DERIVED LENGTH OF ZERO ENTROPY GROUPS ACTING ON COMPACT KÄHLER MANIFOLDS

TIEN-CUONG DINH, KEIJI OGUISO, AND DE-QI ZHANG

Abstract. Let $X$ be a compact Kähler manifold of dimension $n \geq 1$. Let $G$ be a group of zero entropy automorphisms of $X$. Let $G_0$ be the set of elements in $G$ which are isotopic to the identity. We prove that after replacing $G$ by a suitable finite-index subgroup, $G/G_0$ is a unipotent group of derived length at most $n - 1$. This is a corollary of an optimal upper bound of length involving the Kodaira dimension. We also study the algebro-geometric structure of $X$ when it admits a group action with maximal derived length $n - 1$.

1. Introduction

We work over the complex numbers field $\mathbb{C}$. Let $X$ be a compact Kähler manifold of dimension $n \geq 1$ and let $\text{Aut}(X)$ denote the group of all holomorphic automorphisms of $X$. It follows from classical results of Fujiki and Lieberman that $\text{Aut}(X)$ is a complex Lie group of finite dimension, with countably many (and possibly infinitely many) connected components [15, 24]. Denote by $\text{Aut}_0(X)$ the component of the identity in $\text{Aut}(X)$. This is a normal connected Lie subgroup of $\text{Aut}(X)$ whose Lie algebra is the set of holomorphic vector fields on $X$.

Recall that for every $g \in \text{Aut}(X)$, according to theorems of Gromov and Yomdin, the topological entropy of $g$ is equal to the logarithm of the spectral radius of its action on the Hodge cohomology ring of $X$, see [17, 32]. In particular, $g$ has positive entropy if and only if this spectral radius is larger than 1. Groups with positive entropy elements have been intensively studied during the last decade using techniques and ideas from Complex Dynamics and Algebraic Geometry. In this paper, we aim to develop a similar study for groups with zero entropy elements.

It is known that $\text{Aut}(X)$ satisfies a Tits alternative and for solvable subgroups of $\text{Aut}(X)$, the positive entropy part has a “bounded” size. Precisely, the following result was proved

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in [5, 33], see also [7] and, for earlier results, [11, 22]. If \( G \) is a subgroup of \( \text{Aut}(X) \), define
\[
G_0 := G \cap \text{Aut}_0(X)
\]
and denote by \( N(G) \) the set of all elements in \( G \) with zero entropy.

**Theorem 1.1.** Let \( X \) be a compact Kähler manifold of dimension \( n \geq 1 \) and Kodaira dimension \( \kappa(X) \) which is known to be in \( \{-\infty, 0, 1, \ldots, n\} \). Define \( \kappa := \max\{\kappa(X), 0\} \) if \( \kappa(X) < n \) and \( \kappa := n - 1 \) otherwise. With the notation introduced above, the manifold \( X \) satisfies the following Tits alternative: if a subgroup \( G \) of \( \text{Aut}(X) \) contains no non-abelian free subgroup, then it admits a finite-index solvable subgroup \( G' \) such that \( N(G') \) is a normal subgroup of \( G' \) and \( G' / N(G') \) is a free abelian group of rank at most equal to \( n - \kappa - 1 \).

If \( X \) admits a group \( G \) such that the rank of \( G' / N(G') \) is maximal, i.e., equal to \( n - 1 \), we will say that \( X \) is a manifold with MSHA, where MSHA stands for Maximal Solvable Hyperbolic Action. Clearly, for such a manifold, we have \( \kappa(X) \leq 0 \). We refer to [11, 34] for more properties of these manifolds. The problem of classifying manifolds with MSHA is still open when \( X \) is rationally connected.

In this article, we will focus our study on the group \( N(G') \) in the last statement. More generally and in order to simplify the notation, we consider groups \( G \) such that all elements of \( G \) have zero entropy.

Now we state our first main result. Recall that for a group \( G \) and a non-negative integer \( p \), the \( p \)-th derived group \( G^{(p)} \) is defined inductively by
\[
G^{(0)} := G \quad \text{and} \quad G^{(i+1)} := [G^{(i)}, G^{(i)}].
\]
By definition, \( G^{(p)} = \{1\} \) for some non-negative integer \( p \) exactly when \( G \) is solvable. We call the minimum of such \( p \) the derived length of \( G \) (when \( G \) is solvable) and denote it by \( \ell(G) \).

A group \( H \) is said to be unipotent if there is an injective homomorphism \( \rho : H \to \GL(N, \mathbb{R}) \) such that for every \( h \in H \), the image \( \rho(h) \) is upper triangular with all entries on the diagonal being 1. Note that unipotent groups are solvable. It is known that if a group \( H \) is isomorphic to a subgroup of \( \GL(N, \mathbb{R}) \) whose elements have only eigenvalue 1, then \( H \) is unipotent, see [19, §17.5]. Note that in the above discussion, we can replace \( \mathbb{R} \) by \( \mathbb{C} \) as we have the natural inclusions \( \GL(N, \mathbb{R}) \subset \GL(N, \mathbb{C}) \subset \GL(2N, \mathbb{R}) \).

**Theorem 1.2.** Let \( X \) be a compact Kähler manifold of dimension \( n \geq 1 \) and Kodaira dimension \( \kappa(X) \) which is known to be in \( \{-\infty, 0, 1, \ldots, n\} \). Let \( G \) be a subgroup of \( \text{Aut}(X) \) such that all elements of \( G \) have zero entropy. Then

1. \( G \) admits a finite-index subgroup \( G' \) such that, for any \( 1 \leq p \leq n - 1 \), the natural map \( G' / G_0' \to G' |_{H^{2p}(X, \mathbb{R})} \) (resp. \( G' / G_0' \to G' |_{H^{2p}(X, \mathbb{C})} \)) is an isomorphism with image a unipotent subgroup of \( \GL(H^{2p}(X, \mathbb{R})) \) (resp. \( \GL(H^{2p}(X, \mathbb{C})) \));
2. For every finite-index subgroup \( G' \) of \( G \) such that \( G' / G_0' \) is a unipotent group, the derived length of \( G' / G_0' \) does not depend on the choice of \( G' \) and is at most equal to \( n - \max\{\kappa(X), 1\} \);
3. The estimate in (2) is optimal when \( \kappa(X) \geq 0 \): for each integers \( n \geq 2 \) and \( 0 \leq \kappa \leq n \), there are a smooth projective variety \( X \) of dimension \( n \) with \( \kappa(X) = \kappa \), and a subgroup \( G \) of \( \text{Aut}(X) \) such that all elements of \( G \) have zero entropy and the derived length of \( G' / G_0' \) in (2) is equal to \( n - \max\{\kappa(X), 1\} \).
Let $X$ and $G$ be as in Theorem 1.2. According to this theorem, we can define the essential derived length of the action of $G$ on $X$ by

$$\ell_{\text{ess}}(G, X) := \ell(G'/G'_0).$$

Here, $G'$ is any finite-index subgroup of $G$ such that $G'/G'_0$ is a unipotent group. This definition does not depend on the choice of $G'$, see also Lemma 2.7 below.

It follows from Theorem 1.2 that

$$\ell_{\text{ess}}(G, X) \leq \dim X - \max\{\kappa(X), 1\} \leq \dim X - 1.$$

When $X$ admits a group action by $G$ such that the elements of $G$ have zero entropy and $\ell_{\text{ess}}(G, X) = \dim X - 1$ we say that $X$ has MPA by the group $G$, where MPA stands for Maximal Parabolic Action. By Theorem 1.2, if $X$ has MPA, then $\kappa(X) \leq 1$. Moreover, there are such manifolds of dimension $n$ with $\kappa(X) = 0$ or $\kappa(X) = 1$, see Propositions 4.4, 4.5 and 4.6 below. By definition, any compact Riemann surface is a manifold with MPA.

Next, we study algebro-geometric structure of manifolds such that we have an equality for the estimate in Theorem 1.2 (2). It follows that compact Kähler manifolds of Kodaira dimension $0$ with MPA is a main building block to construct manifolds with large unipotent groups of zero entropy automorphisms. See also Corollary 3.2 and Remark 5.9 below.

**Theorem 1.3.** Let $X$ be a compact Kähler manifold of dimension $n \geq 2$ and Kodaira dimension $\kappa(X) \geq 1$. Assume that there is a subgroup $G \leq \text{Aut}(X)$ of zero entropy such that $\ell_{\text{ess}}(G, X) = n - \kappa(X)$, i.e., the derived length of $G'/G'_0$ in Theorem 1.2 (2) is $n - \kappa(X)$. Then there are

(i) a compact Kähler manifold $\tilde{X}$ bimeromorphic to $X$;

(ii) a surjective morphism $f : \tilde{X} \to B$ to a smooth projective variety $B$ of dimension $\kappa(X)$, with connected fibres;

(iii) a biregular action of $G$ on $\tilde{X}$ induced by the one on $X$; and

(iv) a finite-index subgroup $G'$ of $G$;

such that if $F$ is a very general fibre of $f$, the following assertions hold

1. $G'$ descends to a trivial action on the base $B$ of the map $f$;
2. $F$ is a compact Kähler manifold of dimension $n - \kappa(X)$ and with $\kappa(F) = 0$;
3. $F$ has MPA by the group $G'_F$;
4. $\text{Aut}_0(F)$ is a complex torus and $\text{Aut}_0(F) \neq \{1\}$.

A normal complex variety $V$ is a $Q$-torus (resp. $Q$-abelian) if there is a compact complex torus (resp. abelian variety) $A$ and a finite surjective morphism $A \to V$ which is étale in co-dimension one, see [26, Def. 2.13]. A very general fibre $F$ in Theorem 1.3 is likely bimeromorphic to a $Q$-torus. Unfortunately, we can not prove this in the full generality, but we have the following partial answer.

**Theorem 1.4.** Let $X$ be a smooth projective variety of dimension $n \geq 1$ with MPA by a group $G \leq \text{Aut}(X)$. Assume the following three conditions

(i) $\text{Aut}_0(X) \neq \{1\}$;
(ii) $X$ has Kodaira dimension $\kappa(X) = 0$;
(iii) $X$ has a good minimal model, i.e., $X$ is birational to a normal projective variety $X_m$ with at most canonical singularities and semi-ample canonical divisor.
Then

1. $X_m$ is smooth and $G$ acts on $X_m$ biregularly;
2. There is a finite étale Galois cover $A \to X_m$ from an abelian variety $A$ to $X_m$ such that $G|_{X_m}$ lifts to $\tilde{G} \leq \text{Aut}(A)$ with $\tilde{G}/\text{Gal}(A/X_m) = G$;
3. Both $X_m$ and $A$ have MPA by $G$ and $\tilde{G}$, respectively.

Remark 1.5. Note that the above property (iii) is conjectural in the Minimal Model Program and it holds when $n \leq 3$ (cf. [23 §3.13]). We will give in Theorem 5.8 below a general version of Theorem 1.4 where singular projective varieties $X$, with possibly trivial $\text{Aut}_0(X)$, are considered.

The paper is organized as follows. In Section 2, we give some basic properties of linear groups and automorphism groups of compact Kähler manifolds that will be used later. In particular, the proof of Theorem 1.2 (1) is given there. In Section 3, we give an estimate of the derived length for automorphism groups which is a weaker version of Theorem 1.2 (2), see Proposition 3.1 below. The main techniques used in this section are a general version of Fujiki-Lieberman theorem [8, 13, 24], and the mixed Hodge-Riemann theorem [9, 16]. The rest of Theorem 1.2 and Theorem 1.3 will be proved in Section 4. In particular, explicit examples of manifolds with large unipotent groups of automorphisms are given there. The proof of Theorem 1.4 occurs in Section 5, where we use some techniques from the Minimal Model Program.

Finally, in view of examples in Propositions 4.4, 4.5 and 4.6 of smooth projective varieties with MPA each of which either has Kodaira dimension 0 or 1, or is rationally connected of dimension $\leq 5$, we ask the following question.

Question 1.6. Let $n \geq 6$. Is there an $n$-dimensional smooth projective variety $X$ with MPA and of Kodaira dimension $-\infty$ (e.g. a rational, or rationally connected variety)?

Related works. The dynamical study of automorphism groups of algebraic varieties over a base field of positive characteristic is widely open especially in higher dimension. A major difficulty is due to the lack of tools such as the Hodge-Riemann relations, see e.g. Esnault-Srinivas [14]. In the case of surface automorphisms, which we will not elaborate here, the theory of reflection groups is crucial and one can use it, instead of the Hodge-Riemann relations, when working over the base field of arbitrary characteristic, see Dolgachev [13].

Terminology and Notation. For simplicity, if $L$ and $M$ are two cohomology classes, we denote by $L \cdot M$ or $LM$ their cup-product. We also identify $H^0(X, \mathbb{R})$ and $H^{2n}(X, \mathbb{R})$ with $\mathbb{R}$ in the canonical way. So classes in these groups are identified to real numbers. The relation $x_m \simeq y_m$ means $x_m = O(y_m)$ and $y_m = O(x_m)$. An automorphism $g$ on $X$ induces the pull-back operator $g^*$ on cohomology. We often drop the star sign * in order to simplify the notation.

When a group $N$ acts on a space $V$, we denote by $N|_V$ the image of the canonical homomorphism $N \to \text{Aut}(V)$. For instance, $\text{Aut}(X)|_{H^r(X, \mathbb{Z})}$ is the image of the canonical action of the automorphism group $\text{Aut}(X)$ on the cohomology group $H^r(X, \mathbb{Z})$. For a normal subgroup $N_1 \triangleleft N$, we set $(N/N_1)|_V = (N|_V)/(N_1|_V)$.

1 Throughout the paper, this notation stands for the automorphism group of the complex variety $A$ which should be distinguished from the automorphisms of the abelian group $A$. 
We say that a property holds for very general (resp. general) parameters or points if it holds for all parameters or points outside a countable (resp. finite) union of proper closed analytic subvarieties of the space of parameters or points.

Given a proper surjective morphism $f : X \to B$ between complex varieties, we denote by $X_b$ the fibre over $b \in B$ in the category of analytic spaces. We also define the relative automorphism group with respect to $f$ by

$$\text{Aut}(X/B) := \{ g \in \text{Aut}(X) : f \circ g = f \}.$$ 

This is a subgroup of $\text{Aut}(X)$. We have $\text{Aut}(X/B)|_B = \{ \text{id}_B \}$ and $\text{Aut}(X/B)$ acts on each fibre $X_b$ of $f$.

Recall that for a solvable group $H'$, we denote by $\ell(H')$ the derived length of $H'$. For a group $H$ having a finite-index solvable subgroup, we define

$$\ell_{\text{min}}(H) := \min_{H'} \ell(H').$$

Here, $H'$ runs through all finite-index solvable subgroups of $H$, see also Lemma 2.7.

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2. **Group actions and proof of Theorem 1.2 (1)**

In this section, we give some properties of group actions to be used later. We first consider unipotent groups and then groups of automorphisms of a compact Kähler manifold. In particular, we obtain Theorem 1.2 (1) as a direct consequence of Lemma 2.8 below.

**Lemma 2.1.** Let $V$ be a real vector space of finite dimension. Let $\Gamma$ be a subgroup of $\text{GL}(V)$. Assume there is an integer $N \geq 1$ such that $g^N$ is unipotent for every $g \in \Gamma$, i.e., their eigenvalues are 1. Then there is a finite-index solvable subgroup, we define

$$\ell_{\text{min}}(H) := \min_{H'} \ell(H').$$

Here, $H'$ runs through all finite-index solvable subgroups of $H$, see also Lemma 2.7.

**Lemma 2.2.** Let $V$ be a real vector space of finite dimension. Let $\rho : \Gamma \to \text{GL}(V)$ be a group homomorphism such that $\ker(\rho)$ is finite and all elements of $\rho(\Gamma)$ are unipotent in $\text{GL}(V)$. Assume moreover that there is a basis of $V$ on which all elements of $\rho(\Gamma)$ are represented by matrices with integer entries. Then

1. $\Gamma$ is countable, finitely generated, and admits only countably many subgroups;
(2) There is a finite-index subgroup $\Gamma'$ of $\Gamma$ such that $\Gamma' \cap \text{Ker}(\rho) = \{1\}$.

Proof. (1) Without loss of generality, we may assume that $V = \mathbb{R}^N$ and $\rho(\Gamma)$ is contained in $\text{SL}(N, \mathbb{Z})$. So $\Gamma$ is countable. We show that there is a basis $\{v_1, \ldots, v_N\}$ of $\mathbb{Z}^N$ on which the elements of $\rho(\Gamma)$ are represented by unipotent upper triangle matrices with integer entries. This property is well-known for a basis of $\mathbb{R}^N$ and matrices with real entries. In particular, there exists a non-zero vector $v_1$ in $\mathbb{R}^N$, not necessarily with integer coordinates, which is fixed by $\rho(\Gamma)$.

Since $\rho(\Gamma)$ is contained in $\text{SL}(N, \mathbb{Z})$, the invariance of the vector $v_1$ is characterized by a system of equations with integer coefficients. Therefore, there exists such a vector $v_1$ in $\mathbb{Z}^N$, i.e., with integer coordinates. We choose $v_1$ minimal in the sense that it is not a multiple of a vector satisfying the same property. Then $\rho(\Gamma)$ induces an action on $\mathbb{R}^N/\mathbb{R}v_1$ which preserves the lattice $\mathbb{Z}^N/\mathbb{Z}v_1$. Without loss of generality, a change of basis of $\mathbb{Z}^N$ allows us to assume that $v_1 = (1, 0, \ldots, 0)$.

In the same way, we find a vector $v_2 \in \mathbb{Z}^N$ such that its image in $\mathbb{Z}^N/\mathbb{Z}v_1 \simeq \mathbb{Z}^{N-1}$ is a minimal invariant vector with integer coordinates. By induction, we obtain a basis $\{v_1, \ldots, v_N\}$ of $\mathbb{Z}^N$ such that $g(v_j) = v_j$ modulo $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_{j-1}$ for every $j$ and every $g$ in $\rho(\Gamma)$. It is clear that, in this basis, all elements of $\rho(\Gamma)$ are represented by unipotent upper triangle matrices with integer entries. From now on, we work with this basis and matrices. For simplicity, we can assume that $\{v_1, \ldots, v_N\}$ is the standard basis of $\mathbb{Z}^N$. Define $V_j := \mathbb{R}v_1 + \cdots + \mathbb{R}v_j$ and $Z_j := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_j$ for $1 \leq j \leq N$. These sets are invariant by the action of $\Gamma$.

We show that $\Gamma$ is finitely generated. Since Ker$(\rho)$ is finite, it is enough to show that $\rho(\Gamma)$ is finitely generated. Denote by $\rho(\Gamma)_{N-1}$ the restriction of $\rho(\Gamma)$ to $V_{N-1}$. This is a group of upper triangle unipotent matrices of size $N - 1$ with integer entries. The kernel of the natural surjective group morphism

$$\rho(\Gamma) \rightarrow \rho(\Gamma)_{N-1}$$

is the set of matrices in $\rho(\Gamma)$ whose entries are integer, 1 on the diagonal, and 0 outside the diagonal, except on the last column. So we can identify this group with a group of translations on $V_{N-1}$ by vectors in $\mathbb{Z}_{N-1}$. Therefore, this kernel is finitely generated. It follows that $\rho(\Gamma)$ is finitely generated if and only if $\rho(\Gamma)_{N-1}$ is finitely generated. We then easily obtain the finite generation property of $\Gamma$ by induction on the size of the matrices.

We remark that in the same way, we obtain that any subgroup of $\Gamma$ is finitely generated. Since $\Gamma$ is countable, we deduce that it admits only countably many subgroups.

(2) Let $\Gamma'$ be a maximal normal subgroup of $\Gamma$ such that $\Gamma' \cap \text{Ker}(\rho) = \{1\}$. It is enough to show that $\Gamma'$ is of finite-index in $\Gamma$. Assume by contradiction that the group $K := \Gamma/\Gamma'$ is infinite. Denote by $\pi : \Gamma \rightarrow K$ the canonical group morphism. If $L$ is a normal subgroup of $K$, then $\pi^{-1}(L)$ is a normal subgroup of $\Gamma$ containing $\Gamma'$. Since $\Gamma'$ is maximal, in order to complete the proof of the lemma, i.e. to get a contradiction, it is enough to construct such a group $L$ with $L \neq \{1\}$ and $L \cap \pi(\text{Ker}(\rho)) = \{1\}$.

Since $\Gamma$ is unipotent, it admits a finite lower central series

$$\Gamma = \Gamma_0 \triangleright \Gamma_1 \triangleright \cdots \triangleright \Gamma_{m+1} = \{1\} \text{ with } \Gamma_{j+1} := [\Gamma, \Gamma_j].$$

As remarked above, all such subgroups $\Gamma_j$ of $\Gamma$ are finitely generated. Since $K$ is infinite, there is a $j$ such that $\pi(\Gamma_j)$ is infinite but $\pi(\Gamma_{j+1})$ is finite. So $\pi(\Gamma_j)/\pi(\Gamma_{j+1})$ is an infinite
abelian group which is finitely generated. Thus, there is an element $a \in \pi(\Gamma_j)$ which is of infinite order. Clearly, $a^m$ does not belong to $\pi(Ker(\rho))$ for $m \neq 0$ because $\ker(\rho)$ is finite.

**Claim 2.3.** For any $k \in K$, there is an integer $l > 0$ such that $ka^mlk^{-1} = a^ml$ for every $m$.

We prove the claim. Observe that $\pi(\Gamma_j)/\pi(\Gamma_{j+1})$ is contained in the center of $K/\pi(\Gamma_{j+1})$ by definition of $\Gamma$. Therefore, for every $p \geq 1$, there is $b_p \in \pi(\Gamma_{j+1})$ such that $ka^pb^{-1} = b_p a^p$. Since $\pi(\Gamma_{j+1})$ is finite, there are $p < q$ such that $b_p = b_q$. We then have for $l := q - p$

$$ka^l k^{-1} = (ka^p)(k^{-1})^l = a l.$$

This implies the claim for $m = 1$ and then for every $m$.

We return back to the proof of Lemma 2.2. Since $K$ is finitely generated, by the claim, we can find an $l \geq 1$ such that $ka^lk^{-1} = a^l$ for every $k \in K$. It follows that the group generated by $a^l$ is normal in $K$. Since it has a trivial intersection with $\pi(Ker(\rho))$, we get a contradiction. This ends the proof of Lemma 2.2.

Let $V$ be a real vector space of finite dimension and let $H$ be a unipotent Lie subgroup of $GL(V)$.

**Definition 2.4.** Define $V_0 := 0$ and then by induction $V_i$ as the space of all vector $v \in V$ such that $g(v) = v$ modulo $V_{i-1}$ for every $g \in H$. So $V_{i+1}/V_i$ is the set of all vectors in $V/V_i$ which are invariant by $H$ and $V_i$ is the intersection of $\ker(g - id)^i$ for $g \in H$. By Lie-Kolchin theorem, we have $V_i \neq V_{i-1}$ unless $V_{i-1} = V$, see [19, §17.6] or [22, Th.1.1]. Since the dimension of $V$ is finite there is a maximal integer $p$ such that $V_p \neq V$.

**Lemma 2.5.** Under the setting of Definition 2.4, let $i$ be an integer with $1 \leq i \leq p + 1$. Then we have $\|g^m\| = O(m^{-1})$ on $V_i$ for all $g \in H$. Moreover, given any vector $v$ in $V_i \setminus V_{i-1}$, we have $\|g^m(v)\| \simeq m^{-1}$ for general $g \in H$.

**Proof.** Consider a vector $v \in V_i$. Define $v_i := v$ and by induction $v_j := g(v_{j+1}) - v_{j+1}$ for $j = i - 1, i - 2, \ldots, 1$. By definition of $V_j$, we have $v_j \in V_j$ for every $j$. Moreover, a computation by induction gives

$$v_j = (g - id)^{i-j}(v) \quad \text{and} \quad g^m(v) = v_i + \left(\begin{array}{c} m \\ 1 \end{array}\right)v_{i-1} + \cdots + \left(\begin{array}{c} m \\ i - 1 \end{array}\right)v_1.$$

It follows that $\|g^m\| = O(m^{-1})$ on $V_i$.

Fix now a vector $v$ in $V_i \setminus V_{i-1}$. By definition of $V_j$, we have for some $g \in H$ that $(g - id)^{i-1}(v) \neq 0$ or equivalently $v_1 \neq 0$. It follows that the same property holds for general $g$ in $H$. We used here the fact that $H$ is connected because it is a unipotent Lie subgroup of $GL(V)$. The above expansion of $g^m(v)$ implies that $\|g^m(v)\| \simeq m^{-1}$ for general $g \in H$. This completes the proof of the lemma.

**Proposition 2.6.** Let $V$ be a real vector space of finite dimension and let $q \geq 0$ be an integer. Let $\Gamma$ be a unipotent subgroup of $GL(V)$, i.e., a subgroup whose elements are unipotent. Assume also that $\|g^m\| \simeq m^q$ for all $g \in \Gamma \setminus \{1\}$. Then $\Gamma$ is commutative.

**Proof.** Let $H$ denote the algebraic Zariski closure of $\Gamma$ in $GL(V)$. Since the characteristic polynomial of $g \in \Gamma$ is $(\lambda - 1)^{dim V}$, the same property holds for $g \in H$. We deduce that $H$ is a Lie subgroup of $GL(V)$ whose elements are unipotent. Let $p$ and $V_j$ be as in Definition 2.4. We deduce from Lemma 2.5 that $p = q$. 
Consider \( l, h \) in \( \Gamma \) and define \( g := lhl^{-1}h^{-1} \). We want to show that \( g = \text{id} \). By hypothesis, it is enough to check that \( \|g^m\| = o(m^p) \). Let \( v \) and \( v_j \) be as in the proof of Lemma 2.5. Since \( v_2 \) is a vector in \( V_2 \), there are vectors \( u_1 \) and \( u'_1 \) in \( V_1 \) such that \( l(v_2) = v_2 + u_1 \) and \( h(v_2) = v_2 + u'_1 \). Since \( H \) acts trivially on \( V_1 \), we deduce that \( l^{-1}(v_2) = v_2 - u_1 \) and \( h^{-1}(v_2) = v_2 - u'_1 \) and hence \( g(v_2) = v_2 \) or equivalently \( v_1 = 0 \) because \( v_1 = (g - \text{id})(v_2) \). Now, by the expansion of \( g^m(v) \) as in the proof of Lemma 2.5 the last term in this expansion vanishes. Thus, \( \|g^m(v)\| = o(m^p) \) for every \( v \in V \). The proposition follows. \( \square \)

Lemma 2.7. Let \( H \) be a unipotent group and let \( H' \) be a finite-index subgroup of \( H \). Then we have \( \ell(H') = \ell(H) \). In particular, we have \( \ell_{\min}(H) = \ell(H) \), see the notation at the end of Introduction.

Proof. By the definition, \( H \) can be regarded as a subgroup of the group of unipotent upper triangular matrices in \( \text{GL}(N, \mathbb{C}) \) for a suitable \( N \). Let \( \overline{H} \subseteq \text{GL}(N, \mathbb{C}) \) be the algebraic Zariski closure of \( H \). Then \( \overline{H} \) is still unipotent and hence connected since we are working over the characteristic zero, see [19, Exercise 15.5.6].

Let \( H \supseteq H^{(1)} \supseteq H^{(2)} \supseteq \cdots \) be the derived sequence of \( H \). Then, \( \overline{H}^{(i)} \) is equal to \( \overline{H}^{(i)} \), the algebraic Zariski closure of \( H^{(i)} \), see [27, Proof of Lemma 2.1(2)]). It follows that the derived length of \( H \) is equal to that of \( \overline{H} \).

Since \( H' \) is of finite-index in \( H \), so is the group \( \overline{H}' \) in the group \( \overline{H} \). Finally, since the latter two groups are both unipotent and hence connected, they are equal. Thus, we have \( \ell(H) = \ell(\overline{H}) = \ell(\overline{H}') = \ell(H') \). \( \square \)

The following lemma implies Theorem 1.2(1).

Lemma 2.8. Let \( X \) be a compact Kähler manifold of dimension \( n \). Let \( G \) be a subgroup of \( \text{Aut}(X) \) with only zero entropy elements. Then there is a finite-index subgroup \( G' \) of \( G \) satisfying the following properties for every \( 1 \leq p \leq n - 1 \).

1. The kernels of the canonical representations \( \rho_p : G' \to \text{GL}(H^{2p}(X, \mathbb{R})) \) and \( \rho_{p,p} : G' \to \text{GL}(H^{p,p}(X, \mathbb{R})) \) are both equal to \( G_0' \).

2. The images of both \( \rho_p \) and \( \rho_{p,p} \) are unipotent subgroups of \( \text{GL}(H^{2p}(X, \mathbb{R})) \) and \( \text{GL}(H^{p,p}(X, \mathbb{R})) \), respectively.

Proof. It is enough to prove the statement for a fixed \( p \) because we can deduce the lemma using a simple induction on \( p \). Let \( g \) be an automorphism of \( X \) of zero entropy. For every \( m \in \mathbb{Z} \), \( g^m \) also has zero entropy. Therefore, all eigenvalues of \( \rho_p(g^m) \) are of modulus less than or equal to 1. Since \( \rho_p(g^m) \) preserves the image of \( H^{2p}(X, \mathbb{Z}) \) in \( H^{2p}(X, \mathbb{R}) \), we deduce that the characteristic polynomial of \( \rho_p(g^m) \) belongs to a finite family of polynomials, as their coefficients are bounded. If \( \lambda \) is an eigenvalue of \( \rho_p(g) \) then \( \lambda^N \) is an eigenvalue of \( \rho_p(g^m) \) and hence belongs to a finite set which is independent of \( m \). Thus, there is an integer \( N \geq 1 \) such that \( \lambda^N = 1 \) for all eigenvalues of \( \rho_p(g) \).

According to Lemma 2.1, replacing \( G \) by a finite-index subgroup, we have that \( \rho_p(G) \) contains only unipotent elements of \( \text{GL}(H^{2p}(X, \mathbb{R})) \). So we have the property (2) in the lemma for both \( \rho_p \) and \( \rho_{p,p} \).

Observe once again that \( \rho_p(G) \) preserves the image of \( H^{2p}(X, \mathbb{Z}) \) in \( H^{2p}(X, \mathbb{R}) \). Let \( V \) denote the smallest complex vector subspace of \( H^{2p}(X, \mathbb{C}) \), containing \( H^{p,p}(X, \mathbb{R}) \), which
is spanned by vectors in $H^{2p}(X, \mathbb{Z})$. Then the intersection of $V$ with

$$H^{2p}(X, \mathbb{Z}) + iH^{2p}(X, \mathbb{Z})$$

is a lattice of $V$. Both $V$ and this lattice are preserved by the action of $G$.

The kernel $K$ of $\rho_{pp} : G \to \text{GL}(H^{p,p}(X, \mathbb{R}))$ is the set of $g \in G$ whose action on $H^{p,p}(X, \mathbb{R})$ is trivial. In particular, $K$ preserves the $p$-power of a Kähler class. By a generalized version of Fujiki-Lieberman’s theorem [8, Th. 2.1], the quotient $K/G_0$ is a finite group.

Finally, we apply Lemma 2.2 to $G := G/G_0$ and to the above vector space $V$. According to this lemma, we can replace $G$ by a finite-index subgroup and assume that $K = G_0$. This implies the property (1) for both $\rho_p$ and $\rho_{pp}$ because the kernel of $\rho_p$ contains $G_0$ and is contained in $K$. This ends the proof of the lemma. \qed

We now consider some properties of groups acting on compact Kähler manifolds. Lemma 2.9 below should be known and is certainly well-known in the algebraic setting. Indeed, if $X$ and $Y$ are both projective, we may apply [31, Th. 1.0.3] for the smooth morphisms $g : \Gamma \to \Gamma$ in the proof below, in order to obtain a $G$-equivariant projective resolution $\tilde{X}$ as in the lemma. For the sake of the completeness, we include the explicit statement and its proof in the Kähler setting here.

**Lemma 2.9.** Let $X$ and $Y$ be compact Kähler manifolds, $G$ a subgroup of $\text{Aut}(X)$ and $f : X \to Y$ a dominant meromorphic map. Assume that $f$ is $G$-equivariant, in the sense that there is a group homomorphism $\rho : G \to \text{Aut}(Y)$ such that $f \circ g = \rho(g) \circ f$ for all $g \in G$. Then, there are a compact Kähler manifold $\tilde{X}$ and a bimeromorphic morphism $\nu : \tilde{X} \to X$ such that $f \circ \nu : \tilde{X} \to Y$ is a $G$-equivariant surjective morphism. In particular, $\nu^{-1} \circ G \circ \nu$ is a subgroup of $\text{Aut}(\tilde{X})$ and is isomorphic to $G$.

**Proof.** Let $\Gamma \subset X \times Y$ be the Zariski closure of the graph of $f : X \to Y$. By our assumption, we have an inclusion $G \leq \text{Aut}(X \times Y)$ given by $g \mapsto (g, \rho(g))$. Then $g(\Gamma) = \Gamma$ for all $g \in G$. Therefore, by restricting the action of $G$ on $X \times Y$ to $\Gamma$, we obtain an inclusion $G \leq \text{Aut}(\Gamma)$. By construction, the projections $p_1 : \Gamma \to X$ and $p_2 : \Gamma \to Y$ are $G$-equivariant.

By taking a canonical resolution in [11, Th. 13.2], we also obtain a $G$-equivariant resolution $\pi : \tilde{X} \to \Gamma$ of $\Gamma$. Setting $X_0 := \Gamma$ and $\tilde{X} := X_m$, by [11, Th. 13.2], we find that the morphism $\pi$ is of the form

$$\pi = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_1,$$

where $\pi_i : X_i \to X_{i-1} (1 \leq i \leq m)$ is a blow-up along a smooth closed analytic subvariety $C_{i-1}$ of $X_{i-1}$.

Since $X_0 = \Gamma \subset X \times Y$, it follows that $X_1$ is a closed analytic subvariety of the blow-up $Z_1$ of $Z_0 := X \times Y$ along $C_{i-1}$. Here, $Z_0$ is a compact Kähler manifold by our assumption. Therefore, $Z_1$ is also a compact Kähler manifold by [3, Th. II 6]. Similarly, $X_2$ is a closed analytic subvariety of the blow-up $Z_2$ of $Z_1$ along $C_1$ and $Z_2$ is again a compact Kähler manifold by [3, Th. II 6]. Repeating this argument, we find that $\tilde{X} = X_m$ is a closed analytic subvariety of a compact Kähler manifold. Since $\tilde{X}$ is smooth, it is itself a compact Kähler manifold. This manifold $\tilde{X}$, together with the morphisms $\nu := p_1 \circ \pi : \tilde{X} \to X$, satisfy the lemma. \qed
The following lemma allows us to work with suitable bimeromorphically equivalent models. It is also very useful when treating the case of singular varieties.

**Lemma 2.10.** Let $\pi : X_1 \to X_2$ be a dominant meromorphic map between compact Kähler manifolds of the same dimension. Let $G$ be a group acting on both $X_1$ and $X_2$ bimeromorphically and $\pi$-equivariantly. Suppose $G|_{X_1}$ (or equivalently $G|_{X_2}$) is of zero entropy, i.e., their elements are zero entropy automorphisms. Then we have

$$\ell_{\text{ess}}(G|_{X_1}, X_1) = \ell_{\text{ess}}(G|_{X_2}, X_2).$$

Further, replacing $G$ by a finite-index subgroup, we have $(G|_{X_1})/(G|_{X_1})_0 \cong (G|_{X_2})/(G|_{X_2})_0$.

**Proof.** The equivalence of $G|_{X_i}$ being of zero entropy for $i = 1$ or $2$ is by [10, Th. 1.1]. Replacing $\pi : X_1 \to X_2$ by a $G$-equivariant resolution as in Lemma 2.9 we may assume that $\pi$ is a well-defined morphism. By Lemma 2.8 replacing $G$ by a finite-index subgroup, we may assume that the natural map $(G|_{X_i})/(G|_{X_i})_0 \to G|_{H^{1,1}(X_i, \mathbb{R})}$ is an isomorphism for $i = 1, 2$. Since the action of $G$ on $H^{1,1}(X_i, \mathbb{R})$ is unipotent, we deduce that $(G|_{X_i})/(G|_{X_i})_0$ contains no non-trivial element of finite order.

It is enough to show that the group homomorphisms $G \to G|_{H^{1,1}(X_i, \mathbb{R})}$ have the same kernel for $i = 1, 2$. Consider an element $g$ of $G$ and denote by $g_i$ its action as an automorphism on $X_i$ for $i = 1, 2$. These automorphisms are related by the identity

$$\pi \circ g_1 = g_2 \circ \pi.$$

We only need to check that $g_1$ acts trivially on $H^{1,1}(X_1, \mathbb{R})$ if and only if $g_2$ acts trivially on $H^{1,1}(X_2, \mathbb{R})$.

Fix a Kähler form $\omega$ on $X_2$. It follows from the equality above that

$$g_1^* \pi^*(\omega) = \pi^* g_2^*(\omega).$$

Note that $\pi^*(\omega)$ and $\pi^* g_2^*(\omega)$ are smooth positive closed $(1, 1)$-forms as $\pi$ is a well-defined morphism. We deduce from the last identity that

$$g_1^* \pi^* \{\omega\} = \pi^* g_2^* \{\omega\},$$

where $\{\omega\}$ denotes the class of $\omega$ in $H^{1,1}(X_2, \mathbb{R})$.

Assume that the action of $g_2$ on $H^{1,1}(X_2, \mathbb{R})$ is trivial. We have

$$g_1^* \pi^* \{\omega\} = \pi^* \{\omega\}.$$

Since $\omega$ is Kähler, the class $\pi^* \{\omega\}$ is big, see e.g. [6, p. 1253] for this fact and the definition of big class in the Kähler setting. By [8, Cor. 2.2], a power of $g_1$ belongs to $(G|_{X_1})_0$. It follows that $g_1$ belongs to $(G|_{X_1})_0$ and hence acts trivially on $H^{1,1}(X_1, \mathbb{R})$ because $(G|_{X_i})/(G|_{X_i})_0$ contains no non-trivial element of finite order.

Assume now that the action of $g_1$ on $H^{1,1}(X_1, \mathbb{R})$ is trivial. We need to show a similar property for $g_2$. For any class $c \in H^{1,1}(X_2, \mathbb{R})$, we have

$$\pi^*(c) = g_1^*(\pi^*(c)) = \pi^*(g_2^*(c))$$

for the same reason as above. Applying $\pi_*$ to the above equality and noting $\pi_* \pi^* = (\deg \pi) \operatorname{id}$, we get $g_2^*(c) = c$. Hence, $g_2$ acts trivially on $H^{1,1}(X_2, \mathbb{R})$. \hfill $\square$

Finally, we close this section with the following useful result. We refer the reader to [15, Prop. 5.10] or [24] for a proof.
Lemma 2.11. Let $X$ be a compact Kähler manifold which is not uniruled. Then $\text{Aut}_0(X)$ is a compact complex torus.

Note that an algebraic version of this result, for non-ruled smooth projective varieties, is a direct consequence of the structure theorem of algebraic group, due to Chevalley, together with [30, Th. 14.1].

3. Main derived length estimate

In this section, we will give a main estimate of derived length for unipotent groups of automorphisms. This is a weak version of Theorem 1.2 (2). For the proof, we will use in an essential way a construction of special invariant cohomology classes.

Proposition 3.1. Let $X$ and $G$ be as in Theorem 1.2. Then we have

$$\ell_{\text{ess}}(G, X) \leq n - 1.$$  

We first deduce the following corollary, which may have its own interest. Though we will not use this corollary itself in this paper, we will use a similar result in Proposition 5.5 for singular projective varieties in Section 5.

Corollary 3.2. Let $X$ be a compact Kähler manifold of dimension $n \geq 1$. Let $G \leq \text{Aut}(X)$ be a group of zero entropy automorphisms. Suppose that $\text{Aut}_0(X)$ is commutative (for instance, this holds when $X$ is not uniruled, see Lemma 2.11). Then

1. We have the following (in)equalities

$$\ell_{\text{min}}(G) \leq \ell_{\text{min}}(G/\text{G}_0) + 1 = \ell_{\text{min}}(G|_{H^2(X, R)}) + 1 = \ell_{\text{ess}}(G, X) + 1 \leq n.$$  

2. If $\ell_{\text{min}}(G) = n$, then we have

$$G_0 \neq \{1\} \quad \text{and} \quad \ell_{\text{ess}}(G, X) = n - 1.$$  

Proof. By Lemma 2.8 and Proposition 3.1, replacing $G$ by a finite-index subgroup, we may assume that the natural map $G/G_0 \to G|_{H^2(X, R)}$ is an isomorphism with image a unipotent group of derived length $\leq n - 1$. Then (1) follows from the fact that $G_0 \leq \text{Aut}_0(X)$ is commutative. (2) follows from (1) as the inequalities in (1) have to be equalities by the assumption.  

Thanks to Lemmas 2.7 and 2.8, in order to prove Theorem 1.2 (2) and Proposition 3.1, we can, for simplicity, replace $G$ by a suitable finite-index subgroup and assume that the properties (1) and (2) in Lemma 2.8 hold for $G$ instead of $G'$. As mentioned above, we will use in an essential way a construction of special invariant cohomology classes.

Let $\mathcal{K}_i(X)$ denote the set of classes in $H^{i,i}(X, R)$ that can be approximated by classes of smooth positive closed $(i, i)$-forms, i.e., $\mathcal{K}_i(X)$ is the closure of the open convex cone generated by the classes of smooth strictly positive closed $(i, i)$-forms in $H^{i,i}(X, R)$. This is a salient (that is, no line contained) convex closed cone with non-empty interior which is invariant by $\text{Aut}(X)$. We need to study the action of automorphisms on this cone. The following lemma will be used later.

Lemma 3.3. Let $L$ be a class in $\mathcal{K}_i(X) \setminus \{0\}$ with $0 \leq i \leq n - 1$. Let $g$ be an automorphism such that its action on $H^{i+1,i+1}(X, R)$ is unipotent and $g(L) = L$. Assume moreover that $\|g^m\| = O(m)$ on $L \cdot H^{1,1}(X, R)$. Then $g = \text{id}$ on $L \cdot H^{1,1}(X, R)$. 

Proof. By hypothesis, the Jordan canonical form of the action of \( g \) on \( L \cdot H^{1,1}(X, \mathbb{R}) \) contains only Jordan blocks of size \( 1 \times 1 \) or \( 2 \times 2 \). Assume by contradiction that \( g \neq \text{id} \) on \( L \cdot H^{1,1}(X, \mathbb{R}) \). We deduce that \( \|g^m\| \simeq m \) on \( L \cdot H^{1,1}(X, \mathbb{R}) \) and at least one of the above Jordan blocks is of size \( 2 \times 2 \). We also see that \( \|g^m(Lv)\| \simeq m \) for \( v \in H^{1,1}(X, \mathbb{R}) \) such that \( g(Lv) \neq Lv \). In particular, this property holds for \( v \) outside some proper linear subspace of \( H^{1,1}(X, \mathbb{R}) \).

Consider a general Kähler class \( c \). We have \( g(Lc) \neq Lc \) and \( \|g^m(Lc)\| \simeq m \). Using the above description of the Jordan canonical form of the action of \( g \), we can write
\[
g(Lc) = Lc + LM
\]
for some class \( M \in H^{1,1}(X, \mathbb{R}) \) with \( LM \neq 0 \) and \( g(LM) = LM \). Iterating the identity \( g(Lc) = Lc + LM \) gives
\[
g^m(Lc) = Lc + mLM
\]
or equivalently
\[
g^m(c) = c + mM + U_m
\]
for some class \( U_m \) such that \( LU_m = 0 \).

We deduce from the last identity that for every \( k \geq 1 \)
\[
Lg^m(c^k) = m^kLM^k + o(m^k) \quad \text{as} \quad m \to \infty.
\]
Choose the maximal integer \( k \leq n - i \) such that \( LM^k \neq 0 \). The class \( LM^k \) belongs to \( \mathcal{K}_{i+k}(X) \) because it is the limit of the classes \( m^{-k}L \cdot g^m(c^k) \) which are in \( \mathcal{K}_{i+k}(X) \). Finally, using the maximality of \( k \), we get
\[
m^{-k}g^m(Lc^{n-i}) = m^{-k}L \cdot g^m(c^{n-i}) = \left(\frac{n-i}{k}\right)LM^k c^{n-i-k} + o(1).
\]

Observe that \( \dim H^{n,n}(X, \mathbb{R}) = 1 \) and \( \text{Aut}(X) \) preserves \( H^{n,n}(X, \mathbb{Z}) \) and \( \mathcal{K}_n(X) \). We deduce that \( \text{Aut}(X) \) acts trivially on \( H^{n,n}(X, \mathbb{R}) \). So the left hand side of the last sequence of equalities tends to \( 0 \). It follows that \( LM^k c^{n-i-k} = 0 \). On the other hand, since \( LM^k \) is a non-zero class in \( \mathcal{K}_{i+k}(X) \), it can be represented by a non-zero positive closed current. As \( c \) is a Kähler class, \( LM^k c^{n-i-k} \) is represented by a non-zero positive measure and it cannot vanish. This is a contradiction which ends the proof of the lemma. \( \square \)

For any class \( L \in \mathcal{K}_i(X) \setminus \{0\} \), denote by \( \text{Nef}(L) \) the closure of the cone \( L \cdot \text{Nef}(X) \), where \( \text{Nef}(X) := \mathcal{K}_1(X) \) is the closure of the Kähler cone in \( H^{1,1}(X, \mathbb{R}) \). This is a salient convex closed cone in the vector space \( L \cdot H^{1,1}(X, \mathbb{R}) \) which is contained in the cone \( \mathcal{K}_{i+1}(X) \) since the last cone and the space \( L \cdot H^{1,1}(X, \mathbb{R}) \) are closed in \( H^{i+1,i+1}(X, \mathbb{R}) \).

**Lemma 3.4.** There is a sequence of classes \( L_i \in \mathcal{K}_i(X) \setminus \{0\} \) for \( 0 \leq i \leq n \) such that

1. \( L_0 = 1 \) and \( L_{i+1} \in \text{Nef}(L_i) \) for every \( 0 \leq i \leq n - 1 \). In particular, there is a class \( M_{i+1} \in H^{1,1}(X, \mathbb{R}) \) such that \( L_{i+1} = L_iM_{i+1} \);
2. \( L_i \) is invariant by \( G \) for every \( 0 \leq i \leq n \).

**Proof.** We construct the sequence by induction. Assume that \( L_0, \ldots, L_i \) are already constructed and satisfy the above properties (1) and (2). We see that \( \text{Nef}(L_i) \) is invariant by \( G \). Since \( G \) is unipotent, it is solvable. Therefore, by a version of Lie-Kolchin’s theorem for cone [22, Th. 1.1], there is a ray in \( \text{Nef}(L_i) \) which is preserved by \( G \). Choose \( L_{i+1} \) in this ray. Since the eigenvalues of the action of \( g \) on \( H^{i+1,i+1}(X, \mathbb{R}) \) are 1 for \( g \in G \), we deduce that \( L_{i+1} \) is invariant by \( G \). This completes the proof of the lemma. \( \square \)
From now, we fix some classes $L_i$ in Lemma 3.4 for $0 \leq i \leq n$. Denote by $H_i$ the group of all $g \in G$ such that $g = \text{id}$ on $L_i \cdot H^{1,1}(X, \mathbb{R})$. Since $L_{i+1} = L_i M_{i+1}$, we see easily that the sequence $H_i$ is increasing, $H_i$ is normal in $G$ and hence in $H_j$ for $i < j$. Moreover, $G/H_i$ acts faithfully on $L_i \cdot H^{1,1}(X, \mathbb{R})$. Observe that $H_0 = G_0$ as we assume Lemma 2.8 holds for $G$ instead of $G'$. So if $H_0 = G$ or equivalently if $G$ acts trivially on $H^{1,1}(X, \mathbb{R})$, then Theorem 1.2(2) and Proposition 3.1 are obvious.

Lemma 3.5. Assume that $H_0 \neq G$. Then there is an integer $0 \leq l \leq n - 2$ such that $H_l \neq G$ and $H_{l+1} = G$.

Proof. Recall that $\text{Aut}(X)$ acts trivially on $H^{n,n}(X, \mathbb{R})$. It follows that $H_{n-1} = G$. The lemma follows easily. \qed

Proposition 3.6. Let $g$ be an element of $H_{i+1}$ such that $g \notin H_i$ for some $0 \leq i \leq l$. Then we have $\|g^n\| \simeq m^2$ on $L_i \cdot H^{1,1}(X, \mathbb{R})$.

Using this result, we first complete the proof of Proposition 3.4.

End of the proof of Proposition 3.4. Recall that we have already replaced $G$ by a finite-index subgroup so that Properties (1) and (2) in Lemma 2.8 are satisfied for $G$ instead of $G'$. We can also assume that $H_0 \neq G$ as in Lemma 3.5. Since $l \leq n - 2$, it is enough to show that $\ell(G/G_0) \leq l + 1$.

Since $H_0 = G_0$ and $H_{i+1} = G$, in order to complete the proof, it is enough to show that $H_{i+1}/H_i$ is commutative. Recall that this group acts faithfully on the vector space $V := L_i \cdot H^{1,1}(X, \mathbb{R})$. Let $\Gamma$ denote the image of the representation of $H_{i+1}/H_i$ in $\text{GL}(V)$. We need to check that $\Gamma$ is commutative. But by Proposition 3.6 this is a direct consequence of Proposition 2.6 applied to $\Gamma, V$ and $q := 2$. This proves Proposition 3.4 assuming Proposition 3.6. \qed

In the rest of this section, we give the proof of Proposition 3.6. We need to introduce some notation and auxiliary results. Let $F_i$ denote the set of all classes $c \in H^{1,1}(X, \mathbb{R})$ such that $L_i c = 0$. We deduce from the properties of $L_i$ that the sequence $F_i$ is increasing. Moreover, $H^{1,1}(X, \mathbb{R})/F_i$ is naturally isomorphic to $L_i \cdot H^{1,1}(X, \mathbb{R})$.

Consider Kähler classes $c_1, \ldots, c_{n-1}$ and define the quadratic form $Q$ on $H^{1,1}(X, \mathbb{R})$ by

$$Q(M, M') := L_i M M' c_1 \ldots c_{n-1} \quad \text{for} \quad M, M' \in H^{1,1}(X, \mathbb{R}).$$

Here, the right hand side of the identity is a real number since $H^{n,n}(X, \mathbb{R})$ is canonically identified with $\mathbb{R}$. By the definition of $F_i$, this quadratic form induces a quadratic form on $H^{1,1}(X, \mathbb{R})/F_i$ that we still denote by $Q$. Consider another Kähler class $c_{n-i-1}$ and the primitive space

$$P := \{ M \in H^{1,1}(X, \mathbb{R}) : L_i M c_1 \ldots c_{n-1} = 0 \}.$$ 

Since $L_i$ belongs to $K_i(X) \setminus \{0\}$, we have $L_i c_1 \ldots c_{n-1-1} \neq 0$. Therefore, by Poincaré duality, $P$ is a hyperplane of $H^{1,1}(X, \mathbb{R})$. Observe that this hyperplane contains $F_i$, hence $P/F_i$ is a hyperplane of $H^{1,1}(X, \mathbb{R})/F_i$.

Lemma 3.7. The quadratic form $Q$ is negative semi-definite on $P/F_i$. Moreover, $Q$ is also negative semi-definite on $F_{i+1}/F_i$.

Proof. If we replace $L_i$ by a product of Kähler classes, then by the mixed version of Hodge-Riemann theorem, $Q$ is negative definite on $P$, see [9, 16]. By definition, $L_i$ is a limit
of classes which are products of Kähler classes. Thus, by continuity, we obtain the first assertion in the lemma.

Define

\[ P' := \{ M \in H^{1,1}(X, \mathbb{R}) : L_{i+1}Mc_1 \ldots c_{n-i-2} = 0 \}. \]

Since the class \( L_{i+1} \) belongs to \( K_{i+1}(X) \setminus \{0\} \) and the classes \( c_k \) are Kähler, we have \( L_{i+1}c_1 \ldots c_{n-i-2} \neq 0 \). Then, by Poincaré duality, \( P' \) is also a hyperplane of \( H^{1,1}(X, \mathbb{R}) \).

Recall that \( L_{i+1} \) can be approximated by classes of type \( L_i c \) with \( c \) Kähler. Therefore, we can approximate \( P' \) by \( P \) using suitable Kähler class \( c_{n-i-1} \). Finally, since \( Q \) is negative semi-definite on \( P \), by continuity, it is negative semi-definite on \( P' \) as well. This implies the second assertion of the lemma because \( F_{i+1} \) is contained in \( P' \).

Lemma 3.8. Let \( M \) and \( M' \) be linearly independent classes in \( H^{1,1}(X, \mathbb{R})/F_1 \) such that

\[ Q(M, M) = Q(M, M') = Q(M', M') = 0 \quad \text{and} \quad L_iM \in K_{i+1}(X) \setminus \{0\}. \]

Then there is a vector \((a, b) \in \mathbb{R}^2 \setminus \{0\} \), unique up to a multiplicative constant, such that

\[ L_i(aM + bM')c_1 \ldots c_{n-i-2} = 0. \]

Moreover, if \( \Pi \) is the plane spanned by \( M \) and \( M' \), then \( \Pi \cap (P/F_i) \) is the real line containing the vector \( aM + bM' \).

Proof. Since the classes \( c_k \) are Kähler and \( L_iM \in K_{i+1}(X) \setminus \{0\} \), we have

\[ L_iM c_1 \ldots c_{n-i-1} \neq 0 \quad \text{and} \quad L_iM c_1 \ldots c_{n-i-2} \neq 0. \]

Therefore, we have \( M \notin P/F_i \) and \( (a, b) \) is unique up to a multiplicative constant. We also obtain that \( \Pi \cap (P/F_i) \) is a real line. Let \( N := aM + bM' \) be a vector in this real line. It is enough to show that this vector satisfies the identity in the lemma.

By Poincaré duality, we only have to show that \( Q(N, N') = 0 \) for every \( N' \in H^{1,1}(X, \mathbb{R})/F_i \). By hypothesis, we have this property for \( N' \in \Pi \). It suffices now to check the same property for \( N' \in (P/F_i) \). Observe that \( Q(N, N) = 0 \) since \( N \in \Pi \cap (P/F_i) \). Using Lemma 3.7 and Cauchy-Schwarz inequality, we have for \( N' \in (P/F_i) \)

\[ Q(N, N')^2 \leq Q(N, N) \cdot Q(N', N') = 0. \]

The lemma follows.

We continue the proof of Proposition 3.6. In order to simplify the notation, we work with the group \( H_{i+1}/H_i \) which acts faithfully on \( L_i \cdot H^{1,1}(X, \mathbb{R}) \) or equivalently on \( H^{1,1}(X, \mathbb{R})/F_i \). Fix an element \( g \) of \( H_{i+1}/H_i \) which is different from the identity. We need to show that \( \|g^m\| \simeq m^2 \) on \( L_i \cdot H^{1,1}(X, \mathbb{R}) \) or equivalently on \( H^{1,1}(X, \mathbb{R})/F_i \).

Since the action of \( g \) on \( H^{1,1}(X, \mathbb{R})/F_i \) is unipotent, there is an integer \( q \geq 0 \) such that \( \|g^m\| \simeq m^q \) on \( H^{1,1}(X, \mathbb{R})/F_i \). This can be seen using the Jordan canonical form of the action of \( g \). It suffices to check that \( q = 2 \) and by Lemma 3.3, we only have to show that \( q \leq 2 \).

Assume the contrary that \( q \geq 3 \). Fix a general Kähler class \( c \). By definition of \( q \), since \( c \) is general, we have \( \|g^m(c)\| \simeq m^q \). In fact, using the positivity, we can show the last property for every Kähler class \( c \). Observe also that since \( g = id \) on \( H^{1,1}(X, \mathbb{R})/F_{i+1} \), we have that \( g^m(c) - c \) belongs to \( F_{i+1}/F_i \) for every \( m \).
Lemma 3.9. The classes $M, M'$ and $M''$ belong to $F_{i+1}/F_i$. Moreover, we have $L_i M \in K_{i+1}(X) \setminus \{0\}$, $g(M) = M$ and $g(M') = M + M'$. In particular, $M$ and $M'$ are linearly independent.

Proof. Clearly, the last assertion is a consequence of the earlier ones. Since $\|g^m(c)\| \simeq m^q$, we deduce from the above expansion of $g^m(c)$ that $M \neq 0$ in $H^{1,1}(X, \mathbb{R})/F_i$ and hence $L_i M \neq 0$. We also deduce from this expansion that

$$L_i M = q! \lim_{m \to \infty} m^{-q} L_i \cdot g^m(c),$$

which implies that $L_i M \in K_{i+1}(X) \setminus \{0\}$.

Moreover, we have

$$M = q! \lim_{m \to \infty} m^{-q} g^m(c) = q! \lim_{m \to \infty} m^{-q} (g^m(c) - c).$$

Since $g^m(c) - c$ belongs to $F_{i+1}/F_i$, the class $M$ also belongs to $F_{i+1}/F_i$. Similarly, if $q \geq 2$ as in our case, we have

$$M' = (q-1)! \lim_{m \to \infty} m^{-q+1} \left[ g^m(c) - \left( \frac{m}{q} \right) M \right] = (q-1)! \lim_{m \to \infty} m^{-q+1} \left[ g^m(c) - c - \left( \frac{m}{q} \right) M \right]$$

which implies that $M'$ belongs to $F_{i+1}/F_i$ as well. In the same way, if $q \geq 3$ as in our case, we can check that $M''$ belongs to $F_{i+1}/F_i$.

Now, from the above identities for $M$ and $M'$, we have

$$g(M) = q! \lim_{m \to \infty} m^{-q} g^{m+1}(c) = q! \lim_{m \to \infty} \left( \frac{m+1}{m} \right)^q (m+1)^{-q} g^{m+1}(c) = M$$

and using $g(M) = M$ we obtain that $g(M')$ is equal to

$$(q-1)! \lim_{m \to \infty} m^{-q+1} \left[ g^{m+1}(c) - \left( \frac{m}{q} \right) g(M) \right]$$

$$= \lim_{m \to \infty} \left( (q-1)! m^{-q+1} \left[ g^{m+1}(c) - \left( \frac{m+1}{q} \right) M \right] \right) + (q-1)! m^{-q+1} \left( \left( \frac{m+1}{q} \right) - \left( \frac{m}{q} \right) \right)$$

$$= M' + M.$$  

This completes the proof of the lemma.

End of the proof of Proposition 3.6. By Lemmas 3.7 and 3.9 we have

$$Q(M, M) \leq 0 \quad \text{and} \quad Q(M', M') \leq 0.$$  

On the other hand, using the above expansion of $g^m(c)$, we have

$$Q(g^m(c), g^m(c)) = (q!^{-2} + o(1)) m^{2q} Q(M, M) + 2(q!^{-1}(q-1)!^{-1} + o(1)) m^{2q-1} Q(M, M')$$

$$+ m^{2q-2} ((q-1)!^{-2} Q(M', M') + 2q!^{-1}(q-2)!^{-1} Q(M, M'')) + o(m^{2q-2}).$$

Since $g^m(c)$ is a Kähler class, the left hand side is non-negative. Therefore, we necessarily have $Q(M, M) = 0$. Using this, Lemma 3.7 and Cauchy-Schwarz inequality, we obtain that
Q(M, N) = 0 for all \( N \in F_{i+1}/F_i \). In particular, we have \( Q(M, M') = Q(M, M'') = 0 \).

Hence, using again the above expansion of \( Q \) to a multiplicative constant, such that

\[
L_i(aM + bM')c_1 \ldots c_{n-i-2} = 0.
\]

Also by the second assertion of Lemma \ref{lem:3.8}, this \((a, b)\) is the unique (up to a multiplicative constant) solution of the equation

\[
L_i(aM + bM')c_1 \ldots c_{n-i-1} = 0.
\]

So Equation (B) is equivalent to Equation (A) which can be obtained from (B) by removing the factor \( c_{n-1-1} \). Since this property holds for arbitrary Kähler classes \( c_1, \ldots, c_{n-1-1} \), Equation (B) is equivalent to any equation obtained from (B) by removing a factor \( c_j \)

\[
L_i(aM + bM')c_1 \ldots c_{j-1}c_{j+1} \ldots c_{n-2}c_{n-1-1} = 0.
\]

We conclude that (A) and (A') are equivalent. In other words, Equation (A) remains

\[
L_i(aM + bM')c_1 \ldots c_{n-i-2} = 0
\]

for all Kähler classes \( c_1, \ldots, c_{n-i-2} \). Since Kähler classes span \( H^{1,1}(X, \mathbb{R}) \), we obtain

\[
L_i(aM + bM')H^{1,1}(X, \mathbb{R})^{n-i-2} = 0.
\]

Finally, using the action of \( g \) which preserves \( L_i \), we get

\[
L_i((a + b)M + bM')H^{1,1}(X, \mathbb{R})^{n-i-2} = 0
\]
or equivalently

\[
L_i((a + b)M + bM')H^{1,1}(X, \mathbb{R})^{n-i-2} = 0
\]
as \( g(M) = M \) and \( g(M') = M + M' \), according to Lemma \ref{lem:3.9} We conclude that

\[
L_iMH^{1,1}(X, \mathbb{R})^{n-i-2} = 0
\]

This is a contradiction because \( L_i M \) is a class in \( K_{i+1}(X) \) \( \{0\} \) and therefore, we have \( L_i Mc_1 \ldots c_{n-i-2} \neq 0 \) for all Kähler classes \( c_1, \ldots, c_{n-i-2} \). The proof of Proposition \ref{prop:3.6} is now complete.

\section*{4. Proofs of Theorem \ref{thm:1.2}(2)(3) and Theorem \ref{thm:1.3}}

In this section, we complete the proof of Theorem \ref{thm:1.2} and also prove Theorem \ref{thm:1.3}. First, we prove the following general result.

\begin{proposition}
Let \( \varphi : Z \to Y \) be a surjective morphism between compact Kähler manifolds. For all \( y \in Y \), denote by \( F_y := \varphi^{-1}(y) \) the fibre over \( y \) and assume that \( F_y \) is connected. Let \( W \) be the set of all \( y \in Y \) such that \( F_y \) is a regular fibre and \( \text{Aut}_0(F_y) \) is an abelian group. Assume moreover that \( \text{Aut}_0(Z) \) is an abelian group contained in \( \text{Aut}(Z/Y) \) and that \( W \) is not contained in any countable union of proper analytic subsets of \( Y \). Let \( G \) be any subgroup of zero entropy of \( \text{Aut}(Z) \) which is contained in \( \text{Aut}(Z/Y) \). Then for every regular fibre \( F_y \), we have

\[
\ell_{ess}(G, Z) \leq 1 + \ell_{ess}(G|_{F_y}, F_y) \leq \dim Z - \dim Y.
\]
\end{proposition}
Moreover, if \( \ell_{\text{ess}}(G, Z) = \dim Z - \dim Y \), then \( \Aut_0(F_y) \neq \{1\} \) for a very general fibre \( F_y \).

We now prove Proposition 4.1. Recall that \( \Aut(Z) \) has countably many connected components. If \( g \) is an element of \( \Aut(Z) \) then the component of \( g \) is \( \Aut_0(Z) g \), see \[15\] \[24\]. Since \( \Aut_0(Z) \) is contained in \( \Aut(Z/Y) \), the group \( \Aut(Z/Y) \) is also a countable union of connected components of \( \Aut(Z) \).

For any subset \( E \) of \( Y \), define

\[
\Aut_E(Z/Y) := \{ g \in \Aut(Z/Y) : g|_{F_t} \in \Aut_0(F_t) \quad \text{for} \quad t \in E \quad \text{with} \quad F_t \text{ regular} \}.
\]

Observe that the restriction of \( \Aut_0(Z) \) to a regular fibre \( F_t \) is a connected group and hence contained in \( \Aut_0(F_t) \). Therefore, \( \Aut_E(Z/Y) \) is a countable union of connected components of \( \Aut(Z) \). Moreover, by hypotheses, the commutators of \( \Aut_E(Z/Y) \) are identity on \( \varphi^{-1}(E \cap W) \). So if \( E \cap W \) is not contained in a proper analytic subset of \( Y \), then these commutators are identity on \( Z \), or equivalently, \( \Aut_E(Z/Y) \) is an abelian subgroup of \( \Aut(Z/Y) \).

Consider the group

\[
\Aut_0^+(Z/Y) := \{ g \in \Aut(Z/Y) : g|_{H^2(F_t, \mathbb{R})} = \text{id} \quad \text{for some regular fibre} \quad F_t \}.
\]

Since \( g|_{F_t} \) preserves \( H^2(F_t, \mathbb{Z}) \), by continuity, the property \( g|_{H^2(F_t, \mathbb{R})} = \text{id} \) does not depend on the choice of the regular fibre \( F_t \). Equivalently, we have

\[
\Aut_0^+(Z/Y) = \{ g \in \Aut(Z/Y) : g|_{H^2(F_t, \mathbb{R})} = \text{id} \quad \text{for all regular fibres} \quad F_t \}.
\]

This is a group of zero entropy (cf. \[10\]) and is a countable union of some connected components of \( \Aut(Z/Y) \). Note also that if \( E \) contains a point \( t \) such that \( F_t \) is a regular fibre, then \( \Aut_E(Z/Y) \) is a subgroup of \( \Aut_0^+(Z/Y) \). We will consider the sets \( E \) with this property. We need the following result.

**Lemma 4.2.** Under the assumption of Proposition 4.1, there are an increasing sequence of subsets \( E_k \) of \( Y \) and a decreasing sequence of finite-index subgroups \( G_k \) of \( G \), for \( k = 1, 2, \ldots \), such that

1. the union of \( E_k \) is the set of all \( y \in Y \) such that \( F_y \) is a regular fibre; and
2. \( G_k \cap \Aut_0^+(Z/Y) \) is contained in \( \Aut_{E_k}(Z/Y) \).

Assuming Lemma 4.2 for the time being, we first complete the proof of Proposition 4.1.

**Proof of Proposition 4.1.** For the first assertion of this proposition, we fix a regular fibre \( F_y \). By Theorem 1.2(1), applied to \( Z \), we can replace \( G \) by a finite-index subgroup in order to assume that \( \ell_{\text{ess}}(G, Z) = \ell(G/G_0) \), see also Lemma 2.7. Applying again this theorem to \( F_y \), we get a similar identity for \( H := G|_{F_y} \), that is, \( \ell_{\text{ess}}(H, F_y) = \ell(H/H_0) \).

Fix an integer \( k \) large enough so that \( W \cap E_k \) is not contained in a proper analytic subset of \( Y \) and that \( y \) belongs to \( E_k \). We use here the hypothesis that \( W \) is not contained in a countable union of proper analytic subsets of \( Y \). We have seen that for such a choice of \( E_k \), the group \( \Aut_{E_k}(Z/Y) \) is abelian and contained in \( \Aut_0^+(Z/Y) \). Therefore, \( G_{E_k} := G \cap \Aut_{E_k}(Z/Y) \) is an abelian subgroup of \( G \) which contains \( G_0 \).

By Lemma 4.2, replacing \( G \) by the finite-index subgroup \( G_k \), we may assume that

\[
G_{E_k} = G \cap \Aut_0^+(Z/Y) = \pi^{-1}(H_0),
\]
where \( \pi : G \to H \) is the natural homomorphism. We use here that \( y \) belongs to \( E_k \) and \( \text{Aut}_{E_k}(Z/Y) \subset \text{Aut}_{\pm}(Z/Y) \). It follows that
\[
\ell_{\text{ess}}(G, Z) = \ell(G/G_0) \leq 1 + \ell(G/G_{E_k}) = 1 + \ell(H/H_0) = 1 + \ell_{\text{ess}}(H, F_y).
\]
On the other hand, we deduce from Proposition 3.1, applied to \( H \) acting on \( F_y \), that
\[
1 + \ell_{\text{ess}}(H, F_y) \leq \dim F_y = \dim Z - \dim Y.
\]
The first assertion in the proposition follows easily.

For the second assertion, assume that \( \ell_{\text{ess}}(G, Z) = \dim Z - \dim Y \). So the above inequalities are now equalities. Remember this property only holds when we replace \( G \) by \( G_k \) with \( k \) large enough. So we deduce that the groups
\[
G_k \cap \text{Aut}_{E_k}(Z/Y)
\]
are not trivial for \( k \) large enough. Note that this sequence of groups is decreasing. Hence,
\[
V_k := \{ y \in Y : g|_{F_y} = \text{id} \text{ for all } g \in G_k \cap \text{Aut}_{E_k}(Z/Y) \}
\]
is an increasing sequence of proper analytic subsets of \( Y \).

Consider a regular fibre \( F_y \) with \( y \) outside the union of \( V_k \). Fix an integer \( k \) large enough as in the proof of the first assertion. Since \( y \) does not belong to \( V_k \), the restriction of \( G_k \cap \text{Aut}_{E_k}(Z/Y) \) to \( F_y \) is non-trivial. Then, by definition of \( \text{Aut}_{E_k}(Z/Y) \), we get \( \text{Aut}_0(F_y) \neq \{1\} \). This ends the proof of Proposition 4.1 (modulo Lemma 4.2).

**Proof of Lemma 4.2.** For simplicity, we can replace \( G \) by the group generated by \( G \) and \( \text{Aut}_0(Z) \) in order to assume that \( \text{Aut}_0(Z) \subset G \) and therefore, \( G \) is a countable union of connected components of \( \text{Aut}(Z/Y) \). All subgroups of \( \text{Aut}(Z) \) considered below satisfy the same property. We need the following result.

**Claim 4.3.** \( G \) admits only countably many finite-index subgroups.

We first prove Claim 4.3. Recall that \( G \) is a Lie group and its component of the identity is \( \text{Aut}_0(Z) \). So any finite-index subgroup of \( G \) contains \( \text{Aut}_0(Z) \). Consider the natural representation
\[
G/\text{Aut}_0(Z) \to \text{GL}(H^2(Z, \mathbb{R})).
\]
Its kernel is finite, see [15] [24]. The action of \( G \) on \( H^2(Z, \mathbb{R}) \) preserves the lattice \( H^2(Z, \mathbb{Z}) \). Therefore, by Lemma 2.2, \( G/\text{Aut}_0(Z) \) admits only countably many subgroups, or equivalently, \( G \) admits only countably many subgroups containing \( \text{Aut}_0(Z) \). Claim 4.3 follows.

We return back to the proof of Lemma 4.2. Denote the finite-index subgroups of \( G \) by \( \Gamma_1, \Gamma_2, \Gamma_3, \ldots \), and define
\[
G_k := \Gamma_1 \cap \ldots \cap \Gamma_k.
\]
This is a decreasing sequence of finite-index subgroups of \( G \). Define also
\[
E_k := \{ t \in Y : F_t \text{ is a regular fibre and } g|_{F_t} \in \text{Aut}_0(F_t) \text{ for } g \in G_k \cap \text{Aut}_{\pm}(Z/Y) \}.
\]
This is an increasing sequence of subsets of \( Y \) and the assertion (2) in the lemma holds for our choice of \( E_k \) and \( G_k \). It remains to show that given a regular fibre \( F_y \), we have \( y \in E_k \) for some \( k \).

By Theorem 1.3(1), applied to the action of \( G|_{F_y} \) on \( F_y \), there is a finite-index subgroup of \( G \), i.e., a group \( \Gamma_k \) for some \( k \), such that the kernel of the natural representation \( \Gamma_k|_{F_y} \to \text{GL}(H^2(F_y, \mathbb{R})) \) is equal to \( \Gamma_k|_{F_y} \cap \text{Aut}_0(F_y) \). Observe that this kernel is also equal to the
restriction of $\Gamma_k \cap \text{Aut}_0^+(Z/Y)$ to $F_y$. Moreover, the last two properties still hold if we replace $\Gamma_k$ by $G_k$ because $G_k$ is a subgroup of $\Gamma_k$. It follows that $y$ belongs to $E_k$. This ends the proof of Lemma 4.2 and also Proposition 4.1.

\[ \square \]

**Proof of Theorem 1.2 (2) and Theorem 1.3.** Observe that the independence of the derived length stated in Theorem 1.2 (2) is a consequence of Lemma 2.7. By Proposition 3.1, we may assume that $\kappa(X) \geq 1$. Let $f : X \to B$ be an Iitaka fibration of $X$. The base space $B$ is a projective variety with

$$\dim B = \kappa(X) \geq 1.$$ 

By [25, Cor. 2.4] (see also [30, Th. 14.10]), replacing $G$ by a finite-index subgroups, say $G$ the same letter, we may assume that $G$ descends to a trivial action on the base space $B$. The same holds if we replace $B$ by a projective resolution of $B$.

Then by Lemma 2.10, we obtain a $G$-equivariant bimeromorphic morphism $\tilde{X} \to X$ from a compact Kähler manifold $\tilde{X}$ such that $G = G|_{\tilde{X}} \leq \text{Aut}(\tilde{X})$ and the Iitaka fibration $f : \tilde{X} \to B$ is a morphism with all fibres connected, see [30, Lem. 5.6 and Prop. 5.7] for the connectedness of fibres. Moreover, the base $B$ is a smooth projective variety with $\dim B = \kappa(X) \geq 1$ and we have $G|_B = \{1\}$.

According to Lemma 2.10 replacing $G$ by a suitable finite-index subgroup, we have $G|_{\tilde{X}}/(G|_{\tilde{X}})_0 \cong G|_X/(G|_X)_0$. Therefore, in order to show the estimate in Theorem 1.2 (2), we may replace $X$ and $Y$ by $\tilde{X}$ and $G|_{\tilde{X}}$. Thus, we may assume for simplicity that $f : X \to B$ is holomorphic and $G|_B = \{1\}$. Replacing $G$ by $G \cdot \text{Aut}_0(X)$, we may further assume that $G \geq \text{Aut}_0(X)$ because this replacement does not change $G/G_0$.

Since $\kappa(X) \geq 1$, $X$ is non-uniruled by the ramification divisor formula and the adjunction formula, see also [30, Th. 6.10, Prop. 6.13]. Note that the same property holds for all Kähler manifolds $X$ with $\kappa(X) \geq 0$. Therefore, $\text{Aut}_0(X)$ is a torus, according to Lemma 2.11.

Observe also that the regular fibres of the fibration $f : X \to B$ are connected compact Kähler manifolds of dimension $n - \kappa(X)$. If $F$ is a very general fibre of $f$, then we have $\kappa(F) = 0$, see [30, Th. 6.11 and its proof]. Therefore, such a fibre $F$ is non-uniruled and $\text{Aut}_0(F)$ is a torus, thanks to Lemma 2.11. We can now apply Proposition 4.1 to the fibration $f : X \to B$.

Let $F$ be a regular fibre of this fibration. Recall that each reduction process above does not change the value of $\ell_{\min}(G/G_0)$. Then, according to Proposition 4.1, we have

\[ \ell_{\min}(G/G_0) = \ell_{\text{ess}}(G, X) \leq 1 + \ell_{\text{ess}}(G|_F, F) \leq n - \dim B = n - \kappa(X). \]

This is just the estimate in Theorem 1.2 (2).

We now prove Theorem 1.3. With the above construction, we already get the assertions (1) and (2) in Theorem 1.3. From the hypotheses of this theorem, we have $\ell_{\min}(G/G_0) = n - \kappa(X)$. Therefore, all inequalities in (\textbullet) are equalities. In particular, we have $\ell_{\text{ess}}(G|_F, F) = \dim F - 1$. Thus, $F$ has MPA by the group $G|_F$ and we obtain Theorem 1.3 (3). If the fibre $F$ is further assumed to be very general, then we have seen that $\kappa(F) = 0$ and $\text{Aut}_0(F)$ is a torus. By Proposition 4.1 we also have $\text{Aut}_0(F) \neq \{1\}$. This completes the proof of Theorem 1.3.

Now we are going to prove Theorem 1.2 (3). Denote by $U(n, \mathbb{Z})$ the group of $n \times n$ upper triangle matrices whose entries are integers and whose diagonal entries are 1. Let $E_{ij}$ be
the \( n \times n \)-matrix whose \((i, j)\)-entry is 1 and other entries are 0. It is well-known and easy to see directly that \( U(n, \mathbb{Z}) \) is generated by
\[
\tau_{ij} := I_n + E_{ij} \quad \text{with} \quad 1 \leq i < j \leq n ,
\]
and \( U(n, \mathbb{Z}) \) is a unipotent group of derived length \( n - 1 \).

First we construct examples of abelian varieties, especially of Kodaira dimension 0, with MPA. This is needed in our proof of Theorem \([1, 2]\)(3) and also covers the case \( \kappa = 0 \) there.

**Proposition 4.4.** Let \( n \geq 2 \) be an integer and \( E = (E, O) \) an elliptic curve. Then, the product \( E^n \) is a smooth projective variety with MPA by the group \( U(n, \mathbb{Z}) \).

**Proof.** The product \( E^n \) is a smooth projective variety (an abelian variety) and \( \text{Aut}_0(E^n) = E^n \) (the translation group of \( E^n \)). \( E^n \) has a natural faithful action of \( U(n, \mathbb{Z}) \) defined via the addition of \( X = E^n \). For instance, the action of \( \tau_{ij} = I_n + E_{ij} \)
\[
(P_1, \ldots, P_i, \ldots, P_j, \ldots, P_n) \mapsto (P_1, \ldots, P_i + P_j, \ldots, P_j, \ldots, P_n).
\]
The induced action on \( H^2(E^n, \mathbb{Z}) = \wedge^2 H^1(E^n, \mathbb{Z}) \) is then faithful and unipotent. Hence, \( U(n, \mathbb{Z}) \) is of zero entropy.

We have \( U(n, \mathbb{Z}) \cap \text{Aut}_0(X) = \{1\} \) because \( U(n, \mathbb{Z}) \) acts on \( H^2(X, \mathbb{Z}) \) faithfully. Thus, by Lemma \([2, 7]\)
\[
\ell_{\text{ess}}(U(n, \mathbb{Z}), E^n) = \ell_{\text{min}}(U(n, \mathbb{Z})) = \ell(U(n, \mathbb{Z})) = n - 1 .
\]
This completes the proof of the proposition. \( \square \)

**Proof of Theorem \([1, 2]\)(3).** Let \( E = (E, O) \) be any elliptic curve. Let \( n \geq 2 \) and \( \kappa \) be integers such that \( 1 \leq \kappa \leq n - 1 \) (the case \( \kappa = n \) is trivial by \([30, \text{Cor. 14.3}]\)). Fix a smooth projective variety \( B \) of dimension \( \kappa \) having an ample canonical divisor \( K_B \) such that there is a surjective morphism \( \rho : B \to E \). One may find such a \( B \) as a general member of a very ample linear system of the product variety \( E^{\kappa+1} \). Indeed, then, the canonical divisor \( K_B \) is very ample (and hence \( B \) is of general type) by the adjunction formula and the projection \( B \to E \) to the first factor is surjective.

Consider the product \( X := E^{n - \kappa} \times B \). This is a smooth projective variety of dimension \( n \geq 2 \) and the projection
\[
q : X = E^{n - \kappa} \times B \to B
\]
is the Iitaka fibration of \( X \) so that \( \kappa(X) = \kappa \geq 1 \). This is because \( K_{E^{n - \kappa}} \) is trivial while \( K_B \) is ample. The group \( U(n - \kappa, \mathbb{Z}) \) acts faithfully on \( X \) by
\[
U(n - \kappa, \mathbb{Z}) = U(n - \kappa, \mathbb{Z}) \times \{1\} \leq \text{Aut}((E, O)^{n - \kappa}) \times \text{Aut}(B) \leq \text{Aut}(E^{n - \kappa} \times B).
\]
For each \( i \) with \( 1 \leq i \leq n - \kappa \), we define \( \rho_i \in \text{Aut}(X) \) by
\[
E^{n - \kappa} \times B \ni (P_1, \ldots, P_i, \ldots, P_{n - \kappa}, Q) \mapsto (P_1, \ldots, P_i + \rho(Q), \ldots, P_{n - \kappa}, Q) \in E^{n - \kappa} \times B.
\]
Then \( \rho_i \in \text{Aut}(X/B) \) with respect to the projection \( q \). We set
\[
A := \langle \rho_i \mid 1 \leq i \leq n - \kappa \rangle \leq \text{Aut}(X/B) \leq \text{Aut}(X).
\]
This is an abelian subgroup of \( \text{Aut}(X) \). Finally, define
\[
G := \langle U(n - \kappa, \mathbb{Z}), A \rangle \leq \text{Aut}(X/B) \leq \text{Aut}(X).
\]
Since $A$ (resp. $U(n - \kappa, \mathbb{Z})$) acts on each fibre $E^{n-\kappa}$ as a translation (resp. zero entropy) group, by the relative dynamical degree formula in [10, Th. 1.1], the action of $G$ on $X$ is of zero entropy.

Now Proposition 4.5 below will complete the proof of Theorem 1.2(3). Note that the case $\kappa = 1$ provides an example of a smooth projective variety of dimension $n \geq 2$ with MPA by a group $G$ and with the maximum possible Kodaira dimension $\kappa(X) = 1$. □

**Proposition 4.5.** Under the notation above, we have $G_0 = \{1\}$ and

$$\ell_{\text{ess}}(G, X) = \ell(G) = n - \kappa = \dim X - \kappa(X).$$

**Proof.** Using [4, Cor. 2.3], for $X = E^{n-\kappa} \times B$, we obtain the following natural isomorphisms

$$\text{Aut}_0(X) \cong \text{Aut}_0(E^{n-\kappa}) \times \text{Aut}_0(B) \cong E^{n-\kappa} \times \{\text{id}_B\} \cong E^{n-\kappa}.$$ 

Here, we use that $B$ is of general type and hence Aut($B$) is finite, see [30, Cor. 14.3].

Let $t \in B$ be a very general point. Then $\rho(t) \in (E, O)$ is a non-torsion point, i.e., an infinite order element of the abelian group $(E, O)$. Then, the action of $G$ on

$$q^{-1}(t) = E^{n-\kappa} \times \{t\} \cong E^{n-\kappa}$$

is faithful and given, for $z \in E^{n-\kappa}$, by

$$z \mapsto Az + \rho(t)b \text{ with } A \in U(n - \kappa, \mathbb{Z}) \text{ and } b \in \mathbb{Z}^{n-\kappa}.$$ 

If $t' \in B$ is another very general point, the intersection between $\rho(t)\mathbb{Z}^{n-\kappa}$ and $\rho(t')\mathbb{Z}^{n-\kappa}$ is trivial, i.e., equal to $\{0\}$. Therefore, we see that the identity is the only element in $G$ which belongs to Aut$_0(X)$. Thus, we have $G_0 = \{1\}$ and therefore $\ell_{\text{ess}}(G, X) = \ell(G)$ provided that $G$ is unipotent.

It remains to show that $G$ is a unipotent group with $\ell(G) = n - \kappa$. This is now a purely group theoretical problem. From the description of the action of $G$ on $q^{-1}(t)$, we may identify $G$ with the unipotent affine transformation group

$$P := \mathbb{Z}^{n-\kappa} \rtimes U(n - \kappa, \mathbb{Z}) \cong U(n - \kappa + 1, \mathbb{Z}).$$

Here, the last isomorphism is the natural one given by

$$\mathbb{Z}^{n-\kappa} \rtimes U(n - \kappa, \mathbb{Z}) \ni f(x) = Ax + b \leftrightarrow \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in U(n - \kappa + 1, \mathbb{Z}).$$

Thus, for the derived sequence of $P \cong U(n - \kappa + 1, \mathbb{Z})$, by induction on the size $n - \kappa + 1$ of the matrices, we deduce that

$$P^{(n-\kappa-1)} = \mathbb{Z}(1, 0, \ldots, 0) \rtimes \{I_{n-\kappa}\} \neq \{1\}, \quad P^{(n-\kappa)} = \{1\}.$$ 

Hence $G$ is a unipotent group with $\ell(G) = n - \kappa$. This proves Proposition 4.5 and also Theorem 1.2(3). □

**Proposition 4.6.** For each $n \in \{2, 3, 4, 5\}$, there is an $n$-dimensional rationally connected smooth projective variety $X_n$ with MPA. In particular, we have $\kappa(X_n) = -\infty$.

**Proof.** Let $E_\omega$ be an elliptic curve whose period is the primitive third root of unity $\omega = (-1 + \sqrt{-3})/2$ in the upper-half plane. Consider the quotient variety

$$X_n := E_\omega^n / \langle -\omega I_n \rangle$$

and its blow-up $X_n$ on $X_n$ along the maximal ideals of all singular points of $X_n$. Then $X_n$ is a smooth projective variety and the action of $U(n, \mathbb{Z})$ on $E_\omega^n$ descends to a (not only
biregular but also) biholomorphic action on \( X_n \). Moreover, if \( n \in \{2, 3, 4, 5\} \), then \( X_n \) is a rationally connected manifold, see e.g. [28] Proof of Cor. 4.6] for a proof of this fact using complex dynamics. Note then that \( X_2 \) is rational, as a smooth projectively rationally connected surface is rational by Castelnouvo’s criterion for rationality of surfaces. See also [29] for the rationality of \( X_3 \). As the action of \( U(n, \mathbb{Z}) \) on \( E^\omega_n \) is MPA, so is the action of \( U(n, \mathbb{Z}) \) on \( X_n \), according to Lemma 2.10.

5. Proof of Theorem 1.4

In this section, we prove Theorem 5.8 below which includes Theorem 1.4 as a special case. In our approach, it is crucial to treat singular projective varieties. We begin with the following lemma.

**Lemma 5.1.** Let \( X \) be a normal projective variety and let \( \sigma : \hat{X} \to X \) be an \( \text{Aut}(X) \)-equivariant resolution (see [31] Th. 1.0.3], Lemma 2.4 and the remark preceding it). Then

1. We have the natural identification \( \text{Aut}_0(\hat{X}) = \text{Aut}_0(X) \).
2. If \( G \) is a subgroup of \( \text{Aut}(X) \) and \( \hat{G} \) denotes its natural action on \( \hat{X} \), then we have the natural identification \( G/G_0 = \hat{G}/\hat{G}_0 \).

**Proof.** Since \( X \) is normal and \( \sigma \) is birational, we have \( \sigma_*\mathcal{O}_{\hat{X}} = \mathcal{O}_X \). Then (1) follows from [4] Prop. 2.1. (2) is then a consequence of (1).

**Definition 5.2.** Let \( X \) be a projective variety of dimension \( n \) and let \( g \in \text{Aut}(X) \). Define the first dynamical degree \( d_1(g) \) of \( g \) as the spectral radius of \( g^*|_{\text{NS}_R(X)} \). Here and hereafter, \( \text{NS}(X) \) denotes the Néron-Severi group of \( X \) and \( \text{NS}_R(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} \).

If \( \hat{X} \to X \) is a \( g \)-equivariant generically finite morphism with \( \hat{X} \) smooth and projective, then \( d_1(g) = d_1(g|_{\hat{X}}) \) by applying [25] Lem. A.7, Prop. A.2] with \( x \) the pullback of an ample divisor on \( X \) and \( y \) the pullback of the \( (n-1) \)-th power of an ample divisor on \( X \).

For smooth \( X \), our definition of \( d_1(g) \) is just the usual one, as in [11], for (smooth) compact Kähler manifolds, and is independent of the choice of the birational model \( X \) where \( g \) acts biregularly by [12] Cor. 7.

An element \( g \in \text{Aut}(X) \) is of zero entropy if \( d_1(g) = 1 \). A group \( G \leq \text{Aut}(X) \) is of zero entropy if every \( g \in G \) has \( d_1(g) = 1 \).

**Proposition 5.3.** Let \( X \) be a normal projective variety of dimension \( n \geq 1 \) and let \( G \leq \text{Aut}(X) \) be a group of zero entropy. Then there is a finite-index subgroup \( G' \) of \( G \) such that the natural map \( \tau : G'/G'_0 \to G'|_{\text{NS}_R(X)} \) is an isomorphism and the image is a unipotent subgroup of \( \text{GL}(... \mathbb{R}) \) of derived length less than or equal to \( n-1 \).

**Proof.** We use the notation \( \sigma : \hat{X} \to X \), \( \hat{G} = G|_{\hat{X}} \), etc, as in Lemma 5.1. We may identify \( \text{NS}_R(X) \) with the subspace \( \sigma^*(\text{NS}_R(X)) \subseteq \text{NS}_R(\hat{X}) \). By Theorem 1.2, applied to \( \hat{X} \), there is a finite-index subgroup \( G' \) of \( G \), such that, for \( \hat{G}' := G'|_{\hat{X}} \), the natural map \( \hat{G}'/\hat{G}'_0 \to \hat{G}'|_{\text{NS}_R(\hat{X})} \) is injective with image a unipotent subgroup of \( \text{GL}(H^2(\hat{X}, \mathbb{R})) \) of derived length at most equal to \( n-1 \).

Hence, \( \hat{G}'|_{\text{NS}_R(\hat{X})} \) is also a unipotent subgroup of \( \text{GL}(\text{NS}_R(\hat{X})) \). Now, the kernel of the natural map \( \hat{\tau} : G'/G'_0 \to \hat{G}'/\hat{G}'_0 \to \hat{G}'|_{\text{NS}_R(\hat{X})} \) is a finite group by Fujiki [13] Th. 4.8 and Lieberman [24] Prop. 2.2. Hence, \( \text{Ker}(\hat{\tau}) = \{1\} \), i.e., \( \hat{\tau} \) is an isomorphism, because a unipotent group has no non-trivial finite subgroup.
We still need to prove that $\tau$ is injective. Let $g \in G'$ which acts on $\text{NS}_{\hat{X}}(X)$ trivially. Let $h$ be an ample divisor class on $X$. Set $\hat{h} := \sigma^* h$. Since $\sigma$ is a birational morphism, $\hat{h}$ is a nef and big class on $\hat{X}$. Since $g|_{\text{NS}_{\hat{X}}(X)}$ is trivial by our assumption, it follows that $(g|_{\hat{X}})^* (\hat{h}) = \hat{h}$. Thus, by a generalized version of Fujiki-Lieberman’s theorem in [8 Th. 2.1], a power of $g|_{\hat{X}}$ belongs to $G_0^\prime$, that is, in $G'/G_0^\prime = G'/G_0^\prime$, the class $[g|_{\hat{X}}] = [g]$ has to be a torsion element. On the other hand, $G'/G_0^\prime$ is torsion free as it is a unipotent group. Thus $g \in G_0^\prime$. Hence, $\tau$ is injective.

\textbf{Definition 5.4.} For $(X, G)$ as in Proposition 5.3, define the \emph{essential derived length} of the action of $G$ on $X$ by

$$\ell_{\text{ess}}(G, X) := \ell_{\text{min}}(G|_{\text{NS}(X)}).$$

Then, by Proposition 5.3 with its notation and proof, and by Theorem 1.2 we have

$$\ell_{\text{ess}}(G, X) = \ell_{\text{min}}(G/G_0) = \ell(G'|_{\text{NS}(X)}) = \ell(\hat{G}'|_{\text{NS}(\hat{X})}) = \ell_{\text{ess}}(G|_{\hat{X}}, \hat{X}) \leq n - 1.$$  

Also, by Proposition 5.3 for a smooth projective variety $X$, our new definition of $\ell_{\text{ess}}(G, X)$ coincides with the definition given in the introduction.

In general, for $(X, G)$ as in Proposition 5.3 if $X'$ is a normal projective variety and if $X \rightarrow X'$ is generically finite, dominant and $G$-equivariant so that $G \leq \text{Aut}(X)$ and $G \leq \text{Aut}(X')$, then $\ell_{\text{ess}}(G, X) = \ell_{\text{ess}}(G, X')$. This can be verified by reducing to the smooth case. Indeed, we can first take $G$-equivariant resolutions $\hat{X} \rightarrow X$ and $\hat{X}' \rightarrow X'$ as in Lemma 5.1 and then apply Lemma 2.10 for the induced map $\hat{X} \rightarrow \hat{X}'$.

We say that $X$ is MPA by the group $G$ if $\ell_{\text{ess}}(G, X) = \dim X - 1$. So, if $X \rightarrow X'$ is a generically finite and dominant $G$-equivariant map between normal projective varieties, then $X$ has MPA by $G$ if and only if the same property holds for $X'$.

\textbf{Proposition 5.5. (cf. Corollary 3.2)} Let $X$ be a normal projective variety of dimension $n \geq 1$. Let $G \leq \text{Aut}(X)$ be a group of zero entropy. Suppose that $\text{Aut}_0(X)$ is commutative (this holds when $X$ is non-ruled; see Lemma 2.11 and the remark after that). Then

1. We have

$$\ell_{\text{min}}(G) \leq \ell_{\text{min}}(G/G_0) + 1 = \ell_{\text{min}}(G|_{\text{NS}(X)}) + 1 = \ell_{\text{ess}}(G, X) + 1 \leq n. $$

2. Suppose that $\ell_{\text{min}}(G) = n$. Then

$$G_0 \neq \{1\} \quad \text{and} \quad \ell_{\text{ess}}(G, X) = n - 1.$$ 

\textbf{Proof.} The same proof for Corollary 3.2 works, but we use Proposition 5.3 instead of Proposition 3.1. \hfill \square

\textbf{Lemma 5.6.} Let $\pi : X_1 \rightarrow X_2$ be a dominant map of compact Kähler manifolds or normal projective varieties, of the same dimension. Assume a group $G$ acts on both $X_1$ and $X_2$ biregularly so that $\pi$ is $G$-equivariant. Suppose that $(G|_{X_i})_0 := (G|_{X_i}) \cap \text{Aut}_0(X_i)$ is an infinite group for $i = 2$. Then so does for $i = 1$.

\textbf{Proof.} As in Lemma 5.1 replacing the $X_i$ by $G$-equivariant resolutions (when the $X_i$ are projective), we may assume that both $X_1$ and $X_2$ are smooth. Thus, we only need to consider the Kähler case. Take a subgroup $H$ of $G$ such that $H|_{X_2} = (G|_{X_2})_0$.

By Lemma 2.10 for a finite-index subgroup $H'$ of $H$, we have

$$\frac{(H'|_{X_1})}{(H'|_{X_1})_0} \cong \frac{(H'|_{X_2})}{(H'|_{X_2})_0} = \{1\}.$$
By hypothesis, \( H|_{X_2} \) is infinite and hence \( H'|_{X_2} \) is also infinite. It follows that \( H'|_{X_1} \) is infinite. This, together with the isomorphism in \((*)\), imply that \( \langle H'|_{X_1} \rangle_0 \) is infinite. Therefore, \( \langle G|_{X_1} \rangle_0 \) is infinite. \( \square \)

Let \( X \) be a projective variety. Let \( \sigma : X' \to X \) be a projective resolution. We define the \textit{Kodaira dimension} of \( X \) as \( \kappa(X) := \kappa(X') \) and the \textit{albanese map} of \( X \) as

\[
alb_X : X \overset{\sigma^{-1}}{\longrightarrow} X' \overset{\text{alb}_{X'}}{\longrightarrow} \text{Alb}(X') =: \text{Alb}(X).
\]

Our \( \kappa(X) \) and \( \text{Alb}(X) \) do not depend on the choice of a resolution of \( X \), see e.g. \cite{30} Cor.6.4, Prop.9.12.

For a surjective morphism \( \pi : X \to Y \) of varieties, a subgroup \( \widetilde{G} \) of \( \text{Aut}(X) \) (resp. \( \text{Bir}(X) \)) is a \textit{lifting} of a subgroup \( G \) of \( \text{Aut}(Y) \) (resp. \( \text{Bir}(Y) \)) if there is a surjective homomorphism \( \sigma : \widetilde{G} \to G \) such that

\[
\pi(\tilde{g}(x)) = \sigma(\tilde{g})(\pi(x))
\]

for every \( \tilde{g} \) in \( \widetilde{G} \) and every closed point (resp. every general point) \( x \) in \( X \).

Let \( S \) be a normal projective variety. We call \( q(S) := h^1(S, \mathcal{O}_S) \) the \textit{irregularity} of \( S \). The variety \( S \) is called \textit{weak Calabi-Yau} in the sense of \cite{25} §1.2, if \( S \) has only canonical singularities, a canonical divisor \( K_S \sim_{\mathbb{Q}} 0 \) and

\[
q^{\text{max}}(S) := \max\{q(S') \mid S' \to S \text{ is finite étale}\} = 0.
\]

\textbf{Proposition 5.7.} (cf. \cite{25} Th.B) Let \( W \) be a normal projective variety with the property

\( (\dagger) \)

\( W \) has only canonical singularities and \( K_W \sim_{\mathbb{Q}} 0 \).

Then there are an abelian variety \( A \), a weak Calabi-Yau variety \( S \) and a finite étale morphism \( \tau : S \times A \to W \) such that for every \( G \leq \text{Bir}(W) \), there is a lifting \( \widetilde{G} \leq \text{Bir}(S \times A) = \text{Bir}(S) \times \text{Aut}(A) \) of \( G \). In particular, we have \( \widetilde{G} \leq G_S \times G_A \), where \( G_S \leq \text{Bir}(S) \) (resp. \( G_A \leq \text{Aut}(A) \)) is the projection of \( \widetilde{G} \) to \( \text{Bir}(S) \) (resp. \( \text{Aut}(A) \)).

\textbf{Proof.} This is proved in \cite{25} Th.B] when \( G \) is cyclic. The general case is the same. We go through the construction of the lifting for the reader’s convenience, but refer the details to \cite{25}. Let \( V \to W \) be the Albanese closure as defined after \cite{25} Prop.4.3 which is étale and unique up to an isomorphism. The properties in \cite{25} Prop.4.3 and the remark there guarantee the existence of the lifting to \( G_V \leq \text{Bir}(V) \) of \( G \leq \text{Bir}(W) \). Here and hereafter, an element \( g \in G \) may have several liftings in \( \text{Bir}(V) \); we take them all and put them in \( G_V \).

Since \( V \to W \) is étale, \( V \), like \( W \), also has the property \( (\dagger) \). So the albanese morphism

\[
alb_V : V \to \text{Alb}(V) =: A_1
\]

is surjective with connected fibres by \cite{20} Main Th.. Moreover, Kawamata’s splitting theorem \cite{21} Th.8.3] implies that the albanese morphism \( \text{alb}_V \) splits after some base change of \( \text{alb}_V \) by an isogeny \( A_1' \to A_1 \), that is, taking a fibre \( S_1 \) of the albanese morphism \( \text{alb}_V \); we have an isomorphism

\[
V \times_{A_1} A_1' \cong S_1 \times A_1'
\]

\footnote{Recall again that \( \text{Aut}(A) \) stands for the automorphism group of the complex variety \( A \).}
over $A'_1$. Let $A_1 \to A'_1$ be an isogeny so that the composition $A_1 \to A'_1 \to A_1 = \text{Alb}(V)$ equals the multiplication by some integer $m \geq 2$. Denote this map by $m_{A_1}$. By \cite[Lemma 4.9]{25}, $G_V|_{\text{Alb}(V)}$ lifts to some $G_{A_1} \leq \text{Aut}(A_1)$ via $m_{A_1} : A_1 \to A_1 = \text{Alb}(V)$.

By construction, the base change $m_{A_1} : A_1 \to A_1 = \text{Alb}(V)$ of $alb_V$ produces the splitting

$$V \times A_1, A_1 = S_1 \times A_1 =: V_1,$$

with $S_1$ a fibre of $alb_V$ as above. Now $G_V$ lifts to $G_{V_1} \leq \text{Bir}(V_1)$ which consists of all $(g_1, g_2)$ with $g_1 \in G_V$, $g_2 \in G_{A_1}$ so that the descending $g_1|_{\text{Alb}(V)}$ of $g_1$ via $alb_V$ equals the descending of $g_2$ via $m_{A_1}$. Since $m_{A_1}$ is étale, the projection $V_1 \to V$ is étale too. Hence, $V_1$, like $V$, also has the property ($\dagger$). In particular, $q(V_1) \leq \dim V_1 = \dim W$ by \cite[Main Th.]{20}. Applying the same process to $V_1$ (instead of $W$), then to $V_2$ and so on, we get $V_i = S_i \times A_i$ with $A_i$ an abelian variety, finite étale morphisms $V_i \to V_{i-1}$, and liftings $G_{V_i}$ of $G$ on $V_0 := W$ for all integers $i \geq 1$. Here, $V_i$, like $W$, has the property ($\dagger$). So $q(V_i) \leq \dim V_i = \dim W$. Thus, by induction on dimension, we may assume that $V_i = S_i \times A_i$ has maximal irregularity $q(V_i) = 0$ and $S_i$ is a weak Calabi-Yau variety for some $t$. By \cite[Lem. 4.6]{25}, $G := G_{V_1}$ has the required (splitting) property.

**Theorem 5.8.** Let $X$ be a normal projective variety of dimension $n \geq 1$ with MPA by a group $G \leq \text{Aut}(X)$. Assume the following two conditions.

(i) $X$ has Kodaira dimension $\kappa(X) = 0$.

(ii) $X$ has a good minimal model, i.e., $X$ is birational to a normal projective variety $X_m$ with at most canonical singularities and semi-ample canonical divisor.

Then Case (I) (satisfying (Ia) - (Ic)) or Case (II) (satisfying (Ia) - (IId)) below occurs.

(Ia) $X_m$ is smooth and $G$ acts on $X_m$ biregularity.

(Ib) There is a finite étale Galois cover $A \to X_m$ from an abelian variety $A$ such that $G|_{X_m}$ lifts to $\tilde{G} \leq \text{Aut}(A)$ with $\tilde{G}/\text{Gal}(A/X_m) = G$.

(Ic) $X_m$ (resp. $A$) has MPA by the group $G$ (resp. $\tilde{G}$), see Definition 5.4.

(Ii) $\text{Aut}_0(X) = \{1\}$.

(IiI) There is a generically finite surjective morphism $\tilde{X} \to X$ with $\tilde{X}$ smooth such that $G$ lifts to $\tilde{G} \leq \text{Aut}(\tilde{X})$ and $\text{Aut}_0(\tilde{X}) = \{1\}$.

(IId) $\tilde{X}$ (resp. $\tilde{S}$ and $A$ in (IiI), if $\dim \tilde{S} > 0$ and $\dim A > 0$) has MPA by the group $G$ (resp. $N|_{\tilde{S}}$ and $G|_A$) with $N := \text{Ker}(\tilde{G} \to G|_A)$, see also Proposition 5.7.

**Remark 5.9.**

(1) In Theorem 1.3 assume further that $X$ is projective and the very general fibre $X_b$ of an Iitaka fibration $X \dashrightarrow B$ has a good minimal model; for instance, this is the case when $\dim X_b \leq 3$ or when $\dim X \leq 4$ (cf. \cite[§3.13]{23}). Then by Theorem 5.8 $X_b$ is birational to a $Q$-abelian variety.

(2) In view of Theorems 1.3 and 5.8 the building blocks of projective varieties $X$ with MPA and $\kappa(X) \geq 0$ are abelian varieties, weak Calabi-Yau varieties and curves, up to equivariant coverings, under the conjectural assumption (ii) in Theorem 5.8 which holds in dimension up to 3 (cf. \cite[§3.13]{23}).

**Proof of Theorem 5.8.** By the assumption, we have $\kappa(X) = 0$ (hence $X$ is not uniruled) and $X$ is birational to a good minimal model $X_m$. Namely, $X_m$ has only canonical
singularities and \( \text{NS}_\mathbb{R}(X) \leq \text{NS}_\mathbb{R}(S) \), and \( \text{NS}_\mathbb{Q}(S) \sim \text{NS}_\mathbb{Q}(X) \). This holds when \( \text{Aut}_0(S) \neq \{1\} \), see Lemma \[5.6\].

**Claim 5.10.** In this case, we have that \( S \) is a point.

*Proof of the claim.* Replacing \((X, G)\) by \((\bar{X}, \bar{G})\) we may assume that \( X \) is birational to \( A \times S \). Replacing \( X \) by its \( \text{Aut}(X) \)-equivariant resolution as in Lemma \[5.1\] we may assume that \( X \) is smooth. Replacing \( G \) by \( \langle G, \text{Aut}_0(X) \rangle \), we may also assume that \( G \supseteq \text{Aut}_0(X) \neq \{1\} \).

Replacing \( G \) by a finite-index subgroup, we may further assume \( G/\text{Aut}_0(X) \to G|_{\text{NS}_\mathbb{Q}(X)} \) is an isomorphism with image a unipotent group of maximal derived length \( n - 1 \), see Proposition \[5.3\].

Replacing \( S \) by its \( \mathbb{Q} \)-factorial terminalization, we may also assume that \( S \) is \( \mathbb{Q} \)-factorial terminal, see [2 Cor. 1.4.3] for the existence of \( \mathbb{Q} \)-factorial terminalization. Under this modification, we still have \( q(S) = 0 \), hence \( \text{Aut}_0(S) = \{1\} \) by [25 Lem. 4.4]. Moreover, by [18 Cor. 3.8], the birational action of the connected algebraic group \( \text{Aut}_0(X) \) on the terminal minimal model \( S \times A \) of \( X \), is birational. Hence (see also [4 Cor. 2.3])

\[
\text{Aut}_0(X) \leq \text{Aut}_0(S \times A) = \text{Aut}_0(A) \cong A.
\]

Thus, \( \text{Aut}_0(X) \) is a subtorus of \( A \).

By Proposition \[5.7\] \( G \leq G_S \times G_A \), where \( G_S \leq \text{Bir}(S) \) and \( G_A \leq \text{Aut}(A) \), are the projections of \( G \) to \( \text{Bir}(S) \) and \( \text{Aut}(A) \). So, by definition of minimal length, we have

\[
\ell_\text{min}(G) = \max\{\ell_\text{min}(G_S), \ell_\text{min}(G_A)\}.\tag{*}
\]

Note that

\[
G/\text{Aut}_0(X) \leq G_S \times G_A/\text{Aut}_0(X)
\]

and the projections \( G/\text{Aut}_0(X) \to G_S \) and \( G/\text{Aut}_0(X) \to G_A/\text{Aut}_0(X) \) are surjective. So

\[
(\star\star) \quad n - 1 = \ell_\text{min}(G/\text{Aut}_0(X)) = \max\{\ell_\text{min}(G_S), \ell_\text{min}(G_A/\text{Aut}_0(X))\}.
\]

Suppose now the contrary that \( S \) is not a point. So \( \dim S \geq 2 \) and \( \dim A = n - \dim S \leq n - 2 \). Note that

\[
\ell_\text{min}(G_A/\text{Aut}_0(X)) \leq \ell_\text{min}(G_A) \leq \dim A \leq n - 2,
\]

By Proposition \[5.7\] there is a finite étale cover \( \bar{X}_m \to X_m \) such that \( \bar{X}_m = S \times A \), where \( A \) is an abelian variety and \( S \) is a weak Calabi-Yau variety (especially, \( S \) has only canonical singularities, \( \bar{K}_S \sim \mathbb{Q} \), and \( S \) is not uniruled, with the irregularity \( q(S) = 0 \); so \( S \) is a point or \( \dim S \geq 2 \), and \( G \) (indeed, the whole \( \text{Aut}(X) \) and \( \text{Bir}(X) \)) lifts to

\[
\bar{G} \leq \text{Bir}(\bar{X}_m) = \text{Bir}(S) \times \text{Bir}(A) = \text{Bir}(S) \times \text{Aut}(A).
\]

Let \( \bar{X} \to X \) be the normalization of \( X \) in the function field \( \mathbb{C}(\bar{X}_m) \). Then, \( \bar{X}_m = S \times A \) is a good minimal model of \( \bar{X} \).

Recall that \( \bar{G} \leq \text{Bir}(\bar{X}_m) = \text{Bir}(\bar{X}) \) is the lifting of the regular action of \( G \) on \( X \). Hence, \( \bar{G} \leq \text{Aut}(\bar{X}) \) by the uniqueness of the normalization. Since \( G \) is of zero entropy so does \( \bar{G} \), see the remark in Definition \[5.2\]. Since \( G \leq \text{Aut}(X) \) has the maximal \( \ell_\text{ess}(G, X) = \dim X - 1 \) and \( \text{NS}_\mathbb{R}(X) \) is naturally embedded in \( \text{NS}_\mathbb{R}(\bar{X}) \), it follows that \( \bar{G} \) and \( \langle \bar{G}, \text{Aut}_0(\bar{X}) \rangle \) also have maximal essential derived length \( \dim X - 1 = \dim \bar{X} - 1 \).

We divide the proof into two cases, Case (I) and Case (II) below.

**Case (I).** \( \text{Aut}_0(\bar{X}) \neq \{1\} \). This holds when \( \text{Aut}_0(X) \neq \{1\} \), see Lemma \[5.6\].
see Proposition 5.5. Thus, (*) and (**) imply that \( \ell_{\min}(G_S) = n - 1 \) and \( \ell_{\min}(G) = n - 1 \).

By [15] Lem. 4.2 and since \( G \) normalizes \( \Aut_0(X) \), there is a quotient (normal) variety \( X_1 := X/\Aut_0(X) \) such that the action of \( G \) on \( X \) descends to a birational action \( G|_{X_1} \) on \( X_1 \). Let \( A_1 := A/\Aut_0(X) \). Then, \( X_1 \) is birational to \( S \times A_1 \), and we have \( \Aut_0(X_1) \leq A_1 \) and \( G|_{X_1} \leq G_S \times G_{A_1} \), so that the projections \( G|_{X_1} \to G_S \) and \( G|_{X_1} \to G_{A_1} \) are surjective.

As argued above, we have \( \ell_{\min}(G|_{X_1}) = \ell_{\min}(G_S) = n - 1 \). On the other hand, by Proposition 5.5

\[
n - 1 = \ell_{\min}(G|_{X_1}) \leq \dim X_1 = n - \dim \Aut_0(X) \leq n - 1. \tag{1}
\]

Thus, all the inequalities above are actually equalities. So \( \ell_{\min}(G|_{X_1}) = \dim X_1 \) and therefore, \( \Aut_0(X_1) \neq \{1\} \) by Proposition 5.5.

Now, let \( X_2 := X_1/\Aut_0(X_1) \). We have as above \( \ell_{\min}(G|_{X_2}) = \ell_{\min}(G_S) = n - 1 \). This contradicts the inequality

\[
\ell_{\min}(G|_{X_2}) \leq \dim X_2 \leq n - 2
\]
given by Proposition 5.5. Hence, \( S \) is a point as claimed. \(\square\)

We continue the proof of Theorem 5.8 in Case (I). Taking the Galois closure, we may assume that \( \pi : \tilde{X}_m = S \times X = A \to X_m \) (and hence \( \tilde{X} \to X \)) is Galois, and \( \pi \) is still étale. In particular, \( X_m \) is smooth. To complete the proof in Case (I), consider the albanese map

\[
alb_{\tilde{X}} : \tilde{X} \longrightarrow \Alb(\tilde{X}) = \Alb(\tilde{X}_m) = \Alb(S \times A) = A
\]

which is just the initial birational map \( \tilde{X} \longrightarrow \tilde{X}_m = S \times A = A \). For the original \((X,G)\), the lifting \( \tilde{G} \leq \Aut(\tilde{X}) \) of \( G \), acts biregularly on \( \Alb(\tilde{X}) = A \). Now, the \( \tilde{G} \)-equivariant birational map \( \alb_{\tilde{X}} \) descends to a \( G = \tilde{G}/\Gal(\tilde{X}/X) \)-equivariant birational map

\[
X = \tilde{X}/\Gal(\tilde{X}/X) \longrightarrow A/\Gal(\tilde{X}/X) = \tilde{X}_m/\Gal(\tilde{X}/X) = X_m.
\]

Here, note that our original \( G \) acts biregularly on both \( X \) and \( X_m \). Since \( X \) has MPA by the group \( G \), so does \( X_m \) (resp. \( A \) and \( \tilde{X} \)) by the group \( G \) (resp. \( \tilde{G} \)), see Definition 5.4 and the remark there. This completes the proof of Theorem 5.8 in Case (I).

**Case (II).** \( \Aut_0(\tilde{X}) = \{1\} \). Hence \( \Aut_0(X) = \{1\} \), see Lemma 5.6.

Replacing \( \tilde{X} \) by an \( \Aut(\tilde{X}) \)-equivariant resolution, we may assume that \( \tilde{X} \) is smooth and we still have \( \Aut_0(\tilde{X}) = \{1\} \) (cf. Lemma 5.1). We use the same approach as in Case (I). Since \( \Aut_0(\tilde{X}) = \{1\} \), we may assume that the lifting of \( G \) to \( \tilde{G} \) on \( \tilde{X} \) has a finite-index subgroup \( \tilde{G}' \) with \( \tilde{G}' \cong \tilde{G}'|_{\NS(\tilde{X})} \) being unipotent of derived length \( n - 1 \).

Consider the albanese map:

\[
alb_{\tilde{X}} : \tilde{X} \to \Alb(\tilde{X}) = \Alb(S \times A) = A.
\]

Note that \( \tilde{X} \) is smooth and \( \kappa(\tilde{X}) = 0 \). So \( \alb_{\tilde{X}} \) is a surjective morphism with connected fibres by [20] Main Th.. Let \( \tilde{S} \) be a general fibre of \( \alb_{\tilde{X}} \). Then \( \tilde{S} \) is birational to \( S \). We may assume

\[
0 < \dim A < n \quad (= \dim \tilde{X}),
\]

hence \( \dim S > 0 \), as otherwise the conclusion in the theorem is trivial.

Let \( N := \ker(\tilde{G} \to \tilde{G}|_S) \). It has a finite-index subgroup \( N' := N \cap \tilde{G}' \), which is also unipotent like \( \tilde{G}' \). Since \( \tilde{X} \) is birational to \( \tilde{S} \times A \) and \( \Bir(\tilde{X}) = \Bir(\tilde{S}) \times \Aut(A) \) as seen early on, \( N \) acts on the fibre \( \tilde{S} \) faithfully. So the restriction \( N \to N|_{\tilde{S}} \) is an isomorphism.
Recall that \( \tilde{S} \) is smooth and birational to \( S \). Since \( S \) has only canonical singularities and \( K_S \sim_{\mathbb{Q}} 0 \), it follows that

\[
q(\tilde{S}) = q(S) = 0 \quad \text{and} \quad \kappa(\tilde{S}) = \kappa(S) = 0.
\]

Thus, \( \tilde{S} \) is a smooth projective non-uniruled variety (by \( \kappa(\tilde{S}) = 0 \)) with \( q(\tilde{S}) = 0 \). Therefore, we have \( \text{Aut}_0(\tilde{S}) = \{1\} \) by [25, Lem. 4.4].

Since \( G/N \cong G|_A \), \( \text{Aut}_0(\tilde{X}) = \{1\} \) and \( \text{Aut}_0(\tilde{S}) = \{1\} \), by Proposition [5.5](1) applied to \( A \) and \( \tilde{S} \) with \( \dim \tilde{S} > 0 \), we have

\[
n - 1 = \ell_{\min}(\tilde{G}) \leq \ell_{\min}(G|_A) + \ell_{\min}(N|_{\tilde{S}}) \leq \dim A + (\dim \tilde{S} - 1) = n - 1.
\]

Thus, \( \ell_{\min}(N) = \ell_{\min}(N|_{\tilde{S}}) = \ell_{\text{ess}}(N|_{\tilde{S}}, \tilde{S}) = \dim \tilde{S} - 1 \) and \( \ell_{\min}(G|_A) = \dim A > 0 \). So \( \ell_{\text{ess}}(G|_A, A) = \dim A - 1 \) by Proposition [5.5]. This completes the proof of the whole Theorem 5.8 (see also the remark in Definition 5.4).

\[\square\]

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