Abstract. We show that pairs \((X,Y)\) of 1-spherical objects in \(A_\infty\)-categories, such that the morphism space \(\text{Hom}(X,Y)\) is concentrated in degree 0, can be described by certain noncommutative orders over (possibly stacky) curves. In fact, we establish a more precise correspondence at the level of isomorphism of moduli spaces which we show to be affine schemes of finite type over \(\mathbb{Z}\).

Introduction

The study of \(A_\infty\)-categories has become an important part of the study of derived categories in algebraic geometry, especially in connection with the homological mirror symmetry. In [25, 26] we started to develop a systematic approach to the moduli spaces of minimal \(A_\infty\)-structures on a given graded vector space. In [25, 13, 26] we related certain moduli spaces of \(A_\infty\)-structures to appropriate moduli spaces of curves, and in [14] this was applied to proving an arithmetic version of homological mirror symmetry for \(n\)-punctured tori.

Philosophically, when replacing a geometric object by the corresponding \(A_\infty\)-category, one enters the world of noncommutative geometry. Thus, it is natural that objects of noncommutative geometry should appear in descriptions of more general moduli spaces of \(A_\infty\)-structures.

In the present paper we consider examples of such moduli spaces parametrizing \(A_\infty\)-categories generated by pairs of 1-spherical objects \((X,Y)\) such that morphism space \(\text{Hom}(X,Y)\) is concentrated in degree 0. Note that examples of such pairs come from considering simple vector bundles on Calabi-Yau curves, as well as from Fukaya categories of punctured surfaces.

Recall (see [32], [30, I.5]) that an object \(X\) of a \(k\)-linear \(A_\infty\)-category \(\mathcal{C}\), where \(k\) be a field, is called \(n\)-spherical if \(\text{Hom}^i(X,X) = 0\) for \(i \neq 0, n\), \(\text{Hom}^0(X,X) = \text{Hom}^n(X,X) = k\), and for any object \(Y\) of \(\mathcal{C}\) the pairing between the morphism spaces in the cohomology category \(H^*\mathcal{C}\),

\[
\text{Hom}^{n-i}(Y,X) \otimes \text{Hom}^i(X,Y) \to \text{Hom}^1(X,X),
\]

induced by \(m_2\), is perfect.

Note that if we have a pair of 1-spherical objects \((X,Y)\) in a minimal \(A_\infty\)-category, such that \(\text{Hom}(X,Y)\) is concentrated in degree 0, then the only interesting double products involving \(X\) and \(Y\) are the perfect pairings

\[
\text{Hom}^1(Y,X) \otimes \text{Hom}^0(Y,X) \to \text{Hom}^1(X,X) \simeq k, \quad \text{Hom}^0(Y,X) \otimes \text{Hom}^1(Y,X) \to \text{Hom}^1(Y,Y) \simeq k.
\]

Supported in part by the NSF grant DMS-1700642 and by the Russian Academic Excellence Project ‘5-100’.
Thus, up to an isomorphism, the graded associative algebra \( \text{Hom}(X \oplus Y, X \oplus Y) \) is determined by a linear automorphism \( g \) of the finite-dimensional space \( \text{Hom}^0(X, Y) \), which measures the difference between the above two pairings. More precisely, fixing trivializations of \( \text{Hom}^1(X, X) \) and \( \text{Hom}^1(Y, Y) \) and a basis \( \alpha_1, \ldots, \alpha_n \) in \( \text{Hom}^0(X, Y) \), we get an identification of graded associative algebras

\[
\text{Hom}(X \oplus Y, X \oplus Y) \simeq S(k^n, g),
\]

where \( S(g) = S(k^n, g) \) is a certain \((2n + 4)\)-dimensional algebra depending on \( g \in \text{GL}_n(k) \) (see Sec. 1.1). Furthermore, it is easy to see that replacing \( g \) by \( \lambda \cdot g \), where \( \lambda \in k^\times \), leads to an isomorphic algebra.

Since \( X \) and \( Y \) were objects of a minimal \( A_\infty \)-category, we get a minimal \( A_\infty \)-structure on \( S(g) \) extending the given \( m_2 \). Thus, the problem of describing pairs of 1-spherical objects \((X, Y)\) as above with \( \text{Hom}^0(X, Y) \) of dimension \( n \) (in this case we refer to \((X, Y)\) as an \( n \)-pair) fits into the framework of [26, Sec. 2.2]. As in [26] we consider the moduli space of all minimal \( A_\infty \)-structures on the family of algebras \( S(\cdot) \) over \( \text{PGL}_n \) (extending the given \( m_2 \)). The corresponding moduli functor \( \mathcal{M}_\infty(\text{sph}, n) \) associates to a commutative ring \( R \) the set of gauge equivalence classes of minimal \( R \)-linear \( A_\infty \)-structures on an algebra of the form \( S(R^n, g) \), where \( g \in \text{PGL}_n(R) \) (see Sec. 1.1 for the precise definition).

Note that by definition, we have a natural projection

\[
\mathcal{M}_\infty(\text{sph}, n) \to \text{PGL}_n,
\]

where we identify the affine scheme \( \text{PGL}_n \) with the corresponding functor on commutative rings.

In the case \( n = 2 \) we need to restrict possible elements \( g \), so we consider the principal open subscheme \( \text{PGL}_2(\text{tr}^{-1}) \subset \text{PGL}_2 \) where \( \text{tr}(g) \) is invertible. We denote by

\[
\mathcal{M}_\infty(\text{sph}, 2)[\text{tr}^{-1}] \subset \mathcal{M}_\infty(\text{sph}, 2)
\]

the corresponding subfunctor.

Our first main result, Theorem A below, relates \( A_\infty \)-structures in \( \mathcal{M}_\infty(\text{sph}, n) \) for \( n \geq 3 \) (resp., \( \mathcal{M}_\infty(\text{sph}, 2)[\text{tr}^{-1}] \) for \( n = 2 \)) to certain noncommutative projective schemes in the sense of [2]. Recall that for a Noetherian graded algebra \( \mathcal{R} \) one considers the quotient-category \( \text{qgr} \mathcal{R} = \text{grmod} -\mathcal{R}/\text{tors} \) of finitely generated graded \( \mathcal{R} \)-modules by the subcategory of torsion modules (it should be viewed as the category of coherent sheaves on the corresponding noncommutative scheme).

**Definition 0.0.1.** Let \( R \) be a commutative ring, \( V \) a finitely generated projective \( R \)-module, and \( \mathcal{L} \) an invertible \( R \)-module. For \( g \in \text{End}(V) \otimes \mathcal{L} \) we denote by \( \text{End}_g(V) \subset \text{End}(V) \) the \( R \)-submodule of transformations \( a \) such that \( \text{tr}(ga) = 0 \). We define the subalgebra in \( \text{End}(V)[z] \) by

\[
\mathcal{E}(V, g) := \{ a_0 + a_1 z + \ldots \in \text{End}(V)[z] \mid a_0 \in R \cdot \text{id}, a_1 \in \text{End}_g(V) \}.
\]

We view \( \mathcal{E}(V, g) \) as a graded \( R \)-algebra, where \( \text{deg}(z) = 1 \).

**Theorem A.** For \( n \geq 2 \), let us consider the functor \( \mathcal{M}_{\text{filt}}(n) \) associating with a commutative ring \( R \) the following data: a morphism \( g : \text{Spec}(R) \to \text{PGL}_n \) and an isomorphism
class of filtered $R$-algebras $(A, F_{\bullet})$ equipped with an isomorphism
\[ \text{gr}^F A \simeq \mathcal{E}(R^n, g)^{op}, \]
where we denote also by $g$ the pull-back under $g$ of the universal matrix in $H^0(\text{PGL}_n, \text{Mat}_n(O) \otimes O(1))$. Then for $n \geq 3$, there is an isomorphism of functors
\[ \mathcal{M}_\infty(\text{sph}, n) \simeq \mathcal{M}_{\text{filt}}(n) \]
and each of these functors is representable by an affine scheme of finite type over $\mathbb{Z}$.

In the case $n = 2$ we have an isomorphism of modified functors
\[ \mathcal{M}_\infty(\text{sph}, 2)[\text{tr}^{-1}] \simeq \mathcal{M}_{\text{filt}}(n)[\text{tr}^{-1}] \]
where we impose the condition that $\text{tr}(g)$ is invertible. These functors are still representable by an affine scheme of finite type over $\mathbb{Z}$.

In either case, if $(A, F_{\bullet})$ is the filtered $S$-algebra corresponding to an $n$-pair $(E, F)$ of 1-spherical objects over a Noetherian commutative ring $S$, then there exists an $A_\infty$-functor from $\langle E, F \rangle$ to the $A_\infty$-enhancement of the derived category of $\text{gr} R(A)$, where $R(A)$ is the Rees algebra of $A$, inducing a quasi-equivalence with its image.

Note that in the case $n = 1$ the equivalence of Theorem A still holds if we restrict to working over $\mathbb{Z}[1/6]$. Indeed, this follows from the results of [12], where the moduli scheme $\mathcal{M}_\infty(\text{sph}, 1)$ is identified with $\mathbb{A}^2$. The corresponding pairs of spherical objects are realized geometrically as $(O_C, O_p)$, where $C$ is an irreducible curve of arithmetic genus 1 and $p \in C$ is a smooth point.

Another case that has a nice geometric realization is that of $n = 2$, $g = \text{id}$. Namely, working over $\mathbb{C}$, Seidel showed in [31, Sec. 2] that pairs of spherical objects $(X, Y)$ with $\dim \text{Hom}_0(X, Y) = 2$ can be realized as $(O_C, \pi^* O_{\mathbb{P}^1}(1))$, where $\pi : C \to \mathbb{P}^1$ is a (possibly singular) double cover of $\mathbb{P}^1$.

It seems that in the case $n = \dim \text{Hom}_0(X, Y) > 2$ one cannot avoid using some noncommutative geometry to realize pairs of spherical objects $(X, Y)$. However, it is still of a sufficiently simple kind. Namely, working over a field $k$, using the classical result of Small-Warfield [33] on algebras of GK dimension one, we deduce that any filtered algebra appearing in Theorem A is finite over its center. Using this we establish a natural bijection between $\mathcal{M}_\infty(\text{sph}, n)$ and certain orders over integral stacky curves. Here by an order over an integral stacky curve $C$ we mean a coherent sheaf of $O_C$-algebras, torsion free as $O_C$-module, whose stalk at the generic point of $C$ is a simple $k(C)$-algebra.

We make the following definition concerning the types of stacky curves and orders we consider.

**Definition 0.0.2.** A neat pointed stacky curve over $k$ is an integral 1-dimensional proper stack $C$ over $k$ with a smooth stacky point of the form $p = \text{Spec}(k)/\mu_d$, such that $C \setminus \{p\}$ is an affine curve. In addition we assume that the coarse moduli space $\overline{C}$ is a projective curve satisfying $H^0(\overline{C}, O) = k$, and there exists an étale morphism of the form $f : U/\mu_d \to C$, where $U$ is a smooth affine curve with a $\mu_d$-action and $k$-point $q$, such that $\mu_d$ acts faithfully on the tangent space to $q$ and $f(q) = p$.

**Definition 0.0.3.** Let $\mathcal{A}$ be an order over an integral proper stacky curve $C$, such that $h^0(C, O) = k$. We say that $\mathcal{A}$ is spherical if $\mathcal{A}$ is a 1-spherical object in the perfect
derived category of right $\mathcal{A}$-modules, $\text{Perf}(\mathcal{A}^{\text{op}})$. We say that $\mathcal{A}$ is weakly spherical if $h^0(C, \mathcal{A}) = h^1(C, \mathcal{A}) = 1$.

It is easy to see that if $\mathcal{A}$ is spherical then it is weakly spherical. We will prove that an order if spherical if and only if $h^0(C, \mathcal{A}) = 1$ and there exists a morphism of coherent sheaves $\tau: \mathcal{A} \to \omega_C$ such that the pairing

$$\mathcal{A} \otimes \mathcal{A} \to \omega: x \otimes y \mapsto \tau(xy)$$

is perfect (see Sec. 3.2). We say that a spherical order $\mathcal{A}$ is symmetric if the above pairing (which is uniquely defined up to a scalar) is symmetric. The importance of spherical orders is due to the fact that they give to cyclic $A_\infty$-structures (see Corollary C below).

**Theorem B.** Fix a field $k$ and a vector space $V$ over $k$ of dimension $n \geq 2$. Let us consider the following two groupoids:

1. filtered algebras $(A, F)$ with a fixed isomorphism $\text{gr}^p A \simeq \mathcal{E}(V, g)^{op}$ for some $g \in \mathbb{P} \text{End}(V)$ (here morphisms exist only when the corresponding elements $g \in \mathbb{P} \text{End}(V)$ are equal);

2. data $(C, p, v, \mathcal{A}, \tau, \phi)$, where $C$ is a neat pointed stacky curve, $p \simeq \text{Spec}(k)/\mu_3$ a unique (smooth) stacky point on $C$, such that $\mathcal{A}$ is a weakly spherical order over $C$ with the center $\mathcal{O}_C$, such that $h^1(C, \mathcal{A}(p)) = 0$; $v$ is a nonzero tangent vector at $p$; and $\phi: \mathcal{A}|_p \simeq \rho_* \text{End}(V)^{op}$ is an isomorphism of algebras, where $\rho: \text{Spec}(k) \to p$ is the projection.

Then the map associating to $(C, p, \mathcal{A})$ the algebra $A = H^0(C \setminus p, \mathcal{A})$ with its natural filtration extends to an equivalence of groupoids (1) and (2).

Furthermore, the element $g$ in (1) is invertible if and only the corresponding order $\mathcal{A}$ is spherical. In this case, assuming in addition that either $n \geq 3$ or $\text{tr}(g) \neq 0$, we have an equivalence between the perfect derived category $\text{Perf}(\mathcal{A}^{op})$ and the $A_\infty$-category generated by the $n$-spherical pair associated with the data (1) by Theorem A.

If either $n \geq 3$ or $\text{char}(k) \neq 2$, then a spherical order $\mathcal{A}$ is symmetric if and only if $g$ is a scalar multiple of the identity matrix.

As an application of Theorems A and B, we get an equivalence of an $A_\infty$-category (over a field) generated by a pair of $1$-spherical objects $(X, Y)$, such that $\text{Hom}(X, Y) = \text{Hom}^0(X, Y)$, with the category of the form $\text{Perf}(\mathcal{A}^{op})$ for some spherical order $\mathcal{A}$ over a stacky curve $C$. More precisely, such an equivalence sends the pair $(X, Y)$ to the pair $(\mathcal{A}, \rho_* V)$ in $\text{Perf}(\mathcal{A}^{op})$, where $\rho \in C$ is a stacky point such that $\mathcal{A}|_p \simeq \rho_* \text{End}(V)^{op}$.

Note that for all examples of spherical orders over stacky curves $C$ that we were able to construct, $C$ is the usual nonstacky curve. However, we do not know whether this is always the case.

We are particularly interested in the case when $\mathcal{A}$ is symmetric because in this case we get a cyclic $A_\infty$-structure.

**Corollary C.** Let $k$ be a field. Assume that either $n \geq 3$ or $\text{char}(k) \neq 2$. Then every minimal $A_\infty$-structure on the algebra $S(k^n, \text{id})$ can be realized by $A_\infty$-endomorphisms of the generator $\mathcal{A} \oplus \rho_* V$ of $\text{Perf}(\mathcal{A}^{op})$, for a symmetric spherical order $\mathcal{A}$ over a neat pointed stacky curve $(C, p)$ equipped with an isomorphism $\mathcal{A}|_p \simeq \rho_* \text{End}(V)^{op}$. If $\text{char}(k) = 0$ then every such $A_\infty$-structure is gauge equivalent to a cyclic $A_\infty$-structure.
As was shown in [15], cyclic $A_\infty$-structures on $\mathcal{S}(k^n, \text{id})$ correspond to formal solutions of set-theoretical Associative Yang-Baxter Equation (AYBE). Applying Corollary C, we get an algebro-geometric realization of these solutions in the category of modules over some symmetric spherical orders. Elsewhere we will present an explicit construction of such orders giving rise to trigonometric nondegenerate solutions of the AYBE that were classified in [24].

The paper is organized as follows. In Section 1 we study the moduli space $\mathcal{M}_\infty(\text{sph}, n)$ of $A_\infty$-structures corresponding to pairs of 1-spherical objects. Using the criterion from [26] we show that it is an affine scheme of finite type over $\text{PGL}_n$. In Section 2, after reminding some background on spherical objects, we give a construction of a filtered algebra corresponding to an $n$-pair of 1-spherical objects (see Theorem 2.4.1). In Section 3, working over a field, we study the correspondence between filtered algebras with $\text{gr}^F A \simeq \mathcal{E}(R^n, g)^{\text{op}}$ and certain noncommutative orders over curves. In particular, using this correspondence we check the AS-Gorenstein property for the Rees algebra $\mathcal{R}(A)$ of every such filtered algebra $A$. Also, in Section 3.2 we give a criterion for an order to be spherical (see Definition 0.0.3). Next, in Section 4 we show how to associate with a filtered algebra $A$ as above an $n$-spherical pair in the derived category of $\text{qgr} \mathcal{R}(A)$ (i.e., of coherent sheaves on the corresponding noncommutative projective scheme). Another key result of this section is Theorem 4.3.2 generalizing [23, Sec. 3.1], computing some Hochschild cohomology relevant for studying $A_\infty$-structures on certain graded algebras having $\mathcal{R}(A)$ as a degree 0 component and its restricted dual as a degree 1 component. Using this we complete the proof of Theorem A in Sections 4.5 and 4.6, and of Theorem B in Section 4.7. Then in Section 4.8 we show that symmetric spherical orders over curves give rise to cyclic $A_\infty$-structures and use this to prove Corollary C.

Conventions. All $A_\infty$-algebras/categories are strictly unital. We denote by $\text{hom}(\cdot, \cdot)$ the morphism spaces in an $A_\infty$-category and by $\text{Hom}(\cdot, \cdot)$ their cohomology, For a commutative ring $S$, and a graded $S$-module $M = \bigoplus_i M_i$, where each $M_i$ is a projective finite dimensional $S$-module, we denote by $M^*$ the restricted dual of $M$, which is a graded $S$-module with components $(M^*)_i = (M_{-i})^\vee = \text{Hom}_S(M_{-i}, S)$.

Acknowledgments. I am grateful to Yanki Lekili and Riley Casper for useful discussions, and to James Zhang for the help with proving finiteness of injective dimension in Sec. 3.4. The construction of an algebra associated with a pair of spherical objects (see Theorem 2.4.1) is inspired by a construction in the work of Van Roosmalen [28]. Part of this work was done while the author was visiting the Korean Institute for Advanced Study and the ETH Zurich. He would like to thank these institutions for hospitality and excellent working conditions.

1. A moduli space of $A_\infty$-structures

1.1. The moduli functor. For basics on $A_\infty$-structures we refer to [9]. We would like to consider minimal (strictly unital) $A_\infty$-structures on a certain family of categories with two objects (which are generalizations of the category considered in [15]).

Definition 1.1.1. Let $R$ be a commutative ring, $V$ a finitely generated projective $R$-module, $\mathcal{L}$ an invertible $R$-module, and $g : V \to V \otimes \mathcal{L}$ an $R$-linear morphism. The
graded category \( S(V, g) \) has two objects \( X \) and \( Y \) and the morphisms

\[
\text{Hom}(X, Y) = \text{Hom}^0(X, Y) = V, \quad \text{Hom}(Y, X) = \text{Hom}^1(Y, X) = V^\vee = \text{Hom}_R(V, R),
\]

\[
\text{Hom}^0(X, X) = R \text{id}_X, \quad \text{Hom}^0(Y, Y) = R \text{id}_Y, \quad \text{Hom}^1(X, X) = \mathcal{L}, \quad \text{Hom}^1(Y, Y) = R.
\]

The compositions are determined by

\[
v^* \circ v = \langle v^*, g(v) \rangle, \quad v \circ v^* = \langle v^*, v \rangle,
\]

where \( v \in V, v^* \in V^\vee \).

In this paper we will consider minimal unital \( A_\infty \)-structures on the algebras \( S(k^n, g) \), where \( g \in \text{GL}_n(k) \), for fixed \( n \geq 2 \) (with \( m_2 \) given above), up to a gauge equivalence. More precisely, this family of algebras can be viewed as a sheaf of associative algebras \( \tilde{S}_n \) on the scheme \( \text{GL}_n \) (over \( \mathbb{Z} \)). Note that we have the standard action of the central \( \mathbb{G}_m \) on \( \text{GL}_n \) and the universal element \( g \) can be viewed as a morphism of bundles over \( \text{GL}_n \), \( g : \mathcal{O}^n \to \mathcal{O}^n \otimes \chi \), compatible with the \( \mathbb{G}_m \)-action. Here \( \chi \) is the identity character of \( \mathbb{G}_m \), \( \chi(\lambda) = \lambda \). Thus, we can descend \( g \) to a morphism

\[
\overline{\varphi} : \mathcal{O}^n \to \mathcal{O}^n \otimes \mathcal{O}(1)
\]

deal bundles over \( \text{PGL}_n = \text{GL}_n / \mathbb{G}_m \). Now it is easy to see that the sheaf of algebras \( \tilde{S}_n \) over \( \text{GL}_n \) is isomorphic to the pull-back of the sheaf of algebras \( S_n \) over \( \text{PGL}_n \) associated with bundle \( \mathcal{O}^n \) and the morphism \( \overline{\varphi} \) as in Definition 1.1.1.

As in [26], we consider the following functor over \( \text{PGL}_n \).

Definition 1.1.2. The functor \( \mathcal{M}_\infty(S_n) \) associates to a pair \( (R, g) \), where \( R \) is a commutative algebra and \( g \in \text{PGL}_n(R) \), the set of gauge equivalence classes of minimal \( A_\infty \)-structures on \( S(R^n, g) \). Note that here \( g \) is viewed as a morphism \( R^n \to R^n \otimes g^* \mathcal{O}(1) \).

For each \( N \geq 2 \) one can similarly consider the functor \( \mathcal{M}_N \) of minimal \( A'_N \)-structures up to equivalence, where we consider \( (m_2, m_3, \ldots, m_N) \) and impose the \( A_\infty \)-identities up to \( [m_2, m_N] + [m_3, m_{N-1}] + \ldots \). In [26, Thm. 2.2.6] we gave a general criterion for the functors \( \mathcal{M}_N \) to be representable by an affine scheme and for the projection \( \mathcal{M}_\infty \to \mathcal{M}_4 \) to be a closed embedding. In this paper we will apply this study in our situation.

More precisely, in the case \( n = 2 \) we restrict to an open subscheme of \( \text{GL}_2 \). Let us consider the closed embedding

\[
\text{Spec}(\mathbb{Z}/2) \hookrightarrow \text{PGL}_{2,\mathbb{Z}/2} \hookrightarrow \text{PGL}_2
\]
given by the unit over \( \mathbb{Z}/2 \), and let \( U \subset \text{PGL}_2 \) be the complementary open subscheme. Let \( S_{2, U} \) be the restriction of the sheaf of algebras \( S_2 \) to \( U \). Since \( U \) is non-affine, as in [26, Sec. 2.2], we first consider the functor \( \tilde{\mathcal{M}}_\infty(S_{2, U}) \) of the set of gauge equivalence classes of minimal \( A_{\infty} \)-structures, and then define \( \mathcal{M}_\infty(S_{2, U}) \) to be the sheafification of \( \tilde{\mathcal{M}}_\infty(S_{2, U}) \) with respect to the Zariski topology on the base.

Theorem 1.1.3. The functor \( \mathcal{M}_\infty(S_n) \) for \( n \geq 3 \) (resp., \( \mathcal{M}_\infty(S_{2, U}) \)) is representable by an affine scheme of finite type over \( \text{PGL}_n \) (resp., over \( U \)). Furthermore, in both cases the projection \( \mathcal{M}_\infty \to \mathcal{M}_4 \) to the moduli space of minimal \( A'_4 \)-structures is a closed embedding.
We will give a proof in Sec. 1.3 after computing some Hochschild cohomology of algebras $S(k^n, g)$.

Note that in [15] we considered minimal $A_\infty$-structures on $S(R^n, \text{id})$, cyclic with respect to a natural pairing. We postpone the discussion of cyclic structures until Sec. 4.8.

1.2. The dual quadratic algebra to $S(k^n, g)$. Let us fix a field $k$ and an element $g \in \text{GL}_n(k)$, where $n \geq 2$. Let us set for brevity $S = S(k^n, g)$. Note that we can view $S$ as a $K$-algebra, where $K = k \oplus k$, where the idempotents $e_X$ and $e_Y$ correspond to the identity elements $\text{id}_X$ and $\text{id}_Y$. We also denote by $\xi_X$ and $\xi_Y$ the basis elements in $\text{Hom}^1(X, X)$ and $\text{Hom}^1(Y, Y)$.

If $(\alpha_i)$ are the elements of $\text{Hom}(X, Y)$ corresponding to the standard basis in $k^n$ and $(\beta_j)$ are the dual elements of $\text{Hom}(Y, X)$, then the product in $S$ is given by

$$\beta_i \alpha_j = a_{ij} \xi_X, \quad \alpha_j \beta_i = \delta_{ij} \xi_Y,$$

where $g = (a_{ij})$. Note that the $K$-algebra $S$ is generated by elements $(\alpha_i)$ and $(\beta_j)$, so we can view it as a quotient of the path algebra of the quiver with two vertices $X$ and $Y$ and $n$ arrows in each direction corresponding to $\alpha_i$ and $\beta_j$. We will use two different gradings on $S$: deg given by $\deg(\alpha_i) = 0$, $\deg(\beta_i) = 1$, and $\deg_K$ given by $\deg_K(\alpha_i) = \deg_K(\beta_i) = 1$. We denote by $S_j$ the graded components of $S$ with respect to $\deg_K$.

We are going to use the following convention about quadratic (and Koszul) duality over $K = k \cdot \text{id}_X \oplus k \cdot \text{id}_Y$. For a quadratic $K$-algebra $A$ with generators $V_{XY}$ and $V_{YX}$ of degree 1, and quadratic relations $R_{XX} \subset V_{YX} \otimes V_{XY}$, $R_{YY} \subset V_{XY} \otimes V_{YX}$, the dual quadratic algebra has generators $V_{XY}^! = V_{YX}^!$ and $V_{YX}^! = V_{XY}^!$ and quadratic relations

$$R_{XX}^! = A_{2,XX}^! \subset (V_{YX} \otimes V_{XY})^! \simeq V_{YX}^! \otimes V_{XY}^!,$$

and similarly for $R_{YY}^!$.

Thus, we think of the dual algebra $S^!$ as the quotient of the path algebra of the quiver with vertices $X$, $Y$, where the direction of $\alpha_i^*$ (resp., $\beta_i^*$) is opposite to that of $\alpha_i$ (resp., $\beta_i$). We denote by $S^!_j$ the graded components of $S^!$ with respect to the grading $\deg_K(\alpha_i^*) = \deg_K(\beta_i^*) = 1$.

**Lemma 1.2.1.** With respect to the grading $\deg_K$ the algebra $S$ is Koszul and the dual quadratic algebra $S^!$ is generated by the dual generators $(\alpha_i^*)$ and $(\beta_i^*)$ with the only relations

$$\sum_{1 \leq i, j \leq n} a_{ij} \alpha_j^* \beta_i^* = 0, \quad \sum_{i=1}^n \beta_i^* \alpha_i^* = 0.$$

**Proof.** The algebra $S$ is obtained by folding from the following $\mathbb{Z}$-algebra $S^\mathbb{Z} = S^\mathbb{Z}(g)$ (where we use the term $\mathbb{Z}$-algebra in the sense of [4, Sec. 4]):

$$S^\mathbb{Z}_{i,i+1} = \begin{cases} V & i \text{ even} \\ V^* & i \text{ odd} \end{cases},$$

$$S^\mathbb{Z}_{i,i+2} = k, \quad S^\mathbb{Z}_{i,j} = 0 \text{ for } j > i + 2.$$
Here $V$ is the $n$-dimensional space with the basis $(\alpha_i)$. The multiplication is given by the pairings

$$V \otimes V^* \to k : v \otimes v^* \mapsto \langle v^*, v \rangle, \quad V^* \otimes V \to k : v^* \otimes v \mapsto \langle v^*, g(v) \rangle.$$ 

There is a natural isomorphism $S^Z_{i+2,j+2} \simeq S^Z_{i,j}$ compatible with the product, and the algebra $S$ is the corresponding folding of $S^Z$.

The Koszul properties for $S$ (with the grading $\deg_K$) and for $S^Z$ are equivalent. On the other hand, it is easy to construct the isomorphism of $\mathbb{Z}$-algebras between $S^Z$ and the $\mathbb{Z}$-algebra corresponding to the $\mathbb{Z}$-graded algebra

$$B_0 \oplus B_1 \oplus B_2 = k \oplus V \oplus k,$$

where the multiplication $V \otimes V = B_1 \otimes B_1 \to B_2 = k$ is given by a nondegenerate symmetric bilinear form. It is well known that the algebra $B$ is Koszul. Hence, $S$ is also Koszul. \hfill \Box

**Remark 1.2.2.** The algebra $S^l$ for $g = \text{id}$ is closely related to the noncommutative projective line $\mathbb{P}^1_n$ (see [21], [17]). Namely, it is a folded version of the $\mathbb{Z}$-algebra of a natural helix in the derived category of $\mathbb{P}^1_n$.

Let $e_X, e_Y \in K$ be the idempotents corresponding to the vertices $X$ and $Y$, respectively.

**Lemma 1.2.3.** (i) Let $m \geq 1$. If $x \in S^l_m e_X$ satisfies $x\alpha_i^* = 0$ for some $i$ then $x = 0$.

(ii) Assume that either $n \geq 3$, or $g \neq \lambda \cdot \text{id}$ (for any $\lambda \in k^*$), or the characteristic of $k$ is $\neq 2$. If $x^0 \in S^l_2 e_X$, $x^1 \in S^l_2 e_Y$ satisfy

$$\beta_i^* x^0 = - x^1 \beta_i^*, \quad \alpha_i^* x^1 = x^0 \alpha_i^*$$

for all $i$, then $x^0 = 0$ and $x^1 = 0$.

(iii) If a collection of elements $(x_i)$, where $x_i \in S^l_2 e_Y$, satisfies

$$\sum_{i,j} a_{ij} x_j \beta_i^* = 0,$$

then there exists $y \in S^l_1 e_X$ such that $x_j = y \alpha_j^*$ for each $j$.

**Proof.** (i) It is easy to see that the question does not depend on $g$, so we can assume $g = \text{id}$. In this case, we need to check that the element $x_1$ in the algebra $k \langle x_1, \ldots, x_n \rangle / (x_1^2 + \ldots + x_n^2)$ is not a right zero divisor. But in fact, the latter algebra is a domain by [39, Thm. 0.2].

(ii) It is easy to see that we can reformulate the question as follows. Given an $n$-dimensional vector space $V$ (in our case the space spanned by $(\beta_i^*)$) and elements $x^0 \in V^* \otimes V$, $x^1 \in V \otimes V^*$, such that

$$g(v) \otimes x^0 = - x^1 \otimes v \mod (\text{id} \otimes V + V \otimes \text{id}) \text{ for any } v \in V, \quad (1.2.1)$$

$$v^* \otimes \text{Ad}(g^{-1}) x^1 = x^0 \otimes g^*(v^*) \mod (\text{id} \otimes V^* + V^* \otimes \text{id}) \text{ for any } v \in V^*, \quad (1.2.2)$$

we should deduce that $x^0$ and $x^1$ are proportional to $id$. \hfill \Box
Assume first that \( n \geq 3 \). Then we claim that (1.2.1) alone implies that \( x^0 \) and \( x^1 \) are proportional to \( \text{id} \). Indeed, suppose we have
\[
g(v) \otimes x^0 + x^1 \otimes v = \text{id} \otimes A(v) + B(v) \otimes \text{id} \in V \otimes V^* \otimes V
\]
for all \( v \), for some operators \( A, B \in \text{End}(V) \). Taking the contraction in the third tensor component with \( v^* \in V^* \), we get the identity
\[
g(v) \otimes \langle x^0, v^* \rangle + \langle v, v^* \rangle x^1 = \langle A(v), v^* \rangle \text{id} + B(v) \otimes v^* \in V \otimes V^*.
\]
Thus, whenever \( \langle v, v^* \rangle = 0 \), the operator \( \langle A(v), v^* \rangle \text{id} \in \text{End}(V) \) is the sum of two operators of rank 1. Since \( n \geq 3 \), this implies that \( \langle A(v), v^* \rangle = 0 \). Hence, \( A(v) \) is proportional to \( v \) for any \( v \in V \), i.e., \( A = \lambda \cdot \text{id} \) for some \( \lambda \in k \). A similar argument shows that \( B = \mu \cdot g \) for \( \mu \in k \). Thus, subtracting some multiples of \( \text{id} \) from \( x^0 \) and \( x^1 \) we reduce ourselves to the situation when
\[
g(v) \otimes x^0 + x^1 \otimes v = 0 \quad \text{for any} \quad v \in V.
\]
Using contractions as above it is easy to deduce from this that \( x^0 = 0 \) and \( x^1 = 0 \).

Next, assume that \( n = 2 \). Then we are going to rewrite condition (1.2.1) by fixing a symplectic isomorphism \( s : V \to V^* \) and observing that \( (s \otimes \text{id}_V)(\wedge^2 V) \) and \( (\text{id}_V \otimes s)(\wedge^2 V) \) are precisely the lines spanned by the identity elements in \( V^* \otimes V \) and \( V \otimes V^* \). Thus, defining \( y^0, y^1 \in V \otimes V \) by
\[
x^0 = (s \otimes \text{id})(y^0), \quad x^1 = (\text{id} \otimes s)(y^1),
\]
we see that (1.2.1) is equivalent to the equation
\[
g(v) : q^0 = -q^1 \cdot v \quad \text{in} \quad S^2 V,
\]
where \( q^i \) is the image of \( y^i \) in \( S^2 V \). Assume first that \( g \) is not proportional to \( \text{id} \). Then the relation above implies that \( q^0 = q^1 = 0 \) in \( S^2 V \). Indeed, let us pick \( v_0 \neq 0 \) such that \( g(v_0) \) is not a multiple of \( v_0 \), and assume \( q^i \) are nonzero. Then we should have the following equations in the algebra \( SV \):
\[
q^0 = v_0 \cdot v', \quad q^1 = -g(v_0) \cdot v'
\]
for some \( v' \in V \), \( v' \neq 0 \). Then for any \( v \) we should have
\[
g(v) : v_0 = g(v_0) \cdot v \quad \text{in} \quad S^2 V.
\]
But this is a contradiction as soon as \( v \) is not proportional to \( v_0 \).

Finally, let us consider the case \( n = 2 \) and \( g = \lambda \cdot \text{id} \). Then the above argument gives
\[
q^1 = -\lambda q^0.
\]
Similarly, we can rewrite condition (1.2.2) for \( g = \lambda \cdot \text{id} \) as
\[
v \otimes y^1 = \lambda y^0 \otimes v,
\]
where \( v^* = s(v) \). Thus, we get
\[
q^1 = \lambda q^0.
\]
Since the characteristic is \( \neq 2 \), we deduce that \( q^0 = q^1 = 0 \).

(iii) This follows from the exactness of the direct summand of the Koszul complex,
\[
\ldots \to e_v S_1^* \otimes S_2^* e_X \to e_v S_1^* \otimes S_1^* e_X \to e_v S_3^* e_X \to 0
\]
(which holds since the algebra \( S^1 \) is Koszul). \( \square \)
1.3. Calculation of Hochschild cohomology and proof of Theorem 1.1.3. For a 
$\mathbb{Z}$-graded algebra $A$, we use the following convention for the bigrading on its Hochschild 
cohomology $HH(A)$. We denote by $CH^t(A)\{t\}$ the space of linear maps $A^{\otimes s} \to A$ of 
degree $t$. The corresponding bigraded piece in the Hochschild cohomology is denoted by 
$HH^{s+t}(A)\{t\}$ (then the upper grading is derived Morita invariant).

Let us consider the algebra $S = S(k^n, g)$ as above. We denote by $S\{m\}$ (resp., $S'\{m\}$) 
the graded components of $S$ (resp., $S'$) with respect to the grading given by $\deg(\alpha_i) = 0$, 
$\deg(\beta_i) = 1$ (resp., $\deg(\alpha^*_i) = 0$, $\deg(\beta^*_i) = -1$). We are interested in Hochschild 
cohomology of $S$ viewed as a graded algebra with this grading (note that this affects 
some signs).

**Theorem 1.3.1.** For $m \geq 0$, one has

$$HH^m(S)\{<-m\} = 0.$$ 

Assume in addition that either $n \geq 3$, or $g \neq \lambda \cdot \text{id}$ (for any $\lambda \in k^*$), or the characteristic 
of $k$ is $\neq 2$. Then

$$HH^1(S)\{-1\} = 0.$$ 

**Proof.** Recall that to compute the Hochschild cohomology of a Koszul $K$-algebra $A$ (where $K$ is commutative semisimple) we can use the Koszul resolution (see e.g.,[35, Sec. 3]). More precisely, we have a natural embedding

$$(A^m_1)^* \hookrightarrow A_1^{\otimes m}$$

(here and below all tensor products are over $K$), so that the image consists of the intersection of kernels of the partial multiplication maps

$$a_1 \otimes \ldots \otimes a_m \mapsto a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_m.$$ 

The corresponding subcomplex

$$A \otimes (A^1_1)^* \otimes A \subset A \otimes T^*(A_+) \otimes A$$

in the standard bar-resolution of $A$ by free $A - A$-bimodules is still exact. Thus, we get a resolution of the form

$$[\ldots \to A \otimes (A^m_m)^* \otimes A \xrightarrow{d_m} A \otimes (A^m_{m-1})^* \otimes A \to \ldots \to A \otimes (A^1_1)^* \otimes A \to A \otimes A] \to A.$$ 

Let $(v_i)$ be generators in $A_1$, $(v_i^*)$ the dual generators of $A_1^*$. Then the differential is given by

$$d_m(r \otimes \phi \otimes s) = \sum_i rv_i \otimes v_i^* \phi \otimes s + (-1)^m \sum_i r \otimes \phi v_i^* \otimes v_is,$$

where we use the $A^1$-bimodule structure on $(A^1_1)^*$ given by the operators dual to the left 
and right multiplication.

Assume now that $A$ has an additional grading $\deg$, induced by some $\mathbb{Z}$-grading on $A_1$.
If we are interested in the Hochschild cohomology of $A$ as a graded algebra with respect 
to this grading, at this point we need to be careful in using the appropriate signs. Namely, 
to compute the Hochschild cohomology $HH^*(A)$ we apply the functor $\text{Hom}_{A\otimes A^op}(?, A)$ to 
the above resolution and use the identification

$$A^m_1 \otimes A \simeq \text{Hom}((A^1_1)^*, A) \simeq \text{Hom}_{A\otimes A^op}(A \otimes (A^m_1)^* \otimes A, A)$$

Note that this affects some signs. 


under which an element $c \in \text{Hom}((A_m^l)^*, A)$ corresponds to the composition of $\text{id} \otimes c \otimes \text{id}$ with the multiplication $\mu$ in $A$. Now we have to use the Koszul sign convention:

$$\mu(\text{id} \otimes c \otimes \text{id})(r \otimes \phi \otimes s) = (-1)^{\text{deg}(c) \text{deg}(r)} rc(\phi)s.$$ 

Thus, we get the complex computing the Hochschild cohomology of $A$,

$$A \rightarrow A^1 \otimes A \rightarrow \ldots \rightarrow A^m \otimes A \xrightarrow{\delta_m} A^1_{m+1} \otimes A \rightarrow \ldots$$

with the differential

$$\delta_m(\psi \otimes s) = (-1)^{(\text{deg}(\psi) + \text{deg}(s)) \text{deg}(v_i)} \sum_i \psi v_i^* \otimes v_i s + (-1)^{m+1} \sum_i v_i^* \psi \otimes s v_i.$$

Here we assume that the basis $(v_i)$ is homogeneous with respect to deg.

We can apply this procedure in our case since $S$ is Koszul, with the generators $(\alpha_i, \beta_i)$ (see Lemma 1.2.1). We are interested in the components $HH^*(S)$ (see Lemma 1.2.1). As explained above, these spaces can be computed as cohomology of the complex $(S^l \otimes S^l)\{j\}$ with respect to the differential

$$\delta(\psi \otimes s) = \sum_i (\psi \alpha_i^* \otimes \alpha_i s + (-1)^i \psi \beta_i^* \otimes \beta_i s) + (-1)^{m+1} \sum_i (\alpha_i^* \psi \otimes s \alpha_i + \beta_i^* \psi \otimes s \beta_i),$$

where $\psi \in S^l_m$.

Since $S\{j\} = 0$ for $j \neq 0, 1$, we have

$$(S^l \otimes S)\{j\} = S^l\{j\} \otimes S\{0\} \oplus S^l\{j-1\} \otimes S\{1\}.$$ 

Note also that because $\alpha_i^*$ and $\beta_j^*$ have to alternate in any nonzero monomial in $S^l_m$, we have $S^l_m\{j\} = 0$ unless $m \in \{-2j-1, -2j, -2j+1\}$. This immediately implies the vanishing

$$HH^m(S)\{-m-1\} = 0$$

for any $m$.

For $m \geq 0$ the space $HH^m(S)\{-m-1\}$ is identified with the kernel of the map

$$\delta: S^l_{2m+1}\{-m-1\} \otimes S\{0\} \rightarrow S^l_{2m+2}\{-m-1\} \otimes S\{0\}.$$ 

But $S^l_{2m+1}\{-m-1\} \otimes S\{0\} = S^l_{2m+1}e_X \otimes e_X$, and for $x \in S^l_{2m+1}e_X$ we have

$$\delta(x \otimes e_X) = \sum_i x \alpha_i^* \otimes \alpha_i.$$ 

Thus, Lemma 1.2.3(i) implies that this kernel is zero.

Next, $HH^1(S)\{-1\}$ is the cohomology in the middle term of

$$S^l_1\{-1\} \otimes S\{0\} \rightarrow S^l_2\{-1\} \otimes S\{0\} \rightarrow S^l_3\{-1\} \otimes S\{0\} \oplus S^l_3\{-2\} \otimes S\{1\}.$$ 

(1.3.1)

An element of $S^l_2\{-1\} \otimes S\{0\}$ has form

$$x = x^0 \otimes e_X + x^1 \otimes e_Y + \sum_j x_j \otimes \alpha_j,$$
where \( x^0 \in \mathcal{S}_i e_X \), \( x^1 \in \mathcal{S}_i e_Y \) and \( x^i \in \mathcal{S}_i e_Y \). We have
\[
\delta(x) = \sum_i \left( x^0 \alpha_i^* \otimes \alpha_i - x^1 \beta_i^* \otimes \beta_i \right) - \sum_{i,j} a_{ij} x_j \beta_i^* \otimes \xi_0 - \sum_i \left( \beta_i^* x^0 \otimes \beta_i + \alpha_i^* x^1 \otimes \alpha_i \right).
\]
Thus, \( \delta(x) = 0 \) if and only if
\[
\beta_i^* x^0 = -x^1 \beta_i^*, \quad \alpha_i^* x^1 = x^0 \alpha_i^* \text{ for all } i,
\]
\[
\sum_{i,j} a_{ij} x_j \beta_i^* = 0.
\]
The coboundaries come from elements \( \mathcal{S}_i \{-1\} \otimes \mathcal{S}\{0\} = \mathcal{S}_i e_X \otimes e_X \) and have form
\[
\delta(y \otimes e_X) = \sum_i y \alpha_i^* \otimes \alpha_i.
\]
Thus, Lemma 1.2.3(ii)(iii) implies that (1.3.1) is exact (under our assumptions), and hence, \( HH^1(\mathcal{S})\{-1\} = 0 \).

Remark 1.3.2. In [31, Sec. (2c)] the computation similar that of Theorem 1.3.1 is done in the case \( n = 2, g = \text{id}, k = \mathbb{C} \). In this case one also has \( HH^2(\mathcal{S})\{-1\} = 0 \), and \( HH^2(\mathcal{S})\{-2\} \) can be identified with the space of binary quartic polynomials. As explained in [31, Sec. (2f)], this means that all minimal \( A_{\infty} \)-structures on the algebra \( \mathcal{S}(\mathbb{C}^2, \text{id}) \) are realized by double coverings of \( \mathbb{P}^1 \).

Proof of Theorem 1.1.3. We apply [26, Thm. 2.2.6] to the family \( \mathcal{S}_n, n \geq 3 \) (resp., \( \mathcal{S}_{2,U} \)). More precisely we use the following vanishing of components of Hochschild cohomology for any algebra \( \mathcal{S} = \mathcal{S}(k^n, g) \) (implied by Theorem 1.3.1):
\[
HH^{\leq 1}(\mathcal{S})\{< 0\} = HH^2(\mathcal{S})\{< -2\} = 0.
\]

2. PAIRS OF 1-SPHERICAL OBJECTS AND NONCOMMUTATIVE ALGEBRAS

2.1. Spherical objects and spherical twists. Recall (see [32], [30, I.5]) that an object \( X \) of a \( k \)-linear \( A_{\infty} \)-category \( \mathcal{C} \), where \( k \) be a field, is called \( n \)-spherical if \( \text{Hom}^i(X, X) = 0 \) for \( i \neq 0, n \), \( \text{Hom}^0(X, X) = \text{Hom}^n(X, X) = k \), and for any object \( Y \) of \( \mathcal{C} \) the pairing between the morphism spaces in the cohomology category \( H^* \mathcal{C} \),
\[
\text{Hom}^{n-i}(Y, X) \otimes \text{Hom}^i(Y, Y) \to \text{Hom}^1(X, X),
\]
induced by \( m_2 \), is perfect.

We need the following generalization of this notion to the case of an \( S \)-linear \( A_{\infty} \)-category, where \( S \) is a commutative ring (our definition is not the most general possible: we impose rather strong assumptions on the hom-complexes).

Definition 2.1.1. Let \( X \) be an object of an \( S \)-linear \( A_{\infty} \)-category \( \mathcal{C} \). Assume that for any \( Y \) the complexes \( \text{hom}(X, Y) \) and \( \text{hom}(Y, X) \) are bounded complexes of finitely generated projective \( S \)-modules. Then \( X \) is called \( n \)-spherical if \( \text{Hom}^i(X, X) = 0 \) for \( i \neq 0, n \), \( \text{Hom}^0(X, X) = S \cdot \text{id}_X \), \( \mathcal{L}_X := \text{Hom}^n(X, X) \) is a locally free \( S \)-module of rank 1, and
for any $Y$ in $\mathcal{C}$ the following composed chain map of complexes of $S$-modules is a quasi-isomorphism:
\[
\hom(Y, X) \to \hom(X, Y)^\vee \otimes_S \hom(X, X) \to \hom(X, Y)^\vee \otimes \tau_{\geq n} \hom(X, X).
\] (2.1.1)

Here the first arrow induced by $m_2$, while the second comes from the natural map $\hom(X, X) \to \tau_{\geq n} \hom(X, X)$, where $\tau_{\geq n}$ is the truncation functor. Also, $P^\vee = \Hom_S(P, S)$ is the dual of a finitely generated projective module $P$.

Note that the complex $\tau_{\geq n} \hom(X, X)$ is bounded, has the only cohomology at the left-most term and all of its subsequent terms are finitely generated projective $S$-modules. This implies that the left-most term is also finitely generated projective and there exists a homotopy equivalence
\[
\tau_{\geq n} \hom(X, X) \to \Hom^n(X, X)[-n] = \mathcal{L}_X[-n].
\]
Fixing such an equivalence we can view the map (2.1.1) as a chain map
\[
s_Y : \hom(Y, X) \to \hom(X, Y)^\vee \otimes \mathcal{L}_X[-n]
\] (2.1.2)

Let $\Tw(\mathcal{C})$ denote the category of twisted complexes over $\mathcal{C}$ (see e.g., [9, Sec. 7.6]).

**Lemma 2.1.2.** An $n$-spherical object $X$ of $\mathcal{C}$ remains $n$-spherical in $\Tw(\mathcal{C})$.

**Proof.** First, note that for any twisted complex $Y$ we still have that $\hom(X, Y)$ and $\hom(Y, X)$ are bounded complexes of finitely generated projective $S$-modules. Now assume we are given an exact triangle $Y' \to Y \to Y'' \to Y'[1]$ in $\Tw(\mathcal{C})$, such that the maps (2.1.2) for $Y'$ and $Y''$ are quasi-isomorphisms. Then we have a morphism of exact triangles of $S$-modules
\[
\begin{array}{cccccccc}
\hom(Y'', X) & \to & \hom(Y, X) & \to & \hom(Y', X) & \to & \ldots \\
\downarrow s_{Y''} & & \downarrow s_Y & & \downarrow s_{Y'} & & \\
\hom(X, Y'') \otimes \mathcal{L}_X[-n] & \to & \hom(X, Y)^\vee \otimes \mathcal{L}_X[-n] & \to & \hom(X, Y')^\vee \otimes \mathcal{L}_X[-n] & \to & \ldots
\end{array}
\]
so the fact that $s_{Y'}$ and $s_{Y''}$ are quasi-isomorphisms imply that $s_Y$ is also a quasi-isomorphism. Since every object of $\Tw(\mathcal{C})$ is an iterated extension of shifts of objects of $\mathcal{C}$, the assertion follows. \qed

Given an $n$-spherical object $E$, we can define (see [32], [30, I.5]) the twist and the adjoint twist $A_\infty$-functors
\[
T_E, T'_E : \Tw(\mathcal{C}) \to \Tw(\mathcal{C})
\]
by
\[
T_E(X) = \Cone(\hom(E, X) \otimes E \xrightarrow{ev} X), \quad T'_E(X) = \Cone(X \xrightarrow{ev'} \hom(X, E)^\vee \otimes E)[-1].
\]
The proof of [32, Prop. 2.10] extends to our situation to prove that $T'_ET_E$ and $T_ET'_E$ are isomorphic to identity in the homotopy category of functors from $\Tw(\mathcal{C})$ to itself.
2.2. \textit{n-pairs of 1-spherical objects}. $A_\infty$-structures we want to consider are related to the following pairs of objects in $A_\infty$-categories.

\textbf{Definition 2.2.1.} Let $\mathcal{C}$ be a $S$-linear $A_\infty$-category, where $S$ is a commutative ring.

(i) We call a pair of 1-spherical objects $(E, F)$ in $\mathcal{C}$ an \textit{n-pair} if $\text{Hom}^*(E, F)$ is concentrated in degree 0 and is isomorphic to $S^n$. In addition we require that $\text{Hom}^1(F, F) \simeq S$. Note that this implies that $\text{Hom}^1(F, E)$ is a free $S$-module of rank $n$ concentrated in degree 1. We say that $(E, F)$ a \textit{symmetric n-pair} if in addition $\text{Hom}^1(E, E) \simeq S$ and the two perfect pairings

$$\text{Hom}^1(F, E) \otimes_S \text{Hom}^0(E, F) \to \text{Hom}^1(E, E) \quad \text{and} \quad \text{Hom}^0(E, F) \otimes_S \text{Hom}^1(F, E) \to \text{Hom}^1(F, F)$$

(coming from the conditions that $E$ and $F$ are 1-spherical) lead to two dualities $\text{Hom}^1(F, E) \simeq \text{Hom}^0(E, F)^\vee$ that differ by a scalar in $S^*$.

(ii) A \textit{weak n-pair} in $\mathcal{C}$ is a pair of objects $(E, F)$ in $\mathcal{C}$ such that $F$ is 1-spherical with $\text{Hom}^1(F, F) \simeq S$, $E$ satisfies $\text{Hom}^0(E, E) \simeq S$, $\text{Hom}^{\neq 0, 1}(E, E) = 0$, $\mathcal{L}_E := \text{Hom}^1(E, E)$ is a locally free $S$-module of rank 1, and $\text{Hom}^*(E, F) = \text{Hom}^0(E, F) \simeq S^n$. An \textit{enhanced weak n-pair} is a weak n-pair $(E, F)$ together with chosen isomorphisms $\text{Hom}^0(E, F) \simeq S^n$ and $\text{Hom}^1(F, F) \simeq S$.

Note that for an enhanced weak n-pair, the second of the pairings (2.2.1) is perfect, so it defines an isomorphism $\text{Hom}^1(F, E) \simeq V^\vee$, where $V := \text{Hom}^0(E, F)$. Then the first of the pairings (2.2.1) has form

$$\text{Hom}^1(F, E) \otimes_S \text{Hom}^0(E, F) \simeq V^\vee \otimes V \to \mathcal{L}_E : (v^\vee, v) \mapsto \langle v^\vee, gv \rangle$$

(2.2.2)

for a unique $g \in \text{End}_S(V) \otimes \mathcal{L}_E$.

\textbf{Lemma 2.2.2.} Let $(E, F)$ be a weak n-pair in $\mathcal{C}$, and assume that $\mathcal{C}$ is generated by $(E, F)$. Then $(E, F)$ is an n-pair (i.e., $E$ is spherical) in $\mathcal{C}$ if and only if the element $g \in \text{End}_S(V) \otimes \mathcal{L}_E$ defined by (2.2.2) is invertible.

\textbf{Proof.} By Lemma 2.1.2, $E$ is spherical in $\mathcal{C}$ if and only if the pairing (2.2.2) is perfect, which is equivalent to $g$ being invertible. \hfill $\square$

\textbf{Example 2.2.3.} Let $D^b(C)$ be the (enhanced) derived category of coherent sheaves on an elliptic curve $C$ over an algebraically closed field $k$. Then 1-spherical objects in $D^b(C)$ are (up to shift) either simple vector bundles or the skyscraper sheaves $\mathcal{O}_p$. The group of autoequivalences of $D^b(C)$ acts transitively on them, so any n-pair of 1-spherical objects can be transformed by an autoequivalence into a pair $(E, \mathcal{O}_p)$, where $E$ is an simple vector bundle of rank $n$. It is easy to see that any such n-pair is special.

Given $g \in \text{PGL}_n(S)$ and a minimal $A_\infty$-structure on $\mathcal{S} = \mathcal{S}(S^n, g)$ we get an n-pair of 1-spherical objects in the corresponding category of right $A_\infty$-modules over $\mathcal{S}$, $(e_x \cdot \mathcal{S}, e_y \cdot \mathcal{S})$.

Conversely, starting with an enhanced n-pair $(E, F)$ in an $S$-linear $A_\infty$-category, let us consider the full $A_\infty$-subcategory with the objects $E$ and $F$. Due to our assumption that $\text{hom}(E, E)$, $\text{hom}(E, F)$, $\text{hom}(F, F)$ and $\text{hom}(F, E)$ are bounded complexes of projective modules, we can apply the homological perturbation to get an equivalent minimal
$A_{\infty}$-structure on this subcategory. Furthermore, as was explained before, we can identify $\text{Hom}^1(F, E)$ with $V^\vee$, where $V = \text{Hom}^0(E, F) \simeq S^n$, so that the second of the compositions (2.2.1) becomes the standard pairing between $V$ and $V^\vee$, while the first has the form (2.2.2) for some $g \in \text{GL}_n(S) \otimes L_E$. Thus, we get a minimal $A_{\infty}$-structure on $\mathcal{S}(R^n, g)$.

### 2.3 Some properties of the algebra $\mathcal{E}(S^n, g)$

**Lemma 2.3.1.** Let $S$ be a commutative ring, $\mathcal{L}$ a locally free $S$-module of rank 1. Assume that $n \geq 2$ and $g \in \text{Mat}_n(S) \otimes \mathcal{L}$ is such that there exists $h \in \text{Mat}_n(S) \otimes \mathcal{L}^{-1}$ with $\text{tr}(gh) = 1$.

(i) Assume that either $n \geq 3$ or $n = 2$ and $\text{tr}(gh') = 1$ for some $h' \in \text{Mat}_2(S) \otimes \mathcal{L}^{-2}$. Then the algebra $\mathcal{E}(S^n, g)$ is generated over $S$ by degree 1 elements.

(ii) $\mathcal{E}(S^n, g)$ is generated by degree 1 and degree 2 elements.

(iii) Assume that $g$ is invertible. Then the algebra $\mathcal{E}(S^n, g)$ is Koszul.

**Proof.** (i) Recall that $\mathcal{E}(S^n, g)_1$ is the subspace $\text{End}_g(S^n)$ of elements $a \in \text{Mat}_n(S)$ such that $\text{tr}(ga) = 0$. The existence of $h$ such that $\text{tr}(gh) = 1$ implies that $\text{End}_g(S^n)$ is a direct summand in $\text{Mat}_n(S)$.

We have to prove the surjectivity of the map

$$\text{End}_g(S^n) \otimes_S \text{End}_g(S^n) \to \text{Mat}_n(S)$$  \hspace{1cm} (2.3.1)

induced by the product in $\text{Mat}_n(S)$.

First, we claim that it is enough to prove the assertion in the case when $S$ is a field. We can easily reduce to the case when $S$ is local. Now let $M$ denote the cokernel of the product map (2.3.1). Note that the construction of $M$ is compatible with any change of scalars $S \to S'$. Thus, the case of the field implies that $M/mM = 0$, where $m \subset S$ is a maximal ideal. By Nakayama lemma, this gives that $M = 0$.

Thus, it is enough to consider the case when $S = k$, where $k$ is a field. In this case we will prove a more general statement that for any pair of nonzero elements $g_1, g_2$ one has

$$\text{End}_{g_1}(k^n) \cdot \text{End}_{g_2}(k^n) = \text{Mat}_n(k),$$

where in the case $n = 2$ we additionally require that $g_2g_1 \neq 0$. Since the question is that certain vectors generate $\text{Mat}_n(k)$ as a linear space, without loss of generality we can assume $k$ to be algebraically closed.

Note that for any $a, b \in \text{GL}_n(k)$ one has

$$a \cdot \text{End}_{g_1}(k^n) \cdot \text{End}_{g_2}(k^n) \cdot b = \text{End}_{g_1a^{-1}}(k^n) \cdot \text{End}_{b^{-1}g_2}(k^n).$$

Thus, we can replace $g_1$ by $g_1a^{-1}$ and $g_2$ by $b^{-1}g_2$. In the case when $g_1$ and $g_2$ are invertible this reduces the statement to the case $g_1 = g_2 = 1$, which is easy to check.

Next, we observe that for any nonzero $g \in \text{Mat}_n(k)$ there exists $a \in \text{GL}_n(k)$ such that $\text{tr}(ag) = 0$. Indeed, otherwise, the entire hyperplane $\text{End}_g(k^n)$ would be contained in the irreducible hypersurface $\det(a) = 0$. Thus, we can assume that $\text{tr}(g_1) = \text{tr}(g_2) = 0$. In this case we have $1 \in \text{End}_{g_1}(k^n)$ and $1 \in \text{End}_{g_1}(k^n)$. Hence,

$$\text{End}_{g_1}(k^n) \subset \text{End}_{g_1}(k^n) \cdot \text{End}_{g_2}(k^n).$$

Thus, the only case when this subspace is not the entire $\text{Mat}_n(k)$ is when $\text{End}_{g_1}(k^n) = \text{End}_{g_2}(k^n)$, i.e., $g_1$ and $g_2$ are proportional. In this case we get that $\text{End}_{g_1}(k^n)$ is a
subalgebra. As was observed above, we can assume that $g_1$ is degenerate. Let us choose a basis in which the last row of $g_1$ vanishes. Let $g'$ denote the $(n-1) \times (n-1)$-submatrix of $g_1$ obtained by deleting the last row and last column.

Assume first that $g' = 0$. Then $\text{End}_{g_1}(k^n)$ contains the maximal parabolic subalgebra of endomorphisms preserving the hyperplane spanned by the first $n - 1$ basis vectors. In the case $n \geq 3$ this implies that $\text{End}_{g_1}(k^n)$ cannot be a subalgebra, since it would be strictly bigger than the maximal parabolic subalgebra. In the case $n = 2$ if $g' = 0$ then $g_1^2 = 0$ which contradicts the assumption that $g_2g_1 \neq 0$ (recall that $g_1$ and $g_2$ are proportional).

Next, consider the case $g' \neq 0$. Let $e_{ij}$ denote the standard matrices with 1 as the $(i, j)$-entry. Then for every $i$ we have $e_{in} \in \text{End}_{g_1}(k^n)$, and for every $j \leq n - 1$ there exists a matrix $A_j$ with zero last row such that $e_{nj} + A_j \in \text{End}_{g_1}(k^n)$. But then $e_{ij} = e_{in} \cdot (e_{nj} + A_j)$, so we deduce that $\text{End}_{g_1}(k^n) \cdot \text{End}_{g_1}(k^n)$ is $\text{Mat}_n(k)$.

(ii) We have to show that the product map

$$\text{End}_{g}(S^n) \otimes_S \text{Mat}_n(S) \to \text{Mat}_n(S)$$

is surjective. As before, it is enough to consider the case when $S = k$ is a field. Furthermore, as in part (i) we reduce to the case when $\text{tr}(g) = 0$, so that $1 \in \text{End}_g(S^n)$, in which case the assertion is clear.

(iii) It is easy to check that the algebra $\mathcal{E}(S^n, g)$ is quadratic dual to the second Veronese subalgebra of the algebra $\mathcal{S}(S^n, g)$, corresponding to the vertex $X$ (i.e., one considers paths of even length starting from $X$). Since the latter algebra is Koszul by Lemma 1.2.1, the result follows (see [27, Prop. 2.2(i)]).

In the next result we consider derivations of $\mathcal{E}(S^n, g)$ as an ungraded algebra (i.e., there is no Koszul sign in the Leibnitz rule).

**Proposition 2.3.2.** Assume that one of the following two conditions holds:

- $n \geq 3$ and $g$ is invertible;
- $\text{tr}(g)$ is a generator of $\mathcal{L}$, and there exists $h_1 \in \text{GL}_n(S)$ with $\text{tr}(gh_1) = 0$.

Then any derivation $\mathcal{E}(S^n, g) \to \mathcal{E}(S^n, g)$ of degree $m \leq -1$ is zero.

**Proof.** The problem is local, so we can assume that $\mathcal{L} = S$.

Let us set for brevity $\mathcal{E} := \mathcal{E}(S^n, g)$, $\mathcal{E}' := \text{End}(V)[z, z^{-1}]$. By the assumption, we can fix an element $h_1 \in \mathcal{E}_1$ such that the operators $\mathcal{E}'_i \to \mathcal{E}'_{i+1}$ of left and right multiplication by $h_1$ are invertible for $i \in \mathbb{Z}$. Now assume we have a derivation $D : \mathcal{E} \to \mathcal{E}$ of degree $m \leq -1$. First, we are going to extend $D$ to a derivation $D' : \mathcal{E}'_{\geq 1} \to \mathcal{E}'$ of degree $m$. For this we compose $D$ with the embedding $\mathcal{E} \hookrightarrow \mathcal{E}'$ and then set for $x \in \mathcal{E}'_{\geq 1}$,

$$D'(x) = D(xh_1)h_1^{-1} - xD(h_1)h_1^{-1},$$

where we use the operation of multiplication by $h_1^{-1}$ as a degree $-1$ map $\mathcal{E}'_{\geq 1} \to \mathcal{E}'$. Note that the expression in the right-hand side is well-defined since $h_1 \in \mathcal{E}_1$ and $xh_1 \in \mathcal{E}'_{\geq 2} = \mathcal{E}_{\geq 2}$. Also we have $D' = D$ on $\mathcal{E}$. Before checking that $D'$ is a derivation we observe that for $x \in \mathcal{E}'_i$, $i \geq 1$, one has

$$D'(x) = D''(x) = h_1^{-1}D(h_1x) - h_1^{-1}D(h_1)x.$$

Indeed, this can be checked by applying the Leibnitz identity to write $D(h_1xh_1)$ in two ways (note that $h_1x \in \mathcal{E}$ and $xh_1 \in \mathcal{E}$). Now we can prove the Leibnitz identity for
$D' = D''$. Namely, it is enough to check that for $x_1, x_2 \in E_{\geq 1}'$ one has

$$D(x_1x_2) = D''(x_1)x_2 + x_1D'(x_2).$$

It is easy to see that this is equivalent to the identity

$$D(h_1x_1)x_2h_1 + h_1x_1D(x_2h_1) = D(h_1)x_1x_2h_1 + h_1D(x_1x_2)h_1 + h_1x_1x_2D(h_1),$$

obtained by writing $D(h_1x_1x_2h_1)$ in two ways.

Next, we claim that there exists $a \in \text{End}(V)$ and $s \in S$ such that

$$D'(x) = \text{ad}(az^m) + s \cdot z^{m+1} \frac{d}{dz}.$$ 

Indeed, as above, we can extend $D'$ to a derivation on $\text{End}(V)[z, z^{-1}]$ of degree $m$. Using Morita equivalence of the latter ring with $S[z, z^{-1}]$ we get the result. Let us first assume that $m = -1$. Our goal is to show that $s = 0$ and $a$ is proportional to the identity. To this end we investigate the condition

$$D'(\text{End}_g(V)z) \subset S \cdot \text{id},$$

which means that for any $x \in \text{End}_g(V)$ one has

$$(a + s)x - xa \in S \cdot \text{id},$$

Equivalently, for any $y \in \text{End}(V)$ with $\text{tr}(y) = 0$ and any $x \in \text{End}_g(V)$, one has

$$\text{tr}((a + s)xy - xay) = \text{tr}(x[y(a + s) - ay]) = 0.$$ 

Equivalently, we should have

$$y(a + s) - ay \in S \cdot g$$

whenever $\text{tr}(y) = 0$. 

Assume first $n \geq 3$ and $g$ is invertible. Then substituting $y = e_{i_0j_0}$ with $i_0 \neq j_0$ in (2.3.2) we get an equality of the form

$$e_{i_0j_0}(a + s) - ae_{i_0j_0} = \lambda \cdot g.$$ 

We claim that this is possible only when $\lambda = 0$. Indeed, for every $j \neq j_0$ we get

$$\lambda \cdot e_j = \mu_j \cdot g^{-1}e_{i_0}$$

for some $\mu_j \in S$. Let $g^{-1}e_{i_0} = \sum b_ie_i$. Then we have a system of equations

$$\mu_jb_j = \lambda, \quad \mu_jb_i = 0 \text{ for } i \neq j, j \neq j_0.$$ 

Since $g \cdot g^{-1} = \text{id}$, we have some $(a_i)$ in $S$ with $\sum b_ia_i = 1$. Thus, we deduce

$$\mu_j = \lambda a_j.$$ 

Plugging back in the above equation, we get that $\lambda \cdot I = 0$, where $I \subset S$ is the ideal generated by $(a_jb_j - 1)_{j \neq j_0}$ and $(a_i b_i)_{i \neq j, j \neq j_0}$. Now choosing a pair $i \neq j$ in $[1, n] \setminus \{j_0\}$ (this is possible since $n \geq 3$), we obtain

$$I \supset (a_i b_i - 1, a_j b_j - 1, a_j b_i) = (1),$$

and hence, $\lambda = 0$.

Thus, we derive that for every $i \neq j$ one has

$$e_{ij}(a + s) - ae_{ij} = 0$$
This immediately implies that $a$ is diagonal, and the diagonal entries $(a_{ii})$ satisfy $a_{jj} - a_{ii} = s$ for $i \neq j$. Using again the assumption $n \geq 3$, we obtain that $s = 0$ and $a$ is proportional to id.

Next, let us consider the case when $\text{tr}(g)$ is invertible (but $g$ is not necessarily invertible). Considering traces of both sides of (2.3.2) we derive that

$$y(a + s) - ay = 0 \text{ whenever } \text{tr}(y) = 0.$$ 

Now substituting $y = e_{ij}$ for $i \neq j$ one can easily derive that $a$ has to be a diagonal matrix. As we have above, in the case $n \geq 3$ this implies that $s = 0$ and $a$ is proportional to id. If $n = 2$ then taking into account the equation for the diagonal $y$ with entries 1 and $-1$, we get the same conclusion.

In the case $m \leq -2$ we should have

$$D'(\text{End}_g(V)z) = 0,$$

i.e., $(a + s)x - xa = 0$ for any $x \in \text{End}_g(V)$. As we have seen above, this implies that $a = s = 0$.

**Remark 2.3.3.** One can also check that there are no nonzero derivations $\mathcal{E}(k^2, g) \to \mathcal{E}(k^2, g)$ of degree $-1$ provided $k$ is a field of characteristic $\neq 2$ and $g$ is invertible.

**Example 2.3.4.** Assume that $n = 2$ and $2 = 0$ in $S$, and let us take $g = \text{id}$. Then there exist nontrivial derivations of $\mathcal{E}(S^2, \text{id})$ of degree $-1$. More precisely, for any $2 \times 2$ matrix $a$ the derivation $\text{ad}(az^{-1}) + \text{tr}(a) \frac{d}{dz}$ of Mat$_2(S)[z, z^{-1}]$ restricts to a derivation of $\mathcal{E}(S^2, \text{id})$.

This essentially amounts to the identity

$$[y, a] = \text{tr}(a)y + \text{tr}(ay)\text{id}$$

for $2 \times 2$-matrices $a$ and $y$ such that $\text{tr}(y) = 0$ (it only holds because $2 = 0$ in $S$).

2.4. **The algebra associated with a pair of 1-spherical objects.** Let $(E, F)$ be a weak $n$-pair with $\text{Hom}^0(E, F) = V \simeq S^n$, equipped with the trivialization

$$\text{Hom}^1(F, F) \simeq S.$$

We denote by $\xi_F \in \text{Hom}^1(F, F)$ the corresponding generator.

We are going to associate to these data an $S$-algebra with an increasing exhaustive filtration $(F_n A)$, together with an isomorphism of graded $S$-algebras

$$\text{gr}^F A = \bigoplus_{n \geq 0} F_n A / F_{n-1} A \simeq \mathcal{E}(V, g)^{op},$$

where $g \in \text{End}(V) \otimes L_E$ is defined as before. Namely, we use the second of the pairings (2.2.1) to identify $\text{Hom}^1(F, E)$ with $V^\vee$, and define $g$ so that the first pairing takes the form (2.2.2).
We will see also that (under some mild assumptions) the filtered algebra \((A, F_\bullet A)\) is determined by the following higher products:

\[
m_3 : V \otimes V^\vee \otimes V \to V \simeq \Hom^0(E, F) \otimes \Hom^1(F, E) \otimes \Hom^0(E, F) \to \Hom^0(E, F) = V,
\]

\[
m_3 : V^\vee \otimes V \otimes V^\vee \simeq \Hom^1(F, E) \otimes \Hom^0(E, F) \otimes \Hom^1(F, E) \to \Hom^1(F, E) \simeq V^\vee,
\]

\[
m_3 : V^\vee \otimes V \simeq \Hom^1(F, E) \otimes \Hom^1(F, E) \otimes \Hom^0(E, F) \to \Hom^1(E, E) = \mathcal{L}_E,
\]

\[
m_4 : V^\vee \otimes V \otimes V^\vee \otimes V \simeq \Hom^1(F, E) \otimes \Hom^0(E, F) \otimes \Hom^1(F, E) \otimes \Hom^0(E, F) \to \Hom^0(E, E) = S.
\]

Let us define the maps \(r, r' : \End(V) \to \End(V)\) by

\[
r(v \otimes v^*) = \sum_i m_3(e_i, v^*, v) \otimes e_i^*, \quad r'(v \otimes v^*) = e_i \otimes \sum_i m_3(v^*, v, e_i^*),
\]

for \(v \in V, v^* \in V^\vee\), where \((e_i)\) and \((e_i^*)\) are dual bases of \(V\) and \(V^\vee\). Similarly, we define \(s : \End(V) \to \mathcal{L}_E\) by

\[
s(v \otimes v^*) = m_3(v^*, \xi_F, v).
\]

**Theorem 2.4.1.** Let \((E, F)\) be a weak \(n\)-pair in a minimal \(S\)-linear \(A_\infty\)-category \(C\), such that \(\Hom^0(E, F) = V \simeq S^n\), where \(n \geq 2\). Let us fix a trivialization \(\Hom^1(F, F) \simeq S\), and let \(g \in \End(V) \otimes \mathcal{L}_E\) be the corresponding element defined using pairings (2.2.1). Assume in addition that there exists \(h \in \End(V) \otimes \mathcal{L}_E^{-1}\) such that \(\text{tr}(gh) = 1\). Set \(E_i = T^i(E) \in \Tw(C)\), where \(T = T_F\) is the spherical twist with respect to \(F\). Let us consider the graded associative algebra

\[
\mathcal{R} = \mathcal{R}_{T,E} := \bigoplus_{n \geq 0} \Hom(E_0, E_n),
\]

with the product \(ab = T^i(a) \circ b\), where \(b \in \Hom(E_0, E_i), a \in \Hom(E_0, E_j)\). Then

(i) \(\mathcal{R}\) is canonically isomorphic to the Rees algebra of a filtered algebra \((A, F_\bullet A)\) equipped with an isomorphism \(\gr^F(A) \simeq \mathcal{E}(V, g)^{op} \simeq \mathcal{E}(V^\vee, g^*)\). In addition, \(\Hom^{\neq 0}(E_0, E_n) = 0\) for \(n > 0\).

(ii) There exist embeddings \(\End_g(V) \hookrightarrow \mathcal{R}_1\) and \(\End(V) \hookrightarrow \mathcal{R}_2\), such that

\[
\mathcal{R}_1 = \End_g(V) \oplus S \cdot t, \quad \mathcal{R}_2 = \End(V) \oplus \End_g(V) \cdot t \oplus S \cdot t^2,
\]

where \(t\) is the central element of degree 1 corresponding to the isomorphism with the Rees algebra. With respect to these decompositions, for \(a, b \in \End_g(V) \subset \mathcal{R}_1\) one has

\[
a \cdot b = ba + [r(b)a + br'(a) + s(ba)h]t + m_4(a \otimes b)t^2.
\]

Here we use the higher products (2.4.2) and the corresponding maps \(r, r', s\) (see (2.4.3), (2.4.4)).

**Proof.** It will be notationally convenient for a while not to use the trivialization of \(\Hom^1(F, F)\), so let us set \(L := \Ext^1(F, F)\).

For an \(S\)-module \(M\), we set \(ML^i := M \otimes L^\otimes i\). Note that the second of the pairings (2.2.1) induces an identification \(\Hom^1(F, E) \simeq V^\vee L\).
Step 1. We start by finding explicit twisted complexes representing $E_i$. Namely, let us denote by $E_i$, for $i \geq 1$, the following twisted complex:

\[
\text{Hom}^1(F, E)L^{i-1} \otimes F \xrightarrow{\delta_i} \text{Hom}^1(F, E)L^{i-2} \otimes F \xrightarrow{\delta_{i-1}} \ldots \xrightarrow{\delta_2} \text{Hom}^1(F, E) \otimes F \xrightarrow{\delta_1} E.
\]

(2.4.6)

Here the differentials $\delta_i$ with $i > 1$ are induced by the evaluation maps $L \otimes F \to F[1]$, while the differential $\delta_1 : \text{Hom}^1(F, E) \otimes F \to F[1]$ is also the evaluation map.

We are going to construct the homotopy equivalences $T(E_i) \simeq E_{i+1}$. Note that for $i = 0$ we have $T(E_0) = T(E) = E_1$ by the definition of the twist functor $T = T_F$. The complex $\text{Hom}(F, E_i)$ has form

\[
\begin{array}{cccc}
\text{Hom}^1(F, E)L^{i-1} \otimes id_F & \text{Hom}^1(F, E)L^{i-1} \otimes id_F & \ldots & \text{Hom}^1(F, E) \otimes id_F \\
\text{Hom}^1(F, E)L^i & \text{Hom}^1(F, E)L^{i-1} & \ldots & \text{Hom}^1(F, E)L & \text{Hom}^1(F, E)
\end{array}
\]

with the first row in degree 0 and the second row in degree 1 (note that higher products do not appear since we assume our $A_{\infty}$-structures to be strictly unital). Since all the components of the differential are isomorphisms, the natural embedding and the projection give a homotopy equivalence

\[
\text{hom}(F, E_i) \simeq \text{Hom}^1(F, E)L^i[-1].
\]

Hence, we deduce a homotopy equivalence

\[
E_{i+1} = \text{Cone}(\text{Hom}^1(F, E)L^i \otimes F[-1] \xrightarrow{\delta_{i+1}} E_i) \simeq \text{Cone}(\text{hom}(F, E_i) \otimes F \xrightarrow{\text{ev}} E_i) = T(E_i)
\]

(2.4.7)

as claimed.

For what follows we need to know explicitly the maps between $E_{i+1}$ and $T(E_i)$. Note that the embedding of $\text{Hom}^1(F, E)L^i \otimes F[-1]$ into $\text{hom}(F, E_i) \otimes F$ commutes with the maps to $E_i$ used to form the above cones, however, the projection in the other direction only commutes up to homotopy. Namely, we have a homotopy $h$ between the map $\text{ev} : \text{hom}(F, E_i) \otimes F \to E_i$ and the composition

\[
\text{hom}(F, E_i) \otimes F \to \text{Hom}^1(F, E)L^i \otimes F \xrightarrow{\delta_{i+1}} E_i,
\]

with the nonzero components

\[
h_j : \text{Hom}^1(F, E)L^j \otimes F \xrightarrow{id} \text{Hom}^1(F, E)L^j \otimes F \xrightarrow{\delta_j} E_i,
\]

for $0 \leq j \leq i - 1$. Hence, the map $E_{i+1} \to T(E_i)$ is given by the obvious embedding of complexes, while the map $T(E_i) \to E_{i+1}$ is the identity on the summands $E_i$ and all the summands $\text{Hom}^1(F, E)L^j \otimes F$ of $\text{hom}^1(F, E_i) \otimes F$.

Step 2. The complex $\text{hom}(E_0, E_i) = \text{hom}(E, E_i)$ has form

\[
\bigoplus_{j=0}^{i-1} \text{Hom}^1(F, E)L^j \otimes \text{Hom}^0(E, F) \oplus \text{Hom}^0(E, E) \to \text{Hom}^1(E, E),
\]

(2.4.8)
with the differential given by $d(id_E) = 0$,
\[ d(e \otimes \xi^{\otimes j} \otimes x) = m_{j+2}(e, \xi, \ldots, \xi, x). \]

Recall that the map $m_2 : \text{Hom}^1(F, E) \otimes \text{Hom}(E, F) \to \text{Hom}^1(E, E) = \mathcal{L}_E$ can be identified with the map $\text{End}(V) \to S : a \mapsto \text{tr}(ga)$. Thus, we immediately see that for $i \geq 1$ one has $\text{Hom}^0(E_0, E_i) = 0$, while $\text{Hom}^0(E_0, E_i)$ fits into an exact sequence
\[ 0 \to \text{End}_g(V)L \oplus \text{Hom}^0(E, E) \to \text{Hom}^0(E_0, E_i) \to \left( \bigoplus_{j=2}^i \text{End}(V)L^j \right) \to 0, \]
where we use the identification $\text{Hom}^1(F, E) \simeq V^\vee L$.

**Step 3.** For $0 \leq i < j$ let us consider the map of complexes
\[ \text{hom}(E_i, E_j) \xrightarrow{T} \text{hom}(T(E_i), T(E_j)) \to \text{hom}(E_{i+1}, E_{j+1}), \]
where the second arrow is induced by the maps $E_{i+1} \to T(E_i)$ and $T(E_j) \to E_{j+1}$ described in Step 1. Then in the case $i > 0$ the following square is commutative
\[
\begin{array}{ccc}
\text{hom}^0(E_i, E_j) & \longrightarrow & \text{Hom}^0(\text{Hom}^1(F, E)L^{i-1} \otimes F, \text{Hom}^1(F, E)L^{j-1} \otimes F) \\
\downarrow T & & \downarrow \otimes \text{id}_L \\
\text{hom}^0(E_{i+1}, E_{j+1}) & \longrightarrow & \text{Hom}^0(\text{Hom}^1(F, E)L^i \otimes F, \text{Hom}^1(F, E)L^j \otimes F)
\end{array}
\]
while in the case $i = 0$ the following square is commutative
\[
\begin{array}{ccc}
\text{hom}^0(E_0, E_j) & \longrightarrow & \text{Hom}^0(E, \text{Hom}^1(F, E)L^{j-1} \otimes F) \\
\downarrow T & & \downarrow \\
\text{hom}^0(E_1, E_{j+1}) & \longrightarrow & \text{Hom}^0(\text{Hom}^1(F, E) \otimes F, \text{Hom}^1(F, E)L^j \otimes F)
\end{array}
\]
where the horizontal arrows are the natural projections, while the right vertical arrow in the second diagram sends $a \in \text{Hom}^1(F, E)L^j \otimes V \simeq \text{End}(V)L^j$ to $a^* \otimes \text{id}_F \in \text{End}(V^\vee)L^j \otimes \text{hom}^0(F, F)$.

**Step 4.** For each $i \geq 1$, let us consider the natural projection (see Step 2)
\[ \pi_i : \text{Hom}^0(E_0, E_i) \to V^\vee L^i \otimes \text{End}(E, F) \simeq \text{End}(V)L^i. \]

Note that for $i \geq 2$ it is surjective, while for $i = 1$ its image is $\text{End}_g(V)L$. We claim that the map $\pi = (\pi_i)$ is a homomorphism of graded algebras
\[ R \to \mathcal{E}(V, g)^{op}. \]
To prove this let us consider elements $a \in \text{Hom}^0(E_0, E_i)$ and $b \in \text{Hom}(E_0, E_j)$, where $i > 0$, $j > 0$, and set $\bar{a} = \pi_i(a)$, $\bar{b} = \pi_j(b)$. Iterating Step 3 we see the component of $T^j(a) \in \text{hom}^0(E_j, E_{i+j})$ in $\text{Hom}^0(\text{Hom}^1(F, E)L^{i-1} \otimes F, \text{Hom}^1(F, E)L^{i+j-1} \otimes F)$ is $\bar{a} \otimes \text{id}_F$. It is easy to see that the $p_{i+j}(T^j(a) \circ b)$ is obtained by composing the above component
of $T^j(a)$ with $\overline{b} \in \text{End}(V)L^j \simeq \text{Hom}^0(E, \text{Hom}^1(F, E)L^{j-1} \otimes F)$. Thus, if we view $\overline{b}$ as an element of $V^{\vee} \otimes VL^j$ then we get
\[
\pi_{i+j}(T^j(a) \circ b) = (\pi^* \otimes \text{id}_V)(\overline{b}) = \overline{\pi} \in \text{End}(V)L^{i+j}.
\]

**Step 5.** Let us define $t \in \text{Hom}^0(E_0, E_1)$ to be the element represented by the element $\text{id}_E \in \text{Hom}^0(E, E)$ in the complex $\text{hom}(E, E_1)$, so that we have a decomposition
\[
\text{Hom}(E_0, E_1) \simeq \text{End}_g(V)L \oplus S \cdot t.
\] (2.4.8)

We claim that for $i \geq 0$ the element $T^i(t) \in \text{Hom}(E_i, E_{i+1})$ is represented by the closed map of twisted complexes

\[
\begin{array}{cccccccc}
V^{\vee}L^i \otimes F & \overset{\delta}{\longrightarrow} & V^{\vee}L^{i-1} \otimes F & \overset{\delta}{\longrightarrow} & \cdots & \overset{\delta}{\longrightarrow} & V^{\vee}L \otimes F & \overset{\delta}{\longrightarrow} & E \\
\text{id} & & \text{id} & & \cdots & & \text{id} & & \text{id}
\end{array}
\]

Indeed, this can be easily checked by induction by computing the effect of the twist functor $T$ on the above map and taking into account the homotopy equivalence (2.4.7).

**Step 6.** It follows easily from Step 5 that the map of the left multiplication by $t$ in our algebra,
\[
\text{Hom}(E_0, E_i) \overset{t}{\longrightarrow} \text{Hom}(E_0, E_{i+1})
\]
is injective and its image is given by the classes that have representatives in $\text{hom}(E_0, E_{i+1})$ with zero component in
\[
V^{\vee}L^{i+1} \otimes \text{Hom}(E, F) \simeq V^{\vee} \otimes VL^{i+1} \simeq \text{End}(V)L^{i+1}.
\]
Thus, for $i \geq 1$ we get an exact sequence
\[
0 \to \text{Hom}(E_0, E_{i-1}) \overset{t}{\longrightarrow} \text{Hom}(E_0, E_i) \overset{\pi^*}{\longrightarrow} \mathcal{E}(V, g)L^i \to 0.
\] (2.4.9)

From these exact sequences we deduce that the algebra $\mathcal{R}$ is generated by degree 1 and degree 2 elements. Indeed, since $\pi$ is a homomorphism, this follows from the similar property of $\mathcal{E}(V, g)^{op}$ (see Lemma 2.3.1).

**Step 7.** Next, let us consider an element $a \in \text{End}_g(V)L$, and let us view it as a cochain in the term $V^{\vee}L \otimes \text{Hom}(E, F) \simeq \text{End}(V)L$ of the complex $\text{hom}(E_0, E_1)$. We would like to calculate $T(a)$. By definition, $T(a)$ is obtained as the composition
\[
E_1 = \text{Cone}(\text{hom}(F, E_0) \otimes F \xrightarrow{ev} E_0) \to \text{Cone}(\text{hom}(F, E_1) \otimes F \xrightarrow{ev} E_1) \to E_2,
\]
where the arrow between the cones is induced by $a : E_0 \to E_1$, and the last arrow is the projection $T(E_1) \to E_2$ that we computed before. Now the map $\text{hom}(F, E_0) \xrightarrow{a} \text{hom}(F, E_1)$ has two nonzero components: the map $\text{Ext}^1(F, E) \to V^{\vee}L^2$ induced by by the composition with $a$ and the map
\[
\mu_a : \text{Ext}^1(F, E) \to \text{Ext}^1(F, E) : x \mapsto m_3(\delta_1, a, x).
\] (2.4.10)
It follows that $T(a)$ is represented by the map

$$
\begin{array}{c}
V^\vee L \otimes F \\
\downarrow a^* \otimes \text{id}_F \\
V^\vee L^2 \otimes F \\
\end{array}
\quad \begin{array}{c}
\delta_1 \\
\downarrow a \\
\delta \\
\end{array}
\quad \begin{array}{c}
E \\
\end{array}
\quad (2.4.11)
$$

where the diagonal arrow is $\mu_a \otimes \text{id}_F$.

**Step 8.** Now let us check that $t$ is central in $R$. By Step 6, it is enough to check $t$ commutes with elements of degree 1 and 2, lifting arbitrary elements in $E(V,g)_1$ and $E(V,g)_2$ under the homomorphism $\pi$. First, let us check that $at = ta$ in $R_2$, for $a \in \text{End}_g(V)L \subset R_1$. Note that here $at = T(a) \circ t$, $ta = T(t) \circ a$. From our description of $T(t)$, we immediately get that $ta$ is represented by the map

$$
\begin{array}{c}
E \\
\downarrow a \\
V^\vee L^2 \otimes F \\
\end{array}
\quad \begin{array}{c}
\delta_2 \\
\downarrow \delta \\
\delta \\
\end{array}
\quad \begin{array}{c}
V^\vee L \otimes F \\
\end{array}
\quad (2.4.12)
$$

On the other hand, from the description (2.4.11) of $T(a)$ it follows immediately that $at = T(a) \circ t$ is represented by the same chain map (2.4.12) as $ta$.

Similarly, let us consider an element $A \in R_2$ represented by a closed map

$$
\begin{array}{c}
E \\
\downarrow A_0 \\
V^\vee L^2 \otimes F \\
\end{array}
\quad \begin{array}{c}
\delta \\
\downarrow \delta \\
\delta \\
\end{array}
\quad \begin{array}{c}
V^\vee L \otimes F \\
\end{array}
\quad E
$$

where $m_3(\delta, \delta, A_0) + m_2(\delta, A_1) = 0$. Then one can easily check that $T(A) \circ t$ and $T^2(t) \circ A$ are both represented by the map

$$
\begin{array}{c}
E \\
\downarrow A_0 \\
V^\vee L^3 \otimes F \\
\end{array}
\quad \begin{array}{c}
\delta_2 \\
\downarrow \delta \\
\delta \\
\end{array}
\quad \begin{array}{c}
V^\vee L^2 \otimes F \\
V^\vee L \otimes F \\
\end{array}
\quad E
$$

Since $t$ is a central nonzero divisor, it follows that the algebra $R$ is the Rees algebra of a filtered algebra $(A, F*A)$. Furthermore, by Steps 4 and 6 we have

$$
\text{gr}^F(A) \simeq R/(t) \simeq E(V,g)^{op},
$$

which proves (i).
Step 9. For \(a, b \in \text{End}_g(V)L \subset \mathcal{R}_1\), the product \(ab = T(a) \circ b\) in \(\mathcal{R}\) can be easily computed using the representation (2.4.11) for \(T(a)\): we get
\[
T(a) \circ b = (A_0, A_1, A_2) \in \text{hom}^0(E_0, E_2) = \text{End}(V)L^2 \oplus \text{End}(V)L \oplus \text{Hom}^0(E, E),
\]
with
\[
A_0 = ba, \quad A_1 = m_3(a, \delta_1, b) + (\mu_a \otimes \text{id}_F) \circ b, \quad A_2 = m_4(\delta_1, a, \delta_1, b),
\]
where \(\mu_a\) is given by (2.4.10).

From this point on we use the trivialization \(L = S \cdot \xi_F\). Assume for simplicity of notation that \(a = v_a \otimes v_a^*, b = v_b \otimes v_b^*,\) for \(v_a, v_b \in V, v_a^*, v_b^* \in V^\vee\) (the general case is proved similarly).

The evaluation map \(\delta_1 : \text{Hom}^1(F, E) \otimes F \to E[1]\) corresponds to the identity element \(\sum e_i \otimes e_i^*\) in \(\text{Hom}^1(F, E)^\vee \otimes \text{Hom}^1(F, E) \simeq V \otimes V^\vee\). Thus, the map \(\mu_a : V^\vee \to V^\vee\) sends \(v^*\) to
\[
\sum_i \langle v^*_i, e_i \rangle \cdot m_3(e_i^*, v_a, v^*) = m_3(v^*_a, v_a, v^*).
\]
Similarly,
\[
m_3(a, \delta_1, b) = m_3(v_a, v_b, v_b) \otimes v_a^*,
\]
\[
m_4(\delta_1, a, \delta_1, b) = m_4(v_a^*, v_a, v_b^*, v_b).
\]
Thus,
\[
A_1 = m_3(v_a, v_b^*, v_b) \otimes v_a + v_b \otimes m_3(v_a^*, v_a, v_b^*), \quad A_2 = m_4(v_a^*, v_a, v_b^*, v_b).
\]
Using the operators \(r\) and \(r'\) we can rewrite the formula for \(A_1\) as
\[
A_1 = r(b)a + br'(a).
\]

Now recall that \(\mathcal{R}_2 = \text{Hom}^0(E_0, E_2)\) is the subspace of \(\text{hom}^0(E_0, E_2)\) consisting of \((A_0, A_1, A_2)\) such that
\[
s(A_0) + \text{tr}(gA_1) = 0,
\]
where \(s(v \otimes v^*) = m_3(v^*, \xi_F, v)\). Thus, we can define the splitting \(\sigma\) of the projection \(\pi_2 : \mathcal{R}_2 = \text{Hom}^0(E_0, E_2) \to \text{End}(V)\) by setting
\[
\sigma(A) = (A, -s(A)h, 0) \in \text{Hom}^0(E_0, E_2) \subset \text{hom}^0(E_0, E_2),
\]
for \(A \in \text{End}(V)\). Rewriting the element \((A_0, A_1, A_2) \in \text{Hom}^0(E_0, E_2)\) as \(\sigma(A_0) + (A_1 + s(A)h)t + A_2t^2\) we get (2.4.5). \(\square\)

**Corollary 2.4.2.** Under the assumptions of Theorem 2.4.1, if in addition \(g\) is invertible, then the isomorphism class of the corresponding filtered algebra \((A, F_\bullet A)\) is determined by the higher products (2.4.2).

**Proof.** Indeed, in this case by Lemma 2.3.1(iii), \(E(V, g)\) is generated in degree 1 and has quadratic defining relations. Hence, the same is true for \(\mathcal{R}\). But, by Theorem 2.4.1(ii), the product \(\mathcal{R}_1 \otimes \mathcal{R}_1 \to \mathcal{R}_2\) is determined by the higher products (2.4.2). \(\square\)

In the context of Theorem 2.4.1 let us define the structure of a graded \(\mathcal{R} - \mathcal{R}\)-bimodule on
\[
\bigoplus_{i \geq 0} \text{Hom}^1(E_i, E_0) = \bigoplus_{i \geq 0} \text{Hom}^1(E_i, E_0)
\]
as follows. The grading of \( \text{Hom}^1(E_i, E_0) \) is set to be \(-i\). The left and right multiplication of \( x \in \text{Hom}^1(E_i, E_0) \) by \( a \in \text{Hom}^0(E_0, E_j) \), with \( j \leq i \), are given by

\[
a \cdot x = T^{-j}(a \circ x), \quad x \cdot a = x \circ T^{i-j}(a).
\]

We would like to calculate this \( \mathcal{R} - \mathcal{R} \)-bimodule.

**Lemma 2.4.3.** Let \( B \) be a non-negatively graded \( S \)-algebra, such that \( B_0 = S \) and all the graded components \( B_i \) are finitely generated projective \( S \)-modules. Let \( M \) be a graded \( B - B \)-bimodule such that \( M \) is isomorphic to \( B^* \) (the restricted dual of \( B \)) as a graded left \( B \)-module and as a graded right \( B \)-module. Then there exists an automorphism \( \phi : B \to B \), preserving the grading, such that \( M \) is isomorphic to \((\text{id}_B \phi)^*\) as a graded \( B - B \)-bimodule.

**Proof.** Note that \( M_{-i} \cong B_i^\vee \), so all the graded components of \( M \) are finitely generated projective modules. Consider the restricted dual \( M^* \). Then \( M^* \) is isomorphic to \( B \) as a left and as a right graded \( B \)-module. This easily implies the claim. \( \square \)

**Proposition 2.4.4.** Under the assumptions of Theorem 2.4.1, assume in addition that \( g \) is invertible. Then there is an isomorphism of graded \( \mathcal{R} - \mathcal{R} \)-bimodules

\[
\bigoplus_{i \geq 0} \text{Hom}^1(E_i, E_0) \cong (\text{id} \mathcal{R}_\phi)^* \otimes_S \mathcal{L}_E,
\]

where \( \phi \) is a graded automorphism of \( \mathcal{R} \), such that \( \phi(t) = t \) and the automorphism \( \overline{\phi} \) of \( \mathcal{R}/(t) \cong \mathcal{E}(V, g)^{\text{op}} \), induced by \( \phi \), is equal the restriction of the automorphism \( \text{Ad}(g^{-1}) : x \mapsto g^{-1}xg \) of \( \text{End}(V)[z]^{\text{op}} \).

**Proof.** Without loss of generality we can assume that \( \mathcal{C} \) is generated by \( (E, F) \). Then by Lemma 2.2.2, the object \( E_0 = E \) is 1-spherical. It follows that all the objects \( E_i = T^i(E_0) \) are 1-spherical in \( \mathcal{C} \). As before, we fix a trivialization \( \text{Hom}^1(F, F) = S \cdot \xi_F \), and consider the identification \( \text{Hom}^1(F, E) \cong V^\vee \), such that the second of the pairings (2.2.1) gets identified with the the natural pairing \( V \otimes V^\vee \to S \), while the first of these pairings is given by (2.2.2). We also use the isomorphisms \( \text{Hom}^1(E_i, E_i) \cong \text{Hom}^1(E, E) = \mathcal{L}_E \).

**Step 1.** There exists an isomorphism of bimodules (2.4.13) for some \( \phi \).

We can use the perfect pairing (given by the composition)

\[
\text{Hom}^1(X, E_0) \otimes \text{Hom}^0(E_0, X) \to \text{Hom}^1(E_0, E_0) = \mathcal{L}_E
\]

to define an isomorphism

\[
\text{Hom}^1(X, E_0) \otimes \mathcal{L}_E^{-1} \xrightarrow{\sim} \text{Hom}^0(E_0, X)^\vee.
\]

These isomorphisms are functorial in \( X \), which immediately implies that summing these isomorphisms over \( X = E_i \), we get an isomorphism of right \( \mathcal{R} \)-modules

\[
\bigoplus_{i \geq 0} \text{Hom}^1(E_i, E_0) \otimes \mathcal{L}_E^{-1} \cong \bigoplus_{i \geq 0} \text{Hom}^0(E_0, E_i)^\vee = \mathcal{R}^*.
\]

On the other hand, using the perfect pairings

\[
\text{Hom}^0(X, E_i) \otimes \text{Hom}^1(E_i, X) \to \text{Hom}^1(E_i, E_i) \cong \mathcal{L}_E
\]
we similarly get an isomorphism of right \( R \)-modules
\[
\mathcal{R} = \bigoplus_{i \geq 0} \text{Hom}^0(E_0, E_i) \simeq \bigoplus_{i \geq 0} \text{Hom}^1(E_i, E_0)^\vee \otimes \mathcal{L}.
\]
Dualizing we get another isomorphism of the form (2.4.14) which is compatible with the left \( R \)-module structures. By Lemma 2.4.3, there exists an automorphism \( \phi : \mathcal{R} \to \mathcal{R} \) such that (2.4.13) holds.

It remains to check that \( \phi(t) = t \) and to calculate the action of \( \phi \) on \( \mathcal{R}/(t) \). To this end we will calculate some compositions of morphisms between \( E_0 = E \) and \( E_1 = [V^\vee \otimes F \to E] \). Recall that we have a canonical decomposition
\[
\text{Hom}(E_0, E_1) = \mathcal{R}_1 = \text{End}_g(V) \oplus S \cdot t.
\]

**Step 2.** We construct canonical identifications
\[
\text{Hom}^1(E_1, E_0) = \text{End}(V)/(S \cdot \text{id}) \oplus \mathcal{L},
\]
\[\text{Hom}^1(E_1, E_0) \xrightarrow{\tau} \mathcal{L},\]

such that the composition
\[
\text{Hom}^1(E, E) \oplus \text{Hom}^1(V^\vee \otimes F, E) \oplus \text{Hom}^1(V^\vee \otimes F, V^\vee \otimes F) = \text{hom}^1(E_1, E_1) \to \text{Hom}^1(E_1, E_1) \xrightarrow{\tau} S,
\]
where the first arrow is the natural projection to cohomology, is given by
\[
(\xi, x, y \otimes y^*) \mapsto \xi + \langle y^*, gy \rangle,
\]
where \( \xi \in \mathcal{L}, y \in V, y^* \in V^\vee \).

By definition, the complex \( \text{hom}(E_1, E_0) \) has the form
\[
\text{Hom}^0(E, E) \to \text{Hom}^1(V^\vee \otimes F, E) \oplus \text{Hom}^1(E, E),
\]
where the differential maps \( \text{id}_E \) to the evaluation morphism in \( \text{Hom}^1(V^\vee \otimes F, E) \). This immediately leads to the identification (2.4.15).

We have
\[
\text{hom}^0(E_1, E_1) = \text{Hom}^0(E, E) \oplus \text{Hom}^0(E, V^\vee \otimes F) \oplus \text{Hom}^0(V^\vee \otimes F, V^\vee \otimes F)
\]
and the part of the differential \( \text{hom}^0(E_1, E_1) \to \text{hom}^1(E_1, E_1) \) that maps to the summands \( \text{Hom}^1(E, E) \oplus \text{Hom}^1(V^\vee \otimes F, V^\vee \otimes F) \) is the map
\[
\text{Hom}^0(E, V^\vee \otimes F) \xrightarrow{(\tau \otimes \delta) \delta} \text{Hom}^1(E, E) \oplus \text{Hom}^1(V^\vee \otimes F, V^\vee \otimes F),
\]
with both components induced by the evaluation map \( V^\vee \otimes F \to E[1] \). We can identify this map with the map
\[
V \otimes V^\vee \to \mathcal{L} \oplus V \otimes V^\vee : y \otimes y^* \mapsto (-\langle y^*, gy \rangle, y \otimes y^*).
\]
Thus, the map \( \text{hom}^1(E_1, E_1) \to S \) given by (2.4.16) descends to a map on cohomology, \( \tau : \text{Hom}^1(E_1, E_1) \to S \). It is clear from the definition that \( \tau \) is surjective. Since \( E_1 \) is 1-spherical, we deduce that \( \tau \) is an isomorphism.

**Step 3.** Let \( T : \mathcal{L} = \text{Hom}^1(E, E) \to \text{Hom}^1(E_1, E_0) \) be the map induced by the spherical twist \( T \). Then \( \tau \circ T \) is the identity map of \( \mathcal{L} \). Hence, the map
\[
\text{Hom}^0(E_0, E_1) \otimes \text{Hom}^1(E_1, E_0) \to \text{Hom}^0(E_0, E_0) = S : a \otimes x \mapsto a \cdot x = T^{-1}(a \circ x),
\]
sends $a \otimes x$ to $\tau(a \circ x)$.

Indeed, it is easy to see that $T(\xi)$ is given by the element $\xi \in \text{Hom}^1(E, E) \subset \text{hom}^1(E_1, E_1)$.

**Step 4.** For $B \in \text{End}(V)/(S \cdot \text{id}) \subset \text{Hom}^1(E_1, E_0)$ one has $B \circ t = 0$ in $\text{Hom}^1(E_0, E_0)$ and $t \circ B = 0$ in $\text{Hom}^1(E_1, E_1)$. Also, viewing $\xi \in \mathcal{L}_E$ as an element of $\text{Hom}^1(E_1, E_0)$ (see (2.4.15)), we get $\xi \circ t = \xi$ and $\tau(t \circ \xi) = \xi$. Hence, for any $x \in \text{Hom}^1(E_1, E_0)$, we have

$$t \cdot x = x \cdot t$$

in the bimodule $\bigoplus_{i \geq 0} \text{Hom}^1(E_i, E_0)$.

Indeed, recall that $t$ corresponds to the element $\text{id}_E \in \text{Hom}^0(E, E) \subset \text{hom}^0(E, E_1)$. The vanishing of the composition $B \circ t = 0$ is clear from the composition rule:

The composition $t \circ B$ is calculated by the diagram

Thus, it belongs to the summand $\text{Hom}^1(V^\vee \otimes F, E) \subset \text{hom}^1(E_1, E_1)$, which is annihilated by $\tau$, hence zero in cohomology.
The composition $\xi \circ t$ is calculated by the diagram

![](diagram1.png)

Finally, the composition $t \circ \xi$ is calculated by the diagram

![](diagram2.png)

so it is given by the element $\xi \in \text{Hom}^1(E, E) \subset \text{hom}^1(E_1, E_1)$, which implies the assertion.

**Step 5.** For $A \in \text{End}_g(V) \subset \text{Hom}^1(E_0, E_1)$ and $B \in \text{End}(V)/(S \cdot \text{id})$, we have

$$B \circ A = \text{tr}(BgA),$$

$$\tau(A \circ B) = \text{tr}(AgB).$$

Indeed, the composition $B \circ A$ is calculated by the diagram

![](diagram3.png)
We claim that the composition of the vertical arrows is \( \text{tr}(BgA) \). Indeed, it is easy to see using our conventions, that for \( A = v^* \otimes v \) and \( B = w^* \otimes w \), this composition will be \( \langle v^*, w \rangle \cdot \langle w^*, gv \rangle \), which gives our claim.

Finally, the composition \( A \circ B \) is calculated by the diagram

\[
\begin{array}{ccc}
V^\vee \otimes F & \xrightarrow{\delta} & E \\
| & & | \\
B & \downarrow & E \\
| & & | \\
A & \downarrow & \\
V^\vee \otimes F & \xrightarrow{\delta} & E
\end{array}
\]

Note that this composition will have a component in \( \text{Hom}^1(V^\vee \otimes F, V^\vee \otimes F) \) given by the composition \( m_2(A, B) \) of the vertical arrows, as well, as a component in \( \text{Hom}^1(V^\vee \otimes F, E) \) given by \( m_3(\delta, A, B) \). However, the latter component does not give any contribution to the cohomology class, due to formula (2.4.16). For \( A = v^* \otimes v \) and \( B = w^* \otimes w \), we have \( m_2(A, B) = \langle w^*, v \rangle \cdot w \otimes v^* \), so applying \( \tau \) we get

\[
\tau(m_2(A, B)) = \langle w^*, v \rangle \cdot \langle v^*, gw \rangle = \text{tr}(AgB).
\]

**Step 6.** Since \( \mathcal{R} \) is generated in degree 1, the automorphism \( \phi \) is uniquely determined by its restriction to \( \mathcal{R}_1 \), which is in turn uniquely determined by the equation

\[
x \cdot \phi(a) = a \cdot x
\]

in \( \mathcal{L}_E = \text{Hom}^1(E_0, E_0) \), where \( a \in \mathcal{R}_1, x \in \text{Hom}^1(E_1, E_0) \). Hence, by Step 4, we deduce that \( \phi(t) = t \). Furthermore, still by Step 4, for \( a \in \mathcal{R}_1 \), one has \( B \cdot a = 0 \) for all \( B \in \text{End}(V)/(S \cdot \text{id}) \subset \text{Hom}^1(E_1, E_0) \) if and only if \( a \in S \cdot t \). Therefore, for \( A \in \text{End}_g(V) \), the element \( \phi(A) \mod S \cdot t \) is determined by the products \( B \cdot \phi(A) \) in \( \text{Hom}^1(E_0, E_0) \). Now the calculation of Step 5 implies that \( \phi(A) \equiv g^{-1}Ag \mod S \cdot t \). □

**Proposition 2.4.5.** Let \( V = S^n \), where \( n \geq 2 \), and let \( g \in \text{End}_S(V) \otimes \mathcal{L} \) be an invertible element (where \( \mathcal{L} \) is a locally free \( S \)-module of rank 1). Assume that either \( n \geq 3 \) or \( \text{tr}(g) \) is a generator of \( \mathcal{L} \). Let \( (A, F^*_A) \) be a filtered \( S \)-algebra, such that

\[
\text{gr}^F A \simeq \mathcal{E}(V,g)^{op}.
\]  

(2.4.17)

Suppose \( \phi_1 \) and \( \phi_2 \) are automorphisms of \( A \), preserving the filtration (i.e., \( \phi_i F^*_j A = F^*_j A \)), such that the induced automorphisms of \( \text{gr}^F A \) are the same. Then \( \phi_1 = \phi_2 \).

**Proof.** Let us consider the automorphism \( \phi = \phi_1^{-1} \phi_2 \) of \( A \). Then \( \phi \) still preserves the filtration and induces the identity on \( \text{gr}^F A \). It is easy to check that setting for \( a \in F^n A \)

\[
D(a) = \phi(a) - a \mod F_{n-2} A
\]

29
we get a well defined derivation \( D : \text{gr}^F A \to \text{gr}^F A \) of degree \(-1\). Using (2.4.17) and Proposition 2.3.2, we deduce that \( D = 0 \). In particular, \( \phi(a) = a \) for \( a \in F_1 A \). But Lemma 2.3.1(i) implies that \( A \) is generated by \( F_1 A \) (since this is true for \( \text{gr}^F A \simeq \mathcal{E}(V,g)^{\text{op}} \)). Hence, it follows that \( \phi(a) = a \) for all \( a \in A \).

\[ \square \]

3. Connection with noncommutative orders over stacky curves

3.1. Filtered algebras and orders. In this section we work over a fixed ground field \( k \). Let \( A \) be a filtered algebra over \( k \) equipped with an isomorphism (2.4.1) for some \( g \in \mathbb{P} \text{End}(V) \), and let \( Z \subset A \) be its center. We equip \( Z \) with the induced filtration.

Lemma 3.1.1. The algebra \( A \) is finitely generated, prime and of GK-dimension 1. Hence, \( A \) is Noetherian and finite over its center \( Z \), which is a 1-dimensional domain, finitely generated as \( k \)-algebra. Also, \( A \) is an order in a central simple algebra over the quotient field of \( Z \).

Proof. Since the algebra \( \mathcal{E}(V,g) \) is generated by degree 1 elements, we deduce that \( A \) is generated by \( F_1 A \). Given a nonzero ideal \( I \subset A \), let \( I_0 \subset \text{End}(V) \) be the set of all elements \( x \) such that \( xt^n \) appears as an initial form of an element of \( I \) for some \( n \). Then \( I_0 \) is a nonzero ideal, hence, \( I_0 = \text{End}(V) \). Hence, for a pair of nonzero ideals \( I, J \subset A \) we have \( I_0 J_0 \neq 0 \), so \( IJ \neq 0 \), which shows that \( A \) is prime. We have \( \dim F_i A/F_{i-1} A = n^2 \) for \( i > 1 \), so the GK-dimension of \( A \) is one. Now the results of [33] and [29] imply that \( A \) and \( Z \) are Noetherian, \( A \) is finite over \( Z \), and \( Z \) has dimension 1.

Note that the center of \( \text{gr}^F(A) \simeq \mathcal{E}(V,g)^{\text{op}} \) is either \( k[z^2, z^3] \subset k[z] \), in the case when \( \text{tr}(g) \neq 0 \), or \( k[z] \), when \( \text{tr}(g) = 0 \). Thus, \( \text{gr}^F(Z) \) is a graded \( k \)-subalgebra in \( k[z] \), i.e., a group algebra of a subsemigroup in natural numbers. This easily implies that the algebra \( \mathcal{R}(Z) \) is a domain, finitely generated as a \( k \)-algebra. Next, the fact that \( \text{gr}^F(A) \simeq \mathcal{E}(V,g)^{\text{op}} \) is torsion free as a module over \( \text{gr}^F(Z) \subset k[z] \) implies that \( A \) is torsion free as a \( Z \)-module. Let \( K \) be the quotient field of \( Z \). Then \( A \otimes_Z K \) is a finite-dimensional prime algebra over \( K \) with the center \( K \), so it is a central simple algebra over \( K \).

Next, we would like to extend \( A \) to a sheaf of algebras over a projective curve compactifying \( \text{Spec}(Z) \). The first obvious choice is to consider the Rees algebras \( \mathcal{R}(A) = \bigoplus_{m \geq 0} F_mA \) and \( \mathcal{R}(Z) = \bigoplus_{m \geq 0} F_mZ \) and to consider the corresponding Proj-construction. However, the resulting structures are not always easy to analyze. Namely, the problem arises when \( \text{gr}^F(Z) \) is contained in \( k[t^d] \) for some \( d \geq 2 \). It turns out that a better behaved construction is provided by the stacky version of \( \text{Proj} \), which we denote by \( \text{Proj}^{st} \).

Namely, for any commutative non-negatively graded \( k \)-algebra \( B = \bigoplus_{n \geq 0} B_n \), where \( B_0 = k \), one can define a stack

\[ \text{Proj}^{st}(B) := \text{Spec}(B) \setminus \{ B_+ \}/\mathbb{G}_m, \]

where \( B_+ \) is the augmentation ideal. Assuming in addition that \( B \) is finitely generated, we have an equivalence of the category \( \text{Coh}(\text{Proj}^{st}(B)) \) with the quotient of the category of finitely generated graded \( B \)-modules by the subcategory of finite-dimensional modules. Note that we have a natural line bundle \( \mathcal{O}(1) \) on \( \text{Proj}^{st}(B) \) such that elements of \( B_n \) can be viewed as global sections of \( \mathcal{O}(n) \).

30
Now starting with an algebra $A$ as above we define the stacky curve $C$ by

$$C := \text{Proj}^* \mathcal{R}(Z).$$

Let $d \geq 1$ be the maximal such that $\text{gr}^F(Z) \subset k[t^d]$. We will see below that $d$ measures the "stackiness" of $C$ (see Lemma 3.1.2(i)). In particular, $d = 1$ if and only if $C$ is the usual curve.

Let us denote by $t$ the element $1 \in \mathcal{R}_1(Z) = F_1A \cap Z$. Note that $t$ is a non-zero-divisor, and $\mathcal{R}(A)/t\mathcal{R}(A) \simeq \text{gr}^F(A)$, $\mathcal{R}(Z)/t\mathcal{R}(Z) \simeq \text{gr}^F(Z)$. Since $\text{gr}^F(A) \simeq \mathcal{E}(V,g)^\text{op}$ if finitely generated as a $\text{gr}^F(Z)$-module (see the proof of Lemma 3.1.1), we deduce that $\mathcal{R}(A)$ is finitely generated as an $\mathcal{R}(Z)$-module. Thus, localizing $\mathcal{R}(A)$ we get a sheaf of coherent $O$-algebras $\mathcal{A}$ on $C$.

**Lemma 3.1.2.** (i) The pointed stacky curve $C$ is neat, and the divisor $(t = 0)$ is the unique stacky point $p = \overline{p}/\mu_d$ (where $\overline{p} = \text{Spec}(k)$). Thus, we have $O_C(1) \simeq O_C(p)$. There is a natural identification of the cotangent space $T^*_pC$ with $\chi$, the 1-dimensional space on which $\mu_d$ acts with weight 1.

(ii) The sheaf $\mathcal{A}$ is an order on $C$, i.e., a torsion-free coherent sheaf of $O$-algebras, whose stalk at the generic point is a central simple $k(X)$-algebra. Furthermore, the center of $\mathcal{A}$ is $O_C$.

(iii) One has a natural isomorphism of algebras on $p$,

$$\mathcal{A}|_p \simeq \rho_* \text{End}(V)^\text{op},$$

where $\rho : \overline{p} \to p$ is the natural morphism. Similarly, we have a natural isomorphism $\mathcal{A}(mp)|_p \simeq \rho_* \text{End}(V)$ for every $m \in \mathbb{Z}$, compatible with the above isomorphism via the identification $O(mp)|_p \simeq \chi^{-m}$. The rank of $\mathcal{A}$ is equal to $dn^2$.

(iv) The natural map $F_mA = \mathcal{R}_m(A) \to H^0(C, \mathcal{A}(mp))$ is an isomorphism for $m \in \mathbb{Z}$ (where $F_{-1}A = 0$). One has $h^1(A) = 1$ and $h^1(A(p)) = 0$. In particular, $\mathcal{A}$ is a weakly spherical order in the sense of Definition 0.0.3.

**Proof.** (i) We have seen that $\mathcal{R}(Z)$ is a domain, so $C$ is integral. Also, the coarse moduli is $\text{Proj} \mathcal{R}(Z)$ which is a projective curve. Note that the divisor $(t = 0)$ in $C$ can be identified with $\text{Proj}^* (\text{gr}^F(Z))$. Since $\text{gr}^F(Z) \subset k[z^d]$, and these algebras agree in all sufficiently high degrees, we see that $\text{Spec}(\text{gr}^F(Z))$ is an affine line with the pinched origin. In particular,

$$p = (\text{Spec}(\text{gr}^F(Z)) \setminus \{0\})/\mathbb{G}_m \simeq B\mu_d.$$  

Set $S = \text{Spec}(\mathcal{R}(Z)) \setminus \{0\}$. We can view $t$ as a map $S \to \mathbb{A}^1$ and the fiber over 0, $D \subset S$ is a closed $\mathbb{G}_m$-orbit with the stabilizer $\mu_d$. Since $D = \text{Spec}(\text{gr}^F(Z)) \setminus \{0\}$ is smooth, the surface $S$ is smooth near $D$. By the argument of Luna’s étale slice theorem (see [16]), there exists a smooth $\mu_d$-invariant locally closed curve $\Sigma \subset S$ through the point $z = 1$ of $D$ such that the induced map of stacks $\Sigma/\mu_d \to S/\mathbb{G}_m$ is étale.

The identification of the cotangent space at $\overline{p}$ with $\chi$ comes from the fact that $p$ is given by the equation $t = 0$, where $t$ is a section of $O_C(1)$.

(ii) This first assertion follows from Lemma 3.1.1. Also, we know that $\mathcal{R}(Z)$ is the center of $\mathcal{R}(A)$, so $O_C$ is the center of $\mathcal{A}$.

(iii) Note that $\mathcal{A}/\mathcal{A}(-p)$ is the localization of the graded module $\text{gr}^F(A) \simeq \mathcal{E}(V,g)^\text{op}$ over $\text{gr}^F(Z)$. Up to finite-dimensional pieces, this is the same as considering $\text{End}(V)^\text{op}[z]$
as a $k[z^d]$-module, which easily implies the isomorphism (3.1.1). If we identify sheaves on $p$ with $\mu_d$-representations then the functor $\rho_*$ is given by tensoring with the regular representation of $\mu_d$, so the image of $\rho_*$ is stable under tensoring with $\chi$.

Finally, since $p$ is a smooth (stacky) point of $C$, the sheaf $A$ is locally free near $p$. Thus, by considering ranks in the isomorphism (3.1.1), we obtain that the rank of $A$ is $dn^2$.
(iv) Note that the isomorphism

$$\phi : A \xrightarrow{\sim} H^0(C \setminus \{p\}, A)$$

sends $F_mA$ to $H^0(C, A(m))$. It is easy to check that the induced map of the associated graded spaces has as components the natural maps

$$F_mA/F_{m-1}A \simeq \mathcal{E}(V, g)_m \hookrightarrow \text{End}(V) = H^0(p, \rho_\ast \text{End}(V)) \simeq H^0(p, A(mp)|_p),$$

where we use (iii). It follows that $\phi^{-1}H^0(C, A(mp)) = F_mA$ for every $m \in \mathbb{Z}$. In particular, $H^0(C, A) = k$.

Note that for $m \geq 1$ we have $\operatorname{dim} F_mA = mn^2$. Hence, for sufficiently large $m$ we have $\chi(A(mp)) = h^0(A(mp)) = mn^2$. Hence, $\chi(A) = 0$ and $\chi(A(p)) = n^2$. Thus, since $h^0(A) = 1$ and $h^0(A(p)) = n^2$, we get $h^1(A) = 1$ and $h^1(A(p)) = 0$. \hfill $\square$

**Proposition 3.1.3.** The category $\text{Coh}(A^{op})$ of coherent right $A$-modules is equivalent to the category $\text{qgr} R(A)$, the quotient of the category of finitely generated right $R(A)$-modules by the subcategory of torsion modules.

**Proof.** We can apply the formalism of [2] to the triple $(\text{Coh}(A^{op}), A, s)$, where $s$ is the autoequivalence $M \mapsto M(1)$. The fact that that $s$ is ample follows easily from the ampleness of $\mathcal{O}(1)$ on $C$. Since we have an isomorphism of graded algebras

$$R(A) \simeq \bigoplus_{n \geq 0} H^0(C, A(n)) \simeq \bigoplus_{n \geq 0} \text{Hom}_{A^{op}}(A, A(n))$$

(3.1.2)

(see Lemma 3.1.2), this implies the required equivalence. \hfill $\square$

Below we refer to the condition $\chi$ introduced in [2, Def. 3.7] which is useful in the context of noncommutative projective geometry. We also use the notion of the **cohomological dimension of Proj** of a graded Noetherian algebra $B$ defined in terms of the cohomology functor

$$H^i(\cdot) := \text{Ext}^i_{qgr B}(\mathcal{O}, \cdot)$$

(see [2, Sec. 7]). Note that for $i \geq 1$, one has an isomorphism

$$H^i(M) \simeq \lim_{m \to \infty} \text{Ext}^i_{B^{op}}(B/B_{\geq m}, M)$$

(3.1.3)

(see [2, Prop. 7.2]). Thus, finiteness of the cohomological dimension of $\text{Proj} B$ is equivalent to finiteness of the cohomological dimension of the functor $\Gamma_{B,m} = \lim_{m \to \infty} \text{Hom}_{B^{op}}(B/B_{\geq m}, M)$.

**Corollary 3.1.4.** The algebra $R(A)$ is right Noetherian, satisfies the condition $\chi$, and $\text{Proj} R(A)$ has cohomological dimension $\leq 1$. The same is true for the algebra $R(A)^{op}$.

**Proof.** By [2, Thm. 4.5], from Proposition 3.1.3 and isomorphism (3.1.2) we get that $R(A)$ is right Noetherian and satisfies $\chi_1$. 32
Next, we claim that for every coherent right \( \mathcal{A} \)-module \( M \) the spaces \( \text{Ext}^j_{\mathcal{A}^{\op}}(\mathcal{A}, M) \) are finite-dimensional, \( \text{Ext}^{1}_{\mathcal{A}^{\op}}(\mathcal{A}, M) = 0 \), and \( \text{Ext}^{j}_{\mathcal{A}^{\op}}(\mathcal{A}, M(i)) = 0 \) for \( i \gg 0 \). Indeed, this immediately follows from the identification \( \text{Ext}^j_{\mathcal{A}^{\op}}(\mathcal{A}, M) \simeq H^j(\mathcal{C}, M) \) (note that the latter cohomology is isomorphic to the cohomology of the push-forward of \( M \) to the coarse moduli space of \( \mathcal{C} \)).

By Proposition 3.1.3, we deduce a similar statement for the cohomology functor \( H^* \) on the category \( qgr \mathcal{R}(\mathcal{A}) \). In particular, we see that the cohomological dimension of \( \text{Proj} \mathcal{R}(\mathcal{A}) \) is \( \leq 1 \). Now the fact that \( \mathcal{R}(\mathcal{A}) \) satisfies \( \chi \) follows from [2, Thm. 7.4(2)].

The last assertion follows from the fact that \( \text{gr}^F(\mathcal{A}^{\op}) \simeq \mathcal{E}(V, g) \simeq \mathcal{E}(V^*, g^*)^{\op} \), so we can repeat the argument with \( A \) replaced by \( A^{\op} \).

\[ \square \]

3.2. Spherical orders and duality. Let \( \mathcal{A} \) be an order over a proper stacky curve \( \mathcal{C} \) with a stacky point \( p \simeq B\mu_d \) such that \( \mathcal{A}|_p \simeq \rho_* \text{End}(V)^{\op} \), where \( \rho : \tilde{p} \to p \) is the \( \mu_d \)-covering of \( p \) by \( \tilde{p} \simeq \text{Spec}(k) \). Then we can view \( \rho_* V \) as a right \( \mathcal{A} \)-module supported at \( p \). Note that if \( d = 1 \) then this module is \( V \otimes \mathcal{O}_p \).

**Lemma 3.2.1.** Let \( \mathcal{A} \) be an order over a neat pointed stacky curve \( \mathcal{C} \) with the stacky point \( p \simeq B\mu_d \in \mathcal{C} \), such that \( \mathcal{A}|_p \simeq \rho_* \text{End}(V)^{\op} \), where \( V \) is a finite-dimensional vector space. Then the pair of \( \mathcal{A}^{\op} \)-modules \((\mathcal{A}, \rho_* V)\) (resp., \((\mathcal{A}, \mathcal{A}(-p))\)) generates \( \text{Perf}(\mathcal{A}^{\op}) \).

**Proof.** First, we note that the \( \mathcal{A} \)-module \( \mathcal{A}|_p \) is the direct sum of several copies of \( \rho_* V \). In particular, \( \rho_* V \) is in \( \text{Perf}(\mathcal{A}^{\op}) \), and the pairs \((\mathcal{A}, \rho_* V)\) and \((\mathcal{A}, \mathcal{A}(-p))\) generate the same subcategory. Next, using the exact sequences

\[ 0 \to \mathcal{A}(-mp) \to \mathcal{A} \to \mathcal{A}|_p \to 0 \]

for \( m \geq 0 \), we see that the subcategory \( \langle \mathcal{A}, \rho_* V \rangle \) generated by our objects contains all \( \mathcal{A}(-mp) \) for \( m \geq 0 \).

We claim that for any coherent right \( \mathcal{A} \)-module \( M \) there exists a surjection of the form \( \bigoplus_{i=1}^{\infty} \mathcal{A}(-ni) \to M \) for some \( n_i \geq 0 \). Indeed, let \( p = B\mu_d \). Using [19, Prop. 5.2], we see that the bundle \( \mathcal{E} = \bigoplus_{i=1}^d \mathcal{O}(-ip) \) over \( \mathcal{C} \) has the property that the map

\[ \pi^* \pi_* \text{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}, \]

where \( \pi : \mathcal{C} \to \overline{\mathcal{C}} \) is the coarse moduli map, is surjective for every quasicoherent sheaf \( \mathcal{F} \) on \( \mathcal{C} \). Let \( \overline{\mathcal{p}} \in \overline{\mathcal{C}} \) be the image of \( p \). Then \( \pi^* \mathcal{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{p}}) \simeq \mathcal{O}_C(dp) \). Thus, viewing a coherent right \( \mathcal{A} \)-module \( M \) as a coherent sheaf of \( \mathcal{O} \)-modules, we get a surjection of the form

\[ \mathcal{E}(-mp) \otimes M \to \pi_* \text{Hom}(\mathcal{E}, M) \otimes \mathcal{E} \to M. \]

Hence, the induced map of right \( \mathcal{A} \)-modules

\[ \mathcal{E}(-mp) \otimes \mathcal{A} \to M \]

is also surjective, which proves our claim.

Now we can repeat the well known argument for the category of perfect \( \mathcal{O} \)-modules (see e.g., the proof of [20, Thm. 4]): starting with any perfect complex of \( \mathcal{A} \)-modules \( E \), we can find bounded above complex \( P^* \), where each \( P^i \) is a direct sum of modules of the
form $\mathcal{A}(-mp)$, and a quasi-isomorphism $P^\bullet \to E$. Now we consider brutal truncation $\sigma^{\geq n}P^\bullet$ for sufficiently large $n$. The cone of the composition

$$\sigma^{\geq n}P^\bullet \to P^\bullet \to E$$

will be isomorphic in the derived category to $F[n+1]$, where $F$ is a coherent right $\mathcal{A}$-module. Furthermore, for sufficiently large $n$, we will have $\text{Hom}(E, F[n+1]) = 0$, so we deduce that $E$ is a direct summand in $\sigma^{\geq n}P^\bullet$.

We say that a pairing

$$\mathcal{F} \otimes \mathcal{G} \to \mathcal{H},$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are coherent sheaves on a scheme, is perfect on the left (resp. on the right) if the induced map $\mathcal{F} \to \text{Hom}(\mathcal{G}, \mathcal{H})$ (resp., $\mathcal{G} \to \text{Hom}(\mathcal{F}, \mathcal{H})$) is an isomorphism. We say that such a paring is perfect in the derived category (on the left or on the right) if the similar statements hold with $\text{Hom}$ replaced by $R\text{Hom}$.

**Proposition 3.2.2.** Let $\mathcal{A}$ be an order over an integral proper stacky curve $C$, which is smooth near all stacky points and satisfies $H^0(C, \mathcal{O}) = k$.

(i) $\mathcal{A}$ is spherical if and only if $h^0(C, \mathcal{A}) = 1$ and there is an isomorphism of left $\mathcal{A}$-modules

$$\mathcal{A} \simeq \text{Hom}(\mathcal{A}, \omega_C),$$

where $\omega_C$ is the dualizing sheaf on $C$ (equivalently, one can ask for an existence of an isomorphism of right $\mathcal{A}$-modules above). In particular, $\mathcal{A}$ is spherical if and only if $\mathcal{A}^{\text{op}}$ is spherical.

Furthermore, if $\mathcal{A}$ is spherical then $h^0(C, \mathcal{A}) = h^1(C, \mathcal{A}) = 1$ and for a nonzero morphism $\tau : \mathcal{A} \to \omega_C$ (which is unique up to rescaling) the pairing

$$\mathcal{A} \otimes \mathcal{A} \to \omega_C : (x, y) \mapsto \tau(xy)$$

is perfect in the derived category (on both sides).

(ii) Assume now that $(C, p)$ is a neat pointed stacky curve, and $\mathcal{A}$ is a spherical order over it, such that $\mathcal{A}|_p \simeq \rho_* \text{End}(V)$. Let $g \in \text{End}(V)$ be the element such that the morphism

$$\text{End}(V) \simeq H^0(\mathcal{A}(p)|_p) \xrightarrow{\rho_*} H^0(\omega_C(p)|_p) \simeq k$$

is of the form $x \mapsto \text{tr}(gx)$. Then $g$ is invertible, and the boundary homomorphism

$$\text{End}(V) \simeq H^0(\mathcal{A}(p)|_p) \to H^1(\mathcal{A}) \simeq k,$$

associated with the exact sequence $0 \to \mathcal{A} \to \mathcal{A}(p) \to \mathcal{A}(p)|_p \to 0$, is also of the form $x \mapsto \text{tr}(gx)$, for an appropriate choice of an isomorphism $H^1(\mathcal{A}) \simeq k$. In addition, one has $h^1(C, \mathcal{A}(p)) = 0$.

**Proof.** (i) For any vector bundles $\mathcal{V}, \mathcal{V}'$ over $C$ we have an isomorphism

$$\underline{\text{Hom}}_\mathcal{A}(\mathcal{A} \otimes \mathcal{V}, \mathcal{A} \otimes \mathcal{V}') \simeq \underline{\text{Hom}}(\mathcal{V}, \mathcal{A} \otimes \mathcal{V}'),$$

whereas $\text{Ext}^i$ vanish for $i > 0$. Hence, we have isomorphisms

$$\text{Ext}^i_\mathcal{A}(\mathcal{A}, \mathcal{A} \otimes \mathcal{V}) \simeq H^i(C, \mathcal{A} \otimes \mathcal{V}), \quad \text{Ext}^i_\mathcal{A}(\mathcal{A} \otimes \mathcal{V}, \mathcal{A}) \simeq \text{Ext}^i(\mathcal{V}, \mathcal{A}).$$

In particular, $\text{Ext}^i(\mathcal{A}, \mathcal{A}) \simeq H^i(C, \mathcal{A})$. Furthermore, the canonical pairings

$$\text{Ext}^{1-i}_\mathcal{A}(\mathcal{A} \otimes \mathcal{V}, \mathcal{A}) \otimes \text{Ext}^i_\mathcal{A}(\mathcal{A}, \mathcal{A} \otimes \mathcal{V}) \to \text{Ext}^{1-i}_\mathcal{A}(\mathcal{A}, \mathcal{A})$$
get identified with the natural composed maps
\[ \text{Ext}^{1-i}(\mathcal{V}, \mathcal{A}) \otimes H^i(\mathcal{A} \otimes \mathcal{V}) \to H^1(\mathcal{A} \otimes \mathcal{A}) \to H^1(\mathcal{A}), \] (3.2.2)
where the second arrow is induced by the multiplication on \( \mathcal{A} \). Since the modules of the form \( \mathcal{A} \otimes \mathcal{V} \) generate \( \text{Perf}(\mathcal{A}) \), we deduce that \( \mathcal{A} \) is 1-spherical as an object of \( \text{Perf}(\mathcal{A}) \) (i.e., the order \( \mathcal{A}^{op} \) is spherical) if and only if \( h^0(\mathcal{A}) = h^1(\mathcal{A}) = 1 \) and all the pairings (3.2.2) are perfect.

Now assume that \( \mathcal{A} \) is 1-spherical in \( \text{Perf}(\mathcal{A}) \). The Serre duality on \( \mathcal{C} \) gives us perfect pairings
\[ \text{Ext}^{1-i}(\mathcal{A} \otimes \mathcal{V}, \omega_C) \otimes H^i(\mathcal{A} \otimes \mathcal{V}) \to H^1(\omega_C). \]
In particular, we have a nonzero generator \( \tau \) in the 1-dimensional space \( \text{Hom}(\mathcal{A}, \omega_C) \) such that the induced map \( H^1(\mathcal{A}) \xrightarrow{H^1(\tau)} H^1(\omega_C) \) is an isomorphism. It is easy to check that the map (3.2.2) for \( i = 1 \) fits into a commutative diagram
\[ \begin{array}{ccc}
\text{Hom}(\mathcal{V}, \mathcal{A}) \otimes H^1(\mathcal{A} \otimes \mathcal{V}) & \longrightarrow & H^1(\mathcal{A}) \\
\downarrow & & \downarrow H^1(\tau) \\
\text{Hom}(\mathcal{V}, \text{Hom}(\mathcal{A}, \omega_C)) \otimes H^1(\mathcal{A} \otimes \mathcal{V}) & \longrightarrow & H^1(\omega_C)
\end{array} \] (3.2.3)
where the bottom arrow is the Serre duality pairing combined with the isomorphism
\[ \text{Hom}(\mathcal{V}, \text{Hom}(\mathcal{A}, \omega_C)) \simeq \text{Hom}(\mathcal{A} \otimes \mathcal{V}, \omega_C), \]
and the left vertical arrow comes from the morphism of left \( \mathcal{A} \)-modules
\[ \nu = \nu_\tau : \mathcal{A} \to \text{Hom}(\mathcal{A}, \omega_C) : a \mapsto (x \mapsto \tau(\tau a)). \]
Since both horizontal arrows give perfect pairing and \( H^1(\tau) \) is an isomorphism, we deduce that the map
\[ \text{Hom}(\mathcal{V}, \mathcal{A}) \to \text{Hom}(\mathcal{V}, \text{Hom}(\mathcal{A}, \omega_C)), \]
induced by \( \nu \), is an isomorphism for all vector bundles \( \mathcal{V} \). It follows that \( \nu \) is an isomorphism.

Note for an order \( \mathcal{A} \) with \( h^0(\mathcal{A}) = 1 \), the biduality morphism
\[ \mathcal{A} \to \text{Hom}(\text{Hom}(\mathcal{A}, \omega_C), \omega_C) \]
is an isomorphism. Indeed, since \( \omega_C \) is a dualizing sheaf on \( \mathcal{C} \), it suffices to check that \( \text{Ext}^{>0}(\mathcal{A}, \omega_C) = 0 \). Equivalently, we have to check that
\[ \text{Ext}^i(\mathcal{A}, \omega_C \otimes L^m) = H^0(\mathcal{C}, \text{Ext}^i(\mathcal{A}, \omega_C) \otimes L^m) = 0 \]
for \( i > 0 \) and \( m \gg 0 \), where \( L \) be an ample line bundle on \( \mathcal{C} \) (see [7, Prop. 6.9]). By Serre duality, this reduces to the vanishing of \( H^0(\mathcal{C}, \mathcal{A} \otimes L^{-m}) \), which is clear since every global section of \( \mathcal{A} \) is a scalar multiple of the unit.

The above biduality statement easily implies that the bilinear pairing
\[ \mathcal{A} \otimes \mathcal{A} \to \omega_C : a \otimes a' \mapsto \tau(aa') \]
for some \( \tau : A \rightarrow \omega_C \) induces an isomorphism \( \nu \) of left \( A \)-modules as above if and only if the corresponding morphism of right \( A \)-modules

\[
\nu' : A \rightarrow \text{Hom}(A, \omega_C) : a \mapsto (x \mapsto \tau(ax))
\]
is an isomorphism.

Now let us start with an order \( A \) such that \( h^0(A) = 1 \) and there exists an isomorphism of left \( A \)-modules \( A \cong \text{Hom}(A, \omega_C) \). Let \( \tau \in \text{Hom}(A, \omega_C) \) be the element corresponding to the unit global section of \( A \) under this isomorphism. Then the isomorphism is equal to \( \nu_\tau \).

Since \( \tau \) generates the space \( \text{Hom}(A, \omega_C) \), by Serre duality, the map \( H^1(A) \xrightarrow{H^1(\tau)} H^1(\omega_C) \) is an isomorphism. Hence, the diagram (3.2.3) implies that the pairing (3.2.2) for \( i = 0 \) is perfect. Now the pairing (3.2.2) for \( i = 1 \) fits into a similar diagram

\[
\begin{array}{ccc}
\Ext^1(V, A) \otimes H^0(A \otimes V) & \longrightarrow & H^1(A) \\
\downarrow & & \downarrow H^1(\tau) \\
\Ext^1(V, \text{Hom}(A, \omega_C)) \otimes H^0(A \otimes V) & \longrightarrow & H^1(\omega_C)
\end{array}
\]

(3.2.4)

Now we have an isomorphism

\[
\Ext^1(V, \text{Hom}(A, \omega_C)) \cong H^1(\text{Hom}(V, \text{Hom}(A, \omega_C))) \cong H^1(\text{Hom}(A \otimes V, \omega_C)) \cong \Ext^1(A \otimes V, \omega_C)
\]

since \( \Ext^1(A \otimes V, \omega_C) = 0 \). Thus, the pairing given by the bottom horizontal arrow in diagram (3.2.4) is perfect, hence, so is the pairing given by the top horizontal arrow.

For the last assertion, we use the isomorphism

\[
H^1(C, A)^* \cong H^0(C, \text{Hom}(A, \omega_C)) \cong H^0(C, A),
\]

together with the vanishing of \( \Ext^>0(A, \omega_C) \) observed before (which holds since \( h^0(C, A) = 1 \)).

(ii) We can think of \( \rho_* \text{End}(V) \) as the algebra \( \text{End}(V) \otimes R_{\mu_d} \) in the category of \( \mu_d \)-representations, where \( R_{\mu_d} = \bigoplus_{i=0}^{d-1} \chi^i \) is the regular representation of \( \mu_d \). The restriction \( \tau|_p : A|_p \rightarrow \omega_C|_p \) can be viewed as a morphism of \( \mu_d \)-representations,

\[
\text{End}(V) \otimes R_{\mu_d} \rightarrow \chi,
\]

whose unique non-trivial component, \( \text{End}(V) \otimes \chi \rightarrow \chi \), corresponds to the functional \( x \mapsto \text{tr}(gx) \) on \( \text{End}(V) \). The fact that the induced pairing \( \tau|_p(xy) \) on \( \text{End}(V) \otimes R_{\mu_d} \) is nondegenerate easily implies that \( g \) is invertible.

Now let us consider the morphism of exact sequences induced by \( \tau \),

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & A(p) & \longrightarrow & A(p)|_p & \longrightarrow & 0 \\
& & \tau & & \tau & & \tau & & \\
0 & \longrightarrow & \omega_C & \longrightarrow & \omega_C(p) & \longrightarrow & \omega_C(p)|_p & \longrightarrow & 0
\end{array}
\]
Passing to the corresponding exact sequences of cohomology, we get a commutative square

\[
\begin{array}{ccc}
H^0(\mathcal{A}(p)|_p) & \longrightarrow & H^1(\mathcal{A}) \\
\tau|_p & \downarrow & \tau \\
H^0(\omega_C(p)|_p) & \longrightarrow & H^1(\omega_C)
\end{array}
\]

in which the horizontal arrows are boundary homomorphisms. The non-degeneracy of the pairing

\[\text{Hom}(\mathcal{A}, \omega_C) \otimes H^1(\mathcal{A}) \to H^1(\omega_C)\]

implies that the right vertical arrow is an isomorphism. Since the bottom horizontal arrow is also an isomorphism, we deduce that the top horizontal arrow can be identified with \(\tau|_p\).

### 3.3. Special spherical orders over the cuspidal cubic.

Let \(C_{\text{cusp}}\) be a cuspidal curve of arithmetic genus 1 over a field \(k\), \(q\) a singular point, \(p\) a smooth point. Note that the normalization map is a homeomorphism, so we can identify \(C_{\text{cusp}}\) with \(\mathbb{P}^1\) as a topological space. We assume that \(p\) corresponds to \(\infty \in \mathbb{P}^1\), while \(q\) corresponds to \(0 \in \mathbb{P}^1\).

For an \(n\)-dimensional vector space \(V\) and \(g \in \text{GL}(V)\), let us define an order \(\mathcal{A}_{\text{cusp}}^g\) over \(C_{\text{cusp}}\) as the subsheaf of algebras \(\mathcal{A}_{\text{cusp}}^g \subset \text{End}(V) \otimes O_{\mathbb{P}^1}\), consisting of the elements that have an expansion \(a(z) = c \cdot I + a_1 z + \ldots\) near \(0 \in \mathbb{P}^1\), with \(c \in k\) and \(\text{tr}(ga_1) = 0\). Note that \(\mathcal{A}_{\text{cusp}}^g\) is a sheaf of \(O_{\mathbb{P}^1}\)-algebras precisely when \(\text{tr}(g) = 0\).

Let us denote by \(\text{tr}_g : \mathcal{A}_{\text{cusp}}^g \to O_{C_{\text{cusp}}}^g\) the homomorphism induced by the map

\[\text{End}(V) \otimes O_{\mathbb{P}^1} \to O_{\mathbb{P}^1} : A \mapsto \text{tr}(gA)\].

**Lemma 3.3.1.** The \(O\)-bilinear form \(\text{tr}_g(aa')\) on \(\mathcal{A}_{\text{cusp}}^g\) induces an isomorphism of right \(\mathcal{A}_{\text{cusp}}^g\)-modules

\[\mathcal{A}_{\text{cusp}}^g \sim \text{Hom}_{O_{C_{\text{cusp}}}^g}(\mathcal{A}_{\text{cusp}}^g, O_{C_{\text{cusp}}}^g) : a \mapsto (a' \mapsto \text{tr}_g(aa'))\]. (3.3.1)

**Proof.** Away from \(q\) this is clear, so it is enough to consider the completions of the stalks at \(q\). Since the localization \(\widehat{\mathcal{A}_{\text{cusp}}^g}_q[z^{-1}]\) is just the matrix algebra over \(k((z))\), we know that any functional \(\widehat{\mathcal{A}_{\text{cusp}}^g}_q \to k[[z]]\) has form \(a \mapsto \text{tr}_g(ab)\), for some \(b = b_{-n}z^{-n} + b_{-n+1}z^{-n+1} + \ldots\). Now considering the condition that \(\text{tr}_g(ab)\) has to be a formal series of the form \(c_0 + c_2 z^2 + \ldots\), we easily deduce that \(b\) has to be in \(\widehat{\mathcal{A}_{\text{cusp}}^g}_q\).

**Remark 3.3.2.** We also have an isomorphism of left \(\mathcal{A}_{\text{cusp}}^g\)-modules like (3.3.1), given by \(a' \mapsto (a \mapsto \text{tr}_g(aa'))\), which is in general different from (3.3.1).

**Lemma 3.3.3.** Under the construction of Sec. 3.1, the order \((\mathcal{A}_{\text{cusp}}^g)^{op}\) over \(C_{\text{cusp}}^g\) comes from the algebra \(E(V, g)^{op}\), viewed as a filtered algebra. In particular, we have an equivalence

\[\text{mod } -\mathcal{A}_{\text{cusp}}^g \simeq \text{qgr } E(V, g)^[t]\]
and an isomorphism of graded algebras
\[ \mathcal{E}(V, g)[t] \simeq \bigoplus_{n \in \mathbb{Z}} H^0(C^{cusp}, \mathcal{A}^{cusp}_g(n)). \]

Also, the algebra \( \mathcal{E}(V, g)^{op}[t] \) is right Noetherian and satisfies \( \chi \).

Proof. First, we need to identify \( \mathcal{A}^{cusp}_g \) with the sheafification of \( \mathcal{E}(V, g)[t] \), viewed as a graded module over \( k[z^2, z^3][t] \). For this we observe that \( \mathcal{A}^{cusp}_g \) is the subsheaf in \( \text{End}(V) \otimes \mathcal{O}_{P_1} \), which coincides with \( \text{End}(V) \otimes \mathcal{O}_{P_1} \) over the complement to \( q \). The same is true for the sheafification of \( \mathcal{E}(V, g)[t] \). Hence, it is enough to compare the restrictions of the two sheaves to the affine open subset \( C^{cusp} \setminus \{p\} \) (i.e., the open subset \( t \neq 0 \)). It remains to observe that
\[ H^0(C^{cusp} \setminus \{p\}, \mathcal{A}^{cusp}_g) = \mathcal{E}(V, g) \subset \text{End}(V)[z]. \]
This proves the first assertion. The other assertions follow from Lemma 3.1.2 and Corollary 3.1.4.

\[ \square \]

3.4. AS-Gorenstein condition over a field. For graded modules \( M \) and \( N \) over a graded algebra \( B \) we use the notation
\[ \text{Ext}_B^i(M, N) := \bigoplus_{j \in \mathbb{Z}} \text{Ext}_{B - \text{gr}}^i(M, N(j)) \]
where \( B - \text{gr} \) is the category of (left) \( B \)-modules.

Recall that a connected graded algebra \( B \) over a field \( k \) is called left Artin-Schelter Gorenstein (AS-Gorenstein) with the parameter \( (d, m) \) if \( B \) has a finite left injective dimension and \( \text{Ext}_B^1(k, B) \) is 1-dimensional, concentrated in degree \( d \) and internal degree \( m \). Similarly one defines the notion of right AS-Gorenstein.

Proposition 3.4.1. For any \( g \in \text{GL}(V) \) the algebra \( \mathcal{E}(V, g)[x] \) (where \( \deg(x) = 1 \)) is left and right AS-Gorenstein with the parameter \( (2, 0) \).

Proof. Let us set \( B = \mathcal{E}(V, g)[x] \). It is easy to see that \( \mathcal{E}(V, g)^{op} \simeq \mathcal{E}(V^*, g^*) \), so it is enough to check that \( B \) is right Gorenstein. We have a natural identification of graded algebras
\[ B = \bigoplus_{n \in \mathbb{Z}} H^0(C^{cusp}, \mathcal{A}^{cusp}_g(n)), \]
where \( \mathcal{O}(1) = \mathcal{O}(p) \) (see Lemma 3.3.3). Next, by Lemma 3.3.1, for any \( n \in \mathbb{Z} \) the map
\[ \mathcal{A}^{cusp}_g(n) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_{C^{cusp}}} (\mathcal{A}^{cusp}_g(-n), \mathcal{O}_{C^{cusp}}), \]
\[ a \mapsto (a' \mapsto \text{tr}_g(aa')) \]
is an isomorphism of right \( \mathcal{A}^{cusp}_g \)-modules. Since the \( C^{cusp} \) is Gorenstein with \( \omega_{C^{cusp}} \simeq \mathcal{O}_{C^{cusp}} \), by Serre duality, we deduce an isomorphism
\[ \bigoplus_{n \in \mathbb{Z}} H^0(C^{cusp}, \mathcal{A}^{cusp}_g(n)) \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} H^1(C^{cusp}, \mathcal{A}^{cusp}_g(-n))^* \]
of right \( B \)-modules. In other words, we have an isomorphism of left \( B \)-modules
\[ \bigoplus_{n \in \mathbb{Z}} H^1(C^{cusp}, \mathcal{A}^{cusp}_g(n)) \simeq B^*, \quad (3.4.1) \]
where $B^*$ is the restricted dual of $B$.

Next, let us consider the bar-resolution of $k$ by the complex of free right $B$-modules,

$$
\cdots \to B_+ \otimes B \to B_+ \otimes B \to B \tag{3.4.2}
$$

Localizing this sequence on $C^{cusp}$ and twisting, we get for each $m \in \mathbb{Z}$ an exact sequence of left $A^{cusp}$-modules,

$$
\cdots \to B_+ \otimes B_+ \otimes A^{cusp}(m) \to B_+ \otimes A^{cusp}(m) \to A^{cusp}(m) \to 0
$$

Let us consider the spectral sequence computing the hypercohomology of this exact complex, i.e., abutting to zero, with the $E_1$-term given by the cohomology of the terms of this complex. Thus, the $E_1$-term has two rows, corresponding to $H^0$ and $H^1$. The row of $H^0$’s is the degree $m$ component of the complex (3.4.2), which is exact for $m \neq 0$, and has 1-dimensional cohomology in degree 0 for $m = 0$. On the other hand, using the isomorphism (3.4.1) of left $B$-modules we can identify the row of $H^1$’s with the degree $m$ component of the complex

$$
\cdots \to B_+ \otimes B_+ \otimes B^* \to B_+ \otimes B^* \to B^* \tag{3.4.3}
$$

Since the spectral sequence abuts to zero, the row of $H^1$’s should be exact for $m \neq 0$ and has one-dimensional cohomology in the term of degree $-2$ for $m = 0$.

Now the resolution (3.4.2) for $k$ as a right $B$-module shows that the complex (3.4.3) computes $\text{Tor}^B(k, B^*)$, while its restricted dual computes $\text{Ext}^*_{B^{op}}(k, B)$.

In other words, $\text{Ext}^*_{B^{op}}(k, B)$ has the one-dimensional cohomology concentrated in cohomological degree 2 and internal degree 0.

To conclude that $B$ is right Gorenstein it remains to check that $B$ has finite injective dimension as a right module over itself. To this end we use [6, Thm. 4.5] together with [36, Thm. 6.3]. More precisely, by Lemma 3.3.3, both $B$ and $B^{op}$ are right Noetherian, satisfy $\chi$, and their Proj has finite cohomological dimension. Hence, [36, Thm. 6.3] gives existence of a balanced dualizing complex over $A$. Now the same method as in [6, Thm. 4.5] can be used to prove that $B$ has finite injective dimension (see also [38, Thm. 0.3(3)] for a similar proof in the case of local rings).

\[\square\]

**Proposition 3.4.2.** Let $k$ be a field. For any filtered $k$-algebra $(A, F \cdot A)$ satisfying (2.4.1) for some $g \in \text{GL}_n(k)$, the graded algebra $R(A)$ is left and right AS-Gorenstein with parameters $(2, 0)$.

**Proof.** We observe that due to the nature of the Rees algebra construction there is a flat family $R_t(A)$ of graded algebras over $A^1$ such that specializing $t$ to a nonzero value gives an algebra isomorphic to $R(A)$, while $R_0(A)$ is $\text{gr}^F(A)[x] \simeq \mathcal{E}(V, g)[x]$. Therefore, since the algebra $\mathcal{E}(V, g)[x]$ is AS-Gorenstein with parameters $(2, 0)$ (see Proposition 3.4.1), we get that $\text{Ext}^i(k, R(A)) = 0$ for $i \neq 2$ and $\text{Ext}^2(k, R(A))$ is at most one-dimensional, concentrated in the internal degree 0.

Now let us check that $\text{Ext}^*(k, R(A))$ cannot be entirely zero. Indeed, if it were zero then using (3.1.3), we would get that all higher cohomology of $\mathcal{O}(i)$ on Proj vanishes. Now we use the identification of $qgr R(A)$ with coherent modules over the corresponding order $\mathcal{A}$ (see Proposition 3.1.3) and get a contradiction with the fact that $H^1(C, \mathcal{A}) \simeq \text{Ext}^1_{A^{op}}(A, \mathcal{A})$ is 1-dimensional (see Lemma 3.1.2(iv)).
Finally, by Corollary 3.1.4, $\mathcal{R}(A)$ and $\mathcal{R}^{\text{op}}$ are both Noetherian, satisfy $\chi$ and their Proj has finite cohomological dimension. Thus, as in the proof of Proposition 3.4.1 we can deduce that $\mathcal{R}(A)$ has finite injective dimension, so $\mathcal{R}(A)$ is AS-Gorenstein. 

4. Application to algebras associated with pairs of spherical objects

4.1. Gorenstein condition over a base ring. Now we are going to switch to working over an arbitrary Noetherian commutative ring $S$. Throughout this section we fix a filtered $S$-algebra $(A, F_{\bullet} A)$ equipped with an isomorphism (2.4.1) for some invertible $g \in \text{End}_S(V) \otimes \mathcal{L}$ (where $V \simeq S^n$), and let $\mathcal{R} = \mathcal{R}(A)$ be the corresponding Rees algebra. Note that the graded components of $\mathcal{R}$ are locally free $S$-modules of finite rank.

**Lemma 4.1.1.** The algebra $\mathcal{R}$ is right and left Noetherian.

**Proof.** First, we note that the ring $\mathcal{R}/(t) \simeq \mathcal{E}(V, g)^{\text{op}}$ is right and left Noetherian, since $\mathcal{E}(V, g)^{\text{op}}$ is finitely generated over its central subring $S[z^2]$. Since $t$ is a regular central element, the assertion follows. 

**Proposition 4.1.2.** Let $\mathcal{R}^*$ be the restricted dual of $\mathcal{R}$. Then for $i \neq 2$, one has

$$\text{Tor}_i^R(S, \mathcal{R}^*) = \text{Tor}_i^R(\mathcal{R}^*, S) = 0, \quad \text{Ext}_R^i(S, \mathcal{R}) = \text{Ext}_R^i(\mathcal{R}, S) = 0,$$

and $\text{Tor}_2^R(S, \mathcal{R}^*), \text{Tor}_2^R(\mathcal{R}^*, S), \text{Ext}_R^2(S, \mathcal{R})$ and $\text{Ext}_R^2(\mathcal{R}, S)$ are locally free $S$-modules of rank 1, concentrated in the internal degree 0.

**Proof.** **Step 1.** First, we observe that the assertion is true when $S = k$ is a field. Indeed, this follows from Proposition 3.4.2 and from the duality between $\text{Tor}_B^k(k, B^*)$ and $\text{Ext}_{B^\text{op}}^k(k, B)$.

**Step 2.** In the general case, since $\mathcal{R}$ is Noetherian, we can find a free resolution

$$\cdots \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to S,$$

where $\mathcal{P}_i$ are free graded $\mathcal{R}$-modules of finite rank. Let us set $Q_{\bullet} = (\mathcal{P}_{\bullet} \otimes_{\mathcal{R}} \mathcal{R}^*)$, so that $H_i(Q_{\bullet}) \simeq \text{Tor}_i^R(S, \mathcal{R}^*)$, and the graded components of $Q_{\bullet}$ are free $S$-modules of finite rank.

Let us assume that $S$ is local with the maximal ideal $M$, and set $k := S/M, \mathcal{R}_k := \mathcal{R} \otimes_S k$.

We are going to prove $H_i Q_{\bullet} = 0$ for $i \neq 2$, while $H_2 Q_{\bullet} \simeq S$ is concentrated in internal degree 0.

We have

$$Q_\bullet \otimes_S k \simeq (\mathcal{P}_\bullet \otimes_S k) \otimes_{\mathcal{R}_k} \mathcal{R}_k^*.$$

Note that $\mathcal{P}_\bullet \otimes_S k$ is a free graded resolution of $k$ over $\mathcal{R}_k$, so that

$$H_i(Q_{\bullet} \otimes_S k) \simeq \text{Tor}_i^{\mathcal{R}_k}(k, \mathcal{R}_k^*).$$

Note that we know from Step 1 that the latter spaces are zero for $i \neq 2$ and are isomorphic to $k$ in degree 0 for $i = 2$.

Now let us consider the third quadrant spectral sequence

$$E_{p,q}^2 = \text{Tor}_q^S(H_{-p} Q_{\bullet}, k) \implies E_\infty^q = \text{Tor}_q^S(Q_{\bullet}, k),$$

with the differentials $d_r : E_{p,q}^r \to E_{p-r,q+r}^r$. Note that since the terms of the complex $Q_{\bullet}$ are free $S$-modules we have $\text{Tor}_i^S(Q_{\bullet}, k) = H_i(Q_{\bullet} \otimes_S k)$, so $E_\infty^n = 0$ for $n \neq -2$, 40
while \( E_{2,0}^n \) is \( k \) in the internal degree 0. It follows that the term \( E_{0,0}^2 \) survives in the spectral sequence, so we get \( H_0 Q \otimes_S k = 0 \). Since \( H_0 Q \) has finitely generated graded components, by Nakayama lemma, this implies that \( H_0 Q = 0 \). Therefore, \( E_{0,q}^2 = 0 \), so the term \( E_{1,0}^2 \) survives, and we get \( H_1 Q \otimes_S k = 0 \). Hence, \( H_1 Q = 0 \), and so \( E_{2-1,q}^2 = 0 \). Thus, the terms \( E_{2,0}^2 \) and \( E_{2-1}^2 \) survive, and we deduce that \( H_2 Q \otimes_S k \cong k \) (in degree 0) and \( \text{Tor}_1^S(H_2 Q, k) = 0 \). Hence, \( H_2 Q \cong S \) (sitting in degree 0). Now the similar argument will prove by induction in \( n \geq 3 \) that \( H_n Q = 0 \) (for the base we use the vanishing of \( E_{2-q}^2 \) for \( q \leq -1 \)).

**Step 3.** For arbitrary \( \mathcal{R} \), since the construction of the complex \( Q_\bullet \) is compatible with localization, we deduce that \( H_i Q_\bullet = 0 \) for \( i \neq 2 \), while \( H_2 Q_\bullet \) is a projective \( S \)-module of rank 1 sitting in internal degree 0. This finishes the computation of \( \text{Tor}_1^R(S, \mathcal{R}^\ast) \).

Since the complex \( \text{Hom}_S(Q_\bullet, S) \) computes \( \text{Ext}_R^1(S, \mathcal{R}) \), and since \( H_i Q_\bullet \) are projective, we deduce that

\[
\text{Ext}_R^1(S, \mathcal{R}) \cong \text{Hom}_S(\text{Tor}_1^R(S, \mathcal{R}^\ast)),
\]

and the assertion about \( \text{Ext}_R^\ast(S, \mathcal{R}) \) follows. It remains to apply the same argument to \( \mathcal{R}^{op} \).

4.2. Noncommutative projective scheme associated with a filtered algebra.

As before, we consider the noncommutative projective scheme over \( S \) associated with \( \mathcal{R} = \mathcal{R}(A) \), i.e., the category \( \text{qgr} \mathcal{R} \), defined as the quotient of the category of graded finitely-generated right \( \mathcal{R} \)-modules by the subcategory of torsion modules. We denote by \( \mathcal{O}(j) \) the object of \( \text{qgr} \mathcal{R} \) corresponding to the module \( \mathcal{R}(j) \). Recall also that \( H^i(?) := \text{Ext}_R^i(\mathcal{O}, ?) \).

**Proposition 4.2.1.** (i) In the category \( \text{qgr} \mathcal{R}(A) \) one has

\[
H^1(\mathcal{O}(j)) = 0 \quad \text{for} \quad i \neq 0, 1; \quad H^1(\mathcal{O}(j)) = 0 \quad \text{for} \quad j > 0;
\]

and there is a natural isomorphism of graded algebras

\[
\bigoplus_j H^0(\mathcal{O}(j)) \cong \mathcal{R}(A).
\]

(ii) Let \( F \) be the object of \( \text{qgr} \mathcal{R}(A) \) corresponding to the graded right \( \mathcal{R}(A) \)-module \( V[z] \), with the module structure induced by the homomorphism

\[
\mathcal{R}(A) \to \mathcal{R}(A)/t\mathcal{R}(A) \cong \mathcal{E}(V, g)^{op} \hookrightarrow \text{End}(V)^{op}[z].
\]

Then the multiplication by \( z \) induces an isomorphism \( F \cong F(1) \). We have a natural exact sequence in \( \text{qgr} \mathcal{R}(A) \):

\[
0 \to \mathcal{O}(-1) \xrightarrow{t} \mathcal{O} \to V^* \otimes F \to 0 \quad (4.2.1)
\]

and canonical isomorphisms \( H^0(F) \cong V, \text{Ext}^1(F, \mathcal{O}) \cong V^* \). Also, \( \text{Hom}(F, F) = S \cdot \text{id}_F, \text{ } \text{H}^{i>0}(F) = 0 \text{ and } \text{Ext}^i(F, \mathcal{O}) = 0 \text{ for } i \neq 1 \).

(iii) There exist canonical isomorphisms \( H^1(\mathcal{O}) \cong \mathcal{L} \) and \( \text{Ext}^1(F, F) \cong S \) such that the compositions \( \text{Ext}^1(F, \mathcal{O}) \otimes H^0(F) \to H^1(\mathcal{O}) \) and \( H^0(F) \otimes \text{Ext}^1(F, \mathcal{O}) \to \text{Ext}^1(F, F) \) get identified the pairings \( \langle v^*, gv \rangle \) and \( \langle v, v^* \rangle \), where \( v \in V, v^* \in V^* \). Hence, the pair \( (\mathcal{O}, F) \) is an \( n \)-pair of 1-spherical objects with the corresponding element \( g \in \text{End}_S(V) \otimes \mathcal{L} \).
(iv) For every \( n \in \mathbb{Z} \) we have isomorphisms \( \mathcal{O}(n+1) \cong T(\mathcal{O}(n)) \), where \( T = T_F \) is the spherical twist associated with \( F \). Hence, the graded algebra \( \mathcal{R}_{T,\mathcal{O}} \) equipped with its natural central element of degree 1 (see Theorem 2.4.1) is isomorphic to \((\mathcal{R}(A),t)\).

**Proof.** (i) Let us set \( \mathcal{R} = \mathcal{R}(A) \). By [2, Prop. 7.2], we have

\[
H^i(\mathcal{O}(j)) = \lim_{m \to \infty} \text{Ext}^{i+1}(\mathcal{R}/\mathcal{R}_{\geq m}, \mathcal{R}(j)) \quad \text{for } i \geq 1,
\]

and there is an exact sequence

\[
0 \to \tau(\mathcal{R}(j))_0 \to \mathcal{R}_j \to H^0(\mathcal{O}(j)) \to \lim_{m \to \infty} \text{Ext}^1(\mathcal{R}/\mathcal{R}_{\geq m}, \mathcal{R}(j)) \to 0,
\]

where \( \tau(M) \) denotes the torsion submodule of \( M \). Note that \( \tau(\mathcal{R}(j)) = 0 \) since \( t \) is a nonzero divisor. On the other hand, by Proposition 4.1.2, we have

\[
\text{Ext}^2(S(m), \mathcal{R}(j)) = 0 \quad \text{for } m \neq j.
\]

Hence, the above exact sequence implies that the natural map

\[
\mathcal{R}_j \to H^0(\mathcal{O}(j))
\]

is an isomorphism. The compatibility of these maps with the products is well known (see [2, Thm. 4.5(2)]. Similarly, using the above formula for \( H^i(\mathcal{O}(j)) \) with \( i > 0 \) we see that \( H^{>1}(\mathcal{O}(j)) = 0 \) and that \( H^1(\mathcal{O}(j)) = 0 \) for \( j > 0 \).

(ii) The multiplication by \( z \) gives an injection \( V[z] \to V[z](1) \) with finite-dimensional cokernel, hence, it induces an isomorphism \( F \cong F(1) \). The exact sequence (4.2.1) is induced by the sequence of graded \( \mathcal{R} \)-modules

\[
0 \to \mathcal{R}(-1) \to \mathcal{R} \to \mathcal{E}(V,g) \to 0
\]

since we have \( \mathcal{E}(V,g)_{\geq 2} \cong \text{End}(V)[z]_{\geq 2} \). Twisting (4.2.1) by (2) and using the identification \( F \cong F(2) \), we get an exact sequence

\[
0 \to \mathcal{O}(1) \to \mathcal{O}(2) \to V^* \otimes F \to 0 \quad \text{(4.2.2)}
\]

Using part (i) and a long exact sequence of cohomology we immediately deduce that \( H^{>0}(F) = 0 \). Note that the sequence (4.2.2) also immediately implies that \( \text{Hom}(V^* \otimes F, \mathcal{O}) = 0 \), hence, \( \text{Hom}(F, \mathcal{O}) = 0 \).

Next, we claim that the natural map \( V = V[z]_0 \to H^0(F) \) is an isomorphism. Note that there is a morphism of exact sequences

\[
0 \to \mathcal{R}_1 \to \mathcal{R}_2 \to V^* \otimes V \to 0
\]

where the bottom row is obtained from (4.2.2) by passing to \( H^0 \), and all vertical maps are the natural maps of the form \( M_0 \to H^0(\tilde{M}) \), where \( \tilde{M} \) is the object of \( \text{qgr} \mathcal{R} \) associated with a graded module \( M \). Note that the exactness of the bottom row follows the vanishing of \( H^1(\mathcal{O}(1)) \) proved in (i). Since the two left vertical arrows are isomorphisms (again by
In particular, this implies that Ext$_{\ast}$ 1a split exact sequence

$V$ so it gives a canonical morphism

The corresponding extension class is an element of Ext$_{\ast}$ 1 $H$

Furthermore, the induced map

$R$

exact sequence of Ext

connecting homomorphism in the long exact sequence of Ext

Finally, the vanishing of Ext$_{\ast}$ ≥ 2

Again, a useful observation is that we have a morphism of functor s

Another useful observation is that we have a morphism of functor s

where

(iii) First, the long exact sequence of cohomology applied to (4.2.3) has form

$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow V^* \otimes F \rightarrow 0$ (4.2.3)

The corresponding extension class is an element of Ext$_{1}(V^* \otimes F, \mathcal{O}) \simeq V \otimes$ Ext$_{1}(F, \mathcal{O})$

so it gives a canonical morphism $V^* \rightarrow$ Ext$_{1}(F, \mathcal{O})$. In other words, this is precisely the connecting homomorphism in the long exact sequence of Ext$_{\ast}(?, \mathcal{O})$ applied to (4.2.3):

$0 = \text{Hom}(F, \mathcal{O}(1)) \rightarrow V^* \otimes \text{Hom}(F, F) \rightarrow \text{Ext}^{1}(F, \mathcal{O}) \overset{t}{\longrightarrow} \text{Ext}^{1}(F, \mathcal{O}(1)) \rightarrow \ldots$

Since the map on Ext$_{1}$ induced by $t$ is zero, we see that the map $V^* \rightarrow$ Ext$_{1}(F, \mathcal{O})$ is an isomorphism.

Finally, the vanishing of Ext$_{2}$ applied to $t$ is zero, we see that the map $V^* \rightarrow$ Ext$_{1}(F, \mathcal{O})$ is an isomorphism.

Next, we observe that since the sequence Ext$_{1}(F, \mathcal{O}(1)) \overset{t}{\longrightarrow} \text{Ext}^{1}(F, \mathcal{O})$ is zero, the long exact sequence of Ext$_{\ast}(?, \mathcal{O})$ associated with (4.2.1) gives an isomorphism

Ext$_{1}(F, \mathcal{O}) \sim$ Ext$_{1}(F, V^* \otimes F) \simeq V^* \otimes$ Ext$_{1}(F, F)$. (4.2.4)

In particular, this implies that Ext$_{1}(F, F)$ is a locally free $S$-module of rank 1. We have a split exact sequence

$0 \rightarrow \text{End}_{0}(V) \otimes F \rightarrow V \otimes V^* \otimes F \overset{\text{tr} \otimes \text{id}_{F}}{\longrightarrow} F \rightarrow 0$

Furthermore, the natural map $V \otimes \mathcal{O} \rightarrow F$ corresponding to the identification $V = H^{0}(F)$, is the composition

$V \otimes \mathcal{O} \overset{\text{id}_{V} \otimes p}{\longrightarrow} V \otimes V^* \otimes F \overset{\text{tr} \otimes \text{id}_{F}}{\longrightarrow} F$

where $p : \mathcal{O} \rightarrow V^* \otimes F$ is the map from the sequence (4.2.1). Thus, the map Ext$_{1}(F, V \otimes \mathcal{O}) \rightarrow$ Ext$_{1}(F, F)$ can be identified with the composition

Ext$_{1}(F, V \otimes \mathcal{O}) \sim$ Ext$_{1}(F, V \otimes V^* \otimes F) = V \otimes V^* \otimes$ Ext$_{1}(F, F) \overset{\text{tr} \otimes \text{id}}{\longrightarrow}$ Ext$_{1}(F, F)$. 43
where the first arrow is obtained from (4.2.4) by tensoring with $V$.

Thus, we deduce the surjectivity of the composition map

$$V \otimes V^* \simeq \text{Hom}(O, F) \otimes \text{Ext}^1(F, O) \to \text{Ext}^1(F, F). \quad (4.2.5)$$

We claim that $\text{End}_0(V) \subset \text{End}(V) = V \otimes V^*$ is contained in the kernel of this map. Indeed, it is enough to check that for any $v \in V \simeq \text{Hom}(O, V)$ and any $v^* \in V^* \simeq \text{Ext}^1(F, O)$, such that $\langle v^*, v \rangle = 0$, the composition of $v^*$ and $v$ in $\text{Ext}^1(F, F)$ vanishes. Let us consider the push-out of (4.2.3) by $v : O \to F$:

$$0 \longrightarrow O \overset{t}{\longrightarrow} O(1) \longrightarrow V^* \otimes F \longrightarrow 0$$

$$0 \longrightarrow F \overset{v}{\longrightarrow} E_v \longrightarrow V^* \otimes F \longrightarrow 0 \quad (4.2.6)$$

Next, let us compute explicitly the subobject $E'_v := \ker(t : E_v \to E_v(1)) \subset E_v$.

We can represent $E_v$ by the graded $R$-module

$V[z]_{\geq 1} \oplus R(1)_{\geq 1}/\{(v \ast r, tr) \mid r \in R_{\geq 1}\}$,

so that the embedding $F \to E_v$ corresponds to the embedding of the summand $V[z]_{\geq 1}$. Here for $r \in R_m$ we denote by $v \ast r \in V \cdot z^m$ the result of the right action of the image of $r$ in $R_m/R_{m-1} \subset \text{End}(V)^{op} \cdot z^m$ on $v$. Hence, $E'_v$ corresponds to the submodule of pairs $(x, r)$ such that $v \ast r = 0$. This easily implies that the image of the projection $E'_v \to O(1)/O \cdot t \simeq V^* \otimes F$ coincides with $\langle v \rangle^\perp \otimes F$. Thus, the bottom sequence in diagram (4.2.6) contains as a subsequence the exact sequence

$$0 \to F \to E'_v \to \langle v \rangle^\perp \otimes F \to 0 \quad (4.2.7)$$

of objects in the subcategory $\ker(t) \subset \text{qgr} R$. Note that the latter subcategory is naturally identified with $\text{qgr} \text{End}(V)[z]$ and that $F$ is a projective object in this subcategory. Hence, the sequence (4.2.7) splits which proves the required vanishing in $\text{Ext}^1(F, F)$.

It follows that the composition map (4.2.5) factors through a surjective map

$$S \simeq \text{End}(V)/\text{End}_0(V) \to \text{Ext}^1(F, F).$$

Since $\text{Ext}^1(F, F)$ is a locally free $S$-module of rank 1, this map is in fact an isomorphism.

(iv) As we have seen above, the exact sequence (4.2.3) induces an isomorphism $V^* \to \text{Ext}^1(F, O)$. Hence, it gives a canonical isomorphism

$$O(1) \simeq T_F(O).$$

On the other hand, for any $n \in \mathbb{Z}$ we have an isomorphism $F(n) \simeq F$. Hence, applying the autoequivalence $M \mapsto M(n)$ to the above isomorphism we get an isomorphism

$$O(n + 1) \simeq T_{F(n)}(O(n)) \simeq T_F(O(n)).$$ 

\[\square\]
Lemma 4.2.3. Under the isomorphism
\[ H^1(\mathcal{O}) := \bigoplus_{i \in \mathbb{Z}} H^1(\mathcal{O}(i)) \simeq (\text{id}_{\mathcal{R}})^* \otimes S \mathcal{L}. \]

Proof. This follows by combining Proposition 4.2.1 with Proposition 2.4.4. \qed

Next, we are going to construct a certain exact complex in qgr $\mathcal{R}(A)$. Namely, let us consider an element $hz \in \text{End}_g(V)[z] \simeq F_1A/F_0A$, where $h$ is any invertible element of $\text{End}(V) \otimes \mathcal{L}^{-1}$, such that $\text{tr}(gh) = 0$. Note that we get take $h = g^{-1}h_0$ where $h_0$ is an element of $\text{GL}(V)$ with $\text{tr}(h_0) = 0$ (it exists since we assume that $V \simeq S^n$ and $n \geq 2$). Let $\tilde{h} \in F_1A \otimes \mathcal{L}^{-1}$ be any lifting of $hz$ to $F_1A \otimes \mathcal{L}^{-1} = \mathcal{R}_1(A) \otimes \mathcal{L}^{-1}$. The right multiplication by $hz$ induces an injective map $\mathcal{E}(V, g) \to \mathcal{E}(V, g)(1) \otimes \mathcal{L}^{-1}$ (which we can view as a map of right $\mathcal{R}(A)$-modules) with finite-dimensional cokernel, hence, an isomorphism $V^* \otimes F \to V^* \otimes F(1) \otimes \mathcal{L}^{-1}$ in qgr $\mathcal{R}(A)$. Since $t \in \mathcal{R}_1(A)$ is central, we have a commutative diagram in qgr $\mathcal{R}(A)$ with exact rows

\[
\begin{array}{cccccccc}
0 & \to & \mathcal{O} & \xrightarrow{t} & \mathcal{O}(1) & \xrightarrow{h} & V^* \otimes F(1) & \to & 0 \\
| & & | \downarrow \tilde{h} & & \downarrow \tilde{h} & & \downarrow hz & \downarrow & 0 \\
0 & \to & \mathcal{O}(1) \otimes \mathcal{L}^{-1} & \xrightarrow{t} & \mathcal{O}(2) \otimes \mathcal{L}^{-1} & \xrightarrow{h} & V^* \otimes F(2) \otimes \mathcal{L}^{-1} & \to & 0
\end{array}
\]

This implies the exactness of the complex

\[ 0 \to \mathcal{O} \xrightarrow{\alpha} \mathcal{O}(1) \oplus \mathcal{O}(1) \otimes \mathcal{L}^{-1} \xrightarrow{\beta} \mathcal{O}(2) \otimes \mathcal{L}^{-1} \to 0 \quad (4.2.8) \]

where

\[ \alpha = (t, \tilde{h} \cdot), \quad \beta = (\tilde{h} \cdot, (-t) \cdot). \quad (4.2.9) \]

Lemma 4.2.3. Under the isomorphism

\[ \text{Ext}^1(\mathcal{O}(2), \mathcal{O}) \otimes \mathcal{L} \xrightarrow{\sim} \text{Hom}_S(\mathcal{R}_2, \mathcal{L}) \otimes \mathcal{L}: c \mapsto (r \mapsto c \cdot r), \]

where we use the identification of $H^1(\mathcal{O}) \simeq \mathcal{L}$ from Proposition 4.2.1(iii), the class $\gamma \in \text{Ext}^1(\mathcal{O}(2), \mathcal{O}) \otimes \mathcal{L}$ of extension (4.2.8) corresponds to the functional

\[ \mathcal{R}_2 \to \mathcal{L}^2: r \mapsto \text{tr}(\frac{p_2(r)}{z^2} h^{-1} g), \]

where $p : \mathcal{R} \to \mathcal{R}/(t) \simeq \mathcal{E}(V, g)^{op}$ is the natural projection (note that $p_2(r)$ is an element of $\text{End}(V)z^2$).

Proof. Let $\gamma' \in \text{Ext}^1(V^* \otimes F, \mathcal{O}(-1)) \simeq \text{Ext}^1(V^* \otimes F(1), \mathcal{O})$ be the class of the extension (4.2.1), and let $p : \mathcal{O} \to V^* \otimes F$ be the natural projection. We claim that $\gamma$ is equal to the composition

\[ \mathcal{O}(2) \otimes \mathcal{L}^{-1} \xrightarrow{p} V^* \otimes F(2) \otimes \mathcal{L}^{-1} \xrightarrow{(h_2)^{-1}} V^* \otimes F(1) \xrightarrow{\gamma'} \mathcal{O}[1]. \]
Indeed, this follows immediately from the commutative diagram in which rows and columns extend to short exact sequences,

$$
\begin{array}{c}
\mathcal{O} \xrightarrow{t} \mathcal{O}(1) \xrightarrow{\mathcal{P}} V^* \otimes F(1) \\
\uparrow \alpha \downarrow \beta \uparrow \tilde{t} \\
\mathcal{O} \oplus \mathcal{O}(1) \otimes L^{-1} \xrightarrow{-t} \mathcal{O}(1) \otimes L^{-1}
\end{array}
$$

Next, for any \( r \in R_m(A) \) the composition \( \mathcal{O} \xrightarrow{r} \mathcal{O}(m) \xrightarrow{\mathcal{P}} V^* \otimes F(m) \) corresponds to an element \( p(r) \in \mathcal{E}(V)_m \simeq \text{End}(V)z^m \simeq H^0(V^* \otimes F(m)) \). Finally, the extension class \( \gamma' \) corresponds to the identity element in \( V \otimes V^* \) under an isomorphism \( \text{Ext}^1(V^* \otimes F(1), \mathcal{O}) \simeq V \otimes V^* \), and the composition \( \text{Ext}^1(F, \mathcal{O}) \otimes \text{Hom}(\mathcal{O}, F) \rightarrow H^1(\mathcal{O}) \) is given by \( v^* \otimes v \mapsto \langle v^*, gv \rangle \). This easily implies the assertion. \( \square \)

4.3. Digression: minimally non-formal algebras. Let \( A \) be a graded algebra over a commutative ring \( S \). We can consider structures of \((S\text{-linear)}\) minimal \( A_\infty \)-algebras on \( A \) extending the given \( m_2 \), up to (strict) gauge equivalences. If every such structure is gauge equivalent to the one with \( m_i = 0 \) for \( i > 2 \), then \( A \) is called intrinsically formal. We are going to describe a class of algebras \( A \) for which gauge equivalence classes of minimal \( A_\infty \)-structures are classified by elements of a certain \( S \)-module.

Recall that the set of minimal \( A_\infty \)-structures on \( A \) is governed by the Hochschild cohomology \( HH^*(A) \) of the underlying graded associative algebra. More precisely, if we already have products \( m_i \) for \( i \leq n - 1 \), forming an \( A_{n-1} \)-structure, then the set of \( m_n \) extending these to an \( A_n \)-structure is a torsor over \( HH^2(A)_{2-n} \) (we follow the grading convention in which \( m_n \) is a Hochschild 2-cochain of internal degree \( 2 - n \)). So the vanishing of \( HH^2(A)_{<0} \) implies intrinsic formality. The next simplest case after that which occurs in some situations is when \( HH^2(A)_{2-n} \) is nonzero for the unique value \( n = d \geq 3 \). In this case every \( A_\infty \)-structure is gauge equivalent to the one with \( m_i = 0 \) for \( 2 < i < d \). Furthermore, \( m_d \), calculated for such a representative, gives a class in \( HH^2_{2-d} \), which uniquely determines the gauge equivalence class of the \( A_\infty \)-structure.

**Definition 4.3.1.** Let \( B = \bigoplus_{n \geq 0} B_n \) be a graded \( S \)-algebra with \( B_0 = S \), and let \( M = \bigoplus_{n \geq 0} M_n \) be a graded \( B - B \)-bimodule. We define the bigraded algebra \( A(\overline{B}, M, d) \) to be \( B \oplus M \), where \( M \cdot M = 0 \), with the natural internal grading and the homological grading given by \( \text{deg}(S) = 0, \text{deg}(M) = d \).

The next theorem is a slight generalization of the results in [23, Sec. 3.1]. As in [23, Sec. 3.1], for graded \( B - B \)-bimodules \( M_1, \ldots, M_n \) let us consider the bar-complex

\[
\text{Bar}^\bullet(M_1, \ldots, M_n) := M_1 \otimes_S T(B_+) \otimes_S M_2 \otimes_S \ldots \otimes_S T(B_+) \otimes_S M_n,
\]
where $T(B_+)$ is the tensor algebra of $B_+ = \bigoplus_{n \geq 1} B_n$ as an $S$-module. The grading is given by

$$\text{Bar}^{-m}(M_1, \ldots, M_n) := \bigoplus_{m_1 + \cdots + m_{n-1} = m} M_1 \otimes_S T^{m_1}(B_+) \otimes_S M_2 \otimes_S \cdots \otimes_S T^{m_{n-1}}(B_+) \otimes_S M_n.$$ 

Note that the cohomology $H^{-m}$ of the complex $\text{Bar}^\bullet(S, M)$ (resp., $\text{Bar}^\bullet(M, S)$) is isomorphic to $\text{Tor}_m^B(S, M)$ (resp., $\text{Tor}_m^B(M, S)$).

**Theorem 4.3.2.** Assume that $M$ bounded above, i.e., $M_n = 0$ for $n > n_0$, and that $P^l := \text{Tor}_*^B(S, M)$ and $P^r := \text{Tor}_*^B(M, S)$ are both finitely generated projective $S$-modules, concentrated in degree $d + 1$ and the internal degree $0$. Let us fix an embedding

$$P^r \xrightarrow{\varphi} M \otimes_S T^{d+1}(B_+)$$

inducing the isomorphism of $P^r$ with the cohomology of $\text{Bar}^\bullet(M, S)$. Then for the algebra $A = A(B, M, d)$ one has $HH^i_{	ext{mod}}(A) = 0$ for $m \geq 1$ and $i < 2m$, and there is an embedding

$$HH^2_d(A) \hookrightarrow \text{Hom}_S(P^r, S),$$

induced by the evaluation of a Hochschild cochain on the image of $\varphi$. Here the lower index denotes the grading on Hochschild cohomology induced by the homological grading on $A$.

**Proof.** Below we consider graded $B - B$-bimodules $M_i$ which are equipped with a pair of isomorphisms $l : M_i \rightarrow M \otimes_S P_i$, $r : M_i \rightarrow P'_i \otimes_S M$, for some finitely generated projective $S$-modules $P_i$, $P'_i$, where $l$ (resp., $r$) is compatible with the left (resp., right) graded $B$-module structures.

**Step 1.** $H^i(\text{Bar}^\bullet(M_1, M_2)) = 0$ for $i \neq -d - 1$ and we have an isomorphism of right $B$-modules

$$H^{-d-1} \text{Bar}^\bullet(M_1, M_2) \simeq P'_1 \otimes_S P^r \otimes_S M$$

and an isomorphism of left $B$-modules

$$H^{-d-1} \text{Bar}^\bullet(M_1, M_2) \simeq M \otimes_S P^l \otimes_S P_2.$$ 

To prove the first assertion we consider the spectral sequence associated with the filtration on $\text{Bar}^\bullet(M_1, M_2)$ induced by the $\mathbb{Z}$-grading on $M_2$. The corresponding $E_1$-term is

$$E_1 \simeq H^\bullet \text{Bar}^\bullet(M_1, S) \otimes_S M_2 \simeq P'_1 \otimes_S P^r \otimes_S M_2.$$ 

Hence, the spectral sequence degenerates and we obtain the first assertion. Similarly, the second assertion is obtained by considering the spectral sequence associated with the filtration induced by the $\mathbb{Z}$-grading on $M_1$.

**Step 2.** $H^i\text{Bar}^\bullet(M_1, \ldots, M_n) = 0$ for $i \geq -(n - 1)(d + 1)$, $n \geq 2$.

We use induction on $n$. For $n = 2$ the assertion follows from Step 1. Now for $n > 2$ we can consider $\text{Bar}^\bullet(M_1, \ldots, M_n)$ as the total complex associated with a bicomplex, by considering the bigrading given by the sums of the tensor degrees in even and odd factors $T(B_+)$. This leads to a spectral sequence abutting to $H^\bullet \text{Bar}^\bullet(M_1, \ldots, M_n)$ with the $E_1$-term

$$H^\bullet \text{Bar}^\bullet(M_1, M_2) \otimes_S T(B_+) \otimes_S H^\bullet \text{Bar}^\bullet(M_3, M_4) \otimes_S T(B_+) \otimes_S \ldots$$
where the last tensor factor is either $M_n$ or $H^* \text{Bar}^*(M_{n-1}, M_n)$. Thus, the $E_1$-term is isomorphic to the complex of the form

$$\text{Bar}^*(M'_1, \ldots, M'_n)[(n - n')(d + 1)]$$

where $M'_1 = H^* \text{Bar}^*(M_1, M_2)$, $M'_2 = H^* \text{Bar}^*(M_3, M_4)$, etc., satisfy the same assumptions as $(M_i)$ by Step 1. Applying the induction assumption we deduce the result.

**Step 3**. $H^i \text{Bar}^*(S, M_1, M_2, \ldots, M_n, S) = 0$ for $i > -n(d+1)$, and we have isomorphisms

$$H^{d-1} \text{Bar}^*(S, M_1, S) \simeq P^l \otimes_S P_1 \simeq P'_1 \otimes_S P^r$$

of graded $S$-modules.

Consider first the complex $\text{Bar}^*(S, M_1, S) = T(B_+) \otimes_S M_1 \otimes_S T(B_+)$. We can view it naturally as the total complex associated with a bicomplex (where the bigrading is induced by two tensor degrees). Considering the corresponding two spectral sequences we immediately deduce the vanishing of $H^i \text{Bar}^*(S, M_1, S)$ for $i > -d - 1$ and get the required identifications of $H^{d-1} \text{Bar}^*(S, M_1, S)$.

Now we use induction on $n$. As before, we equip $\text{Bar}^*(S, M_1, \ldots, M_n, S)$ with the bigrading using sums of tensor degrees in even and odd factors $T(B_+)$. Thus, we get a spectral sequence of a bicomplex abutting to $H^* \text{Bar}^*(S, M_1, \ldots, M_n, S)$ with the $E_1$-term either of the form

$$T(B_+) \otimes H^* \text{Bar}^*(M_1, M_2) \otimes T(B_+) \otimes \cdots H^* \text{Bar}^*(M_{n-1}, M_n) \otimes T(B_+) = \text{Bar}^*(S, M_1, \ldots, M_{n/2}, S)[n(d + 1)/2]$$

if $n$ is even, or of the form

$$H^* \text{Bar}^*(S, M_1) \otimes T(B_+) \otimes H^* \text{Bar}^*(M_2, M_3) \otimes T(B_+) \otimes \cdots H^* \text{Bar}^*(M_{n-1}, M_n) \otimes T(B_+) \simeq P^l \otimes_S P_1 \otimes S \text{Bar}^*(S, M_1, \ldots, M_{(n-1)/2}, S)[(n + 1)(d + 1)/2]$$

if $n$ is odd. In both cases the required cohomology vanishing follows from the induction assumption.

**Step 4**. Let us fix $m \geq 1$ and denote by

$$C^i_{md} \subset \text{Hom}_S(A_i^+, A)$$

the submodule of degree $-md$ with respect to the homological grading on $A$ and of degree 0 with respect to the internal grading on $A$. Note that the Hochschild differential maps $C^i_{md}$ to $C^{i+1}_{md}$, and $HH^i_{md}(A)$ is the $(i + md)$-th cohomology of the complex $C^i_{md}$. We have an exact sequence of complexes

$$0 \to C^i_{-md}(M) \to C^i_{-md} \to C^i(B) \to 0$$

where $C^i_{-md}(M) \subset C^i_{-md}$ (resp., $C^i_{-md}(B) \subset C^i_{-md}$) consists of maps $A^i_+ \to M$ (resp., $A^i_+ \to B$). Note that $C^i_{-md}(B)$ (resp., $C^i_{-md}(M)$) consists of $S$-linear maps

$$[T(B_+) \otimes_S M \otimes_S T(B_+) \otimes_S \cdots \otimes_S M \otimes T(B_+)i] \to B$$

(resp.,

$$[T(B_+) \otimes_S M \otimes_S T(B_+) \otimes_S \cdots \otimes_S M \otimes T(B_+)i] \to M$$

preserving the internal grading, where there are $m$ (resp., $m+1$) factors of $M$ in the source and the index $i$ refers to the total number of tensor factors (of $M$ and $B_+$).
We claim that
\[ H^i(C_{-md}^\bullet(M)) = H^i(C_{-md}^\bullet(B)) = 0 \text{ for } i < m(d + 2) \]
and in addition,
\[ H^{d+2}(C_{-d}^\bullet(M)) = 0 \]
and there is an embedding
\[ H^{d+2}(C_{-d}^\bullet(B)) \hookrightarrow \text{Hom}_S(P^r, S) \]
induced by the evaluation on the image of \( \varphi \).

For the proof, let us consider the decomposition \( C_{-md}^\bullet(B) = \prod_{j \geq 0} C_{-md}^\bullet(B_j) \), where \( C_{-md}^\bullet(B_j) \subset C_{-md}^\bullet(B) \) denote the maps with the image contained in \( B_j \). Let us consider the corresponding decreasing filtration on \( C_{-md}^\bullet(B) \). Note that the corresponding associated graded complex is
\[ \bigoplus_{j \geq 0} \text{Hom}_S(\text{Bar}^\bullet(S, (M)^m, S)_{j}, B_j)[-m], \]
where the lower index denotes the internal grading (and we grade the dual complex using the convention \( \text{Hom}(K^\bullet, ?)^i = \text{Hom}(K^{-i}, ?) \)). By Step 3, we have the vanishing
\[ H^i \text{Hom}_S(\text{Bar}^\bullet(S, (M)^m, S)_{j}, B_j) = 0 \text{ for } i < m(d + 1), \]
\[ H^{d+1} \text{Hom}_S(\text{Bar}^\bullet(S, M, S)_{j}, B_j) = 0 \text{ for } j > 0, \]
and in addition,
\[ H^{d+1} \text{Hom}_S(\text{Bar}^\bullet(S, M, S)_{0}, B_0) \cong \text{Hom}_S(P^r, S). \]

Hence, using [23, Lem. 3.2], we obtain the required statements about cohomology of \( C_{-md}^\bullet(B) \).

To deal with the cohomology of \( C_{-md}^\bullet(M) \) we consider the decreasing filtration on \( C_{-md}^\bullet(M) \) associated with the grading induced by the sum of the tensor degrees in the first and last factors of the tensor product \( T(B_+) \otimes M \otimes T(B_+) \otimes \ldots \otimes M \otimes T(B_+) \). The associated graded complex can be identified with
\[ \text{Hom}_{gr-S-mod}(T(B_+) \otimes_S \text{Bar}^\bullet((M)^{m+1}) \otimes_S T(B_+), M)[-m - 1]. \]
Thus, the required vanishing follows from Step 2. \( \square \)

**Corollary 4.3.3.** Under the conditions of Theorem 4.3.2, the map \( (m_\bullet) \mapsto m_{d+2} \) induces a bijection between the set of gauge-equivalence classes of minimal \((S\text{-linear})\ A_\infty\)-structures on \( A = A(B, M, d) \) with given \( m_2 \) and the \( S \)-module \( HH^2_{-d}(A) \).

**Proof.** Since the grading on \( A \) is concentrated in degrees 0 and \( d \), the only potentially nonzero higher products are \( m_n \) with \( n \equiv 2 \text{mod}(d) \). Thus, the \( A_\infty \)-identities imply that \( m_{d+2} \) is a Hochschild cocycle. Now the vanishing of \( HH^2_{-md} \) for \( m \geq 2 \) implies by the standard argument that the gauge equivalence class of \( (m_\bullet) \) is determined by the cohomology class of \( m_{d+2} \). Furthermore, since \( HH^3_{-md}(A) = 0 \) for \( m \geq 2 \), by [25, Lem. 3.1.2(ii)], every Hochschild cocycle \( m_{d+2} \) extends to an \( A_\infty \)-structure on \( A \). \( \square \)
Corollary 4.3.4. Under the conditions of Theorem 4.3.2 with \( d = 1 \), assume that we have a finitely generated projective \( S \)-module \( Q \) and a locally free \( S \)-module \( L \) of rank 1, maps of \( S \)-modules \( \alpha : Q \to B_{n_1}, \beta : Q' \to B_{n_2} \otimes L^{-1} \) and an element \( \gamma \in M_{-n_1-n_2} \otimes L \), where \( n_1 > 0, n_2 > 0 \), such that \( m_2(\beta \otimes \alpha)(\text{id}_Q) = 0 \) in \( B_{n_1+n_2} \otimes L^{-1} \) and \( m_2(\gamma \otimes \beta) = 0 \) in \( Q' \otimes M_{-n-1} \), where \( m_2 \) is induced by the product in \( B \) and the \( B \)-bimodule structure on \( M \). Assume in addition that the \( S \)-module \( P^r = \text{Tor}_2(M,S)_0 \) is locally free of rank 1. Then up to a gauge equivalence, there exists at most one minimal \( S \)-linear \( A_{\infty} \)-structure on \( A = A(B,M,1) \) with the given \( m_2 \) and with

\[
m_3((\gamma \otimes \beta \otimes \alpha)(\text{id}_Q)) = 1. \tag{4.3.1}
\]

Furthermore, if such an \( A_{\infty} \)-structure exists then the gauge equivalence classes of all minimal \( A_{\infty} \)-structures with the given \( m_2 \) are classified by elements of \( S \) via the map

\[
(m_{\bullet}) \mapsto m_3((\gamma \otimes \beta \otimes \alpha)(\text{id}_Q)).
\]

Proof. As in Corollary 4.3.3, we see that a minimal \( A_{\infty} \)-structure on \( A \) with the given \( m_2 \) is determined by the class of \( m_3 \) in \( HH^2_{\infty}(A) \), and we have an embedding

\[
HH^2_{\infty}(A) \hookrightarrow \text{Hom}_S(P^r, S).
\]

Next, we have a morphism \( S \to P^r = \text{Tor}_2(M,S)_0 \) given by the cycle \( (\gamma \otimes \beta \otimes \alpha)(\text{id}_Q) \) in the complex \( \text{Bar}^\bullet(M,S) \). Assume that there exists an \( A_{\infty} \)-structure on \( A \) with the given \( m_2 \) such that (4.3.1) holds. Then the composition

\[
HH^2_{\infty}(A) \to \text{Hom}_S(P^r, S) \xrightarrow{f \to f((\gamma \otimes \beta \otimes \alpha)(\text{id}_Q))} \text{Hom}_S(S)
\]

sends the class of \( m_3 \) to 1. It follows that the morphism \( \text{Hom}_S(P^r, S) \to S \) is surjective. Since \( P^r \) is locally free of rank 1, we deduce that this morphism is an isomorphism. Hence, the map \( HH^2_{\infty}(A) \to S \), given by the evaluation on \( (\gamma \otimes \beta \otimes \alpha)(\text{id}_Q) \), is an isomorphism, which implies our assertion. \( \square \)

4.4. Triple product calculation. Now we return to the situation of Section 4.2, so \( S \) is a Noetherian commutative ring, \((A, F, A)\) a filtered \( S \)-algebra equipped with an isomorphism (2.4.1) for some invertible \( g \in \text{End}_S(V) \otimes L \) (where \( V \simeq S^n \)), and \( R = R(A) \) be the corresponding Rees algebra.

We would like to show that Corollary 4.3.4 is applicable to minimal \( S \)-linear \( A_{\infty} \)-structures on the algebra \( A(R, (id_R)^* \otimes L), 1) \), where \( \phi \) is the automorphism of \( R \) defined in Corollary 4.2.2. Recall that we have an isomorphism of bimodules over \( R, \underline{H}^1(O) \simeq (id_R)^* \otimes L \).

Let us fix as before an invertible element \( h \in \text{End}(V) \otimes L^{-1} \) such that \( \text{tr}(gh) = 0 \) and its lifting \( \tilde{h} \in F_1A \otimes L^{-1} = R_1 \otimes L^{-1} \), and let us consider the extension class \( \gamma \in \text{Ext}^1(O(2), O) \otimes L = H^1(O(-2)) \otimes L \) of the exact sequence (4.2.8) in qgr \( R \). We can view the maps \( \alpha \) and \( \beta \) from this exact sequence as maps of \( S \)-modules

\[
\alpha : Q \to R_1, \quad \beta : Q' \to R_1 \otimes L^{-1},
\]

where \( Q = S \oplus L \), satisfying \( m_2(\beta \otimes \alpha)(\text{id}_Q) = 0 \) and \( m_2(\gamma \otimes \beta) = 0 \).
Lemma 4.4.1. Up to a gauge equivalence, there is at most one minimal $S$-linear $A_\infty$-structure on $A(\mathcal{R}, (\text{id}\mathcal{R}_\phi)^*, 1)$ with given $m_2$ and satisfying

$$m_3((\gamma \otimes \beta \otimes \alpha)(\text{id}_Q)) = 1. \quad (4.4.1)$$

Proof. Note that $(\text{id}\mathcal{R}_\phi)^* \otimes \mathcal{L}$ is isomorphic to $\mathcal{R} \otimes \mathcal{L}$ as a left and as a right $\mathcal{R}$-module, so by Proposition 4.1.2, the conditions of Theorem 4.3.2 with $d = 1$ are satisfied, with $P^r$ and $P^l$ locally free over $S$ of rank 1. Thus, the result follows from Corollary 4.3.4. \qed

Remark 4.4.2. Using the compatibility of the higher products with exact triangles (see e.g., [30, Lem. 3.7]) one can check that (4.4.1) holds for the minimal $A_\infty$-structure on $A(\mathcal{R}, \overline{H}^1(O), 1)$ coming from the standard $A_\infty$-enhancement of the derived category $D(\text{qgr} \mathcal{R})$ (defined uniquely up to a gauge equivalence). This explains why this is a natural condition to consider.

In the case when the filtered algebra $(A, F_\bullet, A)$ is associated with a pair of 1-spherical objects as in Theorem 2.4.1, we get an $A_\infty$-structure on the algebra of the form $A(\mathcal{R}, (\text{id}\mathcal{R}_\phi)^* \otimes \mathcal{L}, 1)$. Namely, we can consider the minimal $A_\infty$-structure on the subcategory of twisted complexes $(E_i)$, obtained by homological perturbation, and use Proposition 2.4.4 to identify the resulting algebra with $A(\mathcal{R}, (\text{id}\mathcal{R}_\phi)^* \otimes \mathcal{L}, 1)$. Note that the homological perturbation is applicable here since all the $S$-modules $\text{Hom}(E_i, E_j)$ are finitely generated projective. Furthermore, if we assume in addition that either $n \geq 3$ or $\text{tr}(g)$ is a generator of $\mathcal{L}$, then we deduce that $\phi' = \phi$, since such an automorphism is uniquely determined by its action on $\mathcal{R}/(t) \simeq \mathcal{E}(V, g)^{op}$ by Proposition 2.4.5.

We need to check that this $A_\infty$-structure satisfies our normalization condition on $m_3$.

Proposition 4.4.3. Assume that $g$ is invertible, and let $\phi$ be the unique automorphism of $\mathcal{R}$, such that $\phi(t) = t$ and the induced automorphism of $\text{gr}^F A \simeq \mathcal{E}(V, g)^{op}$ is $\text{Ad}(g^{-1})$ (see Proposition 2.4.5 and Corollary 4.2.2). Now assume that $\mathcal{R}$ arises as the graded algebra associated with an $n$-pair $(E, F)$ as in Theorem 2.4.1. Then the corresponding minimal $A_\infty$-structure on $A(\mathcal{R}, (\text{id}\mathcal{R}_\phi)^* \otimes \mathcal{L}, 1)$ satisfies

$$m_3(\gamma, \beta, \alpha) = \text{id},$$

where $\alpha : E_0 \to E_1 \oplus E_1 \otimes \mathcal{L}^{-1}$, $\beta : E_1 \oplus E_1 \otimes \mathcal{L}^{-1} \to E_2 \otimes \mathcal{L}^{-1}$ and $\gamma : E_2 \otimes \mathcal{L}^{-1} \to E_0$ correspond to the elements (4.2.9) and to the class of the exact sequence (4.2.8).

Proof. (i) It is enough to compute the relevant triple Massey product in the $A_\infty$-category of twisted complexes over the $A_\infty$-category $\mathcal{C}$ generated by our $n$-pair $(E, F)$, i.e., before applying the homological perturbation (this follows from the functoriality of Massey products, see [22, Prop. 1.1]).

We use our representations for $E_1$ and $E_2$ as twisted complexes from the proof of Theorem 2.4.1. Thus, the complex $\text{hom}(E_2, E)$ has form

$$\text{Hom}^0(E, E) \to (\text{Hom}^1(V^\vee L^2 \otimes F, E) \oplus \text{Hom}^1(V^\vee L \otimes F, E) \oplus \text{Hom}^1(E, E))$$

with the differential induced by the maps $\delta_1$ and $\delta_2$ (see (2.4.6)).

Step 1. We claim that the map $\gamma$ is equal to the class $\gamma' \in \text{Hom}^1(E_2, E) \otimes \mathcal{L}_E$ of the closed element $h^{-1} \in \text{End}(V) \otimes \mathcal{L}_E \simeq \text{Hom}^1(V^\vee L^2 \otimes F, E) \otimes \mathcal{L}_E \subset \text{hom}^1(E_2, E) \otimes \mathcal{L}_E$. Indeed, by
Lemma 4.2.3, it is enough to check that the map \( \text{Hom}^0(E, E_2) \to \text{Hom}^1(E, E) \otimes \mathcal{L}_E = \mathcal{L}_E^2 \), given by postcomposing with \( \gamma' \), coincides with the composition

\[
\text{Hom}(E, E_2) \to \text{Hom}(E, V^\vee \mathcal{L}_E^2 \otimes F) \simeq \text{End}(V) \xrightarrow{A \mapsto \text{tr}(Ah^{-1}g)} \mathcal{L}^2.
\]

Indeed, this amounts to checking that the composition

\[
E \to V^\vee \mathcal{L}_E^2 F \xrightarrow{h^{-1}} E \otimes \mathcal{L}_E [1]
\]

is given by \( A \mapsto \text{tr}(Ah^{-1}g) \) which follows from the identification of the composition \( \text{Hom}^1(F, E) \otimes \text{Hom}^0(E, F) \to \text{Hom}^1(E, E) \) with \( v^* \otimes v \mapsto \langle v^*, gv \rangle \).

**Step 2.** Using the computations from the proof of Theorem 2.4.1 we see that the components of \( \alpha \) and \( \beta \) are represented by the following closed maps. The maps \( E_0 \xrightarrow{t} E_1 \) and \( E_1 \xrightarrow{t} E_2 \) are given by

\[
\begin{array}{ccc}
E & \xrightarrow{\text{id}_E} & E \\
V^\vee \otimes F & \xrightarrow{\delta_1} & E \\
& \xrightarrow{\text{id}_E} & \\
& \xrightarrow{\text{id}_E} & \\
V^\vee \otimes F & \xrightarrow{\delta_2 \delta_1} & V^\vee \otimes F \\
& \xrightarrow{\text{id}_E} & \\
& \xrightarrow{\text{id}_E} & \\
& \xrightarrow{\text{id}_E} & \\
V^\vee \otimes F & \xrightarrow{\delta_1} & E
\end{array}
\]

Also, for a certain choice of \( \tilde{h} \), the maps \( E_0 \xrightarrow{\tilde{h}} E_1 \otimes \mathcal{L}_E^{-1} \) and \( E_1 \xrightarrow{\tilde{h}} E_2 \otimes \mathcal{L}_E^{-1} \) are given by

\[
\begin{array}{ccc}
E & \xrightarrow{h} & E \\
V^\vee \otimes F \otimes \mathcal{L}_E^{-1} & \xrightarrow{\delta_1} & E \otimes \mathcal{L}_E^{-1} \\
& \xrightarrow{h} & \\
& \xrightarrow{h} & \\
V^\vee \otimes F & \xrightarrow{h^* \otimes \text{id}_F} & E \\
& \xrightarrow{h} & \\
& \xrightarrow{h} & \\
V^\vee \otimes F \otimes \mathcal{L}_E^{-1} & \xrightarrow{\delta_2 \delta_1} & V^\vee \otimes F \otimes \mathcal{L}_E^{-1}
\end{array}
\]

where the diagonal arrow is \( \mu_h \otimes \text{id}_F \) (recall that \( \mu_a \) is defined by (2.4.10)).
Note that any other choice of \( \tilde{h} \) is of the form \( \tilde{h} + c \cdot t \), for \( c \in \mathcal{L}^{-1} \). Hence, for a different choice of \( \tilde{h} \), the sequence of maps \( \alpha, \beta \) would change by an automorphism of \( E_1 \oplus E_1 \otimes \mathcal{L}^{-1} \), so the Massey product is unaffected by such a change.

**Step 3.** Now one easily checks that \( m_2(\beta, \alpha) = 0 \) and that \( m_3(\gamma, \beta, \alpha) = 0 \). However, the product \( m_2(\gamma, \beta) \) is not zero on the cochain level. In fact, its only nonzero component is given by the composition

\[
\begin{aligned}
V \otimes F \xrightarrow{\delta_1} E \\
\downarrow h^* \otimes \text{id}_F \\
V \otimes F \otimes \mathcal{L}_E^{-1} \xrightarrow{\delta_2} V \otimes F \otimes \mathcal{L}_E^{-1} \\
\downarrow h^{-1} \\
E \otimes \mathcal{L}_E^{-1} \\
\downarrow \delta_1 \\
E
\end{aligned}
\]

It is easy to check that the composition \( h^{-1} \circ (h^* \otimes \text{id}_F) \) in this diagram is equal to \( \delta_1 \). Hence, we obtain

\[
m_2(\gamma, \beta) = d(\text{id}_E),
\]

where we view \( \text{id}_E \) as an element of \( \text{hom}^0(E_1, E_0) \). Thus, by the definition of the Massey product, we get

\[
MP(\gamma, \beta, \alpha) = m_2(\text{id}_E, t) = \text{id}_E.
\]

\[\square\]

4.5. **Proof of Theorem A for Noetherian rings.** For a commutative ring \( S \) we can think of an \( S \)-point of \( \text{PGL}_n \) as isomorphism classes of pairs \((g, \mathcal{L})\), where \( \mathcal{L} \) is an invertible \( S \)-module and \( g \in \text{End}_S(V) \otimes \mathcal{L} \) (where \( V = S^n \)) is an invertible element.

Let us consider the functors on the category of Noetherian commutative rings, that associate to \( S \) the set of

1. \((g, \mathcal{L}) \in \text{PGL}_n(S) \) and minimal \( A_\infty \)-structures on \( S(V, g) \) up to a gauge equivalence;
2. \((g, \mathcal{L}) \in \text{PGL}_n(S) \) and isomorphism classes of \((A, F \bullet A, \iota, \phi; m_\bullet)\), where \((A, F \bullet A)\) a filtered algebra equipped with an isomorphism \( \iota : \text{gr}^F A \simeq \mathcal{E}(V, g)^{op} \) and an automorphism \( \phi : A \to A \) such that the induced automorphism \( \tilde{\phi} \) of \( \text{gr}^F A \simeq \mathcal{E}(V, g)^{op} \) is equal to \( \text{Ad}(g^{-1}) \); and \( m_\bullet \) is a minimal \( A_\infty \)-structure on \( A(\mathcal{R}(A), (\text{id} \circ \mathcal{R}_\phi(A))^* \otimes \mathcal{L}, 1) \) with given \( m_2 \) and such that \( m_3(\gamma, \beta, \alpha) = 1 \) (viewed up to a gauge equivalence);
3. \((g, \mathcal{L}) \in \text{PGL}_n(S) \) and isomorphism classes of \((A, F \bullet A, \iota)\) as in (2).

**Step 1.** Construction of an injective map from (1) to (2).

Starting from a minimal \( A_\infty \)-structure on \( S^n(V, g) \), we consider the corresponding \( n \)-pair of spherical objects \((E, F)\) (see Sec. 2.2). Now we consider twisted objects \( (E_i) \) in Theorem 2.4.1 and use this Theorem and Proposition 2.4.4 to identify the corresponding
algebra
\[ \bigoplus_i \text{Hom}(E_0, E_i) \oplus \bigoplus_i \text{Ext}^1(E_i, E_0) \]
with \( A(\mathcal{R}(A), (\mathcal{R}_\phi(A))^* \otimes \mathcal{L}, 1) \), for some filtered algebra \( (A, F_\bullet A) \) and an automorphism \( \phi \) satisfying the conditions in (2). Note that the condition on \( m_3 \) holds due to Proposition 4.4.3.

Next, let us show injectivity of this map. For a minimal \( A_\infty \)-structure \( m_\bullet \) on \( S^n(V, g) \) let us denote by \( \mathcal{S}(m_\bullet) \) the corresponding \( A_\infty \)-algebra, which can be also viewed as an \( A_\infty \)-category with two objects \((E, F)\). Let \( \Pi TwS(m_\bullet) \) denote the \( A_\infty \) split-closure of the category of twisted complexes over \( \mathcal{S}(m_\bullet) \). The exact triangle
\[ E_0 \rightarrow E_1 \rightarrow V^\vee \otimes F \rightarrow E_0[1] \]  
coming from the definition of \( E_1 = T_F(E_0) \), shows that \( \Pi TwS(m_\bullet) \) is split-generated by \( E_0 \) and \( E_1 \). Thus, by [30, Cor. 4.9], the inclusion of the full subcategory on objects \( \{E_i \mid i \geq 0\} \) implies that the automorphism \( \text{Ad}(h) \) satisfies the conditions in (2).

Thus, if two minimal \( A_\infty \)-structures \( (m_\bullet) \) and \( (m'_\bullet) \) induce gauge-equivalent \( A_\infty \)-structures on \( \langle E_i \mid i \geq 0 \rangle \), then there exists a quasi-equivalence
\[ \Phi : \Pi TwS(m_\bullet) \simeq \Pi TwS(m'_\bullet) \]  
such that \( H^0\Phi \) is the identity on \( \text{Hom}^*(E_i, E_j) \). Since the functor \( H^0\Phi \) is triangulated, the exact triangle (4.5.1) shows that
\[ V^\vee \otimes \Phi(F) \simeq \Phi(V^\vee \otimes F) \simeq V^\vee \otimes F. \]

Such an isomorphism is induced by a unique isomorphism \( \Phi(F) \simeq F \otimes \mathcal{M} \), for some locally free \( S \)-module of rank 1 equipped with an isomorphism \( V^\vee \otimes \mathcal{M} \simeq V^\vee \).

Localizing over an open affine covering \( \text{Spec}(S) \), we can trivialize \( \mathcal{L}_E \) and \( \mathcal{M} \). Then \( \Phi \) gives an \( A_\infty \)-autoequivalence of the subcategory \( \{E, F\} \), identical on objects, and such that the induced autoequivalence of \( \{E_i \mid i \geq 0\} \) is isomorphic to the identity. To prove that \( (m_\bullet) \) and \( (m'_\bullet) \) are gauge equivalent, it is enough to have that \( H^0\Phi|_{E, F} \) is isomorphic to the identity. Just using the condition the the endomorphism of \( \text{Hom}^1(E, E) \) induced by \( H^0\Phi \) is the identity, it is easy to see that \( H^0\Phi \) should have the following form: it is given by the maps
\[ h : \text{Hom}^0(E, F) = V \rightarrow V = \text{Hom}^0(E, F), \quad (h^{-1})^* : V^\vee = \text{Hom}^1(F, E) \rightarrow \text{Hom}^1(F, F) = V^\vee, \]
\[ \text{Hom}^1(F, F) \xrightarrow{\lambda \cdot g} \text{Hom}^1(F, F), \]
where \( h \in \text{GL}(V) \) and \( \lambda \in S^* \) satisfy
\[ h^{-1}gh = \lambda g. \]

Now the condition that the induced automorphism of the graded algebra \( \mathcal{R} = \bigoplus_{i \geq 0} \text{Hom}(E_0, E_i) \) is the identity, implies that the automorphism \( \text{Ad}(h^{-1}) \) of \( \mathcal{R}/(t) \simeq \mathcal{E}(V, g)^{op} \) is the identity. But this is possible only when \( h \) is a scalar matrix, \( h = c \cdot \text{id} \). In this case we can
change Φ by an isomorphic $A_\infty$-equivalence (rescaling at $F$), so that $H^0\Phi$ becomes the identity.

Thus, we deduce that $(m_\bullet)$ and $(m'_\bullet)$ are gauge equivalent locally over Spec($S$). Applying [26, ], we deduce that they are globally gauge equivalent.

**Step 2.** The forgetful map from (2) to (3) is injective. Indeed, this follows immediately from Lemma 4.4.1.

**Step 3.** We have a natural map from (3) to (1): starting from $(A, F \cdot A, \iota)$ we construct an $n$-pair $(E, F)$ by considering the derived category $D^b(qgr\mathcal{R}(A))$ and considering the objects $E = O$ and $F \in qgr\mathcal{R}(A)$ defined in Proposition 4.2.1(iii).

**Step 4.** We claim that the composition $(3) \to (1) \to (2) \to (3)$ is the identity. Together with the injectivity proved in Steps 1 and 2, this would imply that our arrows give bijections between data (1), (2) and (3).

Thus, we start from $(A, F \cdot A, \iota)$, consider the corresponding $n$-pair of spherical objects $(E, F)$ as in Step 3, then look at the twisted complexes $E_i = T^F_i(E)$ and consider the corresponding Hom-algebra $\mathcal{R}_{T^F,E}$. By [30, Lem. 3.34], the inclusion of the full subcategory $\{E, F\}$ extends to an $A_\infty$-functor $\Phi : Tw\{E, F\} \to D(qgr\mathcal{R}(A))$, which is a quasi-equivalence with its image. Thus, Proposition 4.2.1(iv) gives the required isomorphism of the algebra $R_{T^F,E}$ with $R(A)$, preserving the natural central elements $t$.

Note that the last assertion of the theorem follows from the fact that the composition $(1) \to (2) \to (3)$ is the identity. □

### 4.6. Moduli spaces and the proof of Theorem A.

We would like to show that the functor associating to $S$ an isomorphism class of the data

$$(\mathcal{L}, g, A, F \cdot A, \iota : gr^F(A) \simeq \mathcal{E}(V, g)^{op}) \quad (4.6.1)$$

as before (where $\mathcal{L}$ is a locally free $S$-module of rank 1, $g \in \text{End}(S^n) \otimes \mathcal{L}$ is an invertible element, such that $\text{tr}(g)$ is a generator of $\mathcal{L}$ if $n = 2$), is representable by an affine scheme $\text{Spec} S_{filt}$ of finite type over $\mathbb{Z}$.

For $n \geq 3$, let $S_0$ be the algebra of functions on $\text{PGL}_n$ (which is the degree 0 part in the localization $\mathbb{Z}[x_{ij}][\text{det}^{-1}]$), and let $g \in \text{End}(S_0^n) \otimes \mathcal{O}_{S_0}(1)$ be the universal invertible element. In the case $n = 2$, we define $S_0$ as the algebra of functions on the open subset of $\text{PGL}_n$ given by nonvanishing of $\text{tr}(g)^n/\text{det}(g)$.

Recall that the algebra $\mathcal{E} := \mathcal{E}(S_0^n, g)^{op}$ is Koszul (see Lemma 2.3.1(iii)). This means that the filtered algebras $A$ we would like to study are given by nonhomogeneous quadratic relations whose homogeneous quadratic parts are the quadratic relations in $\mathcal{E}$. Let $I_\mathcal{E} \subset \mathcal{E}_1 \otimes S_0 \mathcal{E}_1$ denote the space of quadratic relations.

**Lemma 4.6.1.** The natural morphism

$$\nabla : \mathcal{E}_1^\vee \to \text{Hom}_{S_0}(I_\mathcal{E}, \mathcal{E}_1) : \xi \mapsto (e \otimes e' \mapsto \xi(e)e' + \xi(e')e) \quad (4.6.2)$$

is a split embedding of $S_0$-modules.

**Proof.** Since both $\mathcal{E}_1^\vee$ and $\text{Hom}_{S_0}(I_\mathcal{E}, \mathcal{E}_1)$ are finitely generated projective modules over $S_0$, it is enough to check that for any $S_0$-algebra $S$, the morphism $\nabla_S$, obtained from $\nabla$ by
extension of scalars, is injective. Now we note that since \( \mathcal{E}_S^- = \mathcal{E} \otimes_{S_0} S \) is quadratic, the elements of \( \ker(\nabla_S^-) \) are precisely derivations of \( \mathcal{E}_S^- \) of degree \(-1\). By Proposition 2.3.2, any such derivation is zero. □

**Proposition 4.6.2.** The functor associating to \( S \) the set of isomorphism classes of triples \((4.6.1)\) is representable by an affine scheme \( \text{Spec} S_{filt} \) of finite type over \( S_0 \) (and hence, over \( \mathbb{Z} \)).

**Proof.** As was discussed above, our functor associates to \( S \) the set of nonhomogeneous quadratic algebras deforming \( \mathcal{E}_S^- = \mathcal{E} \otimes_{S_0} S \). For such an algebra we can always choose a splitting \( s : \mathcal{E}_{S,1} \to F_1A \) of the projection \( F_1A \to F_1A/F_0A \cong \mathcal{E}_{S,1} \). Set \( U := s(\mathcal{E}_{S,1}) \subset F_1A \). Then algebra \( A \) can be given by some submodule of nonhomogeneous quadratic relations

\[
I_A \subset U \otimes_S U \oplus U \oplus S,
\]

such that the projection to \( U \otimes_S U \cong \mathcal{E}_{S,1} \otimes_S \mathcal{E}_{S,1} \) induces an isomorphism of \( I_A \) with the submodule \( I_{\mathcal{E}_S^-} \) of quadratic relations in \( \mathcal{E}_S^- \). Thus, \( I_A \) is the graph of an \( S \)-linear map \((\phi, \theta) : I_{\mathcal{E}_S^-} \to U \oplus S\).

Conversely, starting from such data we can construct the algebra \( T(U)/(I_A) \) which is equipped with a map \( \mathcal{E} \to \text{gr}^F(A) \). Since the algebra \( \mathcal{E} \) is Koszul, by [27, Sec. V.2], this correspondence gives a bijection between the set of quadruples \((A, F_\bullet A, \iota, s)\), where \( s : \mathcal{E}_{S,1} \to F_1A \) is a splitting, and pairs of maps \((\phi, \theta)\) satisfying certain quadratic equations (analogs of Jacobi identity). \(^1\)

Now we can change a splitting \( s \to s + \xi \), where \( \xi \in \text{Hom}_S(\mathcal{E}_{S,1}(S), S) \). It is easy to see that this corresponds to a certain action of \( \text{Hom}_S(\mathcal{E}_{S,1}(S), S) \) (viewed as an additive group) on pairs \((\phi, \theta)\). Furthermore, \( \phi \) gets changed to \( \phi + \nabla(\xi) \), with \( \nabla \) given by (4.6.2). Thus, if we fix a complementary \( S_0 \)-submodule \( K \subset \text{Hom}_{S_0}(I_\mathcal{E}, \mathcal{E}_1) \) to the image of \( \nabla \), which is possible by Lemma 4.6.1, then every orbit of the above action has a unique representative \((\phi, \theta)\) with \( \theta \in K_S \). The set of such \((\phi, \theta)\), satisfying the quadratic equations mentioned above, is the required affine scheme of finite type over \( S_0 \). □

**End of proof of Theorem A.** As we have seen in Proposition 4.6.2 and Theorem 1.1.3, we have two finitely generated \( \mathbb{Z} \)-algebras, \( S_{filt} \) and \( S_{ainf} \), that represent the functors \((1)\) and \((3)\). Since both \( S_{ainf} \) and \( S_{filt} \) are Noetherian, by the Noetherian case of Theorem A(i) and by I온eda lemma, we obtain an isomorphism

\[
S_{filt} \cong S_{ainf}.
\]

This gives the required isomorphism of functors. □

4.7. **Proof of Theorem B.** We have discussed in detail the construction of the order on a neat stacky pointed curve associated with a filtered algebra \((A, F_\bullet)\). To go from \((2)\) to \((1)\) let us consider the filtration \( F_iA = H^0(C \setminus p, \mathcal{A}(ip)) \) on the algebra \( A = H^0(C \setminus p, \mathcal{A}) \).

\(^1\)In [27] we work over a field, however, the argument still applies in the case when all the graded components \( \mathcal{E}_{S,i} \) are finitely generated projective modules over \( S \), and \( \mathcal{E}_{S,0} = S \).
From the trivialization of $\mathcal{O}(p)|_p$ (given by the tangent vector) we get a natural injective homomorphism of graded algebras

$$\text{gr}^F A \to \text{End}(V)^{\text{op}} \otimes k[z].$$

We claim that its image is $\mathcal{E}(V,g)^{\text{op}}$ for some $g \in \mathbb{P} \text{End}(V)$. Indeed, the exact sequence

$$0 \to H^0(C,\mathcal{A}) \to H^0(C,\mathcal{A}(p)) \to H^0(p,\mathcal{A}(p)|_p) \to H^1(C,\mathcal{A}) \to 0$$

shows that the image of the restriction map $F_1A \to H^0(p,\mathcal{A}(p)|_p) \simeq \text{End}(V)$ is of codimension 1. Hence, it has form $\text{End}_g(V)$ for a unique $g \in \mathbb{P} \text{End}(V)$. On the other hand, since $H^1(C,\mathcal{A}(ip)) = 0$ for $i \geq 1$, the restriction maps

$$F_mA = H^0(C,\mathcal{A}(mp)) \to H^0(p,\mathcal{A}(mp)|_p) \simeq \text{End}(V)$$

are surjective for $m \geq 2$, which proves our claim. Thus, we get an isomorphism of $\text{gr}^F A$ with $\mathcal{E}(V,g)^{\text{op}}$.

Using Lemma 3.1.2(iv) it is easy to check that starting from a filtered algebra $(A,F_n)$ and constructing an order $\mathcal{A}$ over a stacky curve $C$, we then recover the original filtered algebra by the above construction.

Conversely, if we start with an order $\mathcal{A}$ over a neat pointed stacky curve $(C,p)$ and consider the filtered algebra $(A,F_n)$ with $F_iA = H^0(C,\mathcal{A}(ip))$, then we recover $(C,p,\mathcal{A})$ by the stacky Proj construction, described in Sec. 3.1. Indeed, this is proved similarly to the non-stacky case, using the fact that $\mu_d$ acts faithfully on the fiber of $\mathcal{O}_C(p)$ at $p$ (see [1, Sec. 2.4] for similar results).

Next, by Lemma 3.2.1, the pair $(\mathcal{A},\rho,sV)$ generates Perf$(\mathcal{A}^{\text{op}})$. Thus, by Proposition 3.1.3, we have an equivalence of Perf$(\mathcal{A}^{\text{op}})$ with the full subcategory in $D_{qgr} \mathcal{R}(A)$, which sends $(\mathcal{A},\rho,sV)$ to the pair $(\mathcal{O},F)$.

If $g$ is invertible then, by Proposition 4.2.1(iii), the pair $(\mathcal{O},F)$ in $D_{qgr} \mathcal{R}(A)$ is an $n$-pair of 1-spherical objects, so $\mathcal{A}$ is a spherical order. Conversely, if $\mathcal{A}$ is spherical then $g$ is invertible by Proposition 3.2.2(ii).

Finally, let us prove that $\mathcal{A}$ is symmetric if and only if $g$ is scalar. Note that for any spherical order we have a canonical Nakayama automorphism $\kappa$ defined by the equation

$$\tau(yx) = \tau(x\kappa(y)),$$

where $\tau : \mathcal{A} \to \omega_C$ is a nonzero morphism. Indeed, we have two isomorphisms of coherent sheaves,

$$\nu : \mathcal{A} \to \underline{\text{Hom}}(\mathcal{A},\omega_C) : a \mapsto (x \mapsto \tau(xa)), \quad \nu' : \mathcal{A} \to \underline{\text{Hom}}(\mathcal{A},\omega_C) : a' \mapsto (x \mapsto \tau(a'x))$$

(see Proposition 3.2.2), and we set $\kappa = \nu^{-1} \circ \nu'$. The fact that $\kappa$ is an automorphism of algebras follows from the defining identity.

By Proposition 3.2.2(ii), the restricted functional

$$\tau|_p : \mathcal{A}|_p \simeq \text{End}(V) \otimes R_{\mu_d} \to \chi$$

is given by $x \mapsto \text{tr}(gx)$ on $\text{End}(V) \otimes \chi$. Hence, we have

$$\kappa|_p = \text{Ad}(g) \otimes \text{id} : \text{End}(V) \otimes R_{\mu_d} \to \text{End}(V) \otimes R_{\mu_d}.$$

This immediately shows that if $\mathcal{A}$ is symmetric, i.e., $\kappa = \text{id}$, then $g$ is scalar. Conversely, assume that $g$ is scalar. Then $\kappa|_p = \text{id}$. Now $\kappa$ induces a filtered automorphism of the algebra $A = H^0(C \setminus \{p\},\mathcal{A})$, and the condition that $\kappa|_p = \text{id}$ implies that the induced
automorphism of $gr^F A$ is equal to the identity. Hence, by Proposition 2.4.5, this automorphism of $A$ is equal to the identity, and so $\kappa = id$. \hfill \Box

4.8. A criterion for cyclic $A_\infty$-structures. Recall that an $A_\infty$-algebra over a field $k$ is called cyclic if it is equipped with a bilinear form $\langle \cdot, \cdot \rangle$ such that

$$
\langle m_n(a_1, \ldots, a_n), a_{n+1} \rangle = (-1)^{n(\deg(a_1)+1)} \langle a_1, m_n(a_2, \ldots, a_{n+1}) \rangle.
$$

Kontsevich and Soibelman give a general criterion [10, Thm. 10.2.2] in the case when $\text{char}(k) = 0$ stating that such a cyclic structure exists on a minimal model of an $A_\infty$-algebra $A$ with finite dimensional cohomology $H^\ast(A)$, provided there is a functional $\theta : HC_N(A) \to k$ (where $HC_\ast(A)$ is the cyclic homology of $A$) such that the induced pairing on $H^\ast(A)$,

$$
\langle x, y \rangle = \theta(\iota(xy)),
$$

where $\iota : H^\ast(C) \to HC_\ast(A)$ is the natural map, is perfect.

In the case of algebras of the form $H^\ast(C, A)$, where $A$ is a sheaf of coherent algebras over a curve $C$, we can provide a more direct construction of a cyclic structure, which only uses the assumption that $\text{char}(k) \neq 2$, and relies instead on the cyclic version of the homological perturbation from [11].

Proposition 4.8.1. Let $B = B_0 \oplus B_1$ be a dg-algebra over a field $k$, concentrated in degrees $[0, 1]$, $\langle \cdot, \cdot \rangle$ a pairing of degree 1 on $B$ satisfying

$$
\langle x, y \rangle = (-1)^{\deg(x) \deg(y)} \langle y, x \rangle,
$$

$$
\langle dx, y \rangle + (-1)^{\deg(x)} \langle x, dy \rangle = 0.
$$

Assume also that $H^\ast(B)$ is finite-dimensional and the induced pairing on $H^\ast(B)$ is perfect. Then the data for the homological perturbation can be chosen in such a way that the resulting minimal $A_\infty$-structure on $H^\ast(B)$ is cyclic with respect to the pairing induced by $\langle \cdot, \cdot \rangle$.

Proof. Let $A \subset \ker(d) \subset B$ be any (graded) subspace of cohomology representatives. We claim that there exists a subspace $C \subset B_0$, complementary to $\ker(d)$, such that $\langle C, A_1 \rangle = 0$. Indeed, let us start with an arbitrary such complement $C \subset B_0$. Then the pairing $C \otimes A_1 \to k$ can be interpreted as a map $C \to A_1^1 \simeq A_0$ (where the latter isomorphism is given by the pairing between $A_0$ and $A_1$). Correcting $C$ by this map, we get a new subspace in $C \oplus A_0$, which is still complementary to $\ker(d)$, and which is orthogonal to $A_1$.

Note that we have orthogonalities $\langle C, A \rangle = 0$, $\langle C, C \rangle = 0$. Hence, the standard homotopy operator $Q : B \to B$ associated with the decomposition $B = \text{im}(d) \oplus A \oplus C$ satisfies

$$
\langle Qx, y \rangle = (-1)^{\deg(x)} \langle x, Qy \rangle.
$$

As was observed in [11, Sec. 3.3], this implies that the minimal $A_\infty$-structure on $H^\ast(B)$ given by the tree formula is cyclic. \hfill \Box

We apply this general result in the following geometric setup.
Proposition 4.8.2. Let $C$ be a tame proper DM-stacky curve over a field $k$ of characteristic $\neq 2$, with a Cohen-Macaulay coarse moduli space $\overline{C}$ such that $H^0(\overline{C}, O) = k$. Let $\mathcal{A}$ be a coherent sheaf of $O_C$-algebras, equipped with a morphism $\tau : \mathcal{A} \to \omega_C$. Assume that we have a morphism $\pi : \mathcal{A} \to \omega_C$ such that $\tau(xy) = \tau(yx)$. Assume that the induced pairing

$$\mathcal{A} \otimes \mathcal{A} \to \omega_C$$

induced by $\tau(xy)$, is perfect in the derived category (either on the left or on the right). Then the minimal $A_\infty$-structure on $H^*(C, \mathcal{A})$ obtained by homological perturbation is gauge equivalent to a one, cyclic with respect to the pairing $\theta(xy)$, where $\theta : H^1(C, \mathcal{A}) \to H^n(C, \omega_C) \to k$ is induced by $\tau$.

Proof. First of all, since $C$ is tame, we have an isomorphism of algebras

$$H^*(C, \mathcal{A}) \simeq H^*(\overline{C}, \overline{\mathcal{A}}),$$

where $\overline{\mathcal{A}} := \pi_* \mathcal{A}$ and $\pi : C \to \overline{C}$ to the coarse moduli map. Also, we have an isomorphism

$$\pi_* \omega_C \simeq \omega_{\overline{C}}$$

(see [18, Prop. 2.3.1]). Thus, we can view the morphism $\pi_* \tau$ as a morphism

$$\overline{\tau} : \overline{\mathcal{A}} \to \omega_{\overline{C}}.$$ 

Furthermore, the induced pairing

$$\overline{\mathcal{A}} \otimes \overline{\mathcal{A}} \to \omega_{\overline{C}},$$

given by $\overline{\tau}(xy)$, factors through the natural projection $\overline{\mathcal{A}} \otimes \overline{\mathcal{A}} \to \pi_*(\mathcal{A} \otimes \mathcal{A})$ and hence, is still symmetric. Finally, by the relative duality we have an isomorphism

$$\pi_* R \text{Hom}(\mathcal{A}, \omega_C) \simeq \pi_* R \text{Hom}(\mathcal{A}, \pi_* \omega_C) \simeq R \text{Hom}(\overline{\mathcal{A}}, \omega_{\overline{C}}),$$

which implies the $\overline{\tau}$ still satisfies the required non-degeneracy condition. Thus, replacing $(C, \mathcal{A}, \tau)$ with $(\overline{C}, \overline{\mathcal{A}}, \overline{\tau})$, we can assume that $C$ is a usual (non-stacky) curve.

We can compute $H^*(C, \mathcal{A})$ using the Cech resolution

$$C(\mathcal{A}) : \mathcal{A}(U_1) \oplus \mathcal{A}(U_2) \xrightarrow{\delta} \mathcal{A}(U_{12}),$$

with respect to a covering $C = U_1 \cap U_2$, where $U_i$ are open affine subsets, $U_{12} = U_1 \cap U_2$. Here $\delta(f_1, f_2) = f_2 - f_1$. Since $\text{char}(k) \neq 2$, we can equip $C(\mathcal{A})$ with the following dg-algebra structure: the product on $\mathcal{A}(U_1) \oplus \mathcal{A}(U_2)$ is the one on direct sum of algebras, while for $(f_1, f_2) \in \mathcal{A}(U_1) \oplus \mathcal{A}(U_2)$, $g \in \mathcal{A}(U_1 \cap U_2)$, we set

$$(f_1, f_2)g = \frac{(f_1 + f_2)|_{U_{12}} g}{2}, \quad g(f_1, f_2) = \frac{g(f_1 + f_2)|_{U_{12}}}{2}.$$ 

Note that with respect to this product we have

$$[(f_1, f_2), g] = \frac{1}{2} ([f_1|_{U_{12}}, g] + [f_2|_{U_{12}}, g]).$$

Since the map $\tau$ vanishes on the commutators, the induced map of Cech complexes

$$C(\mathcal{A}) \to C(\omega_C)$$

also does, with respect to the above product. Composing this map with a map $C(\omega_C) \to k[-1]$, realizing the canonical trace morphism $H^1(C, \omega_C) \to k$, we get a map $\theta : C(\mathcal{A})_1 \to k[-1]$.
$k$, vanishing on the image of the differential and on the commutators. Furthermore, Serre duality implies that the map

$$H^i(C, \mathcal{A}) \otimes H^{1-i}(C, \mathcal{A}) \rightarrow H^1(\omega_C) \rightarrow k,$$

induced by $\tau(xy)$ is a perfect pairing. Thus, using Proposition 4.8.1 we get a cyclic $A_\infty$-structure with respect to $\theta(xy)$. 

\(\square\)

**Remark 4.8.3.** In characteristic zero there is a generalization of Proposition 4.8.2 to coherent sheaves of dg-algebras over higher-dimensional schemes over a field of characteristic zero. In this case one has to use Thom-Sullivan construction (see [8, Sec. 5.2], [37, App. A,B]) to get a multiplicative structure on derived global sections, and then apply the criterion of Kontsevich-Soibelman [10, Thm. 10.2.2].

One more observation is that the assumptions of Proposition 4.8.2 are preserved when passing from $\mathcal{A}$ to the endomorphism sheaf of a locally projective $\mathcal{A}$-module.

**Lemma 4.8.4.** Let $\mathcal{A}$ be a coherent sheaf of $\mathcal{O}_C$-algebras, together with a morphism $\tau : \mathcal{A} \rightarrow \omega_C$, satisfying the assumptions of of Proposition 4.8.2, and let $\mathcal{P}$ be a locally projective finitely generated $\mathcal{A}$-module. Consider the sheaf of algebras $\widetilde{\mathcal{A}} := \text{End}_{\mathcal{A}}(\mathcal{P})$. Then the assumptions of Proposition 4.8.2 still hold for $\widetilde{\mathcal{A}}$ and $\widetilde{\tau} : \widetilde{\mathcal{A}} \rightarrow \omega_C$ defined as the composition of $\tau$ and the trace morphism $\text{tr} : \text{End}_{\mathcal{A}}(\mathcal{P}) \rightarrow \mathcal{A}/[\mathcal{A}, \mathcal{A}]$.

**Proof.** Note that as $\mathcal{O}$-module, $\widetilde{\mathcal{A}}$ is locally a summand in $\mathcal{A}^{\otimes n}$ for some $n$. Hence, the assumption that $\text{Ext}^{>0}(\mathcal{A}, \omega_C) = 0$ implies that the same holds for $\widetilde{\mathcal{A}}$, so we only need to check that $\widetilde{\tau}$ vanishes on $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}]$ and that the pairing $\widetilde{\tau}(xy)$ is perfect. The former is the standard fact about traces. For the latter we can assume $\mathcal{P}$ to be a direct summand of $\mathcal{A}^{\otimes n}$. First, we observe that the pairing

$$\tau(\text{tr}(xy)) : \text{Mat}_n(\mathcal{A}) \otimes \mathcal{O} \text{Mat}_n(\mathcal{A}) \rightarrow \mathcal{O}$$

is a direct sum of pairings $\mathcal{A} \cdot e_{ij} \otimes \mathcal{A} \cdot e_{ji} \rightarrow \mathcal{O}$, which are perfect by assumption. Next, consider a direct sum decomposition $\mathcal{A}^{\otimes n} = \mathcal{P} \oplus \mathcal{Q}$, and let $e_\mathcal{P}$ and $e_\mathcal{Q}$ be the corresponding idempotents in $\text{Mat}_n(\mathcal{A})$. Then to deduce that the restriction of $\tau(\text{tr}(xy))$ to $e_\mathcal{P} \text{Mat}_n(\mathcal{A}) e_\mathcal{P}$ it is enough to check that the decomposition

$$\text{Mat}_n(\mathcal{A}) = e_\mathcal{P} \text{Mat}_n(\mathcal{A}) e_\mathcal{P} \oplus (e_\mathcal{Q} \text{Mat}_n(\mathcal{A}) \oplus e_\mathcal{P} \text{Mat}_n(\mathcal{A}) e_\mathcal{Q})$$

is orthogonal with respect to our form. But this immediately follows from the identities $e_\mathcal{P} e_\mathcal{Q} = 0$ and $\text{tr}(yx) \equiv \text{tr}(xy) \mod [\mathcal{A}, \mathcal{A}]$. \(\square\)

**Proof of Corollary C.** The first part follows immediately from Theorem B: we can realize every $A_\infty$-structure on $S(k^n, \text{id})$ by the one coming from a symmetric spherical order $\mathcal{A}$.

For the last assertion, we use the fact that for such $\mathcal{A}$, a nonzero morphism $\tau : \mathcal{A} \rightarrow \omega_C$ induces a symmetric pairing $\mathcal{A} \otimes \mathcal{A} \rightarrow \omega_C$ which is perfect in derived category (see Proposition 3.2.2(ii)). Recall that we want to construct a cyclic minimal $A_\infty$-structure on $\text{Ext}^*(G, G)$, where $G = \mathcal{A} \oplus \rho_* V$. Let $L$ be a sufficiently positive power of an ample line bundle on $C$. Then twisting $G$ through the spherical object $L^{-1} \otimes \mathcal{A}$ gives an $\mathcal{A}$-module $\mathcal{P}$ fitting into an exact sequence

$$0 \rightarrow \mathcal{P} \rightarrow \text{Hom}_\mathcal{A}(L^{-1} \otimes \mathcal{A}, G) \otimes L^{-1} \otimes \mathcal{A} \rightarrow G \rightarrow 0$$
Since local projective dimension of $G$ is 1, this immediately implies that $P$ is locally projective. Furthermore, since the spherical twist can be defined on a dg-level, we can replace $G$ by $P$ when studying the minimal $A_\infty$-structure on $\text{Ext}^*_A(G,G) \simeq \text{Ext}^*_A(P,P)$ obtained by the homological perturbation. Now, combining Proposition 4.8.2 with Lemma 4.8.4, we get that the minimal $A_\infty$-structure on $H^*(C, \text{End}_A(P,P))$ obtained by the homological perturbation can be chosen to be cyclic. □

References

[1] D. Abramovich, B. Hassett, Stable varieties with a twist, in Classification of algebraic varieties, 1–38, Eur. Math. Soc., Zürich, 2011.
[2] M. Artin, J. J. Zhang, Noncommutative projective schemes, Advances Math. 109 (1994), 228–287.
[3] M. Auslander, O. Goldman, Maximal orders, Trans. AMS 97(1960), 1–24.
[4] A. Bondal, A. Polishchuk, Homological properties of associative algebras: method of helices, Russian Acad. Sci., Izvestia Math. 42 (1994), 219–260.
[5] D. Chan, Lectures on orders, available at web.maths.unsw.edu.au/~danielch/
[6] P. Jorgensen, Local cohomology for non-commutative graded algebras, Communications in Algebra 25 (1997), 575–591.
[7] R. Hartshorne, Algebraic Geometry, Springer, 1977.
[8] V. Hinich, V. Schechtman, Deformation theory and Lie algebra homology, I,II, Algebra Colloq. 4 (1997), 213–240, 291–316.
[9] B. Keller, Introduction to $A$-infinity algebras and modules, Homology Homotopy Appl. 3 (2001), 1–35.
[10] M. Kontsevich, Y. Soibelman, Notes on $A_\infty$-algebras, $A_\infty$-categories and non-commutative geometry, I, in Homological mirror symmetry, 153–219, Springer, Berlin, 2009.
[11] C. I. Lazaroiu, Generating the superpotential on a D-brane category: I, preprint arXiv:hep-th/0610120.
[12] Y. Lekili, T. Perutz, Arithmetic mirror symmetry for the 2-torus, preprint arXiv:1211.4632.
[13] Y. Lekili, A. Polishchuk, A modular compactification of $\mathcal{M}_{1,n}$ from $A_\infty$-structures, arXiv:1408.0611, to appear in J. Reine Angew. Math.
[14] Y. Lekili, A. Polishchuk, Arithmetic mirror symmetry for genus 1 curves with $n$ marked points, Selecta Math. 23 (2017), 1851–1907.
[15] Y. Lekili, A. Polishchuk, Associative Yang-Baxter equation and Fukaya categories of square-tiled surfaces, arXiv:1608.08992.
[16] D. Luna, Slices Étales, Bull. Soc. Math. de France 33 (1973), 81–105.
[17] H. Minamoto, A noncommutative version of Beilinson’s theorem, J. Algebra 320 (2008), 238–252.
[18] F. Nironi, Grothendieck duality for Deligne-Mumford stacks, preprint arXiv:0811.1955.
[19] M. Olsson, J. Starr, Quot functors for Deligne-Mumford stacks, Comm. Algebra 31 (2003), no. 8, 4069–4096.
[20] Orlov, Remarks on generators and dimensions of triangulated categories, Mosc. Math. J. 9 (2009), 153–159.
[21] D. Piontkovski, Coherent algebras and noncommutative projective lines, J. Algebra 319 (2008), 3280–3290.
[22] A. Polishchuk, Classical Yang-Baxter equation and the $A_\infty$-constraint, Adv. Math. 168 (2002), 56–95.
[23] A. Polishchuk, Extensions of homogeneous coordinate rings to $A_\infty$-algebras, Homology, Homotopy and Applications 5 (2003), 407–421.
[24] A. Polishchuk, Massey products on cycles of projective lines and trigonometric solutions of the Yang-Baxter equations, in Algebra, Arithmetic and Geometry, Vol.HI: in Honor of Yu. I. Manin, 573–618, Birkhäuser, Boston, 2009.
[25] A. Polishchuk, Moduli of curves as moduli of $A_\infty$-structures, Duke Math J. 166 (2017), 2871–2924.
[26] A. Polishchuk, Moduli of curves with nonspecial divisors and relative moduli of $A_\infty$-structures, arXiv:1511.03797, to appear in Journal of the Inst. Math. Jussieu.
[27] A. Polishchuk, L. Positselskii, *Quadratic algebras*, Amer. Math. Soc., Providence, RI, 2005.
[28] A.-C. Van Roosmalen, *Abelian 1-Calabi-Yau categories*, IMRN 2008, no. 6, Art. ID rnn003.
[29] Schelter, *On the Krull-Akizuki theorem*, J. London Math. Soc. (2) 13 (1976), 263–264.
[30] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, European Math. Soc., Zürich, 2008.
[31] P. Seidel, *Abstract analogues of flux as symplectic invariants*, Mém. Soc. Math. Fr. (N.S.) No. 137 (2014).
[32] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), 37–108.
[33] L. W. Small, R. B. Warfield, Jr., *Prime affine algebras of Gelfand-Kirillov dimension one*, J. Algebra 91 (1984), 386–389.
[34] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves I*, Compos. Math. 147 (2011), no. 3, 877–913.
[35] M. Van den Bergh, *Non-commutative homology of some three-dimensional quantum spaces*, in *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992)*, K-Theory 8 (1994), 213–230.
[36] M. Van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra 195 (1997), 662–679.
[37] M. Van den Bergh, *On global deformation quantization in the algebraic case*, J. Algebra 315 (2007), no. 1, 326–395.
[38] Q. S. Wu and J. J. Zhang, *Dualizing complexes over noncommutative local rings*, J. Algebra 239 (2001), 513–548.
[39] J. J. Zhang, *Non-Noetherian regular rings of dimension 2*, Proc. AMS 126 (1998), 1645–1653.

University of Oregon, National Research University Higher School of Economics, and Korea Institute for Advanced Study