FUNCTIONS WITH SMALL AND LARGE SPECTRA AS (NON)EXTREME POINTS IN SUBSPACES OF $H^\infty$

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Dedicated to Nikolai Kapitonovich Nikolski on the occasion of his 80th birthday

Abstract. Given a subset $\Lambda$ of $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, let $H^\infty(\Lambda)$ denote the space of bounded analytic functions $f$ on the unit disk whose coefficients $\hat{f}(k)$ vanish for $k \notin \Lambda$. Assuming that either $\Lambda$ or $\mathbb{Z}_+ \setminus \Lambda$ is finite, we determine the extreme points of the unit ball in $H^\infty(\Lambda)$.

1. Introduction

Let $H^\infty$ stand for the space of bounded holomorphic functions on the disk $D := \{z \in \mathbb{C} : |z| < 1\}$. As usual, a function $f \in H^\infty$ is identified with its boundary trace on the circle $T := \partial D$, defined almost everywhere in the sense of nontangential convergence. We thus embed $H^\infty$ in $L^\infty = L^\infty(T)$, the space of essentially bounded functions on $T$, bearing in mind that the quantity $\|f\|_\infty := \sup\{|f(z)| : z \in D\}$ agrees, for $f \in H^\infty$, with the $L^\infty$ norm of the boundary function $f|_T$. The underlying theory and other basic facts about $H^\infty$ can be found in any of [11, 12, 13].

We shall be concerned with the geometry of the unit ball—specifically, with the structure of its extreme points—in certain subspaces of $H^\infty$. These will appear shortly, once a bit of terminology and notation is fixed.

Given a (complex) Banach space $X = (X, \| \cdot \|)$, we write $\text{ball}(X) := \{x \in X : \|x\| \leq 1\}$ for the closed unit ball of $X$. Also, we recall that a point in $\text{ball}(X)$ is said to be extreme for the ball if it is not the midpoint of any two distinct points in $\text{ball}(X)$.

Further, with an integrable function $f$ on $T$ we associate the sequence of its Fourier coefficients $\hat{f}(k) := \frac{1}{2\pi} \int_T \overline{\zeta^k} f(\zeta) |d\zeta|$, $k \in \mathbb{Z}$, and the set $\text{spec } f := \{k \in \mathbb{Z} : \hat{f}(k) \neq 0\}$.

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known as the *spectrum* of \( f \). Thus, in particular,

\[
H^\infty = \{ f \in L^\infty : \text{spec } f \subset \mathbb{Z}_+ \},
\]

where \( \mathbb{Z}_+ \) stands for the set of nonnegative integers.

The geometry of the unit ball in \( H^\infty \), let alone \( L^\infty \), seems to be well understood. To begin with, it is worth mentioning that the extreme points of ball(\( L^\infty \)) are precisely the unimodular functions on \( \mathbb{T} \). As to ball(\( H^\infty \)), its extreme points are characterized among the unit-norm functions \( f \in H^\infty \) by the weaker condition that

\[
\int_{\mathbb{T}} \log(1 - |f(\zeta)|) |d\zeta| = -\infty \quad \text{(see, e.g., [3, Section V] or [12, Chapter 9]).}
\]

Our purpose here is to see what happens for subspaces of \( H^\infty \) that are formed by functions with prescribed spectral gaps. Precisely speaking, given a subset \( \Lambda \) of \( \mathbb{Z}_+ \), we consider the space

\[
H^\infty(\Lambda) := \{ f \in H^\infty : \text{spec } f \subset \Lambda \},
\]

with norm \( \| \cdot \|_\infty \), and we seek to characterize the extreme points of ball(\( H^\infty(\Lambda) \)). This will be accomplished in two special cases that represent two “extreme” situations. Namely, it will be assumed that either \( \Lambda \) or \( \mathbb{Z}_+ \setminus \Lambda \) is a finite set (this dichotomy accounts for the phrase “small and large spectra” in the paper’s title). The results pertaining to each of these cases will be stated in Section 2 below, and then proved in Sections 3 and 4.

Meanwhile, we mention that similar questions have already been studied in the context of the Hardy space \( H^1 \). The extreme points of ball(\( H^1 \)) were identified by de Leeuw and Rudin [3] as outer functions of norm 1. The case of

\[
H^1(\Lambda) := \{ f \in H^1 : \text{spec } f \subset \Lambda \}
\]

was recently settled by the author for sets \( \Lambda \subset \mathbb{Z}_+ \) that are either finite (see [8]) or have finite complement in \( \mathbb{Z}_+ \) (see [9, 10]). Among the finite \( \Lambda \)'s, we single out the “gapless” sets of the form

\[
\Lambda_N := \{0, 1, \ldots, N\},
\]

with \( N \) a positive integer, in which case we are dealing with the space of polynomials of degree at most \( N \). For this last space, endowed with the \( L^1 \) norm over \( \mathbb{T} \), the extreme points of the unit ball were described earlier in [5]; alternatively, the description follows from [4, Theorem 6].

Going back to the \( H^\infty(\Lambda) \) setting, we remark that the nonlacunary polynomial case, where \( \Lambda = \Lambda_N \), was treated previously in [6]. When moving to general finite sets \( \Lambda \), however, we have to face new complications. For spaces of trinomials, which arise when \( \#\Lambda = 3 \), a detailed analysis was carried out by Neuwirth in [14]; there, both the extreme and exposed points of the unit ball were determined. (By definition, given a Banach space \( X \), a point \( x \in \text{ball}(X) \) is *exposed* for the ball if there exists a functional \( \phi \in X^* \) of norm 1 such that the set \( \{ y \in \text{ball}(X) : \phi(y) = 1 \} \) equals \( \{ x \} \).) On the other hand, a theorem of Amar and Lederer (see [1]) tells us
that the exposed points of ball($H^\infty$) are precisely the unit-norm functions $f \in H^\infty$ for which the set $\{ \zeta \in \mathbb{T} : |f(\zeta)| = 1 \}$ has positive measure.

Here, we make no attempt to characterize the exposed points of ball($H^\infty$($\Lambda$)). Rather, we mention this as an open problem. When $\# \Lambda < \infty$ or $\#(\mathbb{Z}_+ \setminus \Lambda) < \infty$, one might probably arrive at a solution with relatively light machinery, via a suitable adaptation of our current techniques.

Restricting our attention to the extreme points of ball($H^\infty$($\Lambda$)), as we do here, we are still puzzled by the case where both $\Lambda$ and $\mathbb{Z}_+ \setminus \Lambda$ are infinite sets. It would be nice to gain some understanding of what happens for such $\Lambda$’s. In particular, we wonder which arithmetic properties of $\Lambda$ (if any) are relevant to the problem. A more specific question related to condition (1.1) is raised in Section 2 below, next to Theorem 2.1.

Finally, we mention yet another type of subspaces in $H^\infty$ where the structure of the extreme points remains unclear. Namely, given an inner function $\theta$, we consider the model subspace $K^\infty_\theta := H^\infty \cap \theta z H^\infty$ and ask for a characterization of the extreme points of ball($K^\infty_\theta$). This problem was originally posed in [7]; see also [4] for a treatment of its $L^1$ counterpart, where a simple solution is available. Except for the case of $\theta(z) = z^{N+1}$, when $K^\infty_\theta$ agrees with $H^\infty(\Lambda_N)$, the two types of spaces (i.e., $H^\infty(\Lambda)$ and $K^\infty_\theta$) are rather different in nature, though.

2. Statement of results

We begin with the case where $\mathbb{Z}_+ \setminus \Lambda$ is finite, since a neater formulation is then available and the result is easier to establish. In fact, the extreme point criterion that arises in this case for $H^\infty(\Lambda)$ turns out to be the same as for $H^\infty$.

**Theorem 2.1.** Let $\Lambda \subset \mathbb{Z}_+$ be a set with
\[
\#(\mathbb{Z}_+ \setminus \Lambda) < \infty.
\]
Suppose further that $f \in H^\infty(\Lambda)$ and $\|f\|_\infty = 1$. Then $f$ is an extreme point of ball($H^\infty(\Lambda)$) if and only if it satisfies (1.1).

It would be interesting to find a complete description of the sets $\Lambda \subset \mathbb{Z}_+$ with the property that the extreme points of ball($H^\infty(\Lambda)$) are characterized by (1.1). One feels that such $\Lambda$’s should be suitably “thick” in $\mathbb{Z}_+$, but the sufficient condition (2.1) is certainly far from being necessary. It seems plausible that an appropriate sparseness condition on $\mathbb{Z}_+ \setminus \Lambda$ would actually suffice. At the same time, for a set $\Lambda$ with the desired property, it may well happen that $\mathbb{Z}_+ \setminus \Lambda$ is no thinner (in whatever sense) than $\Lambda$ itself, as we shall now see.

By way of example, take $\Lambda$ to be $2\mathbb{Z}_+$, the set of nonnegative even integers. Now let $f \in H^\infty(2\mathbb{Z}_+)$ be a function with $\|f\|_\infty = 1$. Assuming that
\[
\int_{\mathbb{T}} \log(1 - |f(\zeta)|) |d\zeta| > -\infty,
\]
we put
\[
(2.2) \quad g(z) := \exp \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(1 - |f(\zeta)|) |d\zeta| \right\}, \quad z \in \mathbb{D},
\]
so that $g$ is the outer function with modulus $1 - |f|$ on $\mathbb{T}$. Furthermore, $g$ is an even function in $H^\infty$ (because $f$ is even) and hence $g \in H^\infty(2\mathbb{Z}_+)$. Also, $\|f \pm g\|_\infty \leq 1$. Thus, $f + g$ and $f - g$ are two distinct points of ball$(H^\infty(2\mathbb{Z}_+))$, while $f$ is their midpoint. This proves the necessity of (1.1) in order that $f$ be an extreme point of ball$(H^\infty(2\mathbb{Z}_+))$. The sufficiency is trivial, since $H^\infty(2\mathbb{Z}_+) \subset H^\infty$.

We now mention an analogue of Theorem 2.1 where the underlying space is taken to be the disk algebra $C_A := H^\infty \cap C(\mathbb{T})$ instead of $H^\infty$. This time, $H^\infty(\Lambda)$ gets replaced by

$$C_A(\Lambda) := H^\infty(\Lambda) \cap C_A$$

and we have the following result.

**Proposition 2.2.** Given a set $\Lambda \subset \mathbb{Z}_+$ satisfying (2.1), the extreme points of ball$(C_A(\Lambda))$ are precisely the unit-norm functions $f \in C_A(\Lambda)$ with property (1.1).

Next, we turn to the case where $\Lambda$ is a finite subset of $\mathbb{Z}_+$. The $H^\infty$ functions with spectrum in $\Lambda$ are now polynomials of the form

$$p(z) = \sum_{k \in \Lambda} \hat{p}(k) z^k,$$

and we prefer to denote the set of such polynomials by $\mathcal{P}(\Lambda)$ rather than by $H^\infty(\Lambda)$. Of course, $\mathcal{P}(\Lambda)$ is still endowed with the supremum norm, and we shall occasionally write $\mathcal{P}^\infty(\Lambda)$ for the normed space $(\mathcal{P}(\Lambda), \| \cdot \|_\infty)$ that arises.

We shall henceforth assume (without losing anything of substance) that $0 \in \Lambda$ and $\# \Lambda \geq 2$, so that

$$\Lambda = \{0, 1, \ldots, N\} \setminus \{k_1, \ldots, k_M\}$$

for some positive integers $N$ and $k_j$ ($j = 1, \ldots, M$) with

$$k_1 < k_2 < \cdots < k_M < N.$$

In the special case where $M = 0$, the set $\{k_1, \ldots, k_M\}$ is empty, so $\Lambda$ becomes $\Lambda_N$ (as defined by (1.2)) and $\mathcal{P}(\Lambda)$ reduces to

$$\mathcal{P}_N := \mathcal{P}(\Lambda_N),$$

the space of polynomials of degree at most $N$. In this nonlacunary case, the extreme points of the unit ball were previously characterized in [6]. Here, we refine the method of [6] to deal with spaces of lacunary polynomials (or fewnomials) that arise as $\mathcal{P}(\Lambda)$ for general sets $\Lambda$ of the form (2.3).

Among the unit-norm polynomials in $\mathcal{P}^\infty(\Lambda)$, the simplest examples are provided by the monomials $z \mapsto c z^k$, with $k \in \Lambda$ and $c$ a unimodular constant. Clearly, any such monomial is an extreme point of ball$(\mathcal{P}^\infty(\Lambda))$, so we may exclude these “trivial” extreme points from further consideration.

Now suppose $p \in \mathcal{P}(\Lambda)$ is a polynomial with $\|p\|_\infty = 1$ whose spectrum contains at least two elements. Our criterion for $p$ to be extreme in ball$(\mathcal{P}^\infty(\Lambda))$ will be stated in terms of a certain matrix $\mathcal{M} = \mathcal{M}_\Lambda(p)$ associated with $p$, and we proceed with the construction of $\mathcal{M}$. 
Let $\zeta_1, \ldots, \zeta_n$ be an enumeration of the (finite and nonempty) set $\{\zeta \in \mathbb{T} : |p(\zeta)| = 1\}$. Viewed as zeros of the function

$$\tau(z) := 1 - |p(z)|^2, \quad z \in \mathbb{T}$$

(or equivalently, of the polynomial $z^N\tau$), the $\zeta_j$’s have even multiplicities, which we denote by $2\mu_1, \ldots, 2\mu_n$ respectively; the $\mu_j$’s are therefore positive integers. We then put

$$(2.5) \quad \mu := \sum_{j=1}^n \mu_j \quad \text{and} \quad \gamma := \mu/2.$$ 

Since $z^N\tau \in \mathcal{P}_{2N}$, it follows that $\mu \leq N$.

For each $j \in \{1, \ldots, n\}$, we consider the Wronski-type matrix

$$W_j := \begin{pmatrix} \zeta_j p(\zeta_j) & (\zeta^+)^1 p(\zeta_j) & \cdots & (\zeta^+)^{N-\gamma} p(\zeta_j) \\ \zeta_j p(\zeta_j) & (\zeta^+)^1 p(\zeta_j) & \cdots & (\zeta^+)^{N-\gamma} p(\zeta_j) \\ \vdots & \vdots & \ddots & \vdots \\ (\zeta^+)^{\mu_j-1}(\zeta_j) & (\zeta^+)^{\mu_j-1}(\zeta_j) & \cdots & (\zeta^+)^{\mu_j-1}(\zeta_j) \end{pmatrix},$$

which has $\mu_j$ rows and $N - \mu + 1$ columns (indeed, the exponent $N - \gamma$ in the last column equals $\gamma + N - \mu$). Here, the convention is that the independent variable $z = e^{it}$ lives on $\mathbb{T}$ and that differentiation is with respect to the real parameter $t = \arg z$. More precisely, expressions of the form $(\zeta^+)^{\ell}(\zeta_j)$ with $\ell, s \in \mathbb{Z}_+$ should be interpreted as

$$\frac{ds}{dt} \left. \{e^{-i(\gamma + \ell)t} p(e^{it})\} \right|_{t=t_j},$$

where $t_j \in (-\pi, \pi]$ is defined by $e^{it_j} = \zeta_j$. We also need the real matrices

$$U_j := \Re W_j \quad \text{and} \quad V_j := \Im W_j \quad (j = 1, \ldots, n).$$

The rest of the construction involves the polynomial

$$(2.6) \quad r(z) := \prod_{j=1}^n (z - \zeta_j)^{\mu_j}$$

and its coefficients $\hat{r}(k)$ with $k \in \mathbb{Z}$. (For $k < 0$ and $k > \mu$, we obviously have $\hat{r}(k) = 0$.) From these, some further matrices will be built. Namely, we introduce the $M \times (N - \mu + 1)$ matrix

$$\mathcal{R} := \begin{pmatrix} \hat{r}(k_1) & \hat{r}(k_1 - 1) & \cdots & \hat{r}(k_1 - N + \mu) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{r}(k_M) & \hat{r}(k_M - 1) & \cdots & \hat{r}(k_M - N + \mu) \end{pmatrix},$$

along with the real matrices

$$\mathcal{A} := \Re \mathcal{R} \quad \text{and} \quad \mathcal{B} := \Im \mathcal{R}.$$
Finally, we define the block matrix

$$
\mathcal{M} = \mathcal{M}_\Lambda(p) := \begin{pmatrix}
\mathcal{A} & -\mathcal{B} \\
\mathcal{B} & \mathcal{A} \\
\mathcal{U}_1 & \mathcal{V}_1 \\
\vdots & \vdots \\
\mathcal{U}_n & \mathcal{V}_n
\end{pmatrix},
$$

which has $2M + \mu$ rows and $2(N - \mu + 1)$ columns.

**Theorem 2.3.** Given a set $\Lambda \subset \mathbb{Z}_+$ of the form (2.3), suppose that $p$ is a unit-norm polynomial in $\mathcal{P}^\infty(\Lambda)$ distinct from a monomial. Then $p$ is an extreme point of ball($\mathcal{P}^\infty(\Lambda)$) if and only if $\text{rank} \mathcal{M}_\Lambda(p) = 2(N - \mu + 1)$.

Even though the rank condition above may appear somewhat bizarre, it is unlikely that the criterion could be substantially simplified. In fact, even in the nonlacunary polynomial space (2.4), and already for $N = 2$, one can find unit-norm polynomials $p_1, p_2$ satisfying

$$
1 - |p_1(z)|^2 = 2 \left(1 - |p_2(z)|^2\right), \quad z \in \mathbb{T},
$$

and such that $p_1$ is a non-extreme point of the unit ball, while $p_2$ is extreme; see [6, p. 720] for an example. This means that, even for $\mathcal{P}_2$, the extreme point criterion cannot be stated in terms of the $\zeta_j$’s and $\mu_j$’s alone, so a certain level of complexity seems to be unavoidable.

### 3. Proofs of Theorem 2.1 and Proposition 2.2

**Proof of Theorem 2.1.** Let $f \in H^\infty(\Lambda)$ and $\|f\|_\infty = 1$. Assuming (1.1), we know that $f$ is an extreme point of ball($H^\infty$) and hence also of the smaller set ball($H^\infty(\Lambda)$).

Conversely, assume that (1.1) fails, so that

$$
\int_{\mathbb{T}} \log(1 - |f(\zeta)|) |d\zeta| > -\infty.
$$

Then we can find a function $g \in H^\infty$, $g \not\equiv 0$, satisfying

$$
|g| \leq 1 - |f|
$$

almost everywhere on $\mathbb{T}$ (e.g., take $g$ to be the outer function with modulus $1 - |f|$, as defined by (2.2)). Further, letting

$$
m := \#(\mathbb{Z}_+ \setminus \Lambda)
$$

and recalling the notation $\mathcal{P}_m$ for the set of polynomials of degree at most $m$, we go on to claim that there exists $p_0 \in \mathcal{P}_m$, $p_0 \not\equiv 0$, for which $gp_0 \in H^\infty(\Lambda)$. To see why, write

$$
\mathbb{Z}_+ \setminus \Lambda = \{k_1, \ldots, k_m\},
$$

where $k_1, \ldots, k_m$ are pairwise distinct integers, and consider the linear operator $T : \mathcal{P}_m \to \mathbb{C}^m$ that acts by the rule

$$
Tp := \left((gp)(k_1), \ldots, (gp)(k_m)\right), \quad p \in \mathcal{P}_m.
$$
Because \( \dim \mathcal{P}_m = m + 1 \), while the rank of \( T \) does not exceed \( m \), the rank-nullity theorem (see, e.g., [2, p. 63]) tells us that \( \ker T \), the null-space of \( T \), has dimension at least 1 and is therefore nontrivial.

Now, if \( p_0 \) is any non-null polynomial in \( \ker T \), then
\[
(gp_0)(k_1) = \cdots = (gp_0)(k_m) = 0,
\]
and so \( gp_0 \) is a nontrivial function in \( H^\infty(\Lambda) \). We may also assume that \( \|p_0\|_\infty \leq 1 \), and together with (3.2) this yields
\[
|f \pm gp_0| \leq |f| + |g||p_0| \leq |f| + |g| \leq 1
\]
almost everywhere on \( \mathbb{T} \). Consequently,
\[
f \pm gp_0 \in \text{ball}(H^\infty(\Lambda))
\]
and the identity
\[
f = \frac{1}{2}(f + gp_0) + \frac{1}{2}(f - gp_0)
\]
shows that \( f \) is not an extreme point of \( \text{ball}(H^\infty(\Lambda)) \).

**Proof of Proposition 2.2.** Once again, we only have to check that every unit-norm function \( f \in C_A(\Lambda) \) satisfying (3.1) is non-extreme in \( \text{ball}(C_A(\Lambda)) \).

For any such \( f \) (and actually for any \( f \in C_A \) with \( \|f\|_\infty \leq 1 \)), condition (3.1) enables us to find a non-null function \( g \in C_A \) that obeys (3.2); see [12, Chapter 9]. Now, using this \( g \) in place of its namesake above, while keeping the rest of notation, we can readily adjust the preceding proof to the current situation. Namely, we construct (exactly as before) a polynomial \( p_0 \in \mathcal{P}_m \) with \( 0 < \|p_0\|_\infty \leq 1 \) that makes (3.3) true. The product \( gp_0 \) is then a nontrivial function in \( C_A(\Lambda) \), and since (3.4) is again valid, it follows that
\[
f \pm gp_0 \in \text{ball}(C_A(\Lambda)).
\]
Finally, we infer from (3.5) that \( f \) is a non-extreme point of \( \text{ball}(C_A(\Lambda)) \).

**4. Proof of Theorem 2.3**

We begin by stating and proving a preliminary result.

**Lemma 4.1.** Given a finite set \( \Lambda \subset \mathbb{Z}_+ \), suppose that \( p \in \mathcal{P}(\Lambda) \) and \( \|p\|_\infty = 1 \). The following conditions are equivalent:

(i) \( p \) is not an extreme point of \( \text{ball}(\mathcal{P}^\infty(\Lambda)) \).

(ii) There exist positive constants \( C_1, C_2 \) and a non-null polynomial \( q \in \mathcal{P}(\Lambda) \) such that
\[
|q|^2 \leq C_1 (1 - |p|^2)
\]
and
\[
|\text{Re}(pq)| \leq C_2 (1 - |p|^2)
\]
everwhere on \( \mathbb{T} \).
Proof. Clearly, (i) holds if and only if there exists a non-null polynomial \( q \in \mathcal{P}(\Lambda) \) for which
\[
\| p + q \|_{\infty} \leq 1 \quad \text{and} \quad \| p - q \|_{\infty} \leq 1.
\]
An obvious restatement of (4.3) is that \( |p \pm q|^2 \leq 1 \) on \( T \); and since
\[
|p \pm q|^2 = |p|^2 \pm 2 \text{Re}(pq) + |q|^2,
\]
while \( \max(a, -a) = |a| \) for all \( a \in \mathbb{R} \), we may further rewrite (4.3) in the form
\[
2 |\text{Re}(pq)| + |q|^2 \leq 1 - |p|^2.
\]
Now, if (4.4) is fulfilled for some nontrivial \( q \in \mathcal{P}(\Lambda) \), then (4.1) and (4.2) are sure to hold (for the same \( q \)) with \( C_1 = 1 \) and \( C_2 = \frac{1}{2} \).

Conversely, suppose \( q \in \mathcal{P}(\Lambda) \) is a nontrivial polynomial that satisfies (4.1) and (4.2). Replacing \( q \) by \( \varepsilon q \) with a suitable \( \varepsilon > 0 \) if necessary, we can arrange it for \( C_1 \) and \( C_2 \) to be as small as desired. In particular, we may assume that \( C_1 \leq \frac{1}{2} \) and \( C_2 \leq \frac{1}{4} \). The resulting inequalities
\[
|q|^2 \leq \frac{1}{2} \left( 1 - |p|^2 \right)
\]
and
\[
2|\text{Re}(pq)| \leq \frac{1}{2} \left( 1 - |p|^2 \right)
\]
imply (4.4) and hence (4.3).

Proof of Theorem 2.3. Suppose that \( p \) satisfies the hypotheses of the theorem and fails to be an extreme point of \( \text{ball}(\mathcal{P}^{\infty}(\Lambda)) \). By Lemma 4.1, we can find a polynomial \( q \in \mathcal{P}(\Lambda), q \neq 0 \), that makes (4.1) and (4.2) true for some constants \( C_1, C_2 > 0 \).

Now, for each \( j \in \{1, \ldots, n\} \), we have
\[
1 - |p(z)|^2 = O \left( |z - \zeta_j|^{2\mu_j} \right)
\]
as \( z \in T \) tends to \( \zeta_j \). In conjunction with (4.1), this yields
\[
|q(z)|^2 = O \left( |z - \zeta_j|^{2\mu_j} \right),
\]
or equivalently,
\[
|q(z)| = O \left( |z - \zeta_j|^{\mu_j} \right)
\]
near \( \zeta_j \). Thus, \( q \) has a zero of multiplicity at least \( \mu_j \) at \( \zeta_j \). It follows that \( q \) is divisible by the polynomial \( r \) given by (2.6); and since \( q \in \mathcal{P}_N \), while \( r \in \mathcal{P}_{\mu} \), we see that
\[
q = q_0 r
\]
for some (non-null) \( q_0 \in \mathcal{P}_{N-\mu} \).

Our next step is to exploit (4.2), so as to gain further information about \( q_0 \). But first we need to derive a more convenient expression for \( r \). Given \( j \in \{1, \ldots, n\} \), we write \( \zeta_j = e^{it_j} \) and note that, for \( z = e^{it} \in T \), we have the identity
\[
z - \zeta_j = e^{it/2}e^{it_j/2} \cdot 2i \sin \frac{t - t_j}{2}.
\]
Here and throughout, it is assumed that
\[ t := \arg z \quad \text{and} \quad t_j := \arg \zeta_j, \]
where “arg” stands for the principal branch of the argument (i.e., the one with values in \((-\pi, \pi]\)). In particular, \( t \) (resp., \( t_j \)) is uniquely determined by \( z \) (resp., \( \zeta_j \)), and we put
\[ \varphi_j(z) := 2 \sin \frac{t - t_j}{2}, \quad z \in T. \]
Clearly, \( \varphi_j \) is real-valued and
\[ |\varphi_j(z)| = |z - \zeta_j|, \quad z \in T, \]
this last property being immediate from (4.6). We then rewrite (4.6) in the form
\[ z - \zeta_j = iz^{1/2} \zeta_j^{1/2} \varphi_j(z) \]
(with the appropriate determination of the square root). Raising both sides of (4.9) to the power \( \mu_j \) and taking products yields
\[ r(z) = \lambda z^\gamma \prod_{j=1}^{n} (\varphi_j(z))^{\mu_j}, \quad z \in T, \]
where
\[ \lambda := i^\mu \prod_{j=1}^{n} \zeta_j^{\mu_j/2} \]
and \( \gamma := \mu/2 \), in accordance with (2.5). We note that \( \lambda \) is a unimodular constant depending only on the \( \zeta_j \)'s and \( \mu_j \)'s.

Further, we combine (4.5) and (4.10) to get
\[ \Re \left( \frac{p(z)q(z)}{\overline{p(z)}q_0(z)} \right) = \prod_{j=1}^{n} (\varphi_j(z))^{\mu_j} \Re \left( \lambda z^\gamma \overline{p(z)}q_0(z) \right), \quad z \in T. \]
In view of (4.8), this implies that
\[ \left| \Re \left( \frac{p(z)q(z)}{\overline{p(z)}q_0(z)} \right) \right| = \prod_{j=1}^{n} |z - \zeta_j|^{\mu_j} \left| \Re \left( \lambda z^\gamma \overline{p(z)}q_0(z) \right) \right|. \]
On the other hand,
\[ 1 - |p(z)|^2 \asymp \prod_{j=1}^{n} |z - \zeta_j|^{2\mu_j}, \quad z \in T. \]
(As usual, the sign \( \asymp \) means that the ratio of the two quantities stays in the interval \([C^{-1}, C]\) for some constant \( C > 1 \)). Taking (4.11) and (4.12) into account, we now rewrite (4.2) as
\[ \left| \Re \left( \lambda z^\gamma \overline{p(z)}q_0(z) \right) \right| \leq \text{const} \cdot \prod_{j=1}^{n} |z - \zeta_j|^{\mu_j}, \quad z \in T. \]
Thus, for every $j \in \{1, \ldots, n\}$, the function
\[ z = e^{it} \mapsto \Re \left( \lambda z^\gamma p(z)q_0(z) \right) \]
has a zero of multiplicity at least $\mu_j$ at $\zeta_j$. This fact admits an obvious restatement in terms of derivatives; namely, for $j = 1, \ldots, n$ we have
\[ \Re \left( \lambda z^\gamma p(z)q_0(z) \right)^{(s)} (\zeta_j) = 0, \quad s = 0, \ldots, \mu_j - 1. \] (To keep on the safe side, we recall (4.7) and emphasize that the derivatives in (4.14) are actually taken with respect to $t$ and computed at $t_j$. In particular, differentiation commutes with the real part operator.) Now, we write the polynomial $q_0$ in the form
\[ q_0(z) = \lambda \sum_{l=0}^{N-\mu} (\alpha_l + i\beta_l) z^l, \]
where $\alpha_l$ and $\beta_l$ are real parameters, and plug this expression into (4.14). This done, we obtain for each $j \in \{1, \ldots, n\}$ the $\mu_j$ equations
\[ \sum_{l=0}^{N-\mu} \alpha_l \Re \left( z^{\gamma+l}p \right)^{(s)} (\zeta_j) - \sum_{l=0}^{N-\mu} \beta_l \Im \left( z^{\gamma+l}p \right)^{(s)} (\zeta_j) = 0, \]
or equivalently,
\[ \sum_{l=0}^{N-\mu} \Re \left( z^{\gamma+l}p \right)^{(s)} (\zeta_j) \cdot \alpha_l + \sum_{l=0}^{N-\mu} \Im \left( z^{\gamma+l}p \right)^{(s)} (\zeta_j) \cdot \beta_l = 0, \]
with $s = 0, \ldots, \mu_j - 1$. We have thus a total of $\mu_1 + \cdots + \mu_n = \mu$ equations here.

Furthermore, we want to recast the condition that $q \in \mathcal{P}(\Lambda)$ in terms of our $\alpha_l$’s and $\beta_l$’s. Since $\Lambda$ is given by (2.3), we know that
\[ \hat{q}(k_\nu) = 0 \quad \text{for} \quad \nu = 1, \ldots, M. \]

On the other hand, setting
\[ A_k := \Re \hat{r}(k) \quad \text{and} \quad B_k := \Im \hat{r}(k), \quad k \in \mathbb{Z}, \]
we use (4.5) and (4.15) to find that
\[ \hat{q}(k_\nu) = \sum_{l=0}^{N-\mu} \hat{q}_0(l) \hat{r}(k_\nu - l) \]
\[ = \lambda \sum_{l=0}^{N-\mu} (\alpha_l + i\beta_l) (A_{k_\nu-l} + iB_{k_\nu-l}) \]
for each $\nu$. Consequently, (4.17) can be rephrased by saying that
\[ \sum_{l=0}^{N-\mu} (A_{k_\nu-l} \alpha_l - B_{k_\nu-l} \beta_l) = 0, \quad \nu = 1, \ldots, M, \]
and

\begin{equation}
\sum_{l=0}^{N-\mu} (B_{k^l \nu - l} \alpha_l + A_{k^l \nu - l} \beta_l) = 0, \quad \nu = 1, \ldots, M. \tag{4.19}
\end{equation}

Taken together, the \(2M + \mu\) equations that appear above as (4.16), (4.18), and (4.19) tell us that the vector

\begin{equation}
(\alpha_0, \ldots, \alpha_{N-\mu}, \beta_0, \ldots, \beta_{N-\mu}) \tag{4.20}
\end{equation}

belongs to \(\text{Ker} \mathcal{M}\), the kernel of the linear map

\[ \mathcal{M} : \mathbb{R}^{2(N-\mu+1)} \to \mathbb{R}^{2M+\mu} \]

given by (2.7). The polynomial \(q\) (and hence \(q_0\)) being non-null, we see that the vector (4.20) is nonzero, and so

\begin{equation}
\text{Ker} \mathcal{M} \neq \{0\}. \tag{4.21}
\end{equation}

Now, because

\[ \dim(\text{Ker} \mathcal{M}) + \text{rank} \mathcal{M} = 2(N - \mu + 1) \]

by virtue of the rank-nullity theorem (see [2, p. 63]), we may further restate (4.21) in the form

\begin{equation}
\text{rank} \mathcal{M} < 2(N - \mu + 1). \tag{4.22}
\end{equation}

To summarize, we have proved that if \(p\) is a non-extreme point of \(\text{ball}(P^\infty(\Lambda))\), then (4.22) holds.

The converse is actually true as well, since every step in the above reasoning can be reversed. Indeed, assuming (4.22), we rewrite it as (4.21) and take (4.20) to be any nonzero vector in \(\text{Ker} \mathcal{M}\). Then we define the polynomials \(q_0\) and \(q\), in this order, by means of (4.15) and (4.5). Equations (4.18) and (4.19) yield (4.17), and it follows that \(q\) is a non-null polynomial in \(P(\Lambda)\). Moreover, conditions (4.1) and (4.2) are then fulfilled. In fact, (4.1) is immediate from (4.5) and (4.12), while (4.2) is ensured by (4.16). (One should recall that (4.16) is expressible as (4.14) and implies (4.13), which is equivalent to (4.2).) Finally, we invoke Lemma 4.1 to conclude that \(p\) is not an extreme point of \(\text{ball}(P^\infty(\Lambda))\).

Now we know that a unit-norm polynomial \(p \in P^\infty(\Lambda)\) is a non-extreme point of the unit ball if and only if the associated matrix \(\mathcal{M} = \mathcal{M}_\Lambda(p)\) satisfies (4.22). In other words, the extreme points—other than monomials—are characterized by the condition

\[ \text{rank} \mathcal{M} = 2(N - \mu + 1). \]

The proof is complete. \(\square\)

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