ROOK THEORY OF THE FINITE GENERAL LINEAR GROUP

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Abstract. Matrices over a finite field having fixed rank and restricted support are a natural $q$-analogue of rook placements on a board. We develop this $q$-rook theory by defining a corresponding analogue of the hit numbers. Using tools from coding theory, we show that these $q$-hit and $q$-rook numbers obey a variety of identities analogous to the classical case. We also explore connections to earlier $q$-analogues of rook theory, as well as settling a polynomiality conjecture and finding a counterexample of a positivity conjecture of the authors and Klein.

1. Introduction

Classically, part of rook theory goes like this [KR46]: given a board $B$ contained in the discrete $n \times n$ square grid $[n] \times [n]$, one wishes to find the rook number $r_i(B)$, the number of ways of placing $i$ non-attacking rooks in $B$, or the hit number $h_i(B)$, the number of $n \times n$ permutation matrices with $i$ rooks in $B$. These numbers are difficult to compute in general [Val79], but nevertheless one can say many things about their properties. For any board $B$, the rook and hit numbers are related by the equation

$$
\sum_{i=0}^{n} h_i(B) \cdot t^i = \sum_{i=0}^{n} r_i(B) \cdot (n - i)! \cdot (t - 1)^i.
$$

Moreover, from their definition the hit numbers satisfy the reciprocity relation

$$
h_{n-i}(B) = h_i(B),
$$

where $\overline{B}$ denotes the complement of $B$ with respect to $[n] \times [n]$. The zero hit number $h_0(B) = h_n(B) = r_n(B)$ is of particular interest; setting $t = 0$ in (1.1) gives the inclusion-exclusion formula

$$
h_0(B) = \sum_{i=0}^{n} (-1)^i \cdot (n - i)! \cdot r_i(B).
$$

For example, this relation can be used to find formulas for the number $d_n$ of permutations of size $n$ with no fixed points (derangements) and the number $c_n$ of permutations $w$ of size $n$ such that $w(i) \neq i, i+1 \pmod{n}$ (the famous problème des ménages; see [Sta12, §2.3]). The boards in these cases are the diagonal $\{(1,1), \ldots, (n,n)\}$ and a board consisting of the diagonal, the next upper diagonal and the cell $(n,1)$ (Figure 3).

Garsia and Remmel [GR86] started the study of $q$-analogues of rook numbers by defining $q$-rook numbers and $q$-hit numbers for Ferrers boards. By definition, these $q$-analogues are polynomials in a formal variable $q$ having nonnegative integer coefficients, whose values at $q = 1$ are equal to the corresponding rook numbers and hit numbers. A different kind of $q$-analogue of rook numbers was proposed in [LLM+11], namely, the number of $n \times n$ matrices with entries in the finite field $F_q$ with $q$ elements having rank $i$ and support in $B$. This number, denoted by $m_i(B,q)$, is an enumerative $q$-analogue of $r_i(B)$ in a sense made precise in (2.4) below. When $B$ is a Ferrers board, Haglund [Hag98] had already shown that $m_i(B,q)$ is equivalent to the Garsia–Remmel
$q$-rook numbers. However, for general boards, the function $m_i(B, q)$ need not be a polynomial in $q$ \cite{Ste98} (indeed it can be much more complicated \cite{KLM17}), and if it is a polynomial it might or not have nonnegative integer coefficients.

In the first part of this paper, we continue the study of this new $q$-rook theory. We define a corresponding notion of $q$-hit numbers for an arbitrary board $B$ using a suggestion of Remmel (private communication), and we give a reciprocity relation for $m_i(B, q)$ and $q$-hit numbers using a result of Delsarte \cite{Del78} that is an analogue of the MacWilliams identity \cite{Mac63} on the weights of dual codes. (The connection between this identity and $M_i(B, q)$ has appeared in work of Ravagnani \cite[Rem. 50]{Rav15}.)

Since $m_i(B, q)$ is always divisible (as an integer) by $(q - 1)^i$, it is convenient to define the reduced (or projective) matrix count $M_i(B, q) = m_i(B, q)/(q - 1)^i$. Then the $q$-hit numbers for an arbitrary board are defined as follows.

**Definition.** Given a board $B \subseteq [n] \times [n]$ and a nonnegative integer $i$, define the $q$-hit number $H_i(B, q)$ by the equation

$$
\sum_{i=0}^{n} H_i(B, q) \cdot t^i = q^{n \choose 2} \sum_{i=0}^{n} M_i(B, q) \cdot [n - i]!_q \prod_{k=0}^{i-1} (tq^{-k} - 1),
$$

where $[n - i]!_q$ is a $q$-factorial. Let $P(B, q, t)$ denote the expression on both sides of this equality.

Some properties of the hit numbers are immediate formal consequences of this definition. By taking leading coefficients we have that $H_n(B, q)$ is equal to $M_n(B, q)$, while by setting $t = 1$ we have that the $q$-hit numbers partition $\text{GL}_n(F_q)$, in the sense that

$$(q - 1)^n \sum_{i=0}^{n} H_i(B, q) = \text{GL}_n(F_q).$$

Other properties are less obvious. We show that the functions $H_i(B, q)$ are enumerative $q$-anallogues of the hit numbers (Proposition \[3.3\]), that they coincide with the Garsia–Remmel $q$-hit numbers when $B$ is a Ferrers board (Proposition \[4.8\]), and that their generating function $P(B, q, t)$ has a probabilistic interpretation (Theorem \[3.11\]). Furthermore, using a generalized MacWilliams complement identity for $M_i(B, q)$, we show in Section \[3.2\] the following reciprocity of $q$-hit numbers.

**Theorem.** For every board $B \subseteq [n] \times [n]$ and for $i = 0, \ldots, n$ we have that

$H_{n-i}(B, q) = q^{n-|B|} \cdot H_i(B, q).$

We leave open the problem of giving a combinatorial interpretation to $H_i(B, q)$ (Question \[6.1\]).

As in the classical case, the zeroth $q$-hit number is particularly nice. By the $q$-hit reciprocity, one can show that $H_0(B, q) = q^{|B|}M_n(B, q)$. Moreover, there is an inclusion-exclusion formula for this number (Corollary \[3.10\]).

**Corollary.** For any board $B \subset [n] \times [n]$ we have

$$H_0(B, q) = q^{|B|}M_n(B, q) = q^{n \choose 2} \sum_{i=0}^{n} (-1)^i \cdot [n - i]!_q \cdot M_i(B, q).$$

This formula is used to recover the formula in \cite{LLM11} for the number $D_n(q)$ of $n \times n$ invertible matrices with entries in $F_q$ with zero diagonal (a $q$-analogue of derangements); see \cite[Cor. 52]{Rav15}. We use it to find a $q$-analogue of the ménage problem (Theorem \[4.13\]), settling a question considered
by Rota and Haglund (private communication from Haglund). Our \(q\)-analogue is very similar to Touchard’s classical formula (4.5) for \(c_n\).

The starting point of any nice result in rook theory is the case of Ferrers boards [GJW75, GR86, Hag98]. In [KLM14] and [LM16], we studied the matrix counts \(M_i(B, q)\) for a richer class of boards, the (coinversion) diagrams of permutations (see Section 2 for the definition). Our study included the following conjecture.

**Conjecture 1.1** ([KLM14, Conj. 5.1]). For all permutations \(w \in S_n\) and ranks \(0 \leq r \leq n\), the reduced matrix count \(M_r(I_w, q)\) of \(n \times n\) matrices over \(\mathbb{F}_q\) of rank \(r\) with support in the complement of the diagram \(I_w\) of \(w\) is a polynomial in \(q\) with nonnegative integer coefficients.

We verified the conjecture computationally for \(r \leq n \leq 7\) and for \(r = n\) for \(n \leq 9\) [LM17]. In [LM16], we proved the conjecture for permutations \(w\) avoiding the patterns 4231, 35142, 42513 and 351624 in the case \(r = n\). In the second part of this paper, we use the complement identity to prove the polynomiality part of Conjecture 1.1 (Corollary 4.6).

**Theorem.** For all permutations \(w \in S_n\) and all ranks \(0 \leq r \leq n\), \(M_r(I_w, q)\) is a polynomial in \(q\) with integer coefficients.

We also give a deletion-contraction relation (Corollary 5.9) that allows for the quick computation of \(M_r(I_w, q)\). Using this relation, we find counterexamples to the positivity part of Conjecture 1.1.

**Example 1.2.** For \(w = 6 8 9 10 4 5 7 1 2 3 \in S_{10}\), we have

\[
M_{10}(I_w, q) = q^{77} + 9q^{76} + 44q^{75} + \cdots + 2q^{48} - 8q^{47} - q^{46} + q^{45} \notin \mathbb{N}[q].
\]

It remains open to characterize the permutations \(w\) such that \(M_r(I_w, q)\) is in \(\mathbb{N}[q]\).

**Outline.** Section 2 establishes notation and introduces a \(q\)-analogue of the MacWilliams complement identity from the literature. Section 3 introduces a \(q\)-analogue of the hit numbers and proves a variety of properties analogous to the classical case. Section 4 studies the \(q\)-rook and \(q\)-hit numbers for boards with a certain structural property, including connections to the Garsia–Remmel \(q\)-rook theory and a \(q\)-analogue of the problème des ménages. Section 5 builds on Section 4 to give deletion-contraction style recurrences for \(q\)-rook and \(q\)-hit numbers. Finally, Section 6 includes a number of additional remarks and open questions.

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2. Background and notation

Throughout this paper, \(m\) and \(n\) will be fixed positive integers with \(m \leq n\). Given an integer \(k\), denote by \([k]\) the set \(\{1, \ldots, k\}\) of the first \(k\) positive integers. We use the word board to refer to any subset of \([m] \times [n]\). Given a board \(B\), we denote by \(\overline{B}\) its complement \(\overline{B} \overset{\text{def}}{=} ([m] \times [n]) \setminus B\) in the rectangle \([m] \times [n]\). A rook placement on a board \(B\) is a subset of \(B\) that contains no two elements in the same row (i.e., having the same first coordinate) or in the same column (having the same second coordinate). When drawing boards and rook placements, we always use matrix
coordinates, so that \( \{(1,y) : y \in [n]\} \) is the top-most row and \( \{(x,1) : x \in [m]\} \) is the left-most column. The elements of a board \( B \) will be variously referred to as cells or boxes.

One particularly nice family of boards are the Ferrers boards. Each Ferrers board is associated to an integer partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \), and consists of an aligned collection of \( \lambda_i \) boxes in the \( i \)th row for \( i = 1, \ldots, k \). We take an ecumenical approach and use the name Ferrers board for boards in both English and French notation, as well as their reflections.

We make use of many standard notations for \( q \)-counting functions, including the \( q \)-Pochhammer symbol and \( q \)-factorial

\[
(a;q)_k \overset{\text{def}}{=} \prod_{i=0}^{k-1} (1 - aq^i) = (aq^{k-1}; q^{-1})_k \quad \text{and} \quad [k]!_q \overset{\text{def}}{=} \frac{(q;q)_k}{(1-q)^k} = \prod_{i=1}^{k} \frac{q^i - 1}{q - 1},
\]

and the \( q \)-binomial coefficient, defined by

\[
\left[ \begin{array}{c} k \\ \ell \end{array} \right]_q \overset{\text{def}}{=} \frac{[k]!_q}{[\ell]!_q [k-\ell]!_q} \quad \text{if } 0 \leq \ell \leq k
\]

and \( \left[ \begin{array}{c} k \\ \ell \end{array} \right]_q = 0 \) otherwise. It is not obvious from this definition, but the \( q \)-binomial coefficients are polynomials in \( q \) with positive integer coefficients. They also give the expansion of the \( q \)-Pochhammer as a sum, called the \( q \)-binomial theorem. In its most general form \([GR04\text{ (II.3)}]\), this is the infinite product-sum identity

\[
(az;q)_\infty \over (z;q)_\infty = \sum_{i=0}^\infty (a;q)_i z^i,
\]

but on specializing \( a \mapsto q^{-k} \) and \( z \mapsto q^kz \) for \( k \in \mathbb{N} \) it becomes

\[
(z;q)_k = \sum_{i=0}^k (-1)^i \left[ \begin{array}{c} k \\ i \end{array} \right]_q q^{i(z)} z^i.
\]

The inverse relation

\[
z^k = \sum_{i=0}^k (-1)^i \left[ \begin{array}{c} k \\ i \end{array} \right]_q q^{i(z)} (z;q^{-1})^i
\]

expressing the pure powers of \( z \) in terms of \( q \)-Pochhammer symbols may be proved by expanding the \( q \)-Pochhammer in the right side of (2.3) using (2.2), reversing the order of summation, and re-collecting terms in the inner sum using (2.2).

Given a board \( B \), let \( m_i(B,q) \) be the number of \( m \times n \) matrices of rank \( i \) over \( \mathbb{F}_q \) with support in \( B \) (that is, with all entries outside of \( B \) equal to 0), and let \( M_i(B,q) \overset{\text{def}}{=} m_i(B,q)/(q-1)^i \). In \([LLM+11\text{ Prop. 5.1}]\), we showed that \( M_i(B,q) \) is an enumerative \( q \)-analogue of \( r_i(B) \), in the following sense: for any prime power \( q \),

\[
M_i(B,q) \equiv r_i(B) \pmod{q-1}.
\]

The symmetric group \( \mathfrak{S}_n \) consists of the permutations of the set \( [n] \). These may be represented in various ways: as words \( w = w_1 \cdots w_n \) in one-line notation, or as permutation matrices, having entries 1 at positions \( (i,w_i) \) for \( i \in [n] \) and other entries 0, or as placements of \( n \) rooks on \( [n] \times [n] \).
For a permutation $w = w_1 \cdots w_n$ in $S_n$, define its diagram $I_w$ by $I_w \overset{\text{def}}{=} \{(i, w_j) \mid i < j, w_i < w_j\}$. The cells of $I_w$ are in bijection with the coinversions of $w$, and so $|I_w| = \binom{n}{2} - \ell(w)$ where $\ell(w)$ is the length (or inversion number) of $w$. The permutation boards contain the Ferrers boards as a sub-class: any Ferrers board that fits inside the upper-right-aligned triangle with legs of length $n-1$ is the diagram of a permutation in $S_n$. Conversely, a permutation $w$ has as its diagram an upper-right-aligned Ferrers board if and only if $w$ avoids the permutation pattern 312, in the sense that $w$ has no three entries $w_i > w_k > w_j$ with $i < j < k$ \cite[Ex. 2.2.2]{ManOl}. 

2.1. A MacWilliams-style complement identity. The classical MacWilliams identity expresses the weight of a code (a subspace of a finite vector space) in terms of the weight of the dual code \cite{Mac63, BH13}. In \cite{Del78}, Delsarte introduced rank-metric codes, in which the code is a linear subspace of matrices over a finite field and the weight of an element is the rank. In this context, he proved what may be viewed as a $q$-analogue of the MacWilliams identity \cite[Thms. 3.3, A2]{Del78}. This identity involves a $q$-analogue of the Krawtchouk polynomials, so we begin by recalling some important facts about them from the literature.

For $i, r \leq m$, define the $q$-Krawtchouk polynomial\footnote{There are many possible variations on the diagram $I_w$: different choices of coordinates for $w$ give different correspondences between the set of permutations and the set of their diagrams, or amount to reflecting or rotating the diagrams; recording the pairs $(i, j)$ instead of $(i, w_j)$ produces diagrams with permuted columns; recording inversions instead of coinversions is equivalent to recordinatizing; and so on. None of these differences materially affect our results. Appropriate variations are known in the literature as inversion diagrams or Rothe diagrams of permutations.}:

$$K_r(i) \overset{\text{def}}{=} \sum_s (-1)^{r-s} q^{ns + \binom{r-s}{2}} \left[ \begin{array}{c} m-s \\ r-s \\ s \\ q \\ q \\ i \\ q \end{array} \right],$$

where the sum is over all indices $s$ such that $0 \leq r - s$ and $0 \leq s \leq m-i$. These polynomials form a family of orthogonal polynomials, and consequently have many nice properties. We mention several of these here, following to various degrees Delsarte, Ravagnani, and Stanton \cite{Del78, Rav15, Sta84}. Let

$$v_k \overset{\text{def}}{=} \prod_{i=0}^{k-1} \frac{(q^m - q^i)(q^n - q^i)}{q^k - q^i}$$

denote the number of $m \times n$ matrices of rank $k$ over $F_q$ (see, e.g., \cite[§1.7]{Mor06}). The $q$-Krawtchouk polynomials satisfy the orthogonality relation

$$\sum_{i=0}^{n} v_i \cdot K_k(i) \cdot K_\ell(i) = q^{mn} \cdot v_k \cdot \delta_{k, \ell},$$

where $\delta_{k, \ell}$ represents the usual Kronecker delta function. They can be written in terms of basic hypergeometric functions in various ways; notably,

$$K_r(i) = v_r \cdot {}_3\phi_2(q^{-r}, q^{-i}, 0; q^{-m}, q^{-n}; q).$$

(It is the $3\phi_2$ evaluation on the right side that is called the affine $q$-Krawtchouk polynomial by Stanton \cite[(4.13)]{Sta84}; Delsarte \cite{Del78} gives a similar expression, but it contains an error.)
formula exhibits the symmetry
\[(2.6) \quad \frac{K_r(i)}{v_r} = \frac{K_i(r)}{v_i},\]
which is not obvious from the definition. As orthogonal polynomials, the \(q\)-Krawtchouk polynomials satisfy a three-term recurrence relation [Sta84, (4.14)]:
\[
q^{m+n}(q^{-k} - 1)K_i(k) = q^i(q^{i+1} - 1)K_{i+1}(k) + (q^m - q^{i-1})(q^n - q^{-i-1})K_{i-1}(k) - ((q^m - q^i)(q^n - q^i) + q^{i-1}(q^i - 1))K_i(k).
\]

The relevance of the \(q\)-Krawtchouk polynomials to the present work is their appearance in the following complement identity, expressing the number of matrices of a given rank supported on a board in terms of the same counts for the complementary board.

**Theorem 2.1** (Complement identity [Del78]). For any board \(B \subseteq [m] \times [n]\) with \(m \leq n\) and any rank \(r \leq m\), we have
\[(2.7) \quad m_r(\overline{B}, q) = \frac{1}{q^{|B|}} \sum_{i=0}^{m} K_r(i) \cdot m_i(B, q).\]

In the case \(r = m\) of full-rank matrices, this formula simplifies.

**Corollary 2.2.** For any subset \(B\) of \([m] \times [n]\) with \(m \leq n\), we have
\[(2.8) \quad m_m(\overline{B}, q) = (-1)^m q^{(m)(n)} \sum_{i=0}^{m} m_i(B, q) \cdot (q^{n-m+1}; q)_{m-i}.\]

**Proof.** The case \(r = m\) in (2.7) gives
\[
m_m(\overline{B}, q) = \frac{1}{q^{|B|}} \sum_{i=0}^{m} m_i(B, q) \cdot \sum_s (-1)^{m-s} q^{(m)(n) + (s)(n-m+1)} \begin{bmatrix} m - i \\ s \end{bmatrix}_q
\]
\[
= (-1)^m q^{(m)(n)} \sum_{i=0}^{m} m_i(B, q) \left( \sum_{s=0}^{m-i} (-q^{n-m+1})^s q^{(s)} \begin{bmatrix} m - i \\ s \end{bmatrix}_q \right).
\]

By the \(q\)-binomial theorem (2.2), the inner sum simplifies to
\[
\sum_{s=0}^{m-i} (-q^{n-m+1})^s q^{(s)} \begin{bmatrix} m - i \\ s \end{bmatrix}_q = (q^{n-m+1}; q)_{m-i},
\]
as desired. \(\square\)

**Example 2.3** (\(q\)-analogue of derangements [LLM+11, Rav15]). In the case \(m = n\) and \(B = \{(1, 1), \ldots, (n, n)\}\), we have that \(m_i(B, q) = \begin{bmatrix} n \\ i \end{bmatrix} (q - 1)^i\). Thus, by (2.8), the number of \(n \times n\) invertible matrices with zero diagonal is
\[
m_n(\overline{B}, q) = (-1)^n q^{(n)(n)} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (q - 1)^i (q; q)_{n-i} = q^{(n)(n)} (q - 1)^n \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} [n - i]_q!.
\]
The classical rook theory of the board

Given a board

Definition 3.1.

This relation follows by the same argument as in the square case (see, e.g., [Sta12, §2.3]).

We now define our $q$-hit numbers for general boards $B$. This definition is based on a suggestion of Remmel (private communication) that is related to the construction of the Garsia–Remmel $q$-hit numbers [GR86, §1].

Definition 3.1. Given a board $B \subseteq [m] \times [n]$ and a nonnegative integer $i$, define the $q$-hit number $H_i(B, q)$ by the equation

$$H_i(B, q) \cdot t^i \overset{\text{def}}{=} Q_m \sum_{i=0}^{m} M_i(B, q) \cdot \frac{[n-i]_q!}{[n-m]_q!} \cdot (-1)^i \cdot (t; q^{-1})^i.$$  

For each fixed $q$, both sides of this equation are polynomials in $t$. We call this the $q$-hit polynomial of the board $B$ and denote it by $P(B, q,t)$.

The remainder of this section is devoted to showing that these $q$-hit numbers satisfy a variety of properties that one would expect from an object bearing the name; this culminates in a natural probabilistic interpretation of the $q$-hit numbers in Section 3.3.

Example 3.2. We give three examples of boards in the case $m = n = 2$; these are illustrated in Figure 1.

(i) When $B_1 = [2] \times [2]$ is the entire square board, we have $M_0(B_1, q) = 1$, $M_1(B_1, q) = (q + 1)^2$, and $M_2(B_1, q) = q(q + 1)$. Thus

$$P(B_1, q, t) = q \left( (q + 1) + (q + 1)^2 \cdot (t - 1) + q(q + 1) \cdot (t - 1)(tq^{-1} - 1) \right) = (q^2 + q)t^2,$$

and so $H_0(B_1, q) = H_1(B_1, q) = 0$ and $H_2(B_1, q) = q^2 + q$.

(ii) When $B_2 = \{(1, 2)\}$ is the $2 \times 2$ board with a single square removed, we have $M_0(B_2, q) = 1$, $M_1(B_2, q) = 2q + 1$, and $M_2(B_2, q) = q$. Thus

$$P(B_2, q, t) = q \left( (q + 1) + (2q + 1) \cdot (t - 1) + q \cdot (t - 1)(tq^{-1} - 1) \right) = q^2t + qt^2,$$

and so $H_0(B_2, q) = 0$, $H_1(B_2, q) = q^2$, and $H_2(B_2, q) = q$. 

\[
\begin{array}{ccc}
B_1 & B_2 & B_3 \\
\begin{array}{|c|}
\hline
\square \\
\hline
\end{array} & 
\begin{array}{|c|}
\hline
\square \\
\hline
\end{array} & 
\begin{array}{|c|}
\hline
\square \\
\hline
\end{array}
\end{array}
\]

Figure 1. The three $2 \times 2$ boards mentioned in Example 3.2.
Proposition 3.5. For all $q$ results hold for number in terms of hit numbers and vice versa. The next proposition shows that the analogous in which all rooks land on $t$

By equating coefficients of $q$ both sides of (3.2) using the

Rearranging powers of $q$ we start with (3.2) and take the residue modulo $q - 1$. By [2.4], we have for each $i$ that $M_i(B, q) ≡ r_i(B) \pmod{q - 1}$, and thus

The right side of this equivalence is the generating function (3.1) for the classical hit numbers, so

By equating coefficients of $t^i$ for $i = 0, \ldots, m$, we obtain the desired result. □

Remark 3.4. For the $q$-rook numbers, we have that $M_i(B, q) = 0$ if and only if the usual rook number $r_i(B)$ is also equal to 0. However, this does not hold for $q$-hit numbers. For example, in Example 3.2 (iii) with $m = n = 2$ and $B = \{(1, 1), (2, 2)\}$, we have $h_1(B) = 0$ (every permutation has either 0 or 2 rooks on $B$) but $H_1(B, q) = q - 1$.

In the classical setting, maximal rook placements on $B$ are exactly the same as rook placements in which all rooks land on $B$, and so $r_m(B) = h_m(B)$. Moreover, from (1.1) one can write each rook number in terms of hit numbers and vice versa. The next proposition shows that the analogous results hold for $q$-hit numbers and matrix counts.

Proposition 3.5. For all $B \subseteq [m] \times [n]$ and for all $k = 0, 1, \ldots, m$ we have that

$H_k(B, q) = q^{\binom{k+1}{2} + \binom{m}{2}} \sum_{i=k}^{m} M_i(B, q) \cdot \left[ \frac{(n-i)!_q}{(n-m)!_q} \right] \cdot \left[ \frac{i}{k} \right]_q \cdot (-1)^{i+k} q^{-ik}$

and

$M_k(B, q) = q^{\binom{k}{2} - \binom{m}{2}} \left[ \frac{(n-m)!_q}{(n-k)!_q} \right] \sum_{i=k}^{m} H_i(B, q) \left[ \frac{i}{k} \right]_q$.

In particular,

$H_m(B, q) = M_m(B, q)$.

Proof. The relations follow by extracting the coefficients of $t^k$ and $(t; q^{-1})_k$ respectively from both sides of (3.2) using the $q$-binomial theorem (2.2) and its inverse transformation (2.3), and rearranging powers of $q$. □
Another straightforward result in the classical case is that \( \sum_i h_i(B) = n(n-1) \cdots (n-m+1) \), since both sides count the total number of maximal non-attacking rook placements on \([m] \times [n]\). The next result is the analogue in our setting.

**Corollary 3.6.** For all \( B \subseteq [m] \times [n] \), we have

\[
(q - 1)^m \sum_{i=0}^{m} H_i(B, q) = v_m = (q^n - 1)(q^n - q) \cdots (q^n - q^{m-1}).
\]

**Proof.** Set \( k = 0 \) in the second equation in Proposition 3.5 to obtain

\[
\sum_{i=0}^{m} H_i(B, q) = q^{(m)} \frac{[n]!_q}{[n-m]!_q} = \frac{q^n - 1)(q^n - q) \cdots (q^n - q^{m-1})}{(q - 1)^m},
\]

as claimed.

This proposition is particularly suggestive when \( m = n \), and the right side becomes \( |\text{GL}_n(F_q)| \) – see Question 6.1 below. We end this section with a final property that \( q \)-hit numbers share with classical hit numbers.

**Proposition 3.7.** For all \( B \subseteq [m] \times [n] \) and for all \( i \in [m] \times [n] \), the \( q \)-hit number \( H_i(B, q) \) is invariant under permuting rows and columns of \( B \).

**Proof.** The numbers \( M_i(B, q) \) are invariant under permuting rows and columns of \( B \), so the result follows immediately from (3.2).

### 3.2. Reciprocity

In this section, we use the complement identity (Theorem 2.1) to prove a reciprocity theorem for \( q \)-hit numbers analogous to the classical (1.2). We begin with a technical lemma.

**Lemma 3.8.** For any positive integers \( m \leq n \) and nonnegative integer \( i \leq m \), we have

\[
(3.3) \quad \sum_{r=0}^{m} K_r(i) \cdot (q^{n-m+1}; q)_{m-r} \cdot (t; q^{-1})_r = t^m \cdot (q^{n-m+1}; q)_{m-i} \cdot (t^{-1} q^n; q^{-1})_i.
\]

**Proof.** Denote by \( L \) the left side of the identity to be proved. Using the definition of the \( q \)-Krawtchouk polynomials and reversing the order of summation gives

\[
L = \sum_{r=0}^{m} \sum_{s=0}^{\min(r,m-i)} (-1)^s q^{ns+\binom{r-s}{2}} \sum_{i=0}^{m-i} \binom{m-i}{s} q^{m-i} \cdot (q^{n-m+1}; q)_{m-r} \cdot (t; q^{-1})_r
\]

\[
= \sum_{s=0}^{m-i} q^{ns} \binom{m-i}{s} q^{m-s-1} (q^{n-s}; q^{-1})_{m-s} (t; q^{-1})_s \sum_{r=s}^{m} q^{s-r} (q^{m-s}; q^{-1})_r (t q^{r-s}; q^{-1})_r (q^{-1}; q^{-1})_{r-s} (q^{n-s}; q^{-1})_{r-s}.
\]

The \( q \)-Chu–Vandermonde identity \([\text{GR04}, \text{II.6}]\) asserts

\[
\sum_k \frac{(a; x)_k (x^{-N}; x)_k x^k}{(c; x)_k (x; x)_k} = \frac{(c/a; x)_N a^N}{(c; x)_N}.
\]
Our next result is a

\[ \text{Corollary 3.10.} \]

For any board \( B \subseteq [m] \times [n] \), we have

\[ \sum_{i=0}^{m} (-1)^i \binom{n}{i} [n-m]_q^i \cdot M_i(B, q) \]
Proof. By the $q$-hit reciprocity (3.4) we have that $H_0(B, q) = q^{|B|} \cdot H_{m}(B, q)$. Also, by Proposition 3.5 we have that $H_{m}(B, q) = M_{m}(B, q)$. Combining these two gives the first formula. For the second formula, we use (2.8) to evaluate $M_{m}(B, q)$.

3.3. Probabilistic interpretation. In this section, we give a probabilistic interpretation to the $q$-hit polynomial $P(B, q, t)$. Consider a board $B$ contained in the rectangle $[m] \times [n]$, and let $B_k$ be the board that we get by adding $k$ rows of length $n$ below $B$, as in Figure 2. Let $F_k(B, q) \overset{\text{def}}{=} m_m(B_k, q)/q^{n(k+|B|)}$ be the probability that a random matrix with support on $B_k$ has rank $m$. There is a natural generating function

$$F_{\infty}(B, q, t) \overset{\text{def}}{=} \sum_{k=0}^{\infty} t^k F_k(B, q)$$

for these numbers, which we may think of as counting infinite matrices by the number of their rows at which they first achieve rank $m$. The main result of this section shows a close relation between $F_{\infty}$ and $P(B, q, t)$, and so provides a probabilistic interpretation for the $q$-hit polynomial.

Theorem 3.11. For any board $B$, we have that the generating function of the probabilities $F_k(B, q)$ is given by

$$F_{\infty}(B, q, t) = \frac{q^{-|B|} \cdot t^m (q - 1)^m}{(tq^{-m}; q)_{m+1}} \cdot P(B, q, q^n t^{-1}).$$

Proof. We begin by giving an alternate expression for $F_k(B, q)$. For each matrix $X$ of rank $r$ with support in $B$, the number of ways to extend $X$ to a matrix of rank $m$ with support in $B_k$ is exactly

$$q^r \cdot \#\{k \times (n - r) \text{ matrices } Y \text{ with rank}(Y) = m - r\}.$$

(This fact may be explained in more or less sophisticated language; at its simplest, it follows easily after multiplication on the right by an invertible matrix to put the $m \times n$ block in reduced column echelon form.) The second factor is an instance of the expression we denote $v_{m-r}$. It is equal to 0 if $k - m + r < 0$, and by some easy manipulations of $q$-Pochhammer symbols we may write it as

$$(-1)^{m-r} q^{(m-r)} (q^{n-r}; q^{-1})_{m-r} (q^{m-r+1}; q)_{k-m+r}.$$ 

otherwise. On the other hand, the number of choices of $X$ is simply $m_r(B, q)$, so summing over all choices of $r$ we have

$$F_k(B, q) = q^{-nk-|B|} \sum_{r=m-k}^{m} m_r(B, q) \cdot (-1)^{m-r} q^{rk+\binom{m-r}{2}} (q^{n-r}; q^{-1})_{m-r} (q^{m-r+1}; q)_{k-m+r}.$$ 

$$(q; q)_{k-m+r}.$$ 

However, this may be expressed as
Plugging this in to the definition of $F_\infty$ and rearranging yields

$$F_\infty = \sum_{k=0}^{\infty} t^k q^{-nk-|B|} \left( \sum_{r=m-k}^{m} (-1)^{m-r} m_r(B, q) q^{r(m-r)} \left( q^{n-r}; q^{-1} \right)_{m-r} \cdot \left( \frac{q^{m-r+1}; q}{q}; q \right)_{k-m+r} \right)$$

$$= q^{-|B|} \sum_{r=0}^{m} (-1)^{m-r} m_r(B, q) \left( q^{n-r}; q^{-1} \right)_{m-r} \frac{\left( \sum_{k=m-r}^{\infty} t^k q^{(r-n)k} \frac{q^{m-r+1}; q}{q}; q \right)_{k-m+r}}{\left( q^{n-r}; q \right)_{m-r+1}}.$$ 

Up to a power of $t q^{-n}$, the inner sum is equal to the summation side of (2.1) upon substituting $a \mapsto q^{r-n+1}$, $z \mapsto t q^{-n}$ and $i \mapsto k - m + r$. Thus,

$$F_\infty = q^{-|B|} \sum_{r=0}^{m} (-1)^{m-r} m_r(B, q) \left( q^{n-r}; q^{-1} \right)_{m-r} \frac{t^{m-r} q^{(m-r)+k(r-n)(m-r)}}{\left( t q^{-n}; q \right)_{m-r+1}}.$$ 

Putting this over a common denominator of $(t q^{-n}; q)_{m+1}$ and rearranging powers of $q$ and $t$ gives

$$F_\infty = \frac{q^{(m/2)-|B|-mn}}{(t q^{-n}; q)_{m+1}} \sum_{r=0}^{m} (-1)^{m-r} m_r(B, q) \cdot \frac{t^{m-r} (q^{n-m+1}; q)_{m-r} \cdot q^{-1} t^{m-r+1}}{(t q^{-n}; q)_{m+1}}$$

$$= q^{(m/2)-|B|-mn} t^m (q-1) \frac{\sum_{r=0}^{m} M_r(B, q) \cdot \left[ \frac{[n-r]! q}{[n-m]! q} \cdot (q-1)^r \cdot t^{n-r} \right]}{(t q^{-n}; q)_{m+1}} P(B, q, q^n t^{-1})$$

as claimed. 

\[ \square \]

**Remark 3.12.** One could do the same computation for any rank $R$ between $m$ and $n$, inclusive (above we calculated with $R = m$). The calculations are not substantially different; only the relatively tame factor in front changes.

**Remark 3.13.** Garsia and Remmel obtained an analogue \cite{GR86} (I.12) of $F_\infty(B, q, t)$ when $B$ is a Ferrers board by considering rook placements on an extended board. In fact, their relation can be obtained from Theorem 3.11 using the results from Section 4 below.

4. **Boards with the NE property and their complements**

4.1. **Garsia–Remmel $q$-rook numbers.** Given a placement $c$ of $r$ non-attacking rooks on a board $B$, Garsia and Remmel \cite{GR86} defined a NE inversion of $c$ to be a cell in $B$ that is not directly north or directly east of a rook in $c$. Denote by $\text{inv}^\text{NE}_B(c)$ the number of NE inversions of the rook placement $c$. This statistic gives rise to a $q$-analogue

$$R^\text{NE}_r(B, q) \overset{\text{def}}{=} \sum_c q^{\text{inv}^\text{NE}_B(c)}$$

of the rook number, where the sum is over placements $c$ of $r$ non-attacking rooks on $B$.

A board $B \subseteq [m] \times [n]$ is said to have the NE property if for all $i, i' \in [m]$ and $j, j' \in [n]$ such that $i < i'$ and $j < j'$, we have that if $(i, j), (i', j)$, and $(i', j')$ are in $B$ then $(i, j')$ is also in $B$:
These boards are convenient to work with, for the following reason: they are precisely the boards $B$ such that whenever the product $U_1 \cdot w \cdot U_2$, involving the rook placement $w$ on $[m] \times [n]$ and two upper-triangular matrices $U_1, U_2$ of respective sizes $m \times m$ and $n \times n$, has support on $B$, then $w$ is supported on $B$. This means that the Bruhat decomposition, or equivalently Gaussian elimination, plays very nicely with matrices supported on $B$. This observation may be exploited to give the following result, connecting our $q$-rook numbers with those of Garsia and Remmel.

**Theorem 4.1** ([Hag98, Thm. 1], [KLM14, Thm. 4.2]). Fix any board $B \subseteq [m] \times [n]$ with the NE property and any positive integer $r$. The number of $m \times n$ matrices over $F_q$ of rank $r$ whose support is in $B$ is

$$m_r(B, q) = (q - 1)^r q^{|B| - r} R_{r}^{\text{NE}}(B, q^{-1}).$$

In particular, we have in this case that $M_r(B, q) = q^{|B| - r} R_{r}^{\text{NE}}(B, q^{-1})$ is a polynomial in $q$ with nonnegative integer coefficients.

**Corollary 4.2.** If $B \subseteq [m] \times [n]$ has the NE property then $M_r(B, q) \in \mathbb{Z}[q]$.

**Proof.** The coefficients in the complement identity (2.7) are polynomials in $q$ with integer coefficients, so $M_r(B, q)$ is also a polynomial in $q$ with integer coefficients. \hfill \Box

As a special case, we settle a question from [LLM+11, Ques. 5.6]. Given two partitions $\lambda$ and $\mu$, the skew Ferrers board $S_{\lambda/\mu}$ consists of those elements of the board of shape $\lambda$ that do not belong to the board of shape $\mu$, when the two are aligned together.

**Corollary 4.3.** Let $S_{\lambda/\mu} \subseteq [m] \times [n]$ be a skew shape and $0 \leq r \leq m$. Then $M_r(S_{\lambda/\mu}, q) \in \mathbb{Z}[q]$.

**Proof.** The board $S_{\lambda/\mu}$ (in a suitable orientation) has the NE property and so the result follows from Corollary 4.2. \hfill \Box

We also get a polynomiality result for $q$-hit numbers.

**Corollary 4.4.** If $B \subseteq [m] \times [n]$ has the NE property then $H_i(B, q)$ and $H_i(B, q)$ are in $\mathbb{Z}[q]$.

**Proof.** This follows by combining Corollary 4.2, Proposition 3.5, and Theorem 3.9. \hfill \Box

Recall that every permutation $w$ in $S_n$ has an associated diagram

$$I_w = \{(i, w_j) \mid i < j, w_i < w_j\} \subseteq [n] \times [n].$$

It is easy to see that permutation diagrams have the NE property. Thus by Theorem 4.1 we have the following result.

**Corollary 4.5.** For any $n$ and $r$ and any permutation $w$ of size $n$, the number of $n \times n$ matrices over $F_q$ of rank $r$ whose support is in $I_w$ is a polynomial:

$$m_r(I_w, q) / (q - 1)^r \in \mathbb{N}[q].$$

---

4 When comparing the statement there, note a notational conflict: in [LLM+11], $m_q(n, B, r)$ counts matrices with support in the complement $B$. 
Part of Conjecture 1.1 (originally posed in [KLM13]) states that $m_r(T_w, q)$ is a polynomial in $q$. Combining Corollary 4.5 and the complementation formula (2.7), we can settle this part of the conjecture.

**Corollary 4.6.** For any $n$ and $r$ and any permutation $w$ of size $n$, the number of $n \times n$ matrices over $\mathbb{F}_q$ of rank $r$ whose support is in $T_w$ is a polynomial:

$$m_r(T_w, q)/(q - 1)^r \in \mathbb{Z}[q].$$

The positivity of $m_r(T_w, q)$ is discussed below in Section 5.3.

4.2. **Comparison with Garsia–Remmel $q$-hit numbers.** When $B = S_\lambda$ is a Ferrers board contained in the square $[n] \times [n]$, Garsia and Remmel defined a $q$-hit number $H^\text{GR}_i(S_\lambda, q)$. These numbers are certain polynomials in $q$, defined by the relation

$$\sum_{i=0}^{n} H^\text{GR}_i(S_\lambda, q)t^i = \sum_{i=0}^{n} R^\text{NE}_i(S_\lambda, q)[n - i]_q \prod_{k=n-i+1}^{n} (t - q^k).$$

Garsia–Remmel showed that $H^\text{GR}_i(S_\lambda, q)$ is a polynomial with nonnegative coefficients.

**Theorem 4.7** ([GR86] Thm. 2.1). For any Ferrers board $S_\lambda \subset [n] \times [n]$ and $i = 0, 1, \ldots, n$ we have that $H^\text{GR}_i(S_\lambda, q)$ is in $\mathbb{N}[q]$.

We show that for the case when $B = S_\lambda$, our $q$-hit numbers are equal to the Garsia–Remmel $q$-hit numbers, up to a power of $q$.

**Proposition 4.8.** For any Ferrers board $S_\lambda \subset [n] \times [n]$ and $i = 0, 1, \ldots, n$, we have

$$H_i(S_\lambda, q) = q^{\binom{n}{2}} H^\text{GR}_i(S_\lambda, q).$$

*Proof.* Since $S_\lambda$ has the NE property, Theorem 4.1 allows us to replace $M_i(S_\lambda, q)$ in (1.3) (the defining equation for $P(S_\lambda, q, t)$) with $q^{\mid \lambda \mid - i} R^\text{NE}_i(S_\lambda, q^{-1})$ to obtain

$$P(S_\lambda, q, t) = q^{\binom{n}{2} + \mid \lambda \mid} \sum_{i=0}^{n} R^\text{NE}_i(S_\lambda, q^{-1})q^{-i}[n - i]_q \prod_{k=0}^{i-1} (tq^{-k} - 1) \cdot q^{\binom{n-i}{2}} \cdot [n - i]_{q^{-1}} \cdot \prod_{k=0}^{i-1} (tq^{-n} - q^{-(n-k)}) .$$

By comparing the right side of this equation with that of (4.2) and rearranging powers of $q$, we see that

$$P(S_\lambda, q, t) = q^{\binom{n}{2} + \mid \lambda \mid} \sum_{i=0}^{n} H^\text{GR}_i(S_\lambda, q^{-1})q^{-ni}t^i .$$

Equating the coefficients of $t$ on both sides yields

$$H_i(S_\lambda, q) = q^{\binom{n}{2} + \mid \lambda \mid - in} H^\text{GR}_i(S_\lambda, q^{-1}).$$

Haglund [Hag98 §5] and Dworkin [Dwo98 Thm. 9.22] independently showed that the Garsia–Remmel $q$-hit numbers are symmetric, i.e.,

$$q^{\binom{n}{2} + \mid \lambda \mid - in} H^\text{GR}_i(S_\lambda, q^{-1}) = H^\text{GR}_i(S_\lambda, q).$$

5 We use the definition by Haglund [Hag98 (3)] of these $q$-hit numbers as opposed to the original definition [GR86 (2.1)]. The two definitions are equivalent up to dividing by $t^n$ and replacing $t$ by $1/t$. 
Combining (4.4) and (4.3) gives the desired expression. □

Remark 4.9. Garsia and Remmel proved Theorem 4.7 using recurrences for the generating polynomial of their \(q\)-hit numbers (their analogue of our \(P(B, q, t)\)) that preserve \(q\)-positivity. Later, Haglund and Dworkin gave (different) statistics \(\text{stat}_H\) and \(\text{stat}_D\) on permutations such that

\[
H_i^{\text{GR}}(S_\lambda, q) = \sum_{\sigma} q^{\text{stat}_H(\sigma)} = \sum_{\sigma} q^{\text{stat}_D(\sigma)},
\]

where the sum is over permutations \(\sigma \in \mathfrak{S}_n\) with \(|\sigma \cap S_\lambda| = i\). For the question of giving a combinatorial interpretation to the numbers \(H_i(B, q)\), see Section 6.1.

Remark 4.10. Dworkin also showed that his statistic \(\text{stat}_D(\cdot)\) is invariant under permuting the columns of the Ferrers boards. In contrast, by Proposition 3.7, for any board \(B \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}\) the \(q\)-hit numbers \(H_i(B, q)\) are invariant under permuting rows and columns of the board.

4.3. A \(q\)-analogue of the problème des ménages. In Example 2.3 we gave a \(q\)-analogue of the problem of derangements. Two other classical combinatorial problems involve the boards

\[
B \overset{\text{def}}{=} \{(1, 1), \ldots, (n, n), (1, 2), (2, 3), \ldots, (n-1, n)\} \quad \text{and} \quad B' \overset{\text{def}}{=} B \cup \{(n, 1)\},
\]

illustrated in Figure 3.

The rook numbers

\[
r_i(B) = \binom{2n - i}{i} \quad \text{and} \quad r_i(B') = \frac{2n}{2n - i} \binom{2n - i}{i}
\]

respectively count the number of ways of choosing \(i\) points, no two consecutive, from a linear collection of \(2n - 1\) points and a cyclic collection of \(2n\) points. The hit numbers \(h_0(B)\) and \(h_0(B')\) respectively count permutations \(w\) in \(\mathfrak{S}_n\) such that \(w_i \neq i, i+1\) and \(w_i \neq i, i+1 \pmod{n}\). These numbers have the formulas

\[
(4.5) \quad h_0(B) = \sum_{i=0}^{n} (-1)^i \binom{2n - i}{i} (n - i)! \quad \text{and} \quad h_0(B') = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n - i} \binom{2n - i}{i} (n - i)!
\]

The rook theory of \(B'\) is the famous problème des ménages \(\text{[Sta12 Ex. 2.3.3]}\).

According to Haglund (private communication), he and Rota considered the problem of finding a \(q\)-analogue of the zero hit number \(h_0(B')\). Here, we give formulas for both \(H_0(B, q) = M_n(\overline{B}, q)\) and \(H_0(B', q) = M_n(\overline{B'}, q)\), \(q\)-analogues of \(h_0(B)\) and \(h_0(B')\), respectively. We begin by computing the \(q\)-rook numbers of \(B\) and \(B'\).

Lemma 4.11. The number of \(n \times n\) matrices of rank \(i\) with support on \(B\) is

\[
m_i(B, q) = (q - 1)^i \left(\binom{2n - 1 - i}{i - 1} q^{i-1} + \binom{2n - 1 - i}{i} q^i\right).
\]
Proof. Let $B^*$ be the reflection of $B$ through a horizontal axis, that is, $B^* = \{(n,1),(n-1,2),\ldots,(1,n),(n,2),\ldots\}$. By inspection, $B^*$ has the NE property (in a vacuous way). Since $q$-rook numbers are invariant under permutations of the board, we have $m_i(B, q) = m_i(B^*, q)$. Since $B^*$ has the NE property, we have by Theorem 4.1 that $m_i(B^*, q) = (q-1)^i q^{2n-1-i} R_{i}^{\text{NE}}(B^*, q^{-1})$. Thus, it suffices to compute the Garsia–Remmel $q$-rook number of $B^*$.

Of the $\binom{2n-1}{i}$ placements of $i$ rooks on $B^*$, exactly $\binom{2n-1}{i-1}$ include a rook in position $(1,n)$, while the remaining $\binom{2n-1}{i-1}$ leave this cell empty. By the definition of NE inversion given at the beginning of Section 4, each rook placed on $B^*$ kills two potential NE inversions, except if the rook is placed on $(1,n)$, in which case it kills only one. Thus, a rook placement of $i$ rooks on $B^*$ has $2n-2i$ NE inversions if it includes $(1,n)$, and $2n-2i-1$ NE inversions if not. Combining these statements with the preceding paragraph gives the desired result. □

Unlike $B$, the board $B'$ does not have a rearrangement with the NE property for $n \geq 3$. Thus, we use a different technique to compute the $q$-rook numbers for this board.

**Lemma 4.12.** The number of $n \times n$ matrices of rank $i$ with support on $B'$ is

$$m_i(B', q) = (q-1)^i \left( q^i \frac{2n}{2n-i} \binom{2n-i}{i} + G_{i,n}(q) \right),$$

where $G_{n,n}(q) \defeq -q(q-1)^{n-1}$, $G_{n-1,n}(q) \defeq (q-1)^n$, and $G_{i,n}(q) \defeq 0$ if $i < n - 1$.

**Proof.** Denote $B'_n \defeq B$ and $B''_n \defeq B'$ to make the dependence on the size $n$ explicit. We proceed by induction. When $n = 2$, we have that $B'_2 = [2] \times [2]$ is the entire square, and it is easy to check that the $q$-rook numbers in this case are $1 = \frac{4}{4} \binom{4}{0}$, $(q-1)(q+1)^2 = (q-1)q \cdot \frac{4}{3} \binom{3}{1} + (q-1)^3$, and $(q-1)^2 q(q+1) = (q-1)^2 q^2 - \frac{1}{2} \binom{4}{2} - q(q-1)^3$.

If $n > 2$, we use recurrences from [KLM14, §3.2.3]; for completeness, we sketch the argument here. The set of matrices of rank $i$ with support on $B'_n$ may be written as a disjoint union of three pieces: those with $(n,1)$ entry equal to 0, those, with $(n,1)$ entry nonzero but $(n,n)$ entry equal to 0, and those with both $(n,1)$ and $(n,n)$ entry nonzero. The first of these subsets is the set of matrices of rank $i$ with support on $B_n$ and so has size $m_i(B_n, q)$. By using Gaussian elimination to kill the $(1,1)$-entry, we have that a matrix with the correct support belongs to the second subset if and only if its $[n-1] \times [2,n]$-submatrix is of rank $i-1$. In this case, the submatrix is the transpose of a matrix with support on $B_{n-1}$, and so the second subset has size $(q-1)^i m_{i-1}(B_{n-1}, q)$. Finally, a matrix with the correct support belongs to the third subset if and only if, after using Gaussian elimination to kill the $(1,1)$-entry, the $[n-1] \times [2,n]$-submatrix is of rank $i-1$ with support on the transpose of $B'_{n-1}$. Thus, the third subset has size $(q-1)^2 \cdot m_{i-1}(B'_{n-1}, q)$. Using Lemma 4.11 and the inductive hypothesis, it follows that

$$\frac{m_i(B'_n, q)}{(q-1)^i} = m_i(B_n, q) + (q-1)^i q^{\cdot} m_{i-1}(B_{n-1}, q) + (q-1)^2 \cdot m_{i-1}(B'_{n-1}, q),$$

$$= q^i \frac{2n}{2n-i} \binom{2n-i}{i} + G_{i,n}(q),$$

as desired. □

With these computations in hand, it is straightforward to give a $q$-analogue of the ménage problem.
Theorem 4.13 (q-analogue of ménages). The number of invertible $n \times n$ matrices over the finite field with $q$ elements with zeros on the main and upper diagonal is

$$m_n(B,q) = (q-1)^n q^{(n)_2} - 2n \sum_{i=0}^{n} (-1)^i q^i \left( q \binom{2n-i}{i} + \binom{2n-i}{i-1} \right) [n-i]_q!.$$ 

For $n \geq 2$, the number of invertible $n \times n$ matrices over the finite field with $q$ elements with zeros on the main and upper diagonal and on the entry $(n,1)$ is

$$m_n(\overline{B},q) = (q-1)^n q^{(n)_2} - 2n \left( (-1)^{n-1}(q-1)^{n-1}(2q-1) + \sum_{i=0}^{n} (-1)^i q^i \frac{2n}{2n-i} \binom{2n-i}{i} [n-i]_q! \right).$$

Proof. Applying (2.8) to Lemmas 4.11 and 4.12 in the case $m = n$ and rearranging powers of $q$ and $q-1$ gives the result. \qed

The preceding result should be compared with (4.3). Observe that in the last formula, the anomalous term $(-1)^{n-1}(q-1)^{n-1}(2q-1)$ vanishes modulo $q-1$, in agreement with (2.4) and Proposition 3.3. As with the $q$-analogue of derangements (Example 2.3), these polynomials in general do not have positive coefficients.

5. Deletion-contraction for complements of boards with the NE property

In this section, we give deletion-contraction recurrence relations to compute the matrix count $M_r(B,q)$ and $q$-hit polynomial $P(B,q,t)$ when the board $B$ is the complement of a shape with the NE property. Given a board $B$, say that an element $(i,j) \in B$ is a SW corner if there is no other element $(i',j') \in B$ such that $i' \geq i$ and $j' \leq j$.

5.1. General relations. Given a board $B \subseteq [m] \times [n]$ with the NE property, let $\square$ be a SW corner of $B$. Denote by $B \setminus \square$ the board obtained by deleting $\square$ from $B$, and denote by $B/\square$ the board obtained by deleting the entire row and column of $\square$. For purposes of taking complements, we think of $B/\square$ as living inside the smaller rectangle $[m-1] \times [n-1]$. The following result of Dworkin gives a deletion-contraction relation for the Garsia–Remmel $q$-rook numbers.

Proposition 5.1 ([Dwo98 Thm. 6.10]). For any board $B \subseteq [m] \times [n]$ and SW corner $\square$ of $B$,

$$R_r^\text{NE}(B,q) = q \cdot R_{r-1}^\text{NE}(B \setminus \square, q) + R_{r-1}^\text{NE}(B/\square, q).$$

Corollary 5.2. For any board $B \subseteq [m] \times [n]$ with the NE property and any SW corner $\square$ of $B$,

$$M_r(B,q) = M_r(B \setminus \square, q) + q^{[B]-[\square]-1} \cdot M_{r-1}(B/\square, q).$$

Proof. If $B$ has the NE property and $\square$ is a SW corner then both $B \setminus \square$ and $B/\square$ have the NE property. So using Theorem 4.1 we can rewrite the deletion-contraction in Proposition 5.1 in terms of $M_r$. \qed

The next result shows how to pass this deletion-contraction relation through the complement identity (2.7) to produce a recurrence counting matrices on complements of boards with the NE property. In the proof, it is necessary to consider simultaneously $q$-Krawtchouk polynomials defined with different parameters $m,n$; thus, for the duration of this section we introduce the notation

$$K_{r,m,n}(i) \overset{\text{def}}{=} \sum_s (-1)^{r-s} q^{ns+\binom{r-s}{2}} \binom{m-s}{r-s} \binom{m-i}{s}_q.$$
for the polynomial previously denoted $K_r(i)$. By applying the $q$-Pascal recurrence
\[
\begin{bmatrix} k \\ \ell \end{bmatrix}_q = \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_q + q^\ell \cdot \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_q
\]
to the first $q$-binomial coefficient appearing in (5.2), it is not hard to show that
\[
K_r^{m,n}(j + 1) = q^r \cdot K_r^{m-1,n-1}(j) - q^{r-1} \cdot K_r^{m-1,n}(j).
\]

**Corollary 5.3.** For any board $B \subseteq [m] \times [n]$ with the NE property and any SW corner $\Box$ of $B$,
\[
q \cdot M_r(\overline{B}, q) = M_r(B \setminus \Box, q) + q^r(q - 1) \cdot M_r(B/\Box, q) - q^{r-1} \cdot M_{r-1}(B/\Box, q).
\]

**Proof.** Applying the complement identity (2.7) to the matrix count $m_r(B, q)$ gives
\[
m_r(\overline{B}, q) = \frac{1}{q^{|B|}} \sum_{i=0}^{m} K_r^{m,n}(i) \cdot m_i(B, q).
\]
Choose a SW corner $\Box$ of $B$. Then applying (5.1) to $m_i(B, q) = (q - 1)^i \cdot M_i(B, q)$ gives
\[
m_r(B, q) = q^{-|B|} \sum_{i=0}^{m} K_r^{m,n}(i) \cdot m_i(B \setminus \Box, q) + \frac{q - 1}{q^{|B/\Box|} + 1} \sum_{i=0}^{m} K_r^{m,n}(i) \cdot m_{i-1}(B/\Box, q).
\]
The first summand simplifies by (2.7) (using the board $B \setminus \Box \subseteq [m] \times [n]$) to $\frac{1}{q} m_r(\overline{B \setminus \Box}, q)$. For the second summand, we have by (5.3) that
\[
\sum_{i=0}^{m} K_r^{m,n}(i) m_{i-1}(B/\Box, q) = \sum_{j=0}^{m-1} K_r^{m,n}(j + 1) m_j(B/\Box, q)
\]
\[
= q^r \sum_{j=0}^{m-1} K_r^{m-1,n-1}(j) m_j(B/\Box, q) - q^{r-1} \sum_{j=0}^{m-1} K_r^{m-1,n}(j) m_j(B/\Box, q).
\]
Both of the sums in this last expression can again be transformed using (2.7), the first using the board $B/\Box \subseteq [m - 1] \times [n - 1]$ at rank $r$, and the second using the board $B/\Box \subseteq [m - 1] \times [n - 1]$ and rank $r - 1$. This gives
\[
\sum_{i=0}^{m} K_r^{m,n}(i) \cdot m_{i-1}(B/\Box, q) = q^{r+|B/\Box|} \cdot m_r(B/\Box, q) - q^{r-1+|B/\Box|} \cdot m_{r-1}(B/\Box, q).
\]
Putting everything together, we get
\[
m_r(\overline{S}, q) = \frac{1}{q} \cdot m_r(\overline{B \setminus \Box}, q) + (q - 1) \left( q^{r-1} \cdot m_r(\overline{B/\Box}, q) - q^{r-2} \cdot m_{r-1}(\overline{S/\Box}, q) \right).
\]
Multiplying by $q(q - 1)^{-r}$ on both sides gives the desired result. \qed

We can also transform these deletion-contraction relations in terms of the $q$-hit polynomial.

**Corollary 5.4.** For any board $B \subseteq [m] \times [n]$ with the NE property and any SW corner $\Box$ of $B$,
\[
P(B, q, t) = P(B \setminus \Box, q, t) + q^{m+|B| - |B/\Box| - 2} (t - 1) \cdot P(B/\Box, q, q^{-1} t)
\]
and
\[
q \cdot P(\overline{B}, q, t) = P(\overline{B \setminus \Box}, q, t) - q^{m-1} (t - q^n) \cdot P(\overline{B/\Box}, q, t).
\]
Proof. To show the first relation, we apply (5.1) to the definition (3.2) to get
\[
P(B, q, t) = q^n \sum_{r=0}^{m} M_r(B \setminus \square, q) \left[ \frac{(n-r)!}{(n-m)!} \right] q^{-r} (-1)^r (t; q^{-1})_r +
\]
\[+ q^n \sum_{i=0}^{m-1} M_{r-1}(B/\square, q) \left[ \frac{(n-r)!}{(n-m)!} \right] q^{-1} (-1)^r (t; q^{-1})_r.
\]
The first sum equals \( P(B \setminus \square, q, t) \). For the second sum, changing the index of summation to \( i = r - 1 \) and factoring out the term \( t - 1 \) produces
\[
q^n \sum_{i=0}^{m-1} M_i(B/\square, q) \left[ \frac{(n-1)!}{(n-1) - (m-1)!} \right] q^{-1} (-1)^i (tq^{-1}; q^{-1})_i =
\]
\[
q^{B-|B/\square|-m-2} (t-1) P(B/\square, q, t^{-1}).
\]
Substituting back in gives the desired relation.

To show the second relation, we use the reciprocity formula (3.5) to rewrite the first relation in terms of \( P(\overline{B}, q, t) \):
\[
q^{-|\overline{B}|} t^m \cdot P(\overline{B}, q, q^n t^{-1}) =
\]
\[
q^{-|\overline{B}\setminus\square|} t^m \cdot P(\overline{B} \setminus \square, q, q^n t^{-1}) + q^{B-|B/\square|+m-2-|\overline{B}/\square|} (t-1) t^{m-1} \cdot P(\overline{B}/\square, q, q^n t^{-1}).
\]
Dividing both sides by \( q^{-|\overline{B}|} t^m \) and substituting \( t \mapsto q^n t^{-1} \) gives the result. \( \square \)

5.2. Permutation diagrams and deletion-contraction. In this section, we study deletion-contraction on diagrams \( I_w \subseteq \mathfrak{S}_n \) of permutations \( w \) of size \( n \). We show that the deletion and contraction boards from Corollaries 5.2, 5.3 are actually diagrams of related permutations.

For \( w \in \mathfrak{S}_n \), let \( \square = (i, w_j) \) be a SW corner of \( I_w \subseteq \mathfrak{S}_n \). By definition, there is no other element \( (i', w_{j'}) \) in \( I_w \) (so \( i' < j' \), \( w_{i'} < w_{j'} \)) with \( i \leq i' \) and \( w_{j'} \leq w_j \). In particular, in this case none of the entries \( w_{i+1}, \ldots, w_{i-1} \) have values between \( w_i \) and \( w_j \), and so the permutation
\[
w \cdot (i, j) = w_i \cdots w_{i-1} w_j w_{i+1} \cdots w_j \cdots w_n
\]
has exactly one fewer coinversion than \( w \). Next, we show that in fact, \( I_{w \cdot (i, j)} \) arises by deleting a single cell from \( I_w \).

Proposition 5.5 (deletion). For any \( w \in \mathfrak{S}_n \) and any SW corner \( \square = (i, w_j) \) of \( I_w \), we have \( I_w \setminus \square = I_{w \cdot (i, j)} \).

Proof. Abbreviate \( w' \) \( \overset{\text{def}}{=} \) \( w \cdot (i, j) \). Consider a box \( (a, w_b) \) of \( I_w \) corresponding to a coinversion between entries \( (a, w_a) \) and \( (b, w_b) \) of \( w \) (so necessarily \( a < b \) and \( w_a < w_b \)). If \( a = i \) and \( b = j \) then obviously this box is absent in \( I_{w'} \); we show that all other boxes in \( I_w \) are in \( I_{w'} \).

First, if \( \{a, b\} \) is disjoint from \( \{i, j\} \), then \( (a, w_a) = (a, w'_a) \) and \( (b, w_b) = (b, w'_b) \) are entries of \( w' \) and so \( (a, w_b) \) belongs to \( I_{w'} \).

Second, suppose that \( a = j \), and so \( b > j \) and \( w_b > w_j = w_i \). Then the entries \( (a, w'_a) = (j, w_i) \) and \( (b, w'_b) = (b, w_b) \) of \( w' \) form a coinversion and so \( (a, w_b) \) belongs to \( I_{w'} \).

Third, suppose that \( a = i \) (and so \( b > i \)). Since \( \square \) is a SW corner of \( I_w \), we must have \( w_b > w_j \). Then the entries \( (i, w'_i) = (i, w_j) \) and \( (b, w'_b) = (b, w_b) \) of \( w' \) form a coinversion and so \( (a, w_b) \) belongs to \( I_{w'} \).
Given a permutation $w \in \mathcal{S}_n$ whose diagram $I_w$ has a SW corner in position $(i, w_i)$, let $v \in \mathcal{S}_{n-1}$ be the permutation order-isomorphic to 

$$w_1 \cdots w_{i-1}w_{i+1} \cdots w_{j-1}w_iw_{j+1} \cdots w_n.$$ 

For example, the diagram of the permutation $w = 139547628 \in \mathcal{S}_9$ has a SW corner in position $(5, 7) = (5, 6)$ (see Figure 4), and $v = 13856427 \in \mathcal{S}_8$ is the permutation order-isomorphic to $139547628$.

**Proposition 5.7 (contraction).** For any $w \in \mathcal{S}_n$ and any SW corner $\square = (i, w_j)$ of $I_w$, we have $I_w / \square = I_v$, where $v$ is the permutation defined in Definition 5.6.

**Proof.** Let $r$ and $c$ be the order-preserving bijections $r : [n] \setminus \{i\} \to [n-1]$ and $c : [n] \setminus \{w_j\} \to [n-1]$, so that $f \overset{\text{def}}{=} r \times c$ is the natural bijection between cells of $[n] \times [n]$ not in the row or column of $\square$ and cells of $[n-1] \times [n-1]$. We seek to show that $f$ restricts to a bijection between the relevant cells of $I_w$ and those of $I_v$. Observe that by the definition of $v$, we have $v_{r(a)} = c(w_a)$ for all $a \neq j$, while $v_{r(j)} = c(w_i)$.

Consider a box $(a, w_b)$ of $I_w / \square$ corresponding to a coinversion between entries $(a, w_a)$ and $(b, w_b)$ of $w$ (so $a < b$, $w_a < w_b$, $a \neq i$, $b \neq j$). We have three cases.

First, assume $a \neq j$ and $b \neq i$. Since $r$ and $c$ are order-preserving, we have $r(a) < r(b)$ and $v_{r(a)} = c(w_a) < c(w_b) = v_{r(b)}$. Therefore $f(a, w_b) = (r(a), v_{r(b)})$ belongs to $I_v$. Conversely, if $(r(a), v_{r(b)}) \in I_v$ is such that $a \neq j$ and $b \neq i$ then $f^{-1}(r(a), v_{r(b)}) = (a, w_b) \in I_w$.

Second, suppose $b = i$. Then $a < i < j$ and $w_a < w_i$. Thus $r(a) < r(j)$ and $v_{r(a)} = c(w_a) < c(w_i) = v_{r(j)}$, and therefore $f(a, w_i) = (r(a), v_{r(j)})$ belongs to $I_v$. Conversely, suppose $(r(a), v_{r(j)}) \in I_v$, so that $r(a) < r(j)$ and $c(w_a) = v_{r(a)} < v_{r(j)} = c(w_i)$. Since $\square = (i, w_j)$ is a SW corner in $I_w$, all of the values $w_{i+1}, \ldots, w_{j-1}$ must be larger than $w_j$, and so also larger than $w_i$. Therefore, $a$ is not equal to any of $i+1, \ldots, j-1$, so $a < i$. Thus $f^{-1}(r(a), v_{r(j)}) = (a, w_i)$ belongs to $I_w$.

Fourth, the cases $b = i$ and $b = j$ are symmetric with the last two cases after reflecting everything across the main antidiagonal.

Finally, since $I_w$ has strictly more coinversions than $I_{w'}$, it follows that these are all the elements of $I_{w'}$, as desired. \hfill $\square$

Next, we show that contractions of permutation diagrams are also permutation diagrams.

**Definition 5.6.** Given a permutation $w \in \mathcal{S}_n$ whose diagram $I_w$ has a SW corner in position $(i, w_j)$, let $v \in \mathcal{S}_{n-1}$ be the permutation order-isomorphic to $w_1 \cdots w_{i-1}w_{i+1} \cdots w_{j-1}w_iw_{j+1} \cdots w_n$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure4.png}
\caption{Left: The diagram of the permutation $w = 139547628$, with SW corner $\square = (5, 6)$ involving the entries $(i, w_i) = (5, 4)$ and $(j, w_j) = (7, 6)$. Center: the contracted diagram. Right: the diagram of the permutation $v = 13856427$.}
\end{figure}
Finally, suppose \( a = j \). Then \( b > j \) and \( w_b > w_j > w_i \). Thus \( r(b) > r(j) \) and \( v_{r(b)} = c(w_b) > c(w_j) = v_{r(j)} \), and therefore \( f(j, w_b) = (r(j), v_{r(b)}) \) belongs to \( I_v \). Conversely, suppose \( (r(j), v_{r(b)}) \in I_v \), so that \( r(j) < r(b) \) and \( c(w_j) = v_{r(j)} < v_{r(b)} = c(w_b) \). Since \( \square = (i, w_j) \) is a SW corner in \( I_w \), all of the values \( w_{i+1}, \ldots, w_{j-1} \) must be smaller than \( i \), and so also smaller than \( j \). Therefore, \( b \) is not equal to any of these values, so \( b > j \). Thus \( f^{-1}(r(j), v_{r(b)}) = (i, w_b) \) belongs to \( I_w \).

The three cases cover every coinversion of \( w \) not in row \( i \) or column \( w_j \), and every coinversion of \( v \), and so \( f \) is a bijection between these two sets. The result follows immediately. \(\square\)

**Corollary 5.8.** For any permutation \( w \in S_n \), let \( \square = (i, w_j) \) be any SW corner of \( I_w \) and let \( v \) be the permutation defined in Definition 5.6. For any \( 1 \leq r \leq n \), we have

\[
M_r(I_w, q) = M_r(I_{w-(i,j)}, q) + q^{n-2-\ell(w)+\ell(v)} M_{r-1}(I_v, q).
\]

**Proof.** Apply (5.1) in the case \( m = n \) and \( B = I_w \), using Propositions 5.5, 5.7 to express \( I_w \setminus \square \) and \( I_w/\square \) as \( I_{w-(i,j)} \) and \( I_v \) respectively. Since \( |I_w| = \binom{n}{2} - \ell(w) \), we have \( |I_w| - |I_v| - 1 = n - 2 - \ell(w) + \ell(v) \). \(\square\)

**Corollary 5.9.** For any permutation \( w \in S_n \), let \( \square = (i, w_j) \) be any SW corner of \( I_w \) and let \( v \) be the permutation defined in Definition 5.6. We have

\[
q \cdot M_n(T_w, q) = M_n(T_{w-(i,j)}, q) + q^{n-1} \cdot M_{n-1}(I_v, q).
\]

**Proof.** Apply Corollary 5.3 in the case \( m = n \) and \( B = I_w \), using Propositions 5.5, 5.7 to express \( I_w \setminus \square \) and \( I_w/\square \) as \( I_{w-(i,j)} \) and \( I_v \) respectively. \(\square\)

The preceding result is particularly nice in the full-rank case.

**Corollary 5.10.** For any permutation \( w \in S_n \), let \( \square = (i, w_j) \) be any SW corner of \( I_w \) and let \( v \) be the permutation defined in Definition 5.6. We have

\[
q \cdot M_n(T_w, q) = M_n(T_{w-(i,j)}, q) - q^{n-1} \cdot M_{n-1}(I_v, q).
\]

Finally, we can rewrite these relations in terms of the \( q \)-hit polynomial \( P \).

**Corollary 5.11.** For any permutation \( w \in S_n \), let \( \square = (i, w_j) \) be any SW corner of \( I_w \) and let \( v \) be the permutation defined in Definition 5.6. We have

\[
P(I_w, q, t) = P(I_{w-(i,j)}, q, t) + q^{2n-3+\ell(v)-\ell(w)}(t-1) \cdot P(I_v, q, q^{-1}t)
\]

and

\[
q \cdot P(I_w, q, t) = P(I_{w-(i,j)}, q, t) - q^{n-1}(t-q^n) \cdot P(I_v, q, t).
\]

**Proof.** Combine Corollary 5.4 for \( m = n \), \( B = I_w \) with Propositions 5.5 and 5.7. \(\square\)

### 5.3. Failures of positivity.

There are no permutations in \( w \in S_n \) for \( n < 9 \) for which any coefficient of \( M_r(T_w, q) \) is negative for any \( r \). However, for \( n \geq 9 \), there are counterexamples to the positivity aspect of Conjecture [11].

**Example 5.12.** Let \( w = 789563412 \in S_9 \); its diagram is shown in Figure 5(a). Applying Corollary 5.9 (or the formula in [KLM14] Prop. 3.1) gives

\[
M_1(T_w, q) = 24q^{11} - 4q^{10} + 10q^9 + 9q^8 + 8q^7 + 7q^6 + 6q^5 + 5q^4 + 4q^3 + 3q^2 + 2q + 1.
\]
Figure 5. The diagrams of the two permutations $789563412 \in S_9$ (left) and $68910457123 \in S_{10}$ (right).

There are three other permutations in $S_9$ whose diagrams are trivial rearrangements of the previous example (namely, $895673412$, $896734512$, and $896745123$). These are the only permutations in $S_9$ for which $M_r(T_w, q)$ has some negative coefficients, and they only have negative coefficients in rank $r = 1$.

Example 5.13. Let $w = 68910457123 \in S_{10}$; its diagram is shown in Figure 5(b). Applying Corollary 5.10 gives

$$M_{10}(T_w, q) = q^{77} + 9q^{76} + 44q^{75} + \cdots + 2q^{48} - 8q^{47} - q^{46} + q^{45}.$$ 

In total, there are 37 permutations $w$ in $S_{10}$ for which $M_{10}(T_w, q)$ has negative coefficients and 303 for which $M_1(T_w, q)$ has negative coefficients, including 11 permutations for which both polynomials have negative coefficients. The coefficients of $M_r(T_w, q)$ are nonnegative for all $w \in S_{10}$ if $2 \leq r \leq 9$.

These calculations are also sufficient to disprove another reasonable conjecture: that positivity is a pattern property. Indeed, the permutation $w = 58910673412 \in S_{10}$ has all coefficients of $M_r(T_w, q)$ nonnegative for all ranks $r$, but it contains the permutation $789563412$ as a pattern.

6. Remarks and open problems

6.1. Combinatorial interpretation of $q$-hit numbers. The main problem raised by our work is to give a combinatorial interpretation to the $q$-hit number $H_i(B, q)$.

Question 6.1. Is there a nice choice of a set $S (= S(i, B, q))$ such that the cardinality $|S|$ is equal to the $q$-hit number $H_i(B, q)$?

Corollary 3.6 expressing the number of full-rank matrices of a given shape as a sum of hit numbers (times an easy-to-understand factor), suggests that a nice description would be in terms of a partition of the set of full-rank $m \times n$ matrices.

For the case of Ferrers boards $S_\lambda$ and $n = m$, Haglund [Hag98] gave an interpretation for the Garsia–Remmel $q$-hit numbers in terms of matrices over finite fields. (Recall that in this case the Garsia–Remmel $q$-hit numbers agree with our $q$-hit numbers by Proposition 4.8.) This interpretation roughly goes like this: by Proposition 3.5 the $k$th $q$-hit number can be written as

$$(q - 1)^n q^{-\binom{n}{2}} H_k(S_\lambda, q) = \sum_{i=k}^n m_q(i, S_\lambda) \cdot (q^n - q^i)(q^{n-1} - q^i) \cdots (q^{i+1} - q^i) \times$$

$$\times \left[ \begin{array}{c} i \cr k \end{array} \right]_q (-1)^{i+k} q^{\binom{k+1}{2} - i(n+k-i)}.$$
View the right side as counting the following replacement procedure [Hag98, §2]:
1. start with a matrix $A$ of rank $i$ and support in $S_\lambda$,
2. after doing row elimination on $A$ in a specified order, replace, with certain rules, the $n-i$ rows without pivots to obtain a matrix $A'$ of full rank,
3. assign a certain signed weight to the matrix $A'$, and
4. do row elimination on $A'$ and record the number of pivots $j$ that do not belong to $B$.

Haglud showed that weighted contribution of such matrices with $j < n-k$ is zero whereas the contribution of those with $j = n-k$ is one.

Unfortunately, we were unable to extend this elimination procedure to other families of boards. One possible (but so far unsuccessful) approach is described in the next remark.

Remark 6.2. Equation (1.1) has a natural double-counting proof that one might try to emulate to find an interpretation for $H_i(B, q)$. Consider (3.2) in the case $m = n$; rearranging powers of $q$ and $q - 1$, we have

$$(q - 1)^n \sum_{i=0}^n H_i(B, q) t^i = \sum_{i=0}^n m_i(B, q) \prod_{k=i}^{n-1} (q^n - q^k) \cdot \prod_{j=0}^{i-1} (t - q^j).$$

When $t = q^N$, we can view the right side as counting triples $(A, \beta, \phi)$ where $A$ is a matrix over $\mathbb{F}_q$ with support in $B$, $\beta$ is an ordered relative basis for $\mathbb{F}_q^n$ over the rowspace of $A$, and $\phi$ is an injective linear map from the rowspace to $\mathbb{F}_q^N$.

We also offer a weak conjecture that is certainly a precondition for an affirmative answer to Question 6.1

Conjecture 6.3. Given any board $B \subseteq [m] \times [n]$, rank $r$, and prime power $q$, the $q$-hit number $H_r(B, q)$ is nonnegative.

6.2. Polynomiality and positivity of $q$-hit numbers. Just as is the case for the $q$-rook number $M_r(B, q)$, the $q$-hit number $H_i(B, q)$ need not be a polynomial in $q$. In fact, by Proposition 3.5 for a fixed board $B$, we have that all $M_r(B, q)$ are polynomial in $q$ if and only if all $H_i(B, q)$ are polynomial.

Example 6.4. In [Ste98], Stembridge found a set $F \subseteq [7] \times [7]$ such that $M_7(F, q)$ is not a polynomial in $q$. This set $F$ is the incidence matrix of the Fano plane (see Figure 6). Stembridge found that

$$M_7(F, x + 1) = H_7(F, x + 1) = (x + 1)^3 (x^{11} + 17x^{10} + 135x^9 + 650x^8 + 2043x^7 + 4236x^6 + 5845x^5 + 5386x^4 + 3260x^3 + 1236x^2 + 264x + 24 - Z_2x^6)$$

where $x \overset{\text{def}}{=} q - 1$ and $Z_2$ is zero or one depending on whether $q$ is even or odd.

Even when the $M_r(B, q)$ are polynomials with nonnegative coefficients, the polynomial $H_i(B, q)$ might have negative coefficients; see Example 3.2(iii). Thus it is natural to ask about positivity if, e.g., we restrict to permutation diagrams $I_w$ (where everything is polynomial by Corollary 4.6).

Question 6.5. For which permutations $w$ do the $q$-hit numbers $H_i(I_w, q)$ have positive coefficients?

If $w \in S_n\lambda$ avoids the permutation pattern 3412 then $I_w$ is a rearrangement of a Ferrers board [Man01, Prop. 2.2.7]. By Proposition 4.8 and Remark 4.10 for such permutations $w$, the $q$-hit
numbers $H_i(T_w, q)$ equal the Garsia–Remmel $q$-hit number $H_i^{\text{GR}}(S_\lambda, q)$ of the associated Ferrers board, and so by Theorem 4.7, $H_i(T_w, q)$ is a polynomial with positive coefficients. For $n \leq 9$, these are the only such permutations; it could be interesting to prove that this is true for all $n$.

**Example 6.6.** For $w = 3412$ we have that

\[
H_0 = H_1 = 0, \quad H_2 = q^{11}(q + 1), \quad H_3 = q^7(2q^4 + 4q^3 + 3q^2 - 1), \quad H_4 = q^6(q^4 + 3q^3 + 5q^2 + 4q + 1).
\]

Since $H_r(T_w, q)$ is a polynomial in $q$, one may also make a strengthened version of Conjecture 6.3 in this case.

**Question 6.7.** Is it true for every permutation $w$ and every rank $r$ that the polynomial $H_r(T_w, x+1)$ has positive coefficients in the variable $x$?

The answer is affirmative for $n \leq 8$. The corresponding question for the matrix counts $M_r(T_w, q)$ is posed in [KLM14, Rmk. 3.4].

### 6.3. Positivity for 123-avoiding permutations

In [LLM+11], the motivating example was the board $B = ([n] \times [n]) \setminus \{(1, 1), \ldots, (n, n)\}$; an elegant alternating formula was given for the matrix count $M_n(B, q)$ (see Example 2.3). This formula is not positive. Let $v \in S_{2n}$ be the permutation $v \overset{\text{def}}{=} (2n-1)(2n)(2n-3)(2n-2) \cdots 563412$. The diagram $I_v$ consists of $n$ boxes on the diagonal, and by Corollary 2.2 one can write down a similar alternating sum for the associated matrix count:

\[
M_{2n}(I_v, q) = q^{2n(n-1)} \sum_{i=0}^{n} (-1)^i \binom{n}{i} [2n - i]!q.
\]

It is easy to check on a computer that for $n \leq 40$ these polynomials have nonnegative coefficients.

**Conjecture 6.8.** For $v = (2n-1)(2n)(2n-3)(2n-2) \cdots 3412$ we have that $M_{2n}(T_v, q)$ is in $\mathbb{N}[q]$.

The diagonal board above is an example of a skew Ferrers board. Any skew Ferrers board can be obtained (with our conventions for $I_w$) from the diagram of a 123-avoiding permutation [BJS93].

This family, to which $v$ belongs, contains $\frac{1}{n+1} \binom{2n}{n}$ permutations in $S_n$. Calculations for $n \leq 14$ suggest that for every such permutation, $M_n(w, q)$ has nonnegative coefficients. This suggests the following strengthening of Conjecture 6.8.

**Conjecture 6.9.** For every 123-avoiding permutation $w$ in $S_n$ we have that $M_n(T_w, q)$ is in $\mathbb{N}[q]$.

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