The 4-Adic Complexity of A Class of Quaternary Cyclotomic Sequences with Period \(2p\)

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Abstract

In cryptography, we hope a sequence over \(\mathbb{Z}_m\) with period \(N\) having larger \(m\)-adic complexity. Compared with the binary case, the computation of 4-adic complexity of knowing quaternary sequences has not been well developed. In this paper, we determine the 4-adic complexity of the quaternary cyclotomic sequences with period \(2p\) defined in [6]. The main method we utilized is a quadratic Gauss sum \(G_p\) valued in \(\mathbb{Z}_{4^\nu - 1}\) which can be seen as a version of classical quadratic Gauss sum. Our results show that the 4-adic complexity of this class of quaternary cyclotomic sequences reaches the maximum if \(5 \nmid p - 2\) and close to the maximum otherwise.

Index Terms

4-adic complexity, quaternary cyclotomic sequences, quadratic Gauss sum, cryptography

I. INTRODUCTION

Periodic sequences over finite field \(\mathbb{F}_q\) or finite ring \(\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}\) have many important applications in spread-spectrum multiple-access communication and cryptography. In the stream cipher schemes we

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The work was supported by the National Science Foundation of China (NSFC) under Grant 12031011, 11701553.

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need the sequences having good pseudorandom cryptographic properties and large linear complexity [1].

In the past three decades, many series of such sequences have been investigated, their autocorrelation and linear complexity have been determined or estimated. A sequence with linear complexity $n$ can be generated by a linear shift register of length $n$ and the period of a sequence is an upper bound of $n$. In 1990’s, Klapper, Goresky and Xu [4, 5] described a kind of non-linear shift registers (feedback with carry shift registers (FCSRs)) and raised a new complexity, called $m$-adic complexity.

**Definition 1.** Let $m \geq 2$ be a positive integer, $A = \{a(i)\}_{i=0}^{N-1}$ be a sequence over $\mathbb{Z}_m$ with period $N$, $a(i) \in \{0, 1, \ldots, m-1\}$ for $0 \leq i \leq N-1$. Let $S_A(m) = \sum_{i=0}^{N-1} a(i) m^i \in \mathbb{Z}$ and $d = \gcd(S_A(m), m^N - 1)$. The $m$-adic complexity of the sequence $A$ is defined by

$$C_A(m) = \log_m \left( \frac{m^N - 1}{d} \right).$$

Roughly speaking, a sequence $A$ with period $N$ over $\mathbb{Z}_m$ can be generated by an FCSR of length $\lceil C_A(m) \rceil$. In cryptography, we hope a sequence $A$ over $\mathbb{Z}_m$ with period $N$ having larger $m$-adic complexity $C_A(m)$. By the Definition [1] we know that $\lceil C_A(m) \rceil \leq N$, where for $\alpha > 0$, $\lceil \alpha \rceil$ is the smallest integer $n$ such that $n \geq \alpha$.

In the past decade, the $2$-adic complexity $C_A(2)$ has been determined or estimated for many binary sequences $A$ with good autocorrelation properties. Particularly, for all known binary sequences $A = \{a(i)\}_{i=0}^{N-1} (a(i) \in \{0, 1\})$ with period $N \equiv 3 \pmod{4}$ and ideal autocorrelation ($\sum_{i=0}^{N-1} (-1)^{a(i+\tau) - a(i)} = -1$, for all $1 \leq \tau \leq N-1$), the $2$-adic complexity $C_A(2)$ reaches the maximum value $\log_2(2^N - 1)$ [3]. On the other hand, quaternary sequences (over $\mathbb{Z}_4$) are also important sequences in many practical applications [7]. Comparing with the binary case, the computation of $4$-adic complexity of knowing quaternary sequences has not been well developed. In this paper we determine the $4$-adic complexity of the quaternary cyclotomic sequences give by Kim et al. in [6].

Let $p$ be an odd prime, $(\cdot)_p : \mathbb{Z}_p^* = \{1, 2, \ldots, p-1\} \rightarrow \{\pm 1\}$ be the Legendre symbol. Namely, for $a \in \mathbb{Z}_p^*$,

$$\left( \frac{a}{p} \right) = \begin{cases} 1, & \text{if } a \text{ is a square in } \mathbb{Z}_p^*; \\ -1, & \text{otherwise}. \end{cases}$$

Let $g$ be a primitive element modulo $2p$, $\mathbb{Z}_{2p}^* = \langle g \rangle$ and

$$D_0^{(2p)} = \langle g^2 \rangle \subseteq \mathbb{Z}_{2p}^*, \quad D_1^{(2p)} = gD_0^{(2p)} \subseteq \mathbb{Z}_{2p}^*$$

$$D_0^{(p)} = \{a \in \mathbb{Z}_p^* : (\frac{a}{p}) = 1\}, \quad D_1^{(p)} = \{a \in \mathbb{Z}_p^* : (\frac{a}{p}) = -1\}$$
Then

\[ \mathbb{Z}_{2p} = D_0^{(2p)} \cup D_1^{(2p)} \cup 2D_0^{(p)} \cup 2D_1^{(p)} \cup \{0, p\} \quad \text{(disjoint)} \]

**Definition 2.** ([6]) Define a quaternary sequence \( A = \{a(i)\}_{i=0}^{2p-1} \) over \( \mathbb{Z}_4 = \{0, 1, 2, 3\} \) with period \( N = 2p \) by

\[
a(i) = \begin{cases} 
0, & \text{if } i = 0 \text{ or } i \in D_0^{(2p)} \\
2, & \text{if } i = p \text{ or } i \in 2D_0^{(p)} \\
1, & \text{if } i \in D_1^{(2p)} \\
3, & \text{if } i \in 2D_1^{(p)} 
\end{cases}
\]

The autocorrelation of this quaternary sequence has been computed in [6]. Du and Chen [2] translated this sequence into a sequence \( A' \) over the finite field \( \mathbb{F}_4 \) by the Gray mapping and computed the linear complexity of \( A' \) over \( \mathbb{F}_4 \). The following theorem is our main result which determines the 4-adic complexity of the quaternary sequence \( A \).

**Theorem 1.** Let \( A \) be the quaternary sequence with period \( N = 2p \) defined by Definition 2. Then the 4-adic complexity of \( A \) is

\[
C_A(4) = \begin{cases} 
\log_4 \left( \frac{4^N-1}{5} \right), & \text{if } 5 \mid p - 2; \\
\log_4 \left( 4^N - 1 \right), & \text{otherwise.}
\end{cases}
\]

In Section III we introduce a quadratic Gauss sum \( G_p \) valued in \( \mathbb{Z}_{4^{N-1}} \) as a version of classical quadratic Gauss sum, prove a property of \( G_p \), and show that \( S_A(4) = \sum_{i=0}^{N-1} a(i)4^i \) can be expressed by \( G_p \) modulo \( 4^N - 1 \). In Section III we prove Theorem 1.

**II. Preliminaries**

Let \( p \) be an odd prime, \( N = 2p \). From the fact that \( a \equiv b \pmod{p} \) implies \( 4^a \equiv 4^b \pmod{4^N - 1} \) we can define an element \( G_p \) in \( \mathbb{Z}_{4^{N-1}} \):

\[
G_p = \sum_{a \in \mathbb{Z}_{4^p}} \left( \frac{a}{p} \right) 4^a = \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) 4^a \pmod{4^N - 1}
\]

The following result shows that \( S_A(4) \) can be expressed by \( G_p \) modulo \( 4^N - 1 \) and \( G_p \) has a similar property as classical quadratic Gauss sum.
Lemma 2. Let $A = \{a(i)\}_{i=0}^{N-1}$ be the quaternary sequence over $\mathbb{Z}_4$ with period $N = 2p$ ($p \geq 3$) defined by Definition 2. Then

\begin{align*}
(1) \quad & S_A(4) \equiv \frac{1}{2}(3 \cdot 4^p - 5) + \frac{4^p+5}{2} \cdot 4^{N-1} - \frac{1}{2}(\frac{2}{p})4^p + 1)G_p \pmod{4^N - 1} \\
(2) \quad & G_p^2 \equiv \left( -\frac{1}{p} \right)(p - 4^{N-1}) \pmod{4^N - 1}
\end{align*}

Proof. (1). By the Chinese Remainder Theorem, we have isomorphism of rings

$$\varphi : \mathbb{Z}_{2p} \cong \mathbb{Z}_p \oplus \mathbb{Z}_2$$

by $\varphi(x \pmod{2p}) = (x \pmod{p}, x \pmod{2})$. It is easy to see that for any element $(A, B) \in \mathbb{Z}_p \oplus \mathbb{Z}_2(0 \leq A \leq p - 1, B \in \{0, 1\})$, $\varphi^{-1}(A, B) = A(p + 1) + pB \in \mathbb{Z}_{2p}$. Then

$$\sum_{i \in D_4^{(p)}} A^i \equiv \sum_{i \in D_4^{(p)}} A^i \quad (i = A(p + 1) + p)$$

From Definition 2, we know that

$$S_A(4) = \sum_{i \in D_4^{(p)}} A^i + 2 \cdot 4^p + 2 \sum_{a \in D_4^{(p)}} 4^{2a} + 3 \sum_{a \in D_4^{(p)}} 4^{2a}$$

$$\equiv \sum_{i \in D_4^{(p)}} A^i + 2 \cdot 4^p + \sum_{a \in D_4^{(p)}} 4^{2a} + \sum_{a \in D_4^{(p)}} 4^{2a} \pmod{4^N - 1}$$

$$\equiv 4^p \cdot \sum_{a \in D_4^{(p)}} 4^{2a} + 2 \cdot 4^p + 2 \sum_{a \in D_4^{(p)}} 4^{2a} + \sum_{a \in D_4^{(p)}} 4^{2a} \pmod{4^N - 1}$$

$$\equiv 4^p \cdot \sum_{a \in D_4^{(p)}} (1 - \left( \frac{2}{p} \right)(\frac{a}{p})4^{2a}) + 2 \cdot 4^p + 2 \sum_{a \in D_4^{(p)}} 4^{2a} + \frac{1}{2} \sum_{a \in D_4^{(p)}} (1 - \left( \frac{2}{p} \right)(\frac{a}{p})4^{2a}) \pmod{4^N - 1}$$

$$\equiv \frac{4^p}{2} + 2 \cdot 4^p + \sum_{a \in D_4^{(p)}} 4^{2a} + 2 \cdot 4^p - \frac{1}{2}(\frac{2}{p})G_p \pmod{4^N - 1}$$

$$\equiv \frac{4^p}{2} \cdot (4^N - 1) + 2 \cdot 4^p - \frac{1}{2}(\frac{2}{p})G_p \pmod{4^N - 1}$$

$$\equiv \frac{1}{2}(3 \cdot 4^p - 5) + \frac{4^p + 5}{2} \cdot \frac{4^N - 1}{15} - \frac{1}{2}(\frac{2}{p})G_p \pmod{4^N - 1}$$
(2). By the definition of $G_p$,

$$G_p^2 = \sum_{x,y=1}^{p-1} \left( \frac{xy}{p} \right) 16^{x+y}$$

$$\equiv \sum_{x,t=1}^{p-1} \left( \frac{t}{p} \right) 16^{x(1+t)} \pmod{4^N-1} \quad (y = xt)$$

$$\equiv \left( \frac{-1}{p} \right)(p-1) + \sum_{t=1}^{p-2} \left( \frac{t}{p} \right) \sum_{x=1}^{p-1} 16^{x(1+t)} \pmod{4^N-1}$$

$$\equiv \left( \frac{-1}{p} \right)(p-1) + \sum_{t=1}^{p-2} \left( \frac{t}{p} \right) \sum_{x=1}^{p-1} 16^x \pmod{4^N-1}$$

$$\equiv \left( \frac{-1}{p} \right)(p-1) - \left( \frac{-1}{p} \right) \frac{16^p - 1}{15} \pmod{4^N-1}$$

$$\equiv \left( \frac{-1}{p} \right)(p - \frac{4^N - 1}{15}) \pmod{4^N-1}$$

$$\square$$

III. PROOF OF THEOREM

By Definition 1, $C_A(4) = \log_4 \left( \frac{4^{N-1}}{d} \right)$ where $d = \gcd(S_A(4), 4^N - 1)$. Since $N = 2p$, $4^N - 1 = (4^p + 1)(4^p - 1)$ and $\gcd(4^p + 1, 4^p - 1) = \gcd(4^p + 1, 2) = 1$. We get $d = d_+d_-$ where both of

$$d_+ = \gcd(S_A(4), 4^p + 1) \quad \text{and} \quad d_- = \gcd(S_A(4), 4^p - 1)$$

are odd. We need to determine $d_+$ and $d_-.$

**Lemma 3.**

$$d_+ = \begin{cases} 
5, & \text{if } 5 \mid p - 2; \\
1, & \text{otherwise.}
\end{cases}$$

**Proof.** Let $\ell$ be a prime divisor of $d_+$. Then $S_A(4) \equiv 4^p + 1 \equiv 0 \pmod{\ell}$. From Lemma 2 and $\ell|4^p + 1$ we have

$$S_A(4) \equiv -4 + \frac{2(4^p - 1)}{15} (4^p + 1) - \frac{1}{2} \left( -\left( \frac{2}{p} \right) + 1 \right) G_p \pmod{\ell} \quad (1)$$

Since $4^p + 1 \equiv 2 \pmod{3}$, we know that $\ell \geq 5$. Firstly we consider the case $\ell = 5$. In this case, $4^p + 1 \equiv (-1)^p + 1 \equiv 0 \pmod{5}$ and

$$G_p = \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) 4^{2a} \equiv \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) \equiv 0 \pmod{5}$$

Then by the formula (1) we get $S_A(4) \equiv -4 - \frac{4}{15} (4^p + 1) \pmod{5}$. Therefore,

$$S_A(4) \equiv 0 \pmod{5} \iff 4^p + 1 \equiv -1 \pmod{5} \iff 4^p + 1 \equiv -15 \pmod{25}$$

$$\iff 4^{p-2} \equiv -1 \pmod{25} \iff 10 \mid 2(p-2)$$
The last equivalence can be obtained by the fact that the order of 4 modulo 25 is 10. Therefore, $5|d_+$ if and only if $5|p-2$. Moreover, assume that $5|p-2$. If $25|4^p+1$, then $4^p \equiv 1 \pmod{25}$ and then $10|2p$ which contradicts to $5|p-2$. In summary, $5|d_+$ if and only if $5|p-2$ and when $5|p-2$ we have $25|d_+$.

Now we assume $\ell \geq 7$. The formula (1) becomes

$$0 \equiv S_A(4) \equiv -4 - \frac{1}{2} \left(1 - \left(\frac{2}{p}\right)\right)G_p \pmod{\ell}$$

(2)

and by Lemma (2), $G^2_p \equiv \left(\frac{-1}{p}\right)p \pmod{\ell}$. If $p \equiv \pm 1 \pmod{8}$ then $\left(\frac{2}{p}\right) = 1$ and we get a contradiction $0 \equiv -4 \pmod{\ell}$. If $p \equiv \pm 3 \pmod{8}$ then $\left(\frac{2}{p}\right) = -1$ and $G_p \equiv -4 \pmod{\ell}$ by (2). Therefore $16 \equiv G^2_p \equiv \left(\frac{-1}{p}\right)p \pmod{\ell}$ and then $\ell | p-16$ or $\ell | p+16$. On the other hand, $2^2p = 4^p \equiv -1 \pmod{\ell}$ which means that the order of 2 modulo $\ell$ is $4p$. Therefore $4p | \ell - 1$ and $4p \leq \ell - 1 \leq p+15$ which implies $p \leq 5$. If $p = 3$, then $l = 13$. From $G_3 = \sum_{a=1}^2 (\frac{a}{3})4^{2a}$ we get $G_3 = 4^2 - 4^4 \equiv -6 \pmod{13}$. By (2) we get $G_3 \equiv -4 \pmod{13}$ which is a contradiction. Then by $p \equiv \pm 3 \pmod{8}$ we get $p = 5$ and $\ell = 21$ which contradicts to that $\ell$ is a prime number. In summary, we get $d_+ = 5$ if $5 | p-2$ and $d_+ = 1$ otherwise. This completes the proof of Lemma (3).

Lemma 4. $d_- = 1$

Proof. Let $\ell$ be a prime divisor of $d_-$. Then $S_A(4) \equiv 4^p - 1 \equiv 0 \pmod{\ell}$. From

$$S_A(4) \equiv 2 \cdot 4^p + \sum_{a \in D^2_{o(p)}} 1 + \sum_{a \in D^2_{o(p)}} 2 \equiv 3 \cdot \frac{p-1}{2} + 2 \equiv 2 \pmod{\ell}$$

we know that $\ell \geq 5$. From $4^p \equiv 1 \pmod{\ell}$ and $\ell \geq 5$ we know that the order of 4 modulo $\ell$ is $p$. Therefore $p | \ell - 1$. On the other hand, by Lemma (2) we have

$$0 \equiv S_A(4) \equiv -1 - \frac{1}{2}\left(\left(\frac{2}{p}\right) + 1\right)G_p \pmod{\ell}, \quad G^2_p \equiv \left(\frac{-1}{p}\right)p \pmod{\ell}$$

If $\left(\frac{2}{p}\right) = -1$, we get contradiction $0 \equiv -1 \pmod{\ell}$. If $\left(\frac{2}{p}\right) = 1$ then $1 \equiv -G_p \pmod{\ell}$ and $1 \equiv G^2_p \equiv \left(\frac{-1}{p}\right)p \pmod{\ell}$. We get $\ell | \frac{p-1}{2}$ or $\ell | \frac{p-1}{2}$. Then we have $p \leq \ell - 1 \leq \frac{1}{2}(p+1) - 1 = \frac{p}{2} - \frac{1}{2}$ which is a contradiction. Therefore we get $d_- = 1$.

Proof of Theorem 1. By Lemma (3) and Lemma (4) we get

$$d = d_+ d_1 = \begin{cases} 5, & \text{if } 5 | p-2; \\ 1, & \text{otherwise.} \end{cases}$$

Therefore

$$C_A(4) = \log_4 \left(\frac{4^N - 1}{d}\right) = \begin{cases} \log_4 \left(\frac{4^N - 1}{5}\right), & \text{if } 5 | p-2; \\ \log_4 (4^N - 1), & \text{otherwise.} \end{cases}$$
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