full solution for \( C(x, y) \) for arbitrary \( p \) and \( q \) in a closed form seems difficult. We are however able to solve (55) perturbatively by expanding \( C(x, y) \) in powers of the drive strength \( \epsilon = p - q \).

4.1.1. Perturbative expansion in \( \epsilon = p - q \) Let us consider the following expansions of the functions \( \mathcal{H} \) and \( C \) in powers of \( \epsilon \):

\begin{equation}
\mathcal{H}(x) = 1 - \frac{\epsilon}{p + q} \text{Sign}(x) \text{erfc}(|x|),
\end{equation}

\begin{equation}
C(x, y) = C_0(x, y) + \frac{\epsilon}{2} C_1(x, y) + \frac{\epsilon^2}{4} C_2(x, y) + \ldots .
\end{equation}

This expansion of \( C(x, y) \) provides a systematic way of solving equation (55) order by order in \( \epsilon \). Indeed, inserting the expansions (58) in the evolution equation (55), we obtain equations for each \( C_i(x, y) \) with previous order functions \( C_j(x, y), \ j < i \) appearing as source. In this paper we compute \( C(x, y) \) to first order. However our method can be generalized to obtain higher order solutions. At order \( \epsilon^0 \) we have

\begin{equation}
[\partial_x^2 + \partial_y^2 + 2x\partial_x + 2y\partial_y - 2]C_0(x, y) = \delta(x - y) - \frac{8\mu_2}{\mu_1 - \mu_2}.
\end{equation}

As this equation physically corresponds to a system without drive, we have \( C_0(x, y) \equiv C_{ub}(x, y) \), where \( C_{ub} \) is given in (56).

Let us now focus on order \( \epsilon \). If we choose to keep the \( \delta(x) \) and \( \delta(y) \) source terms of equation (50) instead of taking them as boundary conditions, then using (54) and (58) we obtain

\begin{equation}
\delta C_v(0, v) = -\mathcal{H}(v) \frac{\mu_2}{\mu_1 - \mu_2}.
\end{equation}

\textbf{Figure 7.} Numerical verification of (54) as a function of \( v \) for \( N = 200, t = 700, p = 0.75 \) and \( q = 0.25 \). Circles represent \( \partial_v C(0, v) \) obtained from numerical measurements and the solid line is \( \frac{\mu_2}{2(\mu_1 - \mu_2)} \mathcal{H}(v)^2 \). The jump distribution is uniform, i.e. \( R(\eta) = 1 \).
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\[ [\partial_x^2 + \partial_y^2 + 2x\partial_x + 2y\partial_y - 2] \mathcal{C}(x, y) = \delta(x-y) \frac{8\mu_2}{\mu_1 - \mu_2} \text{Sign}(x) \text{erfc}\left(\frac{|x|}{p+q}\right) - 4\delta(x)\partial_x \mathcal{C}_{\text{ub}}(0, y) - 4\delta(y)\partial_y \mathcal{C}_{\text{ub}}(x, 0). \]  

(60)

Explicit expression of \( \partial_x \mathcal{C}_{\text{ub}}(0, y) \) can be obtained from (56) as

\[ \partial_x \mathcal{C}_{\text{ub}}(0, y) = \frac{\mu_2}{2(\mu_1 - \mu_2)} \text{Sign}(y) \text{erfc}(|y|). \]  

(61)

Similarly \( \partial_y \mathcal{C}_{\text{ub}}(x, 0) \) can also be obtained. Going to (tilted) polar coordinates \( (x, y) = \left(r\cos\left(\theta + \frac{\pi}{4}\right), r\sin\left(\theta + \frac{\pi}{4}\right)\right) \) and using (61), we rewrite equation (60) as

\[
\left[ \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + 2r\partial_r - 2 \right] \mathcal{C}(r, \theta) = \frac{2\mu_2}{\mu_1 - \mu_2} \frac{\text{erfc}(r/\sqrt{2})}{r} \left[ \frac{2\sqrt{2}}{p+q} (\delta(\theta) - \delta(\theta - \pi)) - \delta\left(\theta - \frac{\pi}{4}\right) + \delta\left(\theta + \frac{3\pi}{4}\right) - \delta\left(\theta + \frac{\pi}{4}\right) + \delta\left(\theta - \frac{3\pi}{4}\right) \right].
\]  

(62)

The boundary conditions for the above equation are \( \mathcal{C}(r, \theta)|_{r \to 0} \) is finite and \( \mathcal{C}(r, \theta)|_{r \to \infty} \to 0 \). We observe that \( \mathcal{C}(r, \theta) \) can be written as

\[ \mathcal{C}(r, \theta) = \frac{2\mu_2}{\mu_1 - \mu_2} \left( \frac{2\sqrt{2}}{p+q} \psi(r, \theta) - \psi\left(r, \theta + \frac{\pi}{4}\right) - \psi\left(r, \theta - \frac{\pi}{4}\right) \right), \]  

(63)

where \( \psi(r, \theta) \) satisfies

\[
\left[ \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + 2r\partial_r - 2 \right] \psi(r, \theta) = (\delta(\theta) - \delta(\theta - \pi)) \frac{\text{erfc}(r/\sqrt{2})}{r}.
\]  

(64)

This equation can be solved by expanding both \( \delta(\theta) \) and \( \psi(r, \theta) \) as

\[ \delta(\theta) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{il\theta}, \quad \text{and} \quad \psi(r, \theta) = \sum_{m=0}^{\infty} 2\cos((2m + 1)\theta) \psi_m(r), \]  

(65)

for \( l, m \) integers. Inserting this expansion in (64), one obtains a radial differential equation for each \( m \), which one has to solve with boundary conditions:

\[ \psi_0(r)|_{r \to 0} = \text{finite}, \quad \psi_{m>0}(r)|_{r \to 0} = 0, \quad \text{and} \quad \psi_m(r)|_{r \to \infty} = 0 \quad \forall m. \]  

(66)

Since the calculation of \( \psi_m(r) \) is long and technical, we do not present it in the main body of the paper but rather in appendix A. Inserting \( \psi_m(r) \) (from appendix A) in (65) we obtain \( \psi(r, \theta) \), using which in (63) we evaluate \( \mathcal{C}(r, \theta) \). In figure 8 we compare
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As it seems the explicit expression of $\psi_\theta(r)$ (see appendix A) is not very illuminating, here we look at the asymptotic behavior of $\psi(r, \theta)$ for small and large $r$ values. We find that close to $r = 0$ the function $\psi(r, \theta)$ behaves as (see appendix A.3 for details)

$$
\psi(r, \theta) = \frac{r \log r}{\pi} \cos \theta + \frac{2\gamma_E + \pi - 4}{4\pi} r \cos \theta - \frac{r}{2\pi} (2\theta - \pi \text{Sign}[\theta]) \sin \theta
$$

$$
- \frac{r^2}{2\sqrt{2\pi}} \text{Sign}[\theta] \sin \theta + O(r^3),
$$

where $\theta \in [-\pi, \pi]$ and $\gamma_E = 0.577...$ is Euler’s constant. Clearly the derivative $\partial_\theta \psi$ is discontinuous across $\theta = 0$ and its value at $\theta = \pi$ is different from that at $\theta = -\pi$. Summing the three $\psi$ functions in (63), one clearly sees that $\partial_\theta \mathcal{C}_1$ is discontinuous along the three lines where the sources are located. For large values of $r$, the $\psi$ function concentrates around $\theta = 0$ as it should do,

$$
\psi(r, \theta) = -\frac{1}{\sqrt{2\pi}} \delta(\theta) e^{-\frac{r^2}{2}} \left( 1 + O\left( \frac{1}{r^2} \right) \right).
$$

4.1.2. Correlations for large $|x|$. In the preceding section we have presented a perturbative method to find $\mathcal{C}(x, y)$. This method is rather lengthy and cumbersome. However, when either one of the arguments ($x$ or $y$) is large in magnitude, one can find explicitly a simpler approximate solution for $\mathcal{C}(x, y)$. When, say, $|x| \to \infty$, we simplify the diagonal source term on the RHS of (55) by its large $x$ form, i.e. put $\mathcal{H}(x) \simeq 1$. As a result it now...
becomes easier to solve (55) with boundary conditions (51) and (56). We find the following solutions when $|x| \to \infty$:

$$C_\infty(x, y) = C(x, y)|_{x \to \infty} \approx \frac{\mu_2}{\sqrt{2\pi(\mu_1 - \mu_2)}} \begin{cases} 
  g(x - y) + \frac{p - q}{p + q} g(x + y), & \text{for } x \geq 0, \ y \geq 0, \\
  \frac{2p}{p + q} g(x - y), & \text{for } x \geq 0, \ y < 0, \\
  \frac{2q}{p + q} g(x - y), & \text{for } x < 0, \ y \geq 0, \\
  g(x - y) - \frac{p - q}{p + q} g(x + y), & \text{for } x < 0, \ y < 0,
\end{cases}$$

where $g(u)$ is given in (57). In figure 9 we compare these solutions with numerical results. The red circles represent data obtained from numerical simulation whereas the black solid line represents the theoretical expression (69). For comparison we also have plotted the scaled correlation function $C_{ub}(x, y)$ corresponding to the unbiased system from (56) (dashed blue line). We see that the effect of the biased tracer is maximum when $y$ is close to zero as expected. To visualize this effect better we zoomed in on the region near $y \sim 0$ in the inset.

4.1.3. Special case: $q = 0$ This case is very interesting since for $q = 0$ the particles in front of the biased tracer are not affected by the particles behind it. As a result the boundary conditions in (51) become simpler:

$$\partial_x C(0^+, y) = 0, \ \partial_y C(x, 0^+) = 0.$$

Figure 9. Plot of the theoretical $C(x, y)$ versus $y$ for large $x$ (solid line) compared with numerical measurements (circles). Here $x = 1.89$. The effect of the driven tracer is maximum near $y \sim 0$. To visualize this effect better, we zoomed in on the region $y \sim 0$ in the inset. Other parameters associated with this plot are $p = 0.75$, $q = 0.25$, $t = 700$ and $N = 200$. The jump distribution is uniform, i.e. $R(\eta) = 1$. 
It turns out that now one can solve (55) for \( C(x, y) \) exactly in the first quadrant \((A_{++} = [x > 0, \ y > 0])\) using the image method. Before going into this let us look at the other boundary condition given in (56), which says that at distances far from the origin (i.e. far from the driven tracer) the scaled correlation function should be the same as that of an unbiased system.

At this point one would naturally tend to assume that \( C(x, y) = C_{\text{ub}}(x, y) + \tilde{C}(x, y) \) and then solve for \( \tilde{C}(x, y) \). However, this choice of decomposition of the solution is not useful since \( C_{\text{ub}}(x, y) \) does not satisfy the boundary conditions (70). As a result it will make the boundary conditions for \( \tilde{C}(x, y) \) complicated. However, one can find a better decomposition

\[
C(x, y) = C_\infty(x, y) + \tilde{C}(x, y) = \frac{\mu_2}{\sqrt{2\pi}(\mu_1 - \mu_2)} \left[ g(x - y) + g(x + y) \right] + \tilde{C}(x, y)
\]  

(71)

where \( C_\infty(x, y) \) from (69) with \( q = 0 \) has been used. Note that \( C_\infty(x, y) \) with \( q = 0 \) satisfies both boundary conditions (70) and (56). Hence the boundary conditions for \( \tilde{C}(x, y) \) remain the same, namely (70) and (56). After inserting (71) in (55) we have

\[
[\partial_x^2 + \partial_y^2 + 2x\partial_x + 2y\partial_y - 2]C(x, y) = -\frac{2\mu_2}{\mu_1 - \mu_2}\left(\mathcal{H}(x)^2 - 1\right) [\delta(x - y) + \delta(x + y)].
\]  

(72)

We now proceed to solve (72) for \((x, y) \in A_{++}\). In this domain the sources (or ‘charges’) of the differential equation (72) are distributed along the \( x = y \) line and the normal derivatives of \( \tilde{C}(x, y) \) at its boundaries \((x = 0 \text{ and } y = 0 \text{ lines})\) vanish. Note that the ‘charge’ distribution along the diagonal in \( A_{++} \) is \( \mathcal{H}(x)^2 - 1 = \text{erf}(x)^2 - 1 \) (see (37) for \( q = 0 \)). To solve the differential equation (72) in \( A_{++} \) with these boundary conditions we consider the following image problem. Since the differential operator \( \tilde{D} \) on the left-hand side of (72) is invariant under \( x \to -x \) and/or \( y \to -y \), we consider the problem on the complete two-dimensional plane \( \mathcal{A} = \{-\infty < x < \infty, \ -\infty < y < \infty\} \) with three image ‘charge’ distributions obtained by reflecting the original ‘charge’ distribution with respect to the \( x \)-axis, \( y \)-axis and origin respectively. As a result we automatically satisfy the boundary conditions in (70) by symmetry. Hence we now solve

\[
[\partial_x^2 + \partial_y^2 + 2x\partial_x + 2y\partial_y - 2]C(x, y) = -\frac{2\mu_2}{\mu_1 - \mu_2}\left(\text{erf}(x)^2 - 1\right) [\delta(x - y) + \delta(x + y)],
\]  

(73)

in the full domain \( \mathcal{A} \), with the boundary conditions \( \tilde{C}(x, y) \to 0 \) as \( \sqrt{x^2 + y^2} \to \infty \) and \( \tilde{C}(x, y) \) finite as \( \sqrt{x^2 + y^2} \to 0 \). Once again going to the tilted polar coordinates \((x, y) = \left( r\cos(\theta + \frac{\pi}{4}), r\sin(\theta + \frac{\pi}{4}) \right) \) (as done in (62)) we rewrite (73) as

\[
\left[\partial_r^2 + \frac{1}{r^2} \partial_r + \frac{1}{r^2} \partial_\theta^2 + 2r\partial_\theta - 2\right] \tilde{C}(r, \theta) = -\frac{2\sqrt{2}\mu_2}{\mu_1 - \mu_2} \frac{2}{\pi} \left(\text{erf}(r\sqrt{2})^2 - 1\right) \left[ \delta(\theta) + \delta\left(\theta - \frac{\pi}{2}\right) + \delta(\theta + \pi) + \delta\left(\theta + \frac{\pi}{2}\right) \right],
\]  

(74)

where we have used the explicit expression of \( \mathcal{H}(x) \) for \( q = 0 \) from (37). To solve this equation we consider the following expansions: \( \delta(\theta) = \frac{1}{2\pi}[1 + 2\sum_{m=1}^{\infty} \cos(m\theta)] \) and

\[
\text{doi:10.1088/1742-5468/2016/05/053212}
\]
\( \bar{C}(r, \theta) = -\sqrt{2} \frac{\mu_2}{\mu_1 - \mu_2} \left( \xi_0(r) + \sum_{m=1}^{\infty} 2 \cos(4m\theta) \xi_m(r) \right). \)  

(75)

Inserting this form in (74) we find that the function \( \xi_m(r) \) satisfies

\[
\xi_m''(r) + \left( \frac{1}{r} + 2r \right) \xi_m'(r) - \left( 2 + \frac{16m^2}{r^2} \right) \xi_m(r) = \frac{2}{\pi r} \left[ \text{erf}(r/\sqrt{2}) \right]^2 - 1,
\]

(76)

with boundary conditions

\[
\xi_0(r)|_{r \to 0} = \text{finite}, \quad \xi_m(r)|_{r \to 0} = 0, \quad \text{and} \quad \xi_m(r)|_{r \to \infty} = 0 \quad \forall m.
\]

(77)

Two homogeneous solutions of the above equation are

\[
\xi_m^1(r) = r^{4m} \, _1F_1 \left( \frac{1}{2} (4m - 1); 4m + 1; -r^2 \right),
\]

\[
\xi_m^2(r) = e^{-r^2} r^{-4m} U \left( \frac{3}{2} - 2m, 1 - 4m, r^2 \right),
\]

(78)

where \( _1F_1(a, b, z) \) is the hypergeometric function and \( U(a, b, z) \) is the Kummer hypergeometric function (for details about the hypergeometric functions see [53]). In terms of these homogeneous solutions the total solution is written as

\[
\xi_m(r) = \xi_m^1(r) \int_r^\infty dr' \frac{\xi_m^2(r')}{W_m(r')} \frac{2}{\pi r'} \left[ \text{erf}(r'/\sqrt{2}) \right]^2 - 1 \bigg|_{r' = \infty} - \xi_m^2(r) \int_0^r dr' \frac{\xi_m^1(r')}{W_m(r')} \frac{2}{\pi r'} \left[ \text{erf}(r'/\sqrt{2}) \right]^2 - 1 \bigg|_{r' = 0},
\]

(79)

where \( W_m(r') = \xi_m^1(r') \partial_r \xi_m^2(r) - \xi_m^2(r') \partial_r \xi_m^1(r) \).

(80)

One can numerically (in Mathematica) evaluate the \( \xi_m(r) \) as a function of \( r \) for different values of \( m \) and use them in (75) to obtain \( \bar{C} \), which finally provides \( C \) in equation (71). In figure 10 we compare the numerically evaluated \( C \) using these equations with the simulation results. We observe good agreement, with slight differences which arise because of the fact that the ring used in the simulation may be not in the thermodynamic limit.

From the above analysis we can, in particular, compute the fluctuation of the displacement of the driven TP \( \sigma_0^2(t) = \langle x_0^2(t) \rangle - \langle x_0(t) \rangle^2 \) in the long time limit. In terms of the scaled correlation function this quantity is given by \( \sigma_0^2(t) = \mu_0^{-2} \sqrt{2} \mu_1 \, C(0, 0) \). More explicitly we have

\[
C(0, 0) = \frac{\mu_2}{\sqrt{2\pi (\mu_1 - \mu_2)}} - \frac{\sqrt{2} \mu_2}{\mu_1 - \mu_2} I, \quad \text{where} \quad I = \int_0^\infty dr' \frac{\xi_0^2(r')}{W_0(r')} \frac{2}{\pi r'} \left[ \text{erf}(r'/\sqrt{2}) \right]^2 - 1,
\]

(81)

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Figure 10. Theoretical $C(x, y)$ versus $y$ for $q = 0$ compared to numerical measurements. Here $x = 0.01$. The theoretical curve (solid Green line) is obtained using equations (71), (75) and (79) where the infinite sum in (75) has been truncated at $m = 4$. In the inset we compare the theoretical expression of $\sigma_0^2(t)$ in (82) (solid black line) with numerical measurements (orange circles). Other parameters associated with this plot are $L = 1.0$ and $N = 200$. The jump distribution is uniform, i.e. $R(\eta) = 1$.

with $\xi_0^2(r)$ and $W_0(r)$ given in (78) and (80) respectively. One can perform this integral exactly (see appendix B) to obtain $I = \frac{2}{\sqrt{\pi}}(\sqrt{2} - 1)$, which implies

$$\sigma_0^2(t) = \rho_0 \frac{\mu_2}{(\mu_1 - \mu_2)} \sqrt{\frac{2}{\pi}} \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \sqrt{2\mu_1 t}. \quad (82)$$

In the inset of figure 10 we verify this result numerically.

In the context of SSEP with driven tracer, similar late time growth $\sim \sqrt{t}$ for the fluctuation of the tracer position has been reported [23, 44]. Moreover, other moments of the tracer position have also been computed using different approximations. For example, in [23] SSEP with driven tracer has been studied in the high density regime. In particular, the authors of [23] have found the distribution of the position of the biased particle by mapping the motion of the particles to appropriate random walks of the holes (absence of particles) for high particle density. However, none of the studies has considered computing the position pair correlation function. To our knowledge, the calculation in this paper is the first attempt at computing such a correlation function without any approximation.

4.2. Gap correlations

Let us now focus on the pair gap correlation $d_{ij}(t)$ defined in (44). To compute the evolution equation for $d_{ij}(t)$, it is convenient to consider the dynamics of the stochastic gap variables $g_i(t) = x_{i+1}(t) - x_i(t)$ independently. From the dynamics (5) of $x_i(t)$s, one can write the dynamics of the $g_i(t)$ as

$$\frac{d}{dt} \left[ \frac{\sigma_i^2(t)}{\rho_0} \right] = \sum_{j} \gamma_i^j \left[ \frac{\sigma_j^2(t)}{\rho_0} \right]$$

where $\gamma_i^j$ is the rate of the $j$th reaction to produce a gap $g_i(t)$.

\[\text{doi:10.1088/1742-5468/2016/05/053212}\]
\[ g_i(t + dt) = g_i(t) + \sigma_i^{j+1} \eta_i g_{i+1}(t) + \sigma_i^j \eta_i g_{i-1}(t) - \sigma_i^{j+1} \eta_{i+1} g_i(t) - \sigma_i^j \eta_{i-1} g_i(t), \]  

where the random fraction \( \eta \) is chosen from the jump distribution \( R(\eta) \) and the variables \( \sigma_{i,j} \) are defined after (5). Using this dynamics, it is straightforward to find evolution equations for \( d_{i,j} \), 

\[
\dot{d}_{i,j} = \frac{\mu_1}{2}(d_{i+1,j} + d_{i-1,j} + d_{i+1,j+1} + d_{i,j-1} - 4d_{i,j}) \\
+ \frac{\mu_2}{2}(\delta_{i,j} - \delta_{i,j+1})(d_{i+1,j+1} + d_{i,j} + h_{i+1}^2 + h_i^2) + \frac{\mu_2}{2}(\delta_{i,j} - \delta_{i,j-1})(d_{i-1,j-1} + d_{i,j} + h_{i-1}^2 + h_i^2) \\
+ (\delta_{i-1,j} - \delta_{i,j}) \frac{\mu_1}{2}(2p - 1)d_{i,0} - (2q - 1)d_{j,0} + (\delta_{j-1,j} - \delta_{j,j}) \frac{\mu_2}{2}((2p - 1)d_{i,0} - (2q - 1)d_{j,0}) \\
+ (\delta_{i-1,j} - \delta_{i,j}) (\delta_{j-1,j} - \delta_{j,j}) \frac{\mu_2}{2}(2q - 1)(d_{i-1,j-1} + h_{i-1}^2) + (2p - 1)(d_{i,0} + h_0^2),
\]

where \( h_i(t) = \langle g_i(t) \rangle \). At the beginning of section 4, we argued that in the long time limit diagonal and off-diagonal gap correlations scale differently as (48) and (46) respectively. We are interested in finding solutions of \( d_{i,j}(t) \) in these scaling forms. Once again note that, while off-diagonal correlations are of order \( t^{-1/2} \), the diagonal correlations, i.e. fluctuations, are of order unity. We now insert the scaling forms of \( d_{i,j}(t) \) from (46), \( d_{i,j}(t) \) from (48) and \( h_i(t) \) from (36) in (84) and then expand both sides in powers of \( \frac{1}{\sqrt{t}} \).

Equating coefficients of each power from both sides, we find that orders \( t^{-1/2} \) and \( t^{-1} \) give 

\[ V(x) = \frac{\mu_2}{\mu_1 - \mu_2} H(x)^2 = \frac{\mu_2}{\mu_1 - \mu_2} \left( 1 - \frac{p - q}{p + q} \right) \text{Sign}(x) \text{erfc}(|x|), \]  

Figure 11. Numerical verification of (85) and (86) (inset). Numerical values for \( D(x,x) \) in simulation are obtained from \( d_{i-1,j+1}(t) \). The parameters associated with this plot are \( p = 0.75, q = 0.25, t = 700 \) and \( N = 200 \). The jump distribution is uniform, i.e. \( R(\eta) = 1 \).
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\[ \mathcal{V}_1(x) = \frac{\mu_2}{\mu_1 - \mu_2} D(x, x), \]  

(86)

where \( D(x, y) \) is completely determined from the knowledge of \( C(x, y) \) through (47). In figure 11 we numerically verify (85) whereas in the inset we verify (86). For the plot in the inset, both the quantities \( \mathcal{V}_1(x) \) and \( D(x, x) \) are obtained from numerical measurements. Numerical values for \( D(x, x) \) in simulation are obtained from \( d_{j-1,j+1}(t) \).

5. Conclusion

The motion of a driven particle in a crowded medium (such as a single-file system) is a common problem, which appears in various situations such as active microrheology inside capillaries [54], active transport of a vesicle in a crowded axon [55], directed cellular movements in crowded channels [56] and particle flow in microfluidic devices [57]. So it is important to determine the statistical properties of the motion of particles in a single-file system. In this work we have studied the motion of a DTP in the context of RAP as this process provides an easier analytically solvable setting where the behavior of the tracer is qualitatively similar to that in SEP and many related questions can be computed in a much simpler way than SEP. In addition, the new results on correlations between the particles will provide a benchmark to which results on other models (such as SEP) can be compared.

In the first part, the motion of the tracer and its effects on its environment have been characterized by computing the displacement of the tracer as well as the perturbation of the density profile. For both quantities the results are very similar to those obtained in a one-dimensional SSEP with a single biased tracer, where the velocity of the tracer also vanishes at long times and the density perturbation decays exactly the same way at large distances.

In single-file systems particles are subjected to strong caging effects, which usually have dramatic effects on the fluctuations and correlations of the positions of the particles. Since in our case the particles are also non-overtaking, their motion constitutes a single-file motion. We have shown in this paper that, at long times, the position–position correlations of different particles at the same time support a nice scaling form when the particle labels are rescaled by \( \sqrt{t} \). We showed that the corresponding scaling function \( \mathcal{C}(x, y) \) satisfies a differential equation which can be solved perturbatively around the solution of the unbiased tracer case. We have computed the first two terms of the perturbative expansion. In the case where the tracer is totally asymmetric the problem is more tractable, enabling us to compute the variance of the position of the tracer exactly. Finally, the variances of the gaps between successive particles were obtained and shown to converge to finite values at long times.

There are many interesting extensions of this problem to explore in the future. For example, finding an exact and complete solution of the equation (55) for arbitrary \( p \) and \( q \) would be of interest. In this paper we have considered only quenched (fixed) initial position configuration of the particles. One natural extension would be to consider other initial configurations (ICs) such as an annealed IC. We guess the difference

\[ \text{doi:10.1088/1742-5468/2016/05/053212} \]
between the quenched and the annealed case would be similar as in the homogeneous
single-file systems, i.e. in both cases the variance of the displacement of the TP grows
subdiffusively whereas the prefactor is different [25, 38]. It would be interesting to
check this fact rigorously. In equations (41) and (42) we have obtained the average
density profile. However, the fluctuations of the local density about this average remain
to be calculated. Also, calculating the probability of large deviations, either of the full
density profile or the position of the driven tracer, would be of interest. A different
problem which we would like to explore is the effective interaction between two or more
tracers and the dynamical effect that one tracer has on another.

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Appendix A. Solution of ψ(r, θ)

In this appendix we solve equation (64) for the function ψ(r, θ) in polar coordinates. For
convenience let us rewrite (64) here:

\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + 2r \frac{\partial}{\partial r} - 2 \right] \psi(r, \theta) = \left( \delta(\theta) - \delta(\theta - \pi) \right) \frac{\text{erfc}(r/\sqrt{2})}{r}. \]  

(A.1)

As we noticed in (63), C(r, θ) can be expressed in terms of ψ(r, θ) as

\[ C(r, \theta) = \frac{2\mu_2}{\mu_1 - \mu_2} \left( 2\sqrt{2} \psi(r, \theta) - \psi\left(r, \theta + \frac{\pi}{4}\right) - \psi\left(r, \theta - \frac{\pi}{4}\right) \right). \]  

(A.2)

We now use the representation of the delta function \( \delta(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{i m \theta} \) to write

\[ \delta(\theta) - \delta(\theta - \pi) = \frac{1}{\pi} \sum_{m=0}^{\infty} 2 \cos((2m + 1)\theta), \]  

(A.3)

and to expand the angular part of \( \psi \) as well,

\[ \psi(r, \theta) = \sum_{m=0}^{\infty} 2 \cos((2m + 1)\theta) \psi_m(r). \]  

(A.4)

Using this expansion on both sides of (A.1), we find

\[ \psi_m''(r) + \left( \frac{1}{r} + 2r \right) \psi_m'(r) - \left( 2 + \frac{(2m + 1)^2}{r^2} \right) \psi_m(r) = \frac{\text{erfc}(r/\sqrt{2})}{\pi r}, \]  

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with the conditions that \( \psi_m(r) \) is finite for \( r \to 0 \) and vanishes for \( r \to \infty \).

**A.1. Solution for \( m \geq 1 \)**

For \( m \geq 1 \), the general solution of (A.5) reads

\[
\psi_m(r) = C_m^1 \psi_m^1(r) + C_m^2 \psi_m^2(r) + \psi_m^P(r),
\]

where \( C_m^1 \) and \( C_m^2 \) are constants to be determined, \( \psi_m^1(r) \) and \( \psi_m^2(r) \) are solutions of the homogeneous equation

\[
\psi_m^1(r) = \frac{e^{-r^2}}{r^{2m+1}} \sum_{k=0}^{m-1} \left( \prod_{l=k}^{m-2} \frac{(l+1)(l-2m)}{l+1-m} \right) r^{2k},
\]

\[
\psi_m^2(r) = \frac{1}{r^{2m+1}} \sum_{k=0}^{m+1} \left( \prod_{l=k}^{m} \frac{(l+1)(l-2m)}{m+1-l} \right) r^{2k},
\]

and \( \psi_m^P(r) \) is a particular solution. Looking at the structure of the solutions for small values of \( m \), we try a particular solution of the form

\[
\psi_m^P(r) = \frac{e^{-r^2}}{r^{2m+1}} \chi_m(r), \quad \text{with}
\]

\[
\chi_m(r) = \frac{r^2}{2} P_{1,m}(r) + e^{r^2} \text{erf}\left( \frac{r}{\sqrt{2}} \right) P_{2,m}(r) + e^{r^2} \text{erfc}\left( \frac{r}{\sqrt{2}} \right) P_{3,m}(r) + \text{erfi}\left( \frac{r}{\sqrt{2}} \right) P_{4,m}(r),
\]

where \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \) is the error function, \( \text{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt \) is the ‘imaginary error function’ and the \( P_{l,m}(r) \) are polynomials that depend on \( m \), although this is not emphasized by the notation. Substituting \( \chi_m^P(r) \) in (A.5) we have

\[
\chi_m''(r) - \left( \frac{1 + 4m}{r} + 2r \right) \chi_m'(r) + 4(m-1) \chi_m(r) = \frac{r^{2m}}{\pi} e^{r^2} \text{erfc}\left( \frac{r}{\sqrt{2}} \right).
\]

The exponential functions in the prefactors of the polynomials in equation (A.8) cannot be generated by polynomials. Equation (A.9) is therefore verified if the coefficient of each of the functions \( e^{r^2} \), \( e^{r^2} \text{erf}\left( \frac{r}{\sqrt{2}} \right) \), \( e^{r^2} \text{erfc}\left( \frac{r}{\sqrt{2}} \right) \) and \( \text{erfi}\left( \frac{r}{\sqrt{2}} \right) \) vanishes. The \( e^{r^2} \text{erf}\left( \frac{r}{\sqrt{2}} \right) \), \( e^{r^2} \text{erfc}\left( \frac{r}{\sqrt{2}} \right) \) and \( \text{erfi}\left( \frac{r}{\sqrt{2}} \right) \) involve only \( P_{2,m} \), \( P_{3,m} \) and \( P_{4,m} \) respectively. The equations are

\[
P_{2,m}' + \left( 2r - \frac{1 + 4m}{r} \right) P_{2,m} - 4(p + 1) P_{2,m} = 0,
\]

\[
P_{3,m}' + \left( 2r - \frac{1 + 4m}{r} \right) P_{3,m} - 4(p + 1) P_{3,m} = \frac{r^{2p}}{\pi},
\]

\[
P_{4,m}' - \left( 2r + \frac{1 + 4m}{r} \right) P_{4,m} - 4(p - 1) P_{4,m} = 0.
\]

Polynomial solutions of equations (A.10) are easily found,

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\[ P_{2,m}(r) = \frac{K_{2,m}}{4\pi} \sum_{k=0}^{m+1} \left( \prod_{l=k+1}^{m+1} \frac{l(l-2m-1)}{m+2-l} \right) r^{2k}, \]

\[ P_{3,m}(r) = -\frac{1}{4\pi} \sum_{k=0}^{m} \left( \prod_{l=k+1}^{m} \frac{l(l-2m-1)}{m+2-l} \right) r^{2k}, \]

\[ P_{4,m}(r) = \frac{K_{4,m}}{4\pi} \sum_{k=0}^{m-1} \left( \prod_{l=k+1}^{m-1} \frac{l(l-2m-1)}{l-m} \right) r^{2k}, \]

(A.11)

where \( K_{2,m} \) and \( K_{4,m} \) are a priori arbitrary constants that depend on \( m \).

Now we focus on the equation for \( P_{1,m} \) which involves the other three polynomials,

\[ P_{1,m} - \frac{1+4m}{r} P'_{1,m} - (4 + r^2) P_{1,m} = -\sqrt{\frac{2}{\pi}} \left[ 2P'_{4,m} - \left( \frac{1+4m}{r} + r \right) P_{4,m} \right. \]

\[ \left. + 2(P'_{2,m} - P'_{3,m}) - \left( \frac{1+4m}{r} - r \right) (P_{2,m} - P_{3,m}) \right]. \]

(A.12)

We take \( P_{1,m}(r) = \sum_{k=0}^{m} f_{2k+1} r^{2k+1} \). Identifying the powers of \( r \) in (A.12) gives \( m + 3 \) equations for the \( m + 1 \) coefficients \( f_{2k+1} \), to which we add the unknown constants \( K_{2,m} \) and \( K_{4,m} \). The equations are clearly linear in the \( f_{2k+1} \) and in \( K_{2,m}, K_{4,m} \), so they may be solved by matrix inversion.

There is no simple expression of the inverse matrix, but based on numerical solutions (found using Mathematica) for the first few values of \( m \) it seems reasonable to assume that equation (A.12) has a unique solution in terms of the \( f_{2k+1}, K_{2,m} \) and \( K_{4,m} \).

The polynomials \( P_{1,m} \) for the first few values of \( m \) are given as

\[ P_{1,1}(r) = \frac{\sqrt{2} r}{4\pi^{3/2}} (r^2 - 2), \]

\[ P_{1,2}(r) = \frac{\sqrt{2} r^3}{12\pi^{3/2}} (r^2 - 4), \]

\[ P_{1,3}(r) = \frac{\sqrt{2} r^3}{4\pi^{3/2}} (r^4 - 10r^2 + 60), \]

\[ P_{1,4}(r) = \frac{\sqrt{2} r^3}{4\pi^{3/2}} (r^4 - 14r^2 + 112), \]

\[ P_{1,5}(r) = \frac{\sqrt{2} r}{60\pi^{3/2}} (r^{10} - 16r^8 + 168r^6 - 2520r^4 + 15120r^2 - 151200), \]

\[ P_{1,6}(r) = \frac{\sqrt{2} r}{84\pi^{3/2}} (r^{10} - 22r^8 + 360r^6 - 6840r^4 + 55440r^2 - 665280), \]

\[ P_{1,7}(r) = \frac{\sqrt{2} r}{4\pi^{3/2}} (r^{12} - 32r^{10} + 610r^8 - 4752r^6 + 78408r^4 - 30880r^2 + 4324320), \]

\[ P_{1,8}(r) = \frac{\sqrt{2} r}{4\pi^{3/2}} (r^{12} - 40r^{10} + 970r^8 - 11440r^6 + 223080r^4 - 1201200r^2 + 19219200). \]

(A.13)
We used these explicit forms of the polynomials to generate the theoretical curves in figures 6 and 8. Based on small values of $m$, the values of the constants are conjectured to be

$$K_{2,m} = \begin{cases} 
\frac{2}{m(m+1)} & \text{for } m = 4l + 1 \text{ or } p = 4l + 2, \\
0 & \text{for } m = 4l + 3 \text{ or } m = 4l + 4.
\end{cases}$$

for $l = 0, 1, 2, \ldots$ \hfill (A.14)

$$K_{4,m} = \begin{cases} 
1 & \text{for } m = 4l + 1 \text{ or } m = 4l + 2, \\
-1 & \text{for } m = 4l + 3 \text{ or } m = 4l + 4.
\end{cases}$$

which have been checked up to $m = 8$. Inserting the expressions of the polynomials from (A.11)–(A.13) and the constants from (A.14) in the ansatz (A.8) one obtains the particular solution $\psi^p_m(r)$.

Let us now fix the constants $C^1_m$ and $C^2_m$. For large $r$ the particular solution goes as $\psi^p_m(r) \sim K_{2,m} r$, which must be compensated by $C^2_m r$, giving

$$C^2_m = -\frac{K_{2,m}}{4\pi}. \hfill (A.15)$$

For small $r$ the homogeneous terms go as

$$C^1_m = \frac{(-1)^{m+1}}{4\pi} \frac{(2m+1)!}{(m+1)!}$$

and the particular solution as

$$C^1_m = \frac{(-1)^{m+1}}{4\pi} (m(m+1)K_{2,m} - 1). \hfill (A.16)$$

A.2. Solution for $m = 0$

For $m = 0$ the general $m$ calculation does not hold, as the sum in the definition of $\psi^1_m$ would be empty. One can however find the homogeneous solutions separately as

$$\psi^1_0(r) = \frac{e^{-r^2} + r^2 \text{Ei}(-r^2)}{r},$$

$$\psi^2_0(r) = r,$$

where $\text{Ei}(u) = -\int_{-u}^{\infty} \frac{e^{-t}}{t} \, dt$ is the exponential integral. The particular solution can be obtained from these homogeneous solutions as

$$\psi^p_0(r) = \int_{r'=1}^{r} \frac{\psi^1_0(r')\psi^2_0(r) - \psi^1_0(r)\psi^2_0(r')}{\psi^1_0(r')\psi^2_0(r') - \psi^1_0(r')\psi^2_0(r')} \frac{1}{\pi r'} \text{erfc} \left( \frac{r'}{\sqrt{2}} \right) \, dr'. \hfill (A.18)$$
When $r$ is small we have $\psi^P_0(r) \sim -\frac{1}{4\pi r}$, which must be compensated by $C_0^1 \psi^P_0(r) \sim \frac{C^1_0}{r}$, giving $C^1_0 = \frac{1}{4\pi}$. When $r \to \infty$ we have

$$\psi^P_0(r) \sim r \int_{u=1}^{\infty} (1 + u^2 e^{u^2} \text{Ei}(-u^2)) \frac{1}{2\pi u} \text{erfc}\left(\frac{u}{\sqrt{2}}\right)\,du,$$

(A.19)

that has to be compensated by $C_0^2 \psi^P_0(r) = C_0^2 r$, giving $C^2_0 = -\int_1^{\infty} (1 + u^2 e^{u^2} \text{Ei}(-u^2)) \frac{1}{2\pi u} \text{erfc}\left(\frac{u}{\sqrt{2}}\right)\,du \simeq -0.00589612$.

The function $\psi(r, \theta)$ is obtained after summing (65) over $m$ using $\psi_m(r, \theta)$ from (A.6), and using this $\psi(r, \theta)$ the solution $C_l(r, \theta)$ is obtained from (63).

### A.3. Small and large $r$ behavior of $\psi(r, \theta)$

As a matter of fact, the above choices of constants $C^1_m$ and $C^2_m$ ensure not only that the strongest divergence of $\psi_m(r)$ is canceled, but also that the $\psi_m(r)$ vanish when $r \to 0$. Here we determine an expansion of $\psi(r, \theta)$ around $r = 0$. Each of the $\psi_m(r)$ may be expanded separately. For $m = 0$ we obtain

$$\psi_0(r) = \frac{r \log r}{2\pi} + \frac{2\gamma_E + \pi - 2}{8\pi} r - \frac{\sqrt{2}}{3\pi^{3/2}} r^2 + O(r^3),$$

(A.20)

where $\gamma_E$ is Euler’s constant. On a numerical basis we can conjecture a general form of the expansion for any $m \geq 1$,

$$\psi_m(r) = -\frac{r}{4\pi m(m+1)} + \frac{\sqrt{2}}{\pi^{3/2}} r^2 \frac{1}{4(m+1)m-3} + O(r^3),$$

(A.21)

The small $r$ behavior of $\psi(r, \theta)$ is therefore given by

$$\psi(r, \theta) = \frac{r \log r}{\pi} \cos \theta + \frac{2\gamma_E + \pi - 2}{4\pi} r \cos \theta - \frac{r}{2\pi} S_1(\theta) + \left(\frac{2}{\pi}\right)^{3/2} r^2 S_2(\theta) + O(r^3),$$

(A.22)

where the sums are given by

$$S_1(\theta) = \sum_{m=1}^{\infty} \frac{\cos((2m+1)\theta)}{m(m+1)} = \cos \theta + (2\theta - \pi \text{Sign}[\theta]) \sin \theta,$$

$$S_2(\theta) = \sum_{m=0}^{\infty} \frac{\cos((2m+1)\theta)}{4m(m+1)-3} = -\frac{\pi}{8} \text{Sign}[\theta] \sin \theta,$$

(A.23)

for $-\pi < \theta \leq \pi$. Combining (A.22) and (A.23) we obtain equation (67) as presented in the main text.

For $r \to \infty$ it can be shown that all the $\psi_m$ functions behave as $-\frac{\sqrt{2}}{\pi m^{3/2}} \frac{e^{-u^2}}{r^2}$. The radial part of the $\psi_m$ functions can be factorized out of the sum over $m$ and the angular part gives back a Dirac delta, giving expression (68) from the main text.
Appendix B. Evaluation of the integral $I$ in (81)

Here we perform the integral

$$I = \int_{0}^{\infty} dr \frac{\xi_{0}^{2h}(r)}{W_{0}(r)} \frac{2}{\pi r} \left[ \text{erf}(r/\sqrt{2})^{2} - 1 \right] \quad (B.1)$$

exactly. For $m = 0$ the homogeneous solutions are more explicitly written as

$$\xi_{0}^{1h}(r) = e^{-r^{2}/2} \left[ r^{2}I_{1}\left( \frac{r^{2}}{2} \right) + (r^{2} + 1)I_{0}\left( \frac{r^{2}}{2} \right) \right],$$

$$\xi_{0}^{2h}(r) = \frac{2}{\sqrt{\pi}} e^{-r^{2}/2} \left[ (r^{2} + 1)K_{0}\left( \frac{r^{2}}{2} \right) - r^{2}K_{1}\left( \frac{r^{2}}{2} \right) \right], \quad (B.2)$$

where $I_{n}(x)$ and $K_{n}(x)$ are modified Bessel functions of, respectively, the first and the second kind of order $n$. Using the expressions of the derivatives of the Bessel functions in terms of Bessel functions of higher order and the recurrence relation between successive Bessel functions, the Wronskian can be brought to a very simple form,

$$W_{0}(r) = \frac{4}{\sqrt{\pi}} \frac{e^{-r^{2}}}{r}. \quad (B.3)$$

Hence simplifying (B.4) we have

$$I = \frac{1}{\pi} \int_{0}^{\infty} dr e^{r^{2}} \left[ (r^{2} + 1)K_{0}\left( \frac{r^{2}}{2} \right) - r^{2}K_{1}\left( \frac{r^{2}}{2} \right) \right] \left[ \text{erf}(r/\sqrt{2})^{2} - 1 \right]. \quad (B.4)$$

Next, we take advantage of the identity

$$\frac{d}{dr} \left( e^{r^{2}} K_{0}\left( \frac{r^{2}}{2} \right) \right) = r e^{r^{2}} \left( K_{0}\left( \frac{r^{2}}{2} \right) - K_{1}\left( \frac{r^{2}}{2} \right) \right) \quad (B.5)$$

to integrate by parts

$$I = \frac{1}{\pi} \int_{r=0}^{\infty} dr e^{r^{2}} \left[ (r^{2} + 1)K_{0}\left( \frac{r^{2}}{2} \right) - r^{2}K_{1}\left( \frac{r^{2}}{2} \right) \right] \left[ \text{erf}(r/\sqrt{2})^{2} - 1 \right]$$

$$= \frac{1}{\pi} \int_{r=0}^{\infty} dr e^{r^{2}} K_{0}\left( \frac{r^{2}}{2} \right) \left[ \text{erf}(r/\sqrt{2})^{2} - 1 \right] + \frac{1}{\pi} \int_{r=0}^{\infty} dr e^{r^{2}} K_{0}\left( \frac{r^{2}}{2} \right) \left[ \text{erf}(r/\sqrt{2})^{2} - 1 \right]$$

$$= -\frac{1}{\pi} \int_{r=0}^{\infty} e^{r^{2}} K_{0}\left( \frac{r^{2}}{2} \right) \frac{d}{dr} \left[ r \left[ \text{erf}(r/\sqrt{2})^{2} - 1 \right] \right]$$

$$= \left( \frac{2}{\pi} \right)^{3/2} \int_{r=0}^{\infty} K_{0}\left( \frac{r^{2}}{2} \right) \text{erf}(r/\sqrt{2}) r dr. \quad (B.6)$$

In the second line of (B.6) the second term vanishes and the derivative of the $r$ part in the third term exactly cancels the first term, so that only the term on the last line remains. Finally, we use the definition of erf and an integral representation of the $K_{0}$ function,
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\[ I = \frac{4\sqrt{2}}{\pi^2} \int_{t=0}^{\infty} \int_{v=0}^{\infty} e^{-v^2} dv \int_{t=1}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{t^2-1}} dt dr \]

\[ = \frac{4\sqrt{2}}{\pi^2} \int_{t=1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \int_{v=0}^{\infty} dv e^{-v^2} \int_{u=v^2}^{\infty} e^{-u^2} du \]

\[ = \frac{2\sqrt{2}}{\pi^{3/2}} \int_{t=1}^{\infty} \frac{dt}{(t(t+1)\sqrt{t-1})} = \frac{2}{\sqrt{\pi}} (\sqrt{2} - 1). \quad (B.7) \]

In equation (B.7), after expressing the \( \text{erf} \) and \( K_0 \) functions, we made the change of variables \( u = \frac{t^2}{2} \), then performed the integrals over \( u, v \) and \( t \) in that order. We obtain \( I = \frac{2}{\sqrt{\pi}} (\sqrt{2} - 1) \), as announced in the main text.

References

[1] Harris T E 1965 J. Appl. Probab. 2 323
[2] Jepsen D W 1965 J. Math. Phys. 6 405
[3] Gupta V, Nivarthi S S, McCormick A V and Davis H T 1995 Chem. Phys. Lett. 247 596
[4] Kulka V et al 1996 Science 272 702
[5] Hahn K, Kärger J and Kuksa V 1996 Phys. Rev. Lett. 76 2762
[6] Wei Q-H, Bechinger C and Leiderer P 2000 Science 287 625
[7] Meersmann T, Logan J W, Simonutti R, Caldarelli S, Comotti A, Sozzani P, Kaiser L G and Pines A 2000 J. Phys. Chem. A 104 11665
[8] Lutz C, Kollmann M and Bechinger C 2004 Phys. Rev. Lett. 93 026001
[9] Lin B, Meron M, Cui B, Rice S A and Diamant H 2005 Phys. Rev. Lett. 94 216001
[10] Percus J K 1974 Phys. Rev. A 9 557
[11] Alexander S and Pincus P 1978 Phys. Rev. B 18 2011
[12] van Beijeren H, Keir K W and Kutner R 1983 Phys. Rev. B 28 5711
[13] Arratia R 1983 Ann. Probab. 11 362
[14] Majumdar S N and Barma M 1991 Phys. Rev. B 44 5306
[15] Rödenbeck C, Kärger J and Hahn K 1998 Phys. Rev. E 57 4382
[16] Kollmann M 2003 Phys. Rev. Lett. 90 180602
[17] Gupta S, Majumdar S N, Godrèche C and Barma M 2007 Phys. Rev. E 76 021112
[18] Sabhapandit S 2007 J. Stat. Mech. L05002
[19] Lizana L and Ambjörnsson T 2008 Phys. Rev. Lett 100 200601
Lizana L and Ambjörnsson T 2009 Phys. Rev. E 80 051103
[20] Barkai E and Silbey R 2009 Phys. Rev. Lett. 102 050602
[21] Barkai E and Silbey R 2010 Phys. Rev. E 81 041129
[22] Roy A, Narayan O, Dhar A and Sabhapandit S 2013 J. Stat. Phys. 150 851
[23] Illien P, Bénichou O, Mejia-Monasterio C, Oshanin G and Voituriez R 2013 Phys. Rev. Lett. 111 038102
[24] Bénichou O et al 2013 Phys. Rev. Lett. 111 260601
[25] Krapivsky P L, Mallick K and Sadhuk T 2014 Phys. Rev. Lett. 113 078101
[26] Hegde C, Sabhapandit S and Dhar A 2014 Phys. Rev. Lett. 113 120601
[27] Sabhapandit S and Dhar A 2015 arXiv:1506.01824
[28] Demasi A and Ferrari P A 1985 J. Stat. Phys. 38 603
[29] Kutner R and van Beijeren H 1985 J. Stat. Phys. 39 317
[30] Ferrari P and Fontes L G 1998 Electron. J. Probab. 3 134
[31] Coppersmith S N, Liu C H, Majumdar S, Narayan O and Witten T A 1996 Phys. Rev. E 53 4673–85
[32] Rajesh R and Majumdar S N 2000 J. Stat. Phys. 99 943
[33] Feng S, Iscoe I and Seppäläinen T 1996 J. Stat. Phys. 85 513–7
[34] Krug J and Garcia J 2000 J. Stat. Phys. 99 31
[35] Melzak Z A 1976 Mathematical Ideas, Modeling and Applications (Companion to Concrete Mathematics vol II) (New York: Wiley)

doi:10.1088/1742-5468/2016/05/053212 34
Correlation and fluctuation in a random average process on an infinite line with a driven tracer

[36] Ispolatov S, Krapivsky P L and Redner S 1998 Eur. Phys. J. B 2 267–76
[37] Aldous D and Diaconis P 1995 Probab. Theory Relat. Fields 103 199–213
[38] Rajesh R and Majumdar S N 2001 Phys. Rev. E 64 036103
[39] Gutsche C, Kremer F, Kräger M, Rauscher M, Weeber R and Harting J 2008 J. Chem. Phys. 129 084902
[40] Krüger M and Rauscher M 2009 J. Chem. Phys. 131 094902
[41] Candelier R and Dauchot O 2010 Phys. Rev. E 81 011304
[42] Pesic J, Terdik J Z, Xu X, Tian Y, Lopez A, Rice S A, Dinner A R and Scherer N F 2012 Phys. Rev. E 86 031403
[43] Dullens R P A and Bechinger C 2011 Phys. Rev. Lett. 107 138301
[44] Burlatsky S F, Oshanin G S, Mogutov A V and Moreau M 1992 Phys. Lett. A 166 230
[45] Burlatsky S F, Oshanin G, Moreau M and Reinhardt W P 1996 Phys. Rev. E 54 3165
[46] Landim C, Olla S and Volchan S B 1998 Commun. Math. Phys. 192 287
[47] Bénichou O, Cazabat A M, Lemarchand A, Moreau M and Oshanin G 1999 J. Stat. Phys. 97 351
[48] De Coninck J, Oshanin G and Moreau M 1997 Europhys. Lett. 38 527
[49] Bénichou O, Cazabat A M, De Coninck J, Moreau M and Oshanin G 2001 Phys. Rev. B 63 235413
[50] Bénichou O, Klafert J, Moreau M and Oshanin G 2000 Phys. Rev. E 62 3327
[51] Reichhardt C and Olson C J 2004 Phys. Rev. Lett. 92 108301
[52] Cividini J, Kundu A, Majumdar S N and Mukamel D 2016 J. Phys. A: Math. Theor. 49 085002
[53] http://functions.wolfram.com/07.20.02.0001.01, http://functions.wolfram.com/07.33.02.0001.01
[54] Wilson L G and Poon W C K 2001 Phys. Chem. Chem. Phys. 13 10617
[55] Loverdo C, Bénichou O, Moreau M and Voituriez R 2008 Nat. Phys. 4 134
[56] Hawkins R J, Piel M, Faure-Andre G, Lennon-Dumenil A M, Joanny J F, Prost J and Voituriez R 2009 Phys. Rev. Lett. 102 058103
[57] Wittbracht F, Weddemann A, Auge A and Hütten A 2010 4th IEEE Int. Conf. on Quantum, Nano and Micro Technologies (IEEE) pp 102–6

doi:10.1088/1742-5468/2016/05/053212