Approximate Message Passing with Spectral Initialization for Generalized Linear Models

Marco Mondelli* and Ramji Venkataramanan†

October 8, 2020

Abstract

We consider the problem of estimating a signal from measurements obtained via a generalized linear model. We focus on estimators based on approximate message passing (AMP), a family of iterative algorithms with many appealing features: the performance of AMP in the high-dimensional limit can be succinctly characterized under suitable model assumptions; AMP can also be tailored to the empirical distribution of the signal entries, and for a wide class of estimation problems, AMP is conjectured to be optimal among all polynomial-time algorithms.

However, a major issue of AMP is that in many models (such as phase retrieval), it requires an initialization correlated with the ground-truth signal and independent from the measurement matrix. Assuming that such an initialization is available is typically not realistic. In this paper, we solve this problem by proposing an AMP algorithm initialized with a spectral estimator. With such an initialization, the standard AMP analysis fails since the spectral estimator depends in a complicated way on the design matrix. Our main technical contribution is the construction and analysis of a two-phase artificial AMP algorithm that first produces the spectral estimator, and then closely approximates the iterates of the true AMP. Our analysis yields a rigorous characterization of the performance of AMP with spectral initialization in the high-dimensional limit. We also provide numerical results that demonstrate the validity of the proposed approach.

1 Introduction

We consider the problem of estimating a $d$-dimensional signal $x \in \mathbb{R}^d$ from $n$ i.i.d. measurements of the form

$$y_i \sim p(y \mid \langle x, a_i \rangle), \quad i \in \{1, \ldots, n\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product, $\{a_i\}_{1 \leq i \leq n}$ are given sensing vectors, and the (stochastic) output function $p(\cdot \mid \langle x, a_i \rangle)$ is a given probability distribution. This is known as a \textit{generalized linear model} [McC18], and it encompasses many settings of interest in statistical estimation and signal processing [RG01, BB08, YLSV12, EK12]. One notable example is the problem of phase retrieval [Fie82, SEC+15], where

$$y_i = |\langle x, a_i \rangle|^2 + w_i, \quad i \in \{1, \ldots, n\},$$

with $w_i$ being noise. Phase retrieval appears in several areas of science and engineering, see e.g. [FD87, Mil90, DJ17], and the last few years have witnessed a surge of interest in the design and analysis of efficient algorithms; see the review [FS20] and the discussion at the end of this section.

*Institute of Science and Technology (IST) Austria. Email: marco.mondelli@ist.ac.at.
†Department of Engineering, University of Cambridge. Email: ramji.v@eng.cam.ac.uk.
Here, we consider generalized linear models (GLMs) in the high-dimensional setting where \( n, d \to \infty \), with their ratio tending to a fixed constant, i.e., \( n/d \to \delta \in \mathbb{R} \). We focus on a family of iterative algorithms known as approximate message passing (AMP). AMP algorithms were first proposed for estimation in linear models [DMM09, BM11], and for estimation in GLMs in [Ran11]. AMP has since been applied to a wide range of high-dimensional statistical estimation problems including compressed sensing [KMS+12, BM12, MAYB13], low rank matrix estimation [RF12, DM14, KKM+16], group synchronization [PWBM18], and specific instances of GLMs such as logistic regression [SC19] and phase retrieval [SR14, MXM19, MLKZ20].

An appealing feature of AMP is that, under suitable model assumptions, its performance in the high-dimensional limit can be precisely characterized by a succinct deterministic recursion called state evolution [BM11, Bol14, JM13]. Using the state evolution analysis, it has been shown that AMP provably achieves Bayes-optimal performance in some special cases [DJM13, DM14, MV17]. Indeed, a conjecture from statistical physics posits that AMP is optimal among all polynomial-time algorithms. The optimality of AMP for generalized linear models is discussed in [BKM+19].

However, when used for estimation in GLMs, a major issue of AMP is that in many problems (including phase retrieval) we require an initialization that is correlated with the unknown signal \( \mathbf{x} \) but independent of the sensing vectors \( \{a_i\} \). In many cases, it is not realistic to assume that such a realization is available. For such GLMs, without a correlated initialization, asymptotic state evolution analysis predicts that the AMP estimates will be uninformative, i.e., their normalized correlation with the signal vanishes in the large system limit.

In this paper, we propose an AMP initialized using a spectral estimator. The idea of using a spectral estimator for GLMs was introduced in [Li92], and its performance in the high-dimensional limit was recently characterized in [LL19, MM19]. It was shown that the normalized correlation of the spectral estimator with the signal undergoes a phase transition, and for the special case of phase retrieval, the threshold for strictly positive correlation with the signal matches the information-theoretic threshold [MM19].

Our main technical contribution is a novel analysis of AMP with spectral initialization for GLMs, under the assumption that the sensing vectors \( \{a_i\} \) are i.i.d. Gaussian. This yields a rigorous characterization of the performance in the high-dimensional limit (Theorem 1). The analysis of AMP with spectral initialization is far from obvious since the spectral estimator depends in a non-trivial way on the sensing vectors \( \{a_i\} \). The existing state evolution analysis for GLMs [Ran11, JM13] crucially depends on the AMP initialization being independent of the sensing vectors, and therefore cannot be directly applied.

At the center of our approach is the design and analysis of an artificial AMP algorithm. The artificial AMP operates in two phases: in the first phase, it performs a power method, so that its iterates approach the spectral initialization of the true AMP; in the second phase, its iterates are designed to remain close to the iterates of the true AMP. The initialization of the artificial AMP is correlated with \( \mathbf{x} \), but independent of the sensing vectors \( \{a_i\} \), which allows us to apply the standard state evolution analysis. Note that the initialization of the artificial AMP is impractical (it requires the knowledge of the unknown signal \( \mathbf{x} \)!). However, this is not an issue, since the artificial AMP is employed as a proof technique: we prove a state evolution result for the true AMP by showing that its iterates are close to those in the second phase of the artificial AMP.

Initializing AMP with a (different) spectral method has been recently shown to be effective for low-rank matrix estimation [MV17]. However, our proof technique for analyzing spectral initialization for GLMs is different from [MV17]. The argument in that paper is specific to the spiked
random matrix model and relies on a delicate decoupling argument between the outlier eigenvectors and the bulk. Here, we follow an approach developed in [MTV20], where a specially designed AMP is used to establish the joint empirical distribution of the signal, the spectral estimator, and the linear estimator.

For the case of phase retrieval, in [MXM18] it is proposed a slightly different version of the spectral estimator to initialize AMP. A heuristic justification of the initialization was given, but a rigorous characterization of its performance remained open.

We note that for some GLMs, AMP does not require a special initialization that is correlated with the signal \(x\). In Section 3, we give a condition on the GLM output function that specifies precisely when such a correlated initialization is required (see (3.13)). This condition is satisfied by a wide class of GLMs, including phase retrieval. It is in these cases that AMP with spectral initialization is most useful.

Other related work. For the problem of phase retrieval, several algorithmic solutions have been proposed and analyzed in recent years. An inevitably non-exhaustive list includes semi-definite programming relaxations [CSV13, CESV15, CLS15a, WdM15], a convex relaxation operating in the natural domain of the signal [GS18, BR17], alternating minimization [NJS13], Wirtinger Flow [CLS15b, CC17, MWCC20], iterative projections [LGL15], and the Kaczmarz method [Wei15, TV19]. A generalized AMP (GAMP) algorithm was introduced in [SR14], and an AMP to solve the non-convex problem with \(\ell_2\) regularization was proposed and analyzed in [MXM19]. Most of the algorithms mentioned above require an initialization correlated with the signal and, to obtain such an initialization, spectral methods are widely employed.

Beyond the Gaussian setting, spectral methods for phase retrieval with random orthogonal matrices are analyzed in [DBMM20]. Statistical and computational phase transitions in phase retrieval for a large class of correlated real and complex random sensing matrices are analyzed in [MLKZ20], and a general AMP algorithm for rotationally invariant matrices is studied in [Fan20]. Thus, the extension of our techniques to more general sensing models represents an interesting avenue for future research.

2 Preliminaries

Notation and definitions. Given \(n \in \mathbb{N}\), we use the shorthand \([n] = \{1, \ldots, n\}\). Given a vector \(x\), we denote by \(||x||_2\) its Euclidean norm. The empirical distribution of a vector \(x = (x_1, \ldots, x_d)^T\) is given by \(\frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}\), where \(\delta_{x_i}\) denotes a Dirac delta mass on \(x_i\). Similarly, the empirical joint distribution of vectors \(x, x' \in \mathbb{R}^d\) is \(\frac{1}{d} \sum_{i=1}^{d} \delta_{(x_i, x'_i)}\).

Generalized linear models. Let \(x \in \mathbb{R}^d\) be the signal of interest, and assume that \(||x||_2^2 = d\). The signal is observed via inner products with \(n\) sensing vectors \((a_i)_{i \in [n]}\), with each \(a_i \in \mathbb{R}^d\) having independent Gaussian entries with mean zero and variance \(1/d\), i.e., \((a_i) \sim \text{i.i.d. } \mathcal{N}(0, I_d/d)\). Given \(g_i = \langle x, a_i \rangle\), the components of the observed vector \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\) are independently generated according to a conditional distribution \(p_{Y|G}\), i.e., \(y_i \sim p_{Y|G}(y_i | g_i)\). We stack the sensing vectors as rows to define the \(n \times d\) sensing matrix \(A\), i.e., \(A = [a_1, \ldots, a_n]^T\). For the special case of phase retrieval, the model is \(y = |Ax| + w\), where \(w\) is a noise vector with independent entries. We consider a sequence of problems of growing dimension \(d\), and assume that, as \(d \to \infty\), the sampling ratio \(n/d \to \delta\), for some constant \(\delta \in (0, \infty)\).
**Spectral initialization.** The spectral estimator $\hat{x}^s$ is the principal eigenvector of the $d \times d$ matrix $D_n$, defined as

$$D_n = A^T Z_s A,$$

with $Z_s = \text{diag}(T_s(y_1), \ldots, T_s(y_n))$, \hspace{1cm} (2.1)

where $T_s : \mathbb{R} \to \mathbb{R}$ is a preprocessing function. We now review some results from [MM19, LL19] on the performance of the spectral estimator in the high-dimensional limit.

Let $G \sim N(0, 1)$, $Y \sim p(\cdot \mid G)$, and $Z_s = T_s(Y)$. We will make the following assumptions on $Z_s$.

(A1) $\mathbb{P}(Z_s = 0) < 1$.

(A2) $Z_s$ has bounded support and $\tau$ is the supremum of this support:

$$\tau = \inf \{ z : \mathbb{P}(Z_s \leq z) = 1 \}.$$ \hspace{1cm} (2.2)

(A3) As $\lambda$ approaches $\tau$ from the right, we have

$$\lim_{\lambda \to \tau^+} \mathbb{E} \left\{ \frac{Z_s}{(\lambda - Z_s)^2} \right\} = \lim_{\lambda \to \tau^+} \mathbb{E} \left\{ \frac{Z_s \cdot G^2}{\lambda - Z_s} \right\} = \infty.$$ \hspace{1cm} (2.3)

For $\lambda \in (\tau, \infty)$ and $\delta \in (0, \infty)$, define

$$\phi(\lambda) = \lambda \mathbb{E} \left\{ \frac{Z_s \cdot G^2}{\lambda - Z_s} \right\}, \quad \psi_\delta(\lambda) = \frac{\lambda}{\delta} + \lambda \mathbb{E} \left\{ \frac{Z_s}{\lambda - Z_s} \right\}.$$ \hspace{1cm} (2.4)

Note that $\phi(\lambda)$ is a monotone non-increasing function and that $\psi_\delta(\lambda)$ is a convex function. Let $\bar{\lambda}_\delta$ be the point at which $\psi_\delta$ attains its minimum, i.e., $\bar{\lambda}_\delta = \arg \min_{\lambda \geq \tau} \psi_\delta(\lambda)$. For $\lambda \in (\tau, \infty)$, also define

$$\zeta_\delta(\lambda) = \psi_\delta(\max(\lambda, \bar{\lambda}_\delta)).$$ \hspace{1cm} (2.5)

The following result characterizes the performance of the spectral estimator $\hat{x}^s$. Its proof follows directly from [MM19, Lemma 2].

**Lemma 2.1.** Let $x$ be such that $\|x\|_2^2 = d$, $\{a_i\}_{i \in [n]} \sim_{\text{i.i.d.}} N(0_d, I_d/d)$, and $y = (y_1, \ldots, y_n)$ with $\{y_i\}_{i \in [n]} \sim_{\text{i.i.d.}} p_y \mid G$. Let $n/d \to \delta$, $G \sim N(0, 1)$ and define $Z_s = T_s(Y)$ for $Y \sim p_Y \mid G$. Assume that $Z_s$ satisfies the assumptions (A1)-(A2)-(A3). Let $\hat{x}^s$ be the principal eigenvector of the matrix $D_n$ defined in (2.1), and let $\lambda^*_s$ be the unique solution of $\zeta_\delta(\lambda) = \phi(\lambda)$ for $\lambda > \tau$. Then, as $n \to \infty$,

$$\frac{|\langle \hat{x}^s, x \rangle|^2}{\|\hat{x}^s\|_2^2 \|x\|_2^2} \overset{a.s.}{\to} a^2 \triangleq \begin{cases} 0, & \text{if } \psi_\delta'(\lambda^*_s) \leq 0, \\ \frac{\psi_\delta'(\lambda^*_s)}{\psi_\delta'(\bar{\lambda}_\delta) - \phi'(\lambda^*_s)}, & \text{if } \psi_\delta'(\lambda^*_s) > 0, \end{cases}$$ \hspace{1cm} (2.6)

where $\psi_\delta'$ and $\phi'$ are the derivatives of the respective functions.

**Remark 2.1** (Equivalent characterization). Using the definitions (2.4)-(2.5), the conditions $\zeta_\delta(\lambda^*_s) = \phi(\lambda^*_s)$ and $\psi_\delta'(\lambda^*_s) > 0$ are equivalent to

$$\mathbb{E} \left\{ \frac{Z_s (G^2 - 1)}{\lambda^*_s - Z_s} \right\} = \frac{1}{\delta}, \quad \text{and} \quad \mathbb{E} \left\{ \frac{Z_s^2}{(\lambda^*_s - Z_s)^2} \right\} < \frac{1}{\delta}.$$ \hspace{1cm} (2.7)
When these conditions are satisfied, the limit of the normalized correlation in (2.6) can be expressed as
\[ a^2 = \frac{1}{\delta} - \mathbb{E}\left\{ \frac{Z_s^2}{(\lambda_s - Z_s)^2} \right\} \]
\[ \frac{1}{\delta} + \mathbb{E}\left\{ \frac{Z_s^2(G^2 - 1)}{(\lambda_s - Z_s)^2} \right\} \]  
\hspace{1cm} (2.8)

**Remark 2.2 (Optimal preprocessing function).** In [MM19], the authors derived the preprocessing function minimizing the value of \( \delta \) necessary to achieve weak recovery, i.e., a strictly positive correlation between \( \hat{x}^s \) and \( x \). In particular, let \( \delta_u \) be defined as
\[ \delta_u = \left( \int_{\mathbb{R}} \left( \frac{\mathbb{E}_G \{ p(y \mid G)(G^2 - 1) \}}{\mathbb{E}_G \{ p(y \mid G) \}} \right)^2 \, dy \right)^{-1}, \]  
with \( G \sim \mathcal{N}(0, 1) \). Furthermore, let us also define
\[ \bar{T}(y) = \frac{\sqrt{\delta_u} \cdot T^*(y)}{\sqrt{\delta} - (\sqrt{\delta} - \sqrt{\delta_u})T^*(y)}, \]  
(2.10)
where
\[ T^*(y) = 1 - \frac{\mathbb{E}_G \{ p(y \mid G) \}}{\mathbb{E}_G \{ p(y \mid G) \} \cdot G^2}. \]  
(2.11)

Then, by taking \( T_s = \bar{T} \), for any \( \delta > \delta_u \), we almost surely have
\[ \lim_{n \to \infty} \frac{|\langle \hat{x}^s, x \rangle|}{\| \hat{x}^s \|_2 \| x \|_2} > \epsilon, \]  
(2.12)
for some \( \epsilon > 0 \). Furthermore, for any \( \delta < \delta_u \), there is no pre-processing function \( T \) such that, almost surely, (2.12) holds. For a more formal statement of this result, see Theorem 4 of [MM19].

The preprocessing function that, at a given \( \delta > \delta_u \), maximizes the correlation between \( \hat{x}^s \) and \( x \) is also related to \( T^*(y) \) as defined in (2.11), and it is derived in [LAL19].

### 3 Generalized Approximate Message Passing with Spectral Initialization

We make the following additional assumptions on the signal \( x \), the output distribution \( p_{Y \mid G} \), and the preprocessing function \( T_s \) used for the spectral estimator.

**B1** Let \( \hat{P}_{X,d} \) denote the empirical distribution of \( x \in \mathbb{R}^d \). As \( d \to \infty \), \( \hat{P}_{X,d} \) converges weakly to a distribution \( P_X \) such that \( \lim_{d \to \infty} \mathbb{E}_{\hat{P}_{X,d}} \{|X|^2\} = \mathbb{E}_{P_X} \{|X|^2\} \). We note that \( \mathbb{E}_{P_X} \{|X|^2\} = 1 \), since we assume \( \|x\|_2^2 = d \).

**B2** We have \( \mathbb{E} \{|Y|^2\} < \infty \), for \( Y \sim p_{Y \mid G}(\cdot \mid G) \) and \( G \sim \mathcal{N}(0, 1) \). Furthermore, there exists a function \( q : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and a random variable \( V \) independent of \( G \) such that \( Y = q(G, V) \). More precisely, for any measurable set \( A \subseteq \mathcal{Y} \) and almost every \( g \), we have \( \mathbb{P}(Y \in A \mid G = g) = \mathbb{P}(q(g, V) \in A) \). We also assume that \( \mathbb{E} \{|V|^2\} < \infty \).

**B3** The function \( T_s : \mathbb{R} \to \mathbb{R} \) is bounded and Lipschitz.
Following the terminology of [Ran11], we refer to the AMP for generalized linear models as GAMP. In each iteration \( t \), the proposed GAMP algorithm produces an estimate \( \hat{x}^t \) of the signal \( x \). The algorithm is defined in terms of a sequence of Lipschitz functions \( f_t : \mathbb{R} \to \mathbb{R} \) and \( h_t : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \), for \( t \geq 0 \). We initialize using the spectral estimator \( \hat{x}^0 \):

\[
x^0 = \frac{1}{\sqrt{d}} \frac{1}{\sqrt{\delta}} \hat{x}^s, \quad (3.1)
\]

\[
u^0 = \frac{1}{\sqrt{\delta}} Af_0(x^0) - b_0 \frac{\sqrt{\delta}}{\lambda_0^s} Z_s Ax^0, \quad (3.2)
\]

where \( b_0 = \frac{1}{n} \sum_{i=1}^d f^a_0(x^0_i) \), the diagonal matrix \( Z_s \) is defined in \( (2.1) \), and \( \lambda_0^s \) is given by \( (2.7) \). Then, for \( t \geq 0 \), the algorithm computes:

\[
x^{t+1} = \frac{1}{\sqrt{\delta}} A^T h_t(\nu^t; y) - c_t f_t(x^t), \quad (3.3)
\]

\[
u^{t+1} = \frac{1}{\sqrt{\delta}} A f_{t+1}(x^{t+1}) - b_{t+1} h_t(\nu^t; y), \quad (3.4)
\]

Here the functions \( f_t \) and \( h_t \) are understood to be applied component-wise, i.e., \( f_t(x^t) = (f_t(x^t_1), \ldots, f_t(x^t_d)) \) and \( h_t(\nu^t; y) = (h_t(\nu^t_1; y_1), \ldots, h_t(\nu^t_d; y_d)) \). The scalars \( b_t, c_t \) are defined as

\[
c_t = \frac{1}{n} \sum_{i=1}^n h_t'(u^t_i; y_i), \quad b_{t+1} = \frac{1}{n} \sum_{i=1}^d f^{a}_{t+1}(x^{t+1}_i), \quad (3.5)
\]

where \( h_t'(\cdot; \cdot) \) denotes the derivative with respect to the first argument.

The asymptotic empirical distribution of the GAMP iterates \( x^t, u^t \), for \( t \geq 0 \), can be succinctly characterized via a deterministic recursion, called state evolution. Our main result, Theorem 1, shows that for \( t \geq 0 \), the empirical distributions of \( u^t \) and \( x^t \) converge in Wasserstein distance \( W_2 \) to the laws of the random variables \( U_t \) and \( X_t \), respectively, with

\[
X_t \equiv \mu_{X,t} X + \sigma_{X,t} W_{X,t}, \quad (3.6)
\]

\[
U_t \equiv \mu_{U,t} G + \sigma_{U,t} W_{U,t}, \quad (3.7)
\]

where \( (G, W_{U,t}) \sim_{\text{i.i.d.}} \mathcal{N}(0,1) \). Similarly, \( X \sim P_X \) and \( W_{X,t} \sim \mathcal{N}(0,1) \) are independent. The deterministic parameters \( \mu_{U,t}, \sigma_{U,t}, \mu_{X,t}, \sigma_{X,t} \) are recursively computed as follows, for \( t \geq 0 \):

\[
\mu_{U,t} = \frac{1}{\sqrt{\delta}} \mathbb{E}\{X f_t(X_t)\},
\]

\[
\sigma_{U,t}^2 = \frac{1}{\delta} \mathbb{E}\{f_t(X_t)^2\} - \mu_{U,t}^2,
\]

\[
\mu_{X,t+1} = \sqrt{\delta} \mathbb{E}\{G h_t(U_t; Y)\} - \mathbb{E}\{h_t'(U_t; Y)\} \mathbb{E}\{X f_t(X_t)\},
\]

\[
\sigma_{X,t+1}^2 = \mathbb{E}\{h_t'(U_t; Y)^2\}.
\]

For the spectral initialization in \( (3.1) - (3.2) \), with \( a \) as defined in \( (2.6) \), the recursion is initialized with

\[
\mu_{X,0} = \frac{a}{\sqrt{\delta}}, \quad \sigma_{X,0}^2 = \frac{(1 - a^2)}{\delta}. \quad (3.9)
\]
We state the main result in terms of pseudo-Lipschitz test functions. A function \( \psi : \mathbb{R}^m \to \mathbb{R} \) is pseudo-Lipschitz of order 2, i.e., \( \psi \in \mathcal{PL}(2) \), if there is a constant \( C > 0 \) such that

\[
\| \psi(x) - \psi(y) \|_2 \leq C(1 + \|x\|_2 + \|y\|_2) \|x - y\|_2,
\]

for all \( x, y \in \mathbb{R}^m \). Examples of test functions in \( \mathcal{PL}(2) \) with \( \sigma_x \) are required to ensure that the spectral initialization \( (3.9) \) being the principal eigenvector of \( D_n \), defined as in \( (2.1) \), with the sign of \( \hat{x}^s \) chosen so that \( \langle \hat{x}^s, x \rangle > 0 \).

Theorem 1. Let \( x \) be such that \( \|x\|_2^2 = d \), \( \{a_i\}_{i \in \mathbb{N}} \sim \text{i.i.d. } N(0, I_{d/d}) \), and \( y = (y_1, \ldots, y_n) \) with \( \{y_i\}_{i \in \mathbb{N}} \sim \text{i.i.d. } p_Y|G \). Let \( n/d \to \delta, G \sim N(0, 1) \), and \( Z_s = T_s(Y) \) for \( Y \sim p_Y|G(\cdot | G) \). Assume that \((A1)-(A2)-(A3)\) and \((B1)-(B2)-(B3)\) hold. Assume further that \( \psi_0'(\lambda_0^s) > 0 \), and let \( \hat{x}^s \) be the principal eigenvector of \( D_n \), defined as in \( (2.1) \), with the sign of \( \hat{x}^s \) chosen so that \( \langle \hat{x}^s, x \rangle > 0 \).

Consider the GAMP iteration in Eqs. \((3.4)-(3.3)\) with initialization in Eqs. \((3.1)-(3.2)\). Assume that for \( t \geq 0 \), the functions \( f_t, h_t \) are Lipschitz with derivatives that are continuous almost everywhere. Then, the following limits hold almost surely for any \( \mathcal{PL}(2) \) function \( \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( t \) such that \( \sigma_{X,k}^2 \) is strictly positive for \( 0 \leq k \leq t \):

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, x_{i+1}^{t+1}) = \mathbb{E}\{\psi(X, \mu_{X,t+1}X + \sigma_{X,t+1}W_{X,t+1})\},
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, u_i^t) = \mathbb{E}\{\psi(Y, \mu_{U,t}G + \sigma_{U,t}W_{U,t})\}.
\]

The result \((3.11)\) also holds for \((t + 1) = 0\). In \((3.11)\) (resp. \((3.12)\)), the expectation is over the independent random variables \( X \sim P_X \) and \( W_{X,t} \sim N(0, 1) \) (resp. \( G, W_{U,t} \sim \text{i.i.d. } N(0, 1) \)). The scalars \( (\mu_{X,t}, \mu_{U,t}, \sigma_{X,t}^2, \sigma_{U,t}^2)_{t \geq 0} \) are given by the recursion \((3.8)\) with the initialization \((3.9)\).

We give a sketch of the proof in Section 5 and defer the technical details to the appendices.

We now comment on some of the assumptions in the theorem. The assumption \( \psi_0'(\lambda_0^s) > 0 \) is required to ensure that the spectral initialization \( x^0 \) has non-zero correlation with the signal \( x \). (Lemma 2.1). From Remark 2.2, we also know that for any sampling ratio \( \delta > \delta_0 \) there exists a choice of \( T_s \) such that \( \psi_0'(\lambda_0^s) > 0 \). We also note that, for \( \delta < \delta_0 \), GAMP converges to the “un-informative fixed point” (where the estimate has vanishing correlation with signal) even if the initial condition has non-zero correlation with the signal, see [MM19] Theorem 5.

There is no loss of generality in assuming the sign of \( \hat{x}^s \) to be such that \( \langle \hat{x}^s, x \rangle \geq 0 \). Indeed, if the sign were chosen otherwise, the theorem would hold with the state evolution initialization in \( (3.9) \) being \( \mu_{X,0} = -a/\sqrt{\delta} \), \( \sigma_{X,0}^2 = (1 - a^2)/\delta \).

The assumption that \( \sigma_{X,k}^2 \) is positive for \( k \leq t \) is natural. Indeed, if \( \sigma_{X,k}^2 = 0 \), then the state evolution result for iteration \( k \) implies that \( \|x - \mu_{X,1}^k x^k\|^2/d \to 0 \) as \( d \to \infty \). That is, we can perfectly estimate \( x \) from \( x^k \), and thus terminate the algorithm after iteration \( k \).

Let us finally remark that the result in \((3.11)\) is equivalent to the statement that the empirical joint distribution of \((x, x^{t+1})\) converges almost surely in Wasserstein distance \((W_2)\) to the joint law of \((X, \mu_{X,t+1}X + \sigma_{X,t+1}W)\). This follows from the fact that a sequence of distributions \( P_n \) with finite second moment converges in \( W_2 \) to \( P \) if and only if \( P_n \) converges weakly to \( P \) and \( \int \|a\|^2 dP_n(a) \to \int \|a\|^2 dP(a) \), see [Vil08] Definition 6.7, Theorem 6.8.

When does GAMP require spectral initialization? For the GAMP to give meaningful estimates, we need either \( x^0 \) or \( x^1 \) to have strictly non-zero asymptotic correlation with \( x \). To see
when this can be arranged without a special initialization, consider the linear estimator \( \hat{x}^L(\xi) := \hat{A}^T \xi(y) \), for some function \( \xi : \mathbb{R} \to \mathbb{R} \) that acts component-wise on \( y \). If there exists a function \( \xi \) such that the asymptotic normalized correlation between \( x^L(\xi) \) and \( x \) is strictly non-zero, then AMP does not require a special initialization (spectral or otherwise) that is correlated with \( x \). Indeed, in this case we can replace the initialization in (3.1)-(3.2) by \( x_0 = 0, u_0 = 0 \) (by taking \( f_0 = 0 \)), and let \( h_0(u_0; y) = \sqrt{\delta}(y) \). This gives \( x^1 = A\xi(y) = \hat{x}^L(\xi) \), which has strictly non-zero asymptotic correlation with \( x \). This ensures that \( |\mu_{x,1}| > 0 \), and the standard AMP analysis \( \text{JMI13} \) directly yields a state evolution result similar to Theorem 1.

The output function \( p_{Y|G} \) determines whether a non-trivial linear estimator exists for the GLM. From [MTV20] Appendix C.1, we have that, if

\[
\int_{\mathbb{R}} \frac{\mathbb{E}_{G \sim N(0,1)} \{ G p_{Y|G}(y \mid G) \}^2}{\mathbb{E}_{G \sim N(0,1)} \{ p_{Y|G}(y \mid G) \}} \, dy = 0, \quad (3.13)
\]

then the correlation between \( A^T \xi(y) \) and \( x \) will asymptotically vanish for any choice of \( \xi \). The condition (3.13) holds for many output functions of interest, including all distributions \( p_{Y|G} \) that are even in \( G \) (and, therefore, including phase retrieval). It is for these models that spectral initialization is particularly useful.

**Bayes-optimal GAMP.** Applying Theorem 1 to the PL(2) function \( \psi(x, y) = (x - f_1(y))^2 \), we obtain the asymptotic mean-squared error (MSE) of the GAMP estimate \( f_t(x^t) \). In formulas, for \( t \geq 0 \), almost surely,

\[
\lim_{d \to \infty} \frac{1}{d} \| x - f_t(x^t) \|^2 = \mathbb{E}\{(X - f_t(\mu_{x,t} X + \sigma_{x,t} W))^2\}. \quad (3.14)
\]

If the limiting empirical distribution \( P_X \) of the signal is known, then the choice of \( f_t \) that minimizes the MSE in (3.14) is

\[
f_t^*(s) = \mathbb{E}\{X \mid \mu_{x,t} X + \sigma_{x,t} W = s\}. \quad (3.15)
\]

Similarly, applying the theorem to the PL(2) functions \( \psi(x, y) = x f_1(y) \) and \( \psi(x, y) = f_1(y)^2 \), we obtain the asymptotic normalized correlation with the signal. In formulas, for \( t \geq 0 \), almost surely

\[
\lim_{d \to \infty} \frac{\| (x, f_t(x^t)) \|}{\| x \|_2 \| f_t(x^t) \|_2} = \frac{\mathbb{E}\{X f_t(\mu_{x,t} X + \sigma_{x,t} W)\}}{\sqrt{\mathbb{E}\{f_t(\mu_{x,t} X + \sigma_{x,t} W)^2\}}} \quad (3.16)
\]

For fixed \( (\mu_{x,t}, \sigma_{x,t}^2) \), the normalized correlation in (3.16) is maximized by taking \( f_t = c f_t^* \) for any \( c \neq 0 \). This choice also maximizes the ratio \( \mu_{U,t}^2/\sigma_{U,t}^2 \) in (3.8). For \( f_t = c f_t^* \), from (3.8) we have

\[
\mu_{U,t} = \frac{c}{\sqrt{2}} \mathbb{E}\{f_t^* (X_t)^2\}, \quad \sigma_{U,t}^2 = \frac{c}{\sqrt{2}} \mu_{U,t} - \mu_{U,t}^2. \quad (3.17)
\]

We now specify the choice of \( h_t(u; y) \) that maximizes the ratio \( \mu_{\tilde{X}_{t+1}}^2/\sigma_{\tilde{X}_{t+1}}^2 \) for fixed \( (\mu_{U,t}, \sigma_{U,t}^2) \).

**Proposition 3.1.** Assume the setting of Theorem 1. For a given \( (\mu_{U,t}, \sigma_{U,t}^2) \), the ratio \( \mu_{\tilde{X}_{t+1}}^2/\sigma_{\tilde{X}_{t+1}}^2 \) is maximized when \( h_t(u; y) = c h_t^*(u; y) \) where \( c \neq 0 \) is any constant, and

\[
h_t^*(u; y) \triangleq \frac{\mathbb{E}\{G \mid U_t = u, Y = y\} - \mathbb{E}\{G \mid U_t = u\}}{\text{Var}(G \mid U_t = u)} \quad (3.18)
\]

\[
= \frac{\mathbb{E}_W\{W p_{Y|G}(y) \rho_t u + \sqrt{1 - \rho_t^2 \mu_{U,t}^2 W}\}}{\sqrt{1 - \rho_t^2 \mu_{U,t}^2}} \mathbb{E}_W\{p_{Y|G}(y) \rho_t u + \sqrt{1 - \rho_t^2 \mu_{U,t}^2 W}\}, \quad (3.19)
\]
where $\rho_t = \mu_{U,t}/(\mu_{U,t}^2 + \sigma_{U,t}^2)$ and $W \sim \mathcal{N}(0,1)$. In (3.18), the random variables $U_t$ and $Y$ are conditionally independent given $G$ with
\[ Y \sim p_{Y|G}(\cdot|G), \quad U_t = \mu_{U,t}G + \sigma_{U,t}W_{U,t}, \quad (G, W_{U,t}) \sim \text{i.i.d.} \, \mathcal{N}(0,1). \] (3.20)

The optimal choice for $h^*_t$ in Proposition 3.1 was derived in [Ran11] by approximating the belief propagation equations. For completeness, we provide a self-contained proof in Appendix A. The proof also shows that with $h_t = ch^*_t$,
\[
\mu_{X,t+1} = c \sqrt{\delta} \mathbb{E}\{|h^*_t(U_t; Y)|^2\}, \quad \sigma_{X,t+1}^2 = c \frac{\mu_{X,t+1}}{\sqrt{\delta}}.
\]

As the choices $f^*_t, h^*_t$ maximize the signal-to-noise ratios $\mu_{U,t}^2/\sigma_{U,t}^2$ and $\mu_{X,t+1}^2/\sigma_{X,t+1}^2$, respectively, we refer to this algorithm as Bayes-optimal GAMP. We note that to apply Theorem 3.1 to the Bayes-optimal GAMP, we need $f^*_t, h^*_t$ to be Lipschitz. This holds under relatively mild conditions on $P_X$ and $p_{Y|G}$ [MV17] Lemma F.1.

4 Numerical Simulations

We now illustrate the performance of the GAMP algorithm with spectral initialization via numerical examples. For concreteness, we focus on noiseless phase retrieval, where $y_i = |\langle a_i, x \rangle|^2$, $i \in [n]$.

Gaussian prior. In Figure 1, $x$ is chosen uniformly at random on the $d$-dimensional sphere with radius $\sqrt{d}$ (hence, $P_X$ is Gaussian), and $\{a_i\}_{i \in [n]} \sim \text{i.i.d.} \, \mathcal{N}(0, I_d/d)$. We take $d = 8000$, and the
Figure 2: Performance comparison between two different choices of $f_t$ for a binary prior $P_X(1) = P_X(-1) = \frac{1}{2}$. The Bayes-optimal choice $f_t = f_t^*$ (in red) has a lower threshold compared to $f_t$ equal to identity (in blue).

numerical simulations are averaged over $n_{\text{sample}} = 50$ independent trials. The performance of an estimate $\hat{x}$ is measured via its normalized squared scalar product with the signal $x$. The black points are obtained by estimating $x$ via the spectral method, using the optimal pre-processing function $T_s$ reported in Eq. (137) of [MM19]. The empirical results match the black curve, which gives the best possible squared correlation in the high-dimensional limit, as given by Theorem 1 of [LAL19]. The red points are obtained by running the GAMP algorithm (3.3)-(3.4) with the spectral initialization (3.1)-(3.2). The function $f_t$ is chosen to be the identity, and $h_t = \sqrt{h_t^*}$, for $h_t^*$ given by Proposition 3.1. The algorithm is run until the normalized squared difference between successive iterates is small. As predicted by Theorem 1, the numerical simulations agree well with the state evolution curve in red, which is obtained by computing the fixed point of the recursion (3.8) initialized with (3.9).

Bayes-optimal GAMP for a binary-valued prior. Assume now that each entry of the signal $x$ takes value in $\{-1, 1\}$, with $P_X(1) = 1 - P_X(-1) = p$. In Figure 2, we take $p = \frac{1}{2}$, and compare the performance of the GAMP algorithm with spectral initialization for two different choices of the function $f_t$: $f_t$ equal to identity (in blue) and $f_t = f_t^*$ (in red), where $f_t^*$ is the Bayes-optimal choice (3.15). By computing the conditional expectation, we have

$$f_t^*(s) = 2p(X = 1 \mid \mu_{X,t} X + \sigma_{X,t} W = s) \frac{2}{1 + \frac{1-p}{p} \exp\left(\frac{-2\mu_{X,t}}{\sigma_{X,t}^2}\right)} - 1. \quad (4.1)$$

The rest of the setting is analogous to that of Figure 1. There is a significant performance gap between the Bayes-optimal choice $f_t = f_t^*$ and the choice $f_t(x) = x$. As in the previous experiment, we observe very good agreement between the GAMP algorithm and the state evolution prediction of Theorem 1. We remark that for this setting, the information-theoretically optimal overlap (computed using the formula in [BKM+19]) is 1 for all $\delta > 0$. Since the components of $x$ are...
Figure 3: Visual comparison between the reconstruction of the GAMP algorithm with spectral initialization and that of the spectral method alone for measurements given by coded diffraction patterns.
in \{-1,1\}, there are \(2^d\) choices for \(x\). The information-theoretically optimal estimator picks the choice that is consistent with \(y_i = \langle x, a_i \rangle\), \(i \in [n]\). (Since \(A\) is Gaussian, with high probability this solution is unique.)

**Coded diffraction patterns.** We consider the model of coded diffraction patterns described in Section 7.2 of [MM19]. Here the signal \(x\) is the image of Figure 3a, and it can be viewed as a \(d_1 \times d_2 \times 3\) array with \(d_1 = 820\) and \(d_2 = 1280\). The sensing vectors are given by

\[
a_r(t_1,t_2) = d_\ell(t_1,t_2) \cdot e^{2\pi ik_1/d_1} \cdot e^{2\pi ik_2/d_2},
\]

where \(r \in [n]\), \(t_1 \in [d_1]\), \(t_2 \in [d_2]\), \(i\) denotes the imaginary unit, \(a_r(t_1,t_2)\) is the \((t_1,t_2)\)-th component of \(a_r \in \mathbb{C}^d\), and the \((d_\ell(t_1,t_2))\)'s are i.i.d. and uniform in \(\{1,-1,i,-i\}\). The index \(r \in [n]\) is associated to a pair \((\ell,k)\), with \(\ell \in [L]\); the index \(k \in [d]\) is associated to a pair \((k_1,k_2)\) with \(k_1 \in [d_1]\) and \(k_2 \in [d_2]\). Thus, \(n = L \cdot d\) and, therefore, \(\delta = L \in \mathbb{N}\). To obtain non-integer values of \(\delta\), we set to 0 a suitable fraction of the vectors \(a_r\), chosen uniformly at random.

In this model, the scalar product \(\langle x_j, a_r \rangle\) can be computed with an FFT algorithm. Furthermore, in order to evaluate the principal eigenvector for the spectral initialization, we use a power method which stops if either the number of iterations reaches the maximum value of 100000 or the modulus of the scalar product between the estimate at the current iteration \(T\) and at the iteration \(T-10\) is larger than 1 - \(10^{-7}\).

The GAMP algorithm with spectral initialization for the complex-valued setting is described in Appendix D. Figure 3b shows a visual representation of the results. The improvement achieved by the GAMP algorithm over the spectral estimator is impressive, with GAMP achieving full recovery already at \(\delta = 2.4\). A numerical comparison of the performance of the two methods is given in Figure 5 in Appendix D. We emphasize that the state evolution result of Theorem 1 is only valid for Gaussian sensing matrices. Extending it to structured matrices such as coded diffraction patterns is an interesting direction for future work.

### 5 Sketch of the Proof of Theorem 1

We give an outline of the proof here, and provide the technical details in the appendices.

**The artificial GAMP algorithm.** We construct an artificial GAMP algorithm, whose iterates are denoted by \(\tilde{x}^t, \tilde{u}^t\), for \(t \geq 0\). Starting from an initialization \((\tilde{x}^0, \tilde{u}^0)\), for \(t \geq 0\) we iteratively compute:

\[
\tilde{x}^{t+1} = \frac{1}{\sqrt{\delta}} A^T \tilde{h}_t(\tilde{u}^t; y) - \tilde{c}_t \tilde{f}_t(\tilde{x}^t),
\]

\[
\tilde{u}^{t+1} = \frac{1}{\sqrt{\delta}} A \tilde{f}_{t+1}(\tilde{x}^{t+1}) - \tilde{b}_{t+1} \tilde{h}_t(\tilde{u}^t; y).
\]

For \(t \geq 0\), the functions \(\tilde{f}_t : \mathbb{R} \to \mathbb{R}\) and \(\tilde{h}_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) are Lipschitz, and will be specified below. The scalars \(\tilde{c}_t\) and \(\tilde{b}_{t+1}\) are defined as

\[
\tilde{c}_t = \frac{1}{n} \sum_{i=1}^{n} \tilde{h}_i(\tilde{u}_i^t; y_i), \quad \tilde{b}_{t+1} = \frac{1}{n} \sum_{i=1}^{d} \tilde{f}_{t+1}(\tilde{x}_i^{t+1}),
\]

12
where \( \tilde{h}' \) denotes the derivative with respect to the first argument. The iteration is initialized as follows. Choose any \( \alpha \in (0, 1) \), and a standard Gaussian vector \( \mathbf{n} \sim \mathcal{N}(0, I_d) \) that is independent of \( \mathbf{x} \) and \( A \). Then,

\[
\tilde{x}^0 = \alpha \mathbf{x} + \sqrt{1 - \alpha^2} \mathbf{n}, \quad \tilde{u}^0 = \frac{1}{\sqrt{\delta}} A \tilde{f}_0(\tilde{x}^0) .
\] (5.4)

The artificial GAMP is divided into two phases. In the first phase, which lasts up to iteration \( T \), the functions \( \tilde{f}_t, \tilde{h}_t \) for \( 0 \leq t \leq (T - 1) \), are chosen such that as \( T \to \infty \), the iterate \( \tilde{x}^T \) approaches the initialization \( \tilde{x}^0 \) of the true GAMP algorithm defined in (3.1). In the second phase, the functions \( \tilde{f}_t, \tilde{h}_t \) for \( t \geq T \), are chosen to match those of the true GAMP. The key observation is that a state evolution result for the artificial GAMP follows directly from the standard analysis of GAMP [JM13] since the initialization \( \tilde{x}^0 \) is independent of \( A \). By showing that as \( T \to \infty \), the iterates and the state evolution parameters of the artificial GAMP approach the corresponding quantities of the true GAMP, we prove that the state evolution result of Theorem 1 holds.

We now specify the functions used in the artificial GAMP. For \( 0 \leq t \leq (T - 1) \), we set

\[
\tilde{f}_t(x) = \frac{x}{\beta_t}, \quad \tilde{h}_t(x; y) = \sqrt{\delta} x \frac{T_s(y)}{\lambda_t^* - T_s(y)},
\] (5.5)

where \( T_s \) is the pre-processing function used for the spectral estimator, \( \lambda_t^* \) is the unique solution of \( \zeta_\delta (\lambda) = \phi(\lambda) \) for \( \lambda > \tau \) (also given by (2.7)), and \( (\beta_t)_{t \geq 0} \) are constants coming from the state evolution recursion defined below. Furthermore, for \( t \geq T \), we set

\[
\tilde{f}_t(x) = f_{t-T}(x), \quad \tilde{h}_t(x; y) = h_{t-T}(x; y).
\] (5.6)

With these choices of \( \tilde{f}_t, \tilde{h}_t \), the coefficients \( \tilde{c}_t \) and \( \tilde{b}_t \) in (5.3) become:

\[
\tilde{c}_t = \frac{\sqrt{\delta}}{n} \sum_{i=1}^n \frac{T_s(y_i)}{\lambda_t^* - T_s(y_i)}, \quad \tilde{b}_t = \frac{1}{\delta \beta_t}, \quad 0 \leq t \leq (T - 1),
\] (5.7)

\[
\tilde{c}_t = \frac{1}{n} \sum_{i=1}^n \tilde{h}'_{t-T}(\tilde{u}_i^t; y_i), \quad \tilde{b}_t = \frac{1}{n} \sum_{i=1}^d \tilde{f}'_{t-T}(\tilde{x}_i^t), \quad t \geq T.
\]

Since the initialization \( \tilde{x}^0 \) in (5.4) is independent of \( A \), the state evolution result of [JM13] can be applied to the artificial GAMP. This result, formally stated in Proposition 3.1 in Appendix B.1 implies that for \( t \geq 0 \), the empirical distributions of \( \tilde{x}^t \) and \( \tilde{u}^t \) converge in \( W_2 \) distance to the laws of the random variables \( \tilde{X}_t \) and \( \tilde{U}_t \), respectively, with

\[
\tilde{X}_t \equiv \mu_{\tilde{X}_t} X + \sigma_{\tilde{X}_t} W_{\tilde{X}_t}, \quad \tilde{U}_t \equiv \mu_{\tilde{U}_t} G + \sigma_{\tilde{U}_t} W_{\tilde{U}_t}.
\] (5.8)

Here \( W_{\tilde{X}_t}, W_{\tilde{U}_t} \) are standard normal and independent of \( X \) and \( G \), respectively. The state evolution recursion defining the parameters \((\mu_{\tilde{X}_t}, \sigma_{\tilde{X}_t}, \mu_{\tilde{U}_t}, \sigma_{\tilde{U}_t}, \beta_t)\) has the same form as (3.8), except that we use the functions defined in (5.5) for \( 0 \leq t \leq (T - 1) \), and the functions in (5.6) for \( t \geq T \). The detailed expressions are given in Appendix B.1.

**Analysis of the first phase.** The first phase of the artificial GAMP is designed so that its output vectors after \( T \) iterations \((\tilde{x}^T, \tilde{u}^T)\) are close to the initialization \((x^0, u^0)\) of the true GAMP.
algorithm given by (3.1)-(3.2). This part of the algorithm is similar to the GAMP used in [MTV20] to approximate the spectral estimator \( \hat{x}^s \). In particular, the state evolution recursion of the first phase (given in (B.2)) converges as \( T \to \infty \) to the following fixed point:

\[
\lim_{T \to \infty} \mu_{x,T} = \frac{a}{\sqrt{\delta}}, \quad \lim_{T \to \infty} \sigma_{x,T}^2 = \frac{1 - a^2}{\delta}, \tag{5.9}
\]

where \( a \) is the limit (normalized) correlation between the spectral estimator \( \hat{x}^s \) and the signal, see (2.8). Furthermore, the GAMP iterate \( \tilde{x}^T \) approaches \( \hat{x}^s \), i.e.,

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\| \sqrt{d} \hat{x}^s - \sqrt{\delta} \tilde{x}^T \|_2}{d} = 0 \quad \text{a.s.} \tag{5.10}
\]

These results are formally stated in Lemma B.2 and B.3, respectively, contained in Appendix B.2.

**Analysis of the second phase.** The second phase of the artificial GAMP is designed so that its iterates \( \tilde{x}^{T+t}, \tilde{u}^{T+t} \) are close to \( x^t, u^t \), respectively for \( t \geq 0 \), and the corresponding state evolution parameters are also close. In particular, in order to prove Theorem 1 we first analyze a slightly modified version of the true GAMP algorithm in (3.3)-(3.4) where the ‘memory’ coefficients \( b_t \) and \( c_t \) in (3.5) are replaced by deterministic values obtained from state evolution. The iterates of this modified GAMP, denoted by \( \hat{x}^t, \hat{u}^t \), are as follows. Start with the initialization

\[
\hat{x}^0 = x^0 = \frac{1}{\sqrt{\delta}} \hat{x}^s, \tag{5.11}
\]

\[
\hat{u}^0 = \frac{1}{\sqrt{\delta}} A f_0(\hat{x}^0) - \bar{b}_0 \sqrt{\delta} Z_s A \hat{x}^0, \tag{5.12}
\]

where \( \bar{b}_0 = \frac{1}{\delta} \mathbb{E} \{ f_0(X_0) \} \). Then, for \( t \geq 0 \):

\[
\hat{x}^{t+1} = \frac{1}{\sqrt{\delta}} A^t h_t(\hat{u}^t; y) - \bar{c}_t f_t(\hat{x}^t), \tag{5.13}
\]

\[
\hat{u}^{t+1} = \frac{1}{\sqrt{\delta}} A f_{t+1}(\hat{x}^{t+1}) - \bar{b}_{t+1} h_t(\hat{u}^t; y). \tag{5.14}
\]

Here, for \( t \geq 0 \), the deterministic memory coefficients \( \bar{b}_t \) and \( \bar{c}_t \) are

\[
\bar{c}_t = \mathbb{E} \{ h_t(U_t; Y) \}, \quad \bar{b}_t = \frac{1}{\delta} \mathbb{E} \{ f_t(X_t) \}, \tag{5.15}
\]

where \( X_t, U_t \) are defined in (3.6)-(3.7).

We have now defined three different GAMP iterations: the original one with iterates \( (x^t, u^t) \) given by (3.3)-(3.4), the modified one above with iterates \( (\hat{x}^t, \hat{u}^t) \), and the artificial GAMP iterates \( (\tilde{x}^t, \tilde{u}^t) \) given by (5.1)-(5.2). Lemma B.5 in Appendix B.3 proves that, for each \( t \geq 0 \), (i) the iterates \( (\tilde{x}^{T+t}, \tilde{u}^{T+t}) \) are close to \( (\hat{x}^t, \hat{u}^t) \) for sufficiently large \( T \), and (ii) the corresponding state evolution parameters are also close. We then use this lemma to prove Theorem 1 in Appendix B.4. In particular, we show that, almost surely,

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_i^t) = \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_i^t) = \mathbb{E} \{ \psi(X, \mu_{X,t}X + \sigma_{X,t}W) \}. \tag{5.16}
\]

That is, the iterates in (5.13)-(5.14) have the same asymptotic empirical distribution as the original version in (3.3)-(3.4).
Acknowledgements

The authors would like to thank Andrea Montanari for helpful discussions. M. Mondelli was partially supported by the 2019 Lopez-Loreta Prize. R. Venkataramanan was partially supported by the Alan Turing Institute under the EPSRC grant EP/N510129/1.

References

[AGZ09] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni, *An introduction to random matrices*, Cambridge University Press, 2009.

[BB08] Petros T Boufounos and Richard G Baraniuk, *1-bit compressive sensing*, Conference on Information Sciences and Systems (CISS), 2008, pp. 16–21.

[BKM+19] Jean Barbier, Florent Krzakala, Nicolas Macris, Léo Miolane, and Lenka Zdeborová, *Optimal errors and phase transitions in high-dimensional generalized linear models*, Proceedings of the National Academy of Sciences 116 (2019), no. 12, 5451–5460.

[BM11] Mohsen Bayati and Andrea Montanari, *The dynamics of message passing on dense graphs, with applications to compressed sensing*, IEEE Transactions on Information Theory 57 (2011), 764–785.

[BM12] M. Bayati and A. Montanari, *The LASSO risk for gaussian matrices*, IEEE Transactions on Information Theory 58 (2012), 1997–2017.

[Bol14] Erwin Bolthausen, *An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model*, Communications in Mathematical Physics 325 (2014), no. 1, 333–366.

[BR17] Sohail Bahmani and Justin Romberg, *Phase retrieval meets statistical learning theory: A flexible convex relaxation*, International Conference on Artificial Intelligence and Statistics (AISTATS), 2017, pp. 252–260.

[CC17] Yuxin Chen and Emmanuel J Candès, *Solving random quadratic systems of equations is nearly as easy as solving linear systems*, Communications on Pure and Applied Mathematics 70 (2017), no. 5, 822–883.

[CESV15] Emmanuel J. Candès, Yonina C. Eldar, Thomas Strohmer, and Vladislav Voroninski, *Phase retrieval via matrix completion*, SIAM Review 57 (2015), no. 2, 225–251.

[CLS15a] Emmanuel J. Candès, Xiaodong Li, and Mahdi Soltanolkotabi, *Phase retrieval from coded diffraction patterns*, Applied and Computational Harmonic Analysis 39 (2015), no. 2, 277–299.

[CLS15b] _____, *Phase retrieval via Wirtinger flow: Theory and algorithms*, IEEE Transactions on Information Theory 61 (2015), no. 4, 1985–2007.

[CSV13] Emmanuel J. Candès, Thomas Strohmer, and Vladislav Voroninski, *Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming*, Communications on Pure and Applied Mathematics 66 (2013), no. 8, 1241–1274.

[DBMM20] Rishabh Dudeja, Milad Bakhshizadeh, Junjie Ma, and Arian Maleki, *Analysis of spectral methods for phase retrieval with random orthogonal matrices*, IEEE Transactions on Information Theory 66 (2020), no. 8, 5182–5203.

[DJ17] Laurent Demanet and Vincent Jugnon, *Convex recovery from interferometric measurements*, IEEE Transactions on Computational Imaging 3 (2017), no. 2, 282–295.
[DJM13] David L. Donoho, Adel Javanmard, and Andrea Montanari, *Information-theoretically optimal compressed sensing via spatial coupling and approximate message passing*, IEEE Transactions on Information Theory **59** (2013), no. 11, 7434–7464.

[DM14] Yash Deshpande and Andrea Montanari, *Information-theoretically optimal sparse PCA*, IEEE International Symposium on Information Theory (ISIT), 2014, pp. 2197–2201.

[DMM09] David L. Donoho, Arian Maleki, and Andrea Montanari, *Message Passing Algorithms for Compressed Sensing*, Proceedings of the National Academy of Sciences **106** (2009), 18914–18919.

[EK12] Yonina C Eldar and Gitta Kutyniok, *Compressed sensing: Theory and applications*, Cambridge University Press, 2012.

[Fan20] Zhou Fan, *Approximate message passing algorithms for rotationally invariant matrices*, arXiv:2008.11892 (2020).

[FD87] C Fienup and J Dainty, *Phase retrieval and image reconstruction for astronomy*, Image recovery: theory and application **231** (1987), 275.

[Fie82] J. R. Fienup, *Phase retrieval algorithms: A comparison*, Applied Optics **21** (1982), no. 15, 2758–2769.

[FS20] Albert Fannjiang and Thomas Strohmer, *The numerics of phase retrieval*, arXiv:2004.05788 (2020).

[GS18] Tom Goldstein and Christoph Studer, *Phasemax: Convex phase retrieval via basis pursuit*, IEEE Transactions on Information Theory **64** (2018), no. 4, 2675–2689.

[JM13] Adel Javanmard and Andrea Montanari, *State evolution for general approximate message passing algorithms, with applications to spatial coupling*, Information and Inference (2013), 115–144.

[KKM+16] Yoshiyuki Kabashima, Florent Krzakala, Marc Mézard, Ayaka Sakata, and Lenka Zdeborová, *Phase transitions and sample complexity in Bayes-optimal matrix factorization*, IEEE Transactions on Information Theory **62** (2016), no. 7, 4228–4265.

[KMS+12] Florent Krzakala, Marc Mézard, Francois Sausset, Yifan Sun, and Lenka Zdeborová, *Probabilistic reconstruction in compressed sensing: algorithms, phase diagrams, and threshold achieving matrices*, Journal of Statistical Mechanics: Theory and Experiment **2012** (2012), no. 08, P08009.

[LAL19] Wangyu Luo, Wael Alghamdi, and Yue M. Lu, *Optimal spectral initialization for signal recovery with applications to phase retrieval*, IEEE Transactions on Signal Processing **67** (2019), no. 9, 2347–2356.

[LGL15] Gen Li, Yuantao Gu, and Yue M Lu, *Phase retrieval using iterative projections: Dynamics in the large systems limit*, Allerton Conference on Communication, Control, and Computing (Allerton), 2015, pp. 1114–1118.

[Li92] Ker-Chau Li, *On principal Hessian directions for data visualization and dimension reduction: Another application of Stein’s lemma*, Journal of the American Statistical Association **87** (1992), no. 420, 1025–1039.

[LL19] Yue M. Lu and Gen Li, *Phase transitions of spectral initialization for high-dimensional non-convex estimation*, Information and Inference (2019).

[MAYB13] Arian Maleki, Laura Anitori, Zai Yang, and Richard G Baraniuk, *Asymptotic analysis of complex lasso via complex approximate message passing (CAMP)*, IEEE Transactions on Information Theory **59** (2013), no. 7, 4290–4308.

[McC18] Peter McCullagh, *Generalized linear models*, Routledge, 2018.

[Mil90] Rick P Millane, *Phase retrieval in crystallography and optics*, JOSA A **7** (1990), no. 3, 394–411.
[MLKZ20] Antoine Maillard, Bruno Loureiro, Florent Krzakala, and Lenka Zdeborová, *Phase retrieval in high dimensions: Statistical and computational phase transitions*, arXiv:2006.05228 (2020).

[MM19] Marco Mondelli and Andrea Montanari, *Fundamental limits of weak recovery with applications to phase retrieval*, Foundations of Computational Mathematics 19 (2019), 703–773.

[MTV20] Marco Mondelli, Christos Thrampoulidis, and Ramji Venkataramanan, *Optimal combination of linear and spectral estimators for generalized linear models*, arXiv:2008.03326 (2020).

[MV17] Andrea Montanari and Ramji Venkataramanan, *Estimation of low-rank matrices via approximate message passing*, Annals of Statistics, to appear (2017), arXiv:1711.01682.

[MWCC20] Cong Ma, Kaizheng Wang, Yuejie Chi, and Yuxin Chen, *Implicit regularization in nonconvex statistical estimation: Gradient descent converges linearly for phase retrieval, matrix completion, and blind deconvolution*, Foundations of Computational Mathematics 20 (2020), no. 3, 451–632.

[MXM18] Junjie Ma, Ji Xu, and Arian Maleki, *Approximate message passing for amplitude based optimization*, International Conference on Machine Learning (ICML), 2018, pp. 3371–3380.

[MXM19] Junjie Ma, Ji Xu, and Arian Maleki, *Optimization-based amp for phase retrieval: The impact of initialization and ℓ_2 regularization*, IEEE Transactions on Information Theory 65 (2019), no. 6, 3600–3629.

[NJS13] Praneeth Netrapalli, Prateek Jain, and Sujay Sanghavi, *Phase retrieval using alternating minimization*, Advances in Neural Information Processing Systems (NIPS), 2013, pp. 2796–2804.

[PWM18] Amelia Perry, Alexander S Wein, Afonso S Bandeira, and Ankur Moitra, *Message-passing algorithms for synchronization problems over compact groups*, Communications on Pure and Applied Mathematics 71 (2018), no. 11, 2275–2322.

[Ran11] S. Rangan, *Generalized Approximate Message Passing for Estimation with Random Linear Mixing*, IEEE International Symposium on Information Theory (ISIT), 2011.

[RF12] Sundeep Rangan and Alyson K Fletcher, *Iterative estimation of constrained rank-one matrices in noise*, IEEE International Symposium on Information Theory (ISIT), 2012, pp. 1246–1250.

[RG01] Sundeep Rangan and Vivek K. Goyal, *Recursive consistent estimation with bounded noise*, IEEE Transactions on Information Theory 47 (2001), no. 1, 457–464.

[SC19] Pragya Sur and Emmanuel J Candès, *A modern maximum-likelihood theory for high-dimensional logistic regression*, Proceedings of the National Academy of Sciences 116 (2019), no. 29, 14516–14525.

[SEC+15] Yoav Shechtman, Yonina C. Eldar, Oren Cohen, Henry N. Chapman, Jianwei Miao, and Mordechai Segev, *Phase retrieval with application to optical imaging: a contemporary overview*, IEEE Signal Processing Magazine 32 (2015), no. 3, 87–109.

[SR14] Philip Schniter and Sundeep Rangan, *Compressive phase retrieval via generalized approximate message passing*, IEEE Transactions on Signal Processing 63 (2014), no. 4, 1043–1055.

[TV19] Yan Shuo Tan and Roman Vershynin, *Phase retrieval via randomized kaczmarz: Theoretical guarantees*, Information and Inference: A Journal of the IMA 8 (2019), no. 1, 97–123.

[Vil08] Cédric Villani, *Optimal transport: Old and new*, vol. 338, Springer Science & Business Media, 2008.

[WdM15] Irène Waldspurger, Alexandre d’Aspremont, and Stéphane Mallat, *Phase recovery, maxcut and complex semidefinite programming*, Mathematical Programming 149 (2015), no. 1-2, 47–81.

[Wei15] Ke Wei, *Solving systems of phaseless equations via Kaczmarz methods: A proof of concept study*, Inverse Problems 31 (2015), no. 12.
A Proof of Proposition 3.1

From assumption (B2) on p. 5, we recall that \( h_t(u; y) = h_t(u; q(g, v)) \). We write \( \partial_g h_t(u; q(g, v)) \) for the partial derivative with respect to \( g \). We will show that \( \mu_{X,t+1} \) in (3.8) can be written as:

\[
\mu_{X,t+1} = \sqrt{\delta} \mathbb{E}\{\partial_g h_t(U_t; q(G,V))\}, \tag{A.1}
\]

\[
= \sqrt{\delta} \mathbb{E}\left\{ h_t(U_t; Y) \left( \frac{\mathbb{E}\{G|U_t,Y\} - \mathbb{E}\{G|U_t\}}{\text{Var}\{G|U_t\}} \right) \right\}. \tag{A.2}
\]

From (A.2), we have that

\[
\frac{\mu_{X,t+1}}{\sigma_{X,t+1}} = \frac{\sqrt{\delta}}{\sqrt{\mathbb{E}\{h_t(U_t; Y)^2\}}} \mathbb{E}\left\{ h_t(U_t; Y) \left( \frac{\mathbb{E}\{G|U_t,Y\} - \mathbb{E}\{G|U_t\}}{\text{Var}\{G|U_t\}} \right) \right\}. \tag{A.3}
\]

The absolute value of the RHS is maximized when \( h_t = c h_t^* \), for \( c \neq 0 \) and \( h_t^* \) is given in (3.18). To obtain the alternative expression in (3.19) from (3.18) we recall that \( U_t \) is Gaussian with zero mean and variance \( (\mu_{U,t}^2 + \sigma_{U,t}^2) \). Furthermore, the conditional distribution of \( G \) given \( U_t = u \) is Gaussian with \( \mathbb{E}\{G \mid U_t = u\} = \rho_t u \) and \( \text{Var}(G \mid U_t = u) = (1 - \rho_t \mu_{U,t}) \). Therefore, with \( W \sim \mathcal{N}(0,1) \) we have

\[
\mathbb{E}\{G \mid U_t = u, Y = y\} = \frac{\mathbb{E}_W\{(\rho_t u + \sqrt{1 - \rho_t \mu_{U,t}})W\} p_Y|G(y \mid \rho_t u + \sqrt{1 - \rho_t \mu_{U,t}} W)\}}{\mathbb{E}_W\{p_Y|G(y \mid \rho_t u + \sqrt{1 - \rho_t \mu_{U,t}} W)\}} = \rho_t u + \sqrt{1 - \rho_t \mu_{U,t}} \frac{\mathbb{E}\{W p_Y|G(y \mid \rho_t u + \sqrt{1 - \rho_t \mu_{U,t}} W)\}}{\mathbb{E}_W\{p_Y|G(y \mid \rho_t u + \sqrt{1 - \rho_t \mu_{U,t}} W)\}}. \tag{A.4}
\]

Substituting (A.4) in (3.18) yields (3.19).

It remains to show (A.2), which we do by first showing (A.1). Define \( e_t : \mathbb{R}^3 \to \mathbb{R} \) by

\[
e_t(g,w,v) = h_t(\mu_{U,t} g + \sigma_{U,t} w; q(g,v)). \tag{A.5}
\]

Then, using the chain rule, the partial derivative of \( e_t(g,w,v) \) with respect to \( g \) is

\[
\partial_g e_t(g,w,v) = \mu_{U,t} h_t'(\mu_{U,t} g + \sigma_{U,t} w; q(g,v)) + \partial_g h_t(u; q(g,v)). \tag{A.6}
\]

The parameter \( \mu_{X,t+1} \) in (3.8) can be written as

\[
\mu_{X,t+1} = \sqrt{\delta} \left[ \mathbb{E}\{G e_t(G,W_{U,t},V)\} - \mu_{U,t} \mathbb{E}\{h_t'(\mu_{U,t} G + \sigma_{U,t} W_{U,t}; Y)\} \right]
\]

\[
= \sqrt{\delta} \left[ \mathbb{E}\{\partial_g e_t(G,W_{U,t},V)\} - \mu_{U,t} \mathbb{E}\{h_t'(\mu_{U,t} G + \sigma_{U,t} W_{U,t}; Y)\} \right]
\]

\[
= \sqrt{\delta} \mathbb{E}\{\partial_g h_t(U_t; q(G,V))\}, \tag{A.7}
\]

[YLSV12] Feng Yang, Yue M. Lu, Luciano Sbaiz, and Martin Vetterli, *Bits from photons: Oversampled image acquisition using binary Poisson statistics*, IEEE Transactions on Image Processing 21 (2012), no. 4, 1421–1436.
Here step (ii) holds due to (A.6), and (i) holds due to Stein’s lemma. Finally, we obtain (A.2) from (A.1) as follows:

\[
\mathbb{E}\{\partial_{\theta} h_t(U_t; q(G, V))\} = \mathbb{E} \left\{ \mathbb{E}_{G|U_t} \left[ \partial_{\theta} h_t(U_t; q(G, V)) \mid U_t \right] \right\} \\
\overset{(ii)}{=} \mathbb{E} \left\{ \mathbb{E}_{G|U_t} \left[ h_t(U_t; q(G, V)) \cdot (G - \mathbb{E}\{G|U_t\})/\text{Var}\{G|U_t\} \mid U_t \right] \right\} \\
= \mathbb{E} \left\{ \mathbb{E}_{G|U_t,Y} \left[ h_t(U_t; Y) \cdot (G - \mathbb{E}\{G|U_t\})/\text{Var}\{G|U_t\} \mid U_t, Y \right] \right\} \\
= \mathbb{E} \left\{ h_t(U_t; Y) \cdot ((\mathbb{E}\{G|U_t, Y\} - \mathbb{E}\{G|U_t\})/\text{Var}\{G|U_t\}) \right\}.
\]

Here step (ii) holds due to Stein’s lemma. This completes the proof of the proposition. \( \Box \)

**B  Proof of the Main Result**

**B.1 The Artificial GAMP Algorithm**

The state evolution parameters for the artificial GAMP are recursively defined as follows. Recall from (5.8) that \( \tilde{X}_t = \mu_{\tilde{X},t} X + \sigma_{\tilde{X},t} \tilde{W}_{\tilde{X},t} \) and \( \bar{U}_t = \mu_{\bar{U},t} G + \sigma_{\bar{U},t} \bar{W}_{\bar{U},t} \). Using (5.4), the state evolution initialization is

\[
\mu_{\tilde{X},0} = \alpha, \quad \sigma_{\tilde{X},0}^2 = 1 - \alpha^2, \quad \beta_0 = \sqrt{\mu_{\tilde{X},0}^2 + \sigma_{\tilde{X},0}^2} = 1. \tag{B.1}
\]

For \( 0 \leq t \leq (T - 1) \), the state evolution parameters are iteratively computed by using the functions defined in (5.5) in (3.8):

\[
\mu_{\tilde{U},t} = \frac{\mu_{\tilde{X},t}}{\sqrt{\delta \beta_t}}, \quad \sigma_{\tilde{U},t}^2 = \frac{\sigma_{\tilde{X},t}^2}{\delta \beta_t^2}, \\
\mu_{\tilde{X},t+1} = \frac{\mu_{\tilde{X},t}}{\sqrt{\delta \beta_t}}, \quad \sigma_{\tilde{X},t+1}^2 = \frac{1}{\beta_t} \mathbb{E} \left\{ \frac{Z_s^2(G^2 \mu_{\tilde{X},t}^2 + \sigma_{\tilde{X},t}^2)}{(\lambda_0^* - Z_s)^2} \right\}, \\
\beta_{t+1} = \sqrt{\mu_{\tilde{X},t+1}^2 + \sigma_{\tilde{X},t+1}^2}. \tag{B.2}
\]

Here we recall that \( G \sim \mathcal{N}(0,1) \), \( Y \sim p_Y|G(\cdot \mid G) \), \( Z_s = T_s(Y) \), and the equality in (2.7) which is used to obtain the expression for \( \mu_{\tilde{X},t+1} \). For \( t \geq T \), the state evolution parameters are:

\[
\mu_{\tilde{U},t} = \frac{1}{\sqrt{\delta}} \mathbb{E}\{X f_{t-T}(\tilde{X}_t)\}, \\
\sigma_{\tilde{U},t}^2 = \frac{1}{\delta} \mathbb{E}\{f_{t-T}(\tilde{X}_t)^2\} - \mu_{\tilde{U},t}^2, \\
\mu_{\tilde{X},t+1} = \sqrt{\delta} \mathbb{E}\{G f_{t-T}(\tilde{U}_t; Y)\} - \mathbb{E}\{h_{t-T}(\tilde{U}_t; Y)\} \mathbb{E}\{X f_{t-T}(\tilde{X}_t)\}, \\
\sigma_{\tilde{X},t+1}^2 = \mathbb{E}\{h_{t-T}(\tilde{U}_t; Y)^2\}. \tag{B.3}
\]

**Proposition B.1** (State evolution for artificial GAMP). Consider the setting of Theorem 1 the artificial GAMP iteration described in (5.1) - (5.7), and the corresponding state evolution parameters defined in (B.1) - (B.3).
For any PL(2) function $\psi : \mathbb{R}^2 \to \mathbb{R}$, the following holds almost surely for $t \geq 1$:

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \ddot{x}^t_i) = \mathbb{E}\{\psi(X, \ddot{X}_t)\}, \quad (B.4)
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{U}_t^i) = \mathbb{E}\{\psi(Y, \tilde{U}_t)\}. \quad (B.5)
\]

Here $X \sim P_X$ and $Y \sim P_{Y|G}$, with $G \sim \mathcal{N}(0,1)$. The random variables $\ddot{X}_t, \tilde{U}_t$ are defined in (5.8).

The proposition follows directly from the state evolution result of [JM13] since the initialization $\tilde{x}^0$ of the artificial GAMP is independent of $A$.

### B.2 Analysis of the First Phase

**Lemma B.2** (Fixed point of state evolution for first phase). Consider the setting of Theorem 1. Then, the state evolution recursion for the first phase, given by (B.1)-(B.2), converges as $T \to \infty$ to the following fixed point:

\[
\mu_{\ddot{X}} \triangleq \lim_{T \to \infty} \mu_{\ddot{X},T} = \frac{a}{\sqrt{\delta}}, \quad \sigma^2_{\ddot{X}} \triangleq \lim_{T \to \infty} \sigma^2_{\ddot{X},T} = \frac{1-a^2}{\delta}, \quad (B.6)
\]

where $a$ is defined in (2.8).

**Proof.** Recall that $\lambda^*_\delta$ denotes the unique solution of $\zeta_\delta(\lambda) = \phi(\lambda)$ for $\lambda > \tau$ (also given by (2.7)), and define $Z = Z_s/(\lambda^*_\delta - Z_s)$, where $Z_s = \mathcal{T}_s(Y)$. Note that

\[
\mathbb{E}\{Z(G^2 - 1)\} = \mathbb{E}\left\{\frac{Z_s(G^2 - 1)}{\lambda^*_\delta - Z_s}\right\} = \frac{1}{\delta}, \quad (B.7)
\]

where the second equality follows from the equality in (2.7). Moreover, the inequality in (2.7) implies that

\[
\mathbb{E}\left\{\frac{Z^2_s}{(\mathbb{E}\{Z(G^2 - 1)\})^2}\right\} = \frac{Z^2_s}{(\mathbb{E}\{Z(G^2 - 1)\})^2} < \delta. \quad (B.8)
\]

Thus, by recalling that the state evolution initialization $\mu_{\ddot{X},0} = \alpha$ is strictly positive, the result follows from Lemma 5.2 in [MTV20].

**Lemma B.3** (Convergence to spectral estimator). Consider the setting of Theorem 1, and consider the first phase of the artificial GAMP iteration, given by (5.1)-(5.2) with $\tilde{f}_t$ and $\tilde{h}_t$ defined in (5.5). Then,

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\|\sqrt{d} \tilde{x}^s - \sqrt{\delta} \tilde{x}^T\|}{d} = 0 \quad \text{a.s.} \quad (B.9)
\]

Furthermore, for any PL(2) function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, almost surely we have:

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \sqrt{d} \tilde{x}^s_i) = \lim_{T \to \infty} \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \sqrt{\delta} \tilde{x}^T_i) = \mathbb{E}\{\psi(X, \sqrt{\delta} (\mu_{\ddot{X}} X + \sigma_{\ddot{X}} W))\}. \quad (B.10)
\]

Here $X \sim P_X$ and $W \sim \mathcal{N}(0,1)$ are independent.
Proof. As in the proof of the previous result, let \( Z = Z_s/(\lambda^*_s - Z_s) \) and note that \([B.7]-[B.8]\) hold. Also define

\[
Z' = \frac{Z}{Z + \delta E\{Z(G^2 - 1)\}} = \frac{Z}{Z + 1} = \frac{Z}{\lambda^*_s}.
\]  

(B.11)

Then, the assumptions of Lemma 5.4 in [MTV20] are satisfied, with the only difference of the initialization of the GAMP iteration (cf. (5.4) in this paper and (5.4) in [MTV20]). However, it is straightforward to verify that the difference in the initialization does not affect the proof of Lemma 5.4 in [MTV20]. Thus, \([B.9]\) follows from (5.87) of [MTV20], and \([B.10]\) follows by taking \(k = 2\) in (5.31) of [MTV20].

We will also need the following result on the convergence of the GAMP iterates.

**Lemma B.4 (Convergence of GAMP iterates).** Consider the first phase of the artificial GAMP iteration, given by \([5.1]-[5.2]\) with \(\tilde{f}_t\) and \(\tilde{h}_t\) defined in \([5.5]\). Then, the following limits hold almost surely:

\[
\lim_{T \to \infty} \lim_{n \to \infty} \frac{1}{n} \|\tilde{u}^{T-1} - \tilde{u}^{T-2}\|_2^2 = 0, \quad \lim_{T \to \infty} \lim_{d \to \infty} \frac{1}{d} \|\tilde{x}^T - \tilde{x}^{T-1}\|_2^2 = 0.
\]  

(B.12)

Though the initialization of the GAMP in [MTV20] is different from \([5.4]\), the proof of Lemma B.4 is the same as that of Lemma 5.3 in [MTV20] since it only relies on \(\mu_{\tilde{X},0} = \alpha\) being strictly non-zero.

### B.3 Analysis of the Second Phase

**Lemma B.5.** Assume the setting of Theorem 1. Consider the artificial GAMP algorithm \([5.1]-[5.2]\) with the related state evolution recursion \([B.2]-[B.3]\), and the modified version of the true GAMP algorithm \([5.13]-[5.14]\). Fix any \(\varepsilon > 0\). Then, for \(t \geq 0\) such that \(\sigma^2_{X,k} > 0\) for \(0 \leq k \leq t\), the following statements hold:

1. \[
\lim_{T \to \infty} \left| \mu_{\tilde{U},t+T} - \mu_{U,t} \right| = 0, \quad \lim_{T \to \infty} \left| \sigma^2_{\tilde{U},t+T} - \sigma^2_{U,t} \right| = 0, \quad \lim_{T \to \infty} \left| \mu_{\tilde{X},t+T+1} - \mu_{X,t+1} \right| = 0, \quad \lim_{T \to \infty} \left| \sigma^2_{\tilde{X},t+T+1} - \sigma^2_{X,t+1} \right| = 0.
\]  

(B.13)

(B.14)

2. Let \(\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) be a PL(2) function. Then, almost surely,

\[
\lim_{T \to \infty} \lim_{n \to \infty} \left| \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_{i}^{t+T}) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_{i}^{t}) \right| = 0, \quad \lim_{T \to \infty} \lim_{d \to \infty} \left| \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_{i}^{t+T+1}) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_{i}^{t+1}) \right| = 0.
\]  

(B.15)

(B.16)

The limits in \([B.14]\) and \([B.16]\) also hold for \(t + 1 = 0\).
Proof. We will use \(\kappa_t, \kappa_t', c_t, \gamma_t\) to denote generic positive constants which depend on \(t\), but not on \(n, d, \text{ or } \varepsilon\). The values of these constants may change throughout the proof.

**Proof of (B.13) and (B.14).** We prove the result by induction, starting from the base case \(|\mu_{X,T} - \mu_{X,0}|, |\sigma_{X,T}^2 - \sigma_{X,0}^2|\). From Lemma [B.2] we have

\[
\lim_{T \to \infty} \frac{\mu_{X,T}}{\sqrt{\delta}} = \frac{a}{\sqrt{\delta}}, \quad \lim_{T \to \infty} \frac{\sigma_{X,T}^2}{\delta} = \frac{1 - a^2}{\delta}.
\]

Recalling from (3.9) that \(\mu_{X,0} = \frac{a}{\sqrt{\delta}}, \sigma_{X,0}^2 = \frac{1 - a^2}{\delta}\), (B.17) implies that

\[
\lim_{T \to \infty} \left| \frac{\mu_{X,T}}{\sqrt{\delta}} - \mu_{X,0} \right| = 0, \quad \lim_{T \to \infty} \left| \frac{\sigma_{X,T}^2}{\delta} - \sigma_{X,0}^2 \right| = 0.
\]

Assume towards induction that (B.14) holds with \((t + 1)\) replaced by \(t\), and that \(\sigma_{X,k}^2 > 0\) for \(0 \leq k \leq t\). We will show that (B.13) holds, and then that (B.14) holds.

For brevity, we write \(\Delta_{\mu,t}, \Delta_{\sigma,t}\) for \((\mu_{X,t} - \mu_{X,t+\Delta})\) and \((\sigma_{X,t} - \sigma_{X,t+\Delta})\), respectively. By the induction hypothesis, given any \(\varepsilon > 0\), for \(T\) sufficiently large we have

\[
|\Delta_{\mu,t}| < \kappa_t \varepsilon, \quad |\Delta_{\sigma,t}| < \frac{\kappa_t}{\sigma_{X,t} + \sigma_{X,t+T}} \varepsilon = \kappa_t' \varepsilon.
\]

Since \(\sigma_{X,t}\) is strictly positive, \(\kappa_t'\) is finite and bounded above.

From (3.8) we have

\[
\mu_{U,t} = \frac{1}{\sqrt{\delta}} \mathbb{E}\{X f_t(\mu_{X,t}X + \sigma_{X,t}W_{X,t})\} = \frac{1}{\sqrt{\delta}} \mathbb{E}\{X f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t} + \Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t})\}.
\]

Recalling that \(f_t\) is Lipschitz and letting \(L_t\) denote its Lipschitz constant, we have

\[
\left| f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t}) - f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t})\right| \leq L_t |\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|.
\]

Using (B.21) in (B.20), we obtain

\[
\sqrt{\delta} \mu_{U,t} \geq \mathbb{E}\{X f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t})\} - L_t \mathbb{E}\{|X| |\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|\},
\]

\[
\sqrt{\delta} \mu_{U,t} \leq \mathbb{E}\{X f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t})\} + L_t \mathbb{E}\{|X| |\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|\}.
\]

Since \(W_{X,t} \overset{d}{=} W_{X,t+\Delta}\) and independent of \(X\), we have that \(\mathbb{E}\{X f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t})\} = \mathbb{E}\{X f_t(\mu_{X,T+\Delta}X + \sigma_{X,T+\Delta}W_{X,t})\} = \sqrt{\delta} \mu_{U,t+\Delta}\). Therefore, (B.22) implies

\[
\sqrt{\delta} \left| \mu_{U,t} - \mu_{U,t+\Delta} \right| \leq L_t (\Delta_{\mu,t} + \Delta_{\sigma,t} \mathbb{E}\{|X| W_{X,t}|\}),
\]

where we have used \(\mathbb{E}\{|X| |X|^2| < \sqrt{\mathbb{E}|X|^2} = 1\). Noting that \(\mathbb{E}\{|W_{X,t}|\} = \sqrt{2/\pi}\), from (B.19) it follows that for sufficiently large \(T\):

\[
|\mu_{U,t} - \mu_{U,t+\Delta}| \leq \frac{L_t}{\sqrt{\delta}} (\kappa_t + \kappa_t' \sqrt{2/\pi}) \varepsilon < \gamma_t \varepsilon.
\]
Next consider $\sigma_{U,t}^2$. From (3.8), we have

$$\sigma_{U,t}^2 = \frac{1}{\delta} \mathbb{E}\{f_1(\mu_{X,t}X + \sigma_{X,t}W_{X,t})^2\} - \mu^2_{U,t}. \quad (B.25)$$

Furthermore, as $W_{X,t} \overset{d}{=} W_{\tilde{X},t+T}$ and independent of $X$, we also have that

$$\sigma_{U,t+T}^2 = \frac{1}{\delta} \mathbb{E}\{f_1(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})^2\} - \mu^2_{U,t+T}. \quad (B.26)$$

Using the reverse triangle inequality, we have

$$\left| f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t} + \Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}) \right|$$

$$\geq |f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})|$$

$$- \left| f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t} + \Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}) - f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t}) \right|$$

$$\geq |f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})| - L_t |\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|, \quad (B.27)$$

where the last inequality follows from (B.21). Similarly,

$$\left| f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t} + \Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}) \right|$$

$$\leq |f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})| + L_t |\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|. \quad (B.28)$$

Using (B.27), we obtain the bound

$$\mathbb{E}\{f_t(\mu_{X,t}X + \sigma_{X,t}W_{X,t})^2\} \geq \mathbb{E}\{f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})^2\} - L_t^2 \mathbb{E}\{|\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|^2\}$$

$$- 2L_t \sqrt{\mathbb{E}\{f_t(\mu_{X,t}X + \sigma_{X,t}W_{X,t})^2\} \cdot \mathbb{E}\{|\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|^2\}}. \quad (B.29)$$

Similarly, using (B.28) we get

$$\mathbb{E}\{f_t(\mu_{X,t}X + \sigma_{X,t}W_{X,t})^2\} \leq \mathbb{E}\{f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})^2\} + L_t^2 \mathbb{E}\{|\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|^2\}$$

$$+ 2L_t \sqrt{\mathbb{E}\{f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{X,t})^2\} \cdot \mathbb{E}\{|\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|^2\}}. \quad (B.30)$$

Furthermore,

$$\mathbb{E}\{|\Delta_{\mu,t}X + \Delta_{\sigma,t}W_{X,t}|^2\} \leq 2 |\Delta_{\mu,t}|^2 \mathbb{E}\{X^2\} + 2 |\Delta_{\sigma,t}|^2 \mathbb{E}\{W_{X,t}\}^2 = 2 (|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2).$$

From (3.8) and (B.3), we note that

$$\mathbb{E}\{f_t(\mu_{X,t}X + \sigma_{X,t}W_{X,t})^2\} = \delta(\mu^2_{U,t} + \sigma^2_{U,t}),$$

$$\mathbb{E}\{f_t(\mu_{\tilde{X},t+T}X + \sigma_{\tilde{X},t+T}W_{\tilde{X},t+T})^2\} = \delta(\mu^2_{U,T+t} + \sigma^2_{U,T+t}). \quad (B.31)$$
The induction hypothesis (B.13) implies that for sufficiently large $T$
\[\sigma_{\mu,t}^2 \leq 2L_t^2 (|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2) + 2L_t \sqrt{2\delta (\mu_{\mu,t}^2 + \sigma_{\mu,t}^2 + \mu_{\sigma,t}^2 + \sigma_{\sigma,t}^2)}(|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2).\] (B.32)

Using this in (B.25)-(B.26), we have
\[\left| \sigma_{\mu,t}^2 - \sigma_{\mu,t+T}^2 \right| \leq \left| \mu_{\mu,t+T} - \mu_{\mu,t} \right| \cdot \left| \mu_{\mu,t} + \mu_{\mu,t} \right| + \left( \frac{2}{\delta} L_t^2 (|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2) + 2 \sqrt{\delta} L_t \sqrt{2(\mu_{\mu,t}^2 + \sigma_{\mu,t}^2 + \mu_{\sigma,t}^2 + \sigma_{\sigma,t}^2)}(|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2) \right).\] (B.33)

From (B.19), we obtain
\[|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2 < (\kappa^2 + (\kappa')^2) \varepsilon^2.\] (B.34)

Furthermore, as $f_t$ is Lipschitz, from (B.31) and the induction hypothesis we have
\[|\mu_{\mu,t+T} + \mu_{\mu,t} + \sigma_{\mu,t} + \sigma_{\mu,t+T} \leq c_t,\] (B.35)

for some constant $c_t$. Using (B.24), (B.34) and (B.35) in (B.33), we conclude that for sufficiently large $T$:
\[|\sigma_{\mu,t}^2 - \sigma_{\mu,t+T}^2| < \gamma_t \varepsilon.\] (B.36)

Next, we show that if (B.13) holds for some $t \geq 0$ and $\sigma_{X,k}^2 > 0$ for $k \leq t$, then :
\[\lim_{T \to \infty} \left| \mu_{\mu,t+T+1} - \mu_{X,t+1} \right| = 0, \quad \lim_{T \to \infty} \left| \sigma_{X,t+T+1} \right| = 0.\] (B.37)

We denote the Lipschitz constant of $h_t$ by $\bar{L}_t$, and write $\bar{\Delta}_{\mu,t}, \bar{\Delta}_{\sigma,t}$ for $(\mu_{\mu,t} - \mu_{\bar{\mu},t+T})$ and $(\sigma_{\mu,t} - \sigma_{\bar{\mu},t+T})$, respectively. Using this notation, we have
\[\left| h_t(\mu_{\mu,t+T} + \bar{\Delta}_{\mu,t} G + \bar{\Delta}_{\sigma,t} W_{U,t} - Y) - h_t(\mu_{\bar{\mu},t+T} + \bar{\Delta}_{\mu,t} G + \bar{\Delta}_{\sigma,t} W_{U,t} - Y) \right| \leq \bar{L}_t \left| \bar{\Delta}_{\mu,t} G + \bar{\Delta}_{\sigma,t} W_{U,t} \right|.\] (B.38)

The induction hypothesis (B.13) implies that for sufficiently large $T$:
\[|\bar{\Delta}_{\mu,t}| < \gamma_t \varepsilon, \quad |\bar{\Delta}_{\sigma,t}| < \frac{\gamma_t}{\sigma_{U,t} + \sigma_{\bar{\mu},t+T}} \varepsilon = \gamma_t \varepsilon.\] (B.39)

We note that $\sigma_{U,t} > 0$ since $\sigma_{X,t} > 0$. Indeed, from the discussion leading to (3.17), for a fixed $\mu_{X,t}, \sigma_{X,t}$ the smallest possible ratio $\sigma_{U,t}^2 / \mu_{U,t}^2$ is achieved by the Bayes-optimal choice $f_t = cf_t^*$, where $f_t^*(X_t) = E\{X|X_t\}$. Furthermore, from (3.17), in order for $\sigma_{U,t} = 0$, we need $E\{E\{X|X_t\}^2\} = 1$. From Jensen’s inequality, we also have $E\{E\{X|X_t\}^2\} \leq E\{E\{X^2|X_t\} \} = 1$. Therefore, $E\{E\{X|X_t\}^2\} = 1$ only if $X$ is a deterministic function of $X_t = \mu_{X,t} X + \sigma_{X,t} W$. But this is impossible when $\sigma_{X,t} > 0$. Therefore $\sigma_{U,t} > 0$, and $\gamma_t$ in (B.39) is strictly positive.
From (B.38), we obtain
\[
\mathbb{E}\{G h_t(\mu_{\tilde{T},T+t} G + \sigma_{\tilde{T},T+t} W_{U,t}; Y)\} - \tilde{L}_t \mathbb{E}\{|\Delta_{\mu,t}| G^2 + |\Delta_{\sigma,t}| \cdot |G| \cdot |W_{U,t}|\} \\
\leq \mathbb{E}\{G h_t(\mu_{U,t} G + \sigma_{U,t} W_{U,t}; Y)\} \\
\leq \mathbb{E}\{G h_t(\mu_{\tilde{T},T+t} G + \sigma_{\tilde{T},T+t} W_{U,t}; Y)\} + \tilde{L}_t \mathbb{E}\{|\Delta_{\mu,t}| G^2 + |\Delta_{\sigma,t}| \cdot |G| \cdot |W_{U,t}|\}.
\]
(B.40)

Now, using (3.8) and (B.3), we have:
\[
\frac{1}{\sqrt{\delta}} |\mu_{\tilde{T},T+t+1} - \mu_{X,t+1}| = \left| \mathbb{E}\{G h_t(\tilde{T}, T+t; Y) - h_t(U_t; Y)\} \right| \\
- \mu_{U,t}\left( \mathbb{E}\{h'_t(\tilde{T}, T+t; Y)\} - \mathbb{E}\{h'_t(U_t; Y)\}\right) - \mathbb{E}\{h'_t(\tilde{T}, T+t; Y)\}(\mu_{\tilde{T},T+t} - \mu_{U,t}) \\
\leq \tilde{L}_t (|\Delta_{\mu,t}| + |\Delta_{\sigma,t}| (2/\pi)) + |\mu_{U,t}| \cdot |\mathbb{E}\{h'_t(\tilde{T}, T+t; Y)\} - \mathbb{E}\{h'_t(U_t; Y)\}| + \tilde{L}_t |\Delta_{\mu,t}|.
\]
(B.41)

For the inequality above, we used (B.40) (noting that \( \mathbb{E}\{|W_{U,t}|\} = \mathbb{E}\{|G|\} = \sqrt{2/\pi} \) and \( \mathbb{E}\{G^2\} = 1 \), and the fact that \( |h'_t| \) is bounded by \( \tilde{L}_t \), the Lipschitz constant of \( h_t \). Now,
\[
\left| \mathbb{E}\{h'_t(U_t; Y) - h'_t(\tilde{T}, T+t; Y)\} \right| = \left| \mathbb{E}\{h'_t(\mu_{U,t} G + \sigma_{U,t} W_{U,t}; Y)\} - \mathbb{E}\{h'_t(\mu_{\tilde{T},T+t} G + \sigma_{\tilde{T},T+t} W_{U,t}; Y)\} \right|.
\]
(B.42)

By the induction hypothesis (B.13), we have
\[
\lim_{T \to \infty} \mu_{\tilde{T}, T+t} = \mu_{U,t}, \quad \lim_{T \to \infty} \sigma_{\tilde{T}, T+t} = \sigma_{U,t}.
\]
(B.43)

Thus, as \( T \to \infty \), the random variable \((\mu_{\tilde{T}, T+t} G + \sigma_{\tilde{T}, T+t} W_{U,t})\) converges in distribution to \( \mu_{U,t} G + \sigma_{U,t} W_{U,t} \). Then, Lemma C.1 in Appendix C implies that
\[
\lim_{T \to \infty} \left| \mathbb{E}\{h'_t(U_t; Y) - h'_t(\tilde{T}, T+t; Y)\} \right| = 0.
\]
(B.44)

Using (B.44), (B.39) and (B.35) in (B.41) proves that the first limit in (B.37) holds.

Finally, we prove the second limit in (B.37). From (3.8), (B.3) and arguments along the same lines as (B.29), (B.32), we obtain the bound
\[
|\sigma_{\tilde{T}, T+1}^2 - \sigma_{X,t+1}^2| = \mathbb{E}\{h_t(U_t; Y)^2\} - \mathbb{E}\{h_t(U_{\tilde{T}, T}; Y)^2\} \\
\leq 2\tilde{L}_t^2 (|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2) + 2\tilde{L}_t \left( \sigma_{\tilde{T}, T+1}^2 + \sigma_{X,t+1}^2 \right)^2 (|\Delta_{\mu,t}|^2 + |\Delta_{\sigma,t}|^2).
\]
(B.45)

Furthermore, as \( h_t \) is Lipschitz, the formulas for \( \sigma_{\tilde{T}, T+1}^2 \) and \( \sigma_{X,T+t+1}^2 \) (in (3.8) and (B.3)) along with the induction hypothesis (B.43) imply that
\[
\sigma_{\tilde{T}, T+1}^2 + \sigma_{X,T+t+1}^2 \leq c_t,
\]
(B.46)

for some constant \( c_t \). By using (B.46) and (B.39), we can upper bound the RHS of (B.45) with \( \kappa_{t+1} \varepsilon \), for sufficiently large \( T \). This completes the proof of the second limit in (B.37).
**Proof of (B.15) and (B.16).**

Since $\psi \in \text{PL}(2)$, for $i \in [d]$ we have

$$
\left| \psi(x_i, \hat{x}_i^{T+t+1}) - \psi(x_i, \hat{x}_i^{t+1}) \right| \leq C \left( 1 + |x_i| + |\hat{x}_i^{T+t+1}| + |\hat{x}_i^{t+1}| \right) |\hat{x}_i^{T+t+1} - \hat{x}_i^{t+1}|,
$$

(B.47)

for a universal constant $C > 0$. Therefore,

$$
\frac{1}{d} \sum_{i=1}^{d} \left| \psi(x_i, \hat{x}_i^{T+t+1}) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_i^{t+1}) \right| \leq C \frac{1}{d} \sum_{i=1}^{d} \left( 1 + |x_i| + |\hat{x}_i^{T+t+1}| + |\hat{x}_i^{t+1}| \right) |\hat{x}_i^{T+t+1} - \hat{x}_i^{t+1}|

\leq 4C \left[ 1 + \frac{1}{d} \sum_{i=1}^{d} \left( |x_i|^2 + |\hat{x}_i^{T+t+1}|^2 + |\hat{x}_i^{t+1}|^2 \right) \right]^{1/2} \frac{||\hat{x}_i^{T+t} - \hat{x}_i^{t+1}||_2}{\sqrt{d}},
$$

(B.48)

where the second inequality follows from Cauchy-Schwarz. By the same argument,

$$
\frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{u}_i^{T+t}) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{u}_i^{0}) \leq 4C \left[ 1 + \frac{1}{n} \sum_{i=1}^{n} \left( |y_i|^2 + |\hat{u}_i^{T+t}|^2 + |\hat{u}_i^{0}|^2 \right) \right]^{1/2} \frac{||\hat{u}_i^{T+t} - \hat{u}_i^{0}||_2}{\sqrt{n}}.
$$

(B.49)

We will show via induction that as $d \to \infty$: $i)$ the terms inside the square brackets in (B.48) and (B.49) converge almost surely to finite deterministic values, and $ii)$ as $T \to \infty$ (with the limit in $T$ taken after the limit in $d$), the terms $\frac{||\hat{x}_i^{T+t} - \hat{x}_i^{t+1}||_2}{\sqrt{d}}$ and $\frac{||\hat{u}_i^{T+t} - \hat{u}_i^{0}||_2}{\sqrt{d}}$ converge to 0 almost surely.

**Base case $t = 0$:** The result (B.16) for $t+1 = 0$ directly follows from Lemma B.3. Next, using (B.49), the LHS of (B.15) for $t = 0$ can be bounded as

$$
\frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{u}_i^{T}) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{u}_i^{0}) \leq 4C \left[ 1 + \frac{1}{n} \sum_{i=1}^{n} \left( |y_i|^2 + \frac{||\hat{u}_i^{T}||_2^2}{n} + \frac{||\hat{u}_i^{0}||_2^2}{n} \right) \right]^{1/2} \frac{||\hat{u}_i^{T} - \hat{u}_i^{0}||_2}{\sqrt{n}}.
$$

(B.50)

From the definition of the artificial GAMP (5.1)-(5.6), we have

$$
\hat{u}^T = \frac{1}{\sqrt{\delta}} A f_0(\hat{x}^T) - \sqrt{\delta} b_T Z \hat{u}^{T-1},
$$

(B.51)

where we define

$$
Z = Z_s (\lambda_s^2 I - Z_s)^{-1},
$$

(B.52)

with $Z_s = \text{diag}((T_s(y_1), \ldots, T_s(y_n)))$. Similarly, defining

$$
e_1 := \hat{u}^{T-1} - \hat{u}^{T-2},
$$

(B.53)

we obtain $\hat{u}^{T-1} = \frac{1}{\sqrt{\delta} \beta_{T-1}} [A \hat{x}^{T-1} - Z \hat{u}^{T-1} + Z e_1]$, or

$$
\hat{u}^{T-1} = \frac{1}{\sqrt{\delta} \beta_{T-1}} \left( I + \frac{1}{\sqrt{\delta} \beta_{T-1}} Z \right)^{-1} [A \hat{x}^{T-1} + Z e_1].
$$

(B.54)
Substituting (B.54) in (B.51), we obtain
\[
\tilde{u}^T = \frac{1}{\sqrt{\delta}} A f_0(\tilde{x}^T) - \frac{\tilde{b}_T}{\beta_{T-1}} Z \left( I + \frac{1}{\sqrt{\delta} \beta_{T-1}} Z \right)^{-1} A \tilde{x}^{T-1} - \frac{\tilde{b}_T}{\beta_{T-1}} Z^2 \left( I + \frac{1}{\sqrt{\delta} \beta_{T-1}} Z \right)^{-1} e_1. \tag{B.55}
\]

Using (B.55) and the expression for \( \tilde{u}^0 \) from (5.12), we have
\[
\frac{1}{d} \| \tilde{u}^T - \tilde{u}^0 \|^2 \leq 3 \left( \| A f_0(\tilde{x}^T) - A f_0(\tilde{x}^0) \| \frac{\delta d}{\delta d} + \frac{3}{d} \left( \frac{\tilde{b}_0 \sqrt{\delta}}{\lambda^2(\delta)} Z A \tilde{x}^0 - \frac{\tilde{b}_T}{\beta_{T-1}} Z \left( I + \frac{1}{\sqrt{\delta} \beta_{T-1}} Z \right)^{-1} A \tilde{x}^{T-1} \right) \right)^2
\]
\[
:= 3(S_1 + S_2 + S_3). \tag{B.56}
\]

We now bound each of the three terms. By Cauchy-Schwarz inequality,
\[
S_1 \leq \| A \|^2_{op} \frac{\| f_0(\tilde{x}^T) - f_0(\tilde{x}^0) \|^2}{\delta d} \leq \| A \|^2_{op} \frac{L_0^2 \delta}{\delta d} \| \tilde{x}^T - \tilde{x}^0 \|^2, \tag{B.57}
\]
where \( L_0 \) is the Lipschitz constant of \( f_0 \). Since the entries of \( A \) are i.i.d. \( \mathcal{N}(0, 1/d) \), almost surely the operator norm of \( A \) is bounded by a universal constant for sufficiently large \( d \) [AGZ09]. From Lemma B.3 and the definition of \( \tilde{x}^0 \) in (5.11), we also have
\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\| \tilde{x}^T - \tilde{x}^0 \|^2}{d} = \frac{1}{\delta} \cdot \frac{\| \sqrt{\delta} \tilde{x}^T - \sqrt{\delta} \tilde{x}^0 \|^2}{d} = 0 \quad \text{a.s.} \tag{B.58}
\]

Therefore,
\[
\lim_{T \to \infty} \lim_{d \to \infty} S_1 = 0 \quad \text{a.s.} \tag{B.59}
\]

Next, recalling the definition of \( \tilde{e}_1 \) from (B.53) we bound \( S_2 \) as follows:
\[
S_2 \leq \frac{\tilde{b}_T}{\beta_{T-1}^2} \left( Z \left( I + Z/(\sqrt{\delta} \beta_{T-1}) \right)^{-1} \right)^{\| A \|^2}_{op} \| \tilde{u}^{T-1} - \tilde{u}^{T-2} \|^2 \frac{2}{d}. \tag{B.60}
\]

From Lemma B.4, we know that \( \lim_{T \to \infty} \lim_{d \to \infty} \frac{\| \tilde{u}^{T-1} - \tilde{u}^{T-2} \|^2}{d} = 0 \) almost surely. We now show that the other terms on the RHS of (B.60) are bounded almost surely. Recall from (5.7) that \( \tilde{b}_T = \frac{1}{n} \sum_{i=1}^{d} f_0'(\tilde{x}^T_i) \). Proposition B.1 guarantees that the empirical distribution of \( \tilde{x}^T \) converges to the law of \( \tilde{X}_T \equiv \mu_{\tilde{X}_T} X + \sigma_{\tilde{X}_T} W \). Since \( f_0 \) is Lipschitz, Lemma C.1 in Appendix C therefore implies that almost surely:
\[
\lim_{d \to \infty} \tilde{b}_T = \frac{1}{\delta} \mathbb{E}\{ f_0'(\mu_{\tilde{X}_T} X + \sigma_{\tilde{X}_T} W) \}. \tag{B.61}
\]

From Lemma B.2, we know that \( \lim_{T \to \infty} \mu_{\tilde{X}_T} = \frac{a}{\sqrt{\delta}} \) and \( \lim_{T \to \infty} \sigma_{\tilde{X}_T}^2 = \frac{1-a^2}{\delta} \). Therefore, letting \( T \to \infty \) and applying Lemma C.1 again, we obtain
\[
\lim_{T \to \infty} \lim_{d \to \infty} \tilde{b}_T = \frac{1}{\delta} \mathbb{E}\left\{ f_0'\left( \frac{a}{\sqrt{\delta}} X + \frac{\sqrt{1-a^2}}{\sqrt{\delta}} W \right) \right\} \quad \text{a.s.} \tag{B.62}
\]

27
From Lemma [B.2], we have $\beta_{T-1} \to 1/\sqrt{\delta}$ as $T \to \infty$. Also recall from assumption (A2) on p. 4 that $\tau$ is the supremum of the support of $Z_s$, and that $\lambda^*_\delta > \tau$. Therefore, $Z = \frac{Z_s}{\lambda^*_\delta - Z_s}$ has bounded support, due to which $\|Z^2(I + Z/(\sqrt{\delta}\beta_{T-1}))^{-1}\|_{op}^2 < C$ for a universal constant $C > 0$. Hence,

$$
\lim_{T \to \infty} \lim_{d \to \infty} S_2 = 0 \text{ a.s.} \quad (B.63)
$$

To bound $S_3$, we first write the term inside the norm on the second line of (B.56) as

$$
\sqrt{\delta} \frac{Z_s A \hat{x}^0}{\lambda^*_\delta} (b_0 - b_T) + \frac{\beta_T}{\lambda^*_\delta} Z_s A \left( \sqrt{\delta} \hat{x}^0 - \frac{\hat{x}^{T-1}}{\beta_{T-1}} \right) + \frac{\beta_T}{\lambda^*_\delta} \left( \frac{Z_s}{\lambda^*_\delta} - Z \left( I + \frac{1}{\sqrt{\delta}\beta_{T-1}} Z \right)^{-1} \right) A \hat{x}^{T-1}.
$$

Then, using triangle inequality and Cauchy-Schwarz, we have

$$
S_3 \leq \frac{3\delta}{(\lambda^*_\delta)^2} \frac{\|Z_s A \hat{x}^0\|_d^2}{d} (b_0 - b_T)^2 + \frac{3\delta}{(\lambda^*_\delta)^2} \frac{\|Z_s A\|_d^2}{d} \|\sqrt{\delta} \hat{x}^0 - \frac{\hat{x}^{T-1}}{\beta_{T-1}}\|_2^2 \\
+ \frac{3\delta}{\beta_{T-1}^2} \frac{\|A \hat{x}^{T-1}\|_d^2}{d} \left( \frac{1}{\lambda^*_\delta} Z_s - Z \left( I + \frac{1}{\sqrt{\delta}\beta_{T-1}} Z \right)^{-1} \right)^{-1} \|\|_d^2 := 3(S_{3a} + S_{3b} + S_{3c}).
$$

(B.64)

Using the expression for $\hat{x}^0$ from (5.11) and applying Cauchy-Schwarz, we can bound $S_{3a}$ as:

$$
S_{3a} \leq \frac{1}{(\lambda^*_\delta)^2} \|Z_s\|_d^2 \|A\|_d^2 \|\hat{x}^s\|_2^2 (b_0 - b_T)^2. 
$$

(B.65)

We note that $Z_s$ is bounded, $\|\hat{x}^s\|_2 = 1$, and $\|A\|_d^2$ is bounded almost surely by a universal constant for sufficiently large $d$. Moreover, recalling the definitions of $b_0$ and $X_0 = \mu_{X,0} X + \sigma_{X,0} W_{X,0}$ from (5.15) and (3.9), we see that $b_0 = \frac{1}{\delta} \mathbb{E}\{f'_0(X_0)\}$ is the limit of $\tilde{b}_T$ in (B.62). Therefore, $\lim_{T \to \infty} \lim_{d \to \infty} S_{3a} = 0$ almost surely.

Next, we bound $S_{3b}$. Recalling that $\hat{x}^0 = \sqrt{d} \hat{x}^s / \sqrt{\delta}$, we have

$$
\frac{\|\sqrt{\delta} \hat{x}^0 - \frac{\hat{x}^{T-1}}{\beta_{T-1}}\|_2^2}{d} = \frac{\|\sqrt{d} \hat{x}^s - \sqrt{\delta} \hat{x}^T + \sqrt{\delta} \hat{x}^T - \sqrt{\delta} \hat{x}^{T-1} + \sqrt{\delta} \hat{x}^{T-1} - \hat{x}^{T-1}/\beta_{T-1}\|_2^2}{d} \\
\leq \frac{3\|\sqrt{d} \hat{x}^s - \sqrt{\delta} \hat{x}^T\|_2^2}{d} + \frac{3\|\sqrt{\delta} \hat{x}^T - \sqrt{\delta} \hat{x}^{T-1}\|_2^2}{d} + \frac{3\|\hat{x}^{T-1}\|_2^2}{d} (\sqrt{\delta} - 1/\beta_{T-1})^2.
$$

(B.66)

Lemmas [B.3] and [B.4] imply that the first two terms on the RHS of (B.66) tend to zero in the iterated limit $T \to \infty, d \to \infty$. Furthermore, from Lemma [B.2] we have $\lim_{T \to \infty} \beta_{T-1} = 1/\sqrt{\delta}$.

From Proposition [B.1] we also have

$$
\lim_{d \to \infty} \frac{\|\hat{x}^{T-1}\|_2^2}{d} = \mu^2_{X,T-1} + \sigma^2_{X,T-1} = \beta^2_{T-1} \text{ a.s.} 
$$

(B.67)

Therefore, $\lim_{T \to \infty} \lim_{d \to \infty} S_{3b} = 0$ almost surely.

To bound $S_{3c}$, recalling from (B.52) that $Z = \frac{Z_s}{\lambda^*_\delta - Z_s}$, we have

$$
\frac{1}{\lambda^*_\delta} Z_s - Z \left( I + \frac{1}{\sqrt{\delta}\beta_{T-1}} Z \right)^{-1} = \frac{1}{\beta_{T-1}} Z_s^2 \left( \lambda^*_\delta I + Z_s \left( \frac{1}{\sqrt{\delta}\beta_{T-1}} - 1 \right) \right)^{-1} \left( \frac{1}{\sqrt{\delta}} - \beta_{T-1} \right) \frac{1}{\lambda^*_\delta}. 
$$

(B.68)
Since \( \lim_{T \to \infty} \beta_{T-1} = \frac{1}{\sqrt{\delta}} \), almost surely

\[
\lim_{T \to \infty} \left\| \frac{1}{\lambda^*_T} Z_s - Z \left( I + \frac{1}{\sqrt{\delta} \beta_{T-1}} Z \right)^{-1} \right\|_{\text{op}}^2 = 0.
\]

(B.69)

Thus \( \lim_{T \to \infty} \lim_{d \to \infty} S_{3c} = 0 \) almost surely. Using the results above in (B.64), we have shown that

\[
\lim_{T \to \infty} \lim_{d \to \infty} S_{3c} = 0 \quad \text{a.s. (B.70)}
\]

Using (B.59), (B.63) and (B.70) in (B.56), and recalling that \( n/d \to \delta \), we obtain

\[
\lim_{T \to \infty} \lim_{n \to \infty} \left\| \hat{u}_0 \right\|_2^2/n = \mu_{U,T}^2 + \sigma_{U,T}^2.
\]

(B.72)

Next, using the triangle inequality, we have

\[
\| \hat{u}_0 \|_2^2 - \| \hat{u}_0 - \hat{u}^0 \|_2^2 \leq \| \hat{u}_T \|_2^2 \leq \| \hat{u}_T \|_2^2 + \| \hat{u}_0 - \hat{u}^0 \|_2^2.
\]

(B.73)

Combining this with (B.71), we obtain

\[
\lim_{T \to \infty} \lim_{n \to \infty} \frac{\| \hat{u}_0 \|_2^2}{n} = \lim_{T \to \infty} \mu_{U,T}^2 + \sigma_{U,T}^2 = \mu_{U,0}^2 + \sigma_{U,0}^2 \quad \text{a.s. (B.74)}
\]

Therefore, using (B.50), we have shown that

\[
\lim_{T \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{x}_T^i) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{u}_0^i) = 0 \quad \text{a.s. (B.75)}
\]

Induction step: Assume that (B.15) holds for some \( t \), and that (B.16) holds with \( t+1 \) replaced by \( t \). Also assume towards induction that almost surely

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\| \tilde{x}_T^t - \hat{x}_T^t \|_2^2}{d} = 0, \quad \lim_{T \to \infty} \lim_{n \to \infty} \frac{\| \hat{u}_T^t - \hat{u}_0^t \|_2^2}{n} = 0.
\]

(B.76)

The limits in (B.76) hold for \( t = 0 \), as established in the proof of the base case (see (B.66), (B.71)). From (B.48), we have the bound

\[
\left| \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_i^{T+t+1}) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_i^{T+1}) \right| \leq 4C \left[ 1 + \frac{\| x \|_2^2}{d} + \frac{\| \tilde{x}_i^{T+t+1} \|_2^2}{d} + \frac{\| \hat{x}_i^{T+1} \|_2^2}{d} \right] \frac{1}{\sqrt{d}} \frac{\| \tilde{x}_T^{T+t+1} - \hat{x}_T^{T+1} \|_2}{\sqrt{d}}.
\]

(B.77)
Using [5.1], [5.6], (5.13) and the triangle inequality, we obtain:

\[
\frac{\|\tilde{x}^{T+t+1} - \hat{x}^{t+1}\|^2}{d} \leq \frac{2}{\delta d} \|A^T h_t(\tilde{u}^{T+t}; y) - A^T h_t(\hat{u}^t; y)\|^2 + 2 \frac{\|\tilde{c}^{T+t} f_t(\tilde{x}^{T+t}) - \hat{c}_t f_t(\hat{x}^t)\|^2}{d} \\
\leq \frac{2}{\delta d} \|A^T h_t(\tilde{u}^{T+t}; y) - A^T h_t(\hat{u}^t; y)\|^2 + 4 \frac{\|f_t(\tilde{x}^{T+t})\|^2}{d} (\tilde{c}^{T+t} - \hat{c}_t)^2 + 4 \frac{\|f_t(\tilde{x}^{T+t}) - f_t(\hat{x}^t)\|^2}{d} \\
:= 2 S_1 + 4 S_2 + 4 S_3. 
\]

(B.78)

The term $S_1$ can be bounded as

\[
S_1 \leq \|A\|_{op}^2 \frac{\|h_t(\tilde{u}^{T+t}; y) - h_t(\hat{u}^t; y)\|^2}{\delta d} \leq \|A\|_{op}^2 \|L_t\| \frac{\|\tilde{u}^{T+t} - \hat{u}^t\|^2}{\delta d}, 
\]

where $L_t$ is the Lipschitz constant of the function $h_t$. Since the operator norm of $A$ is bounded almost surely as $d \to \infty$, by the induction hypothesis (B.76) we have $\lim_{T \to \infty} \lim_{n \to \infty} \|\tilde{u}^{T+t} - \hat{u}^t\|^2 = 0$ almost surely. Therefore,

\[
\lim_{T \to \infty} \lim_{n \to \infty} S_1 = 0 \quad \text{a.s.} 
\]

(B.80)

To bound $S_2$, we recall from (5.7) that $\tilde{c}_{T+t} = \frac{1}{n} \sum f_t(\tilde{u}^t; y_i)$. Proposition B.1 guarantees that the joint empirical distribution of $(\tilde{u}^{T+t}, y)$ converges to the law of $(\hat{U}, Y) \equiv (\mu \hat{U}, \sigma \hat{U} W, Y)$. Since $h_t$ is Lipschitz, Lemma C.1 in Appendix C implies that

\[
\lim_{n \to \infty} \tilde{c}_{T+t} = \mathbb{E}\{h_t(\mu \hat{U} + \sigma \hat{U} W, Y)\} \quad \text{a.s.} 
\]

(B.81)

From (B.13), we know that $\lim_{T \to \infty} \mu \hat{U}_{T+t} = \mu \hat{U}$ and $\lim_{T \to \infty} \sigma \hat{U}_{T+t} W = \sigma \hat{U}$. Therefore applying Lemma C.1 in Appendix C again, we obtain:

\[
\lim_{T \to \infty} \lim_{n \to \infty} \tilde{c}_{T+t} = \mathbb{E}\{h_t(\mu \hat{U} + \sigma \hat{U} W, Y)\} = \hat{c}_t \quad \text{a.s.} 
\]

(B.82)

Next, using the result in Proposition B.1 with the test function $\psi(x, \tilde{x}) = (f_t(\tilde{x}))^2$, we almost surely have

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\|f_t(\tilde{x}^{T+t})\|^2}{d} = \lim_{T \to \infty} \mathbb{E}\{f_t(\tilde{X})^2\} = \mathbb{E}\{f_t(X_\epsilon)^2\},
\]

where the last equality follows from (B.13) since $f_t$ is Lipschitz. Combining the above with (B.82), we obtain

\[
\lim_{T \to \infty} \lim_{d \to \infty} S_2 = 0 \quad \text{a.s.} 
\]

(B.84)

For the third term $S_3$ in (B.78), since $f_t$ is Lipschitz (with Lipschitz constant denoted by $L_t$), we have the bound:

\[
S_3 \leq c_t^2 L_t \|\tilde{x}^{T+t} - \hat{x}^t\|^2. 
\]

(B.85)

Thus, by the induction hypothesis (B.76), we obtain

\[
\lim_{T \to \infty} \lim_{d \to \infty} S_3 = 0 \quad \text{a.s.} 
\]

(B.86)
We have therefore shown that

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\|\mathbf{x}_T^t - \bar{x}^t\|^2}{d} = 0 \quad \text{a.s.} \tag{B.87}
\]

Next, we show that the terms inside the brackets on the RHS of \((B.77)\) are finite almost surely as \(d \to \infty\). Using the pseudo-Lipschitz test function \(\psi(x, \tilde{x}) = x^2 + \tilde{x}^2\), Proposition \([B.1]\) implies that almost surely

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \left( |x_i| + |\tilde{x}_i^t + |x_i^T + t| | \right) = \mathbb{E}\{X^2\} + \mu^2_{X,T+t+1} + \sigma^2_{X,T+t+1}. \tag{B.88}
\]

Moreover, \((B.14)\) implies that \(\lim_{T \to \infty} \mu^2_{X,T+t+1} + \sigma^2_{X,T+t+1} = \mu^2_{X,t+1} + \sigma^2_{X,t+1}\). Using the triangle inequality, we have

\[
\|\mathbf{x}_T^t + 1∥_2 \leq \|\mathbf{x}_T^t + 1∥_2 + \|\mathbf{x}_T^t + 1∥_2. \tag{B.89}
\]

Hence, using \((B.87)\) and Proposition \([B.1]\) we almost surely have

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{\|\mathbf{x}_T^t + 1∥_2}{d} = \lim_{T \to \infty} \lim_{d \to \infty} \frac{\|\mathbf{x}_T^t + 1∥_2}{d} = \lim_{T \to \infty} \left( \mu^2_{X,T+t+1} + \sigma^2_{X,T+t+1} \right) = \mu^2_{X,t+1} + \sigma^2_{X,t+1}. \tag{B.90}
\]

We have thus shown via \((B.77)\) that almost surely

\[
\lim_{T \to \infty} \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_i^T + t + 1) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_i^t) \right) = 0. \tag{B.91}
\]

To complete the proof via induction, we need to show that if \((B.87)\) and \((B.91)\) hold with \((t+1)\) replaced by \(t\) for some \(t > 0\), then almost surely

\[
\lim_{T \to \infty} \lim_{n \to \infty} \frac{\|\tilde{u}_T^t - \tilde{u}_i^t\|^2}{n} = 0, \quad \lim_{T \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_i^t) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_i^t) = 0. \tag{B.92}
\]

From \((B.49)\), we have the bound

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_i^t) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_i^t) \right| \leq 4C \left[ 1 + \frac{\|y\|^2}{n} + \frac{\|\tilde{u}^t+i\|^2}{n} + \frac{\|\tilde{u}^t+i\|^2}{n} \right] \frac{1}{\sqrt{n}}. \tag{B.93}
\]

Using \((5.2), (5.6), (5.14)\) and the triangle inequality, we obtain

\[
\frac{\|\tilde{u}_T^t - \tilde{u}_i^t\|^2}{n} \leq \frac{2}{\delta n} \|A_f(\tilde{x}_T^t) - A_f(\tilde{x}_T^t)\|^2 + 2 \|\tilde{b}_T + h_{t-1}(\tilde{u}_T^t; y) - \tilde{b}_t h_{t-1}(\tilde{u}_T^t; y)\|^2 \leq \frac{2}{\delta n} \|A_f(\tilde{x}_T^t) - A_f(\tilde{x}_T^t)\|^2 + 4 \|h_{t-1}(\tilde{u}_T^t; y)\|^2 \frac{1}{\sqrt{n}} (\tilde{b}_T + h_t - b_t)^2
\]

\[
+ 4 b_t \frac{\|h_{t-1}(\tilde{u}_T^t; y) - h_{t-1}(\tilde{u}_T^t; y)\|^2}{n} = 2 S_1 + 4 S_2 + 4 S_3. \tag{B.94}
\]

31
Using arguments along the same lines as (B.80)-(B.86) (omitted for brevity), we can show that almost surely
\[ \lim_{T \to \infty} \lim_{n \to \infty} S_1 = \lim_{T \to \infty} \lim_{n \to \infty} S_2 = \lim_{T \to \infty} \lim_{n \to \infty} S_3 = 0. \]
Hence \( \lim_{T \to \infty} \lim_{n \to \infty} \frac{\|\tilde{u}^{T+t} - \hat{u}^t\|_2}{\sqrt{n}} = 0 \) almost surely. Furthermore, using a triangle inequality argument as in (B.89), we obtain \( \lim_{T \to \infty} \lim_{n \to \infty} \frac{\|\tilde{u}^{T+t}\|^2}{n} = \lim_{T \to \infty} \lim_{n \to \infty} \frac{\|\hat{u}^t\|^2}{n} \) almost surely. By Proposition B.1 and (B.13), the latter limit equals \( \mu^2_{U,t} + \sigma^2_{U,t} \). Using these limits in (B.93) yields the result (B.92), and completes the proof of the lemma.

B.4 Putting Everything Together: Proof of Theorem 1

We will first use Lemma B.5 to show that the result of the theorem holds for the GAMP iteration \((\hat{x}^t, \hat{u}^t)\), i.e., under the assumptions of Theorem 1, we almost surely have
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{u}_i^t) = E \{ \psi(Y, \mu_{U,t} G + \sigma_{U,t} W_{U,t}) \}, \quad t \geq 0, \] (B.95)
\[ \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_{i}^{t+1}) = E \{ \psi(X, \mu_{X,t+1} + \sigma_{X,t+1} \tilde{W}_{X,t+1}) \}, \quad t + 1 \geq 0. \] (B.96)

Consider the LHS of (B.96). Using the triangle inequality, for any \( T > 0 \), we have
\[
\frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_{i}^{t+1}) - E \{ \psi(X, \mu_{X,t+1} X + \sigma_{X,t+1} \tilde{W}_{X,t+1}) \} \\
\leq \left| \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \hat{x}_{i}^{t+1}) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_{i}^{t+t+1}) \right| \\
+ \left| \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_{i}^{t+t+1}) - E \{ \psi(X, \mu_{X,T+t+1} X + \sigma_{X,T+t+1} \tilde{W}_{X,T+t+1}) \} \right| \\
+ |E\{ \psi(X, \mu_{X,T+t+1} X + \sigma_{X,T+t+1} \tilde{W}_{X,T+t+1}) \} - E \{ \psi(X, \mu_{X,t+1} X + \sigma_{X,t+1} \tilde{W}_{X,t+1}) \} | \\
:= T_1 + T_2 + T_3.
\]
We first bound $T_3$ using the pseudo-Lipschitz property of $\psi$, noting that $W_{\tilde{X},T+t}$ and $W_{X,t}$ are both $\sim N(0,1)$:

\[
T_3 \leq \mathbb{E} \left\{ \left| \psi(X, \mu_{\tilde{X},T+t+1}X + \sigma_{\tilde{X},T+t+1}W) - \psi(X, \mu_{X,t+1}X + \sigma_{X,t+1}W) \right| \right\}, \quad W \sim N(0,1)
\]

\[
\leq C \mathbb{E} \left\{ \left( 1 + \left[ X^2 + \mu_{\tilde{X},T+t+1}^2X^2 + \sigma_{\tilde{X},T+t+1}^2W^2 \right]^{1/2} + \left[ X^2 + \mu_{X,t+1}^2X^2 + \sigma_{X,t+1}^2W^2 \right]^{1/2} \right)^2 \right\} \cdot \left( \left( \mu_{\tilde{X},T+t+1} - \mu_{X,t+1} \right)^2 + \left( \sigma_{\tilde{X},T+t+1} - \sigma_{X,t+1} \right)^2 \right)^{1/2}
\]

\[
\leq 3C \left( 3 + \mu_{\tilde{X},T+t+1}^2 + \sigma_{\tilde{X},T+t+1}^2 + \mu_{X,t+1}^2 + \sigma_{X,t+1}^2 \right)^{1/2} \cdot \left( \left( \mu_{\tilde{X},T+t+1} - \mu_{X,t+1} \right)^2 + \left( \sigma_{\tilde{X},T+t+1} - \sigma_{X,t+1} \right)^2 \right)^{1/2},
\]

(B.98)

where we have used Cauchy-Schwarz inequality in the last line. From Lemma B.5 (Eq. B.14), we know that $\lim_{T \to \infty} \left| \mu_{\tilde{X},T+t+1} - \mu_{X,t+1} \right| = 0$ and $\lim_{T \to \infty} \left| \sigma_{\tilde{X},T+t+1} - \sigma_{X,t+1} \right| = 0$. Therefore, $\lim_{T \to \infty} T_3 = 0$. Next, from B.16, we have that $\lim_{T \to \infty} \lim_{d \to \infty} T_1 = 0$ almost surely. Furthermore, by Proposition B.1, for any $T > 0$ we almost surely have $\lim_{d \to \infty} T_2 = 0$. Letting $T, d \to \infty$ (with the limit in $d$ taken first) and noting that the LHS of (B.97) does not depend on $T$, we obtain that (B.96) holds.

The proof of (B.95) uses a bound similar to (B.97) and arguments along the same lines. It is omitted for brevity.

Next, we prove the main result by showing that under the assumptions of the theorem, almost surely

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, u_i^t) - \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \tilde{u}_i^t) = 0, \quad \lim_{n \to \infty} \frac{\|u^t - \tilde{u}^t\|_2^2}{n} = 0, \quad t \geq 0 \tag{B.99}
\]

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, x_{i}^{t+1}) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_{i}^{t+1}) = 0, \quad \lim_{d \to \infty} \frac{\|x^{t+1} - \tilde{x}^{t+1}\|_2^2}{d} = 0, \quad t + 1 \geq 0. \tag{B.100}
\]

Combining (B.100) with (B.99) yields the results in (3.11) and (3.12).

The proof of (B.100) is via induction and uses arguments very similar to those to prove (B.15) and (B.16). To avoid repetition we only provide a few steps. Noting that $x^0 = \tilde{x}^0$, we now show (B.100), under the induction hypothesis that (B.99) holds and also that (B.100) holds with $t + 1$ replaced by $t$.

Since $\psi \in \text{PL}(2)$, we have

\[
\left| \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, x_{i}^{t+1}) - \frac{1}{d} \sum_{i=1}^{d} \psi(x_i, \tilde{x}_{i}^{t+1}) \right| \leq 4C \left[ 1 + \frac{\|x_t\|_2^2}{d} + \frac{\|x_{t+1}^t\|_2^2}{d} + \frac{\|\tilde{x}_{t+1}^t\|_2^2}{d} \right]^{1/2} \frac{\|x_{t+1}^t - \tilde{x}_{t+1}^t\|_2}{\sqrt{d}}. \tag{B.101}
\]
Furthermore, using the definitions of $x^{t+1}$ and $\hat{x}^{t+1}$, and the triangle inequality we have

$$\|x^{t+1} - \hat{x}^{t+1}\|_2^2 \leq \frac{2}{\delta d} \|A^T h_t(u^t; y) - A^T \hat{h}_t(\hat{u}^t; y)\|_2^2 + \frac{4}{\delta d} \frac{\|f_t(x^t)\|_2^2}{d} (c_t - \bar{c}_t)^2 + 4\epsilon_t^2 \frac{\|f_t(x^t) - f_t(\hat{x}^t)\|_2^2}{d},$$

(B.103)

where $L_t, \bar{L}_t$ are the Lipschitz constants of $f_t, h_t$, respectively. By the induction hypothesis and Lemma C.1, the terms $\frac{\|u^t - \hat{u}^t\|_2^2}{d}$, $\frac{\|x^t - \hat{x}^t\|_2^2}{d}$, and $(c_t - \bar{c}_t)^2$ tend to zero. Furthermore, by the induction hypothesis, we almost surely have $\frac{\|f_t(x^t)\|_2^2}{d} \rightarrow E\{f_t(X_t)^2\}$, and by (B.96), $\frac{\|x^{t+1}\|_2^2}{d} \rightarrow (\mu_{X,t+1}^2 + \sigma_{X,t+1}^2)$ as $d \rightarrow \infty$. Finally, by a triangle inequality argument analogous to (B.89), we also have

$$\lim_{d \rightarrow \infty} \frac{\|x^{t+1}\|_2^2}{d} = \lim_{d \rightarrow \infty} \frac{\|\hat{x}^{t+1}\|_2^2}{d} = (\mu_{X,t+1}^2 + \sigma_{X,t+1}^2) \quad \text{a.s.}$$

Using these limits in (B.101) proves (B.100). The proof of (B.99) (under the induction hypothesis that (B.100) holds with $(t+1)$ replaced by $t$) is along the same lines: we use a bound similar to (B.101) and a decomposition of $\frac{\|u^t - \hat{u}^t\|_2^2}{n}$ similar to (B.103). This completes the proof of the theorem. \hfill \Box

C An Auxiliary Lemma

The following result is proved in [BM11] Lemma 6).

**Lemma C.1.** Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz function, and let $F'(u, v)$ denote its derivative with respect to the first argument at $(u, v) \in \mathbb{R}^2$. Assume that $F'(\cdot, v)$ is continuous almost everywhere in the first argument, for each $v \in \mathbb{R}$. Let $(U_m, V_m)$ be a sequence of random vectors in $\mathbb{R}^2$ converging in distribution to the random vector $(U, V)$ as $m \rightarrow \infty$. Furthermore, assume that the distribution of $U$ is absolutely continuous with respect to the Lebesgue measure. Then,

$$\lim_{m \rightarrow \infty} E\{F'(U_m, V_m)\} = E\{F'(U, V)\}.$$

D Complex-valued GAMP

Consider a complex sensing matrix $A$ with rows distributed as $(a_i) \sim_{i.i.d.} \mathcal{CN}(0, I_d/d)$, for $i \in [n]$. The output of the GLM $y \in \mathbb{C}^n$ is generated as $p_{Y|G}(y \mid g)$, where $g = Ax$. The GAMP algorithm for the complex setting has been studied in the context of phase retrieval in [SR14, MXM19]. Here, we briefly review the complex GAMP and present some numerical results for complex GAMP with spectral initialization.

As in Section 4 we take $f_t$ to be the identity function, and $h_t = \sqrt{\delta} h_t^*$, where $h_t^*$ is given in (3.18). To obtain a compact state evolution recursion, we initialize with a scaled version of the spectral estimator $\hat{x}^s$:

$$x^0 = \sqrt{d} \frac{a}{1 - a^2} \hat{x}^s, \quad u^0 = \frac{1}{\sqrt{\delta}} Ax^0 - \frac{1}{\sqrt{\delta} \lambda^s} Z_s Ax^0.$$  \hfill (D.1)
Figure 4: Performance comparison between complex GAMP with spectral initialization (in red) and the spectral method alone (in black) for a Gaussian prior $P_X \sim \text{CN}(0,1)$. On the $x$-axis, we have the sampling ratio $\delta = n/d$; on the $y$-axis, we have the normalized squared scalar product between the signal and the estimate. The experimental results (• and □ markers) are in excellent agreement with the theoretical predictions (solid lines) given by state evolution for GAMP and Lemma 2.1 for the spectral method. Error bars indicate one standard deviation around the empirical mean.

The iterates are then computed as:

\[ x^{t+1} = A^H h_t^*(u_t; y) - c_t f_t(x^t), \]  
\[ u^{t+1} = \frac{1}{\sqrt{\delta}} A x^{t+1} - \frac{1}{\sqrt{\delta}} h_t^*(u_t; y). \]  

Here, the Onsager coefficient $c_t$ is given in [SR14]:

\[ c_t = \frac{\sqrt{\delta}}{\text{Var}(G \mid U_t = u)} \left( \frac{\text{Var}(G \mid U_t = u, Y = y)}{\text{Var}(G \mid U_t = u)} - 1 \right). \]

For this choice of $f_t, h_t$, the state evolution iteration can be written in terms of a single parameter $\mu_t \equiv \mu_{X,t}$. For $t \geq 0$:

\[ \mu_{U,t} = \frac{1}{\sqrt{\delta}} \mu_t, \quad \sigma_{U,t}^2 = \frac{\mu_t}{\delta}, \quad \sigma_{X,t}^2 = \mu_{X,t} = \mu_t, \]

\[ \mu_{t+1} = \sqrt{\delta} \mathbb{E} \left\{ |h_t^*(U_t; Y)|^2 \right\}. \]

The recursion is initialized with $\mu_0 = a_2 a_1^{-2}$. Moreover, the parameter $\mu_{t+1}$ can be consistently estimated from the iterate $u^t$ as $\hat{\mu}_{t+1} = \sqrt{\delta} \|h^*(u^t; y)\|_2^2/n$. It can also be estimated as the positive solution of the quadratic equation $\hat{\mu}_{t+1}^2 + \hat{\mu}_{t+1} = \|x^{t+1}\|_2^2/d$.

We now discuss some numerical results for noiseless (complex) phase retrieval, where $y_i = |(Ax)_i|^2$, for $i \in [n]$. For a given measurement matrix $A$, note that replacing $x$ by $e^{i\theta}x$ leaves
the measurement $\mathbf{y}$ unchanged. Therefore the performance of any estimator is measured up to a constant phase rotation:

$$\min_{\theta \in [0, 2\pi)} \left| \frac{\langle \hat{\mathbf{x}}, e^{i\theta} \mathbf{x} \rangle}{\|\mathbf{y}\|_2^2} \right|^2.$$  \hfill (D.6)

Figure 4 shows the performance of GAMP with spectral initialization when the signal $\mathbf{x}$ is uniform on the $d$-dimensional complex sphere with radius $\sqrt{d}$, and the sensing vectors $(\mathbf{a}_i) \sim \text{i.i.d.} \text{CN}(0, I_{d/d})$.

Figure 5 shows the performance with coded diffraction pattern sensing vectors, given by (4.2). The signal $\mathbf{x}$ is the image in Figure 3a, which is a $d_1 \times d_2 \times 3$ array with $d_1 = 820$ and $d_2 = 1280$. The three components $\mathbf{x}_j \in \mathbb{R}^d$ ($j \in \{1, 2, 3\}$ and $d = d_1 \cdot d_2$) are treated separately, and the performance is measured via the average squared normalized scalar product $\frac{1}{3} \sum_{j=1}^{3} \frac{|\langle \hat{\mathbf{x}}_j, \mathbf{x}_j \rangle|^2}{\|\hat{\mathbf{x}}_j\|_2^2 \|\mathbf{x}_j\|_2^2}$.

The red points in Figure 5 are obtained by running the complex GAMP algorithm with spectral initialization, as given in (D.1)-(D.4). We perform $n_{\text{sample}} = 5$ independent trials and show error bars at 1 standard deviation. For comparison, the black points correspond to the empirical performance of the spectral method alone, and the black curve gives the theoretical prediction for the optimal squared correlation for Gaussian sensing vectors (see Theorem 1 of [LAL19]).