Symmetric Semi-perfect Obstruction Theory Revisited

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Abstract In this paper we survey some results on the symmetric semi-perfect obstruction theory on a Deligne–Mumford stack $X$ constructed by Chang–Li, and Behrend’s theorem equating the weighted Euler characteristic of $X$ and the virtual count of $X$ by symmetric semi-perfect obstruction theories. As an application, we prove that Joyce’s $d$-critical scheme admits a symmetric semi-perfect obstruction theory, which can be applied to the virtual Euler characteristics by Jiang–Thomas.

Keywords Symmetric semi-perfect obstruction theory, the Behrend function, algebraic $d$-critical scheme, virtual signed Euler characteristics

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1 Introduction

1.1 Symmetric Semi-perfect Obstruction Theory

Perfect obstruction theories and virtual fundamental cycles are powerful tools to define counting invariants in modern enumerative geometry. The construction was given by Li–Tian [26], Behrend–Fantechi [2] in algebraic geometry. The techniques apply to both Gromov–Witten theory and Donaldson–Thomas theory. However there exist some moduli problems such that the perfect obstruction theory only locally exists. In [6], Chang–Li generalized and defined semi-perfect obstruction theory which is locally given by a perfect obstruction theory. The local virtual cycles glue to give a global virtual fundamental cycle. Examples of schemes or Deligne–Mumford stacks admitting semi-perfect obstruction theory include the moduli spaces of derived objects in the derived category of coherent sheaves on a Calabi–Yau threefold.

In [1], Behrend introduced the “symmetric obstruction theory” in order to prove that the Donaldson–Thomas invariants on a Calabi–Yau threefold are motivic invariants. The symmetry on the obstruction theory implies special property for the obstruction theory, see §2.2. Behrend proved that the virtual count of the symmetric obstruction theory on a scheme or Deligne–Mumford stack $X$ is a weighted Euler characteristic of $X$ weighted by the “Behrend function”, see §3, and more details on the Behrend function are included in [13].

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In this paper we survey some results for symmetric semi-perfect obstruction theories, and Behrend’s theorem on equating the virtual count to the weighted Euler characteristic by the Behrend function.

In the paper [20], Kiem–Savvas defined the notion of almost perfect obstruction theory on a Deligne–Mumford stack $X$. An almost perfect obstruction theory is a natural semi-perfect obstruction theory, but the inverse is wrong. Many moduli stacks of stable derived objects in the derived category of coherent sheaves on a Calabi–Yau threefold admit almost perfect obstruction theory, see [18, 19, 21] on this research direction.

1.2 Application to Donaldson–Thomas invariants
Let $Y$ be a smooth Calabi–Yau threefold or a smooth threefold Calabi–Yau Deligne–Mumford stack. The Donaldson–Thomas invariants of $Y$ count stable coherent sheaves. Thomas [32] constructed a perfect obstruction theory $E^\bullet$ in the sense of Li–Tian [26], and Behrend–Fantechi [2] on the moduli space $X$ of stable sheaves over $Y$. If $X$ is proper, then the virtual dimension of $X$ is zero, and the integral $\text{DT}_Y = \int_{[X]_{\text{virt}}} 1$ is the Donaldson–Thomas invariant of $Y$. In [1] Behrend proved that the moduli scheme $X$ of stable sheaves on $Y$ admits a symmetric obstruction theory. Let

$$\nu_X : X \to \mathbb{Z}$$

be the Behrend function on $X$. If $X$ is proper, then in [1, Theorem 4.18] Behrend proved that $\text{DT}_Y = \int_{[X]_{\text{virt}}} 1 = \chi(X, \nu_X)$, where $\chi(X, \nu_X)$ is the weighted Euler characteristic weighted by the Behrend function. Same result for a proper Deligne–Mumford stack $X$ with a symmetric perfect obstruction theory is conjectured by Behrend in [1], and is proved in [14].

1.3 $D$-critical Schemes
In [4] and [5], Joyce et al. studied the classical underlying scheme of a $(-1)$-shifted derived symplectic scheme defined in [30]. In [17], Joyce introduced the notion of $d$-critical schemes or $d$-critical analytic spaces, where the underlying scheme of a $(-1)$-shifted derived symplectic scheme defined in [30] is a $d$-critical scheme. For instance, the moduli space $X$ of stable coherent sheaves or simple complexes on a Calabi–Yau threefold $Y$ can be lifted to a $(-1)$-shifted derived symplectic scheme, and is a $d$-critical scheme. But the notion of $d$-critical schemes is more general than the underlying scheme of a $(-1)$-shifted derived symplectic scheme. Kiem–Li [23] introduced “virtual critical manifolds”, and proved that they are equivalent to $d$-critical analytic spaces. The notion of $d$-critical schemes or virtual critical manifolds are important to the categorification of Donaldson–Thomas invariants.

We provide the proof that a $d$-critical scheme $X$ admits a symmetric semi-perfect obstruction theory. Kiem–Li [23] have already proved that a virtual critical manifold admits a symmetric semi-perfect obstruction theory. All results here in this section are due to Kiem–Li.

1.4 Virtual Signed Euler Characteristics
Motivated by the cotangent theory of Costello [9] and Gromov–Witten theory with $p$-fields in [7], in [16] Thomas and the author defined four virtual signed Euler characteristics on the dual obstruction sheaf $N = \text{Ob}_X^*$ of a perfect obstruction theory on a scheme $X$. The abelian cone $N$ locally is given by the critical locus of a regular function on a higher dimensional smooth scheme,
and $N$ is a $d$-critical scheme of Joyce [17]. In [16], the scheme $X$ with a perfect obstruction theory $E_X^\bullet$ is assumed to the underlying scheme of a quasi-smooth derived scheme so that on the cone $N$ there exists a symmetric obstruction theory by pullback from the derived cotangent bundle of $(X, E_X^\bullet)$. Since $N$ is a $d$-critical scheme, $N$ naturally admits a symmetric semi-perfect obstruction theory, and we still can define the four virtual signed Euler characteristics in [16]. Note that in [18], Kiem has already proved that on $N$ there is a semi-perfect obstruction theory. We also talk about a natural compactification $\overline{N}$ of $N$ by taking the projective cone of $\text{Ob}_X$, and show that $\overline{N}$ is also a $d$-critical scheme under some conditions. This may have applications for the Vafa–Witten invariants as developed in [31].

In a different situation, if $X$ is a scheme or a Deligne–Mumford stack admitting a symmetric semi-perfect obstruction theory, and furthermore there exists a $\mathbb{G}_m$-action on $X$ such that the semi-perfect obstruction theory is $\mathbb{G}_m$-equivariant, we show that the Behrend’s weighted Euler characteristic of $X$ is the same as the Kiem–Li cosection localized invariants of the $\mathbb{G}_m$-action. This generalizes the result in [13] to symmetric semi-perfect obstruction theories.

1.5 Outline

The rough structure of the paper is as follows. We review the basic materials about symmetric semi-perfect obstruction theories in §2, where §2.1 goes over the perfect obstruction theory and obstruction space; and in §2.2 the basic notion of symmetric semi-perfect obstruction theory is reviewed. In §3 we prove Theorem 3.8 that equating the weighted Euler characteristic to the virtual count of a symmetric semi-perfect obstruction theory. We apply the result in the theorem to $d$-critical schemes in §4; where in §4.2 we review Joyce’s $d$-critical schemes, in §4.3 we provide a detailed proof for a $d$-critical scheme admitting a symmetric semi-perfect obstruction theory; and finally in §4.4 two special cases of virtual count are generalized to symmetric semi-perfect obstruction theories.

Convention

Throughout the paper we work over an algebraically closed field $\kappa$ of character zero. We use $\mathbb{G}_m$ to represent the one parameter multiplication group of $\mathbb{A}^1_{\kappa}$. For a scheme or a Deligne–Mumford stack $X$, $L_X := \mathbb{L}_X^{\geq -1}$ represents the truncated cotangent complex, where $L_X$ is the full cotangent complex. We use the notation $E^{-i} := (E_i)^*$ for dual vector bundles, and reserve $\vee$ for the derived dual of coherent sheaves and complexes.

All Deligne–Mumford stacks are quasi-projective, which means from Kretch’s equivalence condition, that they can be embedded into a smooth projective Deligne–Mumford stack and have resolution properties. Let $D(X)$ be the derived category of coherent modules on the Deligne–Mumford stack $X$.

2 Symmetric Semi-perfect Obstruction Theory

In this section we review some basic materials about the symmetric semi-perfect obstruction theory in [1, 6].
2.1 Perfect Obstruction Theory and Obstruction Spaces

**Definition 2.1** Let $X_\alpha$ be a scheme. A perfect obstruction theory of $X_\alpha$ consists of a derived category morphism:

$$\phi : E \rightarrow L_{X_\alpha}$$

such that

1. $E \in D(O_{X_\alpha})$ is perfect, of perfect amplitude contained in the interval $[-1, 0]$,

2. $\phi$ induces an isomorphism on $h^0$ and an epimorphism on $h^{-1}$.

The sheaf $\text{Ob}_\phi := h^1(E^\vee)$ is called the “obstruction sheaf”.

From [2], the intrinsic normal cone $c_{X_\alpha}$ is defined as follows: whenever there is a closed immersion $X_\alpha \rightarrow M$ into a smooth scheme $M$, $c_{X_\alpha} \cong [C_{X_\alpha}/M/T_M|_{X_\alpha}]$, where $C_{X_\alpha}/M$ is the normal cone and $T_M|_{X_\alpha}$ is the restriction of the tangent bundle $T_M$ to $X_\alpha$ which acts on the normal cone. The morphism $\phi : E \rightarrow L_{X_\alpha}$ defines a closed immersion of cone stacks

$$c_{X_\alpha} \subset N_{X_\alpha} := h^1/h^0(L_{X_\alpha}^\vee).$$

Let $[c_{X_\alpha}] \in Z_*(N_{X_\alpha})$ be the associated cycle and let

$$h^1/h^0(\phi^\vee) : N_{X_\alpha} \hookrightarrow h^1/h^0(E^\vee)$$

be the morphism induced by the truncated perfect obstruction theory.

2.1.1 Lifting Problem

We recall the classical lifting problem [2, §4], [6, Definition 2.5].

**Definition 2.2** Let $\iota : T \rightarrow \overline{T}$ be a closed subscheme with $T$ local Artinian. Let $I$ be the ideal of $T$ in $\overline{T}$, and let $m$ be the ideal sheaf of the closed point of $\overline{T}$. We call $\iota$ a small extension if $I \cdot m = 0$. Given a small extension $(T, \overline{T}, I, m)$ fitting into the commutative diagram:

$$\begin{array}{ccc}
T & \xrightarrow{g} & X_\alpha \\
\downarrow & & \downarrow \\
\overline{T} & \xrightarrow{\overline{g}} & \text{Spec}(\kappa)
\end{array} \tag{2.1.1}
$$

so that $\text{Im}(g)$ contains a closed point $p \in X_\alpha$.

Finding a morphism $\overline{g} : \overline{T} \rightarrow X_\alpha$ making the diagram (2.1.1) commute is called the “infinitesimal lifting problem of $X_\alpha$ at $p$”.

Standard obstruction theory tells us, see [11, Chapter 3, Theorem 2.1.7], [6, Lemma 2.6], that for an infinitesimal lifting problem of $X_\alpha$ at $p$ as in (2.1.1), there exists a canonical element

$$\omega(g, T, \overline{T}) \in \text{Ext}^1(g^*L_{X_\alpha}, I) = T_{p,X_\alpha} \otimes \kappa I$$

where $T_{p,X_\alpha} = h^1(L_{X_\alpha}^\vee|_p)$ is the intrinsic obstruction space to deforming $p \in X_\alpha$. The vanishing of $\omega(g, T, \overline{T})$ is necessary and sufficient for the lifting problem to be solvable, in which case the collection of the solutions form a torsor under $\text{Ext}^0(g^*L_{X_\alpha}, I) = \text{Hom}(g^*\Omega_{X_\alpha}, I)$.

**Remark 2.3** Recall that if $\phi : E \rightarrow L_{X_\alpha}$ is a perfect obstruction theory, then the obstruction space (of the obstruction theory $\phi$) to deforming $p \in X_\alpha$ is defined to be $\text{Ob}(\phi, p) = h^1(E^\vee|_p)$. 
Definition 2.4  Let $\phi : E \to L_{X_\alpha}$ be a perfect obstruction theory. For an infinitesimal lifting problem (2.1.1), the image
\[ \text{Ob}(\phi, g, T, \overline{T}) := h^1(\phi^\vee)(\omega(g, T, \overline{T})) \in \text{Ext}^1(g^*E, I) = \text{Ob}(\phi, p) \otimes_{\kappa} I \]
is called “the obstruction class” of $\phi$ to the lifting problem (2.1.1).

From [2, Theorem 4.5], the obstruction class $\text{Ob}(\phi, g, T, \overline{T}) = 0$ if and only if the lifting problem (2.1.1) is solvable.

Let $\phi : E \to L_{X_\alpha}$ and $\phi' : E' \to L_{X_\alpha}$ be two perfect obstruction theories.

Definition 2.5  We call $\phi$ is $\nu$-equivalent to $\phi'$ if there exists an isomorphism of sheaves
\[ \psi : h^1(E^\vee) \xrightarrow{\cong} h^1(E'^\vee) \] (2.1.2)
such that for every closed point $p \in X_\alpha$, and for any “infinitesimal lifting problem” of $X_\alpha$ at $p$ as in the diagram (2.1.1), we have
\[ \psi|_p(\text{Ob}(\phi, g, T, \overline{T})) = \text{Ob}(\phi', g, T, \overline{T}) \in \text{Ob}(\phi', p) \otimes_{\kappa} I. \]

2.1.2 Cycles in Sheaf Stacks

We review the integral cycles in sheaf stacks. From [6, §2], for any coherent sheaf $\mathfrak{F}$ on $X_\alpha$, there is a sheaf stack, still denoted by $\mathfrak{F}$, which is defined as the groupoid that associates to any $\rho : S \to X_\alpha$ the set $\Gamma(S, \rho^*\mathfrak{F})$. For the perfect obstruction theory $\phi : E^* \to L_{X_\alpha}$, we have the stack $h^1/h^0((E^*)^\vee)$, the obstruction sheaf $\text{Ob}_\phi := h^1((E^*)^\vee)$, and morphisms
\[ \eta_\phi : N_{X_\alpha} \to h^1/h^0((E^*)^\vee) \to h^1((E^*)^\vee) = \text{Ob}_\phi \]
where the first arrow is an embedding of the intrinsic normal sheaf to the bundle stack for the obstruction theory $\phi : E^* \to L_{X_\alpha}$. We recall the definition of rational cycles in the sheaf stacks.

Definition 2.6 ([6, Definition 2.2])  A substack $A \subset \mathfrak{F}$ is called a reduced cycle if for any locally free sheaf $\mathcal{E}$ of $O_{X_\alpha}$-modules and a surjective morphism $f : \mathcal{E} \to \mathfrak{F}$ (also denoting the morphism of sheaf stacks), $\mathcal{E} \times_{\mathfrak{F}} A$ is a reduced Zariski closed subset of $\mathcal{E}$. If there are three reduced cycles $A_1, A_2$ and $A_3$ of $\mathfrak{F}$, $A_3 = A_1 \cup A_2$ if for any $f : \mathcal{E} \to \mathfrak{F}$ as before,
\[ \mathcal{E} \times_{\mathfrak{F}} A_3 = \mathcal{E} \times_{\mathfrak{F}} A_1 \cup \mathcal{E} \times_{\mathfrak{F}} A_2 \]
as subschemes of $\mathcal{E}$. We say $A$ is integral if it is reduced and is not a union of two distinct non-empty reduced cycles of $\mathfrak{F}$. We define $Z_*(\mathfrak{F})$ to be the rational linear combination of integral cycles of $\mathfrak{F}$.

We apply this definition to the sheaf stack $\text{Ob}_\phi$ on $X_\alpha$. By [6, Lemma 2.3], for any $A \subset h^1/h^0((E^*)^\vee)$ an integral cycle, the image $\eta_\phi(A) \subset \text{Ob}_\phi = h^1((E^*)^\vee)$ is an integral cycle in $\text{Ob}_\phi$. We call $[\eta_\phi(A)]$ the pushforward of $[A]$ and denoted by $\eta_{\phi*}[A] \in Z_*(\text{Ob}_\phi)$. Now for the cycle $c_{X_\alpha}$, which is the intrinsic normal cone in $N_{X_\alpha}$, by writing the cycle $c_{X_\alpha}$ as linear combination of integral cycles of $N_{X_\alpha}$ we have
\[ [c_{\phi}] = \eta_{\phi*}[c_{X_\alpha}] \]
is the image of the intrinsic normal cone $c_{X_\alpha}$.  


**Proposition 2.7** ([6, Proposition 2.10])  If $\phi$ and $\phi'$ are two $\nu$-equivalent perfect obstruction theories, and let 

$$ \eta_{\phi} : N_{X_{\alpha}} \to h^1((E^\vee)^{\vee}) = \text{Ob}_{\phi} $$

be the morphism above, then for any integral cycle $A \subset N_{X_{\alpha}}$, we have 

$$ \psi_*(\eta_{\phi*}[A]) = \eta_{\phi'*}[A] \in Z_*(h^1((E^{\vee})^{\vee})) = Z_*(\text{Ob}_{\phi'}). $$

So for the intrinsic normal cone $c_{X_{\alpha}}$ of $X_{\alpha}$ we have 

$$ \psi_*(\eta_{\phi*}[c_{X_{\alpha}}]) = \psi_*(c_{\phi}) 
= \eta_{\phi'*}[c_{X_{\alpha}}] 
= [c_{\phi'}]. $$

### 2.2 Symmetric Semi-perfect Obstruction Theory

Let $X$ be a Deligne–Mumford stack of locally of finite type. For any two étale morphisms $X_{\alpha}, X_{\beta} \to X$ we let $X_{\alpha\beta} = X_{\alpha} \times_X X_{\beta}$, and for any object $F \in D^b(X_{\alpha})$ in the derived category, $F|_{X_{\alpha\beta}}$ is the pullback of $F$ under the projection $X_{\alpha\beta} \to X_{\alpha}$.

**Definition 2.8** ([6, Definition 3.1])  A semi-perfect obstruction theory on $X$ consists of an étale covering $\{X_{\alpha}\}_{\alpha \in \Lambda}$ of $X$ by affine schemes, and truncated perfect obstruction theories 

$$ \phi_{\alpha} : E_{\alpha} \to L_{X_{\alpha}}, \quad \alpha \in \Lambda $$

such that

1. for any pair $\alpha, \beta \in \Lambda$ there exists an isomorphism 

$$ \psi_{\alpha\beta} : h^1(E_{\alpha}^{\vee})|_{X_{\alpha\beta}} \xrightarrow{\cong} h^1(E_{\beta}^{\vee})|_{X_{\alpha\beta}} \quad (2.2.1) $$

such that the collections $(h^1(E_{\alpha}^{\vee}), \psi_{\alpha\beta})$ form a descent data of sheaves.

2. for any pair $\alpha, \beta \in \Lambda$, the obstruction theories $\phi_{\alpha}|_{X_{\alpha\beta}}$ and $\phi_{\beta}|_{X_{\alpha\beta}}$ are $\nu$-equivalent via $\psi_{\alpha\beta}$.

Of course, a perfect obstruction theory is a semi-perfect obstruction theory.

**Remark 2.9** ([20, Definition 3.1])  In Definition 2.8, if for some refinement of the covers $\{U_{\alpha}\}$ we require the morphisms 

$$ \psi_{\alpha\beta} : h^1/h^0(E_{\alpha}^{\vee})|_{X_{\alpha\beta}} \xrightarrow{\cong} h^1/h^0(E_{\beta}^{\vee})|_{X_{\alpha\beta}} $$

are isomorphisms on bundle stacks, then the above data define an almost perfect obstruction theory in the sense of [20].

Let $\phi = \{\phi_{\alpha}, X_{\alpha}, E_{\alpha}, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$ be a semi-perfect obstruction theory for $X$. We denote by $\text{Ob}_{\phi}$ the resulting obstruction sheaf of the semi-perfect obstruction theory by the gluing of $\text{Ob}_{\phi_{\alpha}}$ for $\alpha \in \Lambda$.

Let $N_X = h^1/h^0(L_X^{\vee})$ be the intrinsic normal sheaf of $X$, and we can think of this sheaf as the gluing of $N_X|_{X_{\alpha}} = N_{X_{\alpha}} = h^1/h^0(L_{X_{\alpha}}^{\vee})$ for $\alpha \in \Lambda$. Then there exists a group homomorphism 

$$ \eta_* : Z_*(N_X) \to Z_*(\text{Ob}_{\phi}) $$

by patching the collection: 

$$ \eta_{\phi_{\alpha}*} : Z_*(N_{X_{\alpha}}) \to Z_*(\text{Ob}_{\phi_{\alpha}}). $$
[6, Lemma 3.3] proves that for any integral Artin stack \([A] \in Z_*(N_X)\), the collection
\[
[A_\alpha] := \eta_{\phi, *}[A \times_X X_\alpha] \in Z_*(\text{Ob}_\phi |_{X_\alpha})
\]
satisfies the descent condition:
\[
A_\alpha \times_X X_\alpha \cap X_\alpha \beta \subset \text{Ob}_\phi |_{X_\alpha \beta}
\]
(2.2.2)
so that it forms an integral cycle in \(Z_*(\text{Ob}_\phi)\). Let \(\eta_{\phi,*} : Z_*(N_X) \to Z_*(\text{Ob}_\phi)\) be the homomorphism by linear extensions.

Let \(c_X\) be the intrinsic normal cone of \(X\) such that étale locally on \(X_\alpha \to X\), there exists a closed immersion
\[
X_\alpha \hookrightarrow M
\]
into a smooth scheme \(M\), \(c_X |_{X_\alpha} = [C_{X_\alpha/M} / T_M |_{X_\alpha}]\). Then \([c_X] \in Z_*(N_X)\) is a cycle, and we define
\[
[cv_X] = \eta_{\phi,*} [c_X] \in Z_*(\text{Ob}_\phi).
\]
Let \(s : X \to \text{Ob}_\phi\) be the zero section, then [6, Proposition 3.4] constructed the Gysin map
\[
s^! : Z_*(\text{Ob}_\phi) \to Z_*(X)
\]
such that
\[
[X, \phi]^\text{virt} := s^!([cv_X]) \in A_*(X)
\]
is the virtual fundamental cycle associated with the semi-perfect obstruction theory \(\phi\).

2.2.1 Obstruction Cone

We generalize the obstruction cone to semi-perfect obstruction theory.

Recall from [1, §2.1], a local resolution of \(\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha \beta}\}_{\alpha \in \Lambda}\) is a derived category homomorphism
\[
F \to E_\alpha^\vee[1]|_U
\]
over some étale open chart \(U\) of \(X_\alpha\), where \(F\) is a vector bundle over \(U\) and the homomorphism \(F \to E_\alpha^\vee[1]|_U\) satisfies the condition that its cone is a locally free sheaf over \(U\) concentrated in degree \(-1\). Or as a local presentation \(F \to h^1/h^0(E_\alpha^\vee)|_U\) of the bundle stack over \(U\) of \(X_\alpha\).

For every local resolution \(F \to h^1/h^0(E_\alpha^\vee)|_U\) there exists an associated cone \(C \subset F\), the obstruction cone, defined via the Cartesian diagram:

\[
\begin{array}{ccc}
C & \longrightarrow & F \\
\downarrow & & \downarrow \\
c_X |_U & \longrightarrow & h^1/h^0(E_\alpha^\vee)|_U
\end{array}
\]

where \(c_X |_U \cong c_{X_\alpha} |_U\). The local resolution \(F \to h^1/h^0(E_\alpha^\vee)|_U\) induces a canonical epimorphism
\[
F \to \text{Ob}_{\phi,|_U} = \text{Ob}_\phi |_U
\]
of coherent sheaves.
Proposition 2.10 ([1, Proposition 2.2]) Let $\Omega$ be a vector bundle over $X$, and $\Omega \to \text{Ob}_\phi$ be a vector bundle over $X$, and $\Omega \to \text{Ob}_\phi$ an epimorphism of coherent sheaves. Then there exists a unique linear combination of integral cycles $C \subset \Omega$ determined by the intrinsic normal cone such that for every local resolution $F \to E^\vee_\alpha[1]|_U$, with obstruction cone $C' \subset F$, and every lift $\gamma$ we have $C|_U = \gamma^{-1}(C')$, in the scheme-theoretic sense.

Proof. Still since for $\alpha \in \Lambda$, $\phi_\alpha : E_\alpha \to L_{X_\alpha}$ is a perfect obstruction theory. Étale locally around $X_\alpha$ (hence around $X$), the presentation $F$ and $\gamma$ always exists. Hence the uniqueness of $C$ is true. We only need to prove its existence.

Let $X_{\alpha\beta} \to X_\alpha$ be the embedding of étale chart. Then from [2, §3], the intrinsic normal cone

$$c_{X_{\alpha\beta}} = c_X|_{X_{\alpha\beta}} \hookrightarrow c_X|_{X_\alpha} = c_X$$

is a closed subcone stack via the embedding of intrinsic normal sheaves $N_{X_{\alpha\beta}} \hookrightarrow N_{X_\alpha}$. On the other hand, étale locally the vector bundle stack

$$h^1/h^0(E^\vee_\alpha)|_{X_{\alpha\beta}} \to h^1/h^0(E^\vee_\beta)$$

is an embedding, and

$$c_{X_\alpha} \hookrightarrow N_{X_\alpha} \hookrightarrow h^1/h^0(E^\vee_\alpha)$$

as embedding diagrams.

From the definition of semi-perfect obstruction theory, for any $\alpha, \beta$,

$$\psi_{\alpha\beta} : h^1(E^\vee_\alpha)|_{X_{\alpha\beta}} \xrightarrow{\cong} h^1(E^\vee_\beta)|_{X_{\alpha\beta}}$$

is an isomorphism so that the data glue to give the obstruction sheaf $\text{Ob}_\phi$ on $X$.

Still let $cv$ be the coarse moduli sheaf of the intrinsic normal cone $c_X$. From diagram (2.2.3) and [6], the coarse moduli space $cv$ of the intrinsic normal cone, which can be taken as a globalized linear combination $\sum \beta m_\beta[A_\beta]$ of integral cycles, is embedded into the obstruction sheaf. Thus $cv \in Z_* (\text{Ob}_\phi)$.

Then the cone $C \hookrightarrow \Omega$ is constructed by the fibre product of sheaves on the big étale site of $X$:

$$C^c \longrightarrow \Omega$$

$$cv \longleftarrow \text{Ob}_\phi$$
which is Cartesian, i.e., the cone cycle $C$ can be taken as the pullback of the cycle $[cv]$, which is a linear combination of integral cycles in $\Omega$. So any $\Omega \to F$ gives a diagram

$$
\begin{array}{ccc}
C & \to & \Omega \\
\downarrow & & \downarrow \\
C' & \to & F
\end{array}
$$

which is Cartesian because of (2.2.4) and the Cartesian diagram:

$$
\begin{array}{ccc}
C' & \to & F \\
\downarrow & & \downarrow \\
(cv') & \to & Ob_{\phi}
\end{array}
$$

by assuming $F$ is a global resolution. □

**Definition 2.11** ([22, Definition 2.27]) A semi-perfect obstruction theory $\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$ for $X$ is symmetric if for any $\alpha \in \Lambda$, $E_\alpha$ is endowed with a non-degenerate symmetric bilinear form $\theta_\alpha : E_\alpha \cong \rightarrow E_\alpha^\vee[1]$. Moreover, the gluing isomorphisms $\psi_{\alpha\beta}$ for the obstruction sheaf $Ob_\phi$ are the identity maps of $\Omega_{X_{\alpha\beta}}$ via the canonical isomorphism $Ob_{\phi_\alpha} \cong \Omega_{X_\alpha}$.

2.2.2 Almost Closed One Form

Recall from §3.4 of [1], any symmetric obstruction theory is locally given by an almost closed 1-form. This is still true for symmetric semi-perfect obstruction theory. Let $\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$ be a symmetric semi-perfect obstruction theory for $X$. The symmetric obstruction theory $E_\alpha$ on any étale local chart $X_\alpha \to X$ gives rise to the following: for each point $p \in X_\alpha$ in an étale neighborhood, there exists an immersion

$$X_\alpha \hookrightarrow M$$

into a smooth scheme $M$, such that there exists an almost closed one form $\omega \in \Omega^1_M$ and an isometry $E_\alpha \to H(\omega)$ such that

$$
\begin{array}{ccc}
E_\alpha & \to & H(\omega) \\
\downarrow & & \downarrow \\
L_{X_\alpha} & \to & \\
\end{array}
$$

where $H(\omega) = [T_M|_{X_\alpha} \xrightarrow{\varpi_\omega} \Omega_M|_{X_\alpha}], \varpi_\omega = d \cdot \omega^\vee$ coming from the following diagram:

$$
\begin{array}{ccc}
T_M|_{X_\alpha} & \xrightarrow{\varpi_\omega} & \Omega_M|_{X_\alpha} \\
\downarrow & & \downarrow \\
I_{X_\alpha}/I_{X_\alpha}^2 & \xrightarrow{d} & \Omega_M|_{X_\alpha}
\end{array}
$$

(2.2.5)

The almost closed one form $\omega \in \Omega^1_M$ means that $d\omega \in I_{X_\alpha} \cdot \Omega^2_M$, where $I_{X_\alpha}$ is the ideal sheaf of the zero locus of $\omega$ ($X_\alpha$ is the zero locus of $\omega$).

**Proposition 2.12** The obstruction sheaf $Ob_\phi$ for a symmetric semi-perfect obstruction theory $\phi$ is the cotangent sheaf $\Omega_X$. 
Proof  Locally the obstruction sheaf $\text{Ob}_{\phi_\alpha} = h^1(E_\alpha^\vee) = \Omega_{X_\alpha}$ is the cotangent sheaf of $X_\alpha$ since the obstruction theory $E_\alpha$ is symmetric, see [1]. We can embed $X$ into a higher dimensional smooth projective scheme $M$ such that the local perfect obstruction theory $\phi_\alpha : E_\alpha \to \mathbb{L}_{X_\alpha}$ is (note that $X_\alpha \subset X \subset M$) given by (2.2.5). The obstruction sheaf $\text{Ob}_{\phi_\alpha} = \text{cok}(d) = \Omega_{X_\alpha}$.

The semi-perfect obstruction theory $\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$ implies that $\psi_{\alpha\beta} : h^1(E_\alpha^\vee)|_{X_{\alpha\beta}} \to h^1(E_\beta^\vee)|_{X_{\alpha\beta}}$ is an isomorphism as sheaves. From Definition 2.11, the transition functions $\psi_{\alpha\beta}$ guarantee that the local cotangent sheaves $\Omega_{X_\alpha}$ glue to give the cotangent sheaf $\Omega_X$, since it will be the cokernel of $d : I/I^2 \to \Omega_M|_X$.

2.2.3 Virtual Fundamental Cycle

Let $\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$ be a symmetric semi-perfect obstruction theory for $X$. Recall that we have the morphism

$$\eta_{\phi^*} : Z_*(N_X) \to Z_*(\Omega_X)$$

constructed before. Let $c_X$ be the intrinsic normal cone and $\text{cv} = \eta_{\phi^*}[c_X]$ is exactly the coarse moduli sheaf of the intrinsic normal cone taken as a cycle in $Z_*(\Omega_X)$.

Let

$$[X, \phi]^\text{virt} = s^!_{\Omega X}([\text{cv}])$$

be the virtual fundamental cycle by applying the Gysin map

$$s^!_{\Omega X} : Z_*(\Omega_X) \to A_*(X)$$

given by

$$[\text{cv}] \mapsto [X, \phi]^\text{virt}$$

as in [6, Proposition 3.4].

We give an alternative construction due to [1]. Note that since $c_X$ and $h^1/h^0(E_\alpha^\vee)$ are all Artin stacks, one can use the intersection theory of Artin stacks in [25] to directly applying the Gysin map on the Chow group $A_*(h^1/h^0)$ on $X$. Let

$$\begin{array}{ccc}
C & \to & \Omega \\
\downarrow & & \downarrow \\
\text{cv} & \to & \text{Ob}_\phi = \Omega_X
\end{array}$$

be the Cartesian diagram in (2.2.4), where $\Omega$ is a vector bundle and $C$ is the obstruction cone in $\Omega$.

**Proposition 2.13**

$$[X, \phi]^\text{virt} = s^!_{\Omega}[C] \in A_*(X)$$

where $s^!_{\Omega}$ is the Gysin map of the vector bundle $\Omega \to X$.

3 Behrend’s Theorem

In this section we survey the Behrend’s theorem equating the virtual count of a symmetric semi-perfect obstruction theory $\phi$ to the weighted Euler characteristic of $X$. 
3.1 The Behrend Function

Let $X$ be a Deligne–Mumford stack. In [1, §2] Behrend introduced on $X$ an integer valued constructible function

$$\nu_X : X \to \mathbb{Z},$$

which is called the “Behrend function”. We briefly recall its construction. More detailed construction can be found in [1], and [13].

There exists a unique integral cycle $c_X$ on $X$ such that for any étale chart $U \to X$ and $U \to M$ an embedding into a smooth scheme $M$,

$$c_X|_U = c_{U/M}$$

and

$$c_{U/M} = \sum_{C'} (-1)^{\dim(C')} \text{mult}(C')[\pi(C')]$$

where

$$\pi : C_{U/M} \to U$$

is the projection from the normal cone $C_{U/M}$ to $U$; $C'$ are all the irreducible components of the normal cone $C$; $\pi(C')$ are the irreducible closed subset (prime cycle) on $U$ by the image of $\pi$; and $\text{mult}(C')$ is the multiplicity of $C'$ at the generic point.

**Definition 3.1** The Behrend function $\nu_X : X \to \mathbb{Z}$ is defined as

$$\nu_X := \text{Eu}(c_X)$$

where $\text{Eu}(-)$ is the local Euler obstruction of MacPherson [27] on integral cycles on $X$.

**Remark 3.2**

1. If $X$ is smooth, $c_X = (-1)^{\dim X}[X]$. In general it is an integral cycle in $Z_*(X)$.

2. More motivation of the local Euler obstruction can be found in [27], see also [1, §2], [13].

3. If $X = \text{Crit}(f)$ is the critical locus of a holomorphic function $f : M \to \mathbb{C}$, then

$$\nu_X(P) = (-1)^{\dim M}(1 - \chi(F_P))$$

where $F_P$ is the Milnor fiber of the function $f$ at $P$.

**Definition 3.3** The weighted Euler characteristic of $X$ by the Behrend function $\nu_X$ is defined as:

$$\chi(X, \nu_X) = \chi(X, \text{Eu}(c_X)) = \sum_i i \cdot \chi(\nu_X^{-1}(i)).$$

We recall the Aluffi class. First the Chern–Mather class is a group homomorphism

$$c^M : Z_*(X) \to A_*(X)$$

by linear extension for any prime cycle $V$ of degree $p$ on $X$,

$$c^M(V) = \mu_*(c(T_V) \cap [\tilde{V}])$$

where $\mu : \tilde{V} \to V$ is the Nash blow-up, $T_V$ is the Nash tangent bundle on $\tilde{V}$. Let $c^M_0(V)$ be the degree zero part of $c^M(V)$. Behrend [1, Definition 1.1] defined the Aluffi class as

$$\alpha_X := c^M(c_X) \in A_*(X).$$

If $X$ is smooth, then $\alpha_X = (-1)^{\dim X} c(T_X) \cap [X] = c(\Omega_X) \cap [X]$. 

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Theorem 3.4 ([1, Proposition 1.12], [14, Theorem 1.1]) Let $X$ be a proper Deligne–Mumford stack. Then
\[ \int_X \alpha_X = \chi(X, \nu_X). \]

3.2 Behrend’ Theorem for Symmetric Semi-perfect Obstruction Theory

3.2.1 Lagrangian Cone

Let $M$ be a smooth scheme or a smooth Deligne–Mumford stack. The cotangent bundle $\Omega_M$ has the tautological one form $\alpha \in \Omega_M$. Any local étale coordinates $\{x_1, \ldots, x_n\}$ on $M$ induces the canonical coordinate system
\[ \{x_1, \ldots, x_n, p_1, \ldots, p_n\} \]
on $\Omega_M$. In such coordinate system $\alpha = \sum p_i dx_i$. The differential $d\alpha = \theta$ defines the tautological symplectic form on $\Omega_M$.

Recall that an irreducible closed subset $C \subset \Omega_M$ is conic and Lagrangian if and only if $\dim C = \dim M$ and $\alpha$ vanishes when restricted to the generic point of $C$. From [1, Lemma 4.2], if $V \subset M$ is an irreducible closed subset, the closure in $\Omega_M$ of the conormal bundle to any smooth dense open subset of $V$ is conic Lagrangian, This way describes all conic Lagrangians.

Definition 3.5 A closed subset of $\Omega_M$ is conic and Lagrangian if every one of its irreducible components is conic and Lagrangian. An algebraic cycle in $\Omega_M$ is conic Lagrangian if its support is conic Lagrangian.

3.2.2 Behrend’s Theorem

Now since $X$ is a quasi-projective separated Deligne–Mumford stack, and let $X \hookrightarrow M$ be the closed immersion into a smooth projective Deligne–Mumford stack $M$ with projective coarse moduli space. Moreover, $X$ admits a symmetric semi-perfect obstruction theory $\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha \beta}\}_{\alpha \in \Lambda}$. Let $C \subset \Omega_M$ be the obstruction cone cycle constructed in Proposition 2.10 by

\[ C \hookrightarrow \Omega_M \]
\[ cv \hookrightarrow \text{Ob}_\phi = \Omega_X \]

Theorem 3.6 The obstruction cone cycle $C \subset \Omega_M$ is Lagrangian.

Proof The problem is local, hence we may assume that $X$ is cut out by an almost closed one form $\omega \in \Omega^1_M$. Then the result is from [1, Theorem 4.9]. \qed

Recall that in [1, §4], let $\mathcal{L}_X(\Omega_M)$ be the subgroup of $\mathcal{Z}_n(\Omega_M)$ generated by the conic Lagrangian prime cycles supported on $X$. Let $V$ be a prime cycle of $M$. One has $N^*_V/M$ the closure of the conormal bundle of smooth part of $V$ inside $M$. So in ([1, Section 4.1]) the following isomorphism of groups is defined:

\[ L : \mathcal{Z}_s(X) \to \mathcal{L}_X(\Omega_M) \]

which is given by
\[ V \mapsto (-1)^{\dim(V)} N^*_V/M. \]
Conversely there is an isomorphism:

$$\pi : \mathcal{L}_X(\Omega_M) \to \mathcal{Z}_*(X)$$

which is given by

$$W \mapsto (-1)^{\dim(\pi(W))}\pi(W),$$

where $\pi : W \to X$ is the projection. The homomorphisms $L$ and $\pi$ are inverse of each other.

Fix an embedding $X \to M$ of the DM stack $X$ into a smooth DM stack $M$. We have the following diagram due to Behrend in Diagram (2) of [1].

$$\begin{array}{ccc}
\mathcal{Z}_*(X) & \xrightarrow{\text{Eu}} & F(X) \\
& \searrow^{c^0_M} & \swarrow^{c^S_M} \\
& & \mathcal{L}_X(\Omega_M)
\end{array}
\begin{array}{c}
\xrightarrow{\text{ch}}
\end{array}
\begin{array}{ccc}
& & A_0(X)
\end{array}
\begin{array}{c}
\xleftarrow{s^0_{\Omega_M}(\cdot)}
\end{array}
\begin{array}{ccc}
& & \mathcal{L}_X(\Omega_M)
\end{array}
\begin{array}{c}
\xleftarrow{\text{Eu}}
\end{array}
\begin{array}{c}
\xleftarrow{\text{ch}}
\end{array}
\begin{array}{ccc}
\mathcal{Z}_*(X) & \xrightarrow{\text{Eu}} & F(X) \\
& \searrow^{c^0_M} & \swarrow^{c^S_M} \\
& & \mathcal{L}_X(\Omega_M)
\end{array}
\begin{array}{c}
\xleftarrow{s^0_{\Omega_M}(\cdot)}
\end{array}
$$

where $\mathcal{Z}_*(X)$ is the group of integral cycles of $X$, $F(X)$ is the group of constructible functions on $X$. The maps $c^0_M$, $c^S_M$ and $I(\cdot, [M])$ are degree zero Chern–Mather class, degree zero Chern-Schwartz-Mather class and the Lagrangian intersection with zero section of $\Omega_M$, respectively. Note that in [1], the notation of Lagrangian intersection with zero section is denoted by $0^!_{\Omega_M}(\cdot)$.

We briefly explain the horizontal morphisms in the diagram. The first map is the local Euler obstruction $\text{Eu}$ and it gives an isomorphism from $\mathcal{Z}_*(X)$ to $F(X)$. Then the morphism $\text{Ch}$ is defined by the isomorphism $\text{Eu}$ and the morphism $L$ defined above.

**Corollary 3.7** ([1, Corollary 4.15]) We have

$$[C] = L(\epsilon_X) = \text{Ch}(\nu_X).$$

**Theorem 3.8** Let $X$ be a proper Deligne–Mumford stack with a symmetric semi-perfect obstruction theory $\phi = \{\phi_\alpha, X_\alpha, E_\alpha, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$. Then

$$\int_{[X, \phi]_{\text{virt}}} 1 = \chi(X, \nu_X).$$

**Proof** From Proposition 2.13, and the fact that the virtual dimension is zero,

$$\int_{[X, \phi]_{\text{virt}}} 1 = \#s^1_{\Omega_M}([C]).$$

While from the diagram (3.2.3) and Corollary 3.7, $\text{Ch}(\nu_X) = [C]$, so

$$\#s^1_{\Omega_M}([C]) = \int_X c^S_M(\nu_X)$$

$$= \int_X c^0_M(\epsilon_X)$$

$$= \chi(X, \nu_X)$$

by Theorem 3.4.

**Remark 3.9** By [6, Proposition 3.2], the invariant $\int_{[X, \phi]_{\text{virt}}} 1$ is deformation invariant.
4 $D$-critical Schemes

4.1 Introduction
Motivated by Donaldson–Thomas theory and the $(-1)$-shifted symplectic derived schemes, Joyce introduced the notion of $d$-critical schemes or $d$-critical analytic spaces. A parallel notion introduced by Kiem and Li [23] is called virtual critical manifolds, and Kiem–Li proved that the $d$-critical analytic spaces and virtual critical manifolds are equivalent. Classical examples include the moduli space of stable simple complexes on Calabi–Yau threefolds, and these moduli spaces are the underlying schemes of the lifted $(-1)$-shifted symplectic derived schemes of [30]. These moduli schemes admits a natural symmetric obstruction theory.

From Joyce [17], it is not known if a $d$-critical scheme admits a symmetric obstruction theory, although locally it does. In this section we prove that Joyce’s $d$-critical scheme admits a symmetric semi-perfect obstruction theory. Since a $d$-critical analytic space is equivalent to a virtual critical manifold in [23], and Kiem–Li already proved that virtual critical manifolds admit a symmetric semi-perfect obstruction theory, hence in the algebraic sense a $d$-critical scheme admits a symmetric semi-perfect obstruction theory. We provide the proof for $d$-critical schemes and all credits belong to Kiem–Li in [23]. Since a $d$-critical scheme is a scheme, we work in Zariski topology for $d$-critical schemes.

As an application we show that the dual obstruction sheaf as in [16] is a $d$-critical scheme. We also show that its compactification is also a $d$-critical scheme under some restrictions. Therefore the dual obstruction sheaf admits naturally a symmetric semi-perfect obstruction theory.

Let $X$ be a scheme which admits a symmetric semi-perfect obstruction theory, and furthermore assume that there exists a $\mathbb{G}_m$ action on $X$ which makes the symmetric semi-perfect obstruction theory $\mathbb{G}_m$-equivariant. The $\mathbb{G}_m$-action naturally gives rise to a cosection and we show that the Kiem–Li localized invariant is equal to the Behrend’s weighted Euler characteristic by the Behrend function. This generalizes the result in [13, Theorem 5.20] to symmetric semi-perfect obstruction theory.

4.2 Joyce’s $d$-critical Schemes
The algebraic $d$-critical scheme is the classical model for the $(-1)$-shifted symplectic derived scheme as developed by PTVV in [30]. In the same paper [30], PTVV proved that the moduli space of stable coherent sheaves or simple complexes over Calabi–Yau threefolds admits a $(-1)$-shifted symplectic derived structure, hence their underlying moduli scheme has an algebraic $d$-critical locus structure. Thus the algebraic $d$-critical locus of Joyce provides the classical scheme framework for the moduli space of stable simple complex over smooth Calabi–Yau threefolds.

To define the algebraic $d$-critical scheme, we first recall the following theorem in [17]:

**Theorem 4.1** ([17]) Let $X$ be a $\kappa$-scheme, which is locally of finite type. Then there exists a sheaf $S_X$ of $\kappa$-vector spaces on $X$, unique up to canonical isomorphism, which is uniquely characterized by the following two properties:

(i) Suppose that $R \subseteq X$ is Zariski open, $U$ is a smooth $\kappa$-scheme, and $i : R \hookrightarrow U$ is a closed embedding. Then there is an exact sequence of sheaves of $\kappa$-vector spaces on $R$:

$$0 \rightarrow I_{R,U} \longrightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{i^\#} \mathcal{O}_X|_R \rightarrow 0,$$
where \( O_X, O_U \) are the structure sheaves of \( X \) and \( U \), and \( i^\# \) is the morphism of sheaves over \( R \). There is also an exact sequence of sheaves of \( \kappa \)-vector spaces over \( R \):

\[
0 \to \mathcal{S}_X|_R \xrightarrow{i_{R,U}} i^{-1}(O_U)|_R \to \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)} \to 0
\]

where \( d \) maps \( f + I_{R,U} \) to \( df + I_{R,U} \cdot i^{-1}(T^*U) \).

(ii) If \( R \subseteq S \subseteq X \) are Zariski open, and \( U, V \) are smooth \( \kappa \)-schemes, and

\[
i : R \hookrightarrow U
\]

\[
j : S \hookrightarrow V
\]

are closed embeddings. Let

\[
\Phi : U \to V
\]

be a morphism with \( \Phi \circ i = j|_R : R \to V \). Then the following diagram of sheaves on \( R \) commutes:

\[
\begin{array}{c}
0 \\
\downarrow \text{id} \\
0
\end{array}
\begin{array}{c}
\xrightarrow{i_{S,V}|_R} \\
\xrightarrow{i^{-1}(O_V)|_R} \\
\xrightarrow{i^{-1}(O_U)|_R} \\
\xrightarrow{d} \\
\xrightarrow{I_{S,V} \cdot j^{-1}(T^*V)|_R} \\
\xrightarrow{I_{S,V} \cdot j^{-1}(T^*V)} \\
\xrightarrow{I_{R,U} \cdot i^{-1}(T^*U)} \\
\xrightarrow{I_{R,U} \cdot i^{-1}(T^*U)} \\
\xrightarrow{0}
\end{array}
\]

(4.2.1)

Here \( \Phi : U \to V \) induces

\[
\Phi^\# : \Phi^{-1}(O_V) \to O_U
\]

on \( U \), and we have:

\[
i^{-1}(\Phi^\#) : j^{-1}(O_V)|_R = i^{-1} \circ \Phi^{-1}(O_V) \to i^{-1}(O_U),
\]

(4.2.2)

a morphism of sheaves of \( \kappa \)-algebras on \( R \). As \( \Phi \circ i = j|_R \), then (4.2.2) maps to \( I_{S,V}|_R \to I_{R,U} \), and \( I_{S,V}|_R \to I_{R,U}^2 \). Thus (4.2.2) induces the morphism in the second column of (4.2.1). Similarly, \( d\Phi : \Phi^{-1}(T^*V) \to T^*U \) induces the third column of (4.2.1).

According to [17], there is a natural decomposition

\[
\mathcal{S}_X = \mathcal{S}_X^0 \oplus \kappa_X
\]

and \( \kappa_X \) is the constant sheaf on \( X \) and \( \mathcal{S}_X \subset \mathcal{S}_X \) is the kernel of the composition:

\[
\mathcal{S}_X \to O_X \xrightarrow{i_X^\#} O_{X^{\text{red}}}
\]

with \( X^{\text{red}} \) the reduced \( \kappa \)-scheme of \( X \), and \( i_X : X^{\text{red}} \hookrightarrow X \) the inclusion.

**Definition 4.2** An algebraic \( d \)-critical scheme over the field \( \kappa \) is a pair \((X, s)\), where \( X \) is a \( \kappa \)-scheme, locally of finite type, and \( s \in H^0(\mathcal{S}_X^0) \) for \( \mathcal{S}_X^0 \) in Theorem 4.1. These data satisfy the following conditions: for any \( x \in X \), there exists a Zariski open neighborhood \( R \) of \( x \) in \( X \), a smooth \( \kappa \)-scheme \( U \), a regular function \( f : U \to \kappa \), and a closed embedding \( i : R \hookrightarrow U \), such that \( i(R) = \text{Crit}(f) \) as \( \kappa \)-subsheaves of \( U \), and \( i_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2 \). We call the quadruple \((R, U, f, i)\) a critical chart on \((X, s)\).

Some properties of \((X, s)\) are as follows:
Theorem 4.3 ([17]) Let \((X, s)\) be an algebraic \(d\)-critical scheme, and \((R, U, f, i), (S, V, g, j)\) be critical charts on \((X, s)\). Then for each \(x \in R \cap S \subseteq X\) there exists subcharts

\[
(R', U', f', i') \subseteq (R, U, f, i),
\]

\[
(S', V', g', j') \subseteq (S, V, g, j)
\]

with \(x \in R' \cap S' \subseteq X\), a critical chart \((T, W, h, k)\) on \((X, s)\), and embeddings

\[
\Phi: (R', U', f', i') \hookrightarrow (T, W, h, k)
\]

and

\[
\Psi: (S', V', g', j') \hookrightarrow (T, W, h, k).
\]

We introduce the canonical line bundle of \((X, s)\):

**Theorem 4.4** ([17, Theorem 2.28]) Let \((X, s)\) be an algebraic \(d\)-critical scheme, and \(X^{\text{red}} \subseteq X\) the associated reduced \(\kappa\)-scheme. Then there exists a line bundle \(K_{X,s}\) on \(X^{\text{red}}\) which we call the canonical line bundle of \((X, s)\), that is natural up to canonical isomorphism, and is characterized by the following properties:

(i) If \((R, U, f, i)\) is a critical chart on \((X, s)\), there is a natural isomorphism

\[
\iota_{R, U, f, i}: (K_{X,s})|_{R^{\text{red}}} \to i^*(K_U^\otimes 2)|_{R^{\text{red}}}
\]

where \(K_U\) is the canonical line bundle of \(U\).

(ii) Let \(\Phi: (R, U, f, i) \hookrightarrow (S, V, g, j)\) be an embedding of critical charts on \((X, s)\). Then there is an isomorphism of line bundles on \(\text{Crit}(f)^{\text{red}}\):

\[
J_\Phi: (K_U^\otimes 2)|_{\text{Crit}(f)} \xrightarrow{\cong} \Phi|_{\text{Crit}(f)^{\text{red}}}^* (K_V^\otimes 2).
\]

Since \(i: R \to \text{Crit}(f)\) is an isomorphism as schemes with \(\Phi \circ i = j|_R\), this gives

\[
i|_{R^{\text{red}}} (J_\Phi) : i^*(K_U^\otimes 2)|_{R^{\text{red}}} \xrightarrow{\cong} j^*(K_V^\otimes 2)|_{R^{\text{red}}},
\]

and we have:

\[
i|_{R^{\text{red}}} (J_\Phi) \circ \iota_{R, U, f, i}: (K_{X,s})|_{R^{\text{red}}} \to j^*(K_V^\otimes 2)|_{R^{\text{red}}}.
\]

**Definition 4.5** Let \((X, s)\) be an algebraic \(d\)-critical scheme, and \(K_{X,s}\) the canonical line bundle of \((X, s)\). An orientation on \((X, s)\) is a choice of square root line bundle \(K_{X,s}^{1/2}\) for \(K_{X,s}\) on \(X^{\text{red}}\), i.e., an orientation of \((X, s)\) is a line bundle \(L\) over \(X^{\text{red}}\) and an isomorphism \(L^\otimes 2 = L \otimes L \cong K_{X,s}\). A \(d\)-critical scheme with an orientation will be called an oriented \(d\)-critical scheme.

Bussi, Brav and Joyce [4] prove the following interesting result: Let \((X, \omega)\) be a \((-1)\)-shifted symplectic derived scheme over \(\kappa\) in the sense of [30], and let \(X := t_0(X)\) be the associated classical \(\kappa\)-scheme of \(X\). Then \(X\) naturally extends to an algebraic \(d\)-critical scheme \((X, s)\). The canonical line bundle \(K_{X,s} \cong \det(L_X)|_{X^{\text{red}}}\) is the determinant line bundle of the cotangent complex \(L_X\) of \(X\).

One of the applications of the \((-1)\)-shifted symplectic derived scheme or stack is on moduli problems. Let \(Y\) be a smooth Calabi–Yau threefold over \(\kappa\), and \(X\) a classical moduli scheme of simple coherent sheaves in \(\text{Coh}(Y)\), the abelian category of coherent sheaves on \(Y\). Then in
[30], the authors proved that there is a natural $(-1)$-shifted derived scheme structure $X$ on the moduli space $X$, such that if

$$i : X \hookrightarrow X$$

is the inclusion, then the pullback $i^*\Omega_X$ of the cotangent complex of $X$ is a perfect obstruction theory of $X$, thus from the result in [4], $X$ has an algebraic $d$-critical locus structure.

**Example 4.6** Consider $X = \text{Crit}(f)$ to be the critical locus for the function

$$f = x^2y : U = \mathbb{A}^2_\kappa \to \mathbb{A}^1_\kappa,$$

and then $X = \text{Spec}(\kappa[x, y]/(xy, x^2))$. Let $i : X \hookrightarrow U$ be the inclusion. Then $X$ is naturally a $d$-critical scheme, with

$$S_X = \ker \left( \frac{i^{-1}(O_U)}{I_X^2} \to \frac{i^{-1}(\Omega_U)}{I_X \cdot \Omega_U} \right).$$

The sheaf $S_X = S_X^0 \oplus \kappa_X$, where on $X^{\text{red}}$ the function $f$ can be written down as $f = f^0 + c$ and $c$ is locally constant. From Joyce’s explanation, $S_X^0$ is the coherent sheaf that locally remembers the closed one form $df$.

We take $\overline{X} = \text{Proj} R[y_0 : y_1]/(x^2, xy_0)$, where $R = \kappa[x]/(x^2)$. Then $\overline{X}$ is a compactification of $X$, which is a $\mathbb{P}^1 = \text{Proj} \kappa[y_0 : y_1]$ with a non-reduced point $0 \in \mathbb{P}^1$. Let $\infty \in \mathbb{P}^1$ be the infinity point. We explain that $\overline{X}$ is also a $d$-critical scheme. Since $\overline{X}\{\infty\} = X$, then we have a $d$-critical chart:

$$(X, U, f, i)$$

as above. The section $s \in S_{\overline{X}}$ satisfies that

$$i(s|_X) = f + I_X^2$$

where $i : S_{\overline{X}}|_X \rightarrow \frac{i^{-1}(O_U)}{I_X^2}$ is the inclusion.

Since $\overline{X}\{0\} \cong \mathbb{A}^1_\kappa$, then we have a $d$-critical chart:

$$(\overline{X}\{0\}, \mathbb{A}^1_\kappa, 0, j).$$

Then $S_{\overline{X}}|_{\overline{X}\{0\}} = 0$. These two $d$-critical charts glue to give the $d$-critical scheme $\overline{X}$. As proved in [17], $\overline{X}$ is non orientable.

4.3 Symmetric Semi-perfect Obstruction Theory of $d$-critical Schemes

In [23], Kiem–Li has proved that there exists a symmetric semi-perfect obstruction theory on a virtual critical manifold defined in [23]. Taking as analytic spaces, virtual critical manifolds are the same as $d$-critical analytic spaces. We provide a proof here for $d$-critical schemes and all credits of the result belong to Kiem–Li in [23].

**Theorem 4.7** Let $(X, s)$ be a $d$-critical scheme in the sense of [17]. Then $(X, s)$ admits a symmetric semi-perfect obstruction theory.

**Proof** The $d$-critical scheme $(X, s)$ is covered by the $d$-critical charts $(X_\alpha, U_\alpha, f_\alpha, i_\alpha)_{\alpha \in \Lambda}$, where

$$i_\alpha : X_\alpha \hookrightarrow X$$

$$f_\alpha : U_\alpha \rightarrow \mathbb{A}^1_\kappa$$
such that \( \text{Crit}(f_\alpha) \cong X_\alpha \). Locally on each critical chart, there exists a symmetric obstruction theory

\[
\begin{array}{ccc}
E_\alpha^\bullet : & [T_U | x_\alpha \xrightarrow{df^\vee} \Omega_U | x_\alpha ] \\
\phi & \downarrow & \downarrow \\
L_{X_\alpha}^\bullet : & [I_\alpha / I_\alpha^2 \xrightarrow{d} \Omega_U | x_\alpha ]
\end{array}
\]

(4.3.1)

For any \( \alpha, \beta \), and two \( d \)-critical charts:

\((X_\alpha, U_\alpha, f_\alpha, i_\alpha), (X_\beta, U_\beta, f_\beta, i_\beta)\)

let \( X_{\alpha \beta} = X_\alpha \cap X_\beta \). From [17], for any \( x \in X_{\alpha \beta} \), there exist sub-critical charts

\((X'_\alpha, U'_\alpha, f'_\alpha, i'_\alpha) \subset (X_\alpha, U_\alpha, f_\alpha, i_\alpha)\)

and

\((X'_\beta, U'_\beta, f'_\beta, i'_\beta) \subset (X_\beta, U_\beta, f_\beta, i_\beta)\)

such that for any \( x \in X'_\alpha \cap X'_\beta \subset X \), there exists a critical chart \((T, W, h, k)\) such that

\( \Phi: (X'_\alpha, U'_\alpha, f'_\alpha, i'_\alpha) \hookrightarrow (T, W, h, k)\)

and

\( \Psi: (X'_\beta, U'_\beta, f'_\beta, i'_\beta) \hookrightarrow (T, W, h, k)\)

are embeddings of the critical charts.

For the local symmetric obstruction theory \( \phi_\alpha : E_\alpha^\bullet \to L_{X_\alpha}^\bullet \) on \( X_\alpha \), we show:

1. For each embedding of critical charts:

\( \Phi: (X'_\alpha, U'_\alpha, f'_\alpha, i'_\alpha) \hookrightarrow (X_\alpha, U_\alpha, f_\alpha, i_\alpha)\)

we have a morphism

\( \Phi_* : E'_\alpha^\bullet \to E_\alpha^\bullet | X'_\alpha \)

such that it induces an isomorphism on the bundle stacks \( h^1/h^0(E'_\alpha^\bullet | X'_\alpha) \cong h^1/h^0(E_\alpha^\bullet | X_\alpha) \).

2. For two embeddings of critical charts:

\( \Phi: (X''_\alpha, U''_\alpha, f''_\alpha, i''_\alpha) \hookrightarrow (X'_\alpha, U'_\alpha, f'_\alpha, i'_\alpha)\)

and

\( \Psi: (X'_\alpha, U'_\alpha, f'_\alpha, i'_\alpha) \hookrightarrow (X_\alpha, U_\alpha, f_\alpha, i_\alpha)\)

we have \( (\Psi \circ \Phi)_* = \Psi_* | X'_\alpha \circ \Phi_* . \)

The claim (2) is from (1). We prove the property (1). The embedding \( \Phi \) induces the following commutative diagram:

\[
\begin{array}{c}
X'_\alpha \xrightarrow{i'_\alpha} U'_\alpha \xrightarrow{\Phi} U_\alpha \\
\cap \downarrow \downarrow \downarrow \downarrow \\
X_\alpha \xrightarrow{i_\alpha} U_\alpha
\end{array}
\]

From [17, Proposition 2.2.2, Proposition 2.2.3], shrink the Zariski open subset \( U'_\alpha \) if necessary, the critical charts satisfy the following properties:

\( U_\alpha \cong U'_\alpha \times \mathbb{A}_n^m \)
where \( n = \dim U_\alpha - \dim U'_\alpha \), such that

\[
f_\alpha = f'_\alpha + z_1^2 + \cdots + z_n^2.
\]

Then \( E_\alpha^* \) is isomorphic to

\[
E_\alpha^* \cong [T_{U' \times \mathbb{A}^n_k} |_{X_\alpha} \to \Omega_{U' \times \mathbb{A}^n_k} |_{X_\alpha}]
\]

and the bundle stack

\[
h^1/h^0(E_\alpha^*)|_{X_{\alpha'}} = [\Omega_{U_{\alpha}} |_{X_{\alpha'}}/T_{U_{\alpha}} |_{X_{\alpha'}}] = [\Omega_{U'_{\alpha}} |_{X_{\alpha'}}/T_{U'_{\alpha}} |_{X_{\alpha'}}]
\]

where the last equality is true since \( \Omega_{\mathbb{A}^n_k}/T_{\mathbb{A}^n_k} \) is trivial. This fact can be proved by the following arguments. Since \( h = z_1^2 + \cdots + z_n^2 \), then the morphism \( dh^\vee \circ d : T_{\mathbb{A}^n_k} \to \Omega_{\mathbb{A}^n_k} \) is an isomorphism, and the morphism sends the basis \( \{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \} \) to \( \{ dz_1, \ldots, dz_n \} \). Therefore the quotient \( [\mathbb{A}^n_k/T_{\mathbb{A}^n_k}] \) is trivial.

**Remark 4.8** We can understand this bundle stack \( [\mathbb{A}^n_k/\mathbb{A}^n_k] \) as follows. For simplicity, let \( n = 1 \), it is known that the stack \( [\mathbb{G}_m/\mathbb{G}_m] \) where \( \lambda \in \mathbb{G}_m \) acts on \( \mathbb{G}_m \) by the multiplication. The stack is just a point since the action has only one orbit. Then we understand the bundle stack \( [\mathbb{A}^n_k/\mathbb{A}^n_k] \) is trivial since there is also only one orbit of \( \mathbb{A}^1_k \) by the action of \( \mathbb{A}^1_k \).

We then use (1) and (2) to prove the following: let \((X_\alpha, U_\alpha, f_\alpha, i_\alpha)\) and \((X_\beta, U_\beta, f_\beta, i_\beta)\) be two critical charts. Then shrinking \( X_\alpha, X_\beta \) if necessary,

\[
h^1/h^0(E_\alpha^*)|_{X_\alpha \cap X_\beta} \cong h^1/h^0(E_\beta^*)|_{X_\alpha \cap X_\beta}. \tag{4.3.2}
\]

From [17, Theorem 2.20], for any \( x \in X_\alpha \cap X_\beta \), shrinking \( X_\alpha, X_\beta \) if necessary, there exists subcritical charts

\[
(X'_\alpha, U'_\alpha, f'_\alpha, i'_\alpha) \leftrightarrow (T_\gamma, W_\gamma, h_\gamma, k_\gamma)
\]

and

\[
(X'_\beta, U'_\beta, f'_\beta, i'_\beta) \leftrightarrow (T_\gamma, W_\gamma, h_\gamma, k_\gamma).
\]

Let \( E_\gamma^* \) be the symmetric obstruction theory of \((T_\gamma, W_\gamma, h_\gamma, k_\gamma)\). Then from Property (1),

\[
h^1/h^0(E_\gamma^*)|_{X_\alpha} \cong h^1/h^0(E_\alpha^*)
\]

and

\[
h^1/h^0(E_\gamma^*)|_{X_\beta} \cong h^1/h^0(E_\beta^*).
\]

Since \( x \in X'_\alpha \cap X'_\beta \subset X_\alpha \cap X_\beta \subset X \), we may choose \( X'_\alpha, X'_\beta \) small enough such that we have subcritical charts:

\[
(X'_\alpha \cap X'_\beta, U'_\alpha \cap U'_\beta, f'_\alpha|_{U'_\alpha \cap U'_\beta}, i'_\alpha|_{X'_\alpha \cap X'_\beta}) \leftrightarrow (T_\gamma, W_\gamma, h_\gamma, k_\gamma)
\]

and

\[
(X'_\alpha \cap X'_\beta, U'_\alpha \cap U'_\beta, f'_\beta|_{U'_\alpha \cap U'_\beta}, i'_\beta|_{X'_\alpha \cap X'_\beta}) \leftrightarrow (T_\gamma, W_\gamma, h_\gamma, k_\gamma)
\]

such that we get:

\[
h^1/h^0(E_\gamma^*)|_{X'_\alpha \cap X'_\beta} \cong h^1/h^0(E_\alpha^*)|_{X'_\alpha \cap X'_\beta}
\]
and
\[ h^1/h^0(E^\gamma_\alpha)_{|X'_\alpha \cap X'_\beta} \cong h^1/h^0(E^\gamma_\beta)_{|X'_\alpha \cap X'_\beta} \]
and
\[ h^1/h^0(E^\alpha_\alpha)_{|X'_\alpha \cap X'_\beta} \cong h^1/h^0(E^\lambda_\beta)_{|X'_\alpha \cap X'_\beta}. \]
The obstruction sheaf \( h^1(E^\gamma_\alpha) = \Omega_{X_\alpha} \) for all \( \alpha \) glue to give the obstruction sheaf \( \Omega_X \). Hence Condition (1) in the definition of symmetric semi-perfect obstruction theory is proved.

Finally we prove that for two local symmetric obstruction theories: \( \phi_\alpha : E^\bullet_\alpha \to L^\bullet_{X_\alpha} \), and \( \phi_\beta : E^\bullet_\beta \to L^\bullet_{X_\beta} \), \( \phi_\alpha_{|X_\alpha \cap X_\beta} \) is \( \nu \)-equivalent to \( \phi_\beta_{|X_\alpha \cap X_\beta} \). Let \( p \in X_\alpha \cap X_\beta = X_{\alpha \beta} \subset X \) be a closed point, and let \( \overline{T} = \text{Spec} A \) be an Artin local ring with maximal ideal \( m_\overline{T} \). Let \( I \) be an ideal in \( \overline{A} \) such that \( I \cdot m_\overline{T} = 0 \) and let
\[ T = \overline{T}/I \quad (A = \overline{A}/I). \]
Consider the following diagram:
\[
\begin{array}{ccc}
\text{Spec}(A) & \xrightarrow{g} & X_{\alpha \beta} \\
\downarrow & & \downarrow \text{i}_\alpha \\
\text{Spec}(\overline{A}) & \xrightarrow{\text{i}_\beta} & U_\beta.
\end{array}
\]
(4.3.3)

Since \( U_\alpha, U_\beta \) are smooth, there are morphisms
\[
g_\alpha : \text{Spec}(\overline{A}) \to U_\alpha, \quad g_\beta : \text{Spec}(\overline{A}) \to U_\beta
\]
(4.3.4)

extending the morphisms in (4.3.3). The functions \( f_\alpha, f_\beta \) give
\[
df_\alpha : U_\alpha \to \Omega_{U_\alpha}; \quad df_\beta : U_\beta \to \Omega_{U_\beta}.
\]
(4.3.5)

Let
\[
\rho_\alpha : I \otimes_{\kappa} \Omega_{U_\alpha}|_p \to I \otimes_{\kappa} \Omega_{X_\alpha}|_p, \quad \rho_\beta : I \otimes_{\kappa} \Omega_{U_\beta}|_p \to I \otimes_{\kappa} \Omega_{X_\beta}|_p.
\]
(4.3.6)

be the canonical morphisms. Then from (4.3.4), (4.3.5) and (4.3.6), we have:
\[
\text{Ob}(\phi_\alpha, g, T, \overline{T})_p = \rho_\alpha(df_\alpha \circ g_\alpha)|_p \in I \otimes_{\kappa} \Omega_{X_{\alpha \beta}}|_p
\]
and
\[
\text{Ob}(\phi_\beta, g, T, \overline{T})_p = \rho_\beta(df_\beta \circ g_\beta)|_p \in I \otimes_{\kappa} \Omega_{X_{\alpha \beta}}|_p
\]
by [22, Lemma 1.28]. So we need to show that
\[
\text{Ob}(\phi_\alpha, g, T, \overline{T})_p = \text{Ob}(\phi_\beta, g, T, \overline{T})_p.
\]

We write down the detail composition morphisms.
\[
\begin{array}{ccc}
\text{Spec}(\overline{A}) & \xrightarrow{g_\alpha} & U_\alpha \\
& \xrightarrow{df_\alpha} & \Omega_{U_\alpha} \\
& \xrightarrow{\rho_\alpha} & I \otimes_{\kappa} \Omega_{X_\alpha}.
\end{array}
\]

\[
\begin{array}{ccc}
\text{Spec}(\overline{A}) & \xrightarrow{g_\beta} & U_\beta \\
& \xrightarrow{df_\beta} & \Omega_{U_\beta} \\
& \xrightarrow{\rho_\beta} & I \otimes_{\kappa} \Omega_{X_\beta}.
\end{array}
\]

and the obstruction space is the composition morphisms restricted to \( U_{\alpha \beta}, p \in X_{\alpha \beta} \). In between we can insert the composition morphism
\[
\begin{array}{ccc}
\text{Spec}(\overline{A}) & \xrightarrow{g_\gamma} & U_\gamma \\
& \xrightarrow{df_\gamma} & \Omega_{U_\gamma} \\
& \xrightarrow{\rho_\gamma} & I \otimes_{\kappa} \Omega_{X_\gamma}.
\end{array}
\]
where when restricted to the point $p$, they are all the same. Therefore
\[ \mathrm{Ob}(\phi_\alpha, g, T, T)_{X_\alpha \beta, p} = \mathrm{Ob}(\phi_\beta, g, T, T)_{X_\alpha \beta, p} \]
and we are done. \hfill \qed

### 4.4 Application to Virtual Signed Euler Characteristics

In this section we apply the former result to the dual obstruction sheaf for a perfect obstruction theory without assuming derived schemes as studied in [16]. Note that this result is already proved in [18]. We also talked about one compactification of the dual obstruction sheaf cone.

#### 4.4.1 The Dual Obstruction Sheaf

For the consistence of notations, we follow [16]. Let $X$ be a scheme with a perfect obstruction theory $E^\bullet_X$ in the sense of [2, 26]. The obstruction sheaf $\mathrm{Ob}_X := h^1(E^\bullet_X)$. Let $N := \text{Spec}(\text{Sym}^\bullet \mathrm{Ob}_X)$ be the abelian cone corresponding to this obstruction sheaf. It is a cone over $X$ together with a $C^*$-action by scaling the fibers of $N$. Let $\pi : N \to X$ be the projection.

From [16], given a locally free resolution $E_0 \xrightarrow{\phi} E_1 \to \mathrm{Ob}_X \to 0$. Let $\tau$ be the tautological section of $\pi^\ast(E_1)$. Therefore, $N = C(\mathrm{Ob}_X)$ is cut out of $\text{Tot}(E^{-1})|_X$ by the section $\pi^\ast(E_1)(\phi)^*(\tau)$ of $\pi^\ast E^{-1}E^0$.

**Local Model**

Locally we choose a presentation of $(X, E^\bullet_X)$ as the zero locus of a section $s$ of a vector bundle $E \to A$ over a smooth scheme $A$, such that the complex
\[ [T_A|_X \xrightarrow{ds} E|_X] \text{ is quasi-isomorphic to } (E^\bullet_X)^\vee. \]

Let $\tau$ be the tautological section of $\pi^\ast E^\ast$. Therefore, $N = C(\mathrm{Ob}_X)$ is cut out of $\text{Tot}(E^\ast)|_X$ by the section $\pi^\ast(E^\ast)(\tau)$ of $\pi^\ast \Omega^1_A$. In turn $\text{Tot}(E^\ast)|_X$ is cut out of $\text{Tot}(E^\ast)$ by $\pi^\ast s$. Therefore the ideal of $N$ in the smooth ambient space $\text{Tot}(E^\ast)$ is
\[ (\pi^\ast s, \pi^\ast(Ds)^*(\tau)), \]
where we have chosen any holomorphic connection $D$ on $E \to A$ by shrinking $A$ if necessary.

Thinking of the section $s$ of $E \to A$ as a linear function $\tilde{s}$ on the fibers of $\text{Tot}(E^\ast)$, we find that its critical locus is $N$. From [16, Proposition 2.6], the function $\tilde{s}$ is:
\[ \tilde{s} : \text{Tot}(E^\ast) \to \mathbb{A}_\mathbb{R}^1. \]

where
\[ \tilde{s} = \langle \pi^\ast s, \tau \rangle = \pi^\ast s^\ast(\tau) \]
in terms of the tautological section $\tau$ of $\pi^\ast E^\ast$. Then
\[ N = \text{Crit}(\tilde{s}). \]

We work in local coordinates $x_i$ for $A$. Trivializing $E$ (with rank $r$) with a basis of sections $e_j$, we get a dual basis $f_j$ for $E^\ast$ and coordinates $y_j$ on the fibers of $\text{Tot}(E^\ast)$. 

Then we can write \( s = \sum_j s_j e_j \), \( \tau = \sum_{j=1}^r y_j f_j \) and

\[
\tilde{s} = \sum_{j=1}^s s_j y_j.
\]

Therefore

\[
d \tilde{s} = \sum_j y_j ds_j + \sum_j s_j dy_j = \langle \tau, \pi_E^* Ds \rangle + \sum_j s_j dy_j
\]

with zero scheme defined by the ideal

\[
(\pi_E^*(Ds)^*(\tau), \pi_E^* s_1, \pi_E^* s_2, \ldots).
\]

This is the same as (4.4.1).

For the scheme \( N \), from [17, Theorem 2.1], there exists a unique coherent sheaf \( S_N \), such that in the local model \( R \subset N \) such that \( N_\alpha \cong \text{Crit}(\tilde{s}) \) for a regular function \( \tilde{s} : \tilde{A} \to \mathbb{C} \), then there is a section \( s \in S_N \) such that

\[
\iota(s|_{N_\alpha}) = \tilde{s} + I_{N_\alpha}^2.
\]

Therefore we get a critical chart \( (R, \tilde{A}, \tilde{s}, i) \), where \( i : R \hookrightarrow \tilde{A} \) is the inclusion. Since \( N \) is covered by open subschemes \( R \) such that they give the local models for \( N \), in [15], we show:

**Proposition 4.9** ([15, Proposition 2.5]) \((N, s)\) is a \( d \)-critical scheme.

Then from Theorem 4.7, \( N \) admits a symmetric semi-perfect obstruction theory \( \phi = \{\phi_{\alpha}, N_\alpha, E_{\alpha}, \psi_{\alpha\beta}\}_{\alpha \in \Lambda} \) with obstruction sheaf \( \Omega_N \).

In [18], Kiem defined the cosection of a semi-perfect obstruction theory and the cosection here

\[
\sigma : \Omega_N \to \mathcal{O}_N
\]

is constructed by taking the dual of the vector field

\[
v_x = \frac{d}{d\lambda}(\lambda \cdot x)|_{\lambda=1}
\]

of the \( \mathbb{G}_m \)-action. We fix a global embedding \( N \hookrightarrow \tilde{A} \) into a higher dimensional smooth scheme \( \tilde{A} \) such that \( \tilde{A} \to A \) is a vector bundle over a smooth scheme \( A \) and \( X \hookrightarrow A \) is the global immersion of \( X \), see [16, §4]. We have a cartesian diagram

\[
\begin{array}{ccc}
C & \to & \Omega_A|_N \\
\downarrow \downarrow & & \downarrow \\
\Omega_N & \to & \Omega_A|_N
\end{array}
\]

where \( cv \) is the coarse moduli sheaf of the intrinsic normal cone, and \( C \) is the unique lifting cone making the diagram commute. The important key point is that

\[
C \hookrightarrow \Omega_A|_N \subset \Omega_A
\]

is Lagrangian inside \( \Omega_A \), which is naturally a symplectic manifold. From [22],

\[
C \subset \Omega_A|_N \sqcup \ker(\Omega_A|_N \to \mathcal{O}_N).
\]
Taking a small perturbation $\xi$ is the zero section $\tilde{A}$ of $\Omega_{\tilde{A}}$ such that $\xi \cap C$ only supports on $X$. Then Kiem [18, Theorem 3.1] constructed the localized virtual cycle

$$[N]_{\text{loc}}^\text{virt} := \xi \cap C \in \mathcal{A}_0(X).$$

**Theorem 4.10** We have:

$$\int_{[N]_{\text{loc}}^\text{virt}} 1 = \chi(N, \nu_N).$$

**Proof** The proof is nearly the same as in [16, §4]. Note that the essential point is that $C \subset \Omega_{\tilde{A}}$ is Lagrangian, which is still true for symmetric semi-perfect obstruction theory $\phi = \{\phi_\alpha, N_\alpha, E_\alpha, \psi_{\alpha\beta}\}_{\alpha \in \Lambda}$ since it is a local property. \qed

### 4.4.2 $\mathbb{G}_m$-localized Invariants vs Fantechi–Goettsche/Ciocan–Fortaní–Kapranov Virtual Euler Characteristics

We also generalize the invariants (1), (2) in [16] to $N$ admitting a symmetric semi-perfect obstruction theory, as studied in [18].

Let $(X, E^*_X)$ be a perfect obstruction theory on a scheme $X$. The virtual Euler number is defined as:

$$e^\text{virt}(X) = \int_{[X]^\text{virt}} c_{\text{vd}}((E^*_X)^\vee)$$

where $(E^*_X)^\vee$ is taken as the virtual tangent bundle on $X$. We use the signed version:

$$e_1(X) = \int_{[X]^\text{virt}} c_{\text{vd}}(E^*_X)$$

which is deformation invariant.

On the other hand, since $\mathbb{G}_m$ acts on $N$ (the abelian cone), one can apply Graber–Pandharipande’s virtual localization on the virtual cycle of $N$ (for semi-perfect obstruction theory on $N$, [18] generalizes it to this situation). We have, over the $\mathbb{G}_m$ fixed $X$,

$$E^*_\alpha|_X \cong E^*_X \oplus (E^*_X)^\vee \otimes t^{-1}[1]$$

for each $\alpha$, where $t$ denotes the standard weight one representation of $\mathbb{G}_m$. The virtual normal bundle is given by the dual of the second summand

$$N^\text{virt} \cong E^*_X \otimes t[-1].$$

The rest if the same as in [16, §3.2],

$$e_2(X) := \int_{[X]^\text{virt}} \frac{1}{e(N^\text{virt})} = e_1(X).$$

Therefore

**Theorem 4.11** The Graber/Pandharipande localized invariants $e_2(X)$ is the same as the signed Euler number $e_1(X)$.

### 4.5 A Compactification of $N$

We provide a compactification $\overline{N}$ of the abelian cone $N$.

The natural compactification $\overline{N}$ of $N$ is defined as:

$$\overline{N} := \overline{C}(\text{Ob}_X) = \text{Proj}(\text{Sym}^\bullet(\text{Ob}_X \oplus \mathcal{O}_X)) \xrightarrow{\pi} X.$$  (4.5.1)
Then $\mathcal{N}$ is a projective cone over $X$ containing $N$ as an open locus. Let $D_N^\infty := \text{Proj}(\text{Sym}^* \text{Ob}_X)$ be the infinity divisor of $\mathcal{N}$.

We review the construction of the projective cone $\mathcal{N}$. Writing $E_1^\bullet$ as $E^{-1} \to E^0$, we get the exact sequence

$$E_0 \xrightarrow{\phi} E_1 \to \text{Ob}_X \to 0.$$ 

Consider the following exact sequence

$$E_0 \xrightarrow{(\phi, 0)} E_1 \oplus \mathcal{O}_X \to \text{Ob}_X \oplus \mathcal{O}_X \to 0. \quad (4.5.2)$$

Let $\tau$ be the tautological section of $\pi^*_E \oplus \mathcal{O}_X (E^{-1} \oplus \mathcal{O}_X)$. From [16, Lemma 2.1], the abelian cone

$$C(\text{Ob}_X \oplus \mathcal{O}_X) \text{ is cut out of } C(E_1 \oplus \mathcal{O}_X) = \text{Tot}(E^{-1} \oplus \mathcal{O}_X) \text{ by the section } \pi^*_E \oplus \mathcal{O}_X \phi^*(\tau) \text{ of } \pi^*_E \oplus \mathcal{O}_X E^0. \quad (4.5.3)$$

From the construction of the projective bundle (cone), this tautological section $\tau$ of $\pi^*_E \oplus \mathcal{O}_X (E^{-1} \oplus \mathcal{O}_X)$ induces a homomorphism from $\mathcal{O}_{\mathbb{P}(E^{-1} \oplus \mathcal{O}_X)}(-1)$ to $\tau$ on the projective bundle $\mathbb{P}(E^{-1} \oplus \mathcal{O}_X)$, which we denote by $\tau(-1)$. Therefore

$$\mathcal{N} \text{ is cut out of } \mathbb{P}(E_1) = \mathbb{P}(E^{-1} \oplus \mathcal{O}_X) \text{ by the section } \pi^*_E (\phi, 0)^*(\tau(1)) \text{ of } \pi^*_E E^0(1). \quad (4.5.4)$$

### 4.5.1 Local Model

We work on the local model of $\mathcal{N}$. We have

$$\{ T_A | X \oplus \mathcal{O}_X \xrightarrow{(ds, 0)} E | X \oplus \mathcal{O}_X \} \text{ is quasi-isomorphic to } \{ E_0 \oplus \mathcal{O}_X \to E_1 \oplus \mathcal{O}_X \}. \text{Therefore the ideal of } \mathcal{N} \text{ in the smooth ambient space } P := \mathbb{P}(E^* \oplus \mathcal{O}_A) \text{ is }$$

$$(\pi^*_E s, \pi^*_E (ds, 0)^*(\tau(1))). \quad (4.5.5)$$

Here we recall that

$$\pi^*_E : \mathbb{P}(E^* \oplus \mathcal{O}_A) \to A$$

is the projection of the projective bundle.

We can take $A = \mathbb{A}^r_\infty$ and the vector bundle $E = A \times \mathbb{A}^r_\infty$ is trivial. We work in local coordinates $x_i$ for $A$ such that $(s_1, \ldots, s_r)$ are functions of $x_i$. Trivializing $E$ (with rank $r$) with a basis of sections $e_j$, we get a dual basis $f_j$ for $E^*$ and coordinates $y_j$ on the fibers of $\text{Tot}(E^*)$. From the construction of $P := \mathbb{P}(E^* \oplus \mathcal{O}_A) = A \times \mathbb{P}^r$, and let $D_\infty = \mathbb{P}(E^*) = A \times \mathbb{P}^{r-1}$ be the infinity divisor. In the local coordinates above, one takes the homogeneous coordinates of the fibre of $A \times \mathbb{P}^r$ by

$$[u_1 : \cdots : u_r : u_{r+1}].$$

Therefore $D_\infty = \{ u_{r+1} = 0 \} \subset A \times \mathbb{P}^r$.

Recall that the log cotangent bundle $\Omega_P^{\log}$ is defined as the sheaf of differential forms on $P$ with logarithmic poles along $D_\infty$, i.e., to be the sheaf of meromorphic 1-forms on $P$ that are holomorphic away from $D_\infty$ and locally on $A \times U_i$ in coordinates $\{ z, z_i = \frac{u_i}{u_1} \}$ along $D_\infty$ can be written as

$$f \frac{dz}{z} + \sum_i f_i dz_i$$
with all \( f, f_i \) holomorphic functions.

Let us look at the meromorphic function \( \tilde{s} = \sum_{j=1}^{r'} s_j u_j \) on \( P \) which only has linear terms on the homogeneous coordinates \( \{ u_i \} \) of \( \mathbb{P}^r \). The function \( \tilde{s} \) extends to \( P \) and has only first order pole along \( D_\infty \), since in local coordinates \( \tilde{s} = \frac{g}{z} \) for some regular function \( g \) where \( z \) is the normal coordinate of \( D_\infty \). Therefore \( \tilde{s} \in \mathcal{O}_P(D_\infty) \). Then the differential \( d\tilde{s} \) has pole along \( D_\infty \) and gives a section of the twisted log cotangent bundle \( \Omega^1_{\mathcal{O}_P}(D_\infty) \).

The differential \( d\tilde{s} \) also gives the ideal

\[
\left( s_1, \ldots, s_r; \sum_i u_i ds_i \right)
\]

in coordinates \( \{ x_i \} \) of \( A \) and homogeneous coordinates \( \{ u_i \} \) of \( \mathbb{P}^r \) which is (4.5.5) for the compactification \( \overline{N} \).

4.5.2 Calculation on Each Affine Charts

We write down the ideal (4.5.5) on each affine charts of \( P \).

\( P \) is covered by \( r + 1 \) affine open subset \( A \times U_i \), where \( U_i = \{ u_i \neq 0 \} \subset \mathbb{P}^r \). On each \( U_i \), the coordinates are given by:

\[
\left\{ y_1 := \frac{u_1}{u_i}, \ldots, \hat{i}, \ldots, y_i = \frac{u_r}{u_i}, y_i' = \frac{u_{r+1}}{u_i} \right\}
\]

if \( i \neq r + 1 \); and

\[
\left\{ y_1^{r+1} := \frac{u_1}{u_{r+1}}, \ldots, y_r^{r+1} = \frac{u_r}{u_{r+1}} \right\}
\]

if \( i = r + 1 \).

The scheme \( \text{Tot}(E^+) = A \times U_{r+1} \) and the fibre coordinates \( \{ y_1, \ldots, y_r \} \) of \( \text{Tot}(E^+) \) are given by \( \{ \frac{u_1}{u_{r+1}}, \ldots, \frac{u_r}{u_{r+1}} \} \). The regular function \( \tilde{s} = \sum_{i=1}^r y_i s_i \) and the critical locus of \( \tilde{s} \) is \( N \).

Let us look at the open affine \( A \times U_i \) for \( i \neq r + 1 \). Let \( z \) (think of \( z \) as \( \frac{u_i}{u_{r+1}} \)) be the local coordinate of \( P \) in the normal direction of \( D_\infty \) such that the zero locus of \( z \) gives \( D_\infty \). The function

\[
\frac{\tilde{s}}{|A \times U_i|} = s_i \frac{1}{z} + \sum_{l \neq i}^r s_l y_l' \frac{1}{z} = \frac{1}{z} \cdot g
\]

where \( g = s_i + \sum_{l \neq i}^r s_l y_l' \) is a regular function on \( A \times U_i \). Then \( d\tilde{s} = \frac{1}{z} \cdot dg - \frac{1}{z^2} \cdot g dz \). Thus

\[
d\tilde{s} = \frac{1}{z} \left( ds_i + \sum_{l \neq i}^r ds_l \cdot y_l' + \sum_{l \neq i}^r s_l \cdot dy_l' \right) - \frac{1}{z^2} \left( \sum_{l \neq i}^r s_l \cdot y_l' + s_i \right) dz.
\]

We use \( (ds_i + \sum_{l \neq i}^r ds_l \cdot y_l') \) to represent the ideal generated by

\[
\left( \frac{\partial s_i}{\partial x_1} + \sum_{l \neq i}^r \frac{\partial s_l}{\partial x_1} \cdot y_l', \ldots, \frac{\partial s_i}{\partial x_r} + \sum_{l \neq i}^r \frac{\partial s_l}{\partial x_r} \cdot y_l' \right).
\]

Lemma 4.12 The affine chart \( \overline{N} \cap (A \times U_i) \) is given by the zero locus of the ideal

\[
\left( s_1, \ldots, s_i, \ldots, s_r; \left( ds_i + \sum_{l \neq i}^r ds_l \cdot y_l' \right) \right). \tag{4.5.6}
\]
Proof. First in this local case the ideal (4.5.5) is given by:

\[
\left( s_1, \ldots, s_r; \sum_i u_i ds_i \right).
\]

(4.5.7)

Here \((\sum_i u_i ds_i)\) means the ideal generated by:

\[
\left( \sum_i u_i \frac{\partial s_i}{\partial x_1}, \ldots, \sum_i u_i \frac{\partial s_i}{\partial x_r} \right).
\]

To write down the ideal (4.5.7) in this affine chart \(N \cap (A \times U_i)\), we first have

\[
\sum_i u_i ds_i = \sum_i \frac{u_i}{u_{r+1}} ds_i
\]

in the affine chart \(N \cap (A \times U_{r+1})\). From the relation between the coordinates on \(U_i\) and \(U_{r+1}\),

\[
u_i u_{r+1} = 1, \quad \nu_{r+1} u_i = 1, \quad z;
\]

and for \(l = 1, \ldots, r\) but \(l \neq i\),

\[
\frac{u_l}{u_{r+1}} = \frac{u_l}{u_i} \cdot \frac{u_i}{u_{r+1}} = \frac{u_l}{u_i} \cdot \frac{1}{z}
\]

then in the open affine chart \(N \cap A \times U_i\), \(\sum_i u_i ds_i\) can be written as

\[
ds_i + \sum_{l=1}^r \frac{u_l}{u_i} \cdot \frac{1}{z} ds_l.
\]

Since in this affine chart \(u_i \neq 0\), the ideal (4.5.7) in this open affine chart is given by \((s_1, \ldots, s_i, \ldots, s_r)\) and

\[
ds_i + \sum_{l=1}^r \frac{u_l}{u_i} \cdot ds_l = ds_i + \sum_{l=1}^r y_l ds_l.
\]

The locus \(N \cap (A \times U_i)\) determined by the ideal (4.5.6) is not given by the critical locus of \(g\) since the ideal \(dg\) is

\[
\left( s_1, \ldots, \tilde{s}_i, \ldots, s_r; \left( ds_i + \sum_{l=1}^r ds_l \cdot y_l^i \right) \right)
\]

where \(\tilde{s}_i\) means this \(s_i\) is not included. If the ideal \((s_i) \subseteq (ds_i)\), then the ideal (4.5.6) is given by the ideal \(dg\).

4.5.3 \(d\)-critical Scheme Structure

We aim to show that \(N\) is also a \(d\)-critical scheme. We first consider \(A \times U_{r+1}\). Then in this case let \(y_l^{r+1} := \frac{u_l}{u_{r+1}}\), then the function \(\tilde{s} = \sum_{l=1}^r s_l y_l\) and we go back to the case of abelian cone \(N\), which is the critical locus of \(\tilde{s}: \text{Tot}(E^*) A \times U_{r+1} \to \mathbb{A}^1_{\kappa}\). Hence

\[
(Cr\tilde{s}(\tilde{s}), A \times U_{r+1}, \tilde{s}, \iota_{r+1})
\]

is a critical chart, where \(\iota_{r+1}: \text{Crit}(\tilde{s}) \hookrightarrow A \times U_{r+1}\) is the closed embedding.
Let us consider the affine scheme $A \times U_i (i \neq r + 1)$. From Lemma 4.12,

$$\tilde{s}|_{A \times U_i} = \frac{s_i}{z} + \sum_{l=1, l \neq i}^{r} s_l \frac{u_l}{u_i} \frac{1}{z} := \frac{1}{z} \cdot g$$

where $g = s_i + \sum_{l=1, l \neq i}^{r} s_l y_l^i$. Suppose that $(s_i) \subset (ds_i)$, we have critical chart

$$(\text{Crit}(g), A \times U_i, g, i_i)$$

where $i_i : \text{Crit}(g) \hookrightarrow A \times U_i$ is the closed embedding. Since the differential

$$dg = ds_i + \sum_{l=1, l \neq i}^{r} (ds_l y_l^i + s_l dy_l^i),$$

one can write down the ideal of $\text{Crit}(g)$:

$$\{(s_l)_{l=1, l \neq i}, ds_1 \cdot y_1^i, \ldots, ds_i, \ldots, ds_r \cdot y_r^i\}.$$  \hspace{1cm} (4.5.8)

Next we consider another affine scheme $A \times U_j (j \neq i, r + 1)$. Let

$$\{y_l^j := \frac{u_1}{u_j}, \ldots, \hat{i}, \ldots, y_l^j = \frac{u_r}{u_j}, y_j^j = \frac{u_{r+1}}{u_j}\}$$

be the local coordinate of $U_j$. Let $z = \frac{u_i}{u_j}$. Then we have

$$\tilde{s}|_{A \times U_j} = \frac{s_j}{z} + \sum_{l=1, l \neq j}^{r} s_l \frac{u_l}{u_j} \frac{1}{z} := \frac{1}{z} \cdot g'$$

where $g' = s_j + \sum_{l=1, l \neq j}^{r} s_l y_l^j$. Therefore we have critical chart

$$(\text{Crit}(g'), A \times U_j, g, i_j)$$

where $i_j : \text{Crit}(g') \hookrightarrow A \times U_j$ is the closed embedding. Since the differential

$$dg' = ds_j + \sum_{l=1, l \neq j}^{r} (ds_l y_l^j + s_l dy_l^j),$$

one can write down the ideal of $\text{Crit}(g')$:

$$\{(s_l)_{l=1, l \neq j}, ds_1 \cdot y_1^j, \ldots, ds_j, \ldots, ds_r \cdot y_r^j\}.$$  \hspace{1cm} (4.5.9)

We check that the critical locus of $\tilde{s}$ and $g$ are the same on the intersection $A \times U_{r+1} \cap U_i$. Let us write down for $\tilde{s} : A \times U_{r+1} \to k_{x}$,

$$d\tilde{s} = \sum_{l=1}^{r} (ds_l y_l^{r+1} + s_l dy_l^{r+1}).$$

Then the ideal of $\text{Crit}(\tilde{s})$ is given by

$$\langle s_1, \ldots, s_r; ds_1 \cdot y_1^{r+1}, \ldots, ds_r \cdot y_r^{r+1} \rangle.$$  

Comparing with the ideal (4.5.8), note that $y_l^{r+1} = \frac{1}{y_l^j} \neq 0$, and the ideal $(ds_i) \subset (s_i)$ (which is an assumption), therefore they are the same. The comparison of the ideal (4.5.8) and the ideal (4.5.9) is similar.
Let $\mathcal{S}_{N}$ be the unique coherent sheaf on $N$ constructed in Theorem 4.1. Then for the critical charts above, we have the following exact sequences of sheaves of $\kappa$-vector spaces:

$$0 \to \mathcal{S}_{\operatorname{Crit}(\tilde{s})} \xrightarrow{\lambda} i_{r+1}^{-1}(\mathcal{O}_{A \times U_{r+1}}) \xrightarrow{d} i_{r+1}^{-1}(T^{*}(A \times U_{r+1})) \xrightarrow{\mathcal{I}_{\operatorname{Crit}(\tilde{s})},A \times U_{r+1}} I_{\operatorname{Crit}(\tilde{s})},A \times U_{r+1} \cdot i_{r+1}^{-1}(T^{*}(A \times U_{r+1}))$$

where $d$ maps $f + I_{\operatorname{Crit}(\tilde{s}),A \times U_{r+1}}^{2}$ to $df + I_{\operatorname{Crit}(\tilde{s}),A \times U_{r+1}} \cdot i_{r+1}^{-1}(T^{*}(A \times U_{r+1}))$; and

$$0 \to \mathcal{S}_{\operatorname{Crit}(g)} \xrightarrow{\lambda} i_{1}^{-1}(\mathcal{O}_{A \times U_{1}}) \xrightarrow{d} i_{1}^{-1}(T^{*}(A \times U_{1})) \xrightarrow{\mathcal{I}_{\operatorname{Crit}(\tilde{s})},A \times U_{1}} I_{\operatorname{Crit}(\tilde{s})},A \times U_{1} \cdot i_{1}^{-1}(T^{*}(A \times U_{1}))$$

where $d$ maps $f + I_{\operatorname{Crit}(g),A \times U_{1}}^{2}$ to $df + I_{\operatorname{Crit}(g),A \times U_{1}} \cdot i_{1}^{-1}(T^{*}(A \times U_{1}))$. Then we can take a section $s \in H^{0}(\mathcal{S}_{N})$ such that: for any $x \in \operatorname{Crit}(\tilde{s})$ in the open affine $A \times U_{r+1}$,

$$\iota_{\operatorname{Crit}(\tilde{s}),A \times U_{r+1}}(s|_{\operatorname{Crit}(\tilde{s})}) = i_{r+1}^{-1}((\tilde{s}) + I_{\operatorname{Crit}(g),A \times U_{r+1}}^{2})$$

And or any $x \in \operatorname{Crit}(g)$ in the open affine $A \times U_{i}$ for $i \neq r+1$,

$$\iota_{\operatorname{Crit}(g),A \times U_{i}}(s|_{\operatorname{Crit}(g)}) = i_{r+1}^{-1}(g) + I_{\operatorname{Crit}(g),A \times U_{i}}^{2}.$$ 

These local sections $s|_{\operatorname{Crit}(\tilde{s})}$ and $s|_{\operatorname{Crit}(g)}$ glue since $\tilde{s}$ and $g$ are the same at the intersection $A \times (U_{r+1} \cap U_{i})$. Thus we show that:

**Theorem 4.13** If on each affine chart, the sections $s_{i}$ satisfy the condition $(s_{i}) \subset (ds_{i})$, then the pair $(\tilde{N}, s)$ is a $d$-critical scheme.

Then from Theorem 4.7, $\tilde{N}$ admits a symmetric semi-perfect obstruction theory $\phi$, therefore a virtual fundamental cycle $[\tilde{N}, \phi]^{\text{virt}}$ of degree zero. The invariant is

$$\int_{[\tilde{N}, \phi]^{\text{virt}}} 1. \quad (4.5.10)$$

**Example 4.14** We change notation here and let $N = \operatorname{Spec} \kappa[x, y]/(x^{2}, xy)$, and $N$ is the critical locus of $f = x^{2}y$. Also $N$ is the abelian cone over $X = \operatorname{Spec} \kappa[x]/(x^{2})$. The scheme $\tilde{N} = \operatorname{Proj} R[y_{0} : y_{1}]/(x^{2}, xy_{0})$ in Example 4.6 is the compactification of $N$, and is also a $d$-critical scheme.

We calculate the invariant. There is a $\mathbb{G}_{m}$ action on $\tilde{N}$ induced from the $\mathbb{G}_{m}$-action on $\mathbb{A}^{1}_{\kappa} \times \mathbb{P}^{1}$ where the action on $\mathbb{A}^{1}_{\kappa}$ trivial. There are two torus fixed points 0 and $\infty$, where 0 is a fat point with multiplicity 2, and $\infty$ is a smooth point. One can calculate the Behrend function on them. Example 3.1 in [16] calculated $\nu(\tilde{N}, 0) = 1$, since the Behrend function depends only on its local neighborhood hence it is the same as the Behrend function of $\nu_{N}$ at 0. Since $\infty$ is a smooth point, $\nu_{\tilde{N}}(\infty) = 1$. So $\chi(\tilde{N}, \nu_{\tilde{N}}) = 2 = \int_{[\tilde{N}, \phi]^{\text{virt}}} 1$.

The compactification $\tilde{N}$ can also be taken as the non-reduced projective subscheme in $\mathbb{P}^{2} = \operatorname{Proj}(\kappa[x : y_{1} : y_{2}])$ by the homogeneous ideal

$$(x^{2}, xy_{1}).$$

Then we can take a deformation family of $\tilde{N}$ by considering the subschemes given by the ideals

$$(x^{2} - t^{2}y_{2}^{2}, xy_{1}).$$

When $t = 0$, this is the same as $\tilde{N}$. When $t \neq 0$, this gives the subscheme determined by the ideal $(x^{2} - t^{2}y_{2}^{2})$ in $\mathbb{P}^{1} = \operatorname{Proj}(\kappa[x : 0 : y_{2}])$, hence it is the subscheme containing two smooth points. This shows that the invariant $\int_{[\tilde{N}, \phi]^{\text{virt}}} 1$ is deformation invariant.
4.6 Application to the $\mathbb{G}_m$-equivariant Symmetric Semi-perfect Obstruction Theory

We generalize the result in [13] to symmetric semi-perfect obstruction theory. We first recall the $\mathbb{G}_m$-equivariant semi-perfect obstruction theory.

**Definition 4.15 ([18])** A $\mathbb{G}_m$-equivariant symmetric semi-perfect obstruction theory on $X$ consists of the following:

1. a $\mathbb{G}_m$-equivariant étale open cover
   \[ \{ X_\alpha \to X \} \]
   of $X$;
2. the symmetric obstruction theory
   \[ \phi_\alpha : E^\bullet_\alpha \to L^\bullet_{X_\alpha} \]
is $\mathbb{G}_m$-equivariant in the equivariant derived category $D^b(X_\alpha)$ of coherent sheaves.

**Remark 4.16** Note that in [3], a $\mathbb{G}_m$-symmetric equivariant obstruction theory is given by a $\mathbb{G}_m$-equivariant morphism

\[ \phi_\alpha : E^\bullet_\alpha \to L^\bullet_{X_\alpha} \]
such that there exists a $\mathbb{G}_m$-equivariant bilinear form

\[ \Theta : E^\bullet_\alpha \xrightarrow{\sim} E^{\bullet \vee}_\alpha [-1] \]
in $D^b(X_\alpha)$ such that $\Theta$ is an isomorphism satisfying $\Theta^\vee[1] = \Theta$. A $\mathbb{G}_m$-symmetric equivariant obstruction theory requires more than just a $\mathbb{G}_m$-equivariant obstruction theory, see [3, §3.4].

The main result in [13, Theorem 5.20] does not require that $X$ admits a “symmetric” equivariant obstruction theory, but rather a $\mathbb{G}_m$-equivariant perfect obstruction theory. First we have the following result, which is the same as Theorem 5.8 in [13].

**Proposition 4.17** Let $X$ be a scheme or a Deligne–Mumford stack which admits a symmetric semi-perfect obstruction theory. Assume that there exists a $\mathbb{G}_m$-action on $X$ with proper fixed locus $F \subseteq X$ such that

\[ \{ E^\bullet_\alpha \to L^\bullet_{X_\alpha} \}_{\alpha \in \Lambda} \]
is a $\mathbb{G}_m$-equivariant obstruction theory. Then

\[ \chi(X, \nu_X) = \chi(F, \nu_X|_F). \]

Moreover, if $X$ is proper,

\[ \int_{[X, \phi]^{\virt}} 1 = \chi(F, \nu_X|_F). \]

**Proof** The first result is from the fact that the Behrend function $\nu_X$ is constant on the nontrivial $\mathbb{G}_m$-orbits. If $X$ is proper, the last result is from Theorem 3.8.

Cosection Localization

It is important to let $X$ non-proper. Still the $\mathbb{G}_m$-action on $X$ defines a cosection

\[ \sigma : \Omega_X \to \mathcal{O}_X \]
by taking the dual of the canonical vector field $v_x \mapsto \frac{d}{d\lambda} \mu(\lambda \cdot x)|_{\lambda=1}$ on $X$ by the $\mathbb{G}_m$-action. The degenerate locus of $\sigma$ is $D(\sigma) = F$. We assume that $X$ is quasi-projective if it is a Deligne–Mumford stack, therefore there exists a closed immersion

$$X \hookrightarrow M$$

into a smooth higher dimensional smooth Deligne–Mumford stack $M$. Hence we have the following cartesian diagram:

$$
\begin{array}{ccc}
C \subset \Omega_X & \longrightarrow & \Omega_M|_X \\
\downarrow & & \downarrow \\
cv \subset \Omega_M & \longrightarrow & \Omega_X
\end{array}
$$

where $C \subset \Omega_M$ is the unique Lagrangian cone cycle in $\Omega_M$ making the diagram commute, see Theorem 3.6. Also [22] proved that

$$C \subset \Omega_M|_F \cup \ker(\Omega_M|_{X\setminus F} \to \mathcal{O}_X).$$

Then Kiem–Li defined the cosection localized virtual cycle by applying the localized Gysin map in [22, §3.3], see also [18]. We use the analytic method in the Appendix of [22], and take a small perturbation $\xi$ is the zero section $M$ of $\Omega_M$ such that $\xi \cap C$ only supports on $F$. Then

$$[X]_{\text{virt}}^\text{loc} := \xi \cap C \in H_0(F).$$

**Theorem 4.18** Let $F$ be proper. We have:

$$\int [X]_{\text{virt}}^\text{loc} 1 = \chi(F, \nu_X|_F).$$

**Proof** We only need to show that

$$\chi(X, \nu_X) = \#(\xi \cap C),$$

the intersection number, but similar to the proof in [16, §4] or [13, §5.8], this is the global index theorem due to Theorem 9.7.11 of Kashiwara–Schapira [24] since the characteristic cycle of $\nu_X$ is $C$. In the Deligne–Mumford stack case, the index theorem is due to Maulik–Treumann in [28].

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**References**

[1] Behrend, K.: Donaldson–Thomas invariants via microlocal geometry. *Ann. Math.*, 170(3), 1307–1338 (2009)

[2] Behrend, K., Fantechi, B.: The intrinsic normal cone. *Invent. Math.*, 128(1), 45–88 (1997)

[3] Behrend, K., Fantechi, B.: Symmetric obstruction theories and Hilbert scheme of point on threefolds. *Algebra Number Theory*, 2(3), 313–345 (2008)

[4] Brav, C., Bussi, V., Joyce, D.: A Darboux theorem for derived schemes with shifted symplectic structure. *J. Amer. Math. Soc.*, 32(2), 399–443 (2019)
Symmetric Semi-perfect Obstruction Theory Revisited

[5] Ben-Bassat, O., Brav, C., Bussi, V., et al.: A “Darboux theorem” for shifted symplectic structures on derived Artin stacks, with applications. *Geometry and Topology*, 19, 1287–1359 (2015)

[6] Chang, H.-L., Li, J.: Semi-perfect obstruction theory and Donaldson–Thomas invariants of derived objects. *Communications in Analysis and Geometry*, 19(4), 807–830 (2011)

[7] Chang, H.-L., Li, J.: Gromov–Witten invariants of stable maps with fields. *Int. Math. Res. Not.*, 2012, 4163–4217 (2012)

[8] Ciocan-Fontanine, I., Kapranov, M.: Virtual fundamental classes via dg-manifolds. *Geom. and Top.*, 13, 1779–1804 (2009)

[9] Costello, K.: Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. *Pure Appl. Math. Quart.*, 9, 7, 3–165 (2013)

[10] Gonzalez-Sprinberg, G.: L’obstruction locale d’Euler et le theoreme de MacPherson. *Asterisque*, 82–83, 7–32 (1981)

[11] Illusie, L.: Complexe Cotangent et Deformations I, II, Lecture Notes in Mathematics, Vols. 239, 283, Springer, Berlin, 1971, 1972

[12] Jiang, Y.: Motivic Milnor fibre of cyclic $L_\infty$-algebras. *Acta Mathematica Sinica, Engl. Ser.*, 33(7), 933–950 (2017)

[13] Jiang, Y.: Note on MacPherson’s local Euler obstruction. *Michigan Mathematical Journal*, 68, 227–250 (2019)

[14] Jiang, Y.: The Pro-Chern–Schwartz–MacPherson class for DM stacks. *Pure and Applied Mathematics Quarterly*, 11(1), 87–114 (2015)

[15] Jiang, Y.: On the motivic virtual signed Euler characteristics and application to Vafa-Witten invariants, preprint, arXiv:1710.08987

[16] Jiang, Y., Thomas, R. P.: Virtual signed Euler characteristics. *Journal of Algebraic Geometry*, 26, 379–397 (2017)

[17] Joyce, D.: A classical model for derived critical locus. *Journal of Differential Geometry*, 101, 289–367 (2015)

[18] Kiem, Y.-H.: Localizing virtual fundamental cycles for semi-perfect obstruction theories. *Int. J. Math.*, 29(4), 1850032, 30 pp. (2018)

[19] Kiem, Y.-H., Li, J., Savvas, M.: Generalized Donaldson–Thomas invariants via Kirwan Blowups, arXiv:1712.02544

[20] Kiem, Y.-H., Savvas, M.: $K$-Theoretic generalized Donaldson–Thomas invariants. *International Mathematics Research Notices*, 2020(3), 2123–2158 (2022)

[21] Kiem, Y.-H., Savvas, M.: Localizing virtual structure sheaves for almost perfect obstruction theories. *Forum of Mathematics, Sigma*, 8, Paper No. e61, 36 pp. (2020)

[22] Kiem, Y.-H., Li, J.: Localizing virtual cycles by cosection. *J. Amer. Math. Soc.*, 26, 1025–1050 (2013)

[23] Kiem, Y.-H., Li, J.: Critical virtual manifolds and perverse sheaves. *J. Korean Math. Soc.*, 55(3), 623–669 (2018)

[24] Kashihara, M., Schapira, P.: Sheaves on manifolds, Grundlehren der Mathematischen Wissenschaften, 292, Springer Verlag, Berlin, 1990

[25] Kretsch, A.: Cycle groups for Artin stacks. *Invent. Math.*, 138, 495–536 (1999)

[26] Li, J., Tian, G.: Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11, 119–174 (1998)

[27] MacPherson, R.: Chern class for singular algebraic varieties. *Ann. Math.*, 100(2), 423–432 (1974)

[28] Maulik, D., Treumann, D.: Constructible functions and Lagrangian cycles on orbifolds, arXiv:1110.3866

[29] Pandharipande, R., Thomas, R. P.: Curve counting via stable pairs in the derived category. *Inventiones Mathematicae*, 178, 407–447 (2009)

[30] Pantev, T., Toen, B., Vaquie, M., Vezzosi, G.: Shifted symplectic structures. *Publ. Math. I.H.E.S.*, 117, 271–328 (2013)

[31] Tanaka, Y., Thomas, R. P.: Vafa–Witten invariants for projective surfaces I: stable case. *J. Algebraic Geom.*, 29(4), 603–608 (2020)

[32] Thomas, R. P.: A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on $K3$ fibrations. *J. Differential Geom.*, 54, 367–438 (2000)