Existence of solution to parabolic equations with mixed boundary condition on non-cylindrical domain

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Abstract

In this paper we are concerned with the initial boundary value problems of linear and semi-linear parabolic equations with mixed boundary conditions on non-cylindrical domains in spatial-temporal space. We obtain the existence of a weak solution to the problem. In the case of the linear equation the parts for every type of boundary condition are any open subsets of the boundary being nonempty the part for Dirichlet condition at any time. Due to this it is difficult to reduce the problem to one on a cylindrical domain by diffeomorphism of the domain. By a transformation of unknown function and penalty method we connect the problem to a monotone operator equation for functions defined on the non-cylindrical domain. In this way a semilinear problem is considered when the part of boundary for Dirichlet condition is cylindrical.

Keywords: Parabolic equation, Non-cylindrical domain, Mixed boundary condition, Existence

MSC: 35K20, 35K55, 35A15

1. Introduction

There are vast literature for parabolic differential equations on non-cylindrical domain and various methods have been used to study them. In \cite{6} the energy inequality for a linear equation with homogeneous Dirichlet boundary condition is proved, thus unique existence of solution is studied. For domains expanded along time existence and uniqueness of solution to initial boundary value problem of the linear(cf. \cite{9,14} and \cite{15}), semilinear(cf. \cite{13}) and nonlinear(cf. \cite{14}) equations with homogeneous Dirichlet boundary condition are studied. For such domains and boundary conditions \cite{13} also deals with attractor; and \cite{2} considers unique existence of solution to a linear Schrödinger-type equation. In \cite{3} dealing with the Dirichlet problem, they assume only Hölder continuity on time-regularity of the boundary. In \cite{25} semigroup theory is improved and the obtained result is applied to the initial boundary value problem of a linear parabolic equation with inhomogeneous Dirichlet condition on non-cylindrical domain. There are some literatures for unique existence of initial boundary value problems of linear equations relying on the method of potentials (see \cite{8} and references therein). Domains in \cite{20} and \cite{21}, where existence, uniqueness and regularity are studied, are more general, that is, ”initial” condition is given on a hypersurface in spatial-temporal space instead of the plane $t = 0$. In \cite{17} optimal regularity of solution to a special kind of 1-dimensional problem is considered. Neumann problem of heat equations (cf. \cite{11}), parabolic equation with Robin type boundary condition (cf. \cite{12}) in non-cylindrical domains and behavior of solutions to the initial-boundary value problems of nonlinear equations (cf. \cite{10} and \cite{24}) are studied. In \cite{23} and \cite{26} optimal control and controllability of parabolic equation with homogenous Dirichlet condition on non-cylindrical domain, respectively are studied.

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Also, there are many literatures for the initial boundary value problems with mixed boundary conditions. Under certain assumptions the non-cylindrical domains are transformed to a cylindrical one. The initial boundary value problems of linear parabolic equations with mixed time dependent lateral boundary condition on cylindrical domains are studied (cf. [4], [5], [28]). The boundary conditions on the lateral surfaces in [4] may be either two of the following classical ones: Dirichlet, Neumann and Robin, but one part for a kind of boundary condition is a connected and relatively open subset of the lateral surface and the boundary of the part is tangent to the plane \( t = 0 \). In [5] the first part is concerned with a classical problem on cylindrical domains and the result is applied to the problem with zero initial condition and lateral mixed boundary conditions on a cylindrical domain, where the non-cylindrical surface for boundary condition is transformed to a cylindrical one by a diffeomorphism. The lateral boundary surface of the cylindrical domain in [5] is also divided into two parts, and one part \( \Gamma_1 \) for a kind of boundary condition is connected and relatively open subset of the boundary surface and at each point is transverse to the hyperplane \( t = \text{const} \). Developing a method in abstract evolution equations, [28] is concerned with linear parabolic problems on cylindrical domains with mixed variable inhomogeneous Dirichlet and Neumann conditions. But, there change along time of distance of the sections of part of boundary for Dirichlet condition must be dominated by a Lipschitz continuous function in time \( t \). In [28] as application of the result, unique existence of solution to a linear parabolic problem with homogeneous Dirichlet boundary condition on non-cylindrical domains is considered.

Section 4 in [22] deals with the linear parabolic problem on non-cylindrical domain with Robin and Dirichlet boundary conditions, where the surface for Dirichlet condition is cylindrical type. [18] and [19] study existence, uniqueness and regularity of solutions to the initial boundary value problems of linear and semilinear parabolic equations on non-cylindrical domains, which is related to the the combustion phenomena. The domains in [18] and [19] are bordered with a part of cylindrical type surface where homogeneous Neumann condition is given, non-cylindrical hypersurfaces where Dirichlet boundary one is given and planes \( t = 0, t = T \). Thus, by change of spatial independent variable they transform the problems to classical problems on cylindrical domains where Dirichlet and Neumann conditions, respectively, are given on cylindrical surfaces. In [24] some differential inclusions are studied and the result is applied to the following problem

\[
\begin{align*}
    u_t - \Delta u &\in F(u) \text{ in } \Omega, \\
    -\frac{\partial}{\partial n} u &\in \beta(u) \text{ on } \gamma, \\
    u &= 0 \text{ on } \Gamma - \gamma, \\
    u(x, 0) &= \xi \text{ in } \Omega_0,
\end{align*}
\]

where \( \Omega \) is a non-cylindrical domain in spatial-temporal space, \( \Gamma \) is its lateral surface, \( \gamma \) is a part of a cylindrical surface, \( \beta = \partial j \) and \( j \) is a proper lower semicontinuous convex function from \( R \) to \([0, +\infty]\) with \( j(0) = 0 \).

On the other hand, in [27] a time-dependent Navier-Stokes problem on a non-cylindrical domain with a mixed boundary condition is considered. In [27] the part of boundary where homogeneous Dirichlet condition is given is cylindrical type and the boundary condition on the other part of boundary is such a special one that guarantee existence of solution to an elliptic operator equation obtained by penalty method.

In this paper we are concerned with linear and semilinear parabolic equations on non-cylindrical domains with mixed boundary conditions which may include inhomogeneous Dirichlet, Neumann and Robin conditions together. In the case of linear equation the parts for every type of boundary condition are any open subsets of the boundary being nonempty the part for Dirichlet condition at any time. This rises difficulty in reducing the problem to one on cylindrical domains in [4], [5] and [28] or one in [18], [19], [22] and [24].

Our idea is to use a transformation of unknown function by which the problem is connected to a monotone operator equation for functions defined on the non-cylindrical domain. In this way we can also consider semilinear equation when the part of boundary for Dirichlet condition is cylindrical.

This paper is composed of 5 sections. In Section 2 notation, the problem, the definition of weak solution and the main result are stated. In Section 3 by a change of unknown function an
2. Problem and main result

Let $\Omega(t)$ be bounded connected domains of $\mathbb{R}^N$, $Q = \bigcup_{t \in (0, T)} \Omega(t) \times \{t\}$, $0 < T < \infty$, $\Sigma = \bigcup_{t \in (0, T)} \partial \Omega(t) \times \{t\}$ and $\Sigma_0, \Sigma_1$ be open subsets of $\Sigma$ such that $\Sigma_0 \cup \Sigma_1 = \Sigma$ and $\Sigma_0(t) = \Sigma_0 \cap \Omega(t) \neq \emptyset \forall t \in (0, T)$. Let $\nu(x,t)$ be outward normal unit vector on the boundary $\Sigma$ and $n(x,t)$ be outward normal unit vector on $\partial \Omega(t)$ for fixed $t$.

Let $\|y\|^2_{\Omega(t)} = \int_{\Omega(t)} |\nabla y|^2 \, dx$ and $|y|^2_{\partial \Omega(t)} = \int_{\partial \Omega(t)} |y|^2 \, dx$. Let $H^1(Q) = W^1_2(Q)$.

For function $y$ defined on $Q$ define $\beta(y)$ by

$$\beta(y) = \left( \int_0^T \|y\|^2_{\Omega(t)} \, dt \right)^{\frac{1}{2}}$$

whenever the integral make sense. Let

$$D(Q) = \{ \varphi : \varphi \in C^\infty(\bar{Q}), \varphi|_{\Sigma_0} = 0 \},$$
$$V(Q) = \{ \text{the completion of } D(Q) \text{ under the norm } \beta(y) \},$$
$$W(Q) = \{ \text{the completion of } D(Q) \text{ in the space } H^1(Q) \}$$

and $\langle , \rangle$ be duality product between $W(Q)$ and $W(Q)^*$. By the condition $\Sigma_0(t) = \Sigma_0 \cap \Omega(t) \neq \emptyset \forall t \in (0, T)$, $\beta(\cdot)$ is a norm in $D(Q)$.

We use the following

**Assumption 2.1.** The hypersurface $\Sigma$ belongs to $C^2$ for $x$ and to $C^1$ for $t$ and for any $t \in [0, T]$ there exist a diffeomorphisms $X(x) = (\Phi(x), \psi(x))$ on $\mathbb{R}^N$ in the class $C^2$ which maps $\Omega(0)$ onto $\Omega(t)$, $X(0) = I$ and is in $C^1$ for $t$, where $I$ is the unit operator.

**Remark 2.1.** Let $\partial \Omega(0) \in C^2$ and $\Phi_i(x_1, \ldots, x_N, t) \in C^2(\mathbb{R}^N \times [0, T])$, $i = 1, \ldots, N$ be any functions such that $\Phi_i(x_1, \ldots, x_N, 0) = x_i$ and Jacobian $\frac{\partial \Phi}{\partial x_i} > 0$, where $\Phi = \{\Phi_1, \ldots, \Phi_N\}$ and $x = \{x_1, \ldots, x_N\}$. Then $\Sigma = \bigcup_{t \in (0, T)} \Sigma(t) \times \{t\}$, where $\Sigma(t) = \{\Phi(x, t)|x \in \partial \Omega(0)\}$ satisfies Assumption 2.1.

We are concerned with the following initial boundary value problem

$$\frac{\partial y}{\partial t} - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial y}{\partial x_j} \right) + \sum_{i=1}^N b_i(x,t) \frac{\partial y}{\partial x_i} + c(x,t,y) = g(x,t),$$

$$y|_{\Sigma_0} = \bar{y}|_{\Sigma_0}, \left( k(x,t)y + \sum_{i,j=1}^N a_{ij}(x,t)n_i \frac{\partial y}{\partial x_j} \right) \big|_{\Sigma_1} = f(x,t),$$

$$y(x,0) = y_0(x) \in L_2(\Omega(0)),$$

where $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t,r)$ are functions satisfying the following conditions

(A) $a_{ij}(x,t) \in W^2_\infty(Q)$, $a_{ij} = a_{ji}$, $\sum_{i,j=1}^N a_{ij}(x,t)\xi_i \xi_j \geq \rho|\xi|^2$, $\exists \rho > 0$, $\forall \xi \in \mathbb{R}^N$, $i = 1, \ldots, N$,

(B) $b_i(x,t) \in L_\infty(Q)$, $i = 1, \ldots, N$,

(C) $c(x,t,r)$ is Lipschitz continuous with respect to $r$ uniformly for $(x,t)$ and measurable with respect to $(x,t)$ for fixed $r$ and $c(x,t,0) \in L_2(Q)$

(D) $\bar{y} \in H^1(Q)$, $k(x,t) \in L_\infty(\Sigma_1)$, $g(x,t) \in L_2(Q)$, $f(x,t) \in L_2(\Sigma_1)$.

**Remark 2.2.** On a part of $\Sigma_1$ where $k(x,t) = 0$ we have Neumann condition.
When \( y \in C^2(\bar{Q}) \), \( u \in D(Q) \), in view of (2.2) we have

\[
\int_Q \frac{\partial y}{\partial t} u \, dx \, dt = (y(x,T),u(x,T))_{\Omega(T)} - (y_0,u(x,0))_{\Omega(0)}
\]

\[
+ \int_{\Sigma_1} yu \cos(\hat{\nu},t) \, d\sigma - \int_Q y \frac{\partial u}{\partial t} \, dx \, dt,
\]

(2.4)

\[
- \int_Q \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial y}{\partial x_j} \right) u \, dx \, dt = \sum_{i,j=1}^N \int_Q a_{ij}(x,t) \frac{\partial y}{\partial x_j} \frac{\partial u}{\partial x_i} \, dx \, dt
\]

\[
+ \int_{\Sigma_1} k(x,t)yu \, d\sigma - \int_{\Sigma_1} f(x,t)u \, d\sigma,
\]

(2.5)

where \((\hat{\nu},t)\) is the angle between \(\nu\) and the positive direction of \(t\)-axis. Also, if \(y \in L^2(Q)\), then \(c(x,t,y) \in L^2(Q)\) (cf. Lemma 2.2, ch. 2 in [10]).

In view of (2.4) and (2.5), we introduce the following

**Definition 2.1.** A function \( y \) is called a solution to (2.1)-(2.3) if \( y \) satisfies the following

\[
y - \bar{y} \in V(Q),
\]

\[
- \int_Q y \frac{\partial v}{\partial t} \, dx \, dt + \sum_{i,j=1}^N \int_Q a_{ij}(x,t) \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \, dt + \sum_{i=1}^N \int_Q b_i(x,t) \frac{\partial y}{\partial x_i} v \, dx \, dt
\]

\[
+ \int_Q c(x,t,y)v \, dx \, dt + \int_{\Sigma_1} [yu \cos(\hat{\nu},t) + k(x,t)yu] \, d\sigma
\]

\[
= (y_0,v(x,0))_{\Omega(0)} + \int_Q g(x,t)v \, dx \, dt + \int_{\Sigma_1} f(x,t)v \, d\sigma
\]

\[
\forall v \in W(Q) \text{ with } v(x,T) = 0.
\]

Our main result of this paper is the following

**Theorem 2.1.** Suppose that conditions (A), (B), (C) hold and that either \(c(x,t,y)\) is linear with respect to \(y\) or \(\Sigma_0 = \Gamma_0 \times (0,T), \Gamma_0 \subset \partial \Omega(0)\) and \(\Gamma_0\) is invariant under the diffeomorphism in Assumption 2.1. Then there exists a solution to problem (2.1)-(2.3) provided that (D) is valid.

3. Transformation of unknown function

For the sake of simplicity, we will use the same constants in the estimates unless confusion will be caused.

It is known that there exists a function \( \psi \in C^2(\bar{\Omega}(0)) \) such that

\[
\psi(x) > 0 \quad \forall x \in \Omega(0), \quad \psi|_{\partial \Omega(0)} = 0 \quad \text{and} \quad |\nabla \psi| > 0 \quad \forall x \in \bar{\Omega}(0) \setminus \bar{\omega}_0,
\]

where \(\bar{\omega}_0 \subset \Omega(0)\) (cf. Lemma 1.1 in [7]).

**Lemma 3.1.** There exists a function \( \varphi(x,t) \in C^{2,1}(\bar{Q}) \) such that

\[
\varphi(x,t) > 0 \quad \text{on} \quad Q, \quad \varphi(x,t) = 0 \quad \text{on} \quad \Sigma \quad \text{and} \quad -\frac{\partial \varphi(x,t)}{\partial n} > \eta > 0 \quad \text{on} \quad \Sigma,
\]

where \(C^{2,1}(\bar{Q})\) is the space of functions which are twice continuously differentiable with respect to \(x\) and continuously differentiable with respect to \(t\) on \(Q\).
Proof Take \( \varphi(x,t) = \psi(X^{-1}(t)x) \), where \( X^{-1}(t) \) is the diffeomorphism from \( \Omega(t) \) onto \( \Omega(0) \), which is the inverse of the one given in Assumption 2.1. Then the conclusion follows from the properties of function \( \psi \) and Assumption 2.1 \( \Box \)

Let us make a change by

\[
u = e^{k_1 t + k_2 \varphi(x,t)} y,
\]

where \( k_1 \) and \( k_2 \) are constants to be determined later. Let \( y \in C^2(\bar{Q}) \) and (2.2) is satisfied. Then,

\[
e^{k_1 t + k_2 \varphi(x,t)} \frac{\partial y}{\partial t} = \frac{\partial u}{\partial t} - (k_1 + k_2 \frac{\partial \varphi}{\partial t}) u,
\]

\[
- e^{k_1 t + k_2 \varphi(x,t)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial y}{\partial x_j} \right) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial u}{\partial x_j} \right) + 2 \sum_{ij} a_{ij} k_2 \frac{\partial \varphi}{\partial x_j} \frac{\partial u}{\partial x_i}
\]

\[- k_2 u \sum_{ij} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial \varphi}{\partial x_j} - k_2 \sum_{ij} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_{ij} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \cdot
\]

\[
e^{k_1 t + k_2 \varphi(x,t)} \sum_{i=1}^{N} b_i(x,t) \frac{\partial y}{\partial x_i} = \sum_{i=1}^{N} \left[ b_i(x,t) \frac{\partial u}{\partial x_i} - b_i(x,t) k_2 \frac{\partial \varphi}{\partial x_i} u \right].
\]

Also, we have that

\[
\sum_{i,j=1}^{N} a_{ij}(x,t) n_i \frac{\partial u}{\partial x_j} \bigg|_{\Sigma_1} = \left( \sum_{i,j=1}^{N} a_{ij}(x,t) n_i \frac{\partial y}{\partial x_j} e^{k_1 t + k_2 \varphi(x,t)} + \sum_{i,j=1}^{N} a_{ij}(x,t) n_i k_2 \frac{\partial \varphi(x,t)}{\partial x_j} u \right) \bigg|_{\Sigma_1}
\]

\[
= \left( \sum_{i,j=1}^{N} a_{ij}(x,t) n_i \frac{\partial y}{\partial x_j} e^{k_1 t + k_2 \varphi(x,t)} + k_2 \frac{\partial \varphi(x,t)}{\partial n} \sum_{i,j=1}^{N} a_{ij}(x,t) n_i n_j u \right) \bigg|_{\Sigma_1}
\]

\[
= \left( f(x,t) e^{k_1 t} - k(x,t) u + k_2 \frac{\partial \varphi(x,t)}{\partial n} \sum_{i,j=1}^{N} a_{ij}(x,t) n_i n_j u \right) \bigg|_{\Sigma_1}
\]

where the fact that \( \varphi(x,t) = 0 \) on \( \Sigma \) and its corollary \( \frac{\partial \varphi(x,t)}{\partial n} n_j |_{\Sigma_1} = \frac{\partial \varphi(x,t)}{\partial n} n_j |_{\Sigma_1} \) have been used.

Taking into account these facts, we have

\[
\frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{N} B_i(x,t) \frac{\partial u}{\partial x_i} + C(x,t,u) = G(x,t),
\]

\[
u|_{\Sigma_0} = \bar{u}|_{\Sigma_0}, \quad \left( K(x,t) u + \sum_{i,j=1}^{N} a_{ij}(x,t) n_i \frac{\partial u}{\partial x_j} \right) |_{\Sigma_1} = F(x,t),
\]

\[
u(x,0) = u_0 \equiv y_0(x)e^{k_2 \varphi(x,0)} \in L_2(\Omega(0)),
\]
Proof

First, let us prove that if

\begin{equation}
B_i(x,t) = b_i(x,t) + 2 \sum_j a_{ij} k_2 \frac{\partial \varphi}{\partial x_j},
\end{equation}

\begin{equation}
C(x,t,u) = e^{k_1 t + k_2 \varphi(x,t)} c(x,t, e^{-k_1 t - k_2 \varphi(x,t)} u) - (k_1 + k_2 \frac{\partial \varphi}{\partial t}) u
\end{equation}

\begin{equation}
+ k_2 u \left[ \sum_{ij} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial \varphi}{\partial x_j} - k_2 \sum_{i} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{j} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial \varphi}{\partial x_i} \right],
\end{equation}

\begin{equation}
G(x,t) = e^{k_1 t + k_2 \varphi(x,t)} g,
\end{equation}

\begin{equation}
F(x,t) = e^{k_1 t} f,
\end{equation}

\begin{equation}
\bar{u} = e^{k_1 t + k_2 \varphi(x,t)} \bar{y},
\end{equation}

and

\begin{equation}
K(x,t) = k(x,t) - k_2 \frac{\partial \varphi}{\partial n} \sum_{i,j} a_{ij} n_i n_j.
\end{equation}

Now, we take

\begin{equation}
k_2 > 0 \text{ as } K(x,t) \geq \frac{1}{2},
\end{equation}

which is possible by Lemma 3.1 and (A). Functions \(B_i(x,t), C(x,t,u), G(x,t), F(x,t), \bar{u}\) and \(K(x,t)\), respectively, satisfy the conditions for \(b_i(x,t), c(x,t,r), g(x,t), f(x,t), \bar{y}\) and \(k(x,t)\) in (B), (C) and (D).

**Lemma 3.2.** In the sense of Definition 2.7 existence of solution to problems (2.1)-(2.3) and (3.2)-(3.4) are equivalent

**Proof** First, let us prove that if \(y\) is a solution in the sense of Definition 2.1 to problem (2.1)-(2.3), then \(u\) is a solution in the sense of Definition 2.1 to problem (3.2)-(3.4).

For \(v \in W(Q)\), put \(\bar{v} = e^{-k_1 t - k_2 \varphi} v\). Then

\[ - \int_Q y \frac{\partial v}{\partial t} \, dx \, dt = \int_Q u e^{-k_1 t - k_2 \varphi} \frac{\partial e^{k_1 t + k_2 \varphi} \bar{v}}{\partial t} \, dx \, dt \]

\[ = \int_Q \frac{\partial \bar{v}}{\partial t} \, dx \, dt - \int_Q (k_1 + k_2 \frac{\partial \varphi}{\partial t}) u \bar{v} \, dx \, dt, \]

\[ \sum_{i,j=1}^N \int_Q a_{ij}(x,t) \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \, dt = \]

\[ = \sum_{i,j=1}^N \int_Q a_{ij}(x,t) \frac{\partial (e^{-k_1 t - k_2 \varphi} u) \frac{\partial (e^{k_1 t + k_2 \varphi} \bar{v})}{\partial x_j}}{\partial x_i} \, dx \, dt \]

\[ = \sum_{i,j=1}^N \int_Q a_{ij} \left[ \frac{\partial u}{\partial x_j} e^{-k_1 t - k_2 \varphi} - uk_2 e^{-k_1 t - k_2 \varphi} \frac{\partial \varphi}{\partial x_j} \right] \times \]

\[ \times \left[ e^{k_1 t + k_2 \varphi} \frac{\partial \bar{v}}{\partial x_i} + e^{k_1 t + k_2 \varphi} k_2 \frac{\partial \varphi}{\partial x_i} \right] \, dx \, dt \]

\[ = \sum_{i,j=1}^N \int_Q a_{ij} \left[ \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} + k_2 \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j} - k_2 \frac{\partial \varphi}{\partial x_j} \frac{\partial u}{\partial x_i} - k_2 \frac{\partial \varphi}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} \right] \, dx \, dt \equiv I. \]
Integrating by parts in the last term above, we have

\[ I = \sum_{i,j=1}^{N} \int_{Q} a_{ij}(x,t) \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \, dt = \sum_{i,j=1}^{N} \int_{Q} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} \, dx \, dt \]

\[ + \sum_{i,j=1}^{N} \int_{Q} 2k_2 a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx \, dt - \int_{\Sigma_1} a_{ij} k_2 \frac{\partial \varphi}{\partial y_j} n_i \, u \bar{v} \, d\sigma \]

\[ + \sum_{i,j=1}^{N} \int_{Q} k_2 \left[ \frac{\partial a_{ij}}{\partial x_i} \frac{\partial \varphi}{\partial x_j} u \bar{v} - k_2 a_{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_i} u \bar{v} + a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} u \bar{v} \right] \, dx \, dt. \]

Also,

\[ \sum_{i} \int_{Q} b_i \frac{\partial u}{\partial x_i} \, v \, dx \, dt = \sum_{i} \int_{Q} b_i \frac{\partial u}{\partial x_i} \bar{v} \, dx \, dt - \sum_{i} \int_{Q} b_i k_2 \frac{\partial \varphi}{\partial x_i} \bar{v} \, dx \, dt, \]

\[ \int_{Q} c(x, t, y) v \, dx \, dt = \int_{Q} c(x, t, e^{k_1 + k_2 \varphi} u) e^ {k_1 t + k_2 \varphi} \bar{v} \, dx \, dt. \]

The facts \( v \in W(Q) \) and \( \bar{v} = e^{k_1 + k_2 \varphi} v \in W(Q) \) are equivalent, and so from above we can see that \( u \) is a solution to (3.2)-(3.4) in the sense of Definition 2.1. In the same way we can see that if \( u \) is a solution to problem (3.2)-(3.4) in the sense of Definition 2.1, then \( y \) is a solution in the sense of Definition 2.1 to (2.1)-(2.3). □

Therefore, in what follows we will consider the existence of a solution to problem (3.2)-(3.4). To this end, in the next section we will consider an auxiliary problem.

4. An auxiliary problem

The main purpose in this section is to find a function \( u^m \in H^1(\Omega) \) satisfying the following

\[ u^m - \bar{u} \in W(Q), \]

\[ \int_{Q} \frac{1}{m} \frac{\partial u^m}{\partial t} \frac{\partial v}{\partial t} \, dx \, dt - \int_{Q} u^m \frac{\partial v}{\partial t} \, dx \, dt + \int_{Q} \sum_{i,j=1}^{N} a_{ij}(x,t) \frac{\partial u^m}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \, dt \]

\[ + \int_{Q} B_i(x,t) \frac{\partial u^m}{\partial x_i} v \, dx \, dt + \int_{Q} C(x,t, u^m) v \, dx \, dt \]

\[ + \int_{\Sigma_1} [u^m v \cos(\hat{\nu}, t) + K(x,t) u^m v] \, d\sigma + (u^m(x,T), v(x,T))_{\Omega(T)} \]

\[ = (u_0, v(x,0))_{\Omega(0)} + \int_{Q} G(x,t) v \, dx \, dt + \int_{\Sigma_1} F(x,t) v \, d\sigma \]

\[ \forall v \in W(Q), \]

where \( m \) are positive integers.

We have the following result.

**Theorem 4.1.** Let \( k_2 \) in (3.1) be as (3.7). Then, for some \( k_1 \) in (3.1), which is taken before (4.7), there exists a unique solution to problem (4.1).

**Proof** Set \( u = w + \bar{u} \), define an operator \( A_m \in (W(Q) \mapsto W(Q)^*) \) and an element \( L \in W(Q)^* \),
respectively, by

$$\forall w, v \in W(Q);$$

$$\langle A_m w, v \rangle = \int_Q \frac{1}{m} \frac{\partial w}{\partial t} \frac{\partial v}{\partial t} \, dx \, dt - \int_Q w \frac{\partial v}{\partial t} \, dx \, dt + \sum_{i,j=1}^N \int_Q a_{ij}(x,t) \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \, dt$$

$$+ \sum_{i=1}^N \int_Q B_i(x,t) \frac{\partial u}{\partial x_i} v \, dx \, dt + \int_Q C(x,t,u) v \, dx \, dt$$

$$+ \int_{\Sigma_1} [uv \cos(\nu,t) + K(x,t)uv] \, d\sigma + (w(x,T), v(x,T))_{\Omega(T)}$$

and

$$\langle L, v \rangle = (u_0, v(x,0))_{\Omega(0)} + \int_Q G(x,t)v \, dx \, dt + \int_{\Sigma_1} F(x,t)v \, d\sigma.$$ 

Now, let us consider problem of finding \( w \) such that

$$A_m w = L. \quad (4.2)$$

By the conditions (A), (B) and (C), operator \( A_m \) is Lipschitz continuous. For any \( w_1, w_2 \in W(Q) \), letting \( w = w_1 - w_2 \), we have that

$$\langle A_m w_1 - A_m w_2, w \rangle =$$

$$\int_Q \frac{1}{m} \frac{\partial w}{\partial t} \frac{\partial w}{\partial t} \, dx \, dt - \int_Q w \frac{\partial w}{\partial t} \, dx \, dt + \sum_{i,j=1}^N \int_Q a_{ij}(x,t) \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx \, dt$$

$$+ \sum_{i=1}^N \int_Q B_i(x,t) \frac{\partial w}{\partial x_i} w \, dx \, dt + \int_Q [C(x,t,w_1 + \bar{w}) - C(x,t,w_2 + \bar{u})] w \, dx \, dt$$

$$+ \int_{\Sigma_1} [w^2 \cos(\nu,t) + K(x,t)w^2] \, d\sigma + |w(x,T)|^2_{\Omega(T)}.$$

On the other hand, by integrating by parts we get

$$- \int_Q w \frac{\partial w}{\partial t} \, dx \, dt = \frac{1}{2} \left[ |w(0)|^2_{\Omega(0)} - |w(T)|^2_{\Omega(T)} - \int_{\Sigma_1} w^2 \cos(\nu,t) \, d\sigma \right]. \quad (4.4)$$

From (4.3) and (4.4) we conclude that for any \( w_1, w_2 \in W(Q) \)

$$\langle A_m w_1 - A_m w_2, w \rangle = \int_Q \frac{1}{m} \frac{\partial w}{\partial t} \frac{\partial w}{\partial t} \, dx \, dt + \int_Q \sum_{i,j=1}^N a_{ij}(x,t) \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx \, dt$$

$$+ \int_Q \sum_{i=1}^N B_i(x,t) \frac{\partial w}{\partial x_i} w \, dx \, dt + \int_Q [C(x,t,w_1 + \bar{w}) - C(x,t,w_2 + \bar{u})] w \, dx \, dt$$

$$+ \int_{\Sigma_1} \left[ \frac{1}{2} w^2 \cos(\nu,t) + K(x,t)w^2 \right] \, d\sigma + \frac{1}{2} \left[ |w(0)|^2_{\Omega(0)} + |w(x,T)|^2_{\Omega(T)} \right]. \quad (4.5)$$

It follows from (3.6) and the choice of \( k_2 \) mentioned above that

$$\int_{\Sigma_1} \left[ \frac{1}{2} w^2 \cos(\nu,t) + K(x,t)w^2 \right] \, d\sigma \geq 0. \quad (4.6)$$

Note that \( B_i(x,t) \) in (3.3) and \( K(x,t) \) in (3.4) are independent of \( k_1 \). Therefore, taking \( k_3 \) in the expression of \( C(x,t,u) \) in (3.3) a negative number small enough independently of \( m \), we have

$$\int_Q \sum_{i,j=1}^N a_{ij}(x,t) \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx \, dt + \int_Q \sum_{i=1}^N B_i(x,t) \frac{\partial w}{\partial x_i} w \, dx \, dt$$

$$+ \int_Q [C(x,t,w_1 + \bar{w}) - C(x,t,w_2 + \bar{u})] w \, dx \, dt \geq \frac{\rho}{2} \int_0^T \| w \|^2_{\Omega(t)} \, dt \quad (4.7)$$
By (4.4)–(4.7), we have
\[
\langle A_m w_1 - A_m w_2, w_1 - w_2 \rangle \geq \alpha \|w\|_{H^1(Q)}^2, \quad \exists \alpha > 0, \forall w_1, w_2 \in W(Q)
\]
(Note that \(\alpha\) depends on \(m\).) Now, by the theory of monotone operator, there exists a unique solution \(w_m\) to problem (4.2) (cf. Theorem 2.2, ch. 3 in [10]). Thus, \(w^m = w^m + \bar{u}\) is the solution asserted in the theorem. \(\square\)

5. Proof of Theorem 2.1

Let \(k_1\) be as in the proof of Theorem 4.1 and \(k_2\) as in (3.7). When \(w^m = w^m + \bar{u}\) is the solution to (4.1) asserted in Theorem 4.1, putting \(w_1 = w^m\), \(w_2 = 0\), by (4.2), (4.3), (4.5)–(4.7) we have that

\[
\int_Q \frac{1}{m} \frac{\partial w^m}{\partial t} \, dx \, dt + \int_Q \sum_i \frac{\partial w^m}{\partial x_i} \, dx \, dt + \|w^m(x,0)\|_{\Omega(0)} + \|w^m(x,T)\|_{\Omega(T)}^2
\]
\[
\leq c \left( \|L\| + \langle A_m \bar{u}, w^m \rangle \right).
\]

Applying Young inequality to the right hand side of (5.1) and taking into account \(u^m = w^m + \bar{u}\), we have

\[
\int_Q \frac{1}{m} \frac{\partial u^m}{\partial t} \, dx \, dt + \int_Q \sum_i \frac{\partial u^m}{\partial x_i} \, dx \, dt + \|u^m(x,0)\|_{\Omega(0)} + \|u^m(x,T)\|_{\Omega(T)}^2 \leq c,
\]

(5.2)

where \(c\) is independent of \(m\).

We claim that for any \(v \in W(Q)\)

\[
\int_Q \frac{1}{m} \frac{\partial u^m}{\partial t} \frac{\partial v}{\partial t} \, dx \, dt \to 0 \quad \text{as} \quad m \to \infty.
\]

Indeed, by Hölder inequality

\[
\left| \int_Q \frac{1}{m} \frac{\partial u^m}{\partial t} \frac{\partial v}{\partial t} \, dx \, dt \right| \leq \frac{1}{\sqrt{m}} \left[ \int_Q \frac{1}{m} \frac{\partial u^m}{\partial t} \, dx \, dt \right]^{\frac{1}{2}} \cdot \left[ \int_Q \frac{\partial v}{\partial t} \, dx \, dt \right]^{\frac{1}{2}},
\]

which shows (5.3) since by (5.2) one has that \(\int_Q \frac{1}{m} \frac{\partial u^m}{\partial t} \, dx \, dt \leq c\).

By (5.2) we can also choose a subsequence, which is still denoted by \(\{u_m\}\), such that

\[
w^m \equiv u^m - \bar{u} \rightharpoonup w, \quad \text{weakly in} \quad V(Q), \quad u_m(T) \rightharpoonup r \quad \text{weakly in} \quad L^2(\Omega(T)).
\]

(5.4)

First, let \(c(x,t,y)\) be linear with respect to \(y\), that is, \(c(x,t,y) = c(x,t)\).

Then, \(C(x,t,u)\) is also linear with respect to \(u\), that is \(C(x,t,u) = C(x,u)\). Now put \(u \equiv w + \bar{u}\).

Then, using (5.3) and (5.4) and passing to the limit in (4.1), we have

\[
w \equiv u - \bar{u} \in V(Q),
\]

\[
- \int_Q \frac{\partial w}{\partial t} \, dx \, dt + \int_Q \sum_{i,j=1}^N a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \, dt + \int_Q \sum_{i=1}^N B_i(x,t) \frac{\partial u}{\partial x_i} v \, dx \, dt
\]

\[
+ \int_Q C(x,t)uv \, dx \, dt + \int_{\Sigma_1} |uv \cos(\nu, t) + K(x,t)uv| \, d\sigma
\]

\[
= (u_0, v(x,0))_{\Omega(0)} + \int_Q G(x,t)v \, dx \, dt + \int_{\Sigma_1} F(x,t)v \, d\sigma
\]

\[\forall v \in D(Q) \; \text{with} \; v(T) = 0,
\]

which shows that \(u\) is a solution to problem (4.2), (4.3).
Next, let \( \Sigma_0 = \Gamma_0 \times (0, T), \Gamma_0 \subset \partial \Omega(0) \) and \( \Gamma_0 \) is invariant under the diffeomorphism in Assumption \([2.1]\).

Following the method in \([27]\), we will prove that the set \( \{u^m\} \) of solutions to problem \((4.1)\) is relatively compact in \( L_2(Q) \). First, let us prove that the following two norms in \( V(Q) \)

\[
\int_0^T \|w(t)\|_{\Omega(t)} dt \quad \text{and} \quad \int_0^T \|w(t)\|_{H^1(\Omega(t))} dt
\]

are equivalent, \((5.6)\)

where \( H^1(\Omega(t)) \equiv W^2_2(\Omega(t)) \). It is enough to show that there exists a constant \( C \) independent of \( t \) such that

\[
\|w(t)\|_{H^1(\Omega(t))} \leq C\|w(t)\|_{\Omega(t)} \quad \forall t \in [0, T]
\]

\((5.7)\)

Let \( w(x) \) be a function defined on \( \Omega(t) \) and \( x' \in \Omega(0) \). Then, \( w'(x') \equiv w(X(t)x') \) is a function defined on \( \Omega(0) \). By Friedrichs inequality

\[
\int_{\Omega(0)} |w'(x')|^2 \, dx' \leq C(\Omega(0), \Gamma_0) \int_{\Omega(0)} |\nabla w'(x')|^2 \, dx'.
\]

Denoting Jacobian of transformation \( x' = X^{-1}(t)x \) by \( J = \frac{\partial X'}{\partial x} \), we have

\[
\int_{\Omega(t)} |w(x)|^2 |J| \, dx = \int_{\Omega(0)} |w'(x')|^2 \, dx' \leq C(\Omega(0), \Gamma_0) \int_{\Omega(0)} |\nabla w'(x')|^2 \, dx' \leq C(\Omega(0), \Gamma_0) \int_{\Omega(t)} |\nabla w(x)|^2 |J|^{-1} \, dx,
\]

where it was considered that in \( \nabla w'(x') \) and \( \nabla w(x) \) operators \( \nabla \) are, respectively, with respect to \( x' \) and \( x \).

From this we get

\[
\int_{\Omega(t)} |w(x)|^2 \, dx \leq C(t) \int_{\Omega(t)} |\nabla w(x)|^2 \, dx,
\]

where \( C(t) \) is continuous in \( t \in [0, T] \). This implies \((5.7)\).

On the other hand, by \((5.2)\)

\[
\int_0^T \|u^m\|_{\Omega(t)}^2 dt \leq c.
\]

\((5.8)\)

Let \( \Omega \subset R^N \) such that \( \Omega(t) \subset \Omega \quad \forall t \in [0, T] \).

For any \( m \) let us make \( \tilde{w}^m(x, t) \in H^1(\Omega \times (0, T)) \), an extension of \( w^m(x, t) \in W(Q) \) as follows. Let \( w^m(x') \equiv w^m(X(t)x') \) on \( \Omega(0) \) and denote bounded extensions in \( H^1(R^N) \) by the same(cf. Lemma 1.29, ch. 2 in \([10]\)). Then,

\[
\int_{R^N} \left( |w^m(x')|^2 + |\nabla w^m(x')|^2 \right) \, dx' \leq c \int_{\Omega(0)} \left( |w^m(x')|^2 + |\nabla w^m(x')|^2 \right) \, dx' \leq c \int_{\Omega(t)} \left( |w^m(x, t)|^2 |J(t)| + |\nabla w^m(x, t)|^2 |J(t)|^{-1} \right) \, dx \]

\[
\leq c \int_{\Omega(t)} \left( |w^m(x, t)|^2 + |\nabla w^m(x, t)|^2 \right) \, dx.
\]

We take the restriction on \( \Omega \) of a function defined by \( \tilde{w}^m(x, t) = w^m(X^{-1}(t)x) \) on \( R^N \). Then, we have

\[
\int_{\Omega} \left( |\tilde{w}^m(x, t)|^2 + |\nabla \tilde{w}^m(x, t)|^2 \right) \, dx \leq c \int_{R^N} \left( |w^m(x')|^2 + |\nabla w^m(x')|^2 \right) \, dx'.
\]

By \((5.6)\), \((5.8)\) and two inequalities above, we get

\[
\int_0^T \|\tilde{w}^m\|_{H^1(\Omega)}^2 dt \leq c.
\]

\((5.9)\)
Also, by (5.2) we get
\[
\int_{\Omega \times (0,T)} \frac{1}{m} \left| \frac{\partial \bar{w}^m}{\partial t} \right|^2 dx \, dt \leq c, \quad \int_{\Omega \times (0,T)} \frac{1}{m} \left| \frac{\partial \bar{u}}{\partial t} \right|^2 dx \, dt \leq c, \tag{5.10}
\]
where $\bar{w}$ is a bounded extension of $\bar{u}$ and $c$ is independent of $m$.

Put $\bar{w}^m(x,t) = 0$ for $-T < t < 0$, $T < t < 2T$. Let
\[
w_h^m(x,t) = \frac{1}{h} \int_{t-h}^{t} \bar{w}^m(x,s) \, ds \quad \text{for } |h| < T.
\]
Then,
\[
\frac{\partial w_h^m(x,t)}{\partial t} = \frac{1}{h} \left( \bar{w}^m(x,t) - \bar{w}^m(x,t-h) \right), \quad w_h^m(x,t)|_{\Sigma_0} = 0,
\]
which means $w_h^m|_{Q} \in W(Q)$. Replacing $v$ by $w_h^m|_{Q}$ in (4.1), we have
\[
\int_{Q} \frac{1}{m} \frac{\partial w_h^m(x,t)}{\partial t} \left[ \bar{w}^m(x,t) - \bar{w}^m(x,t-h) \right] dx \, dt
\]
\[
- \int_{Q} \bar{u}(x,t) \frac{\partial w_h^m(x,t)}{\partial t} dx \, dt + \int_{Q} \sum_{i=1}^{N} a_{ij}(x,t) \frac{\partial u^m}{\partial x_j} \frac{\partial w_h^m}{\partial x_i} dx \, dt
\]
\[
+ \int_{Q} \sum_{i=1}^{N} B_i(x,t) \frac{\partial u^m}{\partial x_i} w_h^m dx \, dt + \int_{Q} C(x,t,u^m) w_h^m dx \, dt
\]
\[
+ \int_{\Sigma_1} [u^m w_h^m \cos(\nu \cdot t) + K(x,t) u^m w_h^m] \, d\sigma + (u^m(x,T), w_h^m(x,T))_{\Omega(T)}
\]
\[
= (u_0, w_h^m(x,0))_{\Omega(0)} + \int_{Q} G(x,t) w_h^m dx \, dt + \int_{\Sigma_1} F(x,t) w_h^m d\sigma.
\]
Assuming $\bar{w}(x,t) \in C^1(\bar{\Omega} \times [0,T])$, let us estimate
\[
I_1 = \int_{\Omega \times (0,T)} \left| \frac{1}{h} \left[ \bar{w}(x,t) - \bar{w}(x,t-h) \right] \right|^2 dx \, dt.
\]
First, let $h > 0$. Then
\[
I_1 \leq \int_{\Omega \times (0,T)} \frac{1}{h^2} \left[ \int_{t-h}^{t} \frac{\partial \bar{w}(x,s)}{\partial s} \, ds \right]^2 dx \, dt
\]
\[
+ \int_{\Omega \times (0,T)} \frac{1}{h^2} \left[ \bar{w}(x,0) + \int_{0}^{t} \frac{\partial \bar{w}(x,s)}{\partial s} \, ds \right]^2 dx \, dt
\]
\[
\leq \int_{\Omega \times (0,T)} \frac{1}{h} \left[ \int_{t-h}^{t} \left| \frac{\partial \bar{w}(x,s)}{\partial s} \right|^2 \, ds \right] dt
\]
\[
+ \frac{2}{h^2} \left. \int_{\Omega \times (0,T)} \left| \frac{\partial \bar{w}(x,s)}{\partial s} \right|^2 \, ds \cdot h \, dx \, dt \right|_{t-h}
\]
\[
\leq \frac{1}{h} \left( (T-h) \int_{\Omega \times (0,T)} \left| \frac{\partial \bar{w}(x,t)}{\partial t} \right|^2 \, dx \, dt + 2|\bar{w}(x,0)|^2_{\Omega} \right.
\]
\[
+ \left. 2h \int_{\Omega \times (0,T)} \left| \frac{\partial \bar{w}(x,t)}{\partial t} \right|^2 \, dx \, dt \right|
\]
\[
\leq \frac{1}{h} \left( (T+h) \int_{\Omega \times (0,T)} \left| \frac{\partial \bar{w}(x,t)}{\partial t} \right|^2 \, dx \, dt + 2|\bar{w}(x,0)|^2_{\Omega} \right). \tag{5.13}
\]
If $h < 0$, using

$$I_1 \leq \int_{\Omega \times (0,T-h]} \frac{1}{h^2} \left( \int_{t-h}^t \frac{\partial}{\partial s} w(x,s) \, ds \right)^2 \, dx \, dt$$

$$+ \int_{\Omega \times (T-h,T]} \frac{1}{h^2} \left[ \tilde{w}(x,T) + \int_T^t \frac{\partial}{\partial s} w(x,s) \, ds \right]^2 \, dx \, dt,$$

in the same way above we get

$$I_1 \leq \frac{1}{|h|} \left\{ (T + |h|) \int_{\Omega \times (0,T)} \left| \frac{\partial}{\partial t} \tilde{w}(x,t) \right|^2 \, dx \, dt + 2|\tilde{w}(x,T)|^2 \right\}. \quad (5.14)$$

Since $C^1(\bar{\Omega} \times (0,T])$ is dense in $H^1(\Omega \times (0,T))$, by (5.10), (5.11), (5.13), (5.14) for any $\tilde{w}^m$ we have

$$\frac{1}{\sqrt{m}} \left\{ \int_{\Omega \times (0,T)} \left| \frac{1}{h} \left[ \tilde{w}^m(x,t) - \tilde{w}^m(x,t-h) \right] \right|^2 \, dx \, dt \right\}^{\frac{1}{2}} \leq \frac{c}{\sqrt{|h|}}. \quad (5.15)$$

By (5.2) and (5.15), we get

$$\left\| \int_{Q} \frac{1}{m} \frac{\partial u^m(x,t)}{\partial t} \frac{1}{h} \left[ \tilde{w}^m(x,t) - \tilde{w}^m(x,t-h) \right] \, dx \, dt \right\| \leq \left( \int_{Q} \frac{1}{m} \left| \frac{\partial u^m(x,t)}{\partial t} \right|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{Q} \frac{1}{h} \left[ \tilde{w}^m(x,t) - \tilde{w}^m(x,t-h) \right]^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \frac{c}{\sqrt{|h|}}. \quad (5.16)$$

We have

$$\left\| \int_{Q} \sum_{i,j=1}^N a_{ij}(x,t) \frac{\partial u^m}{\partial x_j} \frac{\partial w^m}{\partial x_i} \, dx \, dt \right\| \leq c \int_0^T \| u^m \|_{\Omega(t)} \left( \int_{\Omega(t)} \right) \frac{1}{h} \int_{t-h}^t \tilde{w}^m(x,s) \, ds \, dt$$

$$\leq c \int_0^T \| u^m \|_{\Omega(t)} \frac{1}{\sqrt{|h|}} \left( \int_{\Omega(t)} \frac{1}{h} \tilde{w}^m(x,s)^2 \, ds \right)^{\frac{1}{2}} \, dt \leq c/\sqrt{|h|},$$

$$\left\| \int_{Q} \sum_{i=1}^N B_i(x,t) \frac{\partial u^m}{\partial x_i} w^m \, dx \, dt \right\| \leq c \int_0^T \| u^m \|_{\Omega(t)} \frac{1}{\sqrt{|h|}} \left( \int_{\Omega(t)} \frac{1}{h} \tilde{w}^m(x,s)^2 \, ds \right)^{\frac{1}{2}} \, dt \leq c/\sqrt{|h|},$$

$$\left\| \int_{Q} C(x,t,u^m) w^m \, dx \, dt \right\| \leq c \int_{Q} \left( \| u^m \| + |C(x,t,0)| \right) \left| w^m \right| \, dx \, dt \leq c/\sqrt{|h|}. \quad (5.17)$$

By Assumption 2.1 \(\frac{1}{\sin(\nu,t)} \geq \delta > 0\) on $\Sigma_1$. Taking this and the trace theorem into account, we get

$$\left| \int_{\Sigma_1} [u^m w^m_h \cos(\nu,t) + K(x,t) u^m w^m_h] \, d\sigma \right|$$

$$= \left| \int_{\Sigma_1} [u^m w^m_h \cos(\nu,t) + K(x,t) u^m w^m_h] \frac{1}{\sin(\nu,t)} \, d\sigma \right| \quad (5.18)$$

$$\leq c \int_0^T \| u^m \|_{\Omega(t)} \frac{1}{\sqrt{|h|}} \left( \int_{\Omega(t)} \frac{1}{h} \tilde{w}^m(x,s)^2 \, ds \right)^{\frac{1}{2}} \, dt \leq c/\sqrt{|h|},$$

where (5.6) and (5.8) were used. Also

$$\left| (u^m(x,T), w^m_h(x,T))_{\Omega(T)} \right| \leq c \frac{1}{\sqrt{|h|}} \int_{T-h}^T \tilde{w}^m(x,s)^2 \, ds \frac{1}{\sqrt{|h|}} \, dt \leq c/\sqrt{|h|},$$

$$\left| (u_0, w^m_h(x,0))_{\Omega(0)} \right| \leq c/\sqrt{|h|}. \quad (5.19)$$
Similarly to (5.17), (5.19), let us estimate

\[ - \int_Q \ddot{u}(x,t) \frac{\partial w_h^m(x,t)}{\partial t} \, dx \, dt \]

\[ = \left| \int_Q \frac{\partial \ddot{u}(x,t)}{\partial t} w_h^m(x,t) \, dx \, dt - \int_{\Omega(T)} \ddot{u}(x,T) w_h^m(x,T) \, dx \right| \]

\[ + \int_{\Omega(0)} \ddot{u}(x,0) w_h^m(x,0) \, dx - \int_{\Sigma_1} \ddot{u}(x,t) w_h^m(x,t) \cos(\nu \cdot t) \, d\sigma \]

\[ \leq c/\sqrt{|h|}. \]

Also, we get

\[ \left| \int_Q G(x,t) w_h^m \, dx \, dt \right| \leq c \int_0^T \frac{1}{\sqrt{|h|}} \left| \int_{t-h}^t |\ddot{w}^m(x,s)|^2 \, ds \right|^{\frac{1}{2}} \, dt \leq c/\sqrt{|h|}, \]

(5.21)

\[ \left| \int_{\Sigma_1} F(x,t) w_h^m \, d\sigma \right| \leq c \int_0^T \frac{1}{\sqrt{|h|}} \left| \int_{t-h}^t \|\ddot{w}^m(x,s)\|_2^2 \, ds \right|^{\frac{1}{2}} \, dt \leq c/\sqrt{|h|}. \]

Let us estimate

\[ I \equiv -\frac{1}{h} \int_Q [w^m(x,t) - \ddot{w}^m(x,t - h)] \, dx \, dt. \]

Putting \( \Omega(t) = \Omega(T) \) for \( t > T \), \( \Omega(t) = \Omega(0) \) for \( t < 0 \) and using \( -ab = \frac{1}{2}[(a - b)^2 - a^2 - b^2] \), we have the following estimate.

\[ I = -\frac{1}{h} \int_0^T \left( \ddot{w}^m(x,t), \ddot{w}^m(x,t) - \ddot{w}^m(x,t - h) \right)_{\Omega(t)} \, dt \]

\[ = -\frac{1}{2h} \int_0^T \left| w^m(x,t) \right|_{\Omega(t)}^2 \, dt + \frac{1}{2h} \int_0^T \left| \ddot{w}^m(x,t - h) \right|_{\Omega(t)}^2 \, dt \]

\[ - \frac{1}{2h} \int_0^T \left| \ddot{w}^m(x,t) - \ddot{w}^m(x,t - h) \right|_{\Omega(t)}^2 \, dt \]

\[ = -\frac{1}{2h} \int_0^T \left| w^m(x,t) \right|_{\Omega(t) \cap \Omega(t+h)}^2 \, dt - \frac{1}{2h} \int_0^T \left| w^m(x,t) \right|_{\Omega(t) \cap \Omega(t+h)}^2 \, dt \]

\[ + \frac{1}{2h} \int_0^T \left| \ddot{w}^m(x,t - h) \right|_{\Omega(t) \cap \Omega(t+h)}^2 \, dt + \frac{1}{2h} \int_0^T \left| \ddot{w}^m(x,t - h) \right|_{\Omega(t) \cap \Omega(t+h)}^2 \, dt \]

(5.22)

\[ - \frac{1}{2h} \int_0^T \left| \ddot{w}^m(x,t) - \ddot{w}^m(x,t - h) \right|_{\Omega(t)}^2 \, dt \]

Now, put \( p = 4 \) for \( N = 1, \ldots, 4 \), \( p = \frac{2N}{N-2} \) for \( N \geq 5 \) and let \( \frac{2}{p} + \frac{1}{q} = 1 \). Applying H"{o}lder
inequality, (5.9), Assumption 2.1 and the fact that $H^1(\Omega) \hookrightarrow L_p(\Omega)$, we have

\[
\left| \frac{1}{2h} \int_0^T \left| \bar{w}^m(x, t - h) \right|^2_{\Omega(t) \setminus \Omega(t - h)} \, dt \right| \\
\leq \frac{1}{2|h|} \int_0^T \left[ \int_{\Omega(t) \cup (t + h) \setminus \Omega(t)} \left| \bar{w}^m(x, t) \right|^p \, dx \right] \frac{1}{\sqrt[p]{p}} \cdot \left[ \text{mes}(\Omega(t) \cup (t + h) \setminus \Omega(t)) \right]^{\frac{1}{p}} \, dt \\
\leq \frac{c}{2|h|} \int_0^T \|\bar{w}^m(x, t)\|^2_{H^1(\Omega(t))} \, dt \cdot |h|^{1/p} \leq \frac{c}{|h|^{1-1/p}}.
\]

Substituting (5.23) in the right hand side of (5.22), we have that

\[
- \frac{1}{h} \int_Q w^m(x, t) \left[ \bar{w}^m(x, t) - \bar{w}^m(x, t - h) \right] \, dx \, dt \leq \\
\frac{c}{h^{1-1/q}} - \frac{1}{2h} \int_0^T \left| \bar{w}^m(x, t) - \bar{w}^m(x, t - h) \right|^2_{\Omega(t)} \, dt \quad \text{if } h > 0,
\]

\[
- \frac{1}{h} \int_Q w^m(x, t) \left[ \bar{w}^m(x, t) - \bar{w}^m(x, t - h) \right] \, dx \, dt \geq \\
- \frac{c}{|h|^{1-1/q}} + \frac{1}{2|h|} \int_0^T \left| \bar{w}^m(x, t) - \bar{w}^m(x, t - h) \right|^2_{\Omega(t)} \, dt \quad \text{if } h < 0.
\]

Formulas (5.8), (5.12), (5.16)-(5.21) and (5.24) imply

\[
\int_0^T \left| \bar{w}^m(x, t + h) - \bar{w}^m(x, t) \right|^2_{\Omega(t)} \, dt \leq c|h|^{1/q} \quad \text{for } h \in R
\]

and

\[
\int_0^h \left| \bar{w}^m(x, t) \right|^2_{\Omega(t)} \, dt + \int_{T-h}^T \left| \bar{w}^m(x, t) \right|^2_{\Omega(t)} \, dt \leq ch^{1/q} \quad \text{for } h > 0.
\]

Next, let

\[
\tilde{w}^m(x, t) = \begin{cases} 
    w^m(x, t) & \text{if } (x, t) \in Q \\
    0 & \text{otherwise.}
\end{cases}
\]

Then, when $0 < |h| < T$, similarly to (5.24) we have

\[
\int_0^T \left| \tilde{w}^m(x, t + h) - \tilde{w}^m(x, t) \right|^2_{\Omega(t)} \, dt \\
\leq \int_0^T \left| \tilde{w}^m(x, t + h) - \tilde{w}^m(x, t) \right|^2_{\Omega(t) \cup (t + h) \setminus \Omega(t)} \, dt \\
+ \int_0^T \left| \tilde{w}^m(x, t + h) - \tilde{w}^m(x, t) \right|^2_{\Omega(t) \cup (t + h) \setminus (t + h)} \, dt \\
\leq c \int_0^T \left\| \tilde{w}^m(x, t + h) \right\|^2_{\Omega(t)} \, dt \cdot |h|^\frac{1}{q} \leq c|h|^\frac{1}{q}.
\]

From (5.25) and (5.27) we have

\[
\int_0^T \left| \tilde{w}^m(x, t + h) - w^m(x, t) \right|^2_{\Omega(t)} \, dt \leq c|h|^\frac{1}{q} \quad \text{for } 0 < |h| < T.
\]

Let $\tilde{h} \in R^N$ and

\[
\Omega_j(t) = \{ x \in \Omega(t) : \text{dist}(x, \partial \Omega(t)) > 2/j \}, \quad j = 1, 2, 3, \ldots.
\]
Then, by (5.6) and (5.8)
\[
\int_{\Omega(t)} |w^m(x, t)|^2 \, dx = \\
= \left[ \int_{\Omega(t) \setminus \Omega_j(t)} |w^m(x, t)|^p \, dx \right]^{\frac{2}{p}} \cdot \left[ \text{mes}(\Omega(t) \setminus \Omega_j(t)) \right]^{\frac{2}{p}} \, dt
\]  
(5.29)
\[
\leq c\|w^m(x, t)\|_{L^p(\Omega(t))}^2 \cdot \left(1/j\right)^{\frac{2}{p}},
\]
where \(c\) depends only on \(Q\).

Now, if \(|\tilde{h}| < 1/j\), then \(x + s\tilde{h} \in \Omega_j(t)\) provided \(x \in \Omega_j(t)\) and \(s \in [0, 1]\). For \(w^m \in C^\infty(\Omega(t))\),
\[
\int_{\Omega_j(t)} |w^m(x + \tilde{h}, t) - w^m(x, t)|^2 \, dx \leq \int_{\Omega_j(t)} dx \left[ \int_0^1 \left| \frac{d}{ds}w^m(x + s\tilde{h}, t) \right| ds \right]^2
\]
\[
\leq \int_{\Omega_j(t)} dx \left[ \int_0^1 |\nabla w^m(x + s\tilde{h}, t)| \cdot |\tilde{h}| \, ds \right]^2
\]
\[
\leq |\tilde{h}|^2 \int_0^1 \int_{\Omega_j(t)} |\nabla w^m(x + s\tilde{h}, t)|^2 \, dx \, ds
\]
\[
\leq |\tilde{h}|^2 \int_{\Omega_j(t)} |\nabla w^m|^2 \, dx \leq (1/j)^2\|w^m\|_{L^2(\Omega)}^2.
\]  
(5.30)
Since \(C^\infty(\Omega(t))\) is dense in \(H^1(\Omega(t))\), (5.30) is valid for any \(w^m \in H^1(\Omega(t))\). By (5.29), (5.30) and (5.8) we have that
\[
\int_Q |\tilde{w}^m(x + \tilde{h}, t) - \tilde{w}^m(x, t)|^2 \, dx \, dt \leq c(1/j)^{1/q} \quad \text{for } |\tilde{h}| < 1/j
\]  
(5.31)
where \(c\) is independent of \(m\).

From (5.28) and (5.31) we get
\[
\int_Q |\tilde{w}^m(x + h, t + h) - \tilde{w}^m(x, t)|^2 \, dx \, dt \to 0 \quad \text{as } (\tilde{h}, h) \to 0 \text{ in } \mathbb{R}^{N+1}. 
\]  
(5.32)
From (5.28) and (5.29), we get
\[
\forall \varepsilon, \exists Q_{\varepsilon} \text{ such that } \bar{Q}_{\varepsilon} \subset Q : \int_{Q \setminus Q_{\varepsilon}} |w^m(x, t)|^2 \, dx \, dt < \varepsilon.
\]  
(5.33)
By (5.32) and (5.33) we know that the set \(\{w^m\}\) is relatively compact in \(L_2(Q)\) (cf. Theorem 2.32 in [1]). Thus, we can choose a subsequence, which is still denoted by \(\{w^m\}\), such that \(w^m \rightharpoonup w \in L_2(Q)\). Therefore \(C(x, t, w^m) \to C(x, t, u)\) in \(L_2(\Omega(t))\) for a.e. \(t\), where \(u \equiv w + \tilde{u}\).

Therefore, using (5.4) and passing to the limit in (5.1), we have (5.5) which shows that \(u\) is a solution to problem (5.2)-(5.4). \(\square\)

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