IMPROVED BILINEAR STRICHARTZ ESTIMATES WITH APPLICATION TO THE WELL-POSEDNESS OF PERIODIC GENERALIZED KDV TYPE EQUATIONS

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Abstract. We improve our previous result ([31]) on the Cauchy problem for one dimensional dispersive equations with a quite general nonlinearity in the periodic setting. Under the same hypotheses that the dispersive operator behaves for high frequencies as a Fourier multiplier by $i|\xi|^\alpha \xi$, with $1 \leq \alpha \leq 2$, and that the nonlinear term is of the form $\partial_x f(u)$ where $f$ is a real analytic function whose Taylor series around the origin has an infinite radius of convergence, we prove the unconditional LWP of the Cauchy problem in $H^s(\mathbb{T})$ for $s \geq 1 - \frac{\alpha}{4}$ with $s > 1/2$. It is worth noticing that this result is optimal in the case $\alpha = 2$ (generalized KdV equation) in view of the restriction $s > 1/2$ for the continuous injection of $H^s(\mathbb{T})$ into $L^\infty(\mathbb{T})$. Our main new ingredient is the replacement of improved Strichartz estimates by improved bilinear estimates in the treatment of the worst resonant interactions. Such improved bilinear estimates already appeared in the work of Hani [11] in the context of Schrödinger equations on a compact manifold. Finally note that this enables us to derive global existence results for $\alpha \in [4/3, 2]$.

1. Introduction

We continue our study ([31]) of the Cauchy problem associated with dispersive equation of the form

$$
\partial_t u + L_{\alpha+1} u + \partial_x (f(u)) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \quad (1.1)
$$

$$
u(0, x) = u_0(x), \quad x \in \mathbb{T}, \quad (1.2)
$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, under the two following hypotheses on $L_{\alpha+1}$ with $1 \leq \alpha \leq 2$ and $f$.

Hypothesis 1. $L_{\alpha+1}$ is the Fourier multiplier operator by $-ip_{\alpha+1}$ where $p_{\alpha+1} \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{0\})$ is a real-valued odd function satisfying, for some $\xi_0 > 0$, $p'_{\alpha+1}(\xi) \sim \xi^\alpha$ and $p''_{\alpha+1}(\xi) \sim \xi^{\alpha-1}$ for all $\xi \geq \xi_0$.

*Key words and phrases.* generalized KdV equation, nonlinear dispersive equation, well-posedness, unconditional uniqueness, energy method.
Hypothesis 2. $f : \mathbb{R} \to \mathbb{R}$ is an analytic function whose Taylor series around the origin has an infinite radius of convergence.

Recall that this class of equations contains the famous generalized Korteweg-de Vries (gKdV) and Benjamin-Ono equation (gBO) that correspond respectively to the case $\alpha = 2$ and $\alpha = 1$ and read respectively as

$$\partial_t u + \partial_x^3 u + \partial_x(f(u)) = 0$$

and

$$\partial_t u - \mathcal{H}\partial_x^2 u + \partial_x(f(u)) = 0,$$

where $\mathcal{H}$ is the Hilbert transform (Fourier multiplier by $-i\text{sgn}(\xi)$).

The Cauchy problem associated with this kind of dispersive equation has been extensively studied over these last thirty years (see [1, 3, 6, 9, 12, 13, 15, 16, 17, 22, 26, 28, 29, 30, 35, 37]) and we refer the reader to the introduction of [31] for a brief exposition of the main contributions. In [31] we proposed a way, based on the method developed in [28] and [30] to solve this Cauchy problem in the general framework described by Hypotheses 1–2. We showed in particular that the Cauchy problem associated with (1.1)–(1.2) is unconditionally locally well-posed in $H^s(\mathbb{T})$ with $s \geq 1 - \frac{\alpha}{2(\alpha+1)}$.

In this paper, we improved this last result by establishing the unconditional local well-posedness of (1.1) in $H^s(\mathbb{T})$ with $s \geq 1 - \frac{\alpha}{4}$ and $s > 1/2$ for $\alpha \in [1, 2]$. It is worth noticing that this result is optimal in the case $\alpha = 2$ given the restriction $s > 1/2$ for the continuous injection of $H^s(\mathbb{T})$ into $L^\infty(\mathbb{T})$. This also enables us to reach the energy space for $\alpha \in [4/3, 2]$ and thus extend our global existence results to this range of $\alpha$.

Recall that Colliander, Keel, Staffilani, Takaoka and Tao [5] showed that $k$-gKdV ($L_3 = \partial_x^3$ and $f(x) = x^k, k \geq 2$) is locally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$ by performing a contraction mapping argument in Bourgain’s spaces (note also [18, 27] for well-posedness results on $k$-gBO that is the case $L_2 = -\mathcal{H}\partial_x^2$). Although our result with $\alpha = 2$ and $f(x) = x^k$ is weaker than [5] by $\varepsilon > 0$ in terms of regularity, we succeed in proving the unconditional uniqueness, which ensures that the solution does not depend on how it is constructed (see Definition 2 for the notion of unconditional uniqueness).

The main new ingredient in this paper is the use of improved bilinear estimates that are bilinear version of the improved Strichartz estimates introduced by [22]. Recall that this type of improved estimates are obtained by localizing a solution of
the equation in space frequency, then in small intervals of time with length relating to the space frequency and then summing over small intervals of time to obtain an estimate on $[0, T]$. In the context of KdV-like equation, this type of estimates was first introduced by Koch-Tzvetkov [22] in the study of the Benjamin-Ono equation. They were inspired by Burq-Gérard-Tzvetkov’s work [4], where the authors showed Strichartz estimates on compact Riemannian manifolds without boundary for the Schrödinger operator. It is well known that the Strichartz estimates in such manifolds are weaker than the ones on the Euclidean space, but in [4] it is shown that one could however obtain the same Strichartz estimate for a small time interval that depends on the space frequency as in the Euclidean space. This leads to Strichartz estimates with possible loss on compact Riemannian manifolds by re-summing on the small interval of times for each of frequency range and then re-summing in frequency. Later, Hani [11] generalized the argument of [4] to get bilinear Strichartz estimates on compact Riemannian manifolds without boundary for the Schrödinger operator. Our improved bilinear estimates are of the same type as the ones obtained in [11]. Nevertheless, as far as the authors know, it is the first time that such bilinear estimates are shown to be useful in the context of equations with a derivative nonlinearity. Indeed, one of these improved bilinear estimates enables us to improve the treatment of the worst interactions that is the resonant case of 3 high input frequencies of the same order that give rise to an output frequency of the same order. Note that in [31] we used improved Strichartz estimates to close our estimates in this case. The restriction $s \geq 1 - \frac{4}{T}$ follows from those resonant interactions.

Before stating our main result, let us recall our notion of solutions:

**Definition 1.** Let $s > 1/2$. We will say that $u \in L^\infty([0,T];H^s(T))$ is a solution to (1.1) associated with the initial datum $u_0 \in H^s(T)$ if $u$ satisfies (1.1)–(1.2) in the distributional sense, i.e. for any test function $\phi \in C_\infty([-T,T]\times\mathbb{T})$, it holds

$$
\int_0^\infty \int_T \left[ (\phi_t + L_{\alpha+1}\phi)u + \phi_x f(u) \right] dx dt + \int_T \phi(0,\cdot)u_0 dx = 0. \tag{1.3}
$$

**Remark 1.1.** Note that for $u \in L^\infty([0,T];H^s(T))$, with $s > 1/2$, $f(u)$ is well-defined and belongs to $L^\infty([0,T];H^s(T))$. Moreover, Hypothesis [1] forces

$$
L_{\alpha+1}u \in L^\infty([0,T];H^{s-\alpha-1}(T)).
$$

Therefore $u_t \in L^\infty([0,T];H^{s-\alpha-1}(T))$ and (1.3) ensures that (1.1) is satisfied in $L^\infty([0,T];H^{s-\alpha-1}(T))$. In particular, $u \in C([0,T];H^{s-\alpha-1}(T))$ and (1.3) forces the initial condition $u(0) = u_0$. Note that this ensures that $u \in C_w([0,T];H^s(T))$ and
thus \( \|u_0\|_{H^s} \leq \|u\|_{L^\infty H^s} \). Finally, we notice that this also ensures that \( u \) satisfies the Duhamel formula associated with (1.1).

Finally, let us recall the notion of unconditional well-posedness that was introduced by Kato [14], which is, roughly speaking, the local well-posedness with uniqueness of solutions in \( L^\infty([0,T]; H^s(\mathbb{T})) \).

**Definition 2.** We will say that the Cauchy problem associated with (1.1) is unconditionally locally well-posed in \( H^s(\mathbb{T}) \) if for any initial data \( u_0 \in H^s(\mathbb{T}) \) there exists \( T = T(\|u_0\|_{H^s}) > 0 \) and a solution \( u \in C([0,T]; H^s(\mathbb{T})) \) to (1.1) emanating from \( u_0 \). Moreover, \( u \) is the unique solution to (1.1) associated with \( u_0 \) that belongs to \( L^\infty([0,T]; H^s(\mathbb{T})) \). Finally, for any \( R > 0 \), the solution-map \( u_0 \mapsto u \) is continuous from the ball of \( H^s(\mathbb{T}) \) with radius \( R \) centered at the origin into \( C([0,T(R)]; H^s(\mathbb{T})) \).

We mention that Babin, Ilyin and Titi [2] employed integration by parts in time and showed the unconditional uniqueness of the KdV equation in \( L^2(\mathbb{T}) \). This method is actually a normal form reduction and is successfully applied to a variety of dispersive equations (see for instance [7, 19, 21, 23, 24, 33] and references therein).

Our main result is the following one:

**Theorem 1.1** (Unconditional well-posedness). Assume that Hypotheses 1–2 are satisfied with \( 1 \leq \alpha \leq 2 \). Then for any \( s \geq s(\alpha) = 1 - \frac{\alpha}{4} \) with \( s > 1/2 \) the Cauchy problem associated with (1.1)–(1.2) is unconditionally locally well-posed in \( H^s(\mathbb{T}) \) with a maximal time of existence \( T \geq g(\|u_0\|_{H^s(\alpha)}) > 0 \) where \( g \) is a smooth decreasing function depending only on \( L_{\alpha+1} \) and \( f \).

**Remark 1.2.** The above theorem also holds in the real line case by exactly the same approach. This slightly extends the results obtained by the direct method making use of improved Strichartz estimate that gives \( s > 1 - \frac{\alpha}{4} \) (see [34] and [32], Section 5) at least in the case \( \alpha \in [1,2[. \) Indeed, here we have \( s \geq s(\alpha) = 1 - \frac{\alpha}{4} \) with \( s > 1/2 \) and our hypotheses on the Fourier multiplier \( L_{\alpha+1} \) seems more general.

Equation (1.1) enjoys the following conservation laws at the \( L^2 \) and at the \( H^s \)-level:

\[
M(u) = \int_T u^2 \quad \text{and} \quad E(u) = \frac{1}{2} \int_T u \partial_x^{-1} L_{\alpha+1} u + \int_T F(u)
\]

where \( \partial_x^{-1} L_{\alpha+1} \) is the Fourier multiplier by \( \frac{p_{\alpha+1}(k)}{k}1_{k \neq 0} \) and

\[
F(x) := \int_0^x f(y) \, dy.
\]
At this stage it is worth noticing that Hypothesis 1 ensures that the restriction of the quadratic part of the energy $E$ to high frequencies behaves as the $H^{\frac{\alpha}{2}}(\mathbb{T})$-norm whereas its restriction to the low frequencies can be controlled by the $L^2$-norm. Therefore, gathering these conservation laws with the above local well-posedness result, exactly the same arguments as in [31], Section 5 lead to the following GWP results for (1.1):

**Corollary 1.2** (Global existence for small initial data). Assume that Hypotheses 1-2 are satisfied with $\alpha \in [4/3, 2]$. Then there exists $A = A(L_{\alpha+1}, f) > 0$ such that for any initial data $u_0 \in H^s(\mathbb{T})$ with $s \geq \alpha/2$ such that $\|u_0\|_{H^{\frac{\alpha}{2}}} \leq A$, the solution constructed in Theorem 1.1 can be extended for all times. Moreover its trajectory is bounded in $H^{\frac{\alpha}{2}}(\mathbb{T})$.

**Corollary 1.3** (Global existence for arbitrary large initial data).
Assume that Hypotheses 1-2 are satisfied with $\alpha \in [4/3, 2]$. Then the solution constructed in Theorem 1.1 can be extended for all times if the function $F$ defined in (1.4) satisfies one of the following conditions:

1. There exists $C > 0$ such that $|F(x)| \leq C(1+|x|^{p+1})$ for some $0 < p < 2\alpha+1$.
2. There exists $B > 0$ such that $F(x) \leq B, \forall x \in \mathbb{R}$.

Moreover its trajectory is bounded in $H^{\frac{\alpha}{2}}(\mathbb{T})$.

**Remark 1.3.** Typical examples for the case (1) are

- $f(x)$ is a polynomial function of degree strictly less than $2\alpha + 1$.
- $f(x)$ is a polynomial function of $\sin(x)$ and $\cos(x)$.

Whereas typical example for the case (2) are

- $f(x)$ is a polynomial function of odd degree with $\lim_{x \to +\infty} f(x) = -\infty$.
- $f(x) = -\exp(x)$ or $f(x) = -\sinh(x)$.

**Plan of this paper:** In the next section we introduce the notation and the function spaces and recall some basic estimates. Section 3 is devoted to the proof of the main new ingredient of this paper that is the improved bilinear Strichartz estimate. In Sections 4 and 5 we prove the energy estimates we need on a solution and on the difference of two solutions to obtain the unconditional LWP result. In Section 6, we briefly recall how this unconditional LWP result follows from these energy estimates. Finally in Appendix, for sake of completeness, we write the proof of the two estimates we borrow from the framework of short-time $X^{s,b}$-spaces.
2. Notation, function spaces and basic estimates

2.1. Notation. Throughout this paper, \( \mathbb{N} \) denotes the set of non negative integers. For any positive numbers \( a \) and \( b \), we write \( a \lesssim b \) when there exists a positive constant \( C \) such that \( a \leq Cb \). We also write \( a \sim b \) when \( a \lesssim b \) and \( b \lesssim a \) hold. Moreover, we denote \( a \ll b \) if the estimate \( b \lesssim a \) does not hold. For two non negative numbers \( a, b \), we denote \( a \lor b := \max\{a, b\} \) and \( a \land b := \min\{a, b\} \). We also write \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \). Moreover, if \( a \in \mathbb{R} \), \( a^+ \), respectively \( a^- \) denotes a number slightly greater, respectively lesser, than \( a \).

For \( u = u(t, x) \), \( \mathcal{F}u = \hat{u} \) denotes its space-time Fourier transform, whereas \( \mathcal{F}_x u = \hat{u} \) (resp. \( \mathcal{F}_t u \)) denotes its Fourier transform in space (resp. time). We define the Riesz potentials by \( D^s_x g := \mathcal{F}^{-1}_x (|\xi|^s \mathcal{F}_x g) \). We also denote the unitary group associated to the linear part of (1.1) by \( U_{\alpha}(t) = e^{-tL_{\alpha+1}} \), i.e.,

\[
U_{\alpha}(t)u = \mathcal{F}^{-1}_x (e^{itp_{\alpha+1}(\xi)} \mathcal{F}_x u).
\]

Throughout this paper, we fix a smooth even cutoff function \( \chi \): let \( \chi \in C_0^\infty(\mathbb{R}) \) satisfy

\[
0 \leq \chi \leq 1, \quad \chi|_{-1,1} = 1 \quad \text{and} \quad \text{supp} \chi \subset [-2, 2]. \tag{2.1}
\]

We set \( \phi(\xi) := \chi(\xi) - \chi(2\xi) \). For any \( l \in \mathbb{N} \), we define

\[
\phi_{2^l}(\xi) := \phi(2^{-l}\xi), \quad \psi_{2^l}(\tau, \xi) := \phi_{2^l}(\tau - p_{\alpha+1}(\xi)), \tag{2.2}
\]

where \( ip_{\alpha+1}(\xi) \) is the Fourier symbol of \( L_{\alpha+1} \). By convention, we also denote

\[
\phi_0(\xi) = \chi(2\xi) \quad \text{and} \quad \psi_0(\tau, \xi) = \chi(2(\tau - p_{\alpha+1}(\xi))).
\]

Any summations over capitalized variables such as \( K, L, M \) or \( N \) are presumed to be dyadic. We work with non-homogeneous dyadic decompositions, i.e., these variables ranges over numbers of the form \( \{2^k; k \in \mathbb{N} \} \cup \{0\} \). We call those numbers nonhomogeneous dyadic numbers. It is worth pointing out that \( \sum_N \phi_N(\xi) = 1 \) for any \( \xi \in \mathbb{Z} \),

\[
\text{supp}(\phi_N) \subset \{N/2 \leq |\xi| \leq 2N\}, \quad N \geq 1, \quad \text{and} \quad \text{supp}(\phi_0) \subset \{|\xi| \leq 1\}.
\]

Finally, we define the Littlewood–Paley multipliers \( P_N \) and \( Q_L \) by

\[
P_N u = \mathcal{F}^{-1}_x (\phi_N \mathcal{F}_x u) \quad \text{and} \quad Q_L u = \mathcal{F}^{-1}(\psi_L \mathcal{F} u).
\]

We also set \( P^+ u := \mathcal{F}^{-1}_x (1_{\{\xi \geq 1\}} \mathcal{F}_x u) \), \( P^- u := \mathcal{F}^{-1}_x (1_{\{\xi \leq -1\}} \mathcal{F}_x u) \), \( P_{\geq N} := \sum_{K \geq N} P_K \), \( P_{\leq N} := \sum_{K \leq N} P_K \), \( Q_{\geq L} := \sum_{K \geq L} Q_K \) and \( Q_{\leq L} := \sum_{K \leq L} Q_K \).
2.2. Function spaces. For $1 \leq p \leq \infty$, $L^p(T)$ is the standard Lebesgue space with the norm $\| \cdot \|_{L^p}$.

In this paper, we will use the frequency envelope method (see for instance [37] and [22]) in order to show the continuity result with respect to initial data. To this aim, we first introduce the following:

**Definition 3.** Let $\delta > 1$. An acceptable frequency weight $\{\omega_N(\delta)\}_{N \in 2^\mathbb{N} \cup \{0\}}$ is defined as a dyadic sequence satisfying $\omega_N \leq \omega_{2N} \leq \delta \omega_N$ for $N \geq 1$. We simply write $\{\omega_N\}$ when there is no confusion.

With an acceptable frequency weight $\{\omega_N\}$, we slightly modulate the classical Sobolev spaces in the following way: for $s \geq 0$, we define $H^{s}_{\omega}(T)$ with the norm $\|u\|_{H^{s}_{\omega}} := \left( \sum_{N \in 2^\mathbb{N} \cup \{0\}} \omega_N^2 (1 \vee N)^{2s} \|P_N u\|_{L^2}^2 \right)^{\frac{1}{2}}$.

Note that $H^{s}_{\omega}(T) = H^s(T)$ when we choose $\omega_N \equiv 1$. Here, $H^s(T)$ is the usual $L^2$–based Sobolev space. If $B_x$ is one of spaces defined above, for $1 \leq p \leq \infty$ and $T > 0$, we define the space–time spaces $L^p_t B_x := L^p(\mathbb{R}; B_x)$ and $L^p_T B_x := L^p([0, T]; B_x)$ equipped with the norms (with obvious modifications for $p = \infty$)

$$\|u\|_{L^p_t B_x} = \left( \int_{\mathbb{R}} \|u(t, \cdot)\|_{B_x}^p dt \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{L^p_T B_x} = \left( \int_0^T \|u(t, \cdot)\|_{B_x}^p dt \right)^{\frac{1}{p}},$$

respectively. For $s, b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s, b}$ associated to the operator $L^{\alpha+1}$ endowed with the norm

$$\|u\|_{X^{s, b}} := \left( \sum_{\xi = -\infty}^{\infty} \int_{-\infty}^{\infty} \langle \xi \rangle^{2s} \langle \tau - p_{\alpha+1}(\xi) \rangle^{2b} |\tilde{u}(\tau, \xi)|^2 d\tau \right)^{\frac{1}{2}}.$$

We also use a slightly stronger space $X^{s, b}_{\omega}$ with the norm

$$\|u\|_{X^{s, b}_{\omega}} := \left( \sum_{N} \omega_N^2 (1 \vee N)^{2s} \|P_N u\|_{X^{0, b}}^2 \right)^{\frac{1}{2}}.$$

In the proof of the improved bilinear Strichartz estimate, we use the Besov type $X^{s, b}$ spaces: for $b \in \mathbb{R}$ and $1 \leq q < \infty$,

$$\|u\|_{X^{0, b, q}} := \left( \sum_{L} L^q \|Q_L u\|_{L^2_{t,x}}^q \right)^{\frac{1}{q}}$$

with the obvious modifications in the case $q = \infty$. However, we only use $X^{0, \frac{1}{2}, 1}$ throughout this paper. We define the function spaces $Z^s$ (resp. $Z^{s}_{\omega}$), with $s \in \mathbb{R}$, as
\[ Z^s := L_t^\infty H^s \cap X^{s-1,1} \] (resp. \[ Z^s_\omega := L_t^\infty H^s_\omega \cap X^{s-1,1}_\omega \]), endowed with the natural norm
\[ \|u\|_{Z^s} = \|u\|_{L_t^\infty H^s} + \|u\|_{X^{s-1,1}} \] (resp. \[ \|u\|_{Z^s_\omega} = \|u\|_{L_t^\infty H^s_\omega} + \|u\|_{X^{s-1,1}_\omega} \]).

We also use the restriction in time versions of these spaces. Let \( T > 0 \) be a positive time and \( B \) be a normed space of space-time functions. The restriction space \( B_T \) will be the space of functions \( u : ]0, T[ \times \mathbb{T} \to \mathbb{R} \) or \( \mathbb{C} \) satisfying
\[
\|u\|_{B_T} := \inf \{ \|\tilde{u}\|_B \mid \tilde{u} : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \) or \( \mathbb{C}, \quad \tilde{u} = u \) on \( ]0, T[ \times \mathbb{T} \) \.< \infty.
\]

Finally, we introduce a bounded linear operator from \( X^{s-1,1}_{\omega,T} \cap L_t^\infty H^s_\omega \) into \( Z^s_\omega \) with a bound which does not depend on \( s \) and \( T \). The existence of this operator ensures that actually \( Z^s_{\omega,T} = L_t^\infty H^s_\omega \cap X^{s-1,1}_{\omega,T} \). Following [25], we define \( \rho_T \) as
\[
\rho_T(u)(t) := U_\alpha(t)\chi(t)U_{\alpha(-\mu_T(t))}u(\mu_T(t)), \tag{2.3}
\]
where \( \mu_T \) is the continuous piecewise affine function defined by
\[
\mu_T(t) = \begin{cases} 
0 & \text{for } t \notin ]0, 2T[,
\omega \quad \text{for } t \in [0, T],
2T - t & \text{for } t \in [T, 2T].
\end{cases} \tag{2.4}
\]

**Lemma 2.1.** Let \( 0 < T \leq 1 \), \( s \in \mathbb{R} \) and let \( \{\omega_N\} \) be an acceptable frequency weight. Then,
\[
\rho_T : X^{s-1,1}_{\omega,T} \cap L_t^\infty H^s_\omega \to Z^s_\omega
\]
\[
u \mapsto \rho_T(u)
\]
is a bounded linear operator, i.e.,
\[
\|\rho_T(u)\|_{L_t^\infty H^s_\omega} + \|\rho_T(u)\|_{X^{s-1,1}} \lesssim \|u\|_{L_t^\infty H^s} + \|u\|_{X^{s-1,1}_{\omega,T}}, \tag{2.5}
\]
for all \( u \in X^{s-1,1}_{\omega,T} \cap L_t^\infty H^s_\omega \). Moreover, it holds that
\[
\|\rho_T(u)\|_{L_t^\infty H^s_\omega} \lesssim \|u\|_{L_t^\infty H^s_\omega} \tag{2.6}
\]
for all \( u \in L_t^\infty H^s_\omega \). Here, the implicit constants in (2.5) and (2.6) can be chosen independent of \( 0 < T \leq 1 \) and \( s \in \mathbb{R} \).

**Proof.** See Lemma 2.4 in [30] for \( \omega_N \equiv 1 \) but it is obvious that the result does not depend on \( \omega_N \). \( \square \)
2.3. Basic estimates. In this subsection, we collect some fundamental estimates. Well-known estimates are adapted for our setting $H^s_\omega(\mathbb{T})$ and $f(u)$.

**Lemma 2.2.** Let $\{\omega_N\}$ be an acceptable frequency weight. Then we have the estimate

$$\|uv\|_{H^s_\omega} \lesssim \|u\|_{H^s_\omega}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s_\omega},$$

whenever $s > 0$ or $s \geq 0$ and $\omega_N \equiv 1$. In particular for any fixed real smooth function $f$ with $f(0) = 0$, there exists a real smooth function $G = G[f]$ that is increasing and non-negative on $\mathbb{R}_+$ such that

$$\|f(u)\|_{H^s_\omega} \lesssim G(\|u\|_{L^\infty})\|u\|_{H^s_\omega},$$

whenever $s > 0$ or $s \geq 0$ and $\omega_N \equiv 1$.

*Proof.* See [31], Lemma 2.2. □

**Lemma 2.3.** Assume that $s_1 + s_2 \geq 0, s_1 \wedge s_2 \geq s_3, s_3 < s_1 + s_2 - 1/2$. Then

$$\|uv\|_{H^{s_3}} \lesssim \|u\|_{H^{s_1}}\|v\|_{H^{s_2}}.$$  

(2.9)

In particular, for $u, v \in H^s(\mathbb{T})$ with $s > 1/2$ and any fixed real smooth function $f$, there exists a real smooth function $G = G[f]$ that is increasing and non-negative on $\mathbb{R}_+$ such that

$$\|f(u) - f(v)\|_{H^\theta} \leq G(\|u\|_{H^s} + \|v\|_{H^s})\|u - v\|_{H^\theta}.$$  

(2.10)

for $\theta \in \{0, s - 1\}$.

*Proof.* For (2.9), see [8, Lemma 3.4]. The proof of (2.10) can be found in [31, Lemma 2.3]. □

We will frequently use the following lemma, which can be seen as a variant of the integration by parts.

**Lemma 2.4.** Let $N \in 2^N \cup \{0\}$. Then,

$$\left| \int_T \Pi(u, v)wdx \right| \lesssim \|u\|_{L^2_x}\|v\|_{L^2_x}\|\partial_xw\|_{L^\infty_x},$$

where

$$\Pi(u, v) := v\partial_x P^2_Nu + u\partial_x P^2_Nv.$$  

(2.11)

*Proof.* See [31, Lemma 2.4]. □
3. Improved bilinear Strichartz estimates

In this section, we establish improved bilinear Strichartz estimates which play a crucial role in our study. In the proof of the a priori estimate (Proposition 4.6) and estimate for the difference (Proposition 5.1), we would like to simultaneously use the improved bilinear Strichartz estimate (3.4) and integration by parts (Lemma 2.4). However, Lemma 2.4 is involved with a Fourier multiplier, which does not enable us to use (3.4) as we would like. Therefore, we consider the symbol of $\Pi (u, v)$ (which is defined in (2.11)) in more detail and decompose it into two parts. See Case 3 of $A_1$ in the proof of Proposition 4.6. This is the reason why we have to state the bilinear estimate (3.2) with a Fourier multiplier. For that purpose, we introduce the following notation:

**Definition 4.** For $u, v \in L^2(\mathbb{T})$ and $a \in L^\infty(\mathbb{R}^2)$, we set

$$F_x(\Lambda_a(u, v))(\xi) := \sum_{\xi_1 + \xi_2 = \xi} a(\xi_1, \xi_2) \hat{u}(\xi_1) \hat{v}(\xi_2).$$

(3.1)

Remark that when $a \equiv 1$, we have $\Lambda_a(u, v) = uv$.

Let us now state the two main results of this section:

**Proposition 3.1** (Improved bilinear Strichartz I). Let $0 < T < 1$, $\alpha \in [1, 2]$ and $N_1, N_2 \geq 1$. Let also $f_1, f_2 \in L^\infty([0, T]; L^2(\mathbb{T}))$ and let $a \in L^\infty(\mathbb{R}^2)$ such that $\|a\|_{L^\infty} \lesssim 1$. Finally, let $u_1, u_2 \in C([0, T]; L^2(\mathbb{T}))$ satisfying

$$\partial_t u_j + L_{\alpha+1} u_j + \partial_x f_j = 0$$
on $]0, T[ \times \mathbb{T}$ for $j = 1, 2$, with $L_{\alpha+1}$ satisfying Hypothesis \[A\]. Then

$$\|\Lambda_a(P_{N_1} u_1, P_{N_2} u_2)\|_{L^4_{T,x}} \lesssim T^{-\frac{1}{4}} (N_1 \vee N_2)^{\frac{1}{4} - \frac{1}{4 \alpha + 1}} (\|P_{N_1} u_1\|_{L^4_{T,x}} + \|P_{N_1} f_1\|_{L^4_{T,x}}) \times (\|P_{N_2} u_2\|_{L^\infty_{T} L^2_x} + \|P_{N_2} f_2\|_{L^\infty_{T} L^2_x}).$$

(3.2)

**Remark 3.1.** In [31], we used the improved Strichartz estimates (Proposition 3.5 in [31]) for resonant interactions, which roughly speaking claims

$$\|P_N u\|_{L^4_{T,x}} \lesssim N^{\frac{1}{4(\alpha + 1)}} (\|P_N u\|_{L^4_{T} L^2_x} + \|P_N f\|_{L^4_{T} L^2_x}),$$

(3.3)

where $u$ satisfies the assumption in Proposition 3.1. We can view (3.2) as a bilinear improvement of (3.3) since

$$\frac{2}{4(\alpha + 1)} - \left(\frac{1}{2} - \frac{\alpha}{4}\right) = \frac{\alpha(\alpha - 1)}{4(\alpha + 1)} \geq 0$$
for $\alpha \in [1, 2]$. Note that (3.2) does not improve (3.3) when $\alpha = 1$. Recall that Proposition 1.1 with $\alpha = 1$ is exactly the same as our previous result [31, Theorem 1.1] when $\alpha = 1$.

Now, it is well-known that we can obtain a better estimate when one of two frequencies is dominantly large such as $N_1 \gg N_2$.

**Proposition 3.2 (Improved bilinear Strichartz II).** Let $0 < T < 1$ and $\alpha \in [1, 2]$ and $N_1 \vee N_2 \gg N_1 \wedge N_2 \geq 1$. Let $f_1, f_2 \in L^\infty([0, T]; L^2(\mathbb{T}))$ and let $a \in L^\infty(\mathbb{R}^2)$ such that $\|a\|_{L^\infty} \lesssim 1$. Let $u_1, u_2 \in C([0, T]; L^2(\mathbb{T}))$ satisfying

$$\partial_t u_j + L_{\alpha+1} u_j + \partial_x f_j = 0$$

on $]0, T[ \times \mathbb{T}$ for $j = 1, 2$, with $L_{\alpha+1}$ satisfying Hypothesis 7. Then for any $0 \leq \theta \leq 1$ it holds

$$\|A_a(P_{N_1} u_1, P_{N_2} u_2)\|_{L^2_{t,x}} \lesssim T^{\frac{\alpha-1}{2}} (N_1 \wedge N_2)^{\frac{\theta}{2}}$$

$$\times (\|P_{N_1} u_1\|_{L^2_{t,x}} + \|P_{N_1} f_1\|_{L^2_{t,x}})(\|P_{N_2} u_2\|_{L^\infty_{t,x} L^2_x} + \|P_{N_2} f_2\|_{L^\infty_{t,x} L^2_x}).$$

(3.4)

**Remark 3.2.** We need to introduce the parameter $\theta > 0$ in the above proposition since a factor $T^{-\frac{1}{2}}$ in (3.4) will not enable us to get a positive power of $T$ in the right hand side of the energy estimates (4.9) and (5.1). See for instance Case 3 of $J_{AH}^1$ in the proof of Proposition 4.6. It is worth noticing that taking $\theta > 0$ causes a loss of a factor $N_{\min}^\theta$ that is anyway allowed since we work with $s > 1/2$.

**Remark 3.3.** In Proposition 4.6 we will apply Propositions 3.1 and 3.2 with $u_1 = u_2 = u$ and $f_1(t, x) = f_2(t, x) = f(u(t, x)) - f(0)$, where $f$ satisfies Hypothesis 2. Notice that $P_N f(0) = 0$ when $N \geq 1$. This modification allows us to use (2.8) after summing over $N$. See (4.21) and (5.6).

The above improved bilinear estimates are based on the following classical bilinear estimates:

**Proposition 3.3.** Let $\alpha \in [1, 2]$ and $a \in L^\infty(\mathbb{R}^2)$ with $\|a\|_{L^\infty} \leq 1$. Then there exists $C = C(\xi_0) > 0$ such that for any real-valued functions $u_1, u_2 \in L^2(\mathbb{R}_t \times \mathbb{T}_x)$, any $N_1, N_2, L_1, L_2 \geq 1$ it holds

$$\|A_a(Q_{\leq L_1} P_{N_1} u_1, Q_{\leq L_2} P_{N_2} u_2)\|_{L^2_{t,x}}$$

$$\leq C(L_1 \wedge L_2)^{\frac{\alpha}{2}} \left\{ \frac{(L_1 \vee L_2)^{\frac{1}{2}}}{(N_1 \vee N_2)^{\frac{1}{2}}} + 1 \right\} \|Q_{\leq L_1} P_{N_1} u_1\|_{L^2_{t,x}} \|Q_{\leq L_2} P_{N_2} u_2\|_{L^2_{t,x}}.$$
Moreover,
\[
\| \Lambda_\alpha(Q_{\leq L_1} P_N u_1, Q_{\leq L_2} P_N u_2) \|_{L^2_t x} 
\leq C(L_1 \wedge L_2)^{\frac{1}{2}} \min \left( \left( \frac{L_1 \vee L_2}{N_1} \right)^{\frac{1}{2}}, \frac{1}{(N_1 \wedge N_2)^{\frac{1}{2}}} \right) + 1, \frac{1}{(N_1 \wedge N_2)^{\frac{1}{2}}} \right) \times \| Q_{\leq L_1} P_N u_1 \|_{L^2_t x} \| Q_{\leq L_2} P_N u_2 \|_{L^2_t x}.
\]
whenever \( N_1 \vee N_2 \gg N_1 \wedge N_2 \).

To prove Proposition 3.3 we need the two following technical lemmas.

**Lemma 3.4.** Let \( I \) and \( J \) be two intervals on the real line and \( g \in C^1(J; \mathbb{R}) \). Then
\[
\# \{ x \in J \cap \mathbb{Z}; g(x) \in I \} \lesssim \frac{|I|}{\inf_{x \in J} |g'(x)|} + 1.
\]

*Proof.* See Lemma 2 in [36]. \( \square \)

**Lemma 3.5.** Let \( I \) and \( J \) be two intervals on \( \mathbb{R} \) and let \( \theta > 0 \). Assume that \( g \in C^2(\mathbb{R}) \) satisfies \( g''(x) \gtrsim \theta \) for \( x \in J \). Then,
\[
\# \{ x \in J \cap \mathbb{Z}; g(x) \in I \} \lesssim \frac{|I|^{\frac{1}{2}}}{\theta^{\frac{1}{2}}} + 1
\]

*Proof.* Set \( A := \{ x \in J \cap \mathbb{Z}; g(x) \in I \} \). We divide \( \mathbb{R} \) into three parts:
\[
I_1 := \{ x \in \mathbb{R}; |g'(x)| \leq (\theta |I|)^{\frac{1}{2}} \},
I_2 := \{ x \in \mathbb{R}; g'(x) > (\theta |I|)^{\frac{1}{2}} \},
I_3 := \{ x \in \mathbb{R}; g'(x) < -(\theta |I|)^{\frac{1}{2}} \}.
\]
It is clear that \( I_m \cap I_n = \emptyset \) for \( m \neq n \) and \( I_1 \cup I_2 \cup I_3 = \mathbb{R} \). For \( I_1 \), we see from Lemma 3.4 that
\[
\#(A \cap I_1) \lesssim \# \{ x \in J \cap \mathbb{Z}; |g'(x)| \leq (\theta |I|)^{\frac{1}{2}} \} \lesssim \frac{(\theta |I|)^{\frac{1}{2}}}{\inf_{x \in J} g''(x)} + 1 \lesssim \frac{|I|^{\frac{1}{2}}}{\theta^{\frac{1}{2}}} + 1.
\]
On the other hand, for \( I_2 \) we again use Lemma 3.4 so that
\[
\#(A \cap I_2) \lesssim \# \{ x \in J \cap \mathbb{Z}; g(x) \in I, g'(x) > (\theta |I|)^{\frac{1}{2}} \}
\lesssim \frac{|I|}{(\theta |I|)^{\frac{1}{2}}} + 1 = \frac{|I|^{\frac{1}{2}}}{\theta^{\frac{1}{2}}} + 1.
\]
Similarly, we have
\[
\#(A \cap I_3) \lesssim \# \{ x \in J \cap \mathbb{Z}; h(x) \in -I, h'(x) > (\theta |I|)^{\frac{1}{2}} \} \lesssim \frac{|I|^{\frac{1}{2}}}{\theta^{\frac{1}{2}}} + 1,
\]
where we put \( h(x) := -g(x) \). This completes the proof. \( \square \)
Proof of Proposition 3.3. For simplicity, we put \( v_j := \psi_{\leq L} \phi_N \hat{u}_j \) for \( j = 1, 2 \). The Plancherel theorem leads to

\[
\| A_a(P_{N_1}Q_{\leq L_1}u_1, P_{N_2}Q_{\leq L_2}u_2) \|^2_{L_t^2 L_x^2} = \sum_{\xi \in \mathbb{Z}} \sum_{\tau_1} \left( \sum_{\xi_1} \left| a(\xi_1, \xi - \xi_1) v_1(\tau_1, \xi_1) v_2(\tau - \tau_1, \xi - \xi_1) d\tau_1 \right| \right)^2 d\tau \leq \sum_{\xi \in \mathbb{Z}} \sum_{\tau_1} \left( \sum_{\xi_1} |v_1(\tau_1, \xi_1)||v_2(\tau - \tau_1, \xi - \xi_1)| d\tau_1 \right)^2 d\tau = \| w_1 w_2 \|^2_{L_t^2 L_x^2},
\]

where \( w_i = F^{-1}(|v_i|), i = 1, 2 \). Note that since \( u_i \) is real-valued, \( |v_i| \) has to be an even real-valued function that forces \( w_i \) to be also even and real-valued. It follows from \( N_1, N_2 \geq 1 \) that \( P_0 w_1 = P_0 w_2 = 0 \). Then we can use the trick introduced in [3] that consists in rewriting \( w_i \) as \( P^+ w_i + P^- w_i \) (see Subsection 2.1 for the definitions of \( P^+ \) and \( P^- \)), and observing that since \( P^- w_i(t, x) = P^+ w_i(t, x) \), it holds

\[
\| w_1 w_2 \|^2_{L_t^2 L_x^2} \leq \| P^+ w_1 P^+ w_2 \|^2_{L_t^2 L_x^2} + \| P^- w_1 P^- w_2 \|^2_{L_t^2 L_x^2} + \| P^+ w_1 P^+ w_2 \|^2_{L_t^2 L_x^2} + \| P^- w_1 P^- w_2 \|^2_{L_t^2 L_x^2}
\]

since \( \| P^+ w_1 P^- w_2 \|^2_{L_t^2 L_x^2} = \| P^+ w_1 P^+ w_2 \|^2_{L_t^2 L_x^2} = \| P^+ w_1 P^+ w_2 \|^2_{L_t^2 L_x^2} \). By using the Plancherel theorem and the Cauchy-Schwarz inequality, we get

\[
\| P^+ w_1 P^+ w_2 \|^2_{L_t^2 L_x^2} \lesssim \sup_{(\tau, \xi) \in \mathbb{R} \times \mathbb{N}} A_{L_1, L_2, N_1, N_2}(\tau, \xi) \| v_1 \|^2_{L_t^{2/3} L_x^2} \| v_2 \|^2_{L_t^{2/3} L_x^2},
\]

where

\[
A_{L_1, L_2, N_1, N_2}(\tau, \xi) \lesssim \text{mes} \left\{ (\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{N}; \xi - \xi_1 \geq 0, \xi_1 \sim N_1, \xi - \xi_1 \sim N_2, \langle \tau_1 - p_{\alpha+1}(\xi_1) \rangle \lesssim L_1 \text{ and } \langle \tau - \tau_1 - p_{\alpha+1}(\xi - \xi_1) \rangle \lesssim L_2 \right\}
\]

(3.10)

and

\[
B(\tau, \xi) := \{ \xi_1 \geq 0; \xi - \xi_1 \geq 0, \xi_1 \sim N_1, \xi - \xi_1 \sim N_2 \}
\]

We put \( g(\xi_1) := \tau - p_{\alpha+1}(\xi_1) - p_{\alpha+1}(\xi - \xi_1) \). When \( N_1 \lesssim \xi_0 \) or \( N_2 \lesssim \xi_0 \), it holds \( \#B(\tau, \xi) \lesssim \xi_0 \). Now when \( N_1 \wedge N_2 \gg \xi_0 \) then \( \xi_1, \xi - \xi_1 > \xi_0 \) and by Hypothesis [1]
on $p_{\alpha+1}$, we see that $p''_{\alpha+1}(\xi_1), p''_{\alpha+1}(\xi - \xi_1) > 0$ which leads to

$$|g''(\xi_1)| = |p''_{\alpha+1}(\xi_1) + p''_{\alpha+1}(\xi - \xi_1)| \geq p''_{\alpha+1}(\xi_1) \vee p''_{\alpha+1}(\xi - \xi_1) \geq (N_1 \vee N_2)^{\alpha-1}.$$  

Lemma 3.5 then yields

$$\#B(\tau, \xi) \lesssim \#\{\xi_1 \in \mathbb{N}; \xi_1 \leq \xi \text{ and } |g(\xi_1)| \lesssim L_1 \vee L_2\} \lesssim \frac{(L_1 \vee L_2)^{\frac{1}{2}}}{(N_1 \vee N_2)^{\frac{\alpha}{2}}} + 1.$$  

This concludes the proof of (3.5) by combining the above inequality with 3.7–3.10.

The strategy to prove (3.6) is similar. It suffices to show that

$$\#B(\tau, \xi) \lesssim \max\left\{\frac{L_1 \vee L_2}{(N_1 \vee N_2)^{\alpha}}, 1, N_1 \wedge N_2\right\}, \quad (3.11)$$

where

$$B(\tau, \xi) := \{\xi_1 \in \mathbb{N}; \xi - \xi_1 \geq 0, \xi_1 \sim N_1, \xi \sim N_2 \text{ and } |g(\xi_1)| \lesssim L_1 \vee L_2\}.$$  

with $g(\xi_1) := \tau - p_{\alpha+1}(\xi_1) - p_{\alpha+1}(\xi - \xi_1)$.

We first notice that the estimate $\#B(\tau, \xi) \lesssim N_1 \wedge N_2$ is direct in view of the definition of $B(\tau, \xi)$. Now, when $N_1 \lesssim \xi_0$ or $N_2 \lesssim \xi_0$, it holds $\#B(\tau, \xi) \lesssim \xi_0$ and when $N_1, N_2 \gg \xi_0$, since $N_1 \vee N_2 \gg N_1 \wedge N_2$, we notice that we have either $\xi_1 \gg \xi - \xi_1 \gg \xi_0$ or $\xi - \xi_1 \gg \xi_1 \gg \xi_0$. We also have $p''_{\alpha+1}(\theta) > 0$ and $p''_{\alpha+1}(\theta) \sim \theta^{\alpha-1}$ for $\theta \in [\xi - \xi_1, \xi_1]$ by Hypothesis 1. This leads to

$$|g'(\xi_1)| = \left| \int_{\xi-\xi_1}^{\xi_1} p''_{\alpha+1}(\theta) d\theta \right| \sim |\xi_1^\alpha - (\xi - \xi_1)^\alpha| \gtrsim \xi_1^\alpha \vee (\xi - \xi_1)^\alpha \gtrsim (N_1 \vee N_2)^\alpha.$$  

Therefore, Lemma 3.4 shows (3.11), which concludes the proof.  

Now we have to adapt the bilinear estimates (3.3)–(3.6) to short time intervals. To do this we use the framework of the short-time $X^{s,b}$-spaces introduced by Ionescu-Kenig-Tataru [10] (see also [21]). The following lemma contains the two essential estimates we need for our purpose. We give the proof of these estimates in Appendix for sake of completeness.

Lemma 3.6. There exists $C > 0$ such that for any $u \in X^{0,\frac{1}{2},1}$ and $L \geq 1$,

$$\|\chi(Lt)u\|_{L^2_{t,x}} \leq CL^{-\frac{1}{2}}\|u\|_{X^{0,\frac{1}{2},1}} \quad (3.12)$$

and

$$\|\chi(Lt)u\|_{X^{0,\frac{1}{2},1}} \leq C\|u\|_{X^{0,\frac{1}{2},1}} \quad (3.13)$$

where $\chi$ is the smooth bump function defined in (2.1).
Remark 3.4. Note that (3.12) is a direct consequence of the following inequality applied with $g = U_\alpha(-t)u$ : for $g \in B_{2,1}^\frac{3}{2} (\mathbb{R})$ and $L \geq 1$

$$\|\chi (L \cdot) g\|_{L^2(\mathbb{R})} \leq \|\chi (L \cdot)\|_{L^2(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} \lesssim L^{-\frac{1}{2}} \|g\|_{B_{2,1}^\frac{3}{2}(\mathbb{R})},$$

(3.14)

where $B_{2,1}^\frac{3}{2}(\mathbb{R})$ is the Besov space. (3.14) is reminiscent of the Heisenberg uncertainty principle.

With (3.12)–(3.13) in hands, we can prove the following versions of the bilinear estimate (3.5)–(3.6) on short-time intervals :

Proposition 3.7. Let $\alpha \in [1, 2]$, $u_1, u_2 \in X^{0, \frac{1}{2}, 1}$, $N_1, N_2 \geq 1$, $0 < T < 1$ and $a \in L^\infty (\mathbb{R}^2)$ be such that $\|a\|_{L^\infty} \lesssim 1$. Then for $I \subset \mathbb{R}$ satisfying $|I| \sim (N_1 \lor N_2)^{-1} T$ it holds

$$\|\Lambda_a(P_{N_1} u_1, P_{N_2} u_2)\|_{L^2(I; L^2)} \lesssim T^\frac{1}{2} (N_1 \lor N_2)^{-\frac{1}{2}} \|P_{N_1} u_1\|_{X^{0, \frac{1}{2}, 1}} \|P_{N_2} u_2\|_{X^{0, \frac{1}{2}, 1}}.$$

(3.15)

Moreover, in the case $N_1 \lor N_2 \gg N_1 \land N_2$, (3.15) can be improved to

$$\|\Lambda_a(P_{N_1} u_1, P_{N_2} u_2)\|_{L^2(I; L^2)} \lesssim T^{\theta_2} (N_1 \land N_2)^{\theta_2} (N_1 \lor N_2)^{-\frac{1}{2}} \|P_{N_1} u_1\|_{X^{0, \frac{1}{2}, 1}} \|P_{N_2} u_2\|_{X^{0, \frac{1}{2}, 1}},$$

(3.16)

for any $0 \leq \theta \leq 1$.

Proof. By possibly replacing $a(\cdot, \cdot)$ by $\tilde{a}(\cdot, \cdot)$ with $\tilde{a}(\xi_1, \xi_2) = a(\xi_2, \xi_1)$, we see that the desired estimates are symmetric in $N_1$ and $N_2$. We may thus assume that $N_1 \geq N_2$ and also that $I = [0, K^{-1}]$ where we will take $K = N_1 T^{-1}$. Recall that $\chi|_{[-1, 1]} = 1$ and supp $\chi \subset [-2, 2]$ so that we have $u_j = \chi(K t) u_j$ for $j = 1, 2$ on $I := [0, K^{-1}]$. We decompose $\chi(K t) P_{N_1} u_1$ and $\chi(K t) P_{N_2} u_2$ as

$$\chi(K t) P_{N_j} u_j = Q_{\leq K} \chi(K t) P_{N_j} u_j + \sum_{L > K} Q_L \chi(K t) P_{N_j} u_j$$
for \( j = 1, 2 \). Then, the triangle inequality shows

\[
\| \Lambda_a(P_{N_1}u_1, P_{N_2}u_2) \|_{L^2(I; L^2)} \\
\leq \| \Lambda_a(Q_{\leq K}(\chi(Kt)P_{N_1}u_1), Q_{\leq K}(\chi(Kt)P_{N_2}u_2)) \|_{L^2(I; L^2)} \\
+ \sum_{L_2 > K} \| \Lambda_a(Q_{\leq K}(\chi(Kt)u_1), Q_{L_2}(\chi(Kt)P_{N_2}u_2)) \|_{L^2(I; L^2)} \\
+ \sum_{L_1 > K} \| \Lambda_a(Q_{L_1}(\chi(Kt)P_{N_1}u_1), Q_{\leq K}(\chi(Kt)P_{N_2}u_2)) \|_{L^2(I; L^2)} \\
+ \sum_{L_1, L_2 > K} \| \Lambda_a(Q_{L_1}(\chi(Kt)P_{N_1}u_1), Q_{L_2}(\chi(Kt)P_{N_2}u_2)) \|_{L^2(I; L^2)} \\
=: A_1 + A_2 + A_3 + A_4.
\]

First we prove (3.13). For \( A_1 \), Proposition 3.3 and Lemma 3.6 lead to

\[
A_1 \leq K^2 \left( \frac{K^2}{N_1^2} + 1 \right) \| Q_{\leq K}(\chi(Kt)P_{N_1}u_1) \|_{L^2_{t,x}} \| Q_{\leq K}(\chi(Kt)P_{N_2}u_2) \|_{L^2_{t,x}} \\
\leq \left( \frac{K^{-\frac{1}{2}}}{N_1^2} + K^{-\frac{1}{2}} \right) \| P_{N_1}u_1 \|_{X^{\alpha, \frac{1}{2}, 1}} \| P_{N_2}u_2 \|_{X^{\alpha, \frac{1}{2}, 1}} \\
\leq T^\frac{1}{4} N_1^{-\frac{7}{8}} \| P_{N_1}u_1 \|_{X^{\alpha, \frac{1}{2}, 1}} \| P_{N_2}u_2 \|_{X^{\alpha, \frac{1}{2}, 1}}
\]

since \( \alpha \leq 2 \) and \( K = N_1 T^{-1} \). For \( A_2 \), we get, again with Proposition 3.3 and Lemma 3.6 in hands, that

\[
A_2 \leq \sum_{L_2 > K} K^2 \left( \frac{L_2^2}{N_1^2} + 1 \right) \| Q_{\leq K}(\chi(Kt)P_{N_1}u_1) \|_{L^2_{t,x}} \| Q_{L_2}(\chi(Kt)P_{N_2}u_2) \|_{L^2_{t,x}} \\
\leq \left( \frac{K^{-\frac{1}{2}}}{N_1^2} + K^{-\frac{1}{2}} \right) \| P_{N_1}u_1 \|_{X^{\alpha, \frac{1}{2}, 1}} \sum_{L_2 > K} L_2^2 \| Q_{L_2}(\chi(Kt)P_{N_2}u_2) \|_{L^2_{t,x}} \\
\leq T^\frac{1}{4} N_1^{-\frac{7}{8}} \| P_{N_1}u_1 \|_{X^{\alpha, \frac{1}{2}, 1}} \| P_{N_2}u_2 \|_{X^{\alpha, \frac{1}{2}, 1}}
\]

Similarly, \( A_3 \) can be estimated by the same bound as above. Finally, we evaluate the contribution of \( A_4 \). Proposition 3.3 together with Lemma 3.6 lead to

\[
A_4 \leq \sum_{L_1, L_2 > K} (L_1 \wedge L_2)^{\frac{1}{2}} \left( \frac{L_1 \vee L_2}{N_1^2} \right)^{\frac{1}{4}} \prod_{j=1}^2 \| Q_{L_j}(\chi(Kt)P_{N_j}u_j) \|_{L^2_{t,x}} \\
\leq T^\frac{1}{4} N_1^{-\frac{7}{8}} \| \chi(Kt)P_{N_1}u_1 \|_{X^{\alpha, \frac{1}{2}, 1}} \| \chi(Kt)P_{N_2}u_2 \|_{X^{\alpha, \frac{1}{2}, 1}} \\
\leq T^\frac{1}{4} N_1^{-\frac{7}{8}} \| P_{N_1}u_1 \|_{X^{\alpha, \frac{1}{2}, 1}} \| P_{N_2}u_2 \|_{X^{\alpha, \frac{1}{2}, 1}},
\]

which completes the proof of (3.13).
Finally the proof of (3.16) follows exactly the same line by using the following version of (3.6):

\[
\|A_\theta(Q_{\leq L_1}P_{N_1}u_1, Q_{\leq L_2}P_{N_2}u_2)\|_{L^2_{t,x}} \\
\leq C(L_1 \wedge L_2)^{\frac{\gamma}{2}} \left( \frac{(L_1 \vee L_2)^{(1-\theta)}_\gamma}{(N_1 \vee N_2)^{(1-\theta)_\gamma}} + 1 \right) (N_1 \wedge N_2)^{\frac{\gamma}{2}} \\
\times \|Q_{\leq L_1}P_{N_1}u_1\|_{L^2_{t,x}} \|Q_{\leq L_2}P_{N_2}u_2\|_{L^2_{t,x}}. 
\]  
(3.17)

for any \(0 \leq \theta \leq 1\) whenever \(N_1 \vee N_2 \gg \max(N_1 \wedge N_2, 1)\). For instance, since \(N_1 \geq N_2\), (3.17) leads to

\[
A_1 \lesssim N_2^{\frac{\gamma}{2}} K^\frac{\gamma}{2} \left( \frac{K^{\frac{(1-\theta)}{2}}}{N_1^{\frac{(1-\theta)_\gamma}} + 1} \right) \left\|Q_{\leq \xi}(\chi(Kt)P_{N_1}u_1)\right\|_{L^2_{t,x}} \left\|Q_{\leq \xi}(\chi(Kt)P_{N_2}u_2)\right\|_{L^2_{t,x}} \\
\lesssim N_2^{\frac{\gamma}{2}} \left( \frac{(N_1^{-1}T)^{\frac{\gamma}{2}}}{N_1^{\frac{(1-\theta)_\gamma}} + (N_1^{-1}T)^{\frac{\gamma}{2}}} \right) \left\|P_{N_1}u_1\right\|_{X^{0,\frac{1}{2},1}} \left\|P_{N_2}u_2\right\|_{X^{0,\frac{1}{2},1}} \\
\lesssim T^\frac{\gamma}{2} \left( N_2^{\frac{\gamma}{2}} + N_1^{-\frac{1}{2}} \right) \left\|P_{N_1}u_1\right\|_{X^{0,\frac{1}{2},1}} \left\|P_{N_2}u_2\right\|_{X^{0,\frac{1}{2},1}}
\]

since \(1 \leq \alpha \leq 2\). Similarly,

\[
A_2 \lesssim \sum_{L_2 > K} N_2^{\frac{\gamma}{2}} K^\frac{\gamma}{2} \left( \frac{L_2^{\frac{(1-\theta)}{2}}}{L_2^{\frac{(1-\theta)_\gamma}} + 1} \right) \left\|Q_{\leq \xi}(\chi(Kt)P_{N_1}u_1)\right\|_{L^2_{t,x}} \left\|Q_{L_2}(\chi(Kt)P_{N_2}u_2)\right\|_{L^2_{t,x}} \\
\lesssim T^\frac{\gamma}{2} \left( N_2^{\frac{\gamma}{2}} + N_1^{-\frac{1}{2}} \right) \left\|P_{N_1}u_1\right\|_{X^{0,\frac{1}{2},1}} \sum_{L_2 > K} L_2^{\frac{1}{2}} \left\|Q_{L_2}(\chi(Kt)P_{N_2}u_2)\right\|_{L^2_{t,x}} \\
\lesssim T^\frac{\gamma}{2} N_2^{\frac{\gamma}{2}} N_1^{-\frac{1}{2}} \left\|P_{N_1}u_1\right\|_{X^{0,\frac{1}{2},1}} \left\|P_{N_2}u_2\right\|_{X^{0,\frac{1}{2},1}}
\]

and

\[
A_4 \lesssim \sum_{L_1, L_2 > K} (L_1 \wedge L_2)^{\frac{1}{2}} N_2^{\frac{\gamma}{2}} \left( \frac{(L_1 \vee L_2)^{(1-\theta)}_\gamma}{(N_1)^{(1-\theta)_\gamma}} + 1 \right) \prod_{j=1}^{2} \left\|Q_{L_j}(\chi(Kt)P_{N_j}u_j)\right\|_{L^2_{t,x}} \\
\lesssim T^\frac{\gamma}{2} N_2^{\frac{\gamma}{2}} N_1^{-\frac{1}{2}} \left\|\chi(Kt)P_{N_1}u_1\right\|_{X^{0,\frac{1}{2},1}} \left\|\chi(Kt)P_{N_2}u_2\right\|_{X^{0,\frac{1}{2},1}} \\
\lesssim T^\frac{\gamma}{2} N_2^{\frac{\gamma}{2}} N_1^{-\frac{1}{2}} \left\|P_{N_1}u_1\right\|_{X^{0,\frac{1}{2},1}} \left\|P_{N_2}u_2\right\|_{X^{0,\frac{1}{2},1}},
\]

which completes the proof.

Let us now translate (3.15) and (3.16) in terms of bilinear estimates for free solutions of (1.1).
Corollary 3.8. Let $a \in L^\infty(\mathbb{R}^2)$ be such that $\|a\|_{L^\infty} \lesssim 1$, $\alpha \in [1, 2]$, $N_1, N_2 \geq 1$, $0 < T < 1$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{T})$. Suppose that $I \subset \mathbb{R}$ satisfies $|I| \sim (N_1 \vee N_2)^{-\frac{1}{4}}T$. Then it holds
\[
\|\Lambda_a(e^{-tL_{\alpha+1}}P_{N_1}\varphi_1, e^{-tL_{\alpha+1}}P_{N_2}\varphi_2)\|_{L^2(I;L^2)} \lesssim T^{\frac{1}{4}}(N_1 \vee N_2)^{-\frac{3}{2}}\|P_{N_1}\varphi_1\|_{L^2}\|P_{N_2}\varphi_2\|_{L^2}.
\] (3.18)
Moreover, when $N_1 \vee N_2 \gg N_1 \wedge N_2$, (3.18) can be improved to
\[
\|\Lambda_a(e^{-tL_{\alpha+1}}P_{N_1}\varphi_1, e^{-tL_{\alpha+1}}P_{N_2}\varphi_2)\|_{L^2(I;L^2)} \lesssim T^{\frac{1}{2}}(N_1 \wedge N_2)^{-\frac{3}{2}}(N_1 \vee N_2)^{-\frac{1}{4}}\|P_{N_1}\varphi_1\|_{L^2}\|P_{N_2}\varphi_2\|_{L^2},
\] (3.19)
for any $0 \leq \theta \leq 1$.

Proof. Recall that $\hat{\chi} \in \mathcal{S}$. In particular, it holds that
\[
\sum_{L} L^{\frac{1}{2}}\|R_{L\chi}\|_{L^2_t} \lesssim 1,
\] (3.20)
where $R_{L\chi} := F_{L}^{-1}(\phi(\tau/L)\hat{\chi}(\tau))$. Indeed, we have
\[
\|\phi(\tau/L)\hat{\chi}\|^2_{L^2_t} = \int_{\mathbb{R}} (\tau^{-8}(\tau)^8|\phi(\tau/L)\hat{\chi}(\tau)|^2 d\tau \lesssim \int_{\mathbb{R}} (\tau)^{-8}|\phi(\tau/L)|^2 d\tau \lesssim L^{-7},
\]
which shows (3.20). Observe that $Q_L u = e^{-tL_{\alpha+1}}(R_{L} e^{tL_{\alpha+1}}u)$. By this and (3.20), for $j = 1, 2$, we obtain
\[
\|\chi(t)e^{-tL_{\alpha+1}}P_{N_j}\varphi_j\|_{X^{0,\frac{1}{4},1}} = \sum_{L} L^{\frac{1}{2}}\|Q_L(\chi(t)e^{-tL_{\alpha+1}}P_{N_j}\varphi_j)\|_{L^2_t} \lesssim \sum_{L} L^{\frac{1}{2}}\|R_{L\chi}\|_{L^2_t}\|P_{N_j}\varphi_j\|_{L^2_t} \lesssim \|P_{N_j}\varphi_j\|_{L^2_t}.
\] (3.21)
Note that $\chi(t) = 1$ for $t \in I$ since $N_1, N_2 \geq 1$. This together with (3.21) and Proposition 3.7 with $u_j = \chi(t)e^{-tL_{\alpha+1}}\varphi_j$ for $j = 1, 2$ ensure that
\[
\|\Lambda_a(e^{-tL_{\alpha+1}}P_{N_1}\varphi_1, e^{-tL_{\alpha+1}}P_{N_2}\varphi_2)\|_{L^2(I;L^2)} \lesssim T^{\frac{1}{4}}(N_1 \vee N_2)^{-\frac{3}{2}}\|\chi(t)e^{-tL_{\alpha+1}}P_{N_1}\varphi_1\|_{X^{0,\frac{1}{4},1}}\|\chi(t)e^{-tL_{\alpha+1}}P_{N_2}\varphi_2\|_{X^{0,\frac{1}{4},1}} \lesssim T^{\frac{1}{4}}(N_1 \vee N_2)^{-\frac{3}{2}}\|P_{N_1}\varphi_1\|_{L^2_t}\|P_{N_2}\varphi_2\|_{L^2_t}.
\]
and (3.19) is obtained in the same way. $\square$

We are now ready to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Again, by possibly replacing $a(\cdot, \cdot)$ by $\bar{a}(\cdot, \cdot)$ with $\bar{a}(\xi_1, \xi_2) = a(\xi_2, \xi_1)$, we can assume that $N_1 \geq N_2$. We chop the time interval $[0, T]$ into small pieces of length $\sim N_1^{-1}T$, i.e., we define $\{I_{j,N_1}\}_{j \in J_{N_1}}$ with $\#J_{N_1} \sim N_1$ so that $\bigcup_{j \in J_{N_1}} I_{j,N_1} = [0, T]$, $|I_{j,N_1}| \sim N_1^{-1}T$. For $j \in J_{N_1}$, we choose $c_{j,N_1} \in I_{j,N_1}$ at which
\[ \|P_N u_1(t)\|_{L^2_x}^2 \] attains its minimum on \( I_{j,N_i} \). For simplicity, we write \( c_j = c_{j,N_i} \). Since \( u_1 \) and \( u_2 \) satisfy (1.1), on \( I_{j,N_i} \) it holds

\[ P_N u_1(t) = e^{-(t-c_j)\Lambda_{a+1}} P_N u_1(c_j) + F_{1,j}, \quad P_N u_2(t) = e^{-(t-c_j)\Lambda_{a+1}} P_N u_2(c_j) + F_{2,j}, \]
where

\[ F_{i,j} := \int_{c_j}^{t} e^{-(t-\tau)\Lambda_{a+1}} P_N \partial_x f_i(\tau) d\tau = \int_{c_j}^{t} e^{-t\Lambda_{a+1}} g_{N_i}(\tau) d\tau \]

with \( g_{N_i}(\tau) = e^{t\Lambda_{a+1}} P_N \partial_x f_i(\tau) \) for \( i = 1, 2 \). Therefore \( \|\Lambda_a(P_N u_1, P_N u_2)\|_{L^2([0,T];L^2)} \leq \sum_{j \in J_{N_i}} \sum_{m=1}^4 A_{m,j}^2 \), where

\[ 
A_{1,j} := \|\Lambda_a(P_N e^{-(t-c_j)\Lambda_{a+1}} u_1(c_j), P_N e^{-(t-c_j)\Lambda_{a+1}} u_2(c_j))\|_{L^2(I_{j,N_i};L^2)}, \\
A_{2,j} := \|\Lambda_a(P_N e^{-(t-c_j)\Lambda_{a+1}} u_1(c_j), F_{2,j})\|_{L^2(I_{j,N_i};L^2)}, \\
A_{3,j} := \|\Lambda_a(F_{1,j}, P_N e^{-(t-c_j)\Lambda_{a+1}} u_2(c_j))\|_{L^2(I_{j,N_i};L^2)}, \\
A_{4,j} := \|\Lambda_a(F_{1,j}, F_{2,j})\|_{L^2(I_{j,N_i};L^2)}.
\]

For the contribution of \( A_{1,j} \), \( j \in J_{N_i} \), (3.18) and the definition of \( c_j = c_{j,N_i} \) lead to

\[ \sum_{j \in J_{N_i}} A_{1,j}^2 \lesssim \sum_{j \in J_{N_i}} T^{\frac{1}{4}} N_1^{-\frac{\eta}{2}} \|P_N u_1(c_j)\|_{L^2_x}^2 \|P_N u_2(c_j)\|_{L^2_x}^2 \]
\[ \lesssim T^{\frac{1}{4}} \|u_2\|_{L^8_x L^4_x}^2 \sum_{j \in J_N} N_1^{-\frac{\eta}{2}} |I_{j,N_i}|^{-1} \int_{I_{j,N_i}} \|P_N u_1(t)\|_{L^2}^2 dt \]
\[ \lesssim T^{-\frac{1}{4}} N_1^{-\frac{\eta}{2}} \|P_N u_1\|_{L^8_T L^4_x}^2 \|P_N u_2\|_{L^8_T L^4_x}^2 . \]

For the contribution of \( A_{2,j} \), \( j \in J_{N_i} \), we first notice that according to the definition (3.1) of \( \Lambda_a \), using the space Fourier transform, it is easy to check that

\[ \Lambda_a(P_N e^{-(t-c_j)\Lambda_{a+1}} u_1(c_j), F_{2,j}) = \int_{c_j}^{t} \Lambda_a(P_N e^{-(t-c_j)\Lambda_{a+1}} u_1(c_j), e^{-t\Lambda_{a+1}} g_{N_2}(t_2)) dt_2 . \]

Then we see from the Minkowski inequality, (3.18), (2.8) and the unitarity of \( e^{-t\Lambda_{a+1}} \) in \( L^2(\mathbb{T}) \) that

\[ \sum_{j \in J_{N_i}} A_{2,j}^2 \lesssim \sum_{j \in J_{N_i}} \left( \int_{I_{j,N_i}} \|\Lambda_a(P_N e^{-(t-c_j)\Lambda_{a+1}} u_1(c_j), e^{-t\Lambda_{a+1}} g_{N_2}(t_2))\|_{L^2(I_{j,N_i};L^2)} dt_2 \right)^2 \]
\[ \lesssim \sum_{j \in J_{N_i}} T^{\frac{1}{2}} N_1^{-\frac{\eta}{2}} N_2^2 \|P_N u_1(c_j)\|_{L^2_x}^2 \left( \int_{I_{j,N_i}} \|e^{t_2\Lambda_{a+1}} P_{N_2} f_2(t_2)\|_{L^2_x}^2 dt_2 \right)^2 \]
\[ \lesssim T^{\frac{1}{2}} N_1^{-\frac{\eta}{2}} N_2^2 \|P_{N_2} f_2\|_{L^8_x L^4_x}^2 (N_1^{-1} T)^2 \sum_{j \in J_{N_i}} |I_{j,N_i}|^{-1} \int_{I_{j,N_i}} \|P_N u_1(t)\|_{L^2_x}^2 dt \]
\[ \lesssim T^{\frac{3}{2}} N_1^{-\frac{\eta}{2}} \|P_N u_1\|_{L^8_T L^4_x}^2 \|P_{N_2} f_2\|_{L^8_T L^4_x}^2 . \]
Here, we used the definition of $c_j = c_{j, N_1}$ again in the third inequality. The contribution of $A_{3,j}$, $j \in J_{N_1}$, is estimated in the same way as above:

$$
\sum_{j \in J_{N_1}} A_{3,j}^2 \lesssim T^{\frac{1}{2}} N_1^{-\frac{9}{4}} N_1^2 \sum_{j \in J_{N_1}} \|P_{N_2} u_2(c_j)\|_{L_x^2}^2 \left( \int_{I_{j, N_1}} \|P_{N_1} f_1(t_1)\|_{L_x^2} dt_1 \right)^2 \\
\lesssim T^{\frac{1}{2}} \|P_{N_2} u_2\|_{L_x^2 L_t^\infty}^2 N_1^{-\frac{9}{4}} N_1^2 \sum_{j \in J_{N_1}} |I_{j, N_1}| \int_{I_{j, N_1}} \|P_{N_1} f_1(t_1)\|_{L_x^2} dt_1 \\
\lesssim T^{\frac{3}{2}} N_1^{1-\frac{9}{4}} \|P_{N_1} f_1\|_{L_t^\infty L_x^2}^2 \|P_{N_2} u_2\|_{L_x^2 L_t^\infty}^2.
$$

Here, we used the Hölder inequality in $t_2$ in the second inequality. Finally, to estimate the contribution of $A_{4,j}$, $j \in J_{N_1}$, we first notice that

$$
\Lambda_a(F_{1,j}, F_{2,j}) = \int_{c_j} \int_{c_j} \Lambda_a(e^{-t L_{\alpha+1}} g_{N_1}(t_1), e^{-t L_{\alpha+1}} g_{N_2}(t_2)) dt_2 dt_1
$$

and use the Minkowski inequality together with (3.18) and (2.8) to get

$$
\sum_{j \in J_{N_1}} A_{4,j}^2 \lesssim \sum_{j \in J_{N_1}} \left( \int_{I_{j, N_1}} \int_{I_{j, N_1}} \|\Lambda_a(e^{-t L_{\alpha+1}} g_{N_1}(t_1), e^{-t L_{\alpha+1}} g_{N_2}(t_2))\|_{L^2(I_{j, N_1}; L^2)} dt_2 dt_1 \right)^2 \\
\lesssim T^{\frac{1}{2}} N_1^{-\frac{9}{4}} N_2^2 \sum_{j \in J_{N_1}} \left( \int_{I_{j, N_1}} \|P_{N_1} f_1(t_1)\|_{L_x^2} dt_1 \right)^2 \left( \int_{I_{j, N_1}} \|P_{N_2} f_2(t_2)\|_{L_x^2} dt_2 \right)^2 \\
\lesssim T^{\frac{1}{2}} \|P_{N_2} f_2\|_{L_x^2 L_t^\infty}^2 N_1^{-\frac{9}{4}} N_2^2 \sum_{j \in J_{N_1}} |I_{j, N_1}|^3 \int_{I_{j, N_1}} \|P_{N_1} f_1(t_1)\|_{L_x^2}^2 dt_1 \\
\lesssim T^{\frac{3}{2}} N_1^{1-\frac{9}{4}} \|P_{N_1} f_1\|_{L_t^\infty L_x^2}^2 \|P_{N_2} f_2\|_{L_x^2 L_t^\infty}^2
$$

since $N_1 \geq N_2$. This completes the proof. \qed

**Proof of Proposition 3.2.** The strategy of the proof is exactly the same as the one of Proposition 3.1 but based on (3.19) instead of (3.18). \qed

### 4. A priori estimate

#### 4.1. Preliminary technical estimates

Let us denote by $1_T$ the characteristic function of the interval $[0, T]$. For nonresonant interactions, we recover the derivative loss by Bourgain type estimates. For that purpose, we first use $1_T$ to extend a function on $[0, t]$ to a function on $\mathbb{R}$. As pointed out in [28], $1_T$ does not commute with $Q_L$. Moreover, we use $X^{s-1,1}$--norm whereas $1_T$ belongs to $H^s(\mathbb{R})$ for $s < 1/2$. To avoid this difficulty, following [28], we decompose $1_T$ as

$$
1_T = 1_{T, R}^{\text{low}} + 1_{T, R}^{\text{high}}, \quad \text{with } F_t(1_{T, R}^{\text{low}}) = \chi(\tau/R)F_t(1_T)(\tau),
$$

for some $R > 0$ to be fixed later. See also Remark 4.1 in [31].
In what follows, we prepare estimates and fix notation which will be used for nonresonant interactions in the proofs of Propositions 4.6 and 5.1.

**Lemma 4.1** (Lemma 3.5 in [30]). Let \( 1 \leq p \leq \infty \) and let \( L \) be a nonhomogeneous dyadic number. Then the operator \( Q_{\leq L} \) is bounded in \( L^p_t L^2_x \) uniformly in \( L \). In other words,

\[
\| Q_{\leq L} u \|_{L^p_t L^2_x} \lesssim \| u \|_{L^p_t L^2_x},
\]

for all \( u \in L^p_t L^2_x \) and the implicit constant appearing in (4.2) does not depend on \( L \).

**Lemma 4.2** (Lemma 3.6 in [30]). For any \( R > 0 \) and \( T > 0 \), it holds

\[
\| 1_{T,R}^\text{high} \|_{L^1} \lesssim T \wedge R^{-1},
\]

\[
\| 1_{T,R}^\text{low} \|_{L^1} \lesssim T
\]

and

\[
\| 1_{T,R}^\text{high} \|_{L^\infty} + \| 1_{T,R}^\text{low} \|_{L^\infty} \lesssim 1.
\]

**Lemma 4.3** (Lemma 3.7 in [30]). Assume that \( T > 0, R > 0 \), and \( L \gg R \). Then, it holds

\[
\| Q_L(1_{T,R}^\text{low} u) \|_{L^2_{t,x}} \lesssim \| Q_{\sim L} u \|_{L^2_{t,x}},
\]

for all \( u \in L^2(\mathbb{R}_t \times \mathbb{T}_x) \).

**Definition 5.** Let \( j \in \mathbb{N} \). We define \( \Omega_j(\xi_1, \ldots, \xi_{j+1}) : \mathbb{Z}^{j+1} \to \mathbb{R} \) as

\[
\Omega_j(\xi_1, \ldots, \xi_{j+1}) := \sum_{n=1}^{j+1} p_{\alpha+1}(\xi_n)
\]

for \( (\xi_1, \ldots, \xi_{j+1}) \in \mathbb{Z}^{j+1} \), where \( p_{\alpha+1} \) satisfies Hypothesis 1.

**Lemma 4.4.** Let \( k \geq 1 \) and \( (\xi_1, \ldots, \xi_{k+2}) \in \mathbb{Z}^{k+2} \) satisfy \( \sum_{j=1}^{k+2} \xi_j = 0 \). Assume that \( |\xi_1| \sim |\xi_2| \gtrsim |\xi_3| \) if \( k = 1 \) or \( |\xi_1| \sim |\xi_2| \gtrsim |\xi_3| \gg k \max_{j \geq 4} |\xi_j| \) if \( k \geq 2 \). Then,

\[
|\Omega_{k+1}(\xi_1, \ldots, \xi_{k+2})| \gtrsim |\xi_3|^\alpha
\]

for \( |\xi_1| \gg (\max_{\xi \in [0,\xi_0]} |p'_{\alpha+1}(\xi)|)^{\frac{1}{\alpha}} \).

**Proof.** See Lemma 4.4 in [31].
4.2. Estimates on solutions to (1.1).

Lemma 4.5. Let \( \{\omega_N\} \) be an acceptable frequency weight. Let \( 0 < T < 1 \), \( s > 1/2 \) and \( u \in L^\infty_t H^s_\omega \) be a solution to (1.1) associated with an initial datum \( u_0 \in H^s_\omega(\mathbb{T}) \). Then \( u \in Z^s_\omega,T \) and it holds

\[
\|u\|_{Z^s_\omega,T} \lesssim \|u\|_{L^\infty_t H^s_\omega} + G(\|u\|_{L^\infty_t H^s_\omega}),
\]

where \( G = G[f] \) is a smooth function that is increasing and non negative on \( \mathbb{R}_+ \). Moreover, for any couple \( (u,v) \) \( \in (L^\infty_t H^s)^2 \) of solutions to (1.1) associated with a couple of initial data \( (u_0,v_0) \) \( \in (H^s(\mathbb{T}))^2 \) it holds

\[
\|u - v\|_{Z^{s-1}_\omega} \lesssim \|u - v\|_{L^\infty_t H^{s-1}_\omega} + G(\|u\|_{L^\infty_t H^s_\omega} + \|v\|_{L^\infty_t H^s_\omega})\|u - v\|_{L^\infty_t H^{s-1}_\omega}.
\]

Proof. See Lemma 4.7 in [31]. \( \square \)

The following proposition is one of main estimates in the present paper.

Proposition 4.6 (A priori estimate). Let \( \{\omega_N^{(\delta)}\} \) be an acceptable frequency weight with \( \delta \leq 2 \). Let \( 0 < T < 1 \), \( \alpha \in [1,2] \), and \( 2 > s > 1/2 \) with \( s \geq s(\alpha) := 1 - \frac{\alpha}{2} \). Let \( u \in L^\infty_t H^s_\omega \) be a solution to (1.1) emanating from \( u_0 \in H^s_\omega(\mathbb{T}) \) on \([0,T]\). Then there exists a smooth function \( G = G[f] \) that is increasing and non negative on \( \mathbb{R}_+ \) such that

\[
\|u\|_{L^\infty_t H^s_\omega} \leq \|u_0\|_{H^s_\omega} + T^{\nu}G(\|u\|_{Z^{s(\alpha)}_T} + \|u\|_{Z^{s-1}_T})\|u\|_{Z^{s-1}_\omega,T} \|u\|_{L^\infty_t H^s_\omega},
\]

where \( \nu = s(\alpha) - 1/2 \) whenever \( \alpha \in [1,2] \) and \( \nu = 0^+ \) for \( \alpha = 2 \).

Remark 4.1. The strategy to show estimate (4.9) (and also (5.1) on the difference) is two-fold. For the non-resonant cases, we use Bourgain type estimates, which is almost identical to [31]. On the other hand, for the resonant cases, we apply the improved bilinear Strichartz estimates (3.2) and (3.4) instead of the improved Strichartz estimates which were used in [31]. It is worth noticing that (3.2) does not improve linear estimates when \( \alpha = 1 \) (see Remark 3.1). This is the reason why our main result (Theorem 1.1) is the same as our previous result [31, Theorem 1.1] when \( \alpha = 1 \).

Proof. First we notice that according to Lemma 4.5 it holds \( u \in Z^s_\omega,T \). By using (1.1), we have

\[
\frac{d}{dt}\|P_N u(t,\cdot)\|^2_{L^2_\omega} = -2 \int_T P_N \partial_x(f(u))P_N u dx.
\]
Fixing $t \in [0, T]$, integration in time between 0 and $t$, multiplication by $\omega_N^2 (1 + N)^{2s}$ and summation over $N$ yield

$$\|u(t)\|_{H^s_x}^2 \leq \|u_0\|_{H^s_x}^2 + 2 \sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u \, dx \, dt' \right|$$  \hspace{1cm} (4.10)

since $P_0 \partial_x u = 0$. Now we rewrite $f(u) - f(0)$ as $\sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} u^k$ and we notice that for any fixed $N \in 2^N$, 

$$\int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u \, dx \, dt' = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} \int_0^t \int_T u^k P_N^2 \partial_x u \, dx \, dt' \quad \text{(4.11)}$$

Indeed

$$\sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \int_T |u|^k P_N^2 \partial_x u | \, dx \, dt' \lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \|u\|_{L^2_x}^k \|u\|_{L^2_x} \, dt'$$

$$\lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \|u\|_{L^\infty_x}^{k-1} \|u\|_{L^2_x}^2 \, dt'$$

$$\lesssim NTG(\|u\|_{L^\infty_{T,x}}) \|u\|_{L^\infty_x L^2_x}^2 < \infty,$$

that proves (4.11) by Fubini-Lebesgue’s theorem. (4.11) together with Fubini-Tonelli’s theorem then ensure that

$$\sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u \, dx \, dt' \right| \leq \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} I_k^t \quad \text{(4.12)}$$

where

$$I_k^t := \sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T u^k P_N^2 \partial_x u \, dx \, dt' \right|$$

By integration by parts it is easy to check that $I_1^t = 0$. We set

$$C_0 := \|u\|_{Z^{s(\infty)}} + \|u\|_{Z^{s_0}} \quad \text{with} \quad s_0 \in ]1/2, s]. \quad \text{(4.13)}$$

Let us now prove that for any $k \geq 1$ it holds

$$I_{k+1}^t \leq C^k T^4 G(C_0) C_0^k (\|u\|_{X_T^{2,1,1}} + \|u\|_{L^\infty_T H_x^2}) \|u\|_{L^\infty_T H_x^2}, \quad \text{ (4.14)}$$

that clearly leads (4.9) by taking (4.10) and (4.12) into account since $\sum_{k \geq 1} \frac{|f^{(k)}(0)|}{(k+1)!} C^k C_0^k < \infty$.

In the sequel we fix $k \geq 1$. For simplicity, for any positive numbers $a$ and $b$, the notation $a \lesssim_k b$ means there exists a positive constant $C > 0$ independent of $k$ such that

$$a \leq C^k b. \quad \text{(4.15)}$$
Remark that $a \leq k^mb$ for $m \in \mathbb{N}$ can be expressed by $a \lesssim_k b$ too since an elementary calculation shows $k^m \leq mle^k$ for $m \in \mathbb{N}$. Here, $e$ is Napier's constant. The contribution of the sum over $N \lesssim 1$ in $I_{k+1}^t$ is easily estimated by

$$
\sum_{N \leq 1} N^{2s} \left| \int_0^t \int_T u^{k+1} P_N^2 \partial_x u \, dx \, dt \right| \leq T \sum_{N \leq 1} \| u \|^k_{L_T^\infty L_x^2} \| u \|_{L_T^\infty L_x^2} \| P_N^2 u \|_{L_T^\infty L_x^2} \lesssim_k T C_0^k \| u \|^2_{L_T^\infty H_x^s}.
$$

(4.16)

It thus remains to bound the contribution of the sum over $N \gg 1$ in $I_{k+1}^t$. Putting

$$
A(\xi_1, \ldots, \xi_{k+2}) := \sum_{j=1}^{k+2} \phi_N^2(\xi_j) \xi_j,
$$

$$
A_1(\xi_1, \xi_2) := \phi_N^2(\xi_1) \xi_1 + \phi_N^2(\xi_2) \xi_2,
$$

$$
A_2(\xi_3, \ldots, \xi_{k+2}) := \sum_{j=3}^{k+2} \phi_N^2(\xi_j) \xi_j,
$$

so that $A(\xi_1, \ldots, \xi_{k+2}) = A_1(\xi_1, \xi_2) + A_2(\xi_3, \ldots, \xi_{k+2})$. We see from the symmetry that

$$
\int_T u^{k+1} P_N^2 \partial_x u \, dx = \frac{i}{k + 2} \sum_{\xi_1 + \cdots + \xi_{k+2} = 0} A(\xi_1, \ldots, \xi_{k+2}) \prod_{j=1}^{k+2} \hat{u}(\xi_j)
$$

(4.17)

$$
= \frac{i}{k + 2} \sum_{N_1, \ldots, N_{k+2}} \sum_{\xi_1 + \cdots + \xi_{k+2} = 0} A(\xi_1, \ldots, \xi_{k+2}) \prod_{j=1}^{k+2} \phi_{N_j}(\xi_j) \hat{u}(\xi_j).
$$

By symmetry we can assume that $N_1 \geq N_2 \geq N_3$ if $k = 1$, $N_1 \geq N_2 \geq N_3 \geq N_4 = \max_{j \geq 4} N_j$ if $k \geq 2$. We notice that the cost of this choice is a constant factor less than $(k + 2)^5$. It is also worth noticing that the frequency projection operator $P_N$ ensures that the contribution of any $N_1 \leq N/4$ does cancel. We thus can assume that $N_1 \geq N/4$ and that $N_2 \geq N_1/k$ with $N_2 \geq 1$.

First, we consider the contribution of $A_2$. It suffices to consider the contribution of $(\phi_N(\xi_3))^2 \xi_3$ since the contributions of $(\phi_N(\xi_j))^2 \xi_j$ for $j \geq 4$ are clearly simplest. Note that $N_3 \sim N$ in this case. By the Bernstein inequality, we have

$$
\sum_k \| P_k u \|_{L_T^\infty L_x^s} \lesssim \sum_k (1 + K)^{0-} \| u \|_{L_T^\infty H_x^s} \lesssim \| u \|_{L_T^\infty H_x^{s_0}} \lesssim C_0.
$$

(4.18)

where $0- \omega$ denotes a number slightly less than 0 (see Subsection 2.1). We divide the contribution of $(\phi_N(\xi_3))^2 \xi_3$ into two cases: 1. $N_3 \gg kN_4$ or $k = 1$ and 2. $N_3 \lesssim kN_4$. 

Set
\[ J_{t}^{A_{2}} := \sum_{N>1} \sum_{N_{1}, \ldots, N_{k+2}} \omega_{N}^{2} N^{2s} \left| \int_{0}^{t} \int_{T} \partial_{x} P_{N}^{2} P_{\omega} u \prod_{j=1, j \neq 3}^{k+2} P_{N_{j}} u \, dxdt' \right| . \]

Note that \( N \gg 1 \) ensures that \( N_{3} \gg 1 \).

**Case 1:** \( N_{3} \gg kN_{4} \) or \( k = 1 \). By impossible frequency interactions, we obtain \( N_{1} \sim N_{2} \). In this case we make use of Lemma 4.4 to close our estimate. For that purpose, we first take the extensions \( \tilde{u} = \rho_{T}(u) \) of \( u \) defined in (2.3). With a slight abuse of notation, we define the following functional:

\[
J_{\infty}^{A_{2}}(u_{1}, \ldots, u_{k+2}) := \sum_{N>1} \sum_{N_{1}, \ldots, N_{k+2}} \omega_{N}^{2} N^{2s} \left| \int_{R} \int_{T} \partial_{x} P_{N}^{2} u_{3} \prod_{j=1, j \neq 3}^{k+2} u_{j} \, dxdt' \right|. \tag{4.19}
\]

Setting \( R = N_{1}^{3} N_{3}^{3} \), we decompose \( J_{t}^{A_{2}} \) as

\[
J_{t}^{A_{2}} \leq J_{t}^{A_{2}}(P_{N_{1}} 1_{t,R}^{\text{high}}, P_{N_{2}} 1_{t,R}^{\text{high}}, P_{N_{3}} 1_{t,R}^{\text{high}}, \ldots, P_{N_{k+2}} 1_{t,R}^{\text{high}})
+ J_{t}^{A_{2}}(P_{N_{1}} 1_{t,R}^{\text{low}}, P_{N_{2}} 1_{t,R}^{\text{low}}, P_{N_{3}} 1_{t,R}^{\text{low}}, \ldots, P_{N_{k+2}} 1_{t,R}^{\text{low}})
+ J_{t}^{A_{2}}(P_{N_{1}} 1_{t,R}^{\text{high}}, P_{N_{2}} 1_{t,R}^{\text{low}}, P_{N_{3}} 1_{t,R}^{\text{low}}, \ldots, P_{N_{k+2}} 1_{t,R}^{\text{low}})
=: J_{t}^{A_{2,1}} + J_{t}^{A_{2,2}} + J_{t}^{A_{2,3}}.
\]

For \( J_{t}^{A_{2,1}} \), we see from (4.3) that \( \|1_{t,R}^{\text{high}}\|_{L^{1}} \lesssim T^{1/2} N_{1}^{3/4} N_{3}^{-1} \), which together with (2.6) gives

\[
J_{t}^{A_{2,1}} \lesssim \sum_{N_{1}, \ldots, N_{k+2}} \omega_{N_{1}} \omega_{N_{2}} N_{3}^{2s+1} \|1_{t,R}^{\text{high}}\|_{L^{1}} \|P_{N_{1}} \tilde{u}\|_{L^{1}_{t} L_{x}^{\infty}} \|P_{N_{2}} \tilde{u}\|_{L^{1}_{t} L_{x}^{\infty}} \prod_{j=3}^{k+2} \|P_{N_{j}} \tilde{u}\|_{L^{1}_{t} L_{x}^{\infty}}
\lesssim_{k} T^{1/2} \|\tilde{u}\|_{L_{t,x}^{1/2}}^{k} \|\tilde{u}\|_{L_{t,x}^{\infty}}^{2} \sum_{N_{1}} N_{1}^{-1/4} \lesssim_{k} T^{1/2} C_{0}^{k} \|u\|_{L_{t,x}^{1/2}}^{2}
\]

since \( N_{1} \geq N_{2} \geq N_{3} \). Here, we used (1.3) for \( P_{N_{j}} \tilde{u}, j = 3, \ldots, k+2 \). A similar argument with (4.5) yields the same bound for \( J_{t}^{A_{2,2}} \) as that of \( J_{t}^{A_{2,1}} \). For \( J_{t}^{A_{2,3}} \), we see from Lemma 4.4 that \( |\Omega_{k+1}| \gtrsim N_{3} N_{1} \gg R \) since \( N_{3} \gg 1 \). Defining \( L := N_{3} N_{1}^{-\alpha} \), we split \( J_{t}^{A_{2,3}} \) into \( k+2 \) parts:

\[
J_{t}^{A_{2,3}} \leq J_{t}^{A_{2}}(P_{N_{1}} Q_{\leq L} 1_{t,R}^{\text{low}}, P_{N_{2}} 1_{t,R}^{\text{low}}, P_{N_{3}} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})
+ J_{t}^{A_{2}}(P_{N_{1}} Q_{\leq L} 1_{t,R}^{\text{low}}, P_{N_{2}} \tilde{u}, P_{N_{3}} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})
+ J_{t}^{A_{2}}(P_{N_{1}} Q_{\leq L} 1_{t,R}^{\text{low}}, P_{N_{2}} Q_{\leq L} 1_{t,R}^{\text{low}}, P_{N_{3}} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})
+ \cdots
+ J_{t}^{A_{2}}(P_{N_{1}} Q_{\leq L} 1_{t,R}^{\text{low}}, P_{N_{2}} Q_{\leq L} 1_{t,R}^{\text{low}}, P_{N_{3}} Q_{\leq L} \tilde{u}, \ldots, P_{N_{k+2}} Q_{\leq L} \tilde{u})
=: J_{t}^{A_{2,3,1}} + \cdots + J_{t}^{A_{2,3,k+2}}.
\]
We also see from (4.3) that for $K \geq 1$

$$\| P_{K, t, R} \tilde{u} \|_{L^2_t, L^1_x} \leq \| P_{K, t} \tilde{u} \|_{L^2_t, L^1_x} + \| P_{K, t, R} \tilde{u} \|_{L^2_t, L^1_x}$$

$$\lesssim \| P_{K, t} \tilde{u} \|_{L^2_t, L^1_x} + T^2 \delta \| P_{K, t} \tilde{u} \|_{L^\infty L^2_x}. \quad (4.20)$$

For $J_{\infty, 3, 1}^{(2)}$, Lemma 2.1, the Hölder inequality, (4.6) and (4.20) imply that

$$J_{\infty, 3, 1}^{A_2} \lesssim \sum_{N_1, \ldots, N_{k+2}} \omega_{N_1 \omega_{N_2}} N_3^{2s+1} \| P_{N_1} Q \lesssim L (1_{t, \tilde{R}}^\omega \tilde{u}) \|_{L^I_t L^2_x} \| P_{N_2} \tilde{u} \|_{L^2_t, L^1_x} \prod_{j=3}^{k+2} \| P_{N_j} \tilde{u} \|_{L^\infty_t L^2_x}$$

$$\lesssim_k \| \tilde{u} \|_{L^\infty_t H^j_x} \sum_{N_1 \geq 1} N_1^{2s-\alpha} \| P_{N_1} \tilde{u} \|_{X^0, 1} \| P_{N_1} \|_{L^\infty_t L^2_x}$$

$$+ T^2 \| \tilde{u} \|_{L^\infty_t H^j_x} \sum_{N_1 \geq 1} \sum_{N_3 \geq 1} N_1^{-\alpha} \| N_3^{2s-\frac{1}{2}} P_{N_1} \tilde{u} \|_{X^0, 1} \| P_{N_3} \tilde{u} \|_{L^\infty_t L^2_x}$$

$$\lesssim_k T^2 \| \tilde{u} \|_{L^\infty_t H^j_x} \| \tilde{u} \|_{X^0, 1} \| P_{N_1} \|_{L^\infty_t L^2_x} \lesssim_k T \frac{1}{2} C_0 \| u \|_{L^2_t H^j_x} \| u \|_{Z^1_{\infty, T}}.$$

Here we used $\alpha \geq 1$ so that $N_1^{-\alpha} \leq N_1^{-1}$. By the same way, it is easy to check that

$$J_{\infty, 3, 2}^{A_2} \lesssim_k T \frac{1}{2} C_0 \| u \|_{L^\infty_t H^j_x} \| u \|_{Z^1_{\infty, T}}.$$

Next, we consider the contribution $J_{\infty, 3, 3}^{A_2}$. Lemma 2.1, the Hölder inequality, the Bernstein inequality, (4.3) and (4.2) show

$$J_{\infty, 3, 3}^{A_2} \lesssim \sum_{N_1, \ldots, N_{k+2}} \omega_{N_1 \omega_{N_2}} N_3^{2s+1} \| P_{N_1} Q \lesssim L (1_{t, \tilde{R}}^\omega \tilde{u}) \|_{L^I_t L^2_x} \| P_{N_2} \tilde{u} \|_{L^2_t, L^1_x} \prod_{j=4}^{k+2} \| P_{N_j} \tilde{u} \|_{L^\infty_t L^2_x}$$

$$\times \| P_{N_3} Q \lesssim L \tilde{u} \|_{L^2_t L^\infty_x} \prod_{j=4}^{k+2} \| P_{N_j} \tilde{u} \|_{L^\infty_t L^2_x}$$

$$\lesssim_k T^2 \| \tilde{u} \|_{L^\infty_t H^j_x} \sum_{N_1 \geq 1} \sum_{N_3 \geq 1} N_1^{2s-\alpha} \| P_{N_1} \tilde{u} \|_{L^\infty_t H^j_x} \| P_{N_3} \tilde{u} \|_{X^0, 1} \| P_{N_3} \tilde{u} \|_{L^\infty_t H^j_x}$$

$$\lesssim_k T^2 \frac{1}{2} C_0 \| \tilde{u} \|_{X^0, 1} \| P_{N_1} \|_{L^\infty_t L^2_x} \lesssim_k T \frac{1}{2} C_0 \| u \|_{L^\infty_t H^j_x} \| u \|_{Z^1_{\infty, T}}.$$

since $2 - 3\alpha < 0$. In a similar manner, we can evaluate the contribution $J_{\infty, 3, j}^{A_2}$ for $j = 4, \ldots, k + 2$ by the same bound as $J_{\infty, 3, 3}^{A_2}$.

**Case 2:** $N_3 \leq k N_4$. Note that $N_4 \geq 1$ since $N_3 \gg 1$. We use Proposition 3.1 with $a \equiv 1$ for $P_{N_2} \tilde{u} P_{N_3} \tilde{u}$ and $P_{N_1} \tilde{u} \tilde{u}_x P_{N_3} \tilde{u}$. In this case we can share the lost derivative on four functions. For simplicity, we put

$$U_{\omega, K}^{s, p} := \| P_K u \|_{L^p_t H^j_x} + \| P_K (u(f) - f(0)) \|_{L^p_t H^j_x} \quad (4.21)$$
for $s \geq 0$, $2 \leq p \leq \infty$, $\{\tilde{\omega}_N\}_N$ is an acceptable frequency weight. Recall that $H^s_x(\mathbb{T}) = H^s(\mathbb{T})$ when $\tilde{\omega}_N \equiv 1$. By (1.18), Proposition 3.1 with Remark 3.3 Young’s inequality, (1.21) and (2.8), we can see that

$$
J^A_1 \lesssim_k C_0^{t-k-2} \sum_{N \gg 1} \sum_{N_1 \gg N \gg N_3 \gg N_4} \omega_N^2 N_{2s}^2 \|P_{N_1}u \partial_x P_{N_3}u\|_{L^2_{T,x}} \|P_{N_2}uP_{N_4}u\|_{L^2_{T,x}}
$$

$$
\lesssim_k C_0^{t-k-2} T^{-\frac{1}{2}} \sum_{N_1 \gg N_3 \gg N_4} \omega_N^2 N_1^{\frac{1}{2}} - \frac{\alpha}{4} N_2^{\frac{1}{2}} - \frac{\alpha}{4} N_3^{\frac{1}{2}} N_4^{s+1} U_{1,N_1}^{0,2} U_{1,N_3}^{0,\infty} U_{1,N_4}^{0,\infty}
$$

$$
\lesssim_k C_0^{t-k-2} G(C_0) T^{-\frac{1}{2}} \|u\|_{L_{T,x}^\infty H^s_x} \sum_{N_1 \geq N_2} \left( \frac{N_2}{N_1} \right)^{s-s(\alpha) + \frac{1}{2}} U_{\omega,N_1}^{s,2} U_{\omega,N_2}^{s,2}
$$

$$
\lesssim_k T^\frac{1}{2} C_0^k G(C_0) \|u\|_{L_{T,x}^\infty H^s_x}^2.
$$

Here, in the first inequality, we put $L_{T,x}^\infty$ norm on $P_{N_j}u$ for $j = 5, \ldots, k+2$, and used (1.18). Recall $s(\alpha) = 1 - \frac{\alpha}{4}$.

Next, we consider the contribution of $A_1$. With a slight abuse of notation, put

$$
J^A_1 := \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N_{2s}^2 \left| \int_0^T \int_T \Pi(P_{N_1}u, P_{N_2}u) \prod_{j=3}^{k+2} P_{N_j}u \, dx \, dt \right|,
$$

where $\Pi(u, v)$ is defined in (2.11). Note that $P_0 \Pi(P_{N_1}u, P_{N_2}u) = 0$. As in the estimate on the contribution of $A_2$, we divide the study of $J^A_1$ into three cases: 1. $N_3 \gg kN_4$ or $k = 1, 2$. $N_2 \lesssim N_3 \lesssim kN_4$ and 3. $N_2 \gg N_3$ and $N_3 \lesssim kN_4$. Note that $N \gg 1$ ensures that $N_1 \gg 1$.

**Case 1:** $N_3 \gg kN_4$ or $k = 1$. We can use the argument of Case 1 for $A_2$ combining with Lemma 2.4. Since we only use Bourgain type estimates in this configuration, we do not improve the result in [31] here. We omit the proof since the complete proof can be found in Case 2 of the proof of Proposition 4.8 in [31].

**Case 2:** $N_2 \lesssim N_3 \lesssim kN_4$. In this case, we have $N_3 \gtrsim N_1/k$, so that exactly the same proof as in the Case 3 for $A_2$ is applicable to this case (and we do not need Lemma 2.4).

**Case 3:** $N_2 \gg N_3$ and $N_3 \lesssim kN_4$. In this case it holds $N_1 \sim N_2 \sim N \gg N_3$. We also note that $N_3, N_4 \geq 1$ since $J^A_1 = 0$ otherwise. We would like to use Lemma 2.4 that corresponds to integration by parts. The problem we meet here is how to combine Lemma 2.4 for $\Pi(P_{N_1}u, P_{N_2}u)$ with the improved bilinear Strichartz estimate (Proposition 3.1) for $P_{N_1}uP_{N_2}u$.

Therefore, we consider $A_1$ in more detail. The Taylor theorem implies that for any $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $|\xi_1| \sim |\xi_2| \sim N$ there exists $\theta = \theta(\xi_1, \xi_2) \in \mathbb{R}$ such that $|\theta| \sim N$.
and
\[ \phi_N^2(\xi_1)\xi_1 = \phi_N^2(-\xi_2)\xi_2 + (\xi_1 + \xi_2)(\phi_N^2(x)x)'(-\xi_2) + \frac{1}{2}(\xi_1 + \xi_2)^2(\phi_N^2(x)x)''(\theta). \]

Since \( \phi \) and \( \phi'' \) are even and \( \phi' \) is odd, we have
\[ A_1(\xi_1, \xi_2) = \phi_N^2(\xi_1)\xi_1 + \phi_N^2(\xi_2)\xi_2 = a_1(\xi_2)(\xi_1 + \xi_2) + a_2(\xi_1, \xi_2)\frac{(\xi_1 + \xi_2)^2}{N}, \]
where
\[ a_1(\xi_2) = \phi_N^2(\xi_2) + 2\phi_N(\xi_2)\phi'_N(\xi_2)\frac{\xi_2}{N}, \]
\[ a_2(\xi_1, \xi_2) = 2\phi_N(\theta(\xi_1, \xi_2))\phi'_N(\theta(\xi_1, \xi_2)) + (\phi'_N(\theta(\xi_1, \xi_2)))^2\theta(\xi_1, \xi_2) + \phi_N(\theta(\xi_1, \xi_2))\phi''_N(\theta(\xi_1, \xi_2))\frac{\theta(\xi_1, \xi_2)}{N}. \]

Note that the above identities forces \( a_2(\cdot, \cdot) \) to be measurable and that \( \|a_1\|_{L^\infty(\mathbb{R})} + \|a_2\|_{L^\infty(\mathbb{R}^2)} \lesssim 1 \). The advantage of this decomposition of \( A_1(\cdot, \cdot) \) is that \( a_1 \) only depends on the \( \xi_2 \)-variable whereas we gain a factor \( \frac{\xi_1 + \xi_2}{N} \) on the contribution of \( a_2(\cdot, \cdot) \) with respect to the one of \( A_1(\cdot, \cdot) \).

By integration by parts the term involving \( a_1 \) can be rewritten as
\[ \sum_{\xi_1+\ldots+\xi_{k+2}=0} \phi_{N_1}(\xi_1)\tilde{u}(\xi_1)a_1(\xi_2)\phi_{N_2}(\xi_2)\tilde{u}(\xi_2)(\xi_1 + \xi_2)\prod_{j=3}^{k+2} \phi_{N_j}(\xi_j)\tilde{u}(\xi_j) \]
\[ = - \sum_{\xi_1+\ldots+\xi_{k+2}=0} \phi_{N_1}(\xi_1)\tilde{u}(\xi_1)a_1(\xi_2)\phi_{N_2}(\xi_2)\tilde{u}(\xi_2) \left( \sum_{i=3}^{k+2} \xi_i \right) \prod_{j=3}^{k+2} \phi_{N_j}(\xi_j)\tilde{u}(\xi_j). \]

To estimate its contribution, that we will call \( J_{t1}^{a_1} \), we make use of Proposition 3.2. For instance for the most dangerous term (with the derivative on \( u_3 \)) we use Proposition 3.2 with Remark 3.3 and (2.8) on \( \|P_{N_1}uP_{N_2}\partial_x u\|_{L^3_{T,x}} \) and \( \|P_{N_2}(F_x^{-1}(a_1) * u)P_{N_k}u\|_{L^3_{T,x}} \) with the trivial inequality \( \|F_x^{-1}(a_1) * v\|_{L^2} \lesssim \|v\|_{L^2} \), for any \( v \in L^2(\mathbb{T}) \), to get
\[ J_{t1}^{a_1} \lesssim_k C_0^{k-2} \sum_{N_{1} \geq N_{3} \geq N_{4}} \omega_{N_{3}}^{2N_{3}^{2a}} \|P_{N_1}uP_{N_3}\partial_x u\|_{L^3_{T,x}} \|P_{N_2}(\tilde{a}_1 * u)P_{N_4}u\|_{L^2_{T,x}} \]
\[ \lesssim C_0^{k-2} T^{\theta-1} \sum_{N_{1} \geq N_{3} \geq N_{4}, N_{3} \leq kN_{4}} \omega_{N_{3}}^{2N_{3}^{2a}} N_{3}^{\frac{\theta}{2}+1} N_{4}^{\frac{\theta}{2}} U_{1,N_{1}}^{0} U_{1,N_{3}}^{0} U_{1,N_{4}}^{0} \]
\[ \lesssim k^\frac{1}{2} C_0^{k-2} T^{\theta-1} \sum_{N_{1}, N_{3}, N_{4}} N_{3}^{\frac{\theta}{2}} N_{4}^{\frac{\theta}{2}} U_{\omega,N_{1}}^{0} U_{\omega,N_{3}}^{0} U_{\omega,N_{4}}^{0} \]
\[ \lesssim T^{\theta} C_0^{k} G(C_0) \|u\|_{L^\infty_T H^\omega_{x}}^2. \]
where $\hat{a}_1$ is the inverse Fourier transform of $a_1$ (i.e., $F^{-1}_x(a_1)$) and $\theta \in ]0, 1]$. It is important to have a positive $\theta > 0$ in order to close the estimate above. When $\alpha \in [1, 2]$, we can choose $\theta = s(\alpha) - 1/2$. On the other hand, when $\alpha = 2$, we choose $\theta = \min(s_0, 1 - 1/2, 1)$. Here, we used the notation $U^s_{\omega, N}$ defined in (4.21).

Therefore it remains to evaluate the contribution $J^A$ of $a_2(\cdot, \cdot)$, which we denote by $J^A$. It is to evaluate this contribution that we need to prove (3.2) with a Fourier multiplier. We decompose further in $|\xi_1 + \xi_2| \sim M \geq 1$. Noticing that $N_3 \lesssim k N_4$, Proposition 3.1 with Remark 3.3, (4.21), the Young inequality and 2.8 lead to

\[
J^A_t
\]

\[
\lesssim \sum_{N_1 \sim N_3 \sim N_4 \sim N_k \sim N_4} \omega^2 N^{2s - 1} \int_0^t \int \partial_x^2 P_{N} a_2(P_{N} u, P_{N} u) \prod_{j=1}^{k+2} P_{N} u dx dt
\]

\[
\lesssim C_0^{-2} \sum_{N_1 \sim N_3 \sim N_4 \sim N_k \sim N_4} \omega^2 N^{2s - 1} \int_0^t \int \partial_x^2 P_{N} a_2(P_{N} u, P_{N} u) \prod_{j=1}^{k+2} P_{N} u dx dt
\]

\[
\lesssim C_0^{-2} \sum_{N_1 \sim N_3 \sim N_4 \sim N_k \sim N_4} T^{-\frac{1}{4}} N^{2s - 1} \int_0^t \int \partial_x^2 P_{N} a_2(P_{N} u, P_{N} u) \prod_{j=1}^{k+2} P_{N} u dx dt
\]

\[
\lesssim C_0^{-2} T^{-\frac{1}{4}} \sum_{N_1 \sim N_3 \sim N_4 \sim N_k \sim N_4} \omega^2 N^{2s - 1} \int_0^t \int \partial_x^2 P_{N} a_2(P_{N} u, P_{N} u) \prod_{j=1}^{k+2} P_{N} u dx dt
\]

\[
\lesssim T^{-\frac{1}{4}} C_0^{-2} G(C_0) \parallel u \parallel^2_{L^\infty T H^s} \sum_{N_1 \sim N_3 \sim N_k \sim N_4} \left( \frac{N_3}{N} \right)^{1 + \frac{s}{2}} \parallel U^s_{\omega, N} \parallel_{L^2 T H^s}
\]

This completes the proof.

5. ESTIMATE FOR THE DIFFERENCE

We provide the estimate (at the regularity $s - 1$) for the difference $w$ of two solutions $u, v$ of (1.1). In this section, we do not use the frequency envelope, so we always argue on the standard Sobolev space $H^s(\mathbb{T})$.

**Proposition 5.1.** Let $0 < T < 1$, $\alpha \in [1, 2]$ and $2 \geq s > 1/2$ with $s \geq s(\alpha) := 1 - \frac{\alpha}{4}$.

Let $u$ and $v$ be two solutions of (1.1) belonging to $Z^s_T$ and associated with the initial data $u_0 \in H^s(\mathbb{T})$ and $v_0 \in H^s(\mathbb{T})$, respectively. Then there exists a smooth function

\[1 \text{ See [14] for the definition of } s_0.\]
We claim that for any $k \geq 1$ and $z$ defined in (4.15), the contribution of the sum over $N \leq k$ that clearly leads (4.9) by taking (4.10) and (4.12) into account since
\[
\sum_{i=0}^{k-1} u^i v^{k-1-i}
\]
and arguing as in the proof of Proposition 4.6, we see from (5.2) that for $t \in [0, T]
\|w(t)\|_{H^s}^2 \leq \|w_0\|_{H^s}^2 + 2 \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{(k-1)!} \max_{i \in \{0, \ldots, k-1\}} I_{k,i}^t
\]
where $w_0 = u_0 - v_0$ and
\[
I_{k,i}^t := \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} u^i v^{k-1-i} w P_N^2 \partial_x w \, dx dt \right|
\]
It is clear that $I_{k,i}^t = 0$ by the integration by parts. Therefore we are reduced to estimate the contribution of
\[
I_{k+1}^t = \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} z^k w P_N^2 \partial_x w \, dx dt \right|
\]
where $z^k$ stands for $u^i v^{k-i}$ for some $i \in \{0, \ldots, k\}$. We set
\[
C_0 := \|u\|_{Z^s} + \|v\|_{Z^s}
\]
We claim that for any $k \geq 1$ it holds
\[
I_{k+1}^t \leq C^k T^\frac{1}{2} G(C_0) C_0^k \left\|w\right\|_{Z^s} \left\|w\right\|_{L^\infty T^s}.
\]
that clearly leads (4.9) by taking (4.10) and (4.12) into account since $\sum_{k \geq 1} \frac{|f^{(k+1)}(0)|}{k!} C^k C_0^k < \infty$.

In the sequel we fix $k \geq 1$ and we estimate $I_k^t$. We also use the notation $a \leq_k b$ defined in (4.15). The contribution of the sum over $N \leq 1$ in (5.3) is easily estimated thanks to (2.7) by
\[
\sum_{N \leq 1} (1 + N)^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} z^k w P_N^2 \partial_x w \, dx dt \right|
\]
\[
\lesssim T \sum_{N \leq 1} \left\|w\right\|_{L^\infty T^s} \left\|z^k P_N^2 \partial_x w\right\|_{L^\infty T^s} \lesssim_k T \left\|w\right\|_{L^\infty T^s}^2,
\]
since \( s > 1/2 \). In the last inequality, we used \( 1 - s < 1/2 \). Therefore, in what follows, we can assume that \( N \gg 1 \). A similar argument to (4.17) yields

\[
\sum_{N \gg 1} N^{2(s-1)} \left| \int_0^t \int_T z^k w P_N^2 \partial_x w dx dt' \right| \leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_0^t \int_T \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} z_j dx dt' \right| =: J_t,
\]

where \( \Pi(f, g) \) is defined by (2.11) and \( z_i \in \{u, v\} \) for \( i \in \{3, \ldots, k+2\} \). By symmetry, we may assume that \( N_1 \geq N_2 \). Moreover, we may assume that \( N_3 \geq N_4 \) for \( k = 2 \) and \( N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j \) for \( k \geq 3 \). Note again that the cost of this choice is a constant factor less than \( (k+2)^5 \). It is also worth noticing that the frequency projectors in \( \Pi(\cdot, \cdot) \) ensure that \( N_1 \sim N \) or \( N_2 \sim N \) and in particular \( N_1 \gtrsim N \). We also remark that we can assume that \( N_3 \geq 1 \) since the contribution of \( N_3 = 0 \) does vanish by integration by parts. Finally we note that we can also assume that \( N_2 \geq 1 \) since in the case \( N_2 = 0 \) we must have \( N_3 \gtrsim N_1/k \) and it is easy to check that by the Young inequality

\[
J_t \lesssim k T \|w\|_{L^\infty_T H^{-1}_x} \|z_3\|_{L^\infty_T H^s_x} \|P_0 w\|_{L^\infty_T H^s_x} \prod_{j=4}^{k+2} \|z_j\|_{L^\infty_T H^s_x} \lesssim T \|w\|^2_{L^\infty_T H^s_x}.
\]

We consider the following contribution to \( J_t \):

- \( N_4 \gtrsim N_1/k \) (\( k \geq 2 \)) ,
- \( N_1 \gg kN_4 \) and \( N_2 \gtrsim N_3 \) (or \( k = 1 \) and \( N_2 \gtrsim N_3 \)) ,
- \( N_1 \gg kN_4 \) and \( N_2 \ll N_3 \) (or \( k = 1 \) and \( N_2 \ll N_3 \)).

Case 1: \( N_4 \gtrsim N_1/k \). Note that \( N_3, N_4 \geq 1 \) since \( N_1 \gtrsim N \gg 1 \). As an analogous manner to (1.21), we define

\[
W^{s,p}_K := \|P_K w\|_{L^p_T H^{s}_x} + \|P_K(f(u) - f(v))\|_{L^p_T H^{s}_x},
\]

\[
Y^{s,p}_{N_j} := \|P_{N_j} z_j\|_{L^p_T H^{s}_x} + \|P_{N_j}(f(z_j) - f(0))\|_{L^p_T H^{s}_x}.
\]
for $s \in \mathbb{R}$, $2 \leq p \leq \infty$, $K \geq 1$ and $j = 3, \ldots, k + 2$. Proposition 3.1 together with Remark 3.3, (5.5), (5.6), (2.10) and the Young inequality lead to

$$J_{t} \lesssim_{k} C_{0}^{k-2} \sum_{N_{1} \leq N_{2} \leq kN_{3}} N^{2s-2} \left( \| \partial_{x} P_{N_{1}} w P_{N_{2}} w \|_{L_{T,x}^{2}} + \| P_{N_{1}} w \partial_{x} P_{N_{2}} w \|_{L_{T,x}^{2}} \right)$$

$$\lesssim C_{0}^{-2} T^{-\frac{1}{2}} \sum_{N_{2} \leq N_{1} \leq kN_{3}} N_{1}^{s-1} N_{1}^{2} N_{2}^{\frac{3}{2}} \frac{3}{4} W_{N_{1}}^{0,2} W_{N_{2}}^{0,\infty} Y_{N_{3}}^{0,2} Y_{N_{4}}^{0,\infty}$$

$$\lesssim_{k} C_{0}^{k} T^{-\frac{1}{2}} G(C_{0}) \| w \|_{L_{T}^{\infty} H_{x}^{2}} \sum_{N_{1} \leq kN_{3}} \left( \frac{N_{1}}{kN_{3}} \right)^{\frac{1}{2}} W_{N_{1}}^{s-1,2} Y_{N_{3}}^{s(\alpha),2}$$

$$\lesssim_{k} T^{\frac{1}{2}} C_{0}^{k} G(C_{0}) \| w \|^{2}_{L_{T}^{\infty} H_{x}^{s-1}}$$

since $s > 1/2$.

**Case 2:** $N_{1} \gg kN_{4}$ and $N_{2} \gtrsim N_{3}$ (or $k = 1$ and $N_{2} \gtrsim N_{3}$). The contribution of $J_{t}$ in this case can be estimated by the same way as the contribution of $A_{1}$ in Proposition 3.6, replacing $N^{2s}$, $P_{N_{1}} u$, $P_{N_{2}} u$, $P_{N_{j}} u$ for $j = 3, \ldots, k + 2$ by $N^{2(s-1)}$, $P_{N_{1}} w$, $P_{N_{2}} w$, $P_{N_{j}} z_{j}$ for $j = 3, \ldots, k + 2$, respectively.

**Case 3:** $N_{1} \gg kN_{4}$ and $N_{2} \ll N_{3}$ (or $k = 1$ and $N_{2} \ll N_{3}$). Note that in this case $N_{1} \sim N_{3} \sim N \gg N_{2} \vee N_{4}$. We further divide the contribution of $J_{t}$ into two cases:

- $kN_{4} \gtrsim N_{2}$,
- $kN_{4} \ll N_{2}$ or $k = 1$.

However, it suffices to consider the first case $kN_{4} \gtrsim N_{2}$ since the second case is exactly the same as Subcase 3.1 in the proof of Proposition 5.1 in [31]. Its result requires only $s > 1/2$ since we can use Bourgain type estimates in this configuration, which is sufficient for our purpose. Now, we treat the case $kN_{4} \gtrsim N_{2}$. Notice that $N_{3}, N_{4} \geq 1$. We can apply Proposition 3.2 two times with $\theta = \min(s - 1/2, 1/2)$.
the same argument on \( T \) that only depends on \( \|u \| \) and Proposition 5.1 together with (4.8) ensure that of iteration.

a smooth approximation of \( u \) sions 4.6 and 5.1 by constructing a sequence of smooth solutions as sociated with \( u \) to get (see also Remark 3.3) to get

\[
J_t \lesssim \sum_{N \sim N_1 \geq N_2/k} N^{2s-2} \left( \| \partial_x P_N^2 P_{N_1} w P_{N_4} z_4 \|_{L^2_{T,x}} \| P_{N_3} z_3 P_{N_2} w \|_{L^2_{T,x}} \right)
\]

\[
\lesssim T^{-1} C_0^{k-2} \sum_{N_1 \geq N_2/k} N^{2s-1} W_{N_1}^0 W_{N_2}^{0,\infty} Y_{N_4}^{0,\infty}
\]

\[
\lesssim_k T^{\theta-1} C_0^{k-1} G(C_0) \| w \|_{L^\infty_T H_x^{s-1}} \sum_{N_1 \geq N_4} N_4^{-s(2s-1)} W_{N_1}^{s-1,2} Y_{N_4}^{s,\infty}
\]

\[
\lesssim_k T^\theta C_0^k G(C_0) \| w \|_{L^\infty_T H_x^{s-1}}^2
\]

c since \( s > 1/2 \) and \( \theta = \min(s - 1/2, 1/2) \). Here, we used Cauchy-Schwarz to sum on \( N_1 \). Note that the above estimate only requires \( s > 1/2 \).

6. Local and Global well-posedness results

6.1. Local well-posedness. With Lemma 3.3, Propositions 4.6 and 5.1 in hands, the proof of Theorem 1.1 and Corollaries 1.2, 1.3 follow exactly the same lines as in 3.1. For instance, to prove the unconditional uniqueness, we take \( u_0 \in H^s(\mathbb{T}) \) with \( s \geq s(\alpha) \) and \( s > 1/2 \) and \( u, v \) two solutions to the Cauchy problem (1.1) emanating from \( u_0 \) that belong to \( L_T^\infty H^s \). According to Lemma 3.3, we know that \( u, v \in Z_T^s \) and Proposition 5.1 together with (4.8) ensure that \( u \equiv v \) on \([0, T_0]\) with \( 0 < T_0 \leq T \) that only depends on \( \| u_0 \|_{Z_T^s} + \| v_0 \|_{Z_T^s} \). Therefore \( u(T_0) = v(T_0) \) and we can reiterate the same argument on \([T_0, T] \). This proves that \( u \equiv v \) on \([0, T]\) after a finite number of iteration.

Now the existence of a solution in \( H^s(\mathbb{T}) \) follows also from Lemma 3.3, Propositions 4.6 and 5.1 by constructing a sequence of smooth solutions associated with a smooth approximation of \( u_0 \in H^s(\mathbb{T}) \). Using that this sequence of solutions is bounded in \( L^\infty([0, T]; H^s(\mathbb{T})) \) for some \( T > 0 \) that depends only on \( \| u_0 \|_{H^s} \) with \( s_0 = \max(s(\alpha), \frac{1}{2} + ) \) and that it is a Cauchy sequence in \( L^\infty([0, T]; H^s) \) for any \( s' < s \), we can pass to the limit and prove that this sequence converges in some sense to a solution \( u \in L^\infty([0, T]; H^s(\mathbb{T})) \) to (1.1) emanating from \( u_0 \). The continuity of \( u \) with values in \( H^s(\mathbb{T}) \) follows from classical argument involving the reversibility and time translation invariance of the equation together with the estimate (4.9) whereas the continuity of the flow-map follows from the frequency envelop argument introduced in [22]. See also Remark 4.2 in [31].
6.2. **Global existence results.** The proofs of Corollaries 1.2 and 1.3 are exactly the same as the ones of Theorem 1.2 and Theorem 1.3 in [31]. The improvements with respect to these last results are only due to the improvement of the local well-posedness result. We thus omit the proof here and refer to [31].

**APPENDIX**

In this appendix, we give the proof of Lemma 3.6 (cf. [10]).

**Proof of Lemma 3.6.** We see from (3.14) with $g = U_\alpha(-t)u$ that

$$
\|\chi(L_t)u\|_{L^2_{t,x}} = \|\chi(L_t)U_\alpha(-t)u\|_{L^2_{t,x}} \lesssim L^{-\frac{1}{2}} \|U_\alpha(-t)u\|_{L^2_{x}(B^\frac{1}{2}_{2,1})} \lesssim L^{-\frac{1}{2}} \|u\|_{X^{0,\frac{1}{2},1}},
$$

which shows (3.12). In particular, we have $\|Q_L(\chi(L_t)u)\|_{L^2_{t,x}} \leq \|\chi(L_t)u\|_{L^2_{t,x}} \lesssim L^{-\frac{1}{2}} \|u\|_{X^{0,\frac{1}{2},1}}$. Now we show (3.13). From the definition (6.1) of $\psi$ it is easy to check that $\|\psi_L(\tau, \xi)\|_{L^2} \lesssim L^\frac{1}{2}$ so that

$$
\|\psi_L(\tau, \xi)(\tau - p_{\alpha+1}(\xi))^{1/2}\|_{L^2} \lesssim L_1^{-\frac{1}{2}} \|\psi_L\|_{L^2} \lesssim 1. 
$$

(6.1)

We also notice that $\mathcal{F}_t(\chi(L_t))(\tau) = L^{-1}\hat{\chi}(L^{-1}\tau)$. By definition, we have

$$
\|\chi(L(\cdot))u\|_{X^{0,\frac{1}{2},1}} \lesssim L^\frac{1}{2} \|Q_L(\chi(L_t)u)\|_{L^2_{t,x}} + \sum_{L_1 > L} L_1^\frac{1}{2} \|Q_{L_1}(\chi(L_t)u)\|_{L^2_{t,x}}
$$

$$
\lesssim \|u\|_{X^{0,\frac{1}{2},1}} + \sum_{L_1 > L} L_1^\frac{1}{2} \left\|\psi_{L_1}(\tau, \xi) \int_R L^{-1}\hat{\chi}(L^{-1}(\tau - \tau'))\tilde{u}(\tau', \xi) d\tau'\right\|_{L^2_{t,\xi}}.
$$

It thus remains to show that

$$
\sum_{L_1 > L} L_1^\frac{1}{2} \left\|\psi_{L_1}(\tau, \xi) \int_R L^{-1}\hat{\chi}(L^{-1}(\tau - \tau'))\tilde{u}(\tau', \xi) d\tau'\right\|_{L^2_{t,\xi}} \lesssim \|u\|_{X^{0,\frac{1}{2},1}}, 
$$

(6.2)

which will complete the proof of (3.13). The mean value theorem implies that

$$
|\psi_{L_1}(\tau, \xi)| = |\tilde{\psi}_{L_1}(\tau, \xi)| |\psi_{L_1}(\tau, \xi)|
$$

$$
\leq |\tilde{\psi}_{L_1}(\tau, \xi)||\psi_{L_1}(\tau, \xi) - \psi_{L_1}(\tau', \xi)| + |\tilde{\psi}_{L_1}(\tau, \xi)\psi_{L_1}(\tau', \xi)|
$$

$$
\lesssim L_1^{-1} |\tilde{\psi}_{L_1}(\tau, \xi)||\tau - \tau'| + |\psi_{L_1}(\tau', \xi)|
$$

since $\chi \in C_0^\infty(\mathbb{R})$ and $0 \leq \chi \leq 1$ (see the definition of $\psi$ in (6.2)). By using this, we are reduced to bound $A$ and $B$ defined by

$$
A := \sum_{L_1 > L} L_1^\frac{1}{2} \left\|\psi_{L_1}(\cdot, \xi)\|_{L^2_{t,\xi}} \right\|_{L^2_{t,\xi}}
$$

$$
B := \sum_{L_1 > L} L_1^\frac{1}{2} \left\|\psi_{L_1}(\tau, \xi) \int_R L^{-1}(\tau - \tau')\hat{\chi}(L^{-1}(\tau - \tau'))\tilde{u}(\tau', \xi) d\tau'\right\|_{L^2_{t,\xi}}.
$$
For $A$, the Young inequality in $\tau$ gives

$$A \lesssim \sum_{L_1 > L} L_1^{\frac{1}{2}} \| \psi_{L_1} \tilde{u} \|_{L_2 \ell^2} \lesssim \| u \|_{X^{0,\frac{1}{2},1}}.$$ 

On the other hand, for $B$, as in the proof of (3.6) we have

$$B \lesssim L^{-1} \sum_{L_1 > L} L_1^{\frac{1}{2}} \left\| \psi_{L_1} (\tau, \xi) \sum_{L_2} I_{L_2} \tilde{\psi}_{L_2} (\tau', \xi) \tilde{u}(\tau', \xi) \right\|_{L_2'} \left\| \tilde{u} \right\|_{L_2' \ell^2},$$

where

$$I_{L_2} := \| (\tau - \tau') \tilde{\chi} (L^{-1} (\tau - \tau')) \tilde{\psi}_{L_2} (\tau', \xi) (\tau' - p_{a+1}(\xi))^{-\frac{1}{2}} \|_{L_2'}.$$ 

Now we divide the contribution of $I_{L_2}$ into two pieces. On one hand, when $L_2 \ll L_1$, we have $|\tau - \tau'| \geq |\tau - p_{a+1}(\xi)| - |\tau' - p_{a+1}(\xi)| \gtrsim L_1$, which implies that $|\tau - \tau'| \sim L_1$. Since $\tilde{\chi} \in S(\mathbb{R})$, we see that $sup_{\tau \in \mathbb{R}} (\tau^4 |\tilde{\chi}(\tau)| \lesssim 1$. This and (6.1) show that

$$I_{L_2} \lesssim L_1 \| (L^{-1} (\tau - \tau'))^{-\frac{1}{4}} \tilde{\psi}_{L_2} (\tau', \xi) (\tau' - p_{a+1}(\xi))^{-\frac{1}{2}} \|_{L_2'} \lesssim L_1^{-3} L^4.$$ 

On the other hand, when $L_1 \lesssim L_2$, it holds that

$$I_{L_2} \lesssim L_2^{-\frac{1}{2}} \| \tau' \tilde{\chi}(L^{-1} \tau') \|_{L_2'} \lesssim L_1^{\frac{1}{2}} L^\frac{3}{2}.$$ 

Combining the above estimates, we get

$$B \lesssim L^3 \sum_{L_1 > L} L_1^{\frac{1}{2}} \left\| \psi_{L_1} (\tau, \xi) \sum_{L_2 \ll L_1} L_2^{\frac{1}{2}} \| \tilde{\psi}_{L_2} (\tau', \xi) \tilde{u}(\tau', \xi) \|_{L_2'} \right\|_{L_2 \ell^2} + L^{\frac{1}{2}} \sum_{L_1 > L} L_1^{-1} \left\| \psi_{L_1} (\tau, \xi) \sum_{L_2 \gtrsim L_1} L_2^{\frac{1}{2}} \| \tilde{\psi}_{L_2} (\tau', \xi) \tilde{u}(\tau', \xi) \|_{L_2'} \right\|_{L_2' \ell^2} \lesssim L^3 \sum_{L_1 > L} L_1^{-3} \| u \|_{X^{0,\frac{1}{2},1}} + L^{\frac{3}{2}} \sum_{L_1 > L} L_1^{-\frac{1}{2}} \| u \|_{X^{0,\frac{1}{2},1}} \lesssim \| u \|_{X^{0,\frac{1}{2},1}}.$$

where we used (6.1) in the second inequality. This completes the proof of (6.2). \hfill \square

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