WEIGHTED K-STABILITY AND COERCIVITY WITH APPLICATIONS TO EXTREMAL KÄHLER AND SASAKI METRICS

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Abstract. We show that a compact weighted extremal Kähler manifold (as defined by the third named author in [55]) has coercive weighted Mabuchi energy with respect to a maximal complex torus $T_C$ in the reduced group of complex automorphisms. This provides a vast extension and a unification of a number of results concerning Kähler metrics satisfying special curvature conditions, including Kähler metrics with constant scalar curvature [15, 23], extremal Kähler metrics [48], Kähler–Ricci solitons [29] and their weighted extensions [13, 46]. Our result implies the strict positivity of the weighted Donaldson–Futaki invariant of any non-product $T_C$-equivariant smooth Kähler test configuration with reduced central fibre, a property known as $T_C$-equivariant weighted K-polystability on such test configurations. It also yields the $T_C$-uniform weighted K-stability on the class of smooth $T_C$-equivariant polarized test configurations with reduced central fibre. For a class of fibrations constructed from principal torus bundles over a product of Hodge cscK manifolds, we use our results in conjunction with results of Chen–Cheng [23], He [48] and Han–Li [46] in order to characterize the existence of extremal Kähler metrics and Calabi–Yau cones associated to the total space, in terms of the coercivity of the weighted Mabuchi energy of the fibre. This yields a new existence result for Sasaki–Einstein metrics on certain Fano toric fibrations, extending the results of Futaki–Ono–Wang [40] in the toric Fano case, and of Mabuchi–Nakagawa [61] in the case of Fano $P^1$-bundles.

Introduction

This paper is concerned with the existence and obstruction theory of a class of special Kähler metrics, called weighted constant scalar curvature metrics, which were introduced by the third named author in [56], giving a vast extension of the notion of Kähler metrics of constant scalar curvature (cscK for short), and providing a unification for a number of related notions of Kähler metrics satisfying special curvature conditions.

0.1. The weighted cscK problem. Let $X$ be a smooth compact complex $m$-dimensional manifold with a given deRham cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ of Kähler metrics, and let $T \subset Aut_r(X)$ denote a fixed compact torus in the reduced group $Aut_r(X)$ of automorphisms of $X$, i.e. the connected subgroup of automorphisms of $X$ generated by the Lie algebra of real holomorphic vector fields with zeros, see e.g. [41]. It is well-known that $T$ acts in a hamiltonian way with respect to any $T$-invariant Kähler metric $\omega \in \alpha$, and the corresponding momentum map $\mu_\omega$ sends $X$ onto a compact convex polytope $\Delta \subset t^*$ in the dual vector space $t^*$ of the Lie algebra $t$ of $T$ (cf. [9, 45]). Furthermore, up to translations, $\Delta$ is independent of the choice of $\omega \in \alpha$. We shall further fix $\Delta$, giving rise to a normalization of the corresponding momentum maps $\{\mu_\omega, \omega \in \alpha\}$.

Following [56], let $v(\mu) > 0$ and $w(\mu)$ be smooth functions defined on $\Delta$. One can then consider the following condition for $T$-invariant Kähler metrics $\omega$ in $\alpha$ (and fixed polytope $\Delta$), called $(v, w)$-cscK metric:

$$\text{Scal}_\omega(\omega) = w(\mu_\omega),$$

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The relevance of (1) to various geometric conditions is discussed in detail in [56], but we mention function with ρ explained above, we let the general notion of a 0.2.

Relation to metrics allows one to study these cases altogether.

In general, the problem of finding a T-invariant Kähler metric ω ∈ α solving (1) is obstructed in a similar way that the cscK problem is obstructed by the vanishing of the Futaki invariant: for any T-invariant Kähler metric ω ∈ α and any affine-linear function ℓ on t*, one must have

$$\text{Fut}_{v,w}(\ell) := \int_X (\text{Scal}_v(\omega) - w(\mu_\omega)) \ell(\mu_\omega) \omega^m = 0,$$

should a solution to (1) exists. In [56], an unobstructed modification of (1) is proposed, extending Calabi’s notion [21] of extremal Kähler metrics. To this end, suppose that v, w0 > 0 are given positive smooth functions on ∆. One can then find a unique affine-linear function $\ell^\text{ext}_{v,w0}(\mu)$ on t*, called the extremal function, such that (3) holds for the weights (v, w) = (v, $\ell^\text{ext}_{v,w0}(w0)$). In this case, a solution of the (v, w)-cscK problem (1) is referred to as (v, w0)-extremal Kähler metric. We emphasize that (v, w0)-extremal Kähler metrics are (v, w)-cscK metrics with a special property of the weight function w, namely, w = ℓw0 with w0 > 0 on ∆ and ℓ affine-linear. In particular, (v, w)-cscK metrics with w ≠ 0 on ∆ are (v, w)-extremal with $\ell^\text{ext}_{v,w} = \text{sign}(w|_\Delta)$ and (v, 0)-cscK metrics are (v, w)-extremal with $\ell^\text{ext}_{v,w} = 0$ for any w > 0. It follows that all the above listed special cases are examples of (v, w)-extremal Kähler metrics, and thus the setup of (v, w)-extremal Kähler metrics allows one to study these cases altogether.

0.2. Relation to v-solitons. Motivated by works of T. Mabuchi [59, 60] and consequent work by Berman–Witt-Nystrom [13], Y. Han and C. Li [46] have recently introduced and studied the general notion of a weighted v-soliton on a smooth Fano variety X, as follows. In the setup explained above, we let $\alpha = 2\pi c_1(X)$ and consider the natural action of T on $K_X^{-1}$, which fixes the momentum polytope ∆ of (X, $\alpha$, T) and normalizes the momentum map $\mu_\omega$ for any T-invariant Kähler metric $\omega \in \alpha$. For a (smooth) positive weight function v(μ) on ∆, one defines a v-soliton as a T-invariant Kähler metric $\omega \in \alpha$, such that

$$\rho_\omega - \omega = \frac{1}{2} \text{d} v(\mu_\omega),$$

where $\rho_\omega$ denotes the Ricci form of $\omega$. Notice that when $v(\mu) = e^{(\mu,\xi)}$ for some $\xi \in t$, one gets the well-studied class of Kähler–Ricci solitons [72] whereas the case when $v(\mu)$ is a positive affine-linear function on ∆ corresponds to the Mabuchi solitons studied in [59, 60]. As we shall see below other choices for v are also geometrically meaningful. We make the following useful observation.

**Proposition 1.** Let X be a smooth Fano manifold and $T \subset Aut(X)$ a compact torus. A T-invariant Kähler metric $\omega \in 2\pi c_1(X)$ is a v-soliton if and only if $\omega$ is (v, w)-cscK with $w(\mu) := 2(m + (d\log v, \mu))v(\mu)$.
We use the above result in order to make connection with the recent paper \cite{46} (where the authors obtain a complete Yau-Tian-Donaldson type correspondence for the existence of $v$-Ricci solitons) which will play an important role in our present study of $(v, w)$-cscK metrics.

We also notice that $v$-solitons can be viewed as $(\bar{v}, \bar{w})$-cscK metrics for different choices of weights. This is for instance the case when $v(\mu) = \ell(\mu)^{-m-2}$, where $\ell(\mu) = (\xi, \mu) + a$ is a positive affine-linear on $\Delta$. Whereas Proposition \cite{1} identifies the $v$-soliton as a $(v, w)$-cscK metric with

$$ v = \ell^{-(m+2)}, \quad w = 2\ell^{-(m+3)} (-2\ell + (m+2)a), $$

we also observe that

**Proposition 2.** Let $(X, T)$ be a smooth Fano variety and $\ell(\mu) = ((\xi, \mu) + a)$ a positive affine-linear function on its canonical polytope $\Delta$. A $T$-invariant Kähler metric $\omega \in 2\pi c_1(X)$ is an $\ell^{-(m+2)}$-soliton if and only if the lift $\hat{\xi}$ of $\xi = d\ell$ to $K_X$ via $\ell$ is the Reeb vector field of a Sasaki–Einstein structure defined on the unit circle bundle $N \subset K_X$ with respect to the hermitian metric on $K_X$ with curvature $-\omega$. The latter condition is also equivalent to $\omega$ be a $(\ell^{-m-1}, 2ma\ell^{-m-2})$-cscK metric.

0.3. Main results. Similarly to the usual cscK case, it is shown in \cite{56} that the solutions of \cite{1} can be characterized as minimizers of a functional $M_{v, w}$ defined on the space of $T$-invariant Kähler metrics in $\alpha$, extending the Mabuchi energy to the weighted setting (see Section 1 below for the precise definition). After the deep works \cite{15, 23}, it is now well-understood that the coercivity of the Mabuchi energy is equivalent to the existence of a cscK metric in a given cohomology class. Noting that, by the results in \cite{56}, any $(v, w)$-extremal metric is invariant under a maximal compact torus in $Aut_r(X)$, our first main result is an extension of one direction of the correspondence in the cscK case to the weighted setting.

**Theorem 1.** Suppose $T \subset Aut_r(X)$ is a maximal torus in the reduced group of automorphisms of $X$, and $\omega_0 \in \alpha$ a $T$-invariant $(v, w_0)$-extremal Kähler metric. Then the weighted Mabuchi energy $M_{v, w}$ (with $w = w^ext_{v, w_0}$) is coercive relative to the complex torus $T^c$, in the sense of \cite{29}, i.e. there exist positive real constants $\lambda, \delta$ such that for any $T$-invariant Kähler metric $\omega \in \alpha$,

$$ M_{v, w}(\omega) \geq \lambda \inf_{\sigma \in T^c} J(\sigma^* \omega) - \delta, $$

where $J$ denotes the Aubin functional on the space Kähler metrics, see Definition 3.1 below.

Our proof of Theorem 1 also adapts to the case when the torus $T \subset Aut_r(X)$ is not necessarily maximal, but instead of $T^c$ one takes the infimum of $J(\sigma^* \omega)$ over $\hat{G} := Aut_{\hat{T}}(X)$, the connected component of the identity of the centralizer of $T$ in $Aut_r(X)$ (which by \cite{56} is a reductive group if $X$ admits a $(v, w_0)$-extremal $T$-invariant Kähler metric, see Remark 7.7 for more details). Furthermore, we can also consider any reductive connected subgroup group $G = \mathbb{K}^C \subset \hat{G}$ with a compact form $\mathbb{K}$ containing $T$, and restrict $M_{v, w}$ to the space of $\mathbb{K}$-invariant Kähler metrics in $\alpha$ as in \cite{46}.

As noticed in \cite{15} (in the polarized case) and in \cite{66} (in the more general Kähler case), the coercivity of the Mabuchi energy yields a sharp estimate of the sign of the Donaldson–Futaki invariant of a $T$-equivariant test configuration. In our weighted setting, we consider $\mathbb{T}$-equivariant (compactified) Kähler test configurations $(\mathcal{K}, \mathcal{A})$ associated to $(X, \alpha, T)$, which have smooth total space. To any such test-configuration one can associate a weighted Donaldson–Futaki invariant by the formula (cf. \cite{56})

$$ \mathcal{F}_{v, w}(\mathcal{K}, \mathcal{A}) := -\int_{\mathcal{K}} (\text{Scal}(\Omega) - w(\mu_0)) \Omega^{[m+1]} + (8\pi) \int_X v(\mu_0) \omega^{[m]}, $$

where $\Omega \in \mathcal{A}, \omega \in \alpha$ are $\mathbb{T}$-invariant Kähler forms respectively on $\mathcal{K}$ and $X$, with respective $\Delta$-normalized momentum maps $\mu_0, \mu_\omega$, and $\text{Scal}(\Omega)$ is the $v$-scalar curvature of $\Omega$ defined by \cite{2}.

In the above formula, for any 2-form $\psi$ we use the convention $\psi^{[k]} := \frac{\psi^k}{k!}$ so that $\mathcal{F}_{v, w}(\mathcal{K}, \mathcal{A})$ extends to the weighted setting the expression \cite{63, 73} of the Donaldson–Futaki invariant of $(\mathcal{K}, \mathcal{A})$ in terms of intersection numbers.

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\footnote{We are grateful to Chi Li for pointing this out to us.}
Corollary 1. Under the hypotheses of Theorem 1 for any $T$-equivariant smooth Kähler test configuration $(\mathcal{X}, \mathcal{A})$ of $(X, \alpha, T)$ which has a reduced central fibre, we have the inequality
\[
\mathcal{F}_{v,w}(\mathcal{X}, \mathcal{A}) \geq 0,
\]
with equality if and only if $(\mathcal{X}, \mathcal{A})$ is a product test configuration. Furthermore, if $\alpha = 2\pi c_1(L)$ corresponds to a polarization $L$ of $X$ and $(\mathcal{X}, \mathcal{L}, T)$ is a $T$-equivariant smooth polarized test configuration of $(X, L)$ as above, we have the inequality
\[
\mathcal{F}_{v,w}(\mathcal{X}, \mathcal{A}) \geq \lambda J^{NA}_{m} (\mathcal{X}, \mathcal{A}),
\]
where $\mathcal{A} = 2\pi c_1(\mathcal{L})$, $\lambda > 0$ is the constant appearing in Theorem 1 and $J^{NA}_{m}(\mathcal{X}, \mathcal{A})$ is the $T^C$-relative non-Archimedean $J$-functional of the test configuration introduced in [30, 58], see [20].

Corollary 1 improves the ($T$-equivariant) $(v, w)$-$K$-semistability established in [57] Thm. 2 to ($T$-equivariant) $(v, w)$-$K$-polystability on the test configurations as above, and, in the projective case, further to $T^C$-uniform $(v, w)$-$K$-stability in the sense of [50, 58]. As we already mentioned, the first part of Corollary 1 was proved in [63, 15, 66] in the cscK case $(v = 1$ and $w$ is a constant), and in [34, 69] in the unweighted extremal case $(v = 1 = w_0)$. We however notice that in the extremal case our proof uses directly the coercivity of the relative Mabuchi energy (which follows from Theorem 1) whereas the proofs in [34, 69] and [66] are based respectively on the Arezzo-Pacard existence results of extremal metrics on blow-ups [8], and on the coercivity of the unweighted Mabuchi energy $M_{1,c}$ established in [13, 66]. The $T^C$-uniform $(v, w)$-$K$-stability statement in the second part of Corollary 1 is established in the cscK case in [30, 58], and in the case of a $\nu$-soliton in [46]. Our proof of Corollary 1 in the general weighted case follows easily from Theorem 1 by the established techniques in the cscK case, see Section 4.

Another notable special case where our results apply is when $\alpha = c_1(L)$ for an ample line bundle $L$ over $X$, and $v = E - m^{-1}$, $w_0 = E - m^{-1}$ for a positive affine-linear function on $\Delta$. It is observed in [2] that in this case a $(v, 0)$-extremal Kähler metric in $\alpha$ describes an extremal Sasaki metric on the total space $N$ of the unit circle bundle in $L^{-1}$ with respect to the hermitian metric with curvature $-\omega$ and Reeb vector field corresponding to the lift of $\ell$ to $L^{-1}$ via $\ell$. In this special case, the first part of Corollary 1 above was obtained in [3] for polarized test configurations (see Theorem 1, Conjecture 5.8 and Remark 5.9 in [3]), by using the results in [49] which establish an analogue of Theorem 1 in the Sasaki case. Thus, our proofs of Theorem 1 and Corollary 1 presented in this paper allow one to recast and further generalize [3] Thm. 1 entirely within the framework of weighted Kähler geometry of $X$.

0.4. Method of proof. We now discuss briefly the method of proof of Theorem 1 above. It is an application of the general coercivity principle [29, Thm. 3.4], see Section 3 below. The latter is used in the cscK case in [13], and our approach is mainly inspired by these two references. Noting that in the weighted extremal case $M_{v,w}$ is $G$-invariant and $G := T^C$ is reductive, by the results of [29], in order to obtain Theorem 1 one needs to accomplish the following steps: (1) extend $M_{v,w}$ to the space $\mathcal{E}^1(X, \omega_0)$ of $\omega_0$-relative pluri-subharmonic functions of maximal mass and finite energy; (2) show that the extension is convex and continuous along weak $d_1$-geodesics in $\mathcal{E}^1(X, \omega_0)$; (3) establish a compactness result for the extension of $M_{v,w}$, and (4) show the uniqueness modulo the action of $G$ (and in particular the regularity) of the weak minimizers of $M_{v,w}$, under the assumption that a $(v, w_0)$-extremal metric exists. The steps (1), (2) and (3) in the unweighted cscK case are obtained in [13] and follow from the Chen–Tian formula of $M_{1,1}$. The analogous formula for $M_{v,w}$ is obtained in [69], but the presence of weights does not allow for a straightforward generalization of the arguments in [13]. Similar difficulty arises in [13], in the framework of $\nu$-solitons on a Fano variety, where the authors were able to obtain a suitable extension of the weighted Ding functional to the space $\mathcal{E}^1(X, \omega_0)$. The latter functional has milder dependence on the weights than the weighted Mabuchi functional we consider. Indeed, the arguments of [13] yield the existence of a continuous extension to $\mathcal{E}^1(X, \omega_0)$ of one of the three terms in the Chen–Tian decomposition of $M_{v,w}$, which depend on the weight $w$. Building on [13], Han–Li [46] proposed a new approach to the extension problem in the case of $\nu$-solitons, based on an idea going back to Donaldson [36] (see in particular the proof of Proposition 3 in [36]), which amounts to consider suitable fibre-bundles $Y$ over a cscK bases $B$ and fibre $X$, and show that
the weighted quantities on $X$ correspond to the restrictions of unweighted quantities on the total space $Y$. This is the semi-simple principal $(X, T)$-fibration construction which we review in the next subsection. Going further than [46], we express in general the scalar curvature of a bundle-compatible Kähler metric on $Y$ in terms of the weighted scalar curvature of $X$, and show that the usual (unweighted) Mabuchi energy on $Y$ restricts to a suitably weighted Mabuchi energy on $X$. It thus follows that at least for suitable polynomial weights $v$, the remaining terms of the Chen–Tian decomposition of $M_{v,w}$ can be extended to $E^0(Y, \omega_0)$ simply by restricting to the fibres the corresponding (unweighted) extension of the Mabuchi energy of $Y$. The final crucial observation for obtaining the extension for any weights is that $M_{v,w}$ depends linearly and continuously on $(v, w)$, so that one can further use (as in [46]) the Stone–Weierstrass approximation theorem over $C^0(\Delta)$. With this in place, and using the weighted analogue of the uniqueness [11] achieved in [57], we can adapt the arguments from [15].

0.5. Applications to the semi-simple principal fibration construction. We briefly review here the semi-simple principal bundle construction, which is a key tool in our proof of Theorem 1 but also provides a framework for further geometric applications of our results, extending the setting of the generalized Calabi construction in [7].

We denote by $T$ a compact $r$-dimensional torus with Lie algebra $t$ and lattice $\Lambda \subset t$ of generators of $S^1$-subgroups, i.e. $T = \mathbb{R}^r / 2\pi \Lambda$. Let $B = B_1 \times \cdots \times B_k$ be a $2n$-dimensional cscK manifold which is a product of compact cscK Hodge Kähler $2n$-manifolds $(B_a, \omega_{B_a})$, $a = 1, \ldots, k$. We then consider a principal $T$-bundle $\pi : P \to B$ endowed with a connection 1-form $\theta \in \Omega^1(P, t)$ with curvature
\[
d\theta = \sum_{a=1}^{k} (\pi^* \omega_{B_a}) \otimes p_a, \quad p_a \in \Lambda.
\]
For any smooth compact Kähler $2m$-manifold $(X, \omega_X, T)$, endowed with a hamiltonian isometric action of the torus $T$ as in the setup above, we can construct the principal $(X, T)$-fibration
\[
Y := (X \times P)/T \to B,
\]
where the $T$-action on the product is $\sigma(x, p) = (\sigma^{-1}x, \sigma p)$, $x \in X$, $p \in P$, $\sigma \in T$. Using the chosen connection on $P$, the almost complex structures on $X$ and $B$ lift to define a CR structure on the product $X \times P$, and thus endow $Y$ with the structure of a $2(m+n)$-dimensional smooth complex manifold. Furthermore, $Y$ comes equipped with an induced holomorphic fibration $\pi : Y \to B$, with smooth complex fibres $X$, and induced fibre-wise $T$-action. Fixing constants $c_a \in \mathbb{R}$ such that for each $a = 1, \ldots, k$, the affine linear function $(p_a, \mu) + c_a$ on $t^*$ is strictly positive on the momentum image $\Delta$ of $X$, one can define a lifted Kähler metric $\omega_Y$ on $Y$ which, pulled-back to $X \times P$, has the form
\[
\omega_Y := \omega_X + \sum_{a=1}^{k} ((p_a, \mu_\omega) + c_a) \pi^* \omega_{B_a} + \langle dm_\omega \wedge \theta \rangle
\]
where $\langle \cdot, \cdot \rangle$ stands for the natural pairing of $t$ and $t^*$; thus $(p_a, \mu_\omega)$ is a smooth function and $\langle d\mu_\omega \wedge \theta \rangle$ is a 2-form on $X \times P$. As we show in Section 5 below, when $\omega_X$ varies in a given Kähler class of $X$, the corresponding Kähler metric $\omega_Y$ will vary in a fixed Kähler class on $Y$. We also notice that when $(X, \omega_X, T)$ is a smooth toric Kähler manifold, the setup above reduces to the theory of semi-simple rigid toric fibrations studied in [4, 5, 7]. Inspired by the results in the latter works, we show that the scalar curvature of $\omega_Y$ can be expressed in terms of the $p$-weighted scalar curvature of $(X, \omega_X)$, where the weight function $p(\mu)$ is a polynomial depending on the fixed data $(p_a, c_a, n_a)$ of the construction. With this observation in mind, we show that (similarly to the case of semi-simple rigid toric fibrations recently studied in [53]) the recent results [23, 48] can be used to obtain a converse of Theorem 1 in the case of a semi-simple principal fibrations.

Theorem 2. Suppose $Y$ is a semi-simple principal $(X, T)$-fibration, with a Kähler metric $\omega_Y$ induced by a $T$-invariant Kähler metric $\omega_X$ on $X$. We suppose, moreover, that $T$ is a maximal torus in the reduced group of automorphisms $\text{Aut}_r(X)$. Then, the following conditions are equivalent
(i) $Y$ admits an extremal Kähler metric in the Kähler class $[\omega_Y]$;
(ii) $X$ admits a $\mathbb{T}$-invariant $(p, \tilde{w})$-cscK metric in the Kähler class $[\omega_X]$, with weights
\[
p(\mu) = \prod_{a=1}^{k} \left( (p_a, \mu) + c_a \right)^{\alpha_a}, \quad \tilde{w}(\mu) = p(\mu) \left( - \sum_{a=1}^{k} \frac{\text{Scal}(\omega_{B_a})}{(p_a, \mu) + c_a} + e^{\text{ext}}(\mu) \right),
\]
where $e^{\text{ext}}$ is an affine-linear function determined by the condition \((3)\):

(iii) The weighted Mabuchi energy $\mathbf{M}^{X,p,\tilde{w}}_{\omega}$ of $(X, [\omega_X], \mathbb{T})$ is coercive with respect to $\mathbb{T}^C$, where $p, \tilde{w}$ are the weights defined in (ii).

Compared to the general setting of \cite{35}, the semi-simple principal $(X, \mathbb{T})$-fibration (trivially) satisfy the condition of optimal symplectic connection. Accordingly, one can conclude by \cite{35} that $(X, [\omega_X])$ admits an extremal Kähler metric, provided that $(X, \omega_X)$ is cscK, and if we take large enough constants $c_a$. As a matter of fact, the conclusion also follows under the more general assumption that $(X, \omega_X)$ is extremal, by the proof of \cite[Thm. 3]{6}. The novelty of Theorem 2 is therefore in the fact that it gives a precise condition (in terms of $X$) for the existence of an extremal Kähler metric in a given Kähler class $[\omega_Y]$, also revealing that $(X, [\omega_X])$ needs not to be extremal in general. We finally note that in the case of toric fibre, \cite{53} provides a further equivalence with a certain weighted notion of uniform $K$-stability of the corresponding Delzant polytope.

If all the factors $(B_a, \omega_{B_a})$ of the base are positive Kähler–Einstein manifolds, and the fibre $(X, \mathbb{T})$ is a smooth Fano variety, the semi-simple principal $(X, \mathbb{T})$-fibration construction can produce a smooth Fano variety $Y$ for suitable choice of the principal $\mathbb{T}$-bundle over $B$ (see Lemma \ref{lem:5.11} below). In this case, combining \cite[Thm. 3.5]{46} with the results in this paper, we get

**Theorem 3.** Suppose $Y$ is a Fano semi-simple principal $(X, \mathbb{T})$-fibration, obtained from the product of positive Kähler–Einstein Hodge manifolds $(B_a, \omega_{B_a})$ and a smooth Fano fibre $(X, \mathbb{T})$ via Lemma \ref{lem:5.11}. Suppose also that $\mathbb{T}$ is a maximal torus in the automorphism group Aut$(X)$. Then $Y$ admits a $v$-soliton in $2\pi c_1(Y)$, provided that the weighted Mabuchi functional $\mathbf{M}^{X,p,\tilde{w}}_{\omega}$ of $(X, \mathbb{T}, 2\pi c_1(X))$ is coercive with respect to $\mathbb{T}^C$, where $p$ is the weight defined in Theorem 2 (ii) and
\[
w = 2pe \left( m + \langle d \log v, \mu \rangle + \langle d \log p, \mu \rangle \right).
\]

If, furthermore, the fibre $(X, \mathbb{T})$ is a smooth toric Fano variety, then the latter condition is equivalent to the vanishing of the Futaki invariant \((3)\) associated to the weights $(p v, \tilde{w})$ on $X$. In particular, any Fano semi-simple principal $(X, \mathbb{T})$-fibration with smooth toric Fano fibre $(X, \mathbb{T})$ admits a Kähler–Ricci soliton, and the corresponding affine cone $(K_Y)^{\times}$ admits a Calabi–Yau cone metric, given by a Sasaki–Einstein structure on a unit circle bundle associated to the canonical bundle $K_Y$.

The existence of a Kähler–Ricci soliton in the above setting is essentially known even though we didn’t find it explicitly stated in the literature. In the toric case (i.e. when $Y = X$ and $B$ is a point) the result follows by \cite{14} (see also \cite{30}), and for $\mathbb{P}^1$-bundles by \cite{20, 20a, 9}. In the general case, the result can be obtained from \cite{67}, which in turn extend \cite{14} to the framework of multiplicity-free manifolds, but the arguments can be also adapted to the case of semi-simple principal $(X, \mathbb{T})$-fibrations (see \cite[Rem.7]{14} and \cite{37}). Our approach, however, builds on the idea of \cite{30}. There are also related existence results for Kähler–Ricci solitons on spherical manifolds, see \cite{42, 31}. On the other hand, the existence of Sasaki–Einstein metrics seems to be new in the above stated generality. Indeed, in the toric case the claim follows from \cite{40}, and there are known existence results \cite{15, 66} on $\mathbb{P}^1$-bundles. We expect our arguments to extend to spherical manifolds too.

**0.6. Structure of the paper.** In Section 1, we recall the setup of weighted cscK metrics and state the main results we shall need from \cite{56, 57}. In Section 2, we recall the notion of $v$-solitons from \cite{59, 40}, and establish the equivalences stated in Propositions 1 and 2. Sections 3 and 4 review and recast in the weighted setting respectively the coercivity principle of \cite{29} and its application to stability \cite{15, 66}, thus outlining the main steps needed for the proofs of Theorem 1 and deriving Corollary 11 from the latter. In Section 5, we introduce the semi-simple principal
(X, T)-fibration construction, and establish the main geometric properties allowing us to extend the results from [7]. In Section 6, we use an idea from [46] in order to define an extension of the weighted Mabuchi energy to the space $E$, and show its convexity and compactness properties. In Section 7, we extend the arguments of [15] to show that weak minimizers of the weighted Mabuchi energy are smooth. Here, we complete the proof of Theorem 1. In Section 8, we detail the proofs of Theorems 2 and 3. In the Appendix, we present some technical computational results, detailing the linearization of the scalar and the twisted scalar curvature of a semi-simple principal (X, T)-fibre and re-casting the weighted Futaki invariant [3], which are needed for the proofs of Theorem 2 and 3.

1. Preliminaries on the weighted cscK problem

We recall the setup from [56]. Let X be a smooth compact, connected Kähler manifold of (real) dimension $2m$, and let

$$K(X, \omega_0) = \{ \varphi \in C^\infty(X) \mid \omega_\varphi := \omega_0 + dd^c\varphi > 0 \}$$

be the space of $\omega_0$-relative smooth Kähler potentials on X. We let $T \subset Aut_r(X)$ be a fixed compact torus in the reduced group of automorphisms of X, i.e. the connected closed subgroup $Aut_r(X)$ of the group of complex automorphisms $Aut(X)$, whose Lie algebra is the space of holomorphic vector fields of X with zeros (see e.g. [11]). Equivalently, $Aut_r(X)$ is the connected component of the identity of the kernel of the natural group homomorphism from $Aut(X)$ to the Albanese torus, and is known to be isomorphic to the linear algebraic group in the Chevalley-type decomposition of $Aut(X)$, cf. [38]. We denote by $C^\infty_T(X)$ the space of T-invariant smooth functions on X and introduce the space

$$K_T(X, \omega_0) := K(X, \omega_0) \cap C^\infty_T(X),$$

of T-invariant relative Kähler potentials, assuming also that $\omega_0$ is T-invariant.

It is well-known that the action of T on $(X, \omega_0)$ is hamiltonian, and we let $\mu_0 : X \to t^*$ be a momentum map, where t is the Lie algebra of T and $t^*$ the dual vector space. By the convexity theorem [9, 14], the image $\Delta := \mu_0(X) \subset t^*$ is a compact convex polytope. For any $\varphi \in K_T(X, \omega_0)$, the smooth $t^*$-valued function

$$\mu_\varphi = \mu_0 + d^c\varphi$$

is the T-momentum map of $(X, \omega_\varphi)$, normalized by the condition $\mu_\varphi(X) = \Delta$. In the above formula, $d^c\varphi$ is viewed as a smooth $t^*$-valued function via the identity $\langle d^c\varphi, \xi \rangle := d^c\varphi(\xi)$ for any $\xi \in t \subset C^\infty(X, TX)$.

1.1. The $(v, w)$-constant scalar curvature Kähler metrics. Following [55], let $v(\mu) > 0$ and $w(\mu)$ be smooth functions on $\Delta$. One can then consider the condition (1) for a T-invariant Kähler metric $\omega_\varphi$ in $\alpha$ (and the fixed polytope $\Delta$), called $(v, w)$-cscK metric. We thus want to solve the following PDE for $\varphi \in K_T(X, \omega_0)$:

$$\text{Scal}_v(\omega_\varphi) = w(\mu_\varphi),$$

$$\text{Scal}_w(\omega_\varphi) := v(\mu_\varphi) \text{Scal}(\omega_\varphi) + 2\Delta_\omega_\varphi v(\mu_\varphi) + \langle g_\varphi, \mu_\varphi (\text{Hess}(v)) \rangle$$

As we explained in the Introduction, the problem of finding $\omega_\varphi \in \alpha$ solving (6) is obstructed by the condition (3), and in the case when $v, w_0$ are positive weights, this can be resolved (similarly to the approach in [21]) by finding a unique affine-linear function $\mu^\text{ext}_{v, w_0}(\mu)$ on $t^*$, called the extremal function, such that for any $\omega_\varphi$

$$\int_X \left( \text{Scal}_v(\omega_\varphi) - \mu^\text{ext}_{v, w_0}(\mu_\varphi) w_0(\mu_\varphi) \right) \ell(\mu_\varphi) \omega_\varphi^{[m]} = 0, \quad \forall \ell \in \text{Aff}(t^*).$$

Geometrically, the above condition means that the weighted cscK problem with weights $(v, w) = (v, \mu^\text{ext}_{v, w_0})$ is unobstructed in terms of (3), and a solution $\omega_\varphi$ of the $(v, \mu^\text{ext}_{v, w_0})$-cscK problem is referred to as $(v, w_0)$-extremal metric.
1.2. The weighted Mabuchi energy.

**Definition 1.1.** [56] Let \( v, w \) be weight functions on \( \Delta \) with \( v(\mu) > 0 \). The weighted Mabuchi energy \( M_{v,w} \) on \( K^{\natural}(X, \omega_0) \) is defined by

\[
(d_{\omega} M_{v,w})(\varphi) = -\int_X (\text{Scal}_v(\omega_\varphi) - w(\mu_\varphi)) \varphi \omega^{[m]}, \quad M_{v,w}(0) = 0.
\]

**Remark 1.2.** It follows from the above definition and the results in [56] that for a constant \( c, M_{v,w}(\varphi + c) = M_{v,w}(\varphi) \) if and only if \( v, w \) satisfy the integral relation

\[
(7) \quad \int_X \text{Scal}_v(\omega_0)\omega^{[m]} = \int_X w(\mu_0)\omega_0^{[m]}.
\]

Furthermore, by the results in [56], (7) is a necessary condition for the existence of a solution of (6) and it is incorporated in the definition of \( M_{v,w} \), but we do not assume a priori this condition in the current article. It is however automatically satisfied if \( \alpha \) admits a \( T \)-invariant \((v, w)\)-cscK metric, or if we consider the weights \((v, w) = (v, v_{\text{ext}} w_0 \omega_0)\) corresponding to \((v, w_0)\)-extremal Kähler metrics. In these cases, we shall write \( M_{v,w}(\omega_\varphi) \) to emphasize that the weighted Mabuchi functional acts on the space of \( T \)-invariant Kähler metrics in \( \alpha = [\omega_0] \).

The following result is established in [57], generalizing [11] to arbitrary weights \( v > 0, w \).

**Theorem 1.3.** If \( \omega \) is a \( T \)-invariant \((v, w)\)-cscK metric on \((X, \alpha, T, \Delta)\), then for any \( \varphi \in K^{\natural}(X, \omega_0), \ M_{v,w}(\omega_\varphi) \geq M_{v,w}(\omega) \).

1.3. The automorphism group of a \((v, w_0)\)-extremal Kähler manifold. In what follows we will consider connected Lie groups. We recall that we have set \( \text{Aut}_r(X) \) to be the connected component of the identity of the kernel of the Albanese homomorphism and, similarly, we denote by \( \text{Aut}_r^T(X) \) the connected component of the identity of the centralizer of the torus \( T \) in \( \text{Aut}_r(X) \). We shall use the following result, established in [56 Thm. B.1] (cf. also [39]) and [57 Rem. 2]:

**Proposition 1.4.** If \((X, \alpha, T)\) admits a \((v, w_0)\)-extremal Kähler metric \( \omega \), then the connected component of the identity \( \text{Aut}_r^T(X) \) of the subgroup of \( T \)-commuting automorphisms in \( \text{Aut}_r(X) \) is reductive, and \( \omega \) is invariant under the action of a maximal compact connected subgroup of \( \text{Aut}_r^T(X) \). In particular, the isometry group of \((X, \omega)\) contains a maximal torus \( T_{\text{max}} \subset \text{Aut}_r(X) \) with \( T \subset T_{\text{max}} \). If, furthermore, \( T = T_{\text{max}} \), then \( \text{Aut}_r^T(X) = T^{\mathbb{C}} \).

Because of this result, we shall often assume (without loss of generality for solving (6)) that \( T = T_{\text{max}} \subset \text{Aut}_r(X) \) and thus \( \text{Aut}_r^T(X) = T^{\mathbb{C}} \).

1.4. Uniqueness of the \((v, w_0)\)-extremal Kähler metrics. Another key result in the theory is the extension in [57] of the uniqueness results [11][24] to the weighted setting.

**Theorem 1.5.** Suppose \( \omega, \omega' \) are \( T \)-invariant \((v, w_0)\)-extremal Kähler metrics. Then there exists \( \sigma \in \text{Aut}_r^T(X) \) such that \( \sigma^*(\omega') = \omega \). In particular, if \( T \subset \text{Aut}_r(X) \) is maximal, then the uniqueness holds modulo \( T^{\mathbb{C}} \).

2. \( v \)-solitons as weighted cscK metrics.

We review here the definition of \( v \)-solitons on a Fano manifold, following [13][16], and discuss their link with \((v, w)\)-cscK metrics.

We thus suppose throughout this section that that \( X \) is a smooth Fano manifold, \( \alpha := 2\pi c_1(X) \) and \( T \subset \text{Aut}(X) \) a fixed compact torus. (We recall here that on a Fano manifold, \( \text{Aut}_r(X) \) coincides with the connected component of the identity of the full automorphism group.) We further consider the natural action of \( T \) on the anti-canonical bundle \( K_X^{-1} \) of \( X \), which normalizes the momentum map \( \mu_\omega \) of each \( T \)-invariant Kähler metric \( \omega \in \alpha \), and fixes the momentum image \( \Delta \). We shall sometimes refer to this normalization as the canonical normalization of \( \Delta \). In this setup, we recall...
Definition 2.1. Let $v > 0$ be a positive smooth weight function on $\Delta$. A $v$-soliton on $X$ is a $T$-invariant Kähler metric $\omega \in 2\pi c_1(X)$ which satisfies the relation [4]:

$$\rho_\omega - \omega = \frac{1}{2} dd^c \log v(\mu_\omega).$$

In the special case $v = e^{(\xi,i)}$ we obtain a Kähler–Ricci soliton in the sense of [72].

Lemma 2.2. A $T$-invariant Kähler metric $\omega \in 2\pi c_1(X)$ is a $v$-soliton if and only if $\omega$ is a $(v,w)$-cscK metric with weight $w(\mu) = 2v(\mu)\left[m + \langle d \log v(\mu), \mu \rangle \right]$.

Proof. We start by showing that (4) implies that $\omega$ is $(v,w)$-cscK with the weight $w$ specified in the Lemma. Taking the trace in (4) with respect to $\omega$ we get

$$Scal(\omega) - 2m = -\Delta_\omega \left(\log v(\mu_\omega)\right) = - \frac{1}{v(\mu_\omega)} \Delta_\omega (v(\mu_\omega)) - \frac{1}{v(\mu_\omega)^2} g_\omega \left( dv(\mu_\omega), dv(\mu_\omega) \right)$$

(8)

$$= - \sum_{i=1}^{m} \frac{v_i(\mu_\omega)}{v(\mu_\omega)} (\Delta_\omega \mu_\omega^i) + \sum_{i,j=1}^{m} \frac{v_{ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j)$$

$$- \sum_{i,j=1}^{m} \frac{v_i(\mu_\omega) v_j(\mu_\omega)}{v(\mu_\omega)^2} g_\omega(\xi_i, \xi_j)$$

where $(\xi_i)_{i=1,...,r}$ is a basis of $t$ and $v_i$ denotes the partial derivative in direction of $\xi_i$. On the other hand, by taking the interior product of (4) with $\xi_i$ and using that $\xi_i$ is Killing with respect to $\omega$, we get

$$-d\Delta_\omega \mu_\omega^i + 2d\mu_\omega^i = d \left( d^c \left( \log v(\mu_\omega) \right)(\xi_i) \right) = d \left( \sum_{j=1}^{m} \frac{v_{ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j) \right),$$

where $\mu_\omega^i := \langle \mu_\omega, \xi_i \rangle$ is the momentum of $\xi_i$. It follows that

$$-\Delta_\omega \mu_\omega^i + 2\mu_\omega^i = \sum_{j=1}^{m} \frac{v_{ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j) + c$$

(9)

for some constant $c$. As we consider the canonical normalization of $\mu_\omega$ (corresponding to the natural lifted $T$-action on $K_X^{-1}$), one can see that $c = 0$. Indeed, the infinitesimal actions $A_i$ of the elements of the basis $(\xi_i)$ on smooth sections of $K_X^{-1}$ are given by $A_i(s) := \mathcal{L}_{\xi_i}s$. We denote by $H_g$ the induced hermitian metric on $K_X^{-1}$ through the Riemannian metric $g_\omega$ of $\omega$ (so that $H_g$ has curvature $\rho_\omega$) and by $H = v(\mu_\omega)H_g$ the induced hermitian metric with curvature $\omega$ (by using [4]); comparing the actions of the corresponding Chern connections, $\nabla^g_{\xi_i}$ and $\nabla^H_{\xi_i} = \nabla^g_{\xi_i} - \frac{1}{2} d^c \log v(\mu_\omega)(\xi_i) id$ on smooth sections of $K_X^{-1}$ with the infinitesimal actions $A_i$ gives (see e.g. [11] Prop. 8.8.2 & 8.8.3)

$$A_i(s) = \nabla^g_{\xi_i}s + \sqrt{-1} \left( \Delta_\omega \mu_\omega^i \right)s, \quad A_i(s) = \nabla^H_{\xi_i}s + \sqrt{-1} \mu_\omega^i s.$$ 

(10)

We thus deduce $\frac{1}{2} \Delta_\omega \mu_\omega^i = \mu_\omega^i - \frac{1}{2} d^c \left( \log v(\mu_\omega) \right)(\xi_i)$, i.e. $c = 0$ in (9).

Now letting $c = 0$ in (9), multiplying it by $\frac{v_i(\mu_\omega)}{v(\mu_\omega)}$, and taking the sum over $i$, give

$$\sum_{i=1}^{r} \frac{v_i(\mu_\omega)}{v(\mu_\omega)^2} g_\omega(\xi_i, \xi_j) = \sum_{i=1}^{r} \frac{v_i(\mu_\omega)}{v(\mu_\omega)} (\Delta_\omega \mu_\omega^i) - 2 \sum_{i=1}^{r} \frac{v_i(\mu_\omega)}{v(\mu_\omega)} \mu_\omega^i,$$
which substituting back in \(8\) yields

\[
\text{Scal}(\omega) - 2m = -2 \sum_{i=1}^{r} \frac{v_i(\mu_\omega)}{v(\mu_\omega)} \Delta_\omega \mu_\omega^{\xi_i} + \sum_{i,j=1}^{m} \frac{v_{i,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j) + 2 \sum_{i=1}^{r} \frac{v_i(\mu_\omega)}{v(\mu_\omega)} \mu_\omega^{\xi_i}
\]

\[
= -2 \sum_{i=1}^{r} \frac{v_i(\mu_\omega)}{v(\mu_\omega)} \Delta_\omega \mu_\omega^{\xi_i} + 2 \sum_{i,j=1}^{m} \frac{v_{i,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j)
\]

\[
- \sum_{i,j=1}^{m} v_{i,j}(\mu_\omega) g_\omega(\xi_i, \xi_j) + 2\{d \log v, \mu_\omega\}
\]

\[
= -2 \Delta_\omega (v(\mu_\omega)) - \sum_{i,j=1}^{m} \frac{v_{i,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j) + 2\{d \log v, \mu_\omega\}.
\]

Thus \(\text{Scal}_v(\omega) = w(\mu_\omega)\).

Now we show the converse. To this end, let \(\omega \in 2\pi c_1(X)\) be a \(T\)-invariant Kähler metric, \(v > 0\) a positive smooth function on the canonically normalized polytope \(\Delta\) and \(w = 2(m + \{d \log \mu\})v\) the weight defined in Lemma 2.2. Let \(h \in C^\infty_c(X)\) be an \(\omega\)-relative Ricci potential, i.e.

\[
\rho_\omega - \omega = \frac{1}{2} dd^c h.
\]

Taking the trace with respect to \(\omega\) and the interior product with \(\xi \in \mathfrak{t}\) in the above identity we get

\[
\text{Scal}(\omega) = 2m - \Delta_\omega h, \quad \Delta_\omega \mu_\omega^{\xi} + \mathcal{L}_\xi h = 2\mu_\omega^{\xi},
\]

where we have used the canonical normalization of \(\mu_\omega\) to determine the additive constant in the second inequality (as we did for \(9\)). Similar computations as in the first part of the proof (using \(10\)) give

\[
\text{Scal}_v(\omega) - w(\mu_\omega)
\]

\[
= - v(\mu_\omega)(\Delta_\omega h) + 2 \sum_{i=1}^{r} v_i(\mu_\omega)(\Delta_\omega \mu_\omega^{\xi_i}) - \sum_{i,j=1}^{r} v_{i,j}(\mu_\omega)(\xi_i, \xi_j) + 2 \sum_{i=1}^{r} v_i(\mu_\omega) \mu_\omega^{\xi_i}
\]

\[
= - v(\mu_\omega)(\Delta_\omega h) + \sum_{i=1}^{r} v_i(\mu_\omega) g_\omega(dh, d\mu_\omega^{\xi_i})
\]

\[
+ \sum_{i=1}^{r} v_i(\mu_\omega)(\Delta_\omega \mu_\omega^{\xi_i}) - \sum_{i,j=1}^{r} v_{i,j}(\mu_\omega) g_\omega(\xi_i, \xi_j)
\]

\[
= - v(\mu_\omega)(\Delta_{\omega,v} h) + \sum_{i=1}^{r} v_i(\mu_\omega)(\Delta_{\omega,v} \mu_\omega^{\xi_i}) - \sum_{i,j=1}^{r} v_{i,j}(\mu_\omega) g_\omega(\xi_i, \xi_j),
\]

where \(\Delta_{\omega,v} := \frac{1}{v(\mu_\omega)} \delta_\omega v(\mu_\omega) d\) is the weighted Laplacian, see Appendix A. Using the second equality in \(8\), we compute

\[
v(\mu_\omega) \Delta_{\omega,v} (\log v(\mu_\omega)) = v(\mu_\omega)(\Delta_\omega \log v(\mu_\omega)) - \sum_{i=1}^{m} v_i(\mu_\omega) g_\omega(d(\log v(\mu_\omega)), d\mu_\omega^{\xi_i})
\]

\[
= v(\mu_\omega)(\Delta_\omega \log v(\mu_\omega)) - \sum_{i,j=1}^{m} \frac{v_i(\mu_\omega) v_{i,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j)
\]

\[
= \sum_{i=1}^{r} v_i(\mu_\omega)(\Delta_{\omega,v} \mu_\omega^{\xi_i}) - \sum_{i,j=1}^{r} v_{i,j}(\mu_\omega) g_\omega(\xi_i, \xi_j).
\]

Substituting back in \(12\) we obtain

\[
\text{Scal}_v(\omega) - w(\mu_\omega) = v(\mu_\omega) \Delta_{\omega,v} (\log v(\mu_\omega) - h).
\]
It follows that if \( \omega \) is \( (v, w) \)-cscK then \( h = \log v(\mu, \omega) + c \) by the maximum principle, showing that \( \omega \) satisfies (4).

Remark 2.3. Using the second relation in (11) it follows that under the canonical normalization of \( \mu, \omega \) we have

\[
\int_X \mu^\xi e^h \omega^{[m]} = 0, \quad \xi \in t.
\]

This is precisely the normalization of \( \mu, \omega \) used in [22, Sect.2].

**Lemma 2.4.** Let \( v := \ell^{-(m+2)} \) where \( \ell(\mu) = (\xi, \mu) + a \) is a positive affine-linear function on \( \Delta \).

Then \( \omega \in 2\pi c_1(X) \) is a \( v \)-soliton if and only if \( \omega \) is \( (\ell^{-(m+1)}, 2ma(-m+2)) \)-cscK metric.

**Proof.** The proof is similar to the one of Lemma 2.2.

If \( \omega \) is a \( v \)-soliton with \( v := \ell^{-(m+2)} \), specializing (8) and (9) to the specific choice of \( v \), and letting \( f := \ell(\mu, \omega) = \mu^\xi + a \), we get the identities

\[
\text{Scal}(\omega) = 2m + (m+2)\Delta \log f, \quad -\Delta f + 2f = \frac{(m+2)}{f} g_\omega(df, df) + 2a.
\]

Multiplying by \( f^2 \) the first equality and taking the sum with the second equality multiplied by \( mf \) gives

\[
f^2 \text{Scal}(\omega) - 2(m+1)f\Delta f - (m+1)(m+2)g_\omega(df, df) = 2maf.
\]

The RHS is the \((m+2, f)\)-scalar curvature (see [2]) and it is straightforward check that the above equality is equivalent with the condition that \( \omega \) is an \((\ell^{-(m+1)}, 2ma(-m+2))\)-cscK metric.

In the other direction, for any \( T \)-invariant Kähler metric \( \omega \in 2\pi c_1(X) \) we let \( f := \ell(\mu, \omega) = \mu^\xi + a > 0 \) be the corresponding Killing potential and let \( h \in C^\infty_\Gamma(X) \) be such that \( \rho_\omega - \omega = \frac{1}{2} \log h \).

From (11) we have

\[
\text{Scal}(\omega) = 2m - \Delta h, \quad -\Delta f + 2f = -g_\omega(df, dh) + 2a.
\]

Multiplying the first identity by \( f^2 \) and summing with the second identity multiplied by \( mf \) gives

\[
f^2 \text{Scal}(\omega) - 2(m+1)f\Delta f - (m+1)(m+2)g_\omega(df, df) - 2maf
\]

\[
= -f^2 \left( \frac{\Delta f}{f} (h + (m+2) \log f) + mg_\omega(d \log f, dh + (m+2) d \log f) \right).
\]

If we suppose that (15) holds, we conclude again by the maximum principle that \((m+2) \log f + h\) must be constant.

**Remark 2.5.** Lemmas 2.2 and 2.4 give two different realizations of the same \( \ell^{-(m+2)} \)-soliton as a weighted cscK metric, with respective weights \((\ell^{-(m+2)}, 2(2m+2a)\ell^{-(m+3)})\) and \((\ell^{-(m+1)}, 2ma\ell^{-(m+2)})\).

We derive from Lemma 2.4 and the correspondence in [2] the following fact, which does not seem to have been noticed before.

**Lemma 2.6.** On a Fano manifold \((X, T)\), a \( T \)-invariant Kähler metric \( \omega \in 2\pi c_1(X) \) is a \( \ell^{-(m+2)} \)-soliton with respect to a positive affine linear function \( \ell = (\xi, \mu) + a \) if and only if the lift \( \tilde{\xi} \) of the vector field \( \xi \) to \( K_X \), via the hermitian connection \( \nabla^h \) with curvature \( -\omega \) and the \( \omega \)-momentum \( \ell(\mu, \omega) \) of \( \xi \), is a Reeb vector of a Sasaki–Einstein (transversal) structure of transversal scalar curvature \( 2am \), defined on the unit circle bundle \( N \) of \((K_X, h)\).

**Proof.** By Lemma 2.4 we need to show that an \((\ell^{-(m+1)}, 2ma\ell^{-(m+2)})\)-cscK metric in \( 2\pi c_1(X) \) corresponds to a Sasaki–Einstein structure as defined in the statement. By [2] Thm. 1, the condition that \( \omega \) is \((\ell^{-(m+1)}, 2ma\ell^{-(m+2)})\)-cscK is equivalent to the condition that the corresponding Sasaki structure has transversal scalar curvature equal to \( 2ma \) (notice that \( a > 0 \) by the positivity of \( \ell \) over the canonical polytope \( \Delta \)). Any Sasaki structure of constant transversal scalar curvature on \( N \subset K_X \) is transversally Kähler–Einstein as \( c_1(K_X^\times) = 0 \), and therefore the first Chern class
of the CR distribution of $N$ vanishes (see e.g. [19] Cor. 5.3 and [20] Prop. 4.3). This completes the proof. \hfill \Box

Remark 2.7. The correspondence in Lemma 2.6 is, in fact, local and can be deduced directly from the relation between the transversal Ricci tensors of the two Sasaki structures on the CR manifold $N \subset K_X$, respectively defined by $\xi$ and the regular Reeb vector field $\hat{\chi}$ (cf. [22] [12]).

Proofs of Propositions 1 and 2. Propositions 1 and 2 from the introduction follow directly from Lemmas 2.2, 2.4 and 2.6 above. \hfill \Box

3. The coercivity principle: Plan of proof of Theorem 1

We consider the following general setup, based on the results of [24], [27] [71]. As before, we let $T \subset Aut_r(X)$ be a fixed connected compact torus in the reduced group of automorphisms of $X$, and denote by $G = \mathbb{T}^C \subset Aut_r(X)$ the corresponding complex torus.

Following [27], we consider the $L_1$-length function on $K(X, \omega_0)$, introduced on a smooth curve $\psi_t, t \in [0,1]$ by

$$L_1(\psi_t) := \int_0^1 \left( \int_X |\psi_t| \omega_0^m \right) ds,$$

and, for $\varphi_0, \varphi_1 \in K(X, \omega_0)$, we let

$$d_1(\varphi_0, \varphi_1) := \inf \{ L_1(\psi_t) \mid \psi_t \in K(X, \omega_0), t \in [0,1] \mid \psi_0 = \varphi_0, \psi_1 = \varphi_1 \}.$$

Similarly we define $d_1$ on $K_T(M, \omega_0)$ by considering infimum over smooth curves in $K_T(X, \omega_0)$. It is proved in [27] that $(K(X, \omega_0), d_1)$ is a metric space, and it is observed in [29] that $(K_T(X, \omega_0), d_1)$ is a metric subspace of $(K(X, \omega_0), d_1)$.

Recall the following well-known functionals on $K(X, \omega_0)$.

Definition 3.1. Let $I$ denote the functional on $K(X, \omega_0)$ defined by

$$(d_\varphi I)(\varphi) = \int_X \varphi \omega_0^m, \quad I(0) = 0,$$

and let $J(\varphi) := \int_X \varphi \omega_0^m - I(\varphi)$.

Remark 3.2. For any constant $c$, $I(\varphi + c) = I(\varphi) + c \text{Vol}(X, \omega_0)$ (where $\text{Vol}(X, \omega_0) = \int_X \omega_0^m$) stands for the total volume of $(X, \omega_0)$ whereas $J(\varphi + c) = J(\varphi)$, i.e. we can see $J$ as a functional on the space of Kähler metrics in the Kähler class $\alpha = [\omega_0]$, which motivates the notation $J(\omega_\varphi)$. One can further show that $J(\omega_\varphi) \geq 0$ with equality iff $\omega_\varphi = \omega_0$.

By the above remark, for any Kähler metric $\omega_\varphi$ in the Kähler class $[\omega_0]$, there exists a uniquely determined $\omega_0$-relative potential $\varphi \in K(X, \omega_0)$ satisfying

$$I(\varphi) = 0.$$

We shall denote by $\hat{K}(X, \omega_0)$ (resp. $\hat{K}_T(X, \omega_0)$) the subspaces of normalized $\omega_0$-relative Kähler potentials satisfying the above equality. We notice that the group $G = \mathbb{T}^C$ naturally acts on the space of Kähler metrics in $[\omega_0]$, preserving the subspace of $T$-invariant Kähler metrics. This induces an action $[G]$ on the spaces $\hat{K}(X, \omega_0)$ and $\hat{K}_T(X, \omega_0)$, such that

$$\omega_\sigma[\varphi] = \sigma^* (\omega_\varphi), \quad \forall \sigma \in G, \ \varphi \in \hat{K}(X, \omega_0).$$

We introduce the $G$-relative distance on $\hat{K}(X, \omega_0)$ and $\hat{K}_T(X, \omega_0)$ by

$$d_1^{[G]}(\varphi_0, \varphi_1) = \inf_{\sigma_0, \sigma_1 \in G} d_1(\sigma_0[\varphi_0], \sigma_1[\varphi_1]).$$

It is proved in [29] that $d_1^{[G]}$ is $G$-invariant, i.e. $d_1^{[G]}(\sigma[\varphi_0], \sigma[\varphi_1]) = d_1^{[G]}(\varphi_0, \varphi_1)$ and thus

$$d_1^{[G]}(\varphi_0, \varphi_1) = \inf_{\sigma \in G} d_1(\varphi_0, \sigma[\varphi_1]).$$

Definition 3.3. Let $F$ be a functional on $K_T(X, \omega_0)$. We say that $F$ is $G$-coercive if there exist uniform positive constants $(\lambda, \delta)$ such that

$$F(\varphi) \geq \lambda d_1^{[G]}(0, \varphi) - \delta, \quad \forall \varphi \in \hat{K}_T(X, \omega_0).$$
It is sometimes more natural to introduce $G$-coercivity in terms of the functional $J$, via the following result

**Proposition 3.4.** \[29\] $F$ is $G$-coercive if and only if there exist uniform positive constants $(\lambda', \delta')$ such that

$$F(\varphi) \geq \lambda' \inf_{\sigma \in \mathcal{I}_G} J(\sigma^* \omega_{\varphi}) - \delta', \quad \forall \varphi \in \mathcal{K}_T(X, \omega_0).$$

**Remark 3.5.** If $F$ is $G$-coercive, then it is bounded below by \[17\].

Following \[27\], one can consider the metric completion $(\mathcal{E}^1_1(X, \omega_0), d_1)$ of $(\mathcal{K}(X, \omega_0), d_1)$, which can be characterized by a suitable continuously embedded subspace in $L^1(X, \omega_0)$; similarly we let $(\mathcal{E}^1_1(X, \omega_0), d_1)$ be the metric completion of $(\mathcal{K}(X, \omega_0), d_1)$ which, again by the results in \[29\], can be viewed as the closed subspace of $T$-invariant elements of $\mathcal{E}^1_1(X, \omega_0)$. It will be important for us that $(\mathcal{E}^1_1(X, \omega_0), d_1)$ is a geodesic space, i.e. each two elements $\psi_0, \psi_1 \in \mathcal{E}^1_1(X, \omega_0)$ can be connected with a curve $\psi_t, t \in [0, 1]$ in $(\mathcal{E}^1_1(X, \omega_0), d_1)$, called a weak geodesic, obtained as the limit of $C^{1,1}$-geodesics between elements of $\mathcal{K}(X, \omega_0)$, see \[22 \ 27\]. The latter object is a curve $\varphi_t \in \mathcal{E}^1_1(X, \omega_0)$, of regularity $C^{1,1}([0, 1] \times X)$, which is uniquely associated to each $\varphi_0, \varphi_1 \in \mathcal{K}(X, \omega_0)$ (see \[22 \ 16 \ 25\] and the proof of Proposition \[5.8\] below for more details about the weak $C^{1,1}$-geodesics).

In \[29\] Thm. 3.4, the following general principle is established.

**Theorem 3.6 (Coercivity Principle).** Let $F : \mathcal{K}_T(X, \omega_0) \to \mathbb{R}$ be a lower semicontinuous (lsc) functional with respect to $d_1$, and $F : \mathcal{E}^1_1(X, \omega_0) \to \mathbb{R} \cup \{+\infty\}$ be its largest lsc extension. Suppose, furthermore, that $F(\varphi + c) = F(\varphi) = F(\omega_{\varphi})$ and $F(\sigma^* \omega_{\varphi}) = F(\varphi_{\sigma})$ for any $\varphi \in \mathcal{K}_T(X, \omega_0)$ and $\sigma \in G$, and that $F$ satisfies the following properties

(i) (Convexity) For each $\varphi_0, \varphi_1 \in \mathcal{K}_T(X, \omega_0)$ and the $C^{1,1}$-geodesic $\varphi_t$ joining $\varphi_0$ and $\varphi_1$, $t \to F(\varphi_t)$ is continuous and convex.

(ii) (Regularity) If $\psi \in \mathcal{E}^1_1(X, \omega_0)$ is a minimizer of $F$, then $\psi \in \mathcal{K}_T(X, \omega_0)$.

(iii) (Uniqueness) $G$ acts transitively on the set of minimizers of $F$.

(iv) (Compactness) If $\{\psi_j\} \in \mathcal{E}^1_1(X, \omega_0)$ satisfies $\lim_{j \to \infty} F(\psi_j) = \inf_{\mathcal{E}^1_1(X, \omega_0)} F$ and, for some $C > 0$, $d_1(0, \psi_j) \leq C$, then there exists a $\psi \in \mathcal{E}^1_1(X, \omega_0)$ and a subsequence $\{\psi_{j_k}\}_k$ with $\psi_{j_k} \to \psi$ in $(\mathcal{E}^1_1(X, \omega_0), d_1)$.

Then, the following two conditions are equivalent:

- $F$ has minimizer in $\mathcal{K}_T(X, \omega_0)$;
- $F$ is $G$-coercive.

The above result provides a clear framework for achieving the proof of Theorem \[1\] we need to find a suitable largest lsc extension of the weighted Mabuchi functional $M_{v,w}$ to the space $\mathcal{E}^1_1(X, \omega_0)$, and show it satisfies the properties (i)–(iv). Notice that the invariance of $M_{v,w}$ under the action of $G = \mathbb{T}^C$ is equivalent to the necessary condition \[3\] for the existence of a $(v, w)$-cscK metric whereas (iii) will follow from Theorem \[1.5\] once the regularity condition (ii) is established. Furthermore, the property (i) is proved in \[57 \ Thm. 1\], so the core of our arguments is to define the extension of $M_{v,w}$ to $\mathcal{E}^1_1(X, \omega_0)$ and establish the properties (ii) and (iv). These steps will be respectively detailed in Theorems \[6.1 \ 7.1 \ and 6.17\] below.

4. K-stability via coercivity: Deriving Corollary \[1\] from Theorem \[1\]

We consider the following general setup, based on the results of \[10 \ 15 \ 66 \ 71 \ 18 \ 50 \ 58\] which deal with the K-polystability and uniform K-stability in the unweighted cscK case. Let $\mathbb{T} \subset \text{Aut}_v(X)$ be a connected compact torus in the reduced group of automorphisms of $X$.

**Definition 4.1.** A $T$-equivariant Kähler test configuration $(\mathcal{X}, \omega')$ associated to $(X, \alpha, T)$ is a normal compact Kähler space $\mathcal{X}$ endowed with

- a flat morphism $\pi : \mathcal{X} \to \mathbb{P}^1$;
- a $\mathbb{C}^*$-action $\rho$ covering the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$, and a $T$-action commuting with $\rho$ and preserving $\pi$;
\begin{itemize}
\item a $\mathbb{T} \times \mathbb{C}^*$-equivariant biholomorphism $\Pi_0 : (\mathcal{X}, \pi^{-1}(0)) \cong (X \times (\mathbb{P}^1 \setminus \{0\}))$;
\item a Kähler class $\mathcal{A} \in H^{1,1}(\mathcal{X}, \mathbb{R})$ such that $(\Pi_0^{-1})^*(\mathcal{A})|_{X \times \{t\}} = \alpha$.
\end{itemize}

We say that $(\mathcal{X}, \mathcal{A})$ is smooth if $\mathcal{X}$ is smooth and dominating if $\Pi_0$ extends to a $\mathbb{T} \times \mathbb{C}^*$-equivariant morphism

$$\Pi : \mathcal{X} \to X \times \mathbb{P}^1.$$ 

$(\mathcal{X}, \mathcal{A})$ is called trivial if it is dominating and $\Pi$ is an isomorphism; $(\mathcal{X}, \mathcal{A})$ is called product if $\pi^{-1}(0) \cong X$. If $(X, L)$ is a smooth polarized variety and $\alpha = 2\pi c_1(L)$, a polarized test configuration is a normal polarized variety $(\mathcal{X}, \mathcal{L})$ such that for some $r \in \mathbb{N}^*$, $(\mathcal{X}, \frac{1}{r}2\pi c_1(\mathcal{L}))$ defines a Kähler test configuration of $(X, \alpha)$ and, under $\Pi_0$, $(X, \mathcal{L}|_{X \times \{t\}}) \cong (X, L^r)$.

### 4.1. Non-Archimedean functionals

We recall that any $\mathbb{T} \times S^1$-invariant Kähler metric $\Omega \in \mathcal{A}$ on $\mathcal{X}$ gives rise to a smooth ray of $\mathbb{T}$-invariant Kähler metrics $\omega_t \in \alpha$ on $X$ defined by

$$\omega_t := \rho(e^{-t+is})*(\Omega)|_{X \times (1)}.$$ 

**Definition 4.2.** Let $F$ be a functional defined on the space of $\mathbb{T}$-invariant Kähler metrics on $X$ in the class $\alpha$. We say that $F$ admits a non-Archimedean version $F^{NA}$, defined on a subclass $\mathcal{C}$ of $\mathbb{T}$-equivariant Kähler test configurations $(\mathcal{X}, \mathcal{A})$ associated to $(X, \alpha, \mathbb{T})$, if for any $(\mathcal{X}, \mathcal{A}) \in \mathcal{C}$, and any induced smooth ray of $\mathbb{T}$-invariant Kähler metrics $\omega_t \in \alpha$ on $X$, the slope $\text{lim}_{t \to \infty} \frac{F(\omega_t)}{t}$ is well-defined and given by a quantity $F^{NA}(\mathcal{X}, \mathcal{A})$ which is independent of the choice of the $\mathbb{T} \times S^1$-invariant Kähler form $\Omega \in \mathcal{A}$.

We give below two key examples of non-Archimedean versions of known functionals. The first one is established in the polarized case in [18] and in the generality we consider in [33, 65].

**Example 4.3.** The functional $J$ introduced in Definition 3.1 admits a non-Archimedean version defined, up to a positive dimensional multiplicative constant, on the class of smooth $\mathbb{T}$-equivariant dominating Kähler test configurations $(\mathcal{X}, \mathcal{A})$ by

$$J^{NA}(\mathcal{X}, \mathcal{A}) = \frac{\left((\Pi(\alpha)^m \cdot \mathcal{A})\right)_X}{(a^m)_X} - \frac{1}{m+1} \left((\mathcal{A}^{m+1})\right)_X,$$

where $\Pi$ is the morphism (19) and $\alpha$ denotes both the Kähler class on $X$ and its pull back to $X \times \mathbb{P}^1$.

The above expression generalizes to dominating smooth test configurations which are only relatively nef (in the terminology of [66]), thus also providing a non-Archimedean version of $J$ for any Kähler test configuration: indeed, by the equivariant Hironaka resolution, any $\mathbb{T}$-equivariant test configuration can be dominated by a smooth relatively nef Kähler dominating test configuration, and the computation of $J^{NA}$ on the latter does not depend on the choice made.

The non-Archimedean functional $J^{NA}$ defined above is always non-negative and equals to zero precisely when $(\mathcal{X}, \mathcal{A})$ is the trivial test configuration. The latter statement is established in [18, Thm. 7.9] in the polarized case, and follows from the results in [66] in the Kähler case: see in particular [66, Lemma 4.8] with $G$ trivial and recall that the $J$-norm is Lipschitz equivalent to the $d_1$-distance, so that the unique weak geodesic ray associated to a test configuration with vanishing $J^{NA}$-norm must be constant, and hence the test configuration must be trivial by [66, Cor. 3.12]. Thus, $J^{NA}$ can be thought of as a “norm” on the space of Kähler test configurations.

In order to obtain a norm which is zero for more general product test configurations, in [34, 60, 58] the authors consider smooth rays $\tilde{\omega}_t \in \alpha$ of $\mathbb{T}$-invariant Kähler metrics on $X$ which are obtained by composing an induced ray $\omega_t$ from a $\mathbb{T} \times S^1$-invariant Kähler metric $\Omega \in \mathcal{A}$ on $\mathcal{X}$ with the flow of a vector field $J\xi$, where $\xi$ varies in $\mathbb{R}$, i.e., $\tilde{\omega}_t = \exp(tJ\xi)^*(\omega_t)$. They show that the slope

$$\lim_{t \to \infty} \frac{J(\tilde{\omega}_t)}{t} = J^{NA}(\mathcal{X}_\xi, \mathcal{A}_\xi)$$

is well-defined and independent of the choice of induced ray $\omega_t$. We notice that when $\xi \in 2\pi\Lambda$ is a lattice element (or more generally is rational), $\xi$ induces an $\mathbb{C}^*$-action $\rho_\xi$ on $\mathcal{X}$ and $\tilde{\omega}_t$ is an induced smooth ray from another Kähler test configuration $(\mathcal{X}_\xi, \mathcal{A}_\xi)$, called the $\xi$-twist of $(\mathcal{X}, \mathcal{A})$, obtained from $\mathcal{X}$ by composing the initial $\mathbb{C}^*$-action $\rho$ with $\rho_\xi$, and compactifying trivially at...
infinity. (For instance, the product test configurations are precisely the $\xi$-twists of the trivial test configuration.) In this case, $J^{NA}(\mathcal{X}_\xi, \alpha_\xi)$ is just the non-Archimedean $J$-functional computed as in Example 4.3 on $(\mathcal{X}_\xi, \alpha_\xi)$. For a general $\xi$, the quantity $(\mathcal{X}_\xi, \alpha_\xi)$ in the notation is not a test configuration in the usual sense (it is sometimes refereed to as a $\mathbb{R}$-test configuration) but the value $J^{NA}(\mathcal{X}_\xi, \alpha_\xi)$ can be obtained as a continuous extension of the corresponding quantity for rational $\xi$’s. Following [50, 58], we let

\begin{equation}
J^{NA}_{\mathbb{T}^c}(\mathcal{X}, \mathcal{A}) := \inf_{\xi \in \mathbb{T}^c} J^{NA}(\mathcal{X}_\xi, \alpha_\xi) \geq 0.
\end{equation}

A key observation [18, 50, 58] in the polarized case is that the equality in (20) holds if and only if $(\mathcal{X}, \mathcal{L})$ is a product test configuration. Furthermore, according to [50 Thm. B] and [58 Thm. 3.14], we have

Example 4.3-bis. In the polarized case, the quantity $J^{NA}_{\mathbb{T}^c}(\mathcal{X}, \mathcal{A})$ introduced in (20) defines a non-Archimedean version of the functional

\begin{equation}
J_\mathbb{T}^c(\omega) := \inf_{\sigma \in \mathbb{T}^c \sigma^*(\omega)},
\end{equation}

on the class of $\mathbb{T}$-equivariant polarized test configuration of $(X, L, T)$.

Our second example is established in [56, Thm. 7]:

Example 4.4. Consider the weighted Mabuchi functional $M_{v,w}$ introduced in Definition 1.1 and assume that the relation (7) holds, see Remark 1.2. Then $M_{v,w}$ admits a non-Archimedean version defined on smooth $\mathbb{T}$-equivariant Kähler test configurations with reduced central fibre, given by the formula

\begin{equation}
\mathcal{F}_{v,w}(\mathcal{X}, \mathcal{A}) := -\int_{\mathcal{X}} (\text{Scal}(\Omega) - w(\mu_\Omega)) \Omega^{[m+1]} + (8\pi) \int_X v(\mu_\omega) \omega^{[m]},
\end{equation}

where $\Omega \in \mathcal{A}$ is any $\mathbb{T}$-invariant Kähler metric on $\mathcal{X}$ with $\Delta$-normalized $\mathbb{T}$-momentum map $\mu_\Omega : \mathcal{X} \to \Delta$ and $\nu$-scalar curvature $\text{Scal}(\Omega)$, and $\omega \in \alpha$ is any $\mathbb{T}$-invariant Kähler metric on $X$ with $\Delta$-normalized $\mathbb{T}$-momentum map $\mu_\omega : X \to \Delta$.

Definition 4.5. The RHS of (21) is independent of $\Omega \in \mathcal{A}$ and $\omega \in \alpha$ (see [56]) and is referred to as the $(v, w)$-weighted Donaldson-Futaki invariant of a smooth $\mathbb{T}$-equivariant Kähler test configuration $(\mathcal{X}, \mathcal{A})$.

Remark 4.6. In the unweighted case (i.e. $v = 1, w = 4m\pi \frac{c_1(X) \alpha_{m-1}}{\alpha m}$), $\mathcal{F}_{v,w}(\mathcal{X}, \mathcal{A})$ admits an equivalent expression in terms of intersection cohomology numbers on $\mathcal{X}$, see [63, 73]. This allows one to extend the definition of the (unweighted) Donaldson–Futaki invariant to any normal Kähler test configuration. For arbitrary weight functions $v > 0$ and $w$, we don’t have as yet a general definition for $\mathcal{F}_{v,w}$ but (21) can be readily extended to orbifold test configurations. We also notice that the assumption on the central fibre in Example 4.4 is necessary in order to ensure the equality $\mathcal{F}_{v,w} = M^{NA}_{v,w}$ (see [63] for a general formula of the non-Archimedean version of the unweighted Mabuchi energy). It will be interesting to obtain a non-Archimedean version of $M_{v,w}$ for any orbifold $\mathbb{T}$-equivariant Kähler test configuration.

4.2. $F^{NA}$-K-stability.

Definition 4.7. Let $F$ be a functional defined on the space of $\mathbb{T}$-invariant Kähler metrics on $X$ in the Kähler class $\alpha$, and suppose $F$ admits a non-Archimedean version $F^{NA}(\mathcal{X}, \mathcal{A})$ (see Definition 1.2) defined on a class $C$ of $\mathbb{T}$-equivariant Kähler test configurations $(\mathcal{X}, \mathcal{A})$ associated to $(X, \alpha, T)$. We suppose that $C$ contains the product test configurations. We say that:

(i) $(X, \alpha, T)$ is $\mathbb{T}$-equivariant $F^{NA}$-K-semistable (on test configurations of the class $C$) if for any $(\mathcal{X}, \mathcal{A}) \in C$ we have $F^{NA}(\mathcal{X}, \mathcal{A}) \geq 0$.

(ii) $(X, \alpha, T)$ is $\mathbb{T}$-equivariant $F^{NA}$-K-polystable (on test configurations of the class $C$) if it is $\mathbb{T}$-equivariant $F^{NA}$-K-semistable, and, furthermore, $F^{NA}(\mathcal{X}, \mathcal{A}) = 0$ if and only if $(\mathcal{X}, \mathcal{A})$ is a product test configuration.
(iii) \((X, \alpha, T)\) is \(\mathbb{T}^c\)-uniform \(F^{NA}\)-\(K\)-stable (on test configurations of the class \(C\)) if there exists a uniform positive constant \(\lambda > 0\) such that for any test configuration \((\mathcal{X}, \mathcal{A}) \in C\)

\[F^{NA}(\mathcal{X}, \mathcal{A}) \geq \lambda J^{NA}(\mathcal{X}, \mathcal{A}),\]

where \(J^{NA}(\mathcal{X}, \mathcal{L})\) is introduced in \[(20)\].

**Remark 4.8.** If \(F\) is bounded below then \((X, \alpha, T)\) is \(T\)-equivariant \(F^{NA}\)-\(K\)-semistable; furthermore both (ii) and (iii) imply (i) and, in the polarized case, (iii) implies (ii) by the results in \[18, 50, 58, 66\].

**Theorem 4.9.** \[15, 50, 58, 66\] Suppose \(F\) is a functional defined on the space of \(T\)-invariant Kähler metrics in \(\alpha\), which is \(T\)-relatively \(\mathbb{T}^c\)-proper. Suppose, furthermore, that \(F\) admits a non-Archimedean version \(F^{NA}\) defined for a class \(C\) of \(T\)-equivariant Kähler test configurations of \((X, \alpha, T)\). Then \((X, \alpha, T)\) is \(T\)-equivariant \(F^{NA}\)-\(K\)-polystable on \(C\). If, moreover, \((X, \alpha, T)\) is a polarized potential and \(\alpha = 2\pi c_1(L)\), then \((X, \alpha, T)\) is \(\mathbb{T}^c\)-uniform \(F^{NA}\)-\(K\)-stable on polarized test configurations in \(C\).

**Proof.** For the first part, we follow \[66\] with some minor modifications. We want to show that if \(F^{NA}(\mathcal{X}, \mathcal{A}) = 0\), then \((\mathcal{X}, \mathcal{A})\) is a product test configuration.

We fix a \(T \times S^1\)-invariant Kähler form \(\Omega \in \mathcal{A}\) and let \(\omega_t\) be the corresponding ray of smooth \(T\)-invariant Kähler forms in \(\alpha\), and \(\psi_t \in K_T(X, \omega_0)\) the normalized smooth ray of Kähler potentials satisfying \(I(\psi_t) = 0\). According to \[65\], the Kähler test configuration \((\mathcal{X}, \mathcal{A})\) also determines a unique \(C^{1,1}\) weak geodesic ray \(\varphi_t \in K^{1,1}(X, \omega_0)\), emanating from \(\psi_0\). Furthermore, \(\varphi_t\) is invariant under \(T\) (by its uniqueness) provided that we have \(\psi_0 \in K_T(X, \omega_0)\). According to \[66\] Prop.4.2, we can consider instead of \(\mathcal{A}\) the relative Kähler class \(\mathcal{A}_c = \mathcal{A} - c[X_0] = \mathcal{A} - c\pi^*(\mathcal{O}_{\pi^1}(1))\) (for a constant \(c\) determined from \(\mathcal{A}\) and where \(X_0\) denotes the divisor corresponding to the central fibre \(X_0\) of \(\mathcal{X}\), such that the \(C^{1,1}\) weak geodesic ray \(\varphi_t^c\) corresponding to \((\mathcal{X}, \mathcal{A}_c)\) is the projection of \(\varphi_t\) to the slice \(K^{1,1}_T(X, \omega_0) \cap \Gamma^{-1}(0)\). Notice that the smooth \((1,1)\)-form \(\Omega - c\pi^*\omega_{FS} \in \mathcal{A}_c\) defines the same smooth ray \(\omega_t\) of \(T\)-invariant Kähler metrics, and thus the same ray of smooth potentials \(\psi_t \in K_T(X, \omega_0) \cap \Gamma^{-1}(0)\) and \(F^{NA}(\mathcal{X}, \mathcal{A}_c) = F^{NA}(\mathcal{X}, \mathcal{A}) = 0\). The key point is that \[(17)\] and \(\lim_{t \to \infty} F^{NA}_{t}(\omega_{\psi_t}) = F^{NA}(\mathcal{X}, \mathcal{A}_c) = 0\) yield an estimate \(0 \leq d_{1}^{G}(\psi_t) \leq o(t)\), which is shown in \[66\] Lemma 4.8 to be equivalent to \(0 \leq d_{1}^{G}(0, \varphi_t^c) \leq o(t)\). We can now apply the arguments in the proof of the implication ‘(2) \Rightarrow (5)’ of \[66\] Thm. 4.4, by replacing the Mabuchi energy with the abstract functional \(F\) and the group \(Aut_0(X)\) with \(\mathbb{T}^c\), noting that in our \(T\)-relative situation instead of the cscK potential \(\psi_0\) in \[66\] Prop. 4.10 we can take any Kähler potential in \(K_T(X, \omega_0)\) (as \(\omega_0\) is \(T\)-invariant and \(\mathbb{T}^c\) is reductive). We thus deduce the implication (5) of \[66\], namely, that the geodesic ray \(\varphi_t^c\) associated to \((\mathcal{X}, \mathcal{A}_c)\) is given by the \(\omega_0\)-relative Kähler potentials of \(exp(t\xi)\omega_{\psi_t}\) in \(\Gamma^{-1}(0)\), where \(\xi\) is a vector field in the Lie algebra of \(T\); it follows from \[66\] Thm. A.6] that \((\mathcal{X}, \mathcal{A}_c)\) and hence also \((\mathcal{X}, \mathcal{A})\) is a product test configuration.

The second part follows immediately from \[18\] and Example 4.3-bis.

We next apply Theorem 4.9 to \(F = M_{v,w}\) and \(F^{NA} = F_{v,w}\).

**Definition 4.10.** Let \(F^{NA} = F_{v,w}\), where \(F_{v,w}\) is defined on any smooth \(T\)-equivariant test configuration via the formula \[21\], see Definition 4.5. We then refer to the \(F^{NA}\)-\(K\)-stability notions introduced in the Definition 4.7 (ii)–(iii) respectively as \(T\)-equivariant \((v, w)\)-K-semistability, \(T\)-equivariant \((v, w)\)-polystability, and \(\mathbb{T}^c\)-uniform \((v, w)\)-K-stability on \(T\)-invariant dominating smooth Kähler test configurations with reduced central fibre.

**Proof of Corollary 1** modulo Theorem 1. By the definition of \(M_{v,w}\) (see Definition \[11\]), we have

\[M_{v,w}(\varphi + c) = M_{v,w}(\varphi) + c \int_X (\text{Scal}_v(\omega_{\varphi}) - w(\mu_{\varphi}))\omega_{\varphi}^{[m]},\]

showing that if \(M_{v,w}\) is bounded below on \(K_T(X, \omega_0)\) (in particular if \(M_{v,w}\) is \(T\)-relatively \(\mathbb{T}^c\)-proper), then the relation \[17\] holds and \(M_{v,w}\) defines a functional on the space of \(T\)-invariant Kähler metrics in \(\alpha\) (see Remark \[12\]). In this case, Example 4.3 tells us that \(F_{v,w}(\mathcal{X}, \mathcal{A})\) defines a non-Archimedean version of \(M_{v,w}\). We can now apply Theorem 4.9. 

\(\square\)
5. Semi-simple principal fibrations

Let \((X, \omega)\) be a compact Kähler \(2m\)-manifold, endowed with a Hamiltonian isometric action of an \(r\)-dimensional torus \(T\). As \(T\) will act on various spaces, we shall use at times upper and under scripts to emphasize the space, on which \(T\) acts. For instance, \(T_X\) will denote the \(T\)-action on \(X\). Let \(t\) be the Lie algebra of \(T\) and \(\Lambda \subset t\) the lattice of generators of circle groups in \(T\) (i.e. \(\Lambda = t/2\pi \Lambda\)). We denote by \(\mu : X \to T\) the normalized \(T_X\)-momentum map of \(\omega\), i.e. whose image is a fixed compact convex polytope \(\Delta\).

Let \(B = B_1 \times \cdots \times B_k\) be a \(2n\)-dimensional cscK manifold, where each \((B_a, \omega_{B_a})\), \(a = 1, \ldots, k\) is a compact cscK Hodge Kähler \(2n\)-manifold (i.e. \(\frac{1}{2\pi} [\omega_{B_a}] \in H^2(B_a, \mathbb{Z})\)), and \(\pi_B : P \to B\) a principal \(T\)-bundle endowed with a connection 1-form \(\theta \in \Omega^1(P, t)\) with curvature

\[
\sum_{a=1}^{k} (\pi_B^* \omega_{B_a}) \otimes p_a,
\]

\(p_a \in \Lambda\).

Remark 5.1. The principle \(T\)-bundle \(P\) above can be described in terms of \(r\) complex line bundles over \(B\) as follows. Fixing a lattice basis \(\{\xi_1, \ldots, \xi_r\}\) of \(t\), and writing \(p_a = \sum_{i=1}^{r} p_{ai} \xi_i\), \(p_{ai} \in \mathbb{Z}\), \(a = 1, \ldots, k\), \(\sum_{a=1}^{k} p_{ai} \xi_i \otimes p_a\) yields that \(P\) is the (fiber-wise) product of \(r\) principle \(U(1)\)-bundles \(P_i \to B\), where each \(P_i\) is associated to a complex line bundle \(L_i^*\) on \(B\) with Chern class \(2\pi \psi(L_i^*) = -\sum_{a=1}^{k} p_{ai} \xi^*_i \otimes p_a\), i.e. we have

\[
2\pi \psi(P) := -2\pi \sum_{i=1}^{r} \psi(L_i^*) \otimes \xi_i = \sum_{a=1}^{k} \pi_B^* [\omega_{B_a}] \otimes p_a.
\]

Fixing a connection 1-form \(\theta\) on \(P\) as in \(\sum_{a=1}^{k} \pi_B^* [\omega_{B_a}] \otimes p_a\) amounts to introducing a hermitian metric \(h_i^*\) on each \(L_i^*\), with curvature \(-\sum_{a=1}^{k} p_{ai} \pi_B^* [\omega_{B_a}] \otimes p_a\), and identifying \(P_i \subset L_i^*\) with the corresponding unitary \(S^1\)-bundle.

Let \(\mathcal{D} = \text{ann}(\theta) \subset TP\) be the horizontal distribution defined by \(\theta\), leading to a splitting \(TP = \mathcal{D} \oplus t_P\), where \(t_P\) denotes the Lie algebra of \(T_P\) inside \(C^\infty(P, TP)\), corresponding to the \(T\)-action \(T_P\) on \(P\). The lift \(J_B\) of the integrable almost complex structure of \(B\) to \(\mathcal{D}\) gives rise to a CR structure \((\mathcal{D}, J_B)\) on \(P\) (of co-dimension \(r\)).

We further let \(Z := X \times P\) and consider the induced \(T\)-action, denoted \(T_Z\), generated by \((-\xi^*_i + \xi_i^P\)) for any basis of \(\Lambda\) as above. We thus define \(Y := Z/T_Z\).

It follows that \(Y\) is a \((2n+1)\)-dimensional smooth manifold, and \(\pi_Y : Z = X \times P \to Y\) is a principal \(T\)-bundle over \(Y\) whereas \(\pi_B : P \to B\) defines a fibration \(\pi_B : Y \to B\) with smooth fibres \(X\), as summarized in the diagram below.

The \(T_X\)-action on the factor \(X\) in \(Z = X \times P\) descends to a \(T\)-action on \(Y\), denoted \(T_Y\), which preserves each fibre (and thus coincides with the action of \(T_X\)). Notice that the 1-form \(\theta\) also defines a connection 1-form on \(Z\) with horizontal distribution \(\mathcal{H}\):

\[
T(X \times P) = \mathcal{H} \oplus t_P, \quad \mathcal{H} = \pi_B^* T_X \oplus \mathcal{D} = \text{ann}(\theta),
\]

giving rise to an induced CR structure \((\mathcal{H}, J = J_X \oplus J_B)\) of co-dimension \(r\) on \(Z\), which is clearly invariant under the \(T_Z\)-action, and therefore defines a \(T_Y\)-invariant complex structure \(J_Y\) on \(Y\).

We now consider Kähler metrics on \(Y\), compatible with the fibre-bundle construction of the above form. To simplify the notation, we denote by \(\omega_a := \omega_{B_a}\) the (fixed) cscK metric on each
factor \(B_a\), by \(\omega\) a \(\mathbb{T}\)-invariant Kähler structure in the class \(\alpha\) on \(X\), and by \(\tilde{\omega}\) the resulting Kähler structure on \(Y\), which is defined in terms of a basic 2-form on \(Z = X \times P\), depending on \(k\) real constants \(c_a \in \mathbb{R}\) (which will be fixed) such that for each \(a = 1, \ldots, k\), the affine linear function \((p_a, \mu) + c_a\) on \(t^*\) is strictly positive on the momentum image \(\Delta\):

\[
\tilde{\omega} := \omega + \sum_{a=1}^{k} ((p_a, \mu) + c_a) \pi_B^* \omega_a + \langle d\mu, \theta \rangle
\]

(24)

In the above expression, \langle \cdot, \cdot \rangle stands for the natural pairing between \(t\) and \(t^*\): thus \((p_a, \mu)\) is a smooth function, \((\mu, \theta)\) is a 1-form, and \(\langle d\mu, \theta \rangle\) is a 2-form on \(Z\). One can directly check from the above expression that \(\tilde{\omega}\) is closed, \(T_Z\)-basic, and is positive definite on \((\mathcal{H}, J_X \oplus J_B)\), so it is the pullback of a Kähler form on \(Y\). We shall tacitly identify in the sequel the Kähler form on \(Y\) with its pullback \(\tilde{\omega}\) on \(Z = X \times P\). Notice that \(\tilde{\omega}\) is \(T_Y\)-invariant and \(\mu, \omega\), seen as a smooth \(T_Z\)-invariant function on \(Z\), is the \(\Delta\)-normalized momentum map.

**Remark 5.2.** The horizontal part \(\tilde{\omega}_h := \tilde{\omega}_{|_{\mathcal{H}_T}}\) of 2-form \(\tilde{\omega}\) on \(Z = X \times P\) is invariant and basic with respect to the action \(\mathbb{T}_P\) on the factor \(P\), and thus induces a Hermitian (non-Kähler in general) metric on \(X \times B = X \times \prod_{a=1}^{k} B_a\), given by

\[
\tilde{\omega}_h = \omega + \sum_{a=1}^{k} ((p_a, \mu) + c_a) \omega_a,
\]

which is an instance of warped geometry. On can thus think of \((X \times B, \tilde{\omega}_h)\) and \((Y, \tilde{\omega})\) as being related by the twist construction of \([70]\) applied to \((Z, \tilde{\omega}, T_Z)\) and \((Z, \tilde{\omega}, T_P)\).

**Definition 5.3.** The Kähler manifold \((Y, T_Y)\) constructed as above will be called a **semi-simple (\(X, \mathbb{T}\))-principal fibration** associated to the Kähler manifold \((X, \mathbb{T})\) and the product cscK manifold \(B = B_1 \times \cdots \times B_k\). The \(T_Y\)-invariant Kähler metric \(\tilde{\omega}\) on \(Y\) constructed from a \(T_X\)-invariant Kähler metric \(\omega\) on \(X\) (and fixed cscK metrics \(\omega_a\) on \(B_a\)) will be called **bundle-compatible**.

**Remark 5.4.** In the case when \((X, \mathbb{T}, \omega)\) is a toric Kähler manifold, a semi-simple \((X, \mathbb{T})\)-principal fibration endowed with a bundle-compatible Kähler metric is an example of a semi-simple rigid toric fibration in the sense of \([7]\), and is thus described by the **generalized Calabi construction** with a global product structure on the base and no blow-downs.

### 5.1. The space of functions.

The above bundle construction gives rise to a natural embedding of the space \(C^\infty_T(X)\) of \(T_X\)-invariant smooth functions on \(X\) to the space \(C^\infty_T(Y)\) of \(T_Y\)-invariant smooth functions on \(Y\): for any \(\varphi \in C^\infty_T(X)\) we consider the induced function on \(Z = X \times P\), which is clearly \(T_Z\)-invariant, and thus descends to a smooth \(T_Y\)-invariant function on \(Y\). We shall tacitly identify \(\varphi\) and its image in \(C^\infty_T(Y)\), i.e. we shall consider

\[
C^\infty_T(X) \subset C^\infty_T(Y).
\]

Notice that the above embedding is closed in the Fréchet topology, as we can identify a smooth \(T_X\)-invariant function on \(X\) with a smooth \(T_Y\)-invariant function \(\varphi\) on \(Y\), which has the property

\[
d_p(\pi_Y^* \varphi) = 0
\]

on \(Z = X \times P\). More generally, for any \(T_Y\)-invariant smooth function \(\psi \in C^\infty_T(Y)\) its lift \(\pi_Y^* \psi\) to \(Z = X \times P\) is both \(T_Z\) and \(T_X\)-invariant, or equivalently \(T_X\) and \(T_P\) invariant. It thus follows that \(\pi_Y^* \psi\) can be equivalently viewed as a \(T_X\)-invariant smooth function on \(X \times B\), i.e. we have an identification

\[
C^\infty_T(Y) \cong C^\infty_T(X \times B).
\]

(25)
In particular, for any fixed point $x \in X$, we shall denote by $\psi_x \in C^\infty(B)$ the induced smooth function on $B$, and for any fixed point $b \in B$ by $\psi_b \in C^\infty(X)$ the induced function on $X$. We thus have the identification
\[
C^\infty_T(X) \cong \{ \psi \in C^\infty_T(Y) \mid dB\psi_x = 0 \, \forall x \in X \}.
\]

5.2 The space of bundle-compatible Kähler metrics. We shall next use the construction of (24) in order to identify the space $\mathcal{K}_T(X, \omega_0)$ of $T_X$-invariant $\omega_0$-relative Kähler potentials on $X$ as a subset of the space $\mathcal{K}_T(Y, \omega_0)$ of $T_Y$-invariant $\omega_0$-relative Kähler potentials on $Y$.

**Lemma 5.5.** Let $\omega_\varphi = \omega_0 + d_X d^c_X \varphi$ be an $T_X$-invariant Kähler form on $X$ in the Kähler class $\alpha = [\omega_0]$, where $\varphi \in \mathcal{K}_T(X, \omega_0)$ is a $T_X$-invariant smooth function on $X$. Denote by $\mu_\varphi$ the momentum map of $T_X$ with respect to $\omega_\varphi$, satisfying the normalization $\mu_\varphi(X) = \Delta$, and by $\tilde{\omega}_\varphi$ the induced Kähler metric on $Y$, via (24). Then,
\[
\tilde{\omega}_\varphi = \tilde{\omega}_0 + dY d^c_Y \varphi,
\]
where $\varphi$ stands for the induced smooth function on $Y$.

**Proof.** Recall that $\mu_\varphi = \mu_0 + d^c \varphi$ (see (5)). By (24), the pullback of $\tilde{\omega}_\varphi$ to $Z = X \times P$ is
\[
\tilde{\omega}_\varphi = \omega_\varphi + \sum_{a=1}^k c_a (\pi^*_B \omega_a) + d\langle \mu_\varphi, \theta \rangle
\]
\[
= \omega_0 + \sum_{a=1}^k c_a (\pi^*_B \omega_a) + d_X d^c_X \varphi + d\langle \mu_\varphi, \theta \rangle
\]
\[
= \tilde{\omega}_0 + dd^c_X \varphi + d\langle (d^c_X \varphi, \theta) \rangle,
\]
so it is enough to check that
\[
d^c_Y \varphi = d^c_X \varphi + \langle d^c_X \varphi, \theta \rangle,
\]
for any $T_X$-invariant smooth function $\varphi$ on $X$. To this end, let us choose a basis $\{\xi_1, \ldots, \xi_r\}$ of $t$, with dual basis $\{\xi_1^*, \ldots, \xi_r^*\}$ of $t^*$, and write $d^c_X \varphi = \sum_{j=1}^r (d^c_X \varphi)(\xi_j^*) \xi_j^*$ and $\theta = \sum_{j=1}^r \theta_j \xi_j$ for 1-forms $\theta$ on $Z$ such that $\theta_j$ is zero on $\mathcal{H}$ and $\theta_j(\xi_i^p) = \theta_j(-\xi_i^X + \xi_i^p) = \delta_{ij}$. Thus, (26) is equivalent to
\[
d^c_Y \varphi = d^c_X \varphi + \sum_{j=1}^r (d^c_X \varphi)(\xi_j^X) \theta_j.
\]
Evaluating the RHS of the above equality on the generators $(-\xi_j^X + \xi_j^p)$ of $t_Z$, we see that it is a $\pi_Y$-basic 1-form on $Z$, and thus is the pullback of a 1-form on $Y$ via $\pi_Y$. The claim follows easily. \hfill $\square$

Thus, Lemma 5.5 defines an embedding $\mathcal{K}_T(X, \omega_0) \subset \mathcal{K}_T(Y, \tilde{\omega}_0)$ and we have also identified in Sect. 5.1 a natural embedding of the space of $T_X$-invariant functions on $X$ into the space of $T_Y$-invariant functions on $Y$, through their pull-backs to $Z = X \times P$.

Letting $\theta := \sum_{j=1}^r \theta_j \otimes \xi_j^p$ be the decomposition of the connection 1-form $\theta$ on $P$ in a basis $\{\xi_1, \ldots, \xi_r\}$ of the lattice $\Lambda \subset t$, and $\theta^{\Lambda^r} := \theta_1 \wedge \cdots \wedge \theta_r$, it follows from (24) and Lemma 5.5 that for any $\varphi \in \mathcal{K}_T(X, \omega_0) \subset \mathcal{K}_T(Y, \tilde{\omega}_0)$, the measure $\tilde{\omega}_\varphi^{[m+n]}$ on $Y$ is the push-forward of the measure on $Z$:
\[
\left( \frac{1}{(2\pi)^r} \right) \tilde{\omega}_\varphi^{[m+n]} \wedge \theta^{\Lambda^r} = \left( \frac{1}{(2\pi)^r} \right) \left( p(\mu_\varphi) \omega_\varphi^{[m]} \wedge \bigwedge_{a=1}^k \pi^*_B \omega_a^{[n_a]} \right) \wedge \theta^{\Lambda^r},
\]
where
\[
p(\mu) := \prod_{a=1}^k ((p_a, \mu) + c_a)^{n_a}, \quad n_a = \dim_{\mathbb{C}}(B_a)
\]
is a positive polynomial on $\Delta$, determined by the semi-simple $(X, T)$-principal fibration $Y$ and the given bundle-compatible Kähler class on it. It thus follows that any $T_X$-invariant integrable
function $f$ on $X$ defines an integrable $T_Y$-invariant function on $Y$ and, for any $\varphi \in K_T(X, \omega_0) \subset K_Y(Y, \tilde{\omega}_0)$, we have
\begin{equation}
\int_Y f \tilde{\omega}^{[n+m]} = \text{Vol}(B, \omega_B) \int_X p(\mu_\varphi) f \omega^{[m]}.
\end{equation}

**Corollary 5.6.** There exists an embedding $K_T(X, \omega_0) \subset K_Y(Y, \tilde{\omega}_0)$ such that, for any smooth curve $\psi_t \in K_T(X, \omega_0) \subset K_Y(Y, \tilde{\omega}_0)$, we have
\begin{equation}
L^Y_1(\psi_t) = \text{Vol}(B, \omega_B)L^X_{1,p}(\psi_t),
\end{equation}
where $p(\mu)$ is the positive weight function on $\Delta$ defined in (28), $L^X_{1,p}$ is the $p(\mu)$-weighted length function on $K_T(X, \omega_0)$ given by
\begin{equation}
L^X_{1,p}(\psi_t) := \int_0^1 \left( \int_X |\dot{\psi}_t| p(\mu_\psi) \omega^{[m]} \right) dt,
\end{equation}
and $L^Y_t$ is the length function on $K_Y(Y, \tilde{\omega}_0)$ corresponding to the weight $p = 1$. In particular, for any $\varphi_0, \varphi_1 \in K_T(X, \omega_0) \subset K_Y(Y, \tilde{\omega}_0)$, $d^Y_1(\varphi_0, \varphi_1) = \text{Vol}(B, \omega_B)d^X_{1,p}(\varphi_0, \varphi_1)$, where $d^Y_{1,p}$ is the induced distance via the length functional $L^Y_{1,p}$.

**Proof.** A direct consequence of (29). □

**Lemma 5.7.** Let $\varphi$ be a smooth $\mathbb{T}_X$-invariant function on $X$, also considered as a smooth $T_Y$-invariant function on $Y$, and $\omega$ be an $\mathbb{T}_X$-invariant Kähler metric on $X$ with $\tilde{\omega}$ the corresponding $T_Y$-invariant Kähler metric on $Y$ given by (21). Then
\begin{equation}
||d\varphi||^2_\omega = ||d\varphi||^2_{\tilde{\omega}}.
\end{equation}

**Proof.** We use that
\begin{align*}
||d\varphi||^2_\omega &= \frac{d_X\varphi \wedge d_X^*\varphi \wedge \omega^{[m-1]}}{\omega^{[m]}} = \frac{d_X\varphi \wedge d_X^*\varphi \wedge p(\pi_\omega) (\pi_B^*\omega_B)^{[n]} \wedge \theta^\wedge r}{\omega^{[m]} \wedge p(\mu_\omega) (\pi_B^*\omega_B)^{[n]} \wedge \theta^\wedge r}, \\
||d\varphi||^2_{\tilde{\omega}} &= \frac{d_Y\varphi \wedge d_Y^*\varphi \wedge \tilde{\omega}^{[m+n-1]}}{\tilde{\omega}^{[m+n]}} = \frac{d_Y\varphi \wedge d_Y^*\varphi \wedge \tilde{\omega}^{[m+n-1]} \wedge \theta^\wedge r}{\tilde{\omega}^{[m+n]} \wedge \theta^\wedge r},
\end{align*}
(\text{where the RHS are written on } X \times P) \text{ together with } d_X\varphi = d_Y\varphi \text{ and } (27). □

**Proposition 5.8.** The embedding in Corollary 5.6 is totally geodesic with respect to the weak $C^{1,1}$ geodesics.

**Proof.** Let $\varphi_0, \varphi_1 \in K_T(X, \omega_0)$. If $\varphi_0$ and $\varphi_1$ can be connected with a smooth geodesic $\varphi_t$, i.e. with a smooth curve in $K_T(X, \omega_0)$ such that
\begin{equation}
\varphi = \frac{||d\varphi||^2_{\tilde{\omega}}}{||d\varphi||^2_\omega},
\end{equation}
then, by Lemma 5.7, it follows that $\varphi_t$ is also a smooth geodesic in $K_Y(Y, \tilde{\omega}_0)$ connecting $\varphi_0, \varphi_1 \in K_T(Y, \tilde{\omega}_0)$.

In general, by the results in [22], $\varphi_0, \varphi_1$ can be connected only with a weak $C^{1,1}$-geodesic in $K_T^{1,1}(X, \omega_0)$, where $K_T^{1,1}(X, \omega_0)$ stand for the space of $C^1(X)$ functions $\varphi$ on $X$ such that $\omega_0 + d\bar{d}\varphi \geq 0$ and has bounded coefficients as a $(1,1)$-current. More precisely, letting $\Sigma := \{ 1 \leq t \leq \epsilon \} \subset C$, it is shown in [22] that there exists a unique weak solution (i.e. a positive $(1,1)$-current in the sense of Bedford–Taylor) of the homogeneous Monge–Ampère equation
\begin{equation}
(\pi_X^*\omega_0 + d_X d_X^*\Phi)^{m+1} = \pi_X^*\omega_0 + d_X d_X^* \Phi \geq 0, \Phi \in C^{1,\alpha}(X \times \overline{\Sigma}),
\end{equation}
\begin{align*}
\Phi(x, 1) &= \varphi_0(x), \\
\Phi(x, e) &= \varphi_1(x).
\end{align*}
It was later shown in [25] that $\Phi$ is actually of regularity $C^{1,1}(X \times \overline{\Sigma})$. Note that, by the uniqueness, $\Phi$ is $T$-invariant as soon as $\varphi_0$ and $\varphi_1$ are. The link with (30) is (see [64]) that if $\Phi$ were actually smooth, we can recover the smooth geodesic $\varphi_t$ joining $\varphi_0$ and $\varphi_1$ by letting $t := \log |z|$ and $\varphi_t(x) := \Phi(x, \log |z|)$. In the general case, the curve $\varphi_t$ of (weak) $\omega_0$-relative pluri-subharmonic potentials (of regularity $C^{1,1}(X \times [0, 1])$) is referred to as the weak $C^{1,1}$-geodesic joining $\varphi_0$ and $\varphi_1$. 

We are thus going to check that any weak $C^{1,1}$-geodesic on $X$ (invariant under $\mathbb{T}_X$) defines, via Lemma 5.5, a $C^{1,1}$-geodesic on $Y$. To this end, we need to show that $\Phi$ satisfies
\begin{equation}
(\pi_Y^* \bar{\omega}_0 + dY dY^* \Phi)^{m+n+1} = 0, \quad \pi_Y^* \bar{\omega}_0 + dY dY^* \Phi \geq 0,
\end{equation}
the regularity statements being automatically satisfied on $Y$.

By the results in [22] and [10], $\Phi$ can be approximated as $\varepsilon \to 0$, both it the weak sense of currents and in $C^{1,\alpha}(X \times \Sigma)$ (for a fixed $\alpha \in (0,1)$), by smooth functions $\Psi^\varepsilon(x,z)$ on $X \times \Sigma$ which solve
\begin{equation}
(\pi_Y^* \omega_0 + dY dY^* \Psi^\varepsilon)[m+1] = \varepsilon \left( (\pi_Y^* \omega_0)[m] \wedge (dx \wedge dy) \right), \quad \varepsilon > 0,
\end{equation}
\[\pi_Y^* \omega_0 + dY dY^* \Psi^\varepsilon > 0, \quad \Psi^\varepsilon(x,1) = \varphi_0, \quad \Psi^\varepsilon(x,e) = \varphi_1.
\]
By the uniqueness of the smooth solution of (33) (and using that both $\varphi_0, \varphi_1$ are $\mathbb{T}_X$-invariant), we have that $\Psi^\varepsilon(x,z)$ is a $\mathbb{T}_X$-invariant smooth function on $X$ for any $z \in \Sigma$; furthermore, the positivity condition on the second line yields that $\Psi^\varepsilon(x,z) \in K_T(X,\omega_0)$ for any $z \in \Sigma$. We can then also see $\Psi^\varepsilon(x,z)$, via its pull-back to $X \times P \times \Sigma$, as a $\mathbb{T}_Y$-invariant smooth function on $Y \times \Sigma$; the arguments in the proof of Lemma 5.5 yield that $\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon > 0$ on $Y \times \Sigma$. Furthermore, by the same proof, we have the following equality of volume forms on $X \times P \times \Sigma$:
\begin{equation}
(\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon)[m+n+1] \wedge \theta^{\wedge r} = p(\mu_{\Psi^\varepsilon})(\pi_Y^* \omega_0 + dY dY^* \Psi^\varepsilon)[m+1] \wedge \theta^{\wedge r}
\end{equation}
\[= e(p(\mu_{\Psi^\varepsilon}))(\pi_Y^* \omega_0)[m+1] \wedge (\pi_Y^* \omega_B)[n] \wedge \theta^{\wedge r},
\]
where, we recall, $p(\mu) := \prod_{a=1}^k ((p_a, \mu) + c_a)^{n_a}$, $\theta^{\wedge r} := \theta_1 \wedge \cdots \wedge \theta_r$ ($\theta = \sum_{i=1}^r \theta_i \otimes \xi_i^P$ with respect to a basis $\{\xi_1, \ldots, \xi_r\}$ of $\Lambda = 0$), and, for any fixed $z \in \Sigma$, $\mu_{\Psi^\varepsilon}$ denotes the normalized $\mathbb{T}_X$-momentum map $\bar{\psi}$ of $\omega_0 + dX dX^* \Psi^\varepsilon$. Notice that, as $p$ is uniformly bounded on $\Delta$ by positive constants, it follows by (34) that
\[\lim_{\varepsilon \to 0} \left( (\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon)[m+n+1] \wedge \theta^{\wedge r} \right) = 0,
\]
weakly (as measures on $Z \times \Sigma$). The push-forward measure of $(\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon)[m+n+1] \wedge \theta^{\wedge r}$ to $Y$ is the measure $(\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon)[m+n+1]$, so we obtain on $Y$:
\[\lim_{\varepsilon \to 0} \left( (\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon)[m+n+1] \right) = 0.
\]
Furthermore, using the $C^{1,\alpha}$-convergence of $\Psi^\varepsilon$ to $\Phi$, we get the weak convergences (of positive $(1,1)$-currents):
\[\lim_{\varepsilon \to 0} (\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon) = \pi_Y^* \bar{\omega}_0 + dY dY^* \Phi \geq 0;
\]
\[0 = \lim_{\varepsilon \to 0} (\pi_Y^* \bar{\omega}_0 + dY dY^* \Psi^\varepsilon)[m+n+1] = (\pi_Y^* \bar{\omega}_0 + dY dY^* \Phi)[m+n+1].
\]
Thus, (32) follows. \hfill \Box

**Lemma 5.9.** Let $\nu$ be a smooth positive weight function on $\Delta$ and $\omega$, $\bar{\omega}$ be $\mathbb{T}$-invariant Kähler metrics respectively on $X$ and $Y$, given by $\bar{\psi}$, and suppose $(B, \omega_0)$ has constant scalar curvature $\text{Scal}(\omega_0) = s_a$. Then, the $\nu$-scalar curvature $\text{Scal}(\bar{\omega})$, considered as smooth function on $X \times P$, is given by
\begin{equation}
\text{Scal}_\nu(\bar{\omega}) = \frac{1}{p(\mu)} \text{Scal}_\nu(\omega) + v(\mu)q(\mu)
\end{equation}
with $p(\mu) = \prod_{a=1}^k ((p_a, \mu) + c_a)^{n_a}$ and $q(\mu) = \sum_{a=1}^k \frac{s_a}{(p_a, \mu) + c_a}$. In particular, $\omega$ is $(\nu, \bar{\omega})$-cscK metric on $X$ if and only if $\bar{\omega}$ is a $(\nu, \omega)$-cscK metric on $Y$, with
\[\bar{\omega}(\mu) = p(\mu)(w(\mu) - v(\mu)q(\mu)).
\]
**Proof.** We apply the arguments in the proof [3 Prop. 7] to both $(X, \mathbb{T}_X)$ and $(Y, \mathbb{T}_Y)$ to compute the corresponding scalar curvatures, and compare the results.
On $X$, we consider the open dense subset $\hat{X} \subset X$ of stable points of the $T_X$-action, and take the quotient $\pi_S : \hat{X} \to S := \hat{X}/T_X^\C$ under the induced complexified action $T_X^\C \cong (\C^*)^r$ (thus $S$ is a complex $2(m-r)$-dimensional orbifold).

Consider the point-wise $\omega$-orthogonal and $T$-invariant decomposition
\[ T\hat{X} = \mathfrak{H} \oplus t_X \oplus Jt_X, \]
and write the Kähler structure $(g, J, \omega)$ on $X$ as
\[
g = g_\mathfrak{H} + \sum_{i,j=1}^r H_{ij} (\eta_i \otimes \eta_j + J\eta_i \otimes J\eta_j),
\]
\[
\omega = \omega_\mathfrak{H} + \sum_{i,j=1}^r H_{ij} \eta_i \wedge J\eta_j,
\]
where, for a fixed basis $\{\xi_1, \ldots, \xi_r\}$ of $t$, the 1-forms $\eta_j$ on $\hat{X}$ are defined by $(\eta_j)_{|\mathfrak{H}} = 0$, $\eta_j(\xi_i^X) = \delta_{ij}$; $\eta_j(J\xi_i^X) = 0$ and $H_{ij} = g(\xi_i^X, \xi_j^X)$.

We next fix a local volume form $Vol_S$ on $S$ in some holomorphic coordinates, and write point-wisely
\[
\omega^{[m-r]} = Q\pi_S^*(Vol_S),
\]
for some positive (locally defined) smooth function $Q$ on $\hat{X}$ (where both $\omega^{[m-1]}$ and $\pi_S^*(Vol_S)$ are seen as sections of $\wedge^{m-1}\mathfrak{H}^*$). According to [3] Prop. 7, we have that
\[
\kappa := -\frac{1}{2} (\log(Q) + \log \det(H_{ij}))
\]
is a (local) Ricci potential of $\omega$, i.e. $\rho_\omega = d_X d^*_X \kappa$, and thus
\[
Scal(\omega) = -\frac{1}{2} d_X d^*_X \kappa \wedge \omega^{[m-1]} / \omega^{[m]}. \]

We can now make a similar argument on $Y$, noting that the Kähler reduction of $\hat{Y}$ by the induced $T_Y$-action is $S \times B$: taking a local volume form in holomorphic coordinates on $S \times B$ of the form $Vol_S \wedge Vol_{B_1} \wedge \cdots \wedge Vol_{B_k}$, and using [24], we see that a Ricci potential on $Y$ (when pulled back to $X \times P$) is written as
\[
\tilde{\kappa} = \sum_{a=1}^k \kappa_a - \frac{1}{2} \left( \log(\tilde{Q}) + \log \det(H_{ij}) \right),
\]
where $\kappa_a := -\frac{1}{2} \log \left( \frac{\tilde{\omega}^{[m_a]}}{Vol_{m_a}} \right)$ is a Ricci potential of $(B_a, \omega_a)$ and
\[
\tilde{Q} = p(\mu_\omega)Q.
\]
Thus, we obtain
\[
\tilde{\kappa} = \sum_{a=1}^k \kappa_a + \kappa - \frac{1}{2} \log p(\mu_\omega), \tag{38}
\]
as functions on $X \times P$. Introducing a basis $\{\xi_i\}_i$ of $\Lambda$ and writing the connection 1-form $\theta \in \Omega^1(P, t)$ as $\theta = \sum_{j=1}^r \theta_j \otimes \xi_j^P$ (where the 1-forms $\theta_j$ on $P$ are such that $\theta_j$ is zero on $\mathcal{D}$ and $\theta_j(\xi_j^P) = \delta_{ij}$), we compute for the scalar curvature of $\tilde{\omega}$
\[
Scal(\tilde{\omega}) = -\frac{dY d^*_Y \tilde{\kappa} \wedge \tilde{\omega}^{[m+n-1]}}{\tilde{\omega}^{[m+n]}} \quad \text{(on $Y$)}
\]
\[
= -\frac{dY d^*_Y \tilde{\kappa} \wedge \tilde{\omega}^{[m+n-1]} \wedge \theta^{\wedge r}}{\tilde{\omega}^{[m+n]} \wedge \theta^{\wedge r}} \quad \text{(on $X \times P$).}
\]

\footnote{Our argument is actually local, around each point in $\hat{X}$, so one can assume without loss that $S$ is smooth.}
By (26) and (38), the pullback of $d_Y d_Y^\ast \kappa$ to $X \times P$ is given by,

$$d_Y d_Y^\ast \kappa = d_Y d_Y^\ast (\kappa - \frac{1}{2} \log p(\mu_\omega)) + \sum_{a=1}^k d_Y d_Y^\ast \kappa_a$$

$$= d_X d_X^\ast (\kappa - \frac{1}{2} \log p(\mu_\omega)) + \sum_{j=1}^r d_X^\ast (\kappa d_p \theta_j + \sum_{a=1}^k d_P^\ast \kappa_a)$$

$$+ \sum_{j=1}^r d_X^\ast (\kappa - \frac{1}{2} \log p(\mu_\omega)) (\xi_j) d_P \theta_j + \sum_{a=1}^k d_P^\ast \kappa_a$$

$$= d_X d_X^\ast (\kappa - \frac{1}{2} \log p(\mu_\omega)) + \sum_{j=1}^r d_X^\ast (\kappa - \frac{1}{2} \log p(\mu_\omega)) (\xi_j^X) \wedge \theta_j$$

$$+ \sum_{a=1}^k d_X^\ast (\kappa - \frac{1}{2} \log p(\mu_\omega)) (p_a)(\pi_{\mathcal{B}}^\ast \omega_a) + \sum_{a=1}^k d_P^\ast \kappa_a,$$

where in the last equality we used (22) and we have denoted by $p_a$ the induced vector field on $X$ by the element $p_a \in \mathfrak{t}$. We shall compute the term $d_X^\ast \kappa(p_a)$ on $X$: using (37) we get

$$d_X^\ast \kappa(p_a) = \frac{1}{2} \left( \frac{\mathcal{L}_{Jp_a} Q}{Q} + \text{tr} \left( H_{ij}^{-1}(\mathcal{L}_{Jp_a} H_{ij}) \right) \right).$$

Taking the wedge product of both sides of (36) with

$$\left( \sum_{i,j=1}^r H_{ij} \eta_i \wedge J \eta_j \right)^{[r]} = \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J \eta_j),$$

gives

$$\omega^{[m]} = Q \pi_S^\ast \text{Vol}_S \wedge \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J \eta_j).$$

Applying the Lie derivative $\mathcal{L}_{Jp_a}$ to the above equality yields

$$(\Delta_{\omega^p} \omega^{[m]}) = (\mathcal{L}_{Jp_a} Q) \pi_S^\ast \text{Vol}_S \wedge \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J \eta_j)$$

$$+ Q \pi_S^\ast \text{Vol}_S \wedge \mathcal{L}_{Jp_a} \left( \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J \eta_j) \right)$$

$$Q \pi_S^\ast \text{Vol}_S \wedge \mathcal{L}_{Jp_a} \left( \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J \eta_j) \right)$$

$$= \left( \text{tr} \left( H_{ij}^{-1}(\mathcal{L}_{Jp_a} H_{ij}) \right) \right) Q \pi_S^\ast \text{Vol}_S \wedge \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J \eta_j),$$

where we used that $\mathcal{L}_{Jp_a} \eta_j$ is a basic form (since $(\mathcal{L}_{Jp_a} \eta_j)(\xi_i) = -\eta_j([Jp_a, \xi_i]) = 0$). We thus get $\Delta_{\omega^p} \omega^{[m]} = \frac{\mathcal{L}_{Jp_a} Q}{Q} + \text{tr} \left( H_{ij}^{-1}(\mathcal{L}_{Jp_a} H_{ij}) \right)$, or equivalently, in terms of (41)

$$d_X^\ast \kappa(p_a) = \frac{1}{2} (\Delta_{\omega^p} \omega^{[m]}).$$
Using the above equation in (40), we continue the computation
\begin{equation}
\begin{aligned}
dY dY \kappa = dX dX \kappa - \frac{1}{2} dd^c_X (\log p(\mu)) + \sum_{j=1}^{r} dX \left( d^c_X \left( \kappa - \frac{1}{2} \log p(\mu) \right) (\xi^j) \right) \wedge \theta_j \\
+ \frac{1}{2} \sum_{a=1}^{k} \left( \Delta_\omega \mu_{\omega} + \left( \mathcal{L}_{Jp_a} \left( \frac{p(\mu)}{p(\mu)} \right) \right) \left( \pi_{\omega}^a \omega_a \right) + \sum_{a=1}^{k} d_B d_B \kappa_a \right).
\end{aligned}
\end{equation}
(43)

Recall that (27) on Z we have \( \tilde{\omega}^{[m+n]} \wedge \theta^{\wedge r} = p(\mu) \omega^{[m]} \wedge \wedge_{a=1}^{k} \pi_{\omega}^a \omega_a^{[n_a]} \wedge \theta^{\wedge r} \). Similarly, by (24),
\begin{equation}
\begin{aligned}
\tilde{\omega}^{[m+n-1]} \wedge \theta^{\wedge r} = \sum_{b=1}^{k} \left( \left( \frac{p(\mu)}{p(\mu)} \right) \omega^{[m]} \wedge (\pi_{\omega}^a)^{[n_a]} \wedge \wedge_{a=1}^{k} (\pi_{\omega}^a)^{[n_a]} \wedge \theta^{\wedge r} \right) \\
+ \sum_{a=1}^{k} \omega^{[m+n-1]} \wedge \wedge_{a=1}^{k} (\pi_{\omega}^a)^{[n_a]} \wedge \theta^{\wedge r}.
\end{aligned}
\end{equation}
(44)

Using (39), (43), (27) and (44), we obtain
\begin{equation}
\begin{aligned}
\text{Scal}(\tilde{\omega}) = \text{Scal}(\omega) + \Delta_\omega (\log p(\mu)) & \quad + \sum_{a=1}^{k} \left( \delta_a p_a \left[ (\mu, p_a) + \delta_a \right] \left[ \frac{\Delta_\omega \mu_{\omega} + \mathcal{L}_{Jp_a} \left( \frac{p(\mu)}{p(\mu)} \right) \left( \pi_{\omega}^a \omega_a \right)}{(\mu, p_a) + \delta_a} \right] + \frac{\delta_a}{(\mu, p_a) + \delta_a} \right) \\
& \quad = \text{Scal}(\omega) + \sum_{a=1}^{k} \left( \delta_a \omega_a \log (\mu, p_a + \delta_a) \right) \\
& \quad + \frac{\delta_a}{(\mu, p_a) + \delta_a} \right)
\end{aligned}
\end{equation}
(45)

On the other hand, using a basis \( (\xi_i) \) of \( t \) with a dual basis \( (\xi^i) \) of \( t^* \), we compute
\begin{equation}
\begin{aligned}
\text{Scal}_p(\omega) := p(\mu) \text{Scal}(\omega) + 2 \sum_{i=1}^{r} p_i(\mu) \Delta_\omega (\langle \mu, \xi_i \rangle) = \sum_{i,j=1}^{r} p_{ij} (\mu) g_\omega(\xi_i, \xi_j) \\
= p(\mu) \text{Scal}(\omega) + 2 \sum_{i=1}^{r} \Delta_\omega (\langle \mu, \xi_i \rangle) \sum_{a=1}^{k} \frac{\delta_a \xi^i(p_a)}{(\mu, p_a) + \delta_a} \\
+ \sum_{i,j=1}^{r} g_\omega(\xi_i, \xi_j) \left[ \sum_{a=1}^{k} \frac{\delta_a \xi^i(p_a) \xi^j(p_a)}{(\mu, p_a) + \delta_a} - \frac{\delta_a}{(\mu, p_a) + \delta_a} \right] \begin{aligned}
& \quad = p(\mu) \text{Scal}(\omega) + 2 \Delta_\omega (\langle \mu, p_a \rangle) \sum_{a=1}^{k} \frac{\delta_a p_a}{(\mu, p_a) + \delta_a} \\
& \quad + \left[ \left[ \sum_{a=1}^{k} \frac{\delta_a p_a}{(\mu, p_a) + \delta_a} - \sum_{a=1}^{k} \frac{\delta_a g_\omega(p_a)}{(\mu, p_a) + \delta_a} \right] \right].
\end{aligned}
\end{aligned}
\end{equation}
(46)

Comparing the above expression with (45), we obtain
\begin{equation}
\begin{aligned}
\text{Scal}(\tilde{\omega}) = \frac{1}{p(\mu)} \text{Scal}_p(\omega) + \sum_{a=1}^{k} \frac{s_a}{(\mu, p_a) + \delta_a}.
\end{aligned}
\end{equation}
(46)
Using that (as functions on $X \times P$) $\mu_{\tilde{\omega}} = \mu_\omega$ and $g_\omega(\xi_i, \xi_j) = g_{\tilde{\omega}}(\xi_i, \xi_j)$ (see the proof Lemma 5.7), we further compute from (46)

$$Scal_c(\tilde{\omega}) = v(\mu_{\tilde{\omega}})Scal(\tilde{\omega}) + 2 \sum_{i=1}^r v_i(\mu_{\tilde{\omega}})\Delta^X_{\tilde{\omega}}(\langle \mu_{\tilde{\omega}}, \xi_i \rangle) - \sum_{i,j=1}^r v_{i,j}(\mu_{\tilde{\omega}})g_{\tilde{\omega}}(\xi_i, \xi_j)$$

$$= v(\mu_\omega)p(\mu_\omega)Scal(\omega) + 2 \sum_{i=1}^r v_i(\mu_\omega)p_i(\mu_\omega)\Delta^X_\omega(\langle \mu_\omega, \xi_i \rangle) - \sum_{i,j=1}^r v_{i,j}(\mu_\omega)p_{i,j}(\mu_\omega)g_\omega(\xi_i, \xi_j)$$

$$+ 2 \sum_{i=1}^r v_i(\mu_\omega)\Delta^X_\omega(\langle \mu_\omega, \xi_i \rangle) - \sum_{i,j=1}^r v_{i,j}(\mu_\omega)p_{i,j}(\mu_\omega)g_\omega(\xi_i, \xi_j)$$

$$= \frac{1}{p(\mu_\omega)}(pv)(\mu_\omega)Scal(\omega) + 2 \sum_{i=1}^r (pv)_i(\mu_\omega)\Delta^X_\omega(\langle \mu_\omega, \xi_i \rangle) - \sum_{i,j=1}^r (pv)_{i,j}(\mu_\omega)g_\omega(\xi_i, \xi_j)$$

$$= \frac{1}{p(\mu_\omega)}Scal_{p,w}(\omega).$$

The expression (35) follows from the above formulae.

**Lemma 5.10.** The restriction of the weighted Mabuchi energy $M^Y_{p,w}$ on $Y$ to the subspace $K_T(X, \omega_0) \subset K_T(Y, \tilde{\omega}_0)$ is equal to $CM^X_{p,w,\tilde{\omega}}$, where $p, w, \tilde{\omega}$ are given in Lemma 5.9 and $C = \text{Vol}(B, \omega_B)$.

**Proof.** A direct corollary of Lemma 5.9 and Definition 1.1. □

We now specialize to the case when each $(B_a, \omega_a)$ is a Hodge Kähler–Einstein manifold with positive scalar curvature $s_a = 2n_a k_a$, where $k_a \in \mathbb{N}$. Equivalently, $2\pi c_1(B_a) = k_a[\omega_a]$ for a positive integer $k_a$ and an integral Kähler class $\frac{1}{\pi}[\omega_a]$. Notice that $k_a$ must be a positive divisor of the Fano index $\text{Ind}(B_a)$ of $B_a$, which yields the a priori bound $1 \leq k_a \leq \text{Ind}(B_a)$. We also assume that $(X, T)$ is Fano, with canonically normalized momentum polytope $\Delta$. We then have

**Lemma 5.11.** In the setting above, if the affine linear functions $(\langle p_a, \mu \rangle + k_a) > 0$ on $\Delta$, then the bundle-compatible Kähler metric $\tilde{\omega}$ on $Y$ corresponding to the constants $c_a = k_a$ belongs to deRham class $2\pi c_1(Y)$. Furthermore, $\tilde{\omega}$ is a $v$-soliton if and only if $\omega$ is a $pv$-soliton.

**Proof.** By using (38) and rearranging the terms in (40), we have the following relation (written on $Z$):

$$\rho_\omega = \rho_\omega + \sum_{a=1}^k (\langle p_a, \mu_\omega \rangle + c_a)\omega_a + \langle d\mu_{\rho_\omega} \land \theta \rangle$$

$$+ \sum_{a=1}^k (\rho_a - c_a \omega_a) - \frac{1}{2}dy_d\log p(\mu_\omega),$$

(48)
where $\mu_\omega$, $\rho_\omega$ and $\rho_\varpi$ respectively denote the Ricci forms of $(Y, \varpi)$, $(X, \omega)$ and $(B_a, \omega_a)$, pulled back to $Z$, and $\mu_{\rho_\omega} := d^c X_c \kappa$ is the “momentum map” with respect to the Ricci form $\rho_\omega$. As in [12], we have $\mu_{\rho_\omega} = \frac{1}{2} \Delta \omega \mu_\omega$. Suppose $\rho_\omega - \omega = \frac{1}{2} d Y d^c X h$ for some $T$-invariant smooth function $X$; by using that the momentum polytope $\Delta$ is canonically normalized, we have (see [11]) $\mu_{\rho_\omega} - \mu_\omega = d^c h$. A closer look at the proof of Lemma 5.1 and the relation (48) (with $e_a = \frac{2}{2m} = k_a$) show that
\[
\rho_\varpi - \varpi = \frac{1}{2} d Y d^c X \tilde{h}, \quad \tilde{h} := h - \log p(\mu_\omega).
\]
The claim follows from the above. \hfill \Box

Remark 5.12. Lemma 5.1 provides a useful way to construct semi-simple $(X, T)$-principal Fano fibrations. Indeed, for given positive Hodge Kähler–Einstein manifolds $(B_a, \omega_a)$ as above, with corresponding integer constants $k_a$, and a given Fano manifold $(X, T)$ with associated canonical polytope $\Delta$, one can try to find the possible principal $T$-bundles $P$ over $B = \prod_{a=1}^k B_a$, for which the corresponding semi-simple $(X, T)$-principal fibration is Fano. Such principal $T$-bundles $P$ are in correspondence with the choice of lattice elements $p_a \in \Lambda \subset t$ and Lemma 5.11 tells us that for a set of elements $p_a$ to determine a Fano semi-simple $(X, T)$-principal fibration $Y$, it is sufficient to check that all the affine linear functions
\[
\langle p_a, \mu \rangle + k_a > 0 \text{ on } \Delta.
\]

For instance, if we take $B = B_1 = \mathbb{P}^1$ with a Fubini–Study metric $\omega_1$ of scalar curvature 4 (so that $k_1 = 2$ and $\omega_1$ is primitive) and $(X, T) = (\mathbb{P}^1, S^1)$ with canonical polytope $\Delta = [-1, 1]$, then the possible Fano $(\mathbb{P}^1, S^1)$-principal fibrations will correspond to $p_1 \in \mathbb{Z}$ such that $p_1 \mu + 2 > 0$ on $[-1, 1]$, i.e. $p_1 = \pm 1, 0$ are the only possible values. This gives rise to the Fano surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ and $\mathbb{P}^1 \times \mathbb{P}^1$. In general, the isomorphism class of the principal $T$-bundle $P$ over $B$, and hence also the semi-simple $(X, T)$-principal Fano fibration constructed as above, is encoded by the Hodge classes $\frac{1}{k_a} [\omega_a] \otimes p_a = \frac{1}{k_a} c_1(B_a) \otimes p_a \in H^2(B, \mathbb{Z})_r$. The a priori bounds $1 \leq k_a \leq \text{Ind}(B_a)$ for $k_a$ show that for given base $B = \prod_{a=1}^k B_a$ and fibre $(X, T)$, there are only a finite number of semi-simple $(X, T)$-principal Fano fibrations constructed this way.

Remark 5.13. The relationship between the Ricci potentials $\tilde{h}$ and $h$ established in the proof of Lemma 5.11 and (29) yield, via Remark 2.3, that if the momentum map $\mu_\omega$ of $(X, \omega, T_X)$ is canonically normalized, then the momentum map $\mu_\varpi = \mu_\omega$ of the corresponding bundle-compatible Kähler metric $\hat{\omega}$ on $(Y, T_Y)$ is also canonically normalized.

We end up this section with the following straightforward extension of [7, Lemma 5].

Lemma 5.14. Suppose $Y$ is a semi-simple principal $(X, T)$-fibration over $B$, such that $T$ is a maximal torus in the reduced group of automorphisms $\text{Aut}_r(X)$. Let $\hat{\omega}$ be a bundle-compatible Kähler metric on $Y$ corresponding to a $T$-invariant Kähler metric $\omega$ on $X$, and $K_B \subset \text{Aut}_r(B)$ be a maximal compact torus in the reduced group of automorphisms of $B$ which (without loss by Lichnerowicz–Matsushima theorem) belongs to the isometry group of $\omega_B$. Then $\hat{\omega}$ is invariant under the action of a maximal torus $K_Y \subset \text{Aut}_r(Y)$, and we have an exact sequence of Lie algebras

\[
\{0\} \to \text{Lie}(\mathbb{T}_Y) \to \text{Lie}(K_Y) \to \text{Lie}(K_B) \to \{0\}.
\]

Furthermore, for any positive weight functions $v, w_0$ defined on $\Delta \subset t^*$, there exists a unique affine-linear function $\ell_{v, w_0}^\text{ext}$ on $t^*$ such that, when pulled-back to the dual Lie algebra $t_Y^*$ of $K_Y$, $(v, w_0) \ell_{v, w_0}^\text{ext}$ satisfy (3) with respect to $\hat{\omega}$ on $Y$, and any affine-linear function $\ell$ on $t_Y^*$.

Proof. The proof of the above result is not materially different than the proof of [7, Lemma 5] (which is made in the case when $(X, T)$ is toric and $v = w_0 = 1$). We only give a sketch. A Killing potential $f$ for a Killing vector field $K \in \mathfrak{k}_B := \text{Lie}(K_B)$ is of the form $f = \sum_{a=1}^k f_a$, where $f_a$ is a Killing potential of $(B_a, \omega_a)$. Letting $\tilde{K}$ be the horizontal lift of $K$ to $P$ (using the $t_P$-valued connection 1-form $\theta$), one can check that the vector field on $P$

\[
\tilde{K} = \tilde{K} + \sum_{a=1}^k f_a \xi_{p_a}.
\]
is a CR vector field on \((P, D, J_B)\), hence also on \((Z, H, J_B \oplus J_X)\). Furthermore, a direct verification in [24] reveals that

\[
\tau_K \tilde{\omega} = -d \left( \sum_{a=1}^{k} (p_{a} \mu_{\omega}) + c_{a} f_{a} \right)
\]

showing that \(\tilde{K}\) also preserves \(\tilde{\omega}\). We thus obtain a lift \(\tilde{\ell}_{B}\) of the Lie algebra \(\ell_{B} = \text{Lie}(T_{B})\) to \(Z\), which clearly commutes with the action \(T_{Z}\), and preserves both the CR structure of \((Z, H)\) and the 2-form \(\tilde{\omega}\). The Lie algebra \(\ell_{Y}\) of \(\K_{Y}\) is then induced by \(\ell_{X} \oplus \ell_{B} \subset T_{Z}\), which descend to an abelian Lie algebra of Killing fields on \(Y\). The maximality of \(\K_{Y} \subset \text{Aut}_{r}(Y)\) and the exactness of the sequence follow from the maximality of each \(\K_{B} \subset \text{Aut}_{r}(B)\) and \(T \subset \text{Aut}_{r}(X)\), and the fact that (recall that \(Y\) is a locally trivial \(X\)-fibre bundle and therefore the fibres have trivial normal bundle) any holomorphic vector field on \(Y\) projects under \(\pi_{B}\) to a holomorphic vector field on \(B\). For the final claim in Lemma 5.14 notice that by (49) the Killing potentials of all lifted Killing vector fields \(K\) from \(B\) are of the form \(\sum_{a=1}^{k} (p_{a} \mu_{\omega}) + c_{a} f_{a}\). Thus, by Lemma 5.9 and using [27], the integral condition (3) on \((Y, \tilde{\omega})\) will be zero for any such Killing potential, as soon as we normalize \(\int_{B} f_{a} \omega_{\omega_{0}}^{n_{a}} = 0\) and assume \(\rho_{\omega_{0}}^{\text{ext}} \in \text{Aff}(\ell_{X}^{\text{t}})\). On the other hand, examining (3) on \((Y, \tilde{\omega})\) for the Killing potentials \(\ell_{\mu_{\omega}}, \ell \in \text{Aff}(\ell^{t})\) reduces (again by Lemma 5.9 and (29)) to an integral relation on \((X, \omega)\) which defines a unique element \(\rho_{\omega_{0}}^{\text{ext}} \in \text{Aff}(\ell^{t})\).

6. Weighted functionals and distances and their extensions

Let \(\omega_{0}\) a \(T\)-invariant Kähler metric in the Kähler class \(\alpha\). We denote by \(\text{PSH}_{T}(X, \omega_{0})\) the space of \(T\)-invariant \(\omega_{0}\)-pluri-subharmonic functions in \(L^{1}(X, \omega_{0})\), and define the class of potentials of full volume by

\[
\mathcal{E}_{T}(X, \omega_{0}) := \left\{ \varphi \in \text{PSH}_{T}(X, \omega_{0}) \mid \int_{X} \mathcal{M}_{\alpha}(\varphi) = \int_{X} \omega_{0}^{[n]} \right\}
\]

According to [27], the \(d_{1}\)-completion of \(\mathcal{K}_{T}(X, \omega_{0})\) can be identified with the subspace of potentials of finite energy, i.e.

\[
\mathcal{E}_{T}^{1}(X, \omega_{0}) = \left\{ \varphi \in \mathcal{E}_{T}(X, \omega_{0}) \mid \int_{X} |\varphi| \mathcal{M}_{\alpha}(\varphi) < \infty \right\}.
\]

Our main result in this section will be the existence of a lsc extension of the weighted Mabuchi functional (defined in Definition 1.1 on the space \(\mathcal{K}_{T}(X, \omega_{0})\)) to a functional on the space \(\mathcal{E}_{T}^{1}(X, \omega_{0})\). Our starting point is that the weighted Mabuchi energy \(\mathcal{M}_{v,w}\) admits a weighted Chen–Tian decomposition [56 Thm. 5] into energy and entropy parts as follows:

\[
\mathcal{M}_{v,w}(\varphi) = \int_{X} \log \left( \frac{v(\mu_{\omega}) \omega_{\varphi}^{m}}{\omega_{0}^{m}} \right) v(\mu_{\varphi}) \omega_{\varphi}^{[m]} - 2\mathcal{I}_{v,\omega_{0}}(\varphi) + \mathcal{I}_{w}(\varphi)
\]

\[
- \int_{X} \log(v(\mu_{\omega})) v(\mu_{\omega}) \omega_{\omega_{0}}^{[m]},
\]

where \(\rho_{\omega_{0}}\) is the Ricci form of \(\omega_{0}\) and the functionals \(\mathcal{I}_{w}\) and \(\mathcal{I}_{v,\omega_{0}}^{\rho_{\omega_{0}}}\) are introduced in Definition 6.2 below. We want to show the following

**Theorem 6.1.** For smooth weight functions \(v(\mu), w(\mu)\) such that \(v(\mu) > 0\) on \(\Delta\), the weighted Mabuchi energy \(\mathcal{M}_{v,w} : \mathcal{K}_{T}(X, \omega_{0}) \rightarrow \mathbb{R}\) extends using (50) to the largest \(d_{1}\) lsc functional \(\mathcal{M}_{v,w}^{\rho_{\omega_{0}}} : \mathcal{E}_{T}^{1}(X, \omega_{0}) \rightarrow \mathbb{R} \cup \{\infty\}\) which is convex along the finite energy geodesics of \(\mathcal{E}_{T}(X, \omega_{0})\). Additionally, the extended weighted Mabuchi energy \(\mathcal{M}_{v,w}\) is linear in \(v, w\), uniformly continuous in \(w\) in the \(C^{0}(\Delta)\) topology and continuous with respect to \(v\) in the \(C^{1}(\Delta)\) topology.

The above result is well-known for the unweighted case, by the work [13], and we will follow a similar path to get an extension in the weighted case. The proof of Theorem 6.1 will be given at the end of the section, and we detail below the definition and extension of each component of (50).
6.1. The weighted Aubin–Mabuchi functionals.

**Definition 6.2.** For a smooth weight function \( v(\mu) \) on \( \Delta \), we let \( I_v \) denote the functional on \( K_\mu(X, \omega_0) \), defined by
\[
(d_\mu I_v)(\phi) = \int_X \phi v(\mu_\phi) \omega_\phi^{[m]}, \quad I_v(0) = 0,
\]
and let \( J_v := \int_X \phi v(\mu_0) \omega_0^{[m]} - I_v(\phi) \). Furthermore, for a fixed \( T \)-invariant closed (1,1)-form \( \rho \) on \( X \) with momentum \( \mu_\rho : X \to \mathfrak{t}^* \), we define the \( \rho \)-twisted Aubin–Mabuchi functional \( I_v^\rho : K_\mu(X, \omega_0) \to \mathbb{R} \) by
\[
(d_\mu I_v^\rho)(\phi) := \int_X \phi \left( v(\mu_\phi) \rho \wedge \omega_\phi^{[m-1]} + ((dv)(\mu_\phi), \mu_\rho) \omega_\phi^{[m]} \right), \quad I_v^\rho(0) = 0.
\]
For \( v \equiv 1 \), we let \( I_1 = I \), \( J_1 = J \) and \( I_0^0 = I^0 \), and notice that \( I, J \) are the functionals introduced in Definition 3.1.

**Remark 6.3.** It follows from the above definition and the results in [55] that for any weight \( v(x) \) and a constant \( c \), \( J_v(\phi + c) = J_v(\phi) \), allowing one to see \( J_v \) as a functional on the space of \( T \)-invariant Kähler metrics in the Kähler class \( \alpha = [\omega_0] \), and motivates the notation \( J_v(\omega_\alpha) \). Notice also that \( I_v, J_v, I^0_v \) are linear in \( v \). In the case when \( v > 0 \), \( J_v \) is non-negative (see Lemma 6.4 below), whereas \( I_v \) is monotone in the sense that for any \( \varphi_0, \varphi_1 \in K_\mu(X, \omega_0) \) with \( \varphi_1(x) \geq \varphi_0(x) \)
\[
I_v(\varphi_1) - I_v(\varphi_0) \geq \inf_\Delta (v) \int_X (\varphi_1 - \varphi_0) \omega_\phi^{[m]}.
\]
The above inequality follows by Definition 6.2 and integrating the derivative of \( I_v \) along the path \( t \varphi_1 + (1-t) \varphi_0 \in K_\mu(X, \omega_0) \) and integrating by parts.

The following is established in [55] (2.37).

**Lemma 6.4.** Let \( v > 0 \). There exists a uniform constant \( C = C(X, \omega_0, v) > 0 \) such that
\[
\frac{1}{C} J(\varphi) \leq J_v(\varphi) \leq C J(\varphi).
\]

**Proof.** Let \( \varphi_t := \varphi_0 + t \varphi \) with \( \varphi := \varphi_1 - \varphi_0 \) and \( \omega_{\varphi_t} = \omega_{\varphi_0} + tdd^c \varphi, t \in [0,1] \). We compute
\[
J_v(\varphi) = J_v(\varphi_1) - J_v(\varphi_0)
\]
\[
= \int_0^1 \int_X \varphi \left( v(\mu_\varphi) \omega_\varphi^{[m]} - v(\mu_{\varphi_0}) \omega_{\varphi_0}^{[m]} \right) ds
\]
\[
= - \int_0^1 \int_X \varphi \int_0^s \frac{d}{dt} [v(\mu_{\varphi_t}) \omega_{\varphi_t}^{[m]}] dt ds
\]
\[
= - \int_0^1 \int_X \varphi \left( \int_0^s [g_{\varphi_t}(d[\log \circ v(\mu_{\varphi_t})]), \varphi] - \Delta_{\varphi_t}(\varphi) \right) v(\mu_{\varphi_t}) \omega_{\varphi_t}^{[m]} dt ds
\]
\[
= - \int_0^1 \int_0^s \left( \int_X \varphi v(\mu_{\varphi_t}) d\varphi \wedge d^c \varphi \wedge \omega_{\varphi_t}^{[m-1]} + \varphi d^c \varphi \wedge v(\mu_{\varphi_t}) \omega_{\varphi_t}^{[m-1]} \right) dtds
\]
\[
= \int_0^1 \int_X \varphi v(\mu_{\varphi_t}) d\varphi \wedge d^c \varphi \wedge \omega_{\varphi_t}^{[m-1]} dtds
\]
\[
= \int_0^1 \int_X \varphi v(\mu_{\varphi_t}) d\varphi \wedge d^c \varphi \wedge \omega_{\varphi_t}^{[m-1]} dtds
\]
\[
= \sum_{j=0}^{m-1} \int_0^1 \int_0^s \left( \int_X v^j(1-t)^{m-j-1} v(\mu_{\varphi_t}) d\varphi \wedge d^c \varphi \wedge \omega_{\varphi_t}^{[m-j-1]} \right) dtds
\]
where, in the fourth equality, we have used that
\[
\frac{d}{dt} [v(\mu_{\varphi_t})] = \sum_{i=1}^r v_i(\mu_{\varphi_t})(d^c \varphi)(\xi_i) = g_{\varphi_t}(d[\log \circ v(\mu_{\varphi_t})], d\varphi)
\]
(51)
for any basis \((\xi_i)_{i=1,\ldots,r}\) of \(\mathfrak{t}\). It follows that
\[
\frac{1}{C} J(\varphi) \leq J_v(\varphi) \leq C J(\varphi),
\]
where \(C = C(X, \alpha, v)\) is a constant such that \(\frac{1}{C} \leq v \leq C\) on \(\Delta_\alpha\).

**Lemma 6.5.** Suppose \(v, w\) are smooth functions on \(\Delta\). Then
\[
\left\| J_v(\varphi) - J_w(\varphi) \right\| \leq ||v - w||_{C^0(\Delta)} J_1(\varphi);
\]
\[
\left\| I_v(\varphi) - I_w(\varphi) \right\| \leq ||v - w||_{C^0(\Delta)} \left( ||\varphi||_{L^1(\omega_{\alpha})} + J_1(\varphi) \right).
\]
In particular, for a fixed \(\varphi \in K_\Delta(X, \omega_0)\), \(I_v(\varphi)\) and \(J_v(\varphi)\) are uniform continuous in \(v\).

**Proof.** The first relation follows from Lemma 6.4 above whereas the second inequality follows from the first and Definition 6.2. \(\square\)

**Lemma 6.6.** The restrictions of \(I_Y^1\), \(J_Y^1\) to the subspace \(K_T(X, \omega_0) \subset K_T(Y, \tilde{\omega}_0)\) are respectively equal to \(C I_Y^X\) and \(C J_Y^X\), where \(p(\mu)\) is the weight function defined in Lemma 5.9 and \(C = \text{Vol}(B, \omega_B)\). Furthermore, if \(\tilde{\rho}\) is a Kähler form on \(Y\), induced by a Kähler form \(\rho\) on \(X\) using (24), then the restriction of \((I_Y^1)^Y\) to the subspace \(K_T(X, \omega_0)\) equals \(C(I_Y^0)^X\).

**Proof.** The first part follows from the definition of \(I_Y^1\), using that \(\tilde{\omega}_\varphi^{n+m} \wedge \theta^{\wedge r} = p(\mu_\varphi) \omega_\varphi^{[m]} \wedge \omega_B^n \wedge \theta^{\wedge r}\) on \(Z\).

Similarly, if \(\tilde{\rho}\) is a \((1, 1)\)-form \(Y\) whose pull-back to \(Z\) is
\[
(52) \quad \tilde{\rho} := \rho + \sum_{a=1}^k (\langle p_a, \mu_p \rangle + c_a) \pi_B^* \omega_a + \langle d\mu_\rho \wedge \theta \rangle,
\]
we compute
\[
(d_x I_Y^0 X(\varphi) = \int_X \varphi \left[ p(\mu_\varphi) \rho \wedge \omega_\varphi^{[m-1]} + \langle (dp)(\mu_\varphi), \mu_\rho \rangle \omega_\varphi^{[m]} \right]
\]
\[
= \frac{1}{\text{Vol}(B, \omega_B)} \int_Y \varphi \tilde{\rho} \wedge \tilde{\omega}_\varphi^{[n+m-1]} = \frac{1}{\text{Vol}(B, \omega_B)} (d_x \tilde{\rho})^Y(\varphi).
\]
The claim follows as \((I_Y^0)^Y(0) = 0 = (\tilde{I}^0)^Y(0)\). \(\square\)

### 6.2. The weighted \(d_1\)-distance.

**Definition 6.7.** Let \(v > 0\) be a positive function on \(\Delta\). For \(\varphi_0, \varphi_1 \in K_T(X, \omega_0)\) we let
\[
d_{1,v}(\varphi_0, \varphi_1) := \inf_{\psi(t)} \{ L_{1,v}(\psi(t)) \mid \psi(t, x) \in C^\infty_T([0, 1] \times X), \psi(t) \in K_T(X, \omega_0) \}
\]
where
\[
L_{1,v}(\psi(t)) := \int_0^1 \left( \int_X |\dot{\psi}(t)|v(\mu_{\psi(t)})\omega_{\psi(t)}^{[m]} \right) dt.
\]
For \(v \equiv 1\) we have \(d_{1,1} = d_1\) where \(d_1\) is the distance introduced in Section 3.

**Lemma 6.8.** For any weight \(v > 0\), there exists uniform constant \(C = C(X, \omega_0, v) > 0\) such that
\[
(53) \quad \frac{1}{C} d_1(\varphi_0, \varphi_1) \leq d_{1,v}(\varphi_0, \varphi_1) \leq C d_1(\varphi_0, \varphi_1), \quad \forall \varphi_0, \varphi_1 \in K_T(X, \omega_0),
\]
where \(d_1 := d_{1,1}\) is the distance introduced in [27]. In particular, \(d_{1,v}\) is a distance on \(K_T(X, \omega_0)\) which is quasi-isometric with \(d_1\).

**Proof.** The relation (53) follows from the fact that \(v(\mu)\) is positive and uniformly bounded on \(\Delta\). This yields that \(d_{1,v}\) is a distance, as \(d_1\) is a distance according to [27]. \(\square\)

**Lemma 6.9.** For any smooth weight \(v > 0\) we have
\[
|I_v(\varphi_0) - I_v(\varphi_1)| \leq d_{1,v}(\varphi_0, \varphi_1) \leq C d_1(\varphi_0, \varphi_1), \quad \forall \varphi_0, \varphi_1 \in K_T(X, \omega_0).
\]
Proof. For any smooth curve \( \varphi_t \) between \( \varphi_0 \) and \( \varphi_1 \), using Definition 6.2, we have

\[
|I_v(\varphi_0) - I_v(\varphi_1)| = \left| \int_0^1 (d_{\varphi_t} I_v)(\varphi_t)dt \right| \leq L_{1,v}(\varphi_t).
\]

The claim follows from the above and Lemma 6.8.

\[\square\]

6.3. Extensions to \( E^1_\mathbb{T}(X,\omega_0) \).

**Lemma 6.10.** For any smooth weight \( v \), the functionals \( I_v \) and \( J_v \) continuously extend to the space \( E^1_\mathbb{T}(X,\omega_0) \). Furthermore, for any \( \psi \in E^1_\mathbb{T}(X,\omega_0) \), the extended functionals are linear and uniformly continuous in \( v \) in the topology \( C^0(\Delta) \).

Proof. \( I_v \) is \( d_1 \)-Lipschitz by Lemma 6.9 for \( J_v \), we get from Definition 6.2

\[
\left| J_v(\varphi_0) - J_v(\varphi_1) \right| \leq \int_X |\varphi_0 - \varphi_1| |\omega_0|^m + |I_v(\varphi_0) - I_v(\varphi_1)|.
\]

Combining the above inequality with Lemma 6.9 and [27, Cor. 5.7], there exists a uniform positive constant \( C = C(X,\omega_0,v) \) and, for any fixed positive real number \( R > 0 \), an increasing continuous function \( F_R : \mathbb{R}_+ \to \mathbb{R}_+ \), \( F'(0) = 0 \), defined in terms of \( (X,\omega_0,R) \), such that for any \( \varphi_0, \varphi_1 \in K_T(X,\omega_0) \) with \( d_1(0,\varphi_i) \leq R \), we have

\[
\left| J_v(\varphi_0) - J_v(\varphi_1) \right| \leq Cd_1(\varphi_0,\varphi_1) + F_R(d_1(\varphi_0,\varphi_1)),
\]

showing that \( J_v \) is locally uniformly continuous on \( (K_T(X,\omega_0),d_1) \) and thus extends continuously to \( (E^1_\mathbb{T}(X,\omega_0),d_1) \).

The \( v \)-linearity of \( I_v \) and \( J_v \) is clear by continuity, see Remark 6.3. The continuity with respect to \( v \) follows from the continuous extensions of the inequalities in Lemma 6.5, noting that we have already shown that \( J_v, J_w, J_1, I_v, I_w \) all extend continuously, whereas \( \| \cdot \|_{L^1(X,\omega_0)} \) extends continuously by [27, Thm. 5.8].

\[\square\]

**Corollary 6.11.** The metric completion of \( (K_T(X,\omega_0) \cap I^{-1}_v(0),d_1) \) is the complete geodesic metric space \( (E^1_\mathbb{T}(X,\omega_0) \cap I^{-1}_v(0),d_1) \).

Proof. Similarly to [29, Lemma 5.2], one can show that \( I_v \) is linear along finite energy geodesics. As \( I_v : E^1_\mathbb{T}(X,\omega_0) \to \mathbb{R} \) is \( d_1 \)-continuous, it follows that \( E^1_\mathbb{T}(X,\omega_0) \cap I^{-1}_v(0) \) is a \( d_1 \)-closed subspace.

\[\square\]

**Lemma 6.12.** Let \( v \) be a smooth weight function and \( \rho \) a \( \mathbb{T} \)-invariant closed \((1,1)\)-form. The functional \( I^\rho_v : K_T(X,\omega_0) \to \mathbb{R} \) extends to a \( d_1 \)-continuous functional on \( E^1_\mathbb{T}(X,\omega_0) \), which is bounded on \( d_1 \)-bounded subsets of \( E^1_\mathbb{T}(X,\omega_0) \). Furthermore, the extended functional is linear and uniformly continuous in \( v \), in the \( C^1(\Delta) \) topology.

Proof. Following the proof of [15, Prop. 4.4], we show that \( I^\rho_v \) is locally uniformly \( d_1 \)-continuous and bounded on \( d_1 \)-bounded subsets of \( K_T(X,\omega_0) \). Let \( \varphi_0, \varphi_1 \in K_T(X,\omega_0) \), we put \( \varphi_s := s\varphi_1 + (1-s)\varphi_0 \), \( s \in [0,1] \) and compute

\[
I^\rho_v(\varphi_1) - I^\rho_v(\varphi_0) = \int_0^1 \frac{d}{ds}I^\rho_v(\varphi_s)ds
\]

\[
= \int_0^1 \int_X (\varphi_1 - \varphi_0) \left( (v(\varphi_s)\rho) \wedge \omega_0^{[m-1]} + \langle (dv)(\varphi_s),\mu_e \rangle \omega_0^m \right) ds
\]

\[
= \int_X (\varphi_1 - \varphi_0) \sum_{j=0}^{m-1} vj,m-1(\varphi_0,\varphi_1) \rho \wedge \omega_0^j \wedge \omega_0^{m-j-1}
\]

\[
+ \int_X (\varphi_1 - \varphi_0) \sum_{j=0}^{m} \langle (dv)_{j,m} (\varphi_0,\varphi_1,\mu_e) \rangle \omega_0^j \wedge \omega_0^{m-j}
\]

(54)
where \( v_{j,k}(\mu_0, \mu_1) \), \((dv)_{j,k}(\mu_0, \mu_1)\) are defined on \( \Delta \times \Delta \) by
\[
v_{j,k}(\mu_0, \mu_1) := \int_0^1 s^j(1-s)^{k-j} v(s\mu_1 + (1-s)\mu_0),
\]
\[
(dv)_{j,k}(\mu_0, \mu_1) = \int_0^1 s^j(1-s)^{k-j}(dv)(s\mu_1 + (1-s)\mu_0).
\]

Using the computation (54), we get
\[
|l_0^p(\varphi_1) - l_0^p(\varphi_0)| \leq C \int_X |\varphi_1 - \varphi_0| \sum_{j=0}^{m-1} \omega_0 \wedge \omega_{\varphi_1}^{[j]} \wedge \omega_{\varphi_0}^{[m-j-1]}
\]
\[
+ C \int_X |\varphi_1 - \varphi_0| \sum_{j=0}^m \omega_{\varphi_1}^{[j]} \wedge \omega_{\varphi_0}^{[m-j]}
\]
\[
\leq C \int_X |\varphi_1 - \varphi_0| \omega_{\varphi_0}^{[m]+\varphi_1}
\]

in the first inequality we use that the functions \((dv)_{j,k}(\mu_{\varphi_0}, \mu_{\varphi_1}), \mu_\rho\) and \(v_{j,k}(\mu_{\varphi_0}, \mu_{\varphi_1})\) are bounded on \( \Delta \times \Delta \) and \(-C\omega_0 < \rho < C\omega_0\) for some constant \(C > 1\) and in the second inequality we use the observation \(\omega_{\varphi_0+\varphi_1} = \omega_0/2 + \omega_{\varphi_0}/4 + \omega_{\varphi_1}/4\). Using the estimate (55) we can show, similarly to [13] Prop. 4.4, that for any \(R > 0\) there is an increasing continuous function \(F_R : \mathbb{R} \rightarrow \mathbb{R}\) with \(F_R(0) = 0\) such that
\[
|l_0^p(\varphi_1) - l_0^p(\varphi_0)| \leq F_R(d_1(\varphi, \varphi_1))
\]
for any \(\varphi_0, \varphi_1 \in K_T(X, \omega_0) \cap \{\varphi, d_1(0, \varphi) < R\}\). It follows that \(l_0^p\) extends to a \(d_1\)-continuous functional on \(E_0^X(X, \omega_0)\) which is bounded on \(d_1\)-bounded subsets of \(E_0^X(X, \omega_0)\).

For the last statement, let \(v, w\) be two (smooth) positive weight functions and \(\varphi \in K_T(X, \omega_0)\). Taking \(\varphi_1 = \varphi\) and \(\varphi_0 = 0\) in the computation (54)
\[
l_0^p(\varphi) = \int_X \varphi \sum_{j=0}^{m-1} v_{j,m-1}(\mu_0, \mu_\varphi) \rho \wedge \omega_{\varphi}^{[j]} \wedge \omega_{\varphi_0}^{[m-j-1]}
\]
\[
+ \int_X \varphi \sum_{j=0}^m ((dv)_{j,m}(\mu_0, \mu_\varphi), \mu_\rho) \omega_{\varphi}^{[j]} \wedge \omega_{\varphi_0}^{[m-j]}
\]
Let \(C > 1\) such that \(-C\omega_0 < \rho < C\omega_0\), using the above formula we obtain
\[
|l_0^p(\varphi) - l_0^p(\varphi)| = |l_0^{p-w}(\varphi)|
\]
\[
\leq C \int_X |\varphi| \sum_{j=0}^{m-1} |(v - w)_{j,m-1}(\mu_0, \mu_\varphi) | \omega_{\varphi}^{[j]} \wedge \omega_{\varphi_0}^{[m-j]}
\]
\[
+ C \int_X |\varphi| \sum_{j=0}^m |((dv - w)_{j,m}(\mu_0, \mu_\varphi), \mu_\rho) | \omega_{\varphi}^{[j]} \wedge \omega_{\varphi_0}^{[m-j]}
\]
\[
\leq C \| \varphi \|_{C^1(\Delta)} \int_X \sum_{j=0}^{m} |\varphi| \omega_{\varphi}^{[j]} \wedge \omega_{\varphi_0}^{[m-j]}
\]
\[
\leq C \| \varphi \|_{C^1(\Delta)} \int_X |\varphi| (2\omega_0 + d\omega)(d\varphi)[m]
\]
\[
\leq C \| \varphi \|_{C^1(\Delta)} \int_X |\varphi| \omega_{\varphi}^{[m]}.
\]
Using approximation by decreasing sequences in \(K_T(X, \omega_0)\), the above estimate holds for \(E_0^X(X, \omega_0)\).

Following Berman–Witt-Nyström [13] and the recent work of Han–Li [16], we now define the extension of weighted Monge–Ampère measures to the space \(E_T(X, \omega_0)\).
Proposition 6.13. Let $v > 0$ be a smooth weight function. For any $\varphi \in \mathcal{K}_T(X, \omega_0)$, we let

$$\text{MA}_v(\varphi) := v(\mu_\varphi)\omega_\varphi^{|m|}.$$ 

Then $\text{MA}_v(\varphi)$ extends to a well-defined Radon measure defined for any $\varphi \in \mathcal{E}_T(X, \omega_0)$, such that, for any decreasing sequence $(\varphi_j)_j$ of elements in $\mathcal{K}_T(X, \omega_0)$ converging to $\varphi$ (which exists by (17)), we have $\lim_{j \to \infty} \text{MA}_v(\varphi_j) = \text{MA}_v(\varphi)$.

Proof. The result is established in [13] for $\omega_0 = c_1(L)$ a Kähler Hodge class on a projective variety $X$. The method of Han–Li [16] Prop. 2.2, which uses the semi-simple principle fibration construction and polynomial approximations, extends to the case of arbitrary Kähler class $\alpha = [\omega_0]$. Below we give details of this construction, for Reader’s convenience.

Let $\varphi \in \mathcal{E}_T(X, \omega_0)$. Following the proof of [16] Prop. 2.2, we first define $\text{MA}_p(\varphi)$ for a positive polynomial weight of the form $p(\mu) := \prod_{i=1}^{k}(p_{a_i} + c_a)_{n_a}$, and extend the definition linearly on $p$ for finite sums of such polynomials. We can then use the Bernstein approximation theorem of an arbitrary positive $v$ with polynomials of the above form in order to obtain $\text{MA}_v(\varphi)$.

We start with a semi-simple principal $(X, \mathbb{T})$-fibration $Y$ (see Section 5) with corresponding polynomial weight $p(\mu) := \prod_{i=1}^{k}(p_{a_i} + c_a)_{n_a}$ (see (28)). As the choice of the base $B = B_1 \times \cdots \times B_k$ does not matter, we can simply take (as in [16]) $B$ to be the product of projective spaces $(B_a, \omega_a) = (\mathbb{P}^{n_a}, \omega_a)$ endowed with Fubini–Study metrics of scalar curvatures $2n_a(n_a + 1)$, and $P$ be the principal $U(1)$-bundle over $B$, obtained from the tensor products $P_i$ of (the pullbacks to $B$ of) the natural principle $U(1)$-bundles of degrees $p_{a_i}$ over $\mathbb{P}^{n_i}$ (see Remark 5.5).

Using [17] Thm. 1, there is a decreasing sequence $\varphi_j \in \text{PSH}_T(X, \omega_0) \cap C^\infty(X) = \mathcal{K}_T(X, \omega_0)$ converging towards $\varphi$. By Lemma 5.5 we have $\varphi_j \in \mathcal{K}_T(Y, \tilde{\omega}_0)$ and, by (29), for any $T_X$-invariant continuous function $f$ on $X$ we have

$$\int_X f(\mu_{\varphi_j})(\omega_0 + d_X d_X^* \varphi_j)^{|m|} = \frac{1}{\text{Vol}(B, \omega_B)} \int_Y f(\tilde{\omega}_0 + d_Y d_Y^* \varphi_j)^{|m+n|}.$$

Passing to the limit in both sides of the above equation, we can define $\text{MA}_p^X(\varphi)$ on $T$-invariant continuous functions $f$ by

$$\text{MA}_p^X(\varphi) := \lim_{j \to \infty} \frac{1}{\text{Vol}(B, \omega_B)} \int_Y f(\tilde{\omega}_0 + d_Y d_Y^* \varphi_j)^{|m+n|}$$

Notice that by [17] Thm. 1.9] the limit exists and is well-defined on $Y$ (independent of the chosen sequence).

For a continuous function $f$ on $X$ which is not necessarily $T_X$-invariant, we define

$$\int_X f \text{MA}_p^X(\varphi) := \int_X f^\mathbb{T} \text{MA}_p^X(\varphi)$$

where $f^\mathbb{T}$ is the $T_X$-invariant function given by average of $f$ over the $T_X$-action. It follows that $\text{MA}_p^X(\varphi)$ is a well-defined Radon measure by Riesz representation theorem.

We can extend the above definition by linearity in $p$ on polynomials which are linear combinations with positive coefficients of polynomials of the above special form: thus, for $\varphi \in \text{PSH}_T(X, \omega_0)$ and for two polynomials $p, q$ on $\Delta$, we will have

$$\left| \int_X f \text{MA}_p^X(\varphi) - \int_X f \text{MA}_q^X(\varphi) \right| \leq \| p - q \|_{C^0(\Delta)} \int_X |f| \text{MA}_p^X(\varphi)$$

for any $f \in C^0(X)$.

For an arbitrary smooth positive function $v$ on $\Delta$, we can approximate $v$ in $C^0(\Delta)$ by polynomials $p_i$ as above (e.g. by using Bernstein’s Approximation Theorem) and thus, for any continuous function $f$, the limit

$$\lim_{i \to \infty} \lim_{j \to \infty} \int_X f \text{MA}_p^X(\varphi_j)$$

exists independently of the chosen approximation. We then define

$$\int_X f \text{MA}_v^X(\varphi) := \lim_{i \to \infty} \lim_{j \to \infty} \int_X f \text{MA}_{p_i}^X(\varphi_j).$$
By the Riesz representation theorem, \( MA^X_v(\varphi) \) is a well-defined Radon measure.

Remark 6.14. Notice that for any \( \varphi \in \mathcal{E}_T(X, \omega_0) \), the measure \( MA_v(\varphi) \) is absolutely continuous with respect to \( MA(\varphi) \) since \( v \) is bounded on \( \Delta \). In particular, for any positive weight \( v \), we have that

\[
\mathcal{E}^1_T(X, \omega_0) = \left\{ \varphi \in \mathcal{E}_T(X, \omega_0) \mid \int_X |\varphi| MA_v(\varphi) < \infty \right\}.
\]

Lemma 6.15. Let \( v \) be a positive weight function and \( \varphi_j, \varphi \in \mathcal{E}^1_T(X, \omega_0) \) such that \( d_1(\varphi_j, \varphi) \to 0 \). Then, \( MA_v(\varphi_j) \to MA_v(\varphi) \) weakly.

Proof. Let \( v(\mu) \) be a polynomial of the form \( p(\mu) := \prod_{a=1}^k(p_{a,\mu} + c_a) \), \( \varphi_j \in K_T(X, \omega_0) \), and \( f \) any continuous \( T \)-invariant function on \( X \). We then have by the construction in Section 5

\[
\int_X f p(\mu_{\varphi_j})(\omega_0 + d_X d_X^\mu_{\varphi_j})[m] = \frac{1}{\text{Vol}(B, \omega_B)} \int_Y f(\omega_0 + d_Y d_Y^\varphi_{\varphi_j})[m+n].
\]

It follows that for each \( \varphi_j \in \mathcal{E}^1_T(X, \omega_0) \) (using an approximation with a decreasing sequence of smooth potentials \([17]\)), we have

\[
\int_X f MA^X_v(\varphi_j) = \frac{1}{\text{Vol}(B, \omega_B)} \int_Y f MA^Y(\varphi_j).
\]

By \([27], \text{Thm. 5}\), \( MA^Y(\varphi_j) \to MA^Y(\varphi) \) weakly as \( j \to \infty \). It follows that

\[
\lim_{j \to \infty} \int_X f MA^X_v(\varphi_j) = \frac{1}{\text{Vol}(B, \omega_B)} \int_Y f MA^Y(\varphi) = \int_X f MA^X_v(\varphi).
\]

Using \((56)\), we conclude that \( MA^X_v(\varphi_j) \to MA^X_v(\varphi) \) weakly as \( j \to \infty \).

For an arbitrary weight function \( v \in C^0(\Delta) \), we take a sequence of polynomials \( p_i \) of the above form converging to \( v \) in \( C^0(\Delta) \). For any continuous function \( f \) on \( X \), using \((57)\), we have

\[
\left| \int_X f MA_v(\varphi_j) - \int_X f MA_v(\varphi) \right| \leq \int_X f MA_v(\varphi_j) - \int_X f MA_v(\varphi) + \int_X f MA_v(\varphi) - \int_X f MA_v(\varphi) + \int_X f MA_v(\varphi) - \int_X f MA_v(\varphi) \|
\]

\[
+ \left| \int_X f MA_v(\varphi_j) - \int_X f MA_v(\varphi) \right| + \| p_i - v \|_{C^0(\Delta)} \left( \int_X |f| MA(\varphi_j) + \int_X |f| MA(\varphi) \right).
\]

Letting \( j \to \infty \), we get

\[
\lim_{j \to \infty} \left| \int_X f MA_v(\varphi_j) - \int_X f MA_v(\varphi) \right| \leq 2 \| p_i - v \|_{C^0(\Delta)} \int_X |f| MA(\varphi)
\]

where we used the existence of the weak limits \( MA_{p_i}(\varphi_j) \to MA_{p_i}(\varphi) \) and \( MA(\varphi_j) \to MA(\varphi) \) as \( j \to \infty \) (by \([27], \text{Thm. 5}\) ). Taking the limit \( i \to \infty \) in the above inequality, we obtain

\[
\lim_{j \to \infty} \left| \int_X f MA_v(\varphi_j) - \int_X f MA_v(\varphi) \right| = 0.
\]

It follows that \( MA_v(\varphi_j) \to MA_v(\varphi) \) weakly as \( j \to \infty \). \( \Box \)

For a finite measure \( \chi \) on \( X \) we define the entropy of \( \chi \) with respect to \( \omega^m \) by

\[
\text{Ent}(\omega^m, \chi) := \int_X \log \left( \frac{\chi}{\omega^m} \right) \chi.
\]

In the following lemma we show that the elements of \( \mathcal{E}^1_T(X, \omega_0) \) can be approximated in the \( d_1 \) distance by smooth potentials with converging entropy of the corresponding weighted Monge–Ampère measures. This is the weighted analogue of \([15], \text{Lemma 3.1}\) .
Lemma 6.16. If \( v > 0, E_T^1(X, \omega_0) \ni \varphi \mapsto \text{Ent}(\omega_0^m, MA_v(\varphi)) \) is \( d_1 \) lsc. Furthermore, for any \( \varphi \in E_T^1(X, \omega_0) \) there exist a sequence of smooth potentials \( \varphi_j \in K_T(X, \omega_0) \) such that \( d_1(\varphi_j, \varphi) \to 0 \) and \( \text{Ent}(\omega_0^m, MA_v(\varphi_j)) \to \text{Ent}(\omega_0^m, MA_v(\varphi)) \) as \( j \to \infty \).

Proof. The proof follows closely the arguments of [15] Lemma 3.1. By Lemma 6.15 and the fact that the entropy \( \chi \mapsto \text{Ent}(\omega_0^m, \chi) \) is lsc on the space of finite measures, with respect to the weak convergence of measures, (cf. [11] Prop. 3.1), it follows that the entropy \( \varphi \mapsto \text{Ent}(\omega_0^m, MA_v(\varphi)) \) is \( d_1 \) lsc. Let \( \varphi \in E_T^1(X, \omega_0) \). If \( \text{Ent}(\omega_0^m, MA_v(\varphi)) = \infty \) then any sequence \( \varphi_j \in K_T(X, \omega_0) \) such that \( d_1(\varphi_j, \varphi) \to 0 \) satisfies \( \text{Ent}(\omega_0^m, MA_v(\varphi_j)) \to \infty \) as \( j \to \infty \). We suppose that \( \text{Ent}(\omega_0^m, MA_v(\varphi)) < \infty \) and we put \( g := \frac{MA_v(\varphi)}{\omega_0^m} \geq 0 \) the density function of the measure \( MA_v(\varphi) \).

From the proof of [15] Lemma 3.1, there exist a sequence of positive functions \( g_j \in C_0^\infty(X) \) such that \( ||g - g_j||_{L^1} \to 0 \) and

\[
\int_X g_j \log g_j \omega_0^m \to \text{Ent}(\omega_0^m, MA_v(\varphi)).
\]

Using [46] Prop. 3.7, we can find a smooth potential \( \varphi_j \in K_T(X, \omega_0) \) (which is unique up to adding a constant) such that \( MA_v(\varphi_j) = \left( \int_X e^{(f_0)\omega_0^m} \right) g_j \omega_0^m \). By [46] Lemma 2.16, up to a passing to a subsequence of \( \varphi_j \), there exists a \( \psi \in E^1_T(X, \omega_0) \) such that \( d_1(\psi, \varphi_j) \to 0 \). Lemma 6.15 together with \( ||g - g_j||_{L^1} \to 0 \) gives \( MA_v(\psi) = \lim_{j \to \infty} MA_v(\varphi_j) = MA_v(\varphi) \). It follows that \( \varphi = \psi \) (up to a constant) by [13] Thm. 2.18. Thus, \( d_1(\varphi, \varphi_j) \to 0 \) and \( \text{Ent}(\omega_0^m, MA_v(\varphi_j)) \to \text{Ent}(\omega_0^m, MA_v(\varphi)) \) as \( j \to \infty \). \( \square \)

Now we are in position to prove Theorem 6.1.

Proof of Theorem 6.1. By Lemmas 6.10 and 6.12 the functionals \( I_w \) and \( I_v \) extend as continuous functionals on \( E_T^1(X, \omega_0) \). On the other hand, the entropy \( \varphi \mapsto \text{Ent}(\omega_0^m, MA_v(\varphi)) \) is \( d_1 \) lsc by Lemma 6.16. Thus, the weighted Chen–Tian decomposition (50) gives rise to an extension of the \((v, w)\)-Mabuchi energy to a \( d_1 \) lsc functional \( M_{v,w} : E_T^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\} \). Notice that (using the continuity of \( I_w \) and \( I_v \)) the restriction of \( M_{v,w} : E_T^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\} \) on the subspace \( K_{T,1}^1(X, \omega_0) \) is equal to the weighted \((v, w)\)-Mabuchi energy on that space defined in [57] Cor. 3]. By Lemma 6.16 for \( \varphi \in E_T^1(X, \omega_0) \) we can find a sequence \( \varphi_j \in K_T(X, \omega_0) \), such that \( d_1(\varphi_j, \varphi) \to 0 \) and

\[
\lim_{j \to \infty} M_{v,w}(\varphi_j) = M_{v,w}(\varphi).
\]

It follows that the extension \( M_{v,w} : E_T^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\} \) using (50) is the largest \( d_1 \) lsc extension of \( M_{v,w} : K_T(X, \omega_0) \to \mathbb{R} \).

We now show that \( t \mapsto M_{v,w}(\varphi_t), t \in [0, 1] \) is convex and continuous along the finite energy geodesics \( \varphi_t \in E_T(X, \omega_0) \). We follow closely the arguments of [15] Thm. 4.7. Let \( \varphi_t \in E_T^1(X, \omega_0), t \in [0, 1] \) be a finite energy geodesic. Suppose that \( t_0, t_1 \in [0, 1] \) with \( t_0 < t_1 \). Using Lemma 6.16 we can find sequences \( \varphi_{t_0}^j, \varphi_{t_1}^j \in K_T(X, \omega_0) \), such that \( d_1(\varphi_{t_0}^j, \varphi_{t_0}) \to 0 \) and \( d_1(\varphi_{t_1}^j, \varphi_{t_1}) \to 0 \) and

\[
\lim_{j \to \infty} M_{v,w}(\varphi_{t_0}^j) = M_{v,w}(\varphi_{t_0}), \quad \lim_{j \to \infty} M_{v,w}(\varphi_{t_1}^j) = M_{v,w}(\varphi_{t_1}).
\]

Let \( t \mapsto \varphi_t \in K_{T,1}^1(X, \omega_0), t \in [t_0, t_1] \) the \( C^{1,1} \)-weak geodesic segment connecting \( \varphi_{t_0}^j, \varphi_{t_1}^j \). By [57] Thm. 5], the function \( [t_0, t_1] \ni t \mapsto M_{v,w}(\varphi_t) \) is convex. Since \( M_{v,w} : E_T^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\} \) is \( d_1 \) lsc we have

\[
M_{v,w}(\varphi_t) \leq \liminf_{j \to \infty} M_{v,w}(\varphi_{t_0}^j) + \frac{t_1 - t}{t_1 - t_0} \lim_{j \to \infty} M_{v,w}(\varphi_{t_1}^j) - \frac{t - t_0}{t_1 - t_0} \lim_{j \to \infty} M_{v,w}(\varphi_{t_0}) + \frac{t_1 - t}{t_1 - t_0} \lim_{j \to \infty} M_{v,w}(\varphi_{t_1})
\]

where the second inequality uses the convexity of \( t \mapsto M_{v,w}(\varphi_t) \). Thus, \( t \mapsto M_{v,w}(\varphi_t) \) is convex and continuous up to the boundary of \( [t_0, t_1] \) since it is \( d_1 \) lsc.
It remains to show that $M_{v,w} : E_+^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\}$ is linear and continuous in $v,w$. For smooth potentials $\varphi \in K_T(X, \omega_0)$, we have

$$\operatorname{Ent}(\omega_0^{|m|}, MA_v(\varphi)) = \int_X v(\mu_0)v(\mu_0)\omega_0^{|m|} = \int_X \log \frac{MA(\varphi)}{\omega_0^{|m|}} MA_v(\varphi),$$

which is manifestly linear in $v$. For $\varphi \in E_+^1(X, \omega_0)$, the above expression is still linear in $v$ by Proposition 6.13. Substituting back in (50), and using Lemma 6.10 and 6.12, it follows that $M_{v,w} : E_+^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\}$ is linear in $v,w$. From these two lemmas, we know that $\mathbf{I}^0_v : E_+^1(X, \omega_0) \to \mathbb{R}$ and $\mathbf{I}^0_w : E_+^1(X, \omega_0) \to \mathbb{R}$ are uniformly continuous in $v,w$. For the remaining entropy part, we notice that if $\varphi \in E_+^1(X, \omega_0)$, $v,v' \in C^\infty(\Delta)$ and $f \in C^0(\Delta)$, then

$$\left| \int_X fMA_v(\varphi) - \int_X fMA_{v'}(\varphi) \right| \leq \|v - v'\|_{C^0(\Delta)} \int_X |f| MA_v(\varphi),$$

which can be obtained again by approximating $\varphi$ with a monotone sequence of smooth relative potentials and using Proposition 6.13. It follows that $C^\infty(\Delta) \times E_+^1(X, \omega_0) \ni (v, \varphi) \mapsto MA_v(\varphi)$ is uniformly continuous with respect to $v$ for the weak topology on the space of measures. Since the entropy $\chi \mapsto \operatorname{Ent}(\omega_0^{|m|}, \chi)$ is lsc on the space of finite measures with respect to the weak convergence of measures [11, Prop. 3.1], the term $\operatorname{Ent}(\omega_0^{|m|}, MA_v(\varphi))$ is lsc with respect to $v$. The linearity with respect to $v$ in the RHS of (58) shows that $\operatorname{Ent}(\omega_0^{|m|}, MA_v(\varphi))$ is in fact continuous with respect to $v$.

We derive the following weighted version of the key compactness result from [12, 14]:

**Theorem 6.17.** Any sequence $\varphi_j \in E_+^1(X, \omega_0)$ such that

$$d_1(0, \varphi_j) \leq C, \quad M_{v,w}(\varphi_j) \leq C$$

admits a $d_1$-convergent subsequence.

**Proof.** From the formula (50) and Lemmas 6.9 and 6.12, we see that $\operatorname{Ent}(\omega_0^{|m|}, MA_v(\varphi))$ is uniformly bounded under the hypotheses in the Corollary. We conclude using [10, Lemma 2.16].

**7. Regularity of the weak minimizers of the weighted Mabuchi energy**

In this section, we establish the regularity of the weak minimizers of $M_{v,w}$.

**Theorem 7.1.** Suppose $T \subset \text{Aut}_c(X)$ is a maximal torus and $(X,\omega, T)$ admits a $(v,w)$-cscK metric $\omega$ with $w = \rho_{v,w}w_0$, where $v, w_0 > 0$ are two positive smooth weight functions on $\Delta$. If $\psi \in E_+^1(X, \omega_0)$ is a minimizer of the extended $(v,w)$-Mabuchi energy $M_{v,w} : E_+^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\}$, then $\psi \in K_T(X, \omega_0)$ is a smooth potential.

The proof of this result, which is an adaptation of the arguments in [13], will occupy the remainder of the section.

**Definition 7.2.** Let $v(\mu) > 0$, $w(\mu)$ be smooth weight functions on $\Delta$ and $\rho > 0$ a $T$-invariant Kähler form on $X$. We let

$$M_{v,w} := \{ \psi \in E_+^1(X, \omega_0) \cap \Gamma^{-1}(0) \left| M_{v,w}(\psi) = \inf_{\varphi \in E_+^1} M_{v,w}(\varphi) \right. \}$$

and $M_{v,w}^\rho := M_{v,w} + \mathcal{I}^\rho$, where $\mathcal{I}^\rho$ is introduced via Lemma 6.12 and $\rho = 1$.

By [23, Lemma 5.2] and Theorem 6.1, the set $M_{v,w}$ (when non-empty) is totally geodesic with respect to the finite energy geodesics of $E_+^1(X, \omega_0)$. Furthermore, if there exists a $\psi_\rho \in M_{v,w}$ such that $\mathcal{I}^\rho(\psi_\rho) = \inf_{\psi \in M_{v,w}} \mathcal{I}^\rho(\psi)$, then $\psi_\rho$ is unique by the strict convexity of $\mathcal{I}^\rho$ established in [15, Prop. 4.5]. Furthermore, by Theorem 6.1 the functional $M_{v,w}^\rho : E_+^1(X, \omega_0) \to \mathbb{R} \cup \{\infty\}$ will be also strictly convex along finite energy geodesics, showing the uniqueness of an element $\psi \in E_+^1(X, \omega_0) \cap \Gamma^{-1}(0)$ such that $M_{v,w}^\rho(\psi) = \inf_{\varphi \in E_+^1} M_{v,w}^\rho(\varphi)$ (assuming that such minimizer $\psi$ exists).

We then have the following weighted version of the continuity method of [13, Prop. 3.1]:
Proposition 7.5. Let $v, w$ be smooth weight functions on $\Delta$. Suppose that $M_{v,w}$ is nonempty, and $\varphi \in K_T(X, \omega_0) \cap \Gamma^{-1}(0)$. For any $\lambda > 0$, there exists a unique minimizer $\psi_\lambda \in E^I_T(X, \omega_0) \cap \Gamma^{-1}(0)$ of $M_{v,w}^{\lambda \omega_0} := M_{v,w} + \lambda \omega_0$. The curve $[0, \infty) \ni \lambda \mapsto \psi_\lambda \in E^I_T(X, \omega_0) \cap \Gamma^{-1}(0)$ is $d_1$-continuous, $d_1$-bounded and $\psi_0 := \lim_{\lambda \to 0} \psi_\lambda$ is the unique minimizer of $I^{\omega_0}$ on $M_{v,w}$. Furthermore, for any $\psi \in M_{v,w}$ and $\lambda > 0$, we have

\begin{equation}
I(\varphi, \psi_\lambda) \leq m(m + 1)I(\varphi, \psi),
\end{equation}

where $I(\varphi, \psi) := \int_X (\varphi - \psi) \left( \omega^{mn}_\psi - \omega^{mn}_\varphi \right)$.

Proof. The proof follows by a straightforward adaptation of the arguments in [15, Prop. 3.1].

We next need a weighted analogue of [15, Lemma 3.3].

Lemma 7.4. Let $v, w$ be smooth weight functions on $\Delta$, and $\rho > 0$ a smooth $T$-invariant Kähler form on $X$. Let $\varphi_0 \in K_T(X, \omega_0)$, $\varphi_1 \in E^I_T(X, \omega_0)$, and $[0, 1] \ni t \mapsto \varphi_t \in E^I_T(X, \omega_0)$ be a finite energy geodesic connecting $\varphi_0$ and $\varphi_1$. Then,

\[ \lim_{t \to 0^+} \frac{M_{v,w}^t(\varphi_t) - M_{v,w}(\varphi_0)}{t} \geq \int_X (w(\mu_{\varphi_0}) - \text{Scal}_v(\varphi_0))\varphi_0^{m\mu_0} + \int_X \varphi_0^0 \varphi_0^0 w^{m-1} \]

where $M_{v,w}^t(\varphi_t) = M_{v,w} + t^\rho$.

Proof. Using Theorem 6.1 and the fact that $I^\rho$ is $d_1$-continuous (see [15] or Lemma 6.12), for any $t \in [0, 1]$ there exists a sequence $(\varphi^k_t)_{k \in \mathbb{N}} \subseteq K_T(X, \omega_0)$ such that $\lim_{k \to \infty} d_1(\varphi^k_t, \varphi_t) = 0$ and $M_{v,w}^k(\varphi^k_t) \to M_{v,w}(\varphi_t)$. We let $[0, t] \ni s \mapsto \varphi^k_s$ be the weak $C^{1,1}$-geodesic joining $\varphi^k_0 = \varphi_0$ with $\varphi^k_t$. By the proof of [57, Cor. 1], we get

\[ \lim_{t \to 0^+} \frac{M_{v,w}^t(\varphi^k_t) - M_{v,w}(\varphi_0)}{t} \geq \int_X (w(\mu_{\varphi_0}) - \text{Scal}_v(\varphi_0))\varphi_0^{m\mu_0} + \int_X \varphi_0 \varphi_0 w^{m-1} \]

According to [15, Lemma 3.4], we can use the dominated convergence theorem on the RHS of the above inequality to conclude.

The last step is to establish a weighted version of [15, Prop. 3.2].

Proposition 7.5. Suppose $\mathbb{T} \subseteq \text{Aut}_v(X)$ is a maximal torus, and let $v(\mu), w(\mu) > 0$, $w = \rho_{v,w}(\mu)$. Suppose that $\varphi^* \in K_T(X, \omega_0) \cap \Gamma^{-1}(0)$ is a $(v, w)$-cscK potential. Then, for any fixed Kähler form $\omega_\varphi$, $\varphi \in K_T(X, \omega_0)$, there exists a $\sigma \in G := \mathbb{T}_C$ such that

\[ \inf_{\psi \in M_{v,w}} I^{\omega_\varphi}(\psi) = I^{\omega_\varphi}(\sigma[\varphi^*]). \]

Proof. As $G$ is reductive, there exists a unique $\sigma \in G$ such that

\begin{equation}
I^{\omega_\varphi}(\sigma[\varphi^*]) = \inf_{\tau \in G} I^{\omega_\varphi}(\tau[\varphi^*]),
\end{equation}

(see e.g. [29, Sect. 6] or [57, Lemma 11]) where, we recall, the $G$ action on potentials is introduced via the slice $\Gamma^{-1}(0)$. Let $\varphi_0 := \sigma[\varphi^*] \in K_T(X, \omega_0) \cap \Gamma^{-1}(0)$ and $\psi \in M_{v,w}$ be the unique minimizer of $I^{\omega_\varphi}$. We want to show that $\varphi_0 = \psi_0$.

For $\lambda > 0$, let $\psi_\lambda$ be the unique minimizer of $M_{v,w}^{\lambda \omega_0} := M_{v,w} + \lambda \omega_0$ on $E^I_T(X, \omega_0) \cap \Gamma^{-1}(0)$, given by Proposition 7.3. By this proposition, we know that $\lim_{\lambda \to 0} d_1(\psi_\lambda, \psi_0) = 0$. We denote respectively by $V_\lambda$ and $W$ the differentials of $M_{v,w}^{\lambda \omega_0}$ and $I^{\omega_\varphi}$, viewed as 1-forms on the Fréchet space $K(X, \omega_0)$. We thus have $\forall \psi \in K_T(X, \omega_0)$, $\forall \tilde{\psi} \in \mathcal{C}^{\infty}(X)$

\begin{equation}
(V_0)_\psi(\tilde{\psi}) = -\int_X (\text{Scal}_v(\omega_\psi) - w(\mu_\psi))\tilde{\psi}\omega^m_\psi,
\end{equation}

\begin{equation}
W_\psi(\tilde{\psi}) = \int_X \psi \omega^\rho_\varphi \wedge \omega^{m-1}_\psi;
\end{equation}

\begin{equation}
(V_\lambda)_\psi(\tilde{\psi}) = (V_0)_\psi(\tilde{\psi}) + \lambda W_\psi(\tilde{\psi}).
\end{equation}
Recall that the Mabuchi connection $\mathcal{D}$ on the Fréchet space $\mathcal{K}_T(X, \omega_0)$ is introduced by

$$
(\mathcal{D}_\varphi \psi_t)_{\varphi_t} := \psi_t - (d\psi_t, d\varphi_t)_{\omega_{\varphi_t}},
$$

where $\varphi_t$ and $\psi_t$ are smooth paths in $\mathcal{K}_T(X, \omega_0)$. Using [56] Lemma B.1, we compute the covariant derivative of $V_0$ with respect to the Mabuchi connection to be

$$
\left( (\mathcal{D}_\varphi V_0)(\psi_1) \right)_{\psi_1} = \int_X [2v(\mu_\psi) \left( (\nabla^{\omega_\psi} d\psi_1)^-, (\nabla^{\omega_\psi} d\psi_2)^- \right)_{\omega_{\psi}} + (\text{Scal}_v(\omega_{\psi}) - w(\mu_\psi)))(d\psi_1, d\psi_2)_{\omega_{\psi}}] \omega_{\psi}^m],
$$

where $(\nabla^{\omega_\psi} d\psi)^-$ denotes the $(2, 0)+(0, 2)$ part of the Hessian of $\psi$ with respect to the Levi-Civita connection $\nabla^{\omega_\psi}$ of $\omega_\psi$. Taking $\psi = \varphi_0$ to be the $(v, u)$-cscK potential, we get

$$
\left( (\mathcal{D}_\varphi V_0)(\psi_1) \right)_{\varphi_0} = 2 \int_X [(\nabla^{\omega_{\varphi_0}} d\psi_1)^-, (\nabla^{\omega_{\varphi_0}} d\psi_2)^-]_{\omega_{\varphi_0}} v(\mu_{\varphi_0}) \omega_{\varphi_0}^m] = 2 \int_X \mathcal{L}_{\omega_{\varphi_0}, v}(\psi_1) \psi_2 \omega_{\varphi_0}^m],
$$

where the operator $\mathbb{L}_{\omega_{\varphi_0}, v}(\psi) := \delta_{\omega_\psi} \delta_{\omega_{\varphi_0}} (v(\mu_\psi)(\nabla^{\omega_\psi} d\psi)^-)$ is a 4th order elliptic self-adjoint operator on $(X, \omega_\psi)$, with kernel given by the space of Killing potentials in $C^\infty_T(X)$, see Appendix A.

As $\varphi_0$ is a $(v, u)$-cscK potential which satisfies [61], we have by [67] Lemma 10 that $W_{\varphi_0}(\psi) = 0$ for any $T$-invariant Killing potential $\psi$ with respect to $\omega_{\varphi_0}$. It follows that we can solve the linear equation (for a function $\psi \in C^\infty_T(X)$)

$$
\mathbb{L}_{\omega_{\varphi_0}, v}(\psi) = \frac{\omega_{\varphi} \wedge \omega_{\varphi_0}^{m-1}}{\omega_{\varphi_0}^m],
$$

as the RHS is $L^2$-orthogonal (with respect to the measure $\omega_{\varphi_0}^m]$) to the kernel of $\mathcal{L}_{\omega_{\varphi_0}, v}$. Equivalently, there exits a $\psi_0 \in C^\infty_T(X)$, such that we have equality of 1-forms on $\mathcal{K}_T(X, \omega_0)$:

$$
(\mathcal{D}_\varphi V_0)_{\varphi_0} = -W_{\varphi_0}.
$$

Let $\lambda \to \dot{\lambda}_\lambda \in C^\infty_T(X)$ be a smooth curve in the tangent space to $(\varphi_0 + \lambda \dot{\psi}_0) \in \mathcal{K}_T(X, \omega_0)$, defined for $\lambda > 0$ small enough. We compute

$$
\frac{d}{d\lambda |\lambda=0} \left( \mathcal{V}_\lambda \right)_{\varphi_0 + \lambda \dot{\psi}_0} (\dot{\lambda}_\lambda) = W_{\varphi_0} (\dot{\phi}_0) + \left( (\mathcal{D}_{\phi_0} V_0)(\dot{\phi}_0) \right)_{\varphi_0} + (V_0)_{\varphi_0} \frac{d}{d\lambda |\lambda=0} \dot{\lambda}_\lambda = 0,
$$

where we have used (63) and that $(V_0)_{\varphi_0} = 0$ since $\varphi_0$ is a $(v, w)$-cscK potential, see (62). On the other hand, letting

$$
f_\lambda := -\text{Scal}_v(\omega_{\varphi_0} + \lambda \dot{\psi}_0) + w(\mu_{\varphi_0} + \lambda \dot{\psi}_0) + \left( \omega_{\varphi_0} + \lambda \dot{\psi}_0, \omega_{\varphi_0} \right)_{\omega_{\varphi_0}}
$$

it follows from (62) that for any $\dot{\phi} \in C^\infty_T(X)

$$
\left( \mathcal{V}_\lambda \right)_{\varphi_0 + \lambda \dot{\psi}_0} (\dot{\phi}) = \int_X \dot{\phi} f_\lambda \omega_{\varphi_0}^m] + \dot{\psi}_0.
$$

Thus (64) implies that $f_\lambda = O(\lambda^2)$ and

$$
|\left( \mathcal{V}_\lambda \right)_{\varphi_0 + \lambda \dot{\psi}_0} (\dot{\phi})| \leq C \lambda^2 \sup_X \dot{\phi}.
$$

Let $\psi_\lambda(t) \in \mathcal{E}_T^1(X, \omega_0)$ be a finite energy geodesic connecting $\psi_\lambda(0) := \psi_\lambda \in \mathcal{E}_T^1(X, \omega_0)$ with $\psi_\lambda(1) := \varphi_0 + \lambda \dot{\psi}_0 \in \mathcal{K}_T(X, \omega_0)$ for $\lambda > 0$ small enough. By Lemma 7.4, we get

$$
\frac{d}{dt} |t=1| \mathcal{M}_{v,w} (\psi_\lambda(t)) \leq \int_X \dot{\psi}_\lambda(1) f_\lambda \omega_{\varphi_0}^m] + \dot{\psi}_0.
By Proposition 7.3, \( d_1(0, \psi_\lambda(0)) \) is uniformly bounded. As \( \psi_\lambda(1) := \varphi_0 + \lambda \psi_0 \in K_T(\mathcal{X}, \omega_0) \), it follows that \( d_1(0, \psi_\lambda(1)) \) is uniformly bounded for \( \lambda \) small enough. We thus have that both \( d_1(0, \psi_\lambda(0)) \) and \( d_1(0, \psi_\lambda(1)) \) are uniformly bounded and by [15] Lemma 3.4(ii) we get

\[
\int_X |\psi_\lambda(1)|^{m_\omega} d\omega = d_1(0, \psi_\lambda(0)) \leq d_1(0, \psi_\lambda(0)) + d_1(0, \psi_\lambda(1)) \leq C.
\]

From \( f_\lambda = O(\lambda^2) \) we obtain

\[
\frac{d}{dt}_{|t=1^-} M_{v,w}^\omega(\psi_\lambda(t)) \leq O(\lambda^2).
\]

As the unique minimizer of the strictly convex functional \( M_{v,w}^\omega \) on \( \mathcal{E}_T^1(\mathcal{X}, \omega_0) \cap \Gamma^{-1}(0) \) is \( \psi_\lambda(0) = \psi_\lambda \),

\[
\frac{d}{dt}_{|t=1^-} M_{v,w}^\omega(\psi_\lambda(t)) \geq \frac{d}{dt}_{|t=0^+} M_{v,w}^\omega(\psi_\lambda(t)) \geq 0.
\]

Using that the functions \( t \mapsto \Gamma^\omega(\psi_\lambda(t)) \) and \( t \mapsto M_{v,w}(\psi_\lambda(t)) \) are both convex (this follows from [15] Prop. 4.5 and Theorem 6.1), we have

\[
0 \leq t(1-t) \frac{\Gamma^\omega(\psi_\lambda(1)) - \Gamma^\omega(\psi_\lambda(0))}{1-t} - t(1-t) \frac{\Gamma^\omega(\psi_\lambda(0)) - \Gamma^\omega(\psi_\lambda(t))}{1-t} \leq t(1-t)O(\lambda).
\]

Letting \( \lambda \to 0 \), and using the endpoint stability of the finite energy geodesic segments (see [15] Prop. 4.3), together with the \( d_1 \)-continuity of \( \Gamma^\omega \) established in [15] Prop. 4.4, it follows that \( t \mapsto \Gamma^\omega(\psi(t)) \) is linear along the finite energy geodesic \( \psi(t) = \lim_{\lambda \to 0^+} \psi_\lambda(t) \) connecting \( \psi_0(0) = \psi_0 \) and \( \psi_0(1) = \varphi_0 \). The strict convexity of \( \Gamma^\omega \) along finite energy geodesics ([15] Prop. 4.5) then yields \( \psi_0 = \varphi_0 = \sigma[\varphi^*] \).

Now, we are in position to prove Theorem 7.1 by the arguments in [15] Thm. 1.4.

**Proof of Theorem 7.1.** Without loss of generality, we can assume that the \((v,w)\)-extremal metric \( w^* = \omega_0 \) is the initial metric, and we suppose \( \psi_0 \in \mathcal{E}_T^1(\mathcal{X}, \omega_0) \cap \Gamma^{-1}(0) \) is a weak minimizer of \( M_{v,w} : \mathcal{E}_T^1(\mathcal{X}, \omega_0) \to \mathbb{R} \cup \{\infty\} \). We want to show that \( \psi_0 = \sigma[0] \) for some \( \sigma \in \mathbb{G} = \mathbb{T}^\mathbb{C} \). It is well-known (see [28] or Lemma 6.11) that there exists a sequence \( \varphi_j \in K_T(\mathcal{X}, \omega_0) \cap \Gamma^{-1}(0) \) such that \( d_1(\varphi_j, \psi_0) \to 0 \). We set \( \rho_j = \omega_0 + dd^c \varphi_j \) which is a \( \mathbb{T} \)-invariant Kähler form.

Since \( \omega_0 \) is \((v,w)\)-extremal metric, \( M_{v,w} \) is non-empty. By Proposition 7.3, the functional \( M_{v,w}^{\lambda \rho_j} = M_{v,w} + \lambda \rho_j \) admits a unique minimizer \( \psi_{j,\lambda} \in \mathcal{E}_T^1(\mathcal{X}, \omega_0) \cap \Gamma^{-1}(0) \), such that

\[
I(\varphi_j, \psi_{j,\lambda}) \leq m(m+1)I(\varphi_j, \psi_0).
\]

By the quasi-triangle identity [15] (2.16), we get

\[
(65)\quad I(\psi_0, \psi_{j,\lambda}) \leq C \left( I(\psi_0, \varphi_j) + I(\varphi_j, \psi_{j,\lambda}) \right) \leq C(m^2 + m + 1)I(\varphi_j, \psi_0),
\]

where \( C > 0 \) is a uniform constant depending only on \( m \).

Let \( j > 0 \) be fixed. According to Proposition 7.3, \( \psi_{j,0} := \lim_{\lambda \to 0} \psi_{j,\lambda} \) is the unique minimizer of \( \Gamma^{\rho_j} \) on \( M_{v,w} \) whereas Proposition 7.5 yields that there exists a \( \sigma_j \in \mathbb{G} \) such that \( \psi_{j,0} = \sigma_j[0] \). Letting \( \lambda \to 0^+ \) in (65) (and using the \( d_1 \)-continuity of \( I \), see e.g. [14] or Lemma 6.10), we have

\[
I(\psi_0, \sigma_j[0]) \leq C(m^2 + m + 1)I(\varphi_j, \psi_0).
\]

Taking \( j \to \infty \) (and using \( d_1(\varphi_j, \psi_0) \to 0 \)), we get \( I(\psi_0, \sigma_j[0]) \to 0 \). By [12] Prop. 2.3 and [24] Prop. 5.9, the latter limit is equivalent to \( d_1(\sigma_j[0], \psi_0) \to 0 \). Using [15] Lemma 3.7, there exists a \( \sigma \in \mathbb{G} \) such that \( \sigma[0] = \psi_0 \).
Remark 7.6. The arguments in the proofs of Proposition 7.5 and Theorem 7.1 extend if we remove the maximality assumption for $T \subset \text{Aut}_T(X)$, but replace the group $\mathbb{G} = T^C$ with the connected component of the identity $\hat{\mathbb{G}} = \text{Aut}_T^c(X)$ of the centralizer of $T$ in $\text{Aut}_T(X)$. The key points are that $\hat{\mathbb{G}}$ is reductive (see Proposition 1.4), and $\hat{\mathbb{G}}$ acts transitively on the space of $T^C$-invariant $(v, w_0)$-extremal Kähler metrics (see Theorem 1.5).

Proof of Theorem 1. We apply the Coercivity Principle of [29], see Theorem 3.6. By Theorem 6.1, the extension of the weighted Mabuchi energy $\rho$ satisfies the hypotheses of Theorem 3.6 (the invariance of $M_{v, w}$ under the action of $G = T^C$ is equivalent to the necessary condition (3) for the existence of a $(v, w)$-cscK metric). We thus need to ensure that $M_{v, w}$ further satisfies the properties (i)-(iv) of Theorem 3.6. Theorem 6.1 also yields the convexity property (i) whereas the regularity property (ii) is established in Theorem 7.1. This last result also yields the uniqueness property (iii), via Theorem 1.5. Finally, the compactness property (iv) is established in Theorem 6.17.

Remark 7.7. By virtue of Theorem 1.5 and Remark 7.6, the conclusion of Theorem 1 holds true if one drops the assumption that $T \subset \text{Aut}_T(X)$ is a maximal torus, but instead of $T^C$ one considers the larger reductive group $\hat{\mathbb{G}} = \text{Aut}_T^c(X)$ (see Proposition 1.4).

8. PROOFS OF THEOREMS 2 AND 3

Proof of Theorem 2. The implication (ii) $\Rightarrow$ (i) follows from Lemma 5.9 whereas (ii) $\Rightarrow$ (iii) is established in Theorem 1. We shall prove below (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (ii). The arguments are very similar to the ones in the proof of [33, Thm. 1] where the case when $(X, T)$ is toric is studied. The main idea is to show that on a semi-simple principal $(X, T)$-bundle $Y \subset X$, the pull-back of $\omega_0$ to the vector space $T_Y^* = (\text{Lie}(\mathbb{T}_Y))^*$ of the extremal affine-linear function $\rho_{\text{ext}}(\mu)$ on $t$ defined in Theorem 2 (ii). Furthermore, by Lemma 5.10, we have that the restriction of $M_{\rho_{\text{ext}}}^X$ to the subspace $\mathcal{K}_T(X, \omega_0) \subset \mathcal{K}_{\mathbb{T}_Y}(Y, \tilde{\omega}_0)$ (see Corollary 3.6 and Lemma 5.14) is a positive multiple of $M_{\rho_{\text{ext}}}^X$, where the weights are the one defined in Theorem 2 (ii). In this setup, the main ingredients of the proof are as follows.

Step 1. Following [23, 47, 48], one considers the continuity path $\varphi_t \in \mathcal{K}_{\mathbb{T}_Y}(Y, \tilde{\omega}_0)$, determined by the solution of the PDE

$$
(66) \quad t \left( \text{Scal}(\tilde{\omega}_0) - \rho_{\text{ext}}(\mu_{\tilde{\omega}_0}) \right) = (1 - t) \left( \text{tr}_{\tilde{\omega}_0} (\hat{\rho}) - (n + m) \right), \quad t \in (0, 1),
$$

where $\hat{\rho}$ is a suitable (fixed) $\mathbb{T}_Y$-invariant Kähler metric on $Y$ in the class $[\tilde{\omega}_0]$. By [23, 48], there exits $\tilde{\rho} \in [\tilde{\omega}_0]$ and a $t_0 \in (0, 1)$, such that a solution $\varphi_t$ of (66) exits for $t$ in the interval $[t_0, 1)$; furthermore, the solution $\varphi_t(y)$ is smooth as a function on $[t_0, 1) \times Y$. The main observation of [52] is that, with a suitable choice for $\tilde{\rho}$, the path (66) can in fact be reduced to a continuity path on $Y$. To see this, we observe that, by [48, Prop. 3.1], one can take $\tilde{\rho}$ in (66) to be of the form $\rho = \omega_0 + \frac{1}{s_0} \omega$ with $s_0$ large enough, where $f$ is the smooth function on $Y$ with zero mean with respect to $\tilde{\omega}_0$, which solves the Laplace equation

$$
\Delta_{\tilde{\omega}_0} f = \left( \text{Scal}(\tilde{\omega}_0) - \rho_{\text{ext}}(\mu_{\tilde{\omega}_0}) \right).
$$

By Lemmas 5.9 and 5.3, $f \in C^\infty_{\rho}(X)$, whereas by Lemma 5.5 $\hat{\rho}$ is bundle-compatible, i.e.

$$
\hat{\rho} = \rho + \sum_{a=1}^k \left( (p_a, \mu_{\rho}) + c_a) \right) \pi^*_B \omega_a + (d \mu_{\rho} \wedge \theta),
$$
where $\rho = \omega_0 + \frac{1}{2} dd^c f$ is a $T$-invariant Kähler metric on $X$, see (24). Using Lemma 5.9 and that both $\tilde{\omega}_\varphi$ and $\tilde{\rho}$ are of the form (24), we get a path of PDE’s on $X$ of the form

$$(67) \quad t \left( \text{Scal}_p(\omega_\varphi) - \bar{w}(\mu_{\omega_\varphi}) \right) = (1 - t)H(\varphi_t), \quad t \in (t_0, 1),$$

where $\varphi_t \in K_Y(X, \omega_0)$ and $H(\varphi_t) := (\text{tr}_{\omega_\varphi}(\tilde{\rho}) - (n + m))$ is manifestly a second order differential operator on $X$ for $\varphi_t \in K_Y(X, \omega_0) \subset K_{K_Y}(Y, \omega_0)$. It follows that the solution $\varphi_t$, $t \in [t_0, 1]$ of (66) will actually belong to $K_Y(X, \omega_0) \subset K_{K_Y}(Y, \omega_0)$. This last point is a consequence of the implicit function theorem (used in [17] to establish the openness) which can be applied directly to (67); to find the linearization of (67), we use [47] that the linearization of $H(\varphi)$ on $Y$ is the operator $H_{\varphi}^p \in L^p$ so that, by virtue of Lemma A.3, the linearization of $H(\varphi)$ when restricted to $K_Y(X, \omega_0) \subset K_{K_Y}(Y, \omega_0)$ is given by the $p$-weighted operator $H_{\varphi}^p \in L^p$ introduced in Appendix A. Similar argument allows us to identify the linearization of $\text{Scal}_p(\omega_\varphi)$ (see also [26, Lemma B1]). We refer the Reader to [55, Sect. 6] for further details.

Step 2. The next ingredient is a deep result from [23] with a complement in [48], showing that if $M^Y_{1,ext}$ is $G$-coercive along the continuity path $\varphi_t$ with respect to a reductive subgroup $G \subset Aut_Y(Y)$ containing the torus generated by the extremal vector field $\xi_{ext}^Y = d\omega_{ext} \in T_Y$ in its center, then there exists a subsequence of times $j \to 1$ and elements $\sigma_j \in G$, such that $\sigma_j^*(\omega_{\varphi_j})$ converges in $C^\infty(Y)$ to an extremal Kähler metric $\check{\omega}$. In our case, assuming (iii), we have that $M^Y_{1,ext}(\varphi_t) = \text{Vol}(B, \omega_B)M^X_{p,\tilde{w}}(\varphi_t)$ (see Lemma 5.10) and $G = T_Y$-coercive (see Lemmas 6.6, 6.4 and Proposition 3.4). We can thus find $\sigma_j \in T_Y$ and $\varphi_j$ as above. The Kähler metrics $\sigma_j^*(\omega_{\varphi_j})$ are bundle-compatible in the sense of Definition 5.3 and thus are of the form $\sigma_j^*(\omega_{\varphi_j}) = \omega_0 + dv d^c d_j^* \sigma_j^* \rho_j$ with $\rho_j \in K_Y(X, \omega_0) \subset K_{K_Y}(Y, \omega_0)$. It follows that $\omega_j$ is bundle-compatible extremal Kähler metric on $Y$ (as $K_Y(X, \omega_0)$ is closed in $K_{K_Y}(Y, \omega_0)$). By Lemma 5.9 the corresponding Kähler metric $\omega_j$ on $X$ is then $(p, \tilde{w})$-cscK.

Proof of (i) $\Rightarrow$ (ii). The proof is very similar to the proof of (iii) $\Rightarrow$ (ii). As in the Step 1 of the latter, we consider the continuity path (66) which defines potentials $\varphi_t \in K_Y(X, \omega_0) \subset K_{K_Y}(Y, \omega_0)$ for $t \in [t_0, 1]$. We can assume without loss [21] that $Y$ admits a $K_Y$-invariant extremal Kähler metric in $[\omega_0]$, where $K_Y \subset T_Y$ is the maximal torus given by Lemma 5.14. This implies that $M^Y_{1,ext}$ is $G$-coercive for $G = K_Y$. Indeed, this can be justified for instance by applying Theorem 1 and Proposition 3.4 in the case $(v, w) = (1, \ell_{ext})$. As in the Step 2 of the proof of (iii) $\Rightarrow$ (ii), we use [23, 48] and the G-coercivity of $M^Y_{1,ext}$ along the path in order to find a sub-sequence of times $j \to 1$ and elements $\sigma_j \in G$, such that $\sigma_j^*(\omega_{\varphi_j})$ converges in $C^\infty(Y)$ to a $K_Y$-invariant extremal Kähler metric $\check{\omega_j} \in [\omega_0]$. However, unlike the proof of (iii) $\Rightarrow$ (ii), in general $\sigma_j^*(\omega_{\varphi_j})$ and hence $\check{\omega_j}$ are not bundle-compatible, as $\sigma_j$ can act non-trivially on $B$ (see the proof of Lemma 5.14). We thus need to modify slightly the argument in order to show that $\check{\omega_j}$ still induces a $(p, \tilde{w})$-cscK metric on any given fibre $X_b = \pi_B^{-1}(b) \subset Y$. We denote by $\omega_j(b) := (\sigma_j^*(\omega_{\varphi_j}))|_{X_b}$ and $\sigma_j(b) := (\sigma_j^*(\omega_{\varphi_j}))|_{X_b}$ the induced $T$-invariant metrics on $X_b$. As $\sigma_j$ is bundle-compatible, Lemma 5.9 yields

$$\text{Scal}_p(\omega_j(b)) = \left[ p(\mu_{\omega_{\varphi_j}}) \text{Scal}(\omega_{\varphi_j}) - p(\mu_{\omega_{\varphi_j}})q(\mu_{\omega_{\varphi_j}}) \right]|_{X_b}$$

Using that $\sigma_j \in K_Y^C$ sends the fibre $X_b$ to the fibre $X_{\sigma_j(b)}$ (this follows from the construction of $K_Y$ in the proof of Lemma 5.14), the above equality holds true for the metrics $\sigma_j(b)$, where in the RHS we replace the metric $\omega_{\varphi_j}$ on $Y$ with $\check{\omega_j} := \sigma_j^*(\omega_{\varphi_j})$. It thus follows by the smooth convergence of $\sigma_j(b)$ to $\omega_1(b)$ that

$$\text{Scal}_p(\omega_1(b)) = \left[ p(\mu_{\omega_1}) \text{Scal}(\check{\omega_1}) - p(\mu_{\omega_1})q(\mu_{\omega_1}) \right]|_{X_b}$$

$$= \left[ p(\mu_{\omega_1}) (\ell_{ext}(\mu_{\omega_1}) - q(\mu_{\omega_1})) \right]|_{X_b} = \bar{w}(\mu_{\omega_1(b)}),$$

where for the equalities on the second line we have used that the $K_Y$-extremal function $\ell_{ext} \in \text{Aff}(t_X^*)$ (see Lemma 5.14). Thus $\omega_1(b)$ is a $(v, \tilde{w})$-cscK metric on $X$. \[\square\]
Proof of Theorem 3 In [46], Han–Li introduced a functional $M_{PL}^\omega : K_t(X, \omega_0) \to \mathbb{R}$ whose critical points are the $v$-solitons, see [46] Lemma 4.4. A careful inspection using [50] shows that $M_{PL}^\omega(\omega) = M_{v,\omega}(\omega) - \int_X \log(v(\mu_v))v(\mu_v)\omega^{[m]}$, where $w$ is the weight function defined in Proposition 1. Thus, the difference of the two functionals is a constant independent of the choice of a $T$-invariant Kähler metric $\omega \in 2\pi c_1(X)$, see e.g. [50]. Thus, by [46] Thm. 3.3 applied to $(X, 2\pi c_1(X), T)$ (and weights $pv, \tilde{w}$), the $T^C$-coercivity of $M_{pv,\tilde{w}}^\omega$ is equivalent to the existence of a $vp$-soliton on $X$. By Lemma 5.11 this implies that $Y$ admits a (bundle-compatible) $v$-soliton.

By [46] Thm. 1.7, the $T^C$-coercivity of $M_{pv,\tilde{w}}^\omega$ is also equivalent to the uniform $vp$-$K$-stability on $T$-equivariant special test configurations. When $(X, T)$ is a toric Fano variety, the only such test configurations are the product test configurations, and thus by [50] Prop.3, the condition is reduced to verifying (3) on $X$ with respect to the weights $(pv, \tilde{w})$.

By the above conclusion, in order to show the existence of a Kähler–Ricci soliton, it is sufficient to find $\xi_0 \in t$, such (3) is satisfied for the weights functions $v(\mu) = e^{(\xi_0, \mu)}(\xi)$ and $\tilde{w}(\mu) = 2p(\mu)e^{(\xi_0, \mu)}(m + (\xi_0, \mu) + (d\log p, \mu))$. We detail the proof of this fact below.

Let $\omega \in 2\pi c_1(X)$ be any $T$-invariant Kähler metric with canonically normalized momentum map $\mu_\omega : X \to \Delta$. We then consider the following $p$-weighted version of a functional on $t$, defined originally by Tian–Zhu [72], Lemma 2.2:

$$\xi \to \int_X e^{(\xi, \mu_\omega)} p(\mu_\omega)\omega^{[m]}, \quad \xi \in t.$$  \hspace{1cm} (68)

The convexity and properness of the above functional follow by the arguments in [72] Lemma 2.2, but under our toric assumption these can also be seen directly by rewriting the RHS in (68) as an integral over the Delzant polytope:

$$\xi \to (2\pi)^m \int_\Delta e^{(\xi, \mu)}p(\mu)d\mu.$$  

The properness of the latter follows by the fact that the origin is in the interior of $\Delta$ (by the canonical normalization condition of $\Delta$, see Remark 2.3). Let $\xi_0 \in t$ be the unique critical point of (68). We have that

$$\int_X \langle \zeta, \mu_\omega \rangle e^{(\xi_0, \mu_\omega)} p(\mu_\omega)\omega^{[m]} = 0,$$

which is precisely the condition $\text{Fut}_{pv,\tilde{w}} = 0$ according to Lemma 2.1 in the Appendix 2.

The existence of a Sasaki–Einstein structure follows by a similar argument: By Proposition 2, Lemma 5.11 and Proposition 1 in that order, we want to find $\xi_0 \in t$ such that (3) holds true for the weights given as in Proposition 1 with $v(\mu) = p(\mu)(\langle \xi_0, \mu \rangle + a)^{-(m+n+2)}$. (This will be enough to conclude the existence of a $pv$-soliton on the toric Fano manifold $(X, T)$ and hence a $v$-soliton on $Y$ by the general arguments evoked above.) We argue based on [62] who introduced the volume functional on the space of normalized positive affine-linear functions on $\Delta$. Strictly speaking, the functional in [62] Sect.3] is introduced on the principle $\mathbb{S}^1$-bundle $N$ over $(X, \omega)$ (which admits a natural strictly pseudo-convex CR structure $(\mathcal{D}, J)$ coming from $X$), and is then defined as the Sasaki volume of a $(\mathcal{D}, J)$-compatible normalized Sasaki–Reeb vector field $\xi$ on $N$; using the point of view of [3] (see in particular Lemma 1.4), the volume functional can also be written on $X$, noting that positive affine-linear functions $\xi_\zeta = (\xi, \mu) + a$ over $\Delta$ are in bijection with Sasaki–Reeb vector fields $\xi$ on $(N, \mathcal{D}, J)$, and the normalization condition used in [62] is equivalent to requiring $\xi_\zeta(0) = a = 1$. Specifically, in our toric weighted setting, we let

$$\xi \to \int_X ((\xi, \mu_\omega) + 1)^{-(m+n+1)}p(\mu_\omega)\omega^{[m]} = (2\pi)^m \int_\Delta ((\xi, \mu) + 1)^{-(m+n+1)}p(\mu)d\mu,$$

which is defined for $\xi \in t$ such that $(\langle \xi, \mu \rangle + 1) > 0$ on $\Delta$; the properness of the functional follows by the fact that a canonically normalized Delzant polytope of a Fano toric manifold is determined by $\Delta = \{ \mu : L_j(\mu) \geq 0 \}$ where the affine-linear functions $L_j(\mu)$ satisfy $L_j(0) = 1$, see e.g. [11] Sect. 7.4]. The unique critical point $\xi_0 \in t$ of the above convex functional then satisfies

$$\int_X \langle \zeta, \mu_\omega \rangle ((\xi_0, \mu_\omega) + 1)^{-(m+n+2)}p(\mu_\omega)\omega^{[m]} = 0, \quad \zeta \in t.$$
which, by Lemma 2.1, is precisely the condition 3 for the weight functions considered. This concludes the proof of Theorem 3. 

Appendix A. Weighted differential operators

Let \((X, \omega, T)\) be as in Section 1 and \(v > 0\) be a positive smooth weight function defined over the polytope \(\Delta\). We denote by \(\nabla^\omega\) the Levi–Civita connection of the Riemannian metric \(g_\omega\), and by \(\delta_\omega\) the formal adjoint of \(\nabla^\omega\). We define the following weighted differential operators which are self-adjoint with respect to the volume form \(v(\mu_\omega)\omega^{[m]}\) on \(X\).

**Definition A.1.** The \(v\)-weighted Laplacian of \(\psi\) is the second order operator acting of smooth functions defined by

\[
\Delta_{\omega,v}(\psi) = \frac{1}{v(\mu_\omega)}\delta_\omega(v(\mu_\omega)d\psi).
\]

The \(v\)-weighted linear Lichnerowicz operator is the forth-order operator given by

\[
L_{\omega,v}(\psi) := \frac{\delta_\omega \delta_\omega(v(\mu_\omega)(\nabla^\omega d\psi)^-)}{v(\mu_\omega)},
\]

where \((\nabla^\omega d\phi)^-\) stands for the \((0, 2)\)-symmetric tensor of type \((2, 0) + (0, 2)\) with respect to the complex structure of \(X\). For any \(T\)-invariant Kähler form \(\rho\) on \(X\), we define the second-order operator given by

\[
\mathbb{H}^p_{\omega,v}(\psi) := \langle \rho, dd^c\psi \rangle_\omega + \langle \text{tr}_\omega(\rho), d\psi \rangle_\omega + \frac{1}{v(\mu_\omega)}\langle \rho, dv(\mu_\omega) \wedge d^c\psi \rangle_\omega,
\]

where \(\text{tr}_\omega(\rho) := (\rho \wedge \omega^{(m-1)}/\omega^m) = \langle \rho, \omega \rangle_\omega\). The operator \(\mathbb{H}^p_{\omega,v}\) is a \(v\)-weighted version of the linear operator used in [47].

A straightforward computation shows that

**Lemma A.2.** The \(v\)-weighted Lichnerowicz’s operator can be written as

\[
\mathbb{L}^p_{\omega,v}(\psi) = \frac{1}{2}(\Delta_{\omega,v})^2(\psi) + \delta_{\omega,v}\left((d^c\psi)^2\right),
\]

where \(\delta_{\omega,v} := \frac{1}{v(\mu_\omega)}\delta_\omega(v(\mu_\omega))\) is the formal adjoint of the exterior derivative \(d\) on functions with respect to the weighted volume form \(v(\mu_\omega)\omega^m\), \(\rho_{\omega,v} := \rho_\omega - \frac{1}{2}dd^c(\log v(\mu_\omega))\) is the Ricci form of the weighted volume form \(v(\mu_\omega)\omega^m\), and \(\sharp = g_\omega^{-1}\) stands for the riemannian duality between \(TM\) and \(T^*M\) by using the Kähler metric \(\omega\).

We now specialize to the case when \((Y, \tilde{\omega}, T_Y)\) is a semi-simple principal \((X, \omega, T_X)\)-fibration over \(B\), as in Section 5. We then denote by \(\Delta^Y_{\omega,v}\), \(\mathbb{L}^p_{\omega,v}^Y\) and \((\mathbb{H}^p_{\omega,v}^Y)^Y\) the corresponding unweighted operators on \((Y, \tilde{\omega})\), where the Kähler form \(\tilde{\rho}\) in the definition of \(\mathbb{H}^p_{\omega,v}^Y\) is bundle-compatible, i.e. given by (24) for a \(T_X\)-invariant Kähler form \(\rho\) on \(X\). We further let \(\Delta_{\omega,v}^B\) denote the Laplacian on \((B, \omega_B)\), and \(\Delta^B_{\omega,v}\) and \(\mathbb{L}^p_{\omega,v}^B\) respectively the Laplacian and Lichnerowicz operators on \(B\) with respect to the Kähler metric \(\omega_B(x) := \sum_{a=1}^k(\langle p_a, \mu_\omega(x) \rangle + c_a)\omega_a\). We thus have the following result.

**Lemma A.3.** Let \(\psi\) be a \(T_Y\)-invariant smooth function on \(Y\), seen as a \(T_X\)-invariant function on \(X \times B\) via (25), and \(\tilde{\omega}\) a bundle-compatible \(T_Y\)-invariant Kähler metric on \(Y\) associated to a \(T_X\)-invariant Kähler metric \(\omega\) on \(X\). We then have

\[
\Delta^Y_{\omega,v}\psi = \Delta^X_{\omega,v}\psi_b + \Delta^B_{\omega,v}\psi_x, \\
\mathbb{L}^p_{\omega,v}^Y\psi = \mathbb{L}^p_{\omega,v}^X\psi_b + \mathbb{L}^p_{\omega,v}^B\psi_x + \Delta^B_{\omega,v}\left(\Delta^X_{\omega,v}\psi_b\right)_x + \Delta^X_{\omega,v}\left(\Delta^B_{\omega,v}\psi_x\right)_b
\]

\[
+ \sum_{a=1}^k Q_a(x)\Delta^B_{\omega_a}\psi_x,
\]

\[
(\mathbb{H}^p_{\omega,v}^Y)^Y\psi = (\mathbb{H}^p_{\omega,v}^X)^X\psi_b + \sum_{a=1}^k P_a(x)\Delta^B_{\omega_a}\psi_x,
\]
where \(P_a(x), Q_a(x)\) are smooth \(T\)-invariant functions on \(X\), and \(\psi_x\) and \(\psi_b\) are respectively the induced smooth functions on \(B\) and \(X\) via \([25]\).

**Proof.** This first two equalities are established in \([7]\) (see the proof of Lemma 8) in the special case when \((X, \omega, T_X)\) is a toric variety whereas the third identity is proved in \([53]\) (also in the case when \((X, T_X)\) is toric). These computations extend to the general setting with no substantial additional difficulty (by using Lemma \([A.2]\) above for the second identity), but we include them below for the sake of self-containedness.

In the notation of Sect. \([5]\)

\[
\Delta^Y_\omega(\psi) = -\frac{d_Y d_Y^c \psi \wedge \tilde{\omega}^{[n+m-1]}}{\tilde{\omega}^{[n+m]}} = -\frac{d_Y d_Y^c \psi \wedge \tilde{\omega}^{[n+m-1]} \wedge \theta^{\text{tr}}}{\tilde{\omega}^{[n+m]} \wedge \theta^{\text{tr}}} \quad \text{(on } Z = X \times P),
\]

where \(\theta^{\text{tr}} := \bigwedge_{i=1}^r \theta_i\) with respect to any lattice basis \((\xi_i)\) of \(\Lambda \subset \mathfrak{t}\). Viewing \(d_Y^{c}X \times B\psi\) as a 1-form on \(Z\), it admits the following decomposition with respect to \([23]\)

\[
d_X \times B = (d_X \times B)^c \omega \n + \sum_{i=1}^r (d_X \times B)_{\xi_i} (\xi_i - \xi_i^X) \theta_i = d_X \psi = \langle d_X \psi, \theta \rangle.
\]

We thus compute on \(Z\):

\[
(d_Y d_Y^c \psi)_{(x,b)} = d_Z \left( d_X \psi + \sum_{j=1}^r d_X^c \psi (\xi_j^X) \theta_j + d_B^c \psi \right)
\]

\[
= d_Z d_X^c \psi + \sum_{j=1}^r d_Z (d_X \psi (\xi_j^X)) \theta_j
\]

\[
+ \sum_{j=1}^r d_X^c \psi_b (\xi_j^X) \left( \sum_{a=1}^k \xi_j (p_a) \pi_a^\star \omega_a \right) + d_Z d_B^c \psi
\]

\[
= d_X d_X^c \psi_b + d_B d_B^c \psi_b \n + \sum_{j=1}^r d_Z (d_X \psi (\xi_j^X)) \wedge \theta_j + \sum_{a=1}^k d_X \psi_b (p_a^X) \pi_a^\star \omega_a
\]

\[
+ d_B d_B^c \psi_b + d_X d_B^c \psi,
\]

where for getting the third equality we used \([22]\), as well as the identities \(d_Y d_X^c \psi = d_B d_B^c \psi\) and \(d_Y^c d_X \psi = d_B d_B^c \psi\) (which follow from the identification \([25]\)). Using \([27]\) and \([44]\), we derive from \([72]\) and \([74]\)

\[
\Delta^Y_\omega(\psi)(x, b) = (\Delta^X_\omega \psi_b)(x) + (\Delta^B_{\mu_B}(x) \psi_x)(b) - \sum_{a=1}^k \frac{n_a}{(\mu_a, p_a) + c_a} (d_X \psi_b p_a^X),
\]

where, we recall, for a fixed \(x \in X\), we have set \(\omega_B(x) := \sum_{a=1}^k (p_a, \mu_a) + c_a \omega_a\), and \(p_a^X\) denotes the vector field field on \(X\) corresponding to \(p_a \in \mathfrak{t}\). The first equality in the Lemma follows from the identity \([17]\), taking in mind that for any smooth function on \(u\) on \(\Delta\) and any \(T\)-invariant smooth function \(\phi\) on \(X\), \(g_\omega(d(u(\mu_\phi)), d\phi) = \sum_{i=1}^r u_{\mu_\phi} \delta_i^\mu d^c \phi(\xi_i)\).

Now, we establish the expression of the corresponding Lichnerowicz operators. Recall that (see e. g. \([11]\))

\[
L^Y_\omega \psi := \frac{1}{2} (\Delta^Y_\omega) (\psi) + \delta_\omega (d^c_\omega \psi).
\]

Using the decomposition of \(\Delta^Y_\omega\) we have just established, we have

\[
(\Delta^Y_\omega)^2 (\psi) = (\Delta^X_{\omega,B})^2 (\psi_b) + (\Delta^B_{\mu_B})^2 (\psi_x) + \Delta^X_{\omega,B} \Delta^B_{\mu_B} (\psi_x) + \Delta^B_{\mu_B} (\Delta^X_{\omega,B} (\psi_b)).
\]
It remains to compute the Ricci term in (75). From (43), we have
\[
\rho_\omega = \rho_{\omega, p} + \pi^*_B \rho_{\omega_B} + \frac{1}{2} \sum_{a=1}^k \Delta^X_{\omega, p}((\mu_\omega, p_a^X)) \pi^*_B \omega_a \\
+ \sum_{j=1}^r d\omega \left( d\omega \left( \kappa - \frac{1}{2} \log p(\mu_\omega) \right) (\xi_j^X) \right) \wedge \theta_j.
\] (77)
where \(\rho_{\omega, p} := \rho_\omega - \frac{1}{2} d\omega d\omega \log p(\mu_\omega)\) is the Ricci form of the weighted volume form \(p(\mu_\omega) \omega^{[m]}\).
Using integration by parts, for any \(T\)-invariant smooth test function \(\phi\) on \(Y\), seen as a \(T_X\) and \(T_P\)-invariant function on \(Z = X \times P\) via [25], we have
\[
\int_Z \phi_\omega(d\omega (d\omega Y)) \omega^{[n+m]} \wedge \theta^{\wedge r} = - \int_Z \phi_\omega(d\omega (d\omega Y)) \omega^{[n+m]} \wedge \theta^{\wedge r} \\
- \frac{1}{2} \int_Z \text{Scal}(\omega) \tilde{g}_\omega(d\omega Y) d\omega^{[n+m]} \wedge \theta^{\wedge r} \\
- \frac{1}{2} \int_Z \left( \frac{\text{Scal}_B(\omega)}{p(\mu_\omega)} + q(\mu_\omega) \right) d\omega Y \wedge d\omega Y \wedge \omega^{[n+m-1]} \wedge \theta^{\wedge r}.
\] (78)
From the above formula, using (44), (73) and (77), we compute (after some straightforward but long algebraic manipulations and integration by parts over \(X\) and \(B\))
\[
\delta^Y_\omega(\rho_{\omega, p}(d\omega Y)) = \delta^X_\omega(\rho_{\omega, p}(d\omega Y)) + \delta^B_{\omega_B}(\rho_{\omega, p}(d\omega Y)) \\
+ \frac{1}{2} \sum_{a=1}^k \frac{q(\mu_\omega)}{((\mu_\omega, p_a) + c_a)} \Delta^B_{\omega, p}(\psi_a) \\
+ \frac{1}{2} \sum_{a=1}^k \frac{(n_a - 1)}{((\mu_\omega, p_a) + c_a)} \Delta^X_{\omega, p}((\mu_\omega, p_a)) \Delta^B_{\omega, p}(\psi_a) \\
+ \sum_{a,b=1}^k \frac{n_b}{((\mu_\omega, p_a) + c_a)((\mu_\omega, p_b) + c_b)} \Delta^X_{\omega, p}((\mu_\omega, p_b)) \Delta^B_{\omega, p}(\psi_a).
\] (79)
Combining (75), (76) and (79) yields the desired expression.

The expression for \(\langle \mathbb{H}^Y_{\omega, 1} \rangle (\psi)\) is obtained by similar arguments, using that
\[
\langle \mathbb{H}^Y_{\omega, 1} \rangle (\psi) = \langle \rho, d\omega d\omega Y \rangle + \langle d\omega Y \text{tr}_\omega(\tilde{\rho}), d\omega Y \rangle \\
- \text{tr}_\omega(\tilde{\rho}) \Delta^Y_\omega(\psi) - \frac{\tilde{\rho} \wedge d\omega d\omega Y \wedge \omega^{[n+m-2]} \wedge \omega^{[n+m-1]}}{\omega^{[n+m]}}. 
\]

2. Weighted Futaki invariants

On a smooth Fano manifold \((X, T)\) as in the setting and notation of Section 2 we further relate the weighted Futaki obstruction \(\text{Fut}_{v, w}(0)\) (see [3]) with weights \(v(\mu), w(\mu)\) as in Proposition 1 with the Futaki-type obstructions studied by Tian–Zhu [22] in the case of Kähler–Ricci solitons (i.e. when \(v = e^{\langle \xi, \mu \rangle}\)):

**Lemma 2.1.** Let \((X, T)\) be a smooth Fano manifold \((X, T)\) with canonically normalized momentum polytope \(\Delta\), and \(v > 0, w\) smooth functions on \(\Delta\) as in Proposition 7. Then, for any \(T\)-invariant Kähler metric \(\omega \in 2\pi c_1(X)\) with momentum map \(\mu_\omega\) and \(T\)-invariant Ricci potential \(h\) (i.e. \(\rho_\omega - \omega = \frac{1}{2} dd^c h\)), the weighted Futaki invariant \(\text{Fut}_{v, w}(\ell_\zeta)\) introduced in [3] satisfies
\[
\text{Fut}_{v, w}(\ell_\zeta) = \int_X \left( L_{\xi}(\log v(\mu_\omega) - h) \right) v(\mu_\omega) \omega^{[m]} = -2 \int_X (\zeta, (\zeta, \mu_\omega) v(\mu_\omega) \omega^{[m]}, \quad \ell_\zeta = (\zeta, \mu) + a, \ z \in \mathfrak{t}.
\]
Proof. We have
\[ \int_X \left( \mathcal{L}_{\xi}(\log v(\mu_\omega) - h) \right) v(\mu_\omega) \omega^m = \int_X g_\omega \left( d\ell_{\xi}, d\log(v(\mu_\omega) - h) \right) v(\mu_\omega) \omega^m \]
\[ = \int_X \ell_\xi \left( \Delta_{\omega, v}(\log v(\mu_\omega) - h) \right) v(\mu_\omega) \omega^m = \int_X \ell_\xi \left( \text{Scal}_v(\omega) - w(\mu_\omega) \right) \omega^m = \text{Fut}_{v, w}(\ell_\xi), \]
where for the last equality we have used (13). The second equality in the Lemma follows from K-polystability of Q-Fano varieties [11] R. Berman and B. Berndtsson, Convexity of the K-energy on the space of Kähler metrics and uniqueness of Kähler geometry [12] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, The space of Kähler metrics [22] X.X. Chen, The Mabuchi geometry of finite energy class [25] J. Chu, V. Tosatti, and B. Weikove, On the C^{1,1} regularity of geodesics in the space of Kähler metrics, Annals of Partial Differential Equations 3 (2017), art. no 15, arXiv:1611.02390

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