SO(2n, \mathbb{C})-character varieties are not varieties of characters

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Abstract. We prove that the coordinate rings of SO(2n, \mathbb{C})-character varieties are not generated by trace functions nor generalized trace functions for \( n \geq 2 \) and all groups \( \Gamma \) of corank \( \geq 2 \). Furthermore, we give examples of non-conjugate completely reducible representations undistinguishable by generalized trace functions. Hence, SO(2n, \mathbb{C})-character varieties are not varieties of characters!

However, we also prove that any generic SO(2n, \mathbb{C})-representation of a free group can be distinguished from all non-equivalent representations by trace functions and by a single generalized trace function.

For a complex, reductive algebraic group \( G \), and a finitely generated discrete group \( \Gamma \), the \( G \)-character variety of \( \Gamma \), \( X_G(\Gamma) \), is the categorical quotient of the representation variety, \( \text{Rep}(\Gamma, G) \), by the action of \( G \) by conjugation, cf. \cite{LM, S1} and the references within\(^1\). (In this paper \( \text{Rep}(\Gamma, G) \) and \( X_G(\Gamma) \) are affine algebraic sets rather than possibly non-reduced schemes of \cite{LM, S1}).

If \( G \) is a matrix group, then every \( \gamma \in \Gamma \) defines a trace function

\[ \tau_\gamma : X_G(\Gamma) \rightarrow \mathbb{C}, \quad \tau_\gamma([\rho]) = \text{tr}\rho(\gamma). \]

The \( G \)-trace algebra of \( \Gamma \), denoted by \( T_G(\Gamma) \), is a subalgebra of \( \mathbb{C}[X_G(\Gamma)] \) generated by \( \tau_\gamma \), for all \( \gamma \in \Gamma \).

For \( G = SL(n, \mathbb{C}), Sp(n, \mathbb{C}) \) (symplectic groups), and \( SO(2n + 1, \mathbb{C}) \) (odd special orthogonal groups) the coordinate rings, \( \mathbb{C}[X_G(\Gamma)] \), are generated by trace functions, \cite{S2, FL}, and although one might expect that for other groups \( G \) as well, it is not the case for \( G = SO(2n, \mathbb{C}) \). Indeed, if \( \rho : \Gamma \rightarrow SO(2n, \mathbb{C}) \) is irreducible and \( \rho' \) is obtained from \( \rho \) by a conjugation by a matrix \( M \in O(2n, \mathbb{C}) \), \( \det M = -1 \), then \( \rho \) and \( \rho' \) are indistinguishable by trace functions even though it is not difficult to see that they are not \( SO(2n, \mathbb{C}) \)-conjugate and, furthermore, they are distinct in \( X_{SO(2n, \mathbb{C})}(\Gamma) \), cf. \cite{S4, Sec. 4}.

\(^1\)Throughout the paper, the field of complex numbers can be replaced an arbitrary algebraically closed field of characteristic zero.

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Pursuing better conjugacy invariants of representations, one is naturally lead to the notion of generalized trace functions, which (as an added bonus) are defined irrespectively of specific realizations of $G$ as matrix groups. More specifically, associated with any $\gamma \in \Gamma$ and any (finite dimensional) representation $\phi$ of $G$ there is the generalized trace function

$$\tau_{\phi, \gamma} : X_G(\Gamma) \to \mathbb{C}, \quad \tau_{\phi, \gamma}(\rho) = \text{tr}_\phi \rho(\gamma).$$

The full $G$-trace algebra of $\Gamma$, $FT_G(\Gamma)$, is a subalgebra of $\mathbb{C}[X_G(\Gamma)]$ generated by $\tau_{\phi, \gamma}$, for all $\gamma \in \Gamma$ and all $\phi$, cf. [S2].

We proved in [S4] however that even these functions do not generate $\mathbb{C}[X_G(\Gamma)]$. Specifically, for any group $\Gamma$ of corank $\geq 2$ (ie. having the free group on two generators, $F_2$, as a quotient) $FT_{SO(4, \mathbb{C})}(\Gamma)$ is a proper subalgebra of $\mathbb{C}[X_{SO(4, \mathbb{C})}(\Gamma)]$. The main result of this paper is a generalization of this statement to even orthogonal groups of higher rank.

**Theorem 1.**

(1) \hspace{1cm} $FT_{SO(2n, \mathbb{C})}(\Gamma) \subset \mathbb{C}[X_{SO(2n, \mathbb{C})}(\Gamma)]$

for every group $\Gamma$ of corank $\geq 2$ and all $n \geq 2$.

Let $\iota : SO(4, \mathbb{C}) \to SO(2n, \mathbb{C})$ for $n > 2$ be the “obvious” embedding (extending $4 \times 4$ matrices by the $(2n - 4) \times (2n - 4)$ identity matrix). The above result would be an easy consequence of our earlier result for $SO(4, \mathbb{C})$ if the induced homomorphism

$$\iota_* : \mathbb{C}[X_{SO(2n, \mathbb{C})}(\Gamma)] \to \mathbb{C}[X_{SO(4, \mathbb{C})}(\Gamma)]$$

was onto. However, one can prove that the image of $\iota_*$ is $T_{SO(4, \mathbb{C})}(\Gamma)$, cf. Corollary [S4].

It remains an open question whether the proper inclusion (1) holds if $F_2$ is replaced by some of its quotients, like $\mathbb{Z}_p$. Our proof of the result for the free group $\Gamma = F_2$ ([S4]) relies on the fact that $\mathbb{C}[X_{SL(2, \mathbb{C})}(\Gamma)]$ has an $N$-grading, which is not the case for $\Gamma = \mathbb{Z}^2$ and, therefore, it does not generalize to that case.

It is worth mentioning though that we can prove that

$$FT_{SO(4, \mathbb{C})}(\Gamma) = \mathbb{C}[X_{SO(4, \mathbb{C})}(\Gamma)]$$

for $\Gamma = \mathbb{Z}_p^2$, for $p = 2, 3, 4, 5$ by a direct computation. (We use the topologist’s notation, $\mathbb{Z}_p$, for the cyclic group of order $p$.)

Note that the proper inclusion (1) does not imply that the generalized trace functions fail to distinguish non-equivalent $SO(2n, \mathbb{C})$-representations of $\Gamma$. To see that consider as an example representations into $GL(n, \mathbb{C})$. Their equivalence classes in $X_{GL(n, \mathbb{C})}(\Gamma)$ are distinguished by trace functions, even though

$$T_{GL(n, \mathbb{C})}(\Gamma) \subset \mathbb{C}[X_{GL(n, \mathbb{C})}(\Gamma)],$$

since it is easy to show that

$$f_\gamma([\rho]) = \text{det}(\rho(\gamma))^{-1}$$
is a regular function on $X_{GL(n,\mathbb{C})}(\Gamma)$ which is not a polynomial in trace functions for example for $\Gamma = \mathbb{Z}$ and $\gamma \neq 0$.

Here is another useful analogy: $x^2$ and $x^3$ distinguish all points of $\mathbb{C}$ despite the fact that $x^2$ and $x^3$ do not generate $\mathbb{C}[x] = \mathbb{C}[x]$. A reader may note that this “paradox” is related to the fact that $\mathbb{C}[x^2, x^3]$ is not integrally closed. A related issue of integral closeness of $\mathcal{F}T_{SO(2n,\mathbb{C})}(\Gamma)$ is considered in Proposition 3 below.

Nonetheless, we can strengthen our above results for higher $n$ as follows.

**Theorem 2.** For any $n \geq 7$, $n \neq 8$, any group $\Gamma$ projecting onto $\mathbb{Z}_p \ast \mathbb{Z}_q$ for $p, q > \max(2n - 14, 16)$ has two non-conjugate irreducible representations into $SO(2n, \mathbb{C})$ which are undistinguishable by the generalized trace functions.

These representations are related one to another through the involution $\sigma$ of $[S4]$, Sec. 4).

Since the elements of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ distinguish points of $X_{SO(2n,\mathbb{C})}(\Gamma)$, the proper inclusion (1) holds for all $\Gamma$ satisfying the assumptions of Theorem 2.

The embedding

$$\mathcal{F}T_{SO(2n,\mathbb{C})}(\Gamma) \subset \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$$

and the dual map to it which we denote by $\psi$,

$$\psi : X_{SO(2n,\mathbb{C})}(\Gamma) \rightarrow \text{Spec} \mathcal{F}T_{SO(2n,\mathbb{C})}(\Gamma),$$

is analyzed in the next statement.

**Proposition 3.** Let $n \geq 1$ and let $\Gamma$ be a group such that $X_{SO(2n,\mathbb{C})}(\Gamma)$ is irreducible and $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma) \neq \mathcal{F}T_{SO(2n,\mathbb{C})}(\Gamma)$. Then

1. $\psi$ is finite and birational.
   In particular, $\psi$ is generically 1-1. It is also a normalization map if $X_{SO(2n,\mathbb{C})}(\Gamma)$ is normal.

2. Let $\eta$ be one of the two irreducible representations $D_n^{\pm}$ of $[FH]$ §23.2, with the highest weight being twice that of $\pm$-half spin representation. Then there exists $\gamma \in \Gamma$ such that for a generic $[\rho] \in X_{SO(2n,\mathbb{C})}(\Gamma)$ the trace functions together with $\tau_{\eta,\gamma}$ distinguish $[\rho]$ from any other point on the $SO(2n,\mathbb{C})$-character variety of $\Gamma$. (Obviously, only a finite set of trace functions generating the trace algebra is sufficient here.)

Note that the above assumptions are satisfied by free groups of rank $\geq 2$ for $n \geq 2$.

1. **Proof of Theorem 1**

In $[S2]$, we have introduced a function $Q$ of $n$ arguments $A_1, \ldots, A_n \in M(2n, \mathbb{C})$ defined as follows:

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2 For simplicity, we skip the index $2n$ used in $[S2]$. 

\[ Q(A_1, ..., A_n) = \sum_{\sigma \in S_{2n}} sn(\sigma)(A_1,\sigma(1),\sigma(2) - A_1,\sigma(2),\sigma(1)) \]

\[ (A_{n,\sigma(2n-1)},\sigma(2n)) = A_{n,\sigma(2n)},(\sigma(2n-1)) \]

where \( A_{i,j,k} \) is the \((j,k)\)-th entry of \( A_i \) and \( sn(\sigma) = \pm 1 \) is the sign of \( \sigma \).

By abuse of notation, for any \( \gamma_1, ..., \gamma_n \in \Gamma \) we have also defined a function \( Q(\gamma_1, ..., \gamma_n) \) on \( X_{SO(2n,\mathbb{C})}(\Gamma) \), sending \([\rho]\) to \( Q(\rho(\gamma_1), ..., \rho(\gamma_n)) \).

Let \( SO(2n,\mathbb{C}) = \{ A : AA^T = I, \ det(A) = 1 \} \).

By [S2], \( \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)] \) is generated by the trace functions, \( \tau_\gamma \), and the functions \( Q(\gamma_1, ..., \gamma_n) \), for \( \gamma, \gamma_1, ..., \gamma_n \in \Gamma \).

Any homomorphism \( \pi : \Gamma \to F_2 \) induces

\[ \pi_* : \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)] \to \mathbb{C}[X_{SO(2n,\mathbb{C})}(F_2)] \]

which restricts to

\[ \pi_* : FT_{SO(2n,\mathbb{C})}(\Gamma) \to FT_{SO(2n,\mathbb{C})}(F_2). \]

Furthermore, we have

\[ \pi_* \tau_{\phi,\gamma} = \tau_{\phi,\pi(\gamma)} \]

and

\[ \pi_* Q_n(\gamma_1, ..., \gamma_n) = Q_n(\pi(\gamma_1), ..., \pi(\gamma_n)). \]

Consequently, if \( \pi \) is an epimorphism then all generators of \( \mathbb{C}[X_{SO(2n,\mathbb{C})}(F_2)] \) are in the image of \( \pi_* \). Hence, it is sufficient to prove Theorem \( \Box \) for \( \Gamma = F_2 \) only.

Therefore, assume \( \Gamma = F_2 \) from now on, with a generating set denoted by \( \gamma_1, \gamma_2 \).

Note that there is a group isomorphism \( \mathbb{C}^* \to SO(2,\mathbb{C}) \) sending \( c \in \mathbb{C}^* \) to

\[ D_c = \begin{pmatrix} (c + c^{-1})/2 & i(c - c^{-1})/2 \\ -i(c - c^{-1})/2 & (c + c^{-1})/2 \end{pmatrix}, \]

where \( i^2 = -1 \). For \( c \in \mathbb{C}^* \), consider a map \( \iota_c : SO(4,\mathbb{C}) \to SO(2n,\mathbb{C}) \) sending \( A \in SO(4,\mathbb{C}) \) to a block matrix composed of \( A \) (in the top left corner) and of \( n-1 \) blocks \( D_c \) along the diagonal. Using the identification of \( \text{Hom}(F_2, SO(2n,\mathbb{C})) \) with \( SO(2n,\mathbb{C})^2 \) through a map sending a representation \( \rho \) to \( (\rho(\gamma_1), \rho(\gamma_2)) \), we have a map

\[ \alpha_{c_1,c_2} : \text{Hom}(F_2, SO(4,\mathbb{C})) = SO(4,\mathbb{C})^2 \to SO(2n,\mathbb{C})^2 = \text{Hom}(F_2, SO(2n,\mathbb{C})), \]

sends \( (A_1, A_2) \in SO(4,\mathbb{C})^2 \) to \( (\iota_{c_1}(A_1), \iota_{c_2}(A_2)) \). Note that the above map is not induced by a homomorphism from \( SO(4,\mathbb{C}) \) to \( SO(2n,\mathbb{C}) \) unless \( c_1 = c_2 = 1 \).

Alternatively, \( \alpha_{c_1,c_2} \) can be defined as follows: Let

\[ w = (w_1, w_2) : F_2 \to \mathbb{Z}^2 \]

be the abelianization map. Then \( \alpha_{c_1,c_2} \) maps \( \rho \) to \( \alpha_{c_1,c_2}(\rho) \) such that

\[ \alpha_{c_1,c_2}(\rho)(\gamma) = \iota_c(\rho(\gamma)), \]

for \( c = c_1, c_2 \).
where \( c = \frac{w_1(\gamma)}{c_1} \frac{w_2(\gamma)}{c_2}. \)

It is easy to see that \( \alpha_{c_1,c_2} \) factors through

\[
\alpha_{c_1,c_2} : X_{SO(4,\mathbb{C})}(F_2) \to X_{SO(2n,\mathbb{C})}(F_2).
\]

**Proposition 4.** \( \alpha_{c_1,c_2} : \mathbb{C}[X_{SO(2n,\mathbb{C})}(F_2)] \to \mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)] \) is onto for \( c_1, c_2 \neq \{-1, 0, 1\}, c_1 \neq \pm c_2^{-1}. \)

The proof of this statement follows. Afterwards, we are also going to prove (Theorem 9) asserting that

\[
\alpha_{c_1,c_2}((FT_{SO(2n,\mathbb{C})}(F_2)) \subset FT_{SO(4,\mathbb{C})}(F_2).
\]

These two statements together with the proper inclusion

\[
FT_{SO(4,\mathbb{C})}(F_2) \subsetneq \mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]
\]

proven in \([4]\) imply that \( FT_{SO(2n,\mathbb{C})}(F_2) \) is a proper subalgebra of \( \mathbb{C}[X_{SO(2n,\mathbb{C})}(F_2)] \), completing the proof of Theorem 1.

We precede the proof of Proposition 4 with a few lemmas.

**Lemma 5.** Let \( A_1, ..., A_n \in M(2n, \mathbb{C}). \) If each \( A_i \) is a block matrix built of an upper left diagonal block \( B_i \) of dimension \((2n-2) \times (2n-2)\) and of lower right \( 2 \times 2 \) block \( C_i \), then \( Q(A_1, ..., A_n) \) equals

\[
\sum_{i=1}^n Q(B_1, ..., \hat{B}_i, ..., B_n) \cdot Q(C_i),
\]

where \( \hat{\cdot} \) denotes an omitted symbol.

Note that \( Q(D_c) = i(c - c^{-1}) \).

**Proof.** By our assumptions, the a summand in (2) for \( \sigma \in S_n \) is not zero only if for every \( i = 1, ..., n \), both of \( \sigma(2i-1) \) and \( \sigma(2i) \) belong to \( \{1, ..., 2n-2\} \) or to \( \{2n-1, 2n\} \). Therefore, if we denote the set of \( \sigma \in S_n \) such that \( \sigma(2i-1), \sigma(2i) \in \{2n-1, 2n\} \) by \( S_n^i \), then

\[
Q(A_1, ..., A_n) = \sum_{i=1}^n S_i(A_1, ..., A_n),
\]

where \( S_i \) is the sum of terms of (2) for \( \sigma \in S_n^i \). By [2], \( Q \) is a symmetric function with respect to its \( n \) arguments. Therefore, we can write \( S_i(A_1, ..., A_n) \) as \( S_i(A_1, ..., \hat{A}_i, ..., A_n, A_i) \), which by (2) coincides with \( Q(B_1, ..., \hat{B}_i, ..., B_n) \cdot Q(C_i). \)

**Corollary 6.** The “obvious” embedding, \( \iota_1 : SO(4, \mathbb{C}) \hookrightarrow SO(2n, \mathbb{C}) \) extending matrices by the \((2n-4) \times (2n-4)\) identity matrix, induces the map \( \alpha_{1,1} : \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)] \to \mathbb{C}[X_{SO(4,\mathbb{C})}(\Gamma)] \) whose image is \( \mathcal{T}_{SO(4,\mathbb{C})}(\Gamma) \).

**Proof.** The \( SO(4,\mathbb{C}) \) matrices embedded into \( SO(2n,\mathbb{C}) \) as above contain a lower right block \( D_i \). Since \( Q(D_i) = 0 \), the above lemma implies that \( Q(\gamma_1, ..., \gamma_n) = 0 \) for all \( \gamma_1, ..., \gamma_n \in \Gamma \).
On the other hand, since
\[ \alpha_{1,1} \tau_{\gamma} = \tau_{\gamma} + 2n - 4, \]
the homomorphism \( \alpha_{1,1} \) maps the \( SO(2n, \mathbb{C}) \)-trace algebra onto \( SO(4, \mathbb{C}) \)-trace algebra, implying the statement. \( \square \)

Denote by \( Q_{k,l}(A_1, A_2) \) the value of \( Q \) evaluated at \( k \) matrices \( A_1 \) and \( l \) matrices \( A_2 \). For convenience, assume that \( Q_{k,l}(A_1, A_2) = 0 \) for \( k < 0 \) or \( l < 0 \).

**Corollary 7.** If \( A_1, A_2 \) are block matrices, as in Lemma 5 then
\[ Q_{k,l}(A_1, A_2) = k \cdot Q_{k-1,l}(B_1, B_2)Q(C_1) + l \cdot Q_{k,l-1}(B_1, B_2)Q(C_2). \]

Let \( Q_n(A) \) denote the value of \( Q \) for all its \( n \) arguments equal \( A \in \mathbb{M}(2n, \mathbb{C}) \).

**Lemma 8.** For any \( A_1, A_2 \in \mathbb{M}(4, \mathbb{C}) \),
\[
(1) \quad Q_n(t_c(A)) = \frac{1}{2^n} \left[ \frac{1}{2^n} - (c - c^{-1}) \right]^{n-2} n! Q_{2}(A), \quad \text{for any } A \in \mathbb{M}(4, \mathbb{C}).
\]
\[
(2) \quad Q_{n-1,1}(t_{c_1}(A_1), t_{c_2}(A_2)) = [i(c_1 - c_1^{-1})]^{n-2} (n - 1)! Q_{1,1}(A_1, A_2) + \frac{1}{2^n} [i(c_2 - c_2^{-1})]^{n-2} (n - 1)! Q_{1,2}(A_1, A_2)
\]
\[
+ \frac{1}{2^n} [i(c_1 - c_1^{-1})]^{n-2} (n - 1)! Q_{2,1}(A_1, A_2) + \frac{1}{2^n} [i(c_2 - c_2^{-1})]^{n-2} (n - 1)! Q_{2,2}(A_1, A_2)
\]
\[
+ \sum_{k=2}^{n-1} \frac{1}{2^n} [i(c_1 - c_1^{-1})]^{k-2} k! Q_{k,1}(A_1) + \sum_{k=2}^{n-1} \frac{1}{2^n} [i(c_2 - c_2^{-1})]^{k-2} k! Q_{k,2}(A_2).
\]

**Proof.** We have \( Q(D_c) = i(c - c^{-1}) \). Hence, by Corollary 7 for any \( A_1, A_2 \in \mathbb{M}(4, \mathbb{C}) \),
\[
(4) \quad Q_{k,l}(t_{c_1}(A_1), t_{c_2}(A_2)) = k \cdot Q_{k-1,l}(t_{c_1}(A_1), t_{c_2}(A_2)) \cdot i(c_1 - c_1^{-1})
\]
\[
+ l \cdot Q_{k,l-1}(t_{c_1}(A_1), t_{c_2}(A_2)) \cdot i(c_2 - c_2^{-1}),
\]
where \( t_{c_i} \) are embeddings of \( \mathbb{M}(4, \mathbb{C}) \) into \( \mathbb{M}(2(k + l), \mathbb{C}) \) on the left and into \( \mathbb{M}(2(k + l) - 2, \mathbb{C}) \) on the right.

In particular,
\[ Q_k(t_c(A)) = i(c - c^{-1}) \cdot kQ_{k-1}(t_c(A)), \]
implies part (1).

Similarly, by the above formula,
\[
(5) \quad Q_{k,1}(t_{c_1}(A_1), t_{c_2}(A_2)) = i(c_1 - c_1^{-1})\cdot k Q_{k-1,1}(t_{c_1}(A_1), t_{c_2}(A_2))
\]
\[
+ i(c_2 - c_2^{-1})Q_k(t_{c_1}(A_1))
\]
which by part (1) equals to
\[ i(c_1 - c_1^{-1})\cdot k Q_{k-1,1}(t_{c_1}(A_1), t_{c_2}(A_2)) + \frac{1}{2} i^{k-1}(c_2 - c_2^{-1})(c_1 - c_1^{-1})^{k-2} k! Q_{2}(A_1). \]
Now we obtain part (2) of the statement by induction on \( n \). \( \square \)

**Proof of Proposition 4.** By [S3] Thm. 2, \( \mathbb{C}[X_{SO(4, \mathbb{C})}(F_2)] \) is generated by trace functions and by the functions
\[ Q(\gamma; \gamma), \quad \text{for } \gamma \in \{ \gamma_1, \gamma_2, \gamma_1\gamma_2, \gamma_1\gamma_2^{-1} \} \]
and by
\[
(6) \quad Q(\gamma_1, \gamma_2), \quad Q(\gamma_1, \gamma_2\gamma_1^{-1}), \quad Q(\gamma_2, \gamma_1\gamma_2^{-1}),
\]
where $\gamma_1, \gamma_2$ generate $F_2$, as before. (We used here the fact that $Q$ is symmetric in its arguments.)

Since
\[ \alpha_{c_1,c_2*}(\tau_\gamma) = \tau_\gamma + (c + c^{-1})(n-2), \]
for $c = c_1^{u_1(\gamma)} c_2^{u_2(\gamma)}$, the function $\alpha_{c_1,c_2*}$ maps the $SO(2n, \mathbb{C})$-trace algebra of $F_2$ onto the $SO(4, \mathbb{C})$-trace algebra. Therefore, it is enough to show that all of the above $Q$ functions belong to the image of $\alpha_{c_1,c_2*}$ as well.

Note that
\[ \alpha_{c_1,c_2*}Q_n(\gamma)[\rho] = Q_n((\iota_{c_1,c_2}\rho)(\gamma)). \]
By (6) and by Lemma 8(2), that equals to
\[ \frac{1}{2} [i(c - c^{-1})]^{n-2} n! Q_2(\rho(\gamma)) \]
for $c = c_1^{u_1(\gamma)} c_2^{u_2(\gamma)}$. Therefore, up to a constant, $\alpha_{c_1,c_2*}Q_n(\gamma)$ equals $Q(\gamma, \gamma)$. By the assumptions about $c_1$ and $c_2$, that constant is non-zero for $\gamma \in \{\gamma_1, \gamma_2, \gamma_1\gamma_2, \gamma_1\gamma_2^{-1}\}$. Therefore, $Q(\gamma, \gamma)$ belongs to the image of $\alpha_{c_1,c_2*}$ for $\gamma$ as above.

Finally, it remains to be shown that the three functions of (6) belong to the image of $\alpha_{c_1,c_2*}$ as well.

By (6) and by Lemma 8(2),
\[ \alpha_{c_1,c_2*}Q_{n-1,1}(\gamma, \gamma')[\rho] = Q_{n-1,1}((\iota_{c_1,c_2}\rho)(\gamma), (\iota_{c_1,c_2}\rho)(\gamma')) \]
equals to
\[ \frac{1}{2} [i(c - c^{-1})]^{n-2} (n-1)! Q(\rho(\gamma), \rho(\gamma')) + d \cdot Q(\rho(\gamma), \rho(\gamma)), \]
for $c = c_1^{u_1(\gamma)} c_2^{u_2(\gamma)}$. (Note that this “$c$” is the “$c_1$” in Lemma 8(2) here.) Consequently,
\[ \alpha_{c_1,c_2*}Q_{n-1,1}(\gamma, \gamma') = \frac{1}{2} [i(c - c^{-1})]^{n-2} n! Q(\gamma, \gamma') + d \cdot Q(\gamma, \gamma). \]
By taking
\[ (\gamma, \gamma') \in \{ \gamma_1, \gamma_2, \gamma_1\gamma_2^{-1}, \gamma_2, \gamma_1\gamma_2^{-1} \} \]
we see that linear combinations of the desired elements of (6) (with non-zero coefficients) and of $Q(\gamma_1, \gamma_1), Q(\gamma_2, \gamma_2)$ belong to the image of $\alpha_{c_1,c_2*}$. Since we proved already that $Q(\gamma_1, \gamma_1), Q(\gamma_2, \gamma_2)$ belong to the image of $\alpha_{c_1,c_2*}$, the proof is complete.

**Theorem 9.** For any group $\Gamma$ and any $c_1, c_2 \in \mathbb{C}^*$, $\alpha_{c_1,c_2*}$ maps $FT_{SO(2n, \mathbb{C})}(\Gamma)$ to $FT_{SO(4, \mathbb{C})}(\Gamma)$.

For the sake of the proof of the above theorem, it will be convenient to consider another matrix realization of $SO(2n, \mathbb{C})$: Let $J_{2n}$ be a $2n \times 2n$ matrix composed of $n$ diagonal blocks of the form
\[ J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
and let
\[ SO_J(2n, \mathbb{C}) = \{ A : AJ_{2n}A^T = J_{2n}, \ det(A) = 1 \}. \]
Since \( J_{2n} = K_{2n} \cdot K_{2n}^T \), where \( K_{2n} \) is composed of \( n \) diagonal blocks of the form
\[ K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \]
the isomorphism
\[ \Phi(A) = K_2^{-1} AK_{2n}, \quad \Phi : SO_J(2n, \mathbb{C}) \rightarrow SO(2n, \mathbb{C}) \]
defines these two groups.

**Proof of Theorem 9**: It is enough to show that for every \( \gamma \in \Gamma \), for every representation \( \phi : SO(2n, \mathbb{C}) \rightarrow GL(N, \mathbb{C}) \) and for every \( c_1, c_2 \in \mathbb{C}^* \), \( \alpha_{c_1, c_2}(\tau_{\phi, \gamma}) \) can be expressed by a polynomial in variables \( \tau_{\psi_i, \gamma} \) for some \( SO(4, \mathbb{C}) \)-representations \( \psi_i \). We will show (a seemingly stronger, but equivalent statement) that under the above assumptions,

\[ \alpha_{c_1, c_2}(\tau_{\phi, \gamma}) = \sum_{i=1}^{k} a_i \tau_{\psi_i, \gamma}, \]
for some \( SO(4, \mathbb{C}) \)-representations \( \psi_1, ..., \psi_k \) and some \( a_1, ..., a_k \in \mathbb{C} \).

The map \( \alpha_{c_1, c_2} \) sends a regular function \( f \) on \( X_{SO(2n, \mathbb{C})}(\Gamma) \) to a function on \( X_{SO(4, \mathbb{C})}(\Gamma) \), assigning to the equivalence class of \( \rho : \Gamma \rightarrow SO(4, \mathbb{C}) \) the value \( f(\alpha_{c_1, c_2}(\rho)) \). Therefore, (8) can be restated as

\[ tr \phi(\alpha_{c_1, c_2}(\rho))(\gamma) = \sum_{i=1}^{k} a_i tr(\psi_i(\rho(\gamma))). \]

From now on, we will identify \( SO(2n, \mathbb{C}) \) with \( SO_J(2n, \mathbb{C}) \) via (7) and, hence, assume that \( \rho : \Gamma \rightarrow SO(4, \mathbb{C}) \).

Let \( T_n \) be the group of diagonal matrices in \( SO_J(2n, \mathbb{C}) \) such that the \((2i+1, 2i+1)\) entry is the inverse of the \((2i, 2i)\) entry for \( i = 1, ..., n \). It is a maximal torus in \( SO_J(2n, \mathbb{C}) \) (and its Cartan subgroup). Since equation (7) depends on the equivalence class of \( \rho \) in \( Hom(\Gamma, SO_J(4, \mathbb{C})) \) only and since this class contains a representative \( \rho' \) such that \( \rho'(\gamma) \) belongs to \( T_2 \subset SO_J(4, \mathbb{C}) \), we can assume for simplicity that \( \rho(\gamma) \) belongs to \( T_2 \).

Through the isomorphism (7), the embedding \( \iota_c : SO(4, \mathbb{C}) \rightarrow SO_J(2n, \mathbb{C}) \) corresponds to \( \iota_c : SO_J(4, \mathbb{C}) \rightarrow SO_J(2n, \mathbb{C}) \) sending \( A \) to a \( 2n \times 2n \) matrix composed of \( A \) followed by \((n - 2)\) diagonal blocks of the form
\[
\begin{pmatrix}
c & 0 \\
0 & c^{-1}
\end{pmatrix}.
\]

As in (8),\n\[
\alpha_{c_1, c_2}(\rho)(\gamma) = \iota_c \rho(\gamma),
\]
where \( c = c_1^{w_1(\gamma)} c_2^{w_2(\gamma)} \). Hence, \( \alpha_{c_1, c_2}(\rho)(\gamma) \) belongs to \( T_n \).
Therefore, we can complete the proof of Theorem \[9\] by taking $x = \rho(\gamma) \in T_2$ and showing the following.

**Lemma 10.** For every representation $\phi$ of $SO(2n, \mathbb{C})$ and every $c \in \mathbb{C}^*$, there are representations $\psi_1, \ldots, \psi_k$ of $SO(4, \mathbb{C})$ and $a_1, \ldots, a_k \in \mathbb{C}$ such that for

\[
(10) \quad tr\phi(\iota_c(x)) = \sum_{i=1}^{k} a_i tr(\psi_i(x)),
\]

for all $x \in T_2$.

**Proof:** Note that the left hand side of the above equation is a polynomial function on $T_2$ and the right hand side is a formal character of $SO(4, \mathbb{C})$ with complex coefficients.

Note also that the weight lattice, $\Lambda_2$, of $SO(4, \mathbb{C})$ embeds into $\mathbb{C}[T_2]$ and that this embedding extends to $\mathbb{C}[\Lambda_2] \to \mathbb{C}[T_2]$ which is an embedding as well, cf. [Bo, III.8]. The image of this map is an algebra generated by functions sending a diagonal matrix with diagonal entries $d_1, d_1^{-1}, d_2, d_2^{-1}$ to $d_1^{\pm 2}$, $d_2^{\pm 2}$, and $d_1 d_2$. Therefore, the image is of index 2 in $\mathbb{C}[T_2]$.

For complex reductive groups, the algebra of formal characters (over $\mathbb{C}$) coincides with the algebra $(\mathbb{C}\Lambda_2)^W$ of linear combinations of weights invariant under the Weyl group action, cf. [FH, §23.2].

Therefore, the statement follows from the following lemma.

**Lemma 11.** (1) $tr\phi(\iota_c(x))$ is $W_2$-invariant function of $x \in T_2$.

(2) $tr\phi(\iota_c(x))$ is a polynomial function on $T_2$ which belongs to $\mathbb{C}\Lambda_2$.

**Proof.** (1) The Lie algebra $t_n$ of $T_n$ has a basis $H_1, \ldots, H_n$, where $H_i$ is a $2n \times 2n$ matrix whose all entries are zero except for $(2i-1, 2i-1)$-th entry 1 and $(2i, 2i)$-th entry $-1$. Since the exponential map $exp : t_n \to T_n$ is onto, $x = exp(z)$, for some $z \in t_2$. Then $\iota_c x = e^{z+d}$ for $v = H_3 + \ldots + H_n$ and for any $d$ such that $e^d = c$.

Suppose that $\phi$ has weights $\alpha_1, \ldots, \alpha_N \in \Lambda_n$ with multiplicities $m_1, \ldots, m_N$. Then

\[
tr\phi(\iota_c(x)) = tr\phi(e^{z+d}) = \sum_{i=1}^{N} m_i e^{\alpha_i(z+d)},
\]

by [FH, (23.40)].

Since the exponential map $exp : t_2 \to T_2$ is $W_2$-equivariant and onto, it is enough to show that the above expression is a $W_2$ invariant function of $z \in t_2$.

Since $\alpha_i$ and $m_i$ are weights and their multiplicities of an $SO(2n, \mathbb{C})$-representation, $\sum_{i=1}^{N} m_i e^{\alpha_i(\gamma)}$ is $W_n$-invariant function on $t_n$. Hence, it is a linear combination of orbits of $W_n$, i.e. a linear combination of expressions $\sum_{w \in W_n} e^{\omega \alpha_i(\gamma)}$, for some $\omega \in t_n^*$. Therefore, it is enough to prove that for every $\omega \in t_n^*$ and for every $d \in \mathbb{C}$, $\sum_{w \in W_n} e^{\omega \alpha(z+d)}$ is $W_2$ invariant function of $z \in t_2$. 
Let \( L_1, \ldots, L_n \in t_n^* \) be the dual basis to \( H_1, \ldots, H_n \). Assume that \( \alpha = \sum_{i=1}^n d_i L_i \), for some \( d_1, \ldots, d_n \in \mathbb{C} \). Since \( \alpha \) is a weight of an \( SO(2n, \mathbb{C}) \)-representation, it belongs to the weight lattice, implying that either \( d_1, \ldots, d_n \in \mathbb{Z} \) or \( d_1, \ldots, d_n \in \mathbb{Z} + \frac{1}{2} \). Since elements of \( W_n \) are signed permutations with even numbers of sign changes, elements of \( W_n \alpha \) are of the form

\[
\varepsilon_1 d_1 L_{\sigma(1)} + \ldots + \varepsilon_n d_n L_{\sigma(n)},
\]

for some \( \sigma \in S_n \) and \( \varepsilon_1, \ldots, \varepsilon_n \in \{ \pm 1 \} \), such that \( \varepsilon_i = -1 \) for an even number of indices \( i \). By relabeling the indices by \( \sigma^{-1} \), this expression can be written as

\[
\varepsilon_1' d_{\tau(1)} L_1 + \ldots + \varepsilon_n' d_{\tau(n)} L_n,
\]

where \( \tau = \sigma^{-1} \) and \( \varepsilon_i' = \varepsilon_{\tau(i)} \) for simplicity.

Since \( v = H_3 + \ldots + H_n \) and \( L_i(H_j) = \delta_{ij} \),

\[
(11) \quad \sum_{w \in W_n} e^{w \cdot \alpha(z + d \tau)} = \sum_{\varepsilon \in \{ \pm 1 \}} e^{\varepsilon_1' d_{\tau(1)} L_1(z) + \varepsilon_2' d_{\tau(2)} L_2(z) + \ldots + \varepsilon_n' d_{\tau(n)} L_n(z)},
\]

where the sum on the right is over all \( \tau \in S_n \) and all \( \varepsilon_1', \ldots, \varepsilon_n' \in \{ \pm 1 \} \) such that \( \varepsilon_i' = -1 \) for an even number of indices \( i \).

Denote by \( S_{k,l,r}(d) \) the sum

\[
\sum_{\nu} e^{d(\varepsilon_1' d_\nu(1) + \ldots + \varepsilon_{n-1} d_\nu(n-2))},
\]

over all bijections \( \nu : \{1, \ldots, n-2\} \to \{1, \ldots, n\} \setminus \{k, l\} \), and all \( \varepsilon_1, \ldots, \varepsilon_{n-2} \in \{ \pm 1 \} \), such that \( \# \{ i : \varepsilon_i = -1 \} = r \) mod 2.

Then by grouping the terms of (11) by the values of \( \tau(1) \) and of \( \tau(2) \), the sum (11) can be written as

\[
(12) \quad \sum_{k \neq l} \left( \sum_{\varepsilon = \pm 1} e^{d_k L_1(z) + d_l L_2(z)} S_{k,l,0}(d) + \sum_{\varepsilon = \pm 1} e^{d_k L_1(z) - d_l L_2(z)} S_{k,l,1}(d) \right).
\]

Since we can choose two generators of \( W_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) such that the first one applied to \( z \) interchanges the values of \( L_1(z) \) and of \( L_2(z) \) and the second negates \( L_1(z) \) and \( L_2(z) \), it is easy to see that

\[
(13) \quad \sum_{\varepsilon = \pm 1} e^{d_k L_1(z) + d_l L_2(z)} \quad \text{and} \quad \sum_{\varepsilon = \pm 1} e^{d_k L_1(z) - d_l L_2(z)}
\]

are \( W_2 \)-invariant functions of \( z \). Consequently, (11) is a \( W_2 \)-invariant function of \( z \) and the first statement of lemma follows.

(2) By the above argument \( tr(\phi_{\nu}(x)) \) can be written as a linear combination of expressions (13), for \( d_k, d_l \in \mathbb{Z} \) or \( d_k, d_l \in \mathbb{Z} + \frac{1}{2} \). Since \( L_1, L_2, \frac{1}{2}(L_1 + L_2) \) are weights of \( t_2 \), the above expressions belong to \( \mathbb{C}[\Lambda_2] \subset \mathbb{C}[T_2] \).

This completes the proof of Lemma 111 and of Theorem 11.
2. Proof of Theorem 2

Consider the symmetric square, \( Sym^2(\mathbb{C}^5) \), which is a 15-dimensional space composed of vectors

\[
v \odot w = v \otimes w + w \otimes v \in \mathbb{C}^5 \otimes \mathbb{C}^5.
\]

The standard inner product on \( \mathbb{C}^5 \) yields an inner product on \( \mathbb{C}^5 \otimes \mathbb{C}^5 \),

\[
(v_1 \otimes v_2, w_1 \otimes w_2) = (v_1, w_1)(v_2, w_2)
\]

which restricts to \( Sym(\mathbb{C}^5) \). If \( e_1, \ldots, e_5 \) is the standard orthonormal basis of \( \mathbb{C}^5 \) then \( e_i \odot e_j \), for \( i \leq j \), is a basis of \( Sym^2(\mathbb{C}^5) \) and an easy computation shows that

\[
(e_i \odot e_j, e_k \odot e_l) = \begin{cases}
4 & \text{for } i = k = l = j \\
2 & \text{for } i = k \neq j = l \\
0 & \text{otherwise}
\end{cases}
\]

(14)

Note that the natural action of \( SO(5, \mathbb{C}) \) on \( Sym^2(\mathbb{C}^5) \) preserves the above inner product.

Let \( e^1, \ldots, e^5 \) the dual basis of \( (\mathbb{C}^5)^* \). Then \( tr : \mathbb{C}^5 \otimes (\mathbb{C}^5)^* \to \mathbb{C} \) which can be represented by the tensor

\[
\sum_{i=1}^{5} e^i \otimes e_i \in (\mathbb{C}^5 \otimes (\mathbb{C}^5)^*)^* = (\mathbb{C}^5)^* \otimes \mathbb{C}^5
\]

is \( SO(5, \mathbb{C}) \) invariant. Since the map from \( \mathbb{C}^5 \) to \( (\mathbb{C}^5)^* \) sending \( e_i \) to \( e^i \) is a \( SO(5, \mathbb{C}) \)-module isomorphism, we can identify this tensor with

\[
z = \sum_{i=1}^{5} e_i \otimes e_i.
\]

Therefore, as an \( SO(5, \mathbb{C}) \)-module, \( Sym^2(\mathbb{C}^5) \) splits into \( \mathbb{C}z \) and \( z^\perp \). By [FH, Ex 24.32], the representation \( SO(5, \mathbb{C}) \to SO(\mathbb{C}^5) \) is irreducible. We will denote that representation by \( \alpha \).

For the proof of Theorem 2 it is enough to assume that \( \Gamma = \mathbb{Z}_p \ast \mathbb{Z}_q \).

By Corollary 13 below there is a representation \( \psi : \Gamma \to SO(5, \mathbb{C}) \) such that \( \rho = \alpha \psi : \Gamma \to SO(14, \mathbb{C}) \) is irreducible. For \( n = 7 \), let \( \rho = \alpha \psi \). If \( n \geq 9 \) then let \( \rho = \alpha \psi \oplus \eta \), where \( \eta : \mathbb{Z}_p \ast \mathbb{Z}_q \to SO(2n - 14) \) is an irreducible representation, whose existence is implied by Corollary 13.

Let \( \rho' \) be obtained from \( \rho \) by the involution \( \sigma \) of \( \mathbb{S}_n \), that is by a conjugation by a matrix \( M \in O(14, \mathbb{C}) \) of determinant \( -1 \). Clearly, \( \rho \) and \( \rho' \) are undistinguishable by trace functions. Furthermore, we claim that \( Q_n(\gamma) = 0 \) for all \( \gamma \in \Gamma \). To see that, note that \( \psi(\gamma) \), being an element of \( SO(5, \mathbb{C}) \), has eigenvalues \( 1, x^\pm 1, y^\pm 1 \), for some \( x, y \in \mathbb{C}^* \). The eigenvalues of the action of \( \psi(\gamma) \) on \( Sym(\mathbb{C}^5) \) are products of the above ones and, in particular, the eigenvalue 1 appears with multiplicity at least three – coming
from $1 \cdot 1, x \cdot x^{-1}$, and $y \cdot y^{-1}$. Consequently, the action of $\psi(\gamma)$ on $z^3$ has multiplicity at least $2$. Now, by Lemma 5 and the fact that $Q_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0$, we have $Q_n(\gamma) = 0$ indeed for all $\gamma \in \Gamma$.

Now, it remains to be shown that $\rho$ and $\rho'$ are not equivalent. Since they are both completely reducible, it is enough to show that they are not conjugate in $SO(2n, \mathbb{C})$. (By [S1, Thm. 30], completely reducible representations have closed conjugacy orbits in the representation variety and, therefore, are non-equivalent in the character variety if they are not conjugate.) Suppose $\rho = MPM^{-1}$ equals $APA^{-1}$, for some $A \in SO(2n, \mathbb{C})$. Then $MA^{-1}$ commutes with the image of $\rho$, which by Shur’s Lemma spans the entire matrix algebra $M(14, \mathbb{C})$ if $n = 7$ or, spans $M(14, \mathbb{C}) \oplus M(2n - 14, \mathbb{C})$, for $n \geq 9$. Hence, for $n = 7$, $MA^{-1}$ is a scalar matrix and the only such matrices in $O(2n, \mathbb{C})$ are $\pm I$. That however implies that $\det(MA^{-1}) = 1$ and, hence,

$$\det(A) = \det(M) = -1,$$

contradicting the assumption that $\rho$ and $\rho'$ are conjugate in $SO(2n, \mathbb{C})$.

For $n \geq 9$, $MA^{-1}$ is composed of a $14 \times 14$ scalar matrix and of another $(2n - 14) \times (2n - 14)$ scalar matrix. The only such matrices in $O(2n, \mathbb{C})$ are $(\pm I_{14}) \oplus (\pm I_{2n-14})$. Their determinants are $1$, implying that $\det(A) = -1$ again. \qed

The above proof relies on two statements about irreducible representations of $\mathbb{Z}_p \ast \mathbb{Z}_q$. We will start with the proof of the easier one, asserting the existence of an irreducible representations of $\mathbb{Z}_p \ast \mathbb{Z}_q$ into $SO(2m, \mathbb{C})$ for any $m > 1$. The method of proof of this statement will be extended to show the second statement (Lemma 14 and Corollary 15) afterwards.

Let $B_1$ be a block matrix composed of $D_{\xi_p}, D_{\xi_q^2}, ..., D_{\xi_q^n}$, where $\xi_p$ is a primitive $p$-th root of 1, and, similarly, let $B_2$ be a block matrix composed of $D_{\eta_q}, D_{\eta_q^2}, ..., D_{\eta_q^n}$, where $\eta_q$ is a primitive $q$-th root of 1. Since $B_1, B_2$ are of order $p$ and $q$ respectively, for every $A \in SO(2m, \mathbb{C})$ there is a representation

$$\eta_A : \mathbb{Z}_p \ast \mathbb{Z}_q \to SO(2m, \mathbb{C})$$

sending a generator of $\mathbb{Z}_p$ to $B_1$ and a generator of $\mathbb{Z}_q$ to $AB_2A^{-1}$. Assume that $p, q > 2m$. Since $B_1, B_2$ commute and have $2m$ distinct eigenvalues (each), they share $2m$ distinct eigenvectors $v_1, ..., v_{2m}$. Note that since $B_1, B_2$ are orthogonal transformations, $v_1, ..., v_{2m}$ form an orthogonal basis of $\mathbb{C}^{2m}$.

Given $v, w \in \mathbb{C}^{2m}$, let $Z_{v,w}$ denote the set of matrices $A \in SO(2m, \mathbb{C})$ such that $\alpha(A)v \subset w$.

**Proposition 12.** (1) For every $m > 2$ and $p, q > 2m$, if $\eta_A : \mathbb{Z}_p \ast \mathbb{Z}_q \to SO(2m, \mathbb{C})$ is reducible then $A \in \bigcup_{k,l=1}^{2m} Z_{v_k,v_l}$.

(2) For every non-zero $v, w \in \mathbb{C}^{2m}$, $Z_{v,w}$ is Zariski closed proper subspace of $SO(2m, \mathbb{C})$.

**Corollary 13.** For every $m > 2$ and $p, q > 2m$, $\eta_A : \mathbb{Z}_p \ast \mathbb{Z}_q \to SO(2m, \mathbb{C})$ is irreducible for non-empty Zariski open set of matrices $A$ in $SO(2m, \mathbb{C})$. 

Proof of Proposition 12

(1) Suppose that \( \eta_A \) is reducible. Then \( B_1 \) and \( AB_2 A^{-1} \) preserve a certain proper non-empty subspace \( V \subset \mathbb{C}^{2m} \). Since \( V \) is preserved by \( B_1 \), it must have a basis given by a subset of \( \{v_1, \ldots, v_{2m}\} \). Suppose then that \( v_k \in V \).

Since \( AB_2 A^{-1} \) preserves \( V \), the operator \( B_2 \) preserves \( AV \) implying that \( AV \) has a basis which is also a subset of \( \{v_1, \ldots, v_{2m}\} \) (the eigenvectors of \( B_2 \)). Since this basis is a proper subset of \( \{v_1, \ldots, v_{2m}\} \), \( v_l \not\in AV \) for some \( l \) and since all \( v_i \), for \( i \neq l \), are orthogonal to \( v_l \), we have \( A \in Z_{v_k,v_l} \).

(2) It is clear that \( Z_{v,w} \) is Zariski closed. Since \( SO(2m, \mathbb{C}) \) acts transitively on \( \mathbb{C}^{2m} \), the set \( Z_{v,w} \) is a proper subset of \( SO(2m, \mathbb{C}) \). \( \square \)

Recall that \( z^\perp \) is a 14-dimensional subspace of \( Sym^2(\mathbb{C}^5) \) and that \( SO(5, \mathbb{C}) \) acts on it through an irreducible representation \( \alpha \). The reminder of this section is devoted to proving that there is a representation \( \psi : \Gamma \to SO(5, \mathbb{C}) \) such that \( \rho = \alpha \psi : \Gamma \to SO(z^\perp) \) is irreducible. (That is the last component of the proof of Theorem 2 which needs to be established).

Let \( B_c \in SO(5, \mathbb{C}) \) for \( c \in \mathbb{C}^* \) be a block matrix composed of \( D_c, D_c^* \), and of the \( 1 \times 1 \) identity matrix along the diagonal. (Note that the meaning of \( B \) here is different, but analogous to that of the previous proof.)

Let \( \psi_A : \mathbb{Z}_p * \mathbb{Z}_q \to SO(5, \mathbb{C}) \) be a representation sending a generator of \( \mathbb{Z}_p \) to \( B_{\xi_p} \) and a generator of \( \mathbb{Z}_q \) to \( AB_{\xi_q} A^{-1} \), for some \( A \in SO(5, \mathbb{C}) \), where \( \xi_p \) and \( \xi_q \) are primitive \( p \)-th and \( q \)-th roots of 1 respectively, as before. (These matrices have order \( p \) and \( q \), respectively, and, therefore, \( \psi_A \) is well defined.)

Since \( B_{\xi_p} \) has eigenvalues \( \xi_p^{\pm 1}, \xi_p^{\pm 4} \), and 1, the eigenvalues of \( B_{\xi_p} \) acting on \( Sym^2(\mathbb{C}^5) \) are monomials of degree two in the above ones, that is

\[
\xi_p^{\pm 2}, \xi_p^{\pm 4}, \xi_p^{\pm 1}, 1, \xi_p^{\pm 4} \cdot 1,
\]

and

\[
\xi_p \cdot \xi_p^{-1} = \xi_p \cdot \xi_p^{-d} = 1 \cdot 1 = 1.
\]

Assume now that \( p, q > 16 \). Since \( \xi_p \) is a primitive \( p \)-th root of unity, all of these eigenvalues appear once, except for 1 which appears with multiplicity 3. Let \( v_1, \ldots, v_{12} \) be eigenvectors for the eigenvalues \( \xi_p \). Denote the eigenspace of 1 inside \( z^\perp \) by \( F \). Then \( \dim F = 2 \) and \( v_1, \ldots, v_{12}, z \) and \( F \) are all orthogonal to each other.

Note that \( v_1, \ldots, v_{12} \) are eigenvectors of \( B_{\xi_p} \) as well and that \( F \) is also the eigenspace of 1 for the \( B_{\xi_p} \)-action on \( z^\perp \).

Denote by \( \mathcal{R}_{p,q} \subset SO(5, \mathbb{C}) \) the set of matrices \( A \) such that the representation \( \alpha \psi_A : \mathbb{Z}_p * \mathbb{Z}_q \to SO(z^\perp) \) is reducible.

Let \( Z_{v,w} \), for \( v, w \subset z^\perp \), denote the set of matrices \( A \in SO(5, \mathbb{C}) \) such that \( \alpha(A)v \subset w^\perp \). Let \( Y \) denote the set of matrices \( A \in SO(5, \mathbb{C}) \) such that the dimension of the subspace \( F + \alpha(A)F \) of \( z^\perp \) is at most 3.

Lemma 14. Under the above assumptions

(1) For every nonzero \( v, w \in \mathbb{C}^5 \), \( Z_{v,w} \) is a Zariski closed proper subset of \( SO(5, \mathbb{C}) \).
(2) $Y$ is a Zariski closed proper subset of $SO(5, \mathbb{C})$.

$$\mathcal{R}_{p,q} \subset \bigcup_{1 \leq i,j \leq 12} Z_{v_i,v_j} \cup Y.$$ 

**Corollary 15.** $\mathcal{R}_{p,q}$ is a subset of a proper closed subset of $SO(5, \mathbb{C})$ and, hence, $\alpha \psi_A$ is irreducible for a non-empty Zariski-open set of matrices $A \in SO(5, \mathbb{C})$.

**Proof of Lemma 14.** (1) Since $\alpha$ is irreducible, for every nonzero $v \in z^\perp$ the set $\alpha(SO(5, \mathbb{C}))v$ spans $z^\perp$. Therefore, $Z_{v,w}$ is a proper subset of $SO(5, \mathbb{C})$. Clearly, it is Zariski closed.

(2) Since $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ are eigenvectors of $D_c$ with eigenvalues $c$ and $c^{-1}$ respectively,

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

are eigenvectors with eigenvalue 1 of $\alpha(D_{c_1})$ and of $\alpha(D_{c_2})$. By [14], these vectors are orthogonal to each other and one can also easily check that they are orthogonal to $z$ as well. Therefore, $f_1, f_2$ form an orthogonal basis of $F$.

The condition

$$\dim F + \alpha(A)F \leq 3$$

is equivalent to the rank of the $4 \times 14$ matrix built of coordinates of $f_1, f_2$, $\alpha(A)(f_1), \alpha(A)(f_2)$ (with respect to some basis of $z^\perp$) being at most 3. Since that corresponds to vanishing of one of a finite number of $4 \times 4$ minors of that matrix, $Y$ is algebraically closed. Therefore, it is remains to be shown that $A \not\in Y$ for some $A \in SO(5, \mathbb{C})$. We claim that the cyclic permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

will do the job.

Note that the action of $A$ on $Sym^2(\mathbb{C}^5)$ sends $f_1$ to

$$g_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$
which are also orthogonal to $z$. Since $f_1, f_2, g_1, g_2$ are linearly independent,
\[ \dim F + \alpha(A)F = 4. \]

(3) Let $A \in R_{p,q}$. Then $\alpha \psi_A$ is reducible and, hence, $\alpha(B_{\xi_p})$ and $\alpha(AB_{\xi_q}A^{-1})$ preserve some non-zero, proper subspace $V$ of $z^\perp$ and its orthogonal complement $V^\perp$ in $z^\perp$. Since $V$ is preserved by $\alpha(B_{\xi_p})$, it is a sum of a subspace of $F$ and of a space with a basis being a subset of \{v_1, ..., v_{12}\}.

By replacing $V$ by $V^\perp$ if necessary, we can assume without loss of generality that $\dim V \geq \dim V^\perp$, i.e. $\dim V \geq 7$. Therefore, $V$ contains $v_i$ for some $i = 1, ..., 12$. If $\alpha(A)V$ does not contain some $v_j$ then $A \in Z_{v_i,v_j}$. Therefore, it is enough to assume that $\alpha(A)V$ contains all $v_i$, $i = 1, ..., 12$.

Furthermore, if $V$ does not contain some $v_i$ then $v_i \perp V$ and, consequently, $A$ sends it to a vector orthogonal to $AV$. Hence, by the above argument, $A \in Z_{v_i,v_j}$, for any $j$.

Therefore, we can assume that both $V$ and $\alpha(A)V$ contain $v_1, ..., v_{12}$. Then $V^\perp$ is either $F$ or a 1-dimensional subspace of $F$ and $\alpha(A)$ maps it to some subspace of $F$. That implies that
\[ \dim F + \alpha(A)F \leq 3. \]
(Note that $F + \alpha(A)F$ is indeed three-dimensional if $\dim V^\perp = 1$ and $\alpha(A)$ maps a complementary space to it in $F$ to outside of $F$.)

\[ \square \]

3. Proof of Proposition \[ \[ \]

\textbf{Proof.} (1) By [S4, Prop 18], $\mathcal{F}T_{SO(2n,\mathbb{C})}(\Gamma)$ is generated by $Q(\gamma, ..., \gamma)$ for $\gamma \in \Gamma$ as an algebra over $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$. Therefore, by the assumption of the proposition,
\[ Q(\gamma, ..., \gamma) \notin \mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma) \]
for some $\gamma$. By [S4],
\[ Q(\gamma_1, ..., \gamma_n)Q(\gamma, ..., \gamma) \in \mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma) \]
for any $\gamma_1, ..., \gamma_n \in \Gamma$ implying that there is an embedding
\[ \alpha : \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)] \to \frac{1}{Q(\gamma, ..., \gamma)}\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma), \]
defined on generators of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ by
\[ \alpha(\tau_\gamma) = \tau_\gamma, \quad \alpha(Q(\gamma_1, ..., \gamma_n)) = \frac{Q(\gamma_1, ..., \gamma_n)Q(\gamma, ..., \gamma)}{Q(\gamma, ..., \gamma)} \]
which extends to an isomorphism between the field of fractions of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ and the field of fractions of $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$. This proves the birationality of $\psi$.

The finiteness of $\psi$ follows from the fact that the elements $Q(\gamma_1, ..., \gamma_n)$, for $\gamma_1, ..., \gamma_n \in \Gamma$, which generate $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ as an $\mathcal{F}T_{SO(2n,\mathbb{C})}(\Gamma)$-algebra, satisfy
\[ Q(\gamma_1, ..., \gamma_n)^2 \in \mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma), \]
by Corollary 14 and Corollary 17 of [S4], and, therefore, are integral over $T_{SO(2n,\mathbb{C})}(\Gamma)$.

Every finite, birational map is generically 1-1 and it is a normalization map if its domain is normal.

(2) Since the involution $\sigma$ of [S4] fixes $T_{SO(2n,\mathbb{C})}(\Gamma)$ and negates every $Q(\gamma,...,\gamma)$, condition (16) is equivalent to

$$Q(\gamma,...,\gamma) \neq 0.$$ 

Since

$$Q_n(\gamma) = (2i)^n n! (\tau_{D^+_n,\gamma} - \tau_{D^-_n,\gamma})$$

by [S2, Prop. 10], and,

$$\tau_{D^+_n,\gamma} + \tau_{D^-_n,\gamma} \in T_{SO(2n,\mathbb{C})}(\Gamma),$$

we can substitute $Q_n(\gamma)$ for $\tau_{D^+_n,\gamma}$ in the statement of the proposition without loss of generality. Since the elements of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ distinguish non-equivalent representations and $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is generated by functions $Q(\gamma_1,...,\gamma_n)$ as an $T_{SO(2n,\mathbb{C})}(\Gamma)$-algebra and since

$$Q(\gamma_1,...,\gamma_n)([\rho]) = \frac{f([\rho])}{Q(\gamma,...,\gamma)([\rho])},$$

for some $f \in T_{SO(2n,\mathbb{C})}(\Gamma)$, as long as $Q(\gamma,...,\gamma)([\rho]) \neq 0$, the trace functions together with $Q(\gamma,...,\gamma)$ distinguish $\rho$ from all non-equivalent representations as long as $Q(\gamma,...,\gamma)([\rho])$ does not vanish. Since $Q(\gamma,...,\gamma) \neq 0$, that is the case for a generic $\rho$. \qed

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