An improvement on the number of simplices in $\mathbb{F}_q^d$

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Abstract

Let $\mathcal{E}$ be a set of points in $\mathbb{F}_q^d$. Bennett, Hart, Iosevich, Pakianathan, and Rudnev (2016) proved that if $|\mathcal{E}| \gg q^{d-\frac{d+1}{k+1}}$ then $\mathcal{E}$ determines a positive proportion of all $k$-simplices. In this paper, we give an improvement of this result in the case when $\mathcal{E}$ is the Cartesian product of sets. Namely, we show that if $\mathcal{E}$ is the Cartesian product of sets and $q^{\frac{kd}{k+1}-\frac{1}{d}} = o(|\mathcal{E}|)$, the number of congruence classes of $k$-simplices determined by $\mathcal{E}$ is at least $(1-o(1))q^{\frac{k+1}{2}}$, and in some cases our result is sharp.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of order $q$ with $q = p^r$ for some prime $p$ and positive integer $r$. Denote by $O(d, \mathbb{F}_q)$ the orthogonal group in $\mathbb{F}_q^d$. We say that two $k$-simplices in $\mathbb{F}_q^d$ with vertices $(x_1, \ldots, x_{k+1}), (y_1, \ldots, y_{k+1})$ are in the same congruence class if there exist $\theta \in O(d, \mathbb{F}_q)$ and $z \in \mathbb{F}_q^d$ so that $z + \theta(x_i) = y_i$ for all $i = 1, 2, \ldots, k+1$.

Hart and Iosevich [8] made the first investigation on counting the number of congruence classes of simplices determined by a point set in $\mathbb{F}_q^d$. More precisely, they proved that if $|\mathcal{E}| \gg q^{d-\frac{d+1}{2}+k}$, then $\mathcal{E}$ contains a copy of all $k$-simplices with non-zero edges. Here and throughout, $X \ll Y$ means that there exists $C > 0$ such that $X \leq CY$, and $X = o(Y)$ means that $X/Y \to 0$ as $q \to \infty$, where $X, Y$ are viewed as functions in $q$.

Using methods from spectral graph theory, the third listed author [13] improved this result. In particular, he showed that the same result also holds when $d \geq 2k$ and $|\mathcal{E}| \gg q^{(d-1)/2+k}$. It follows from the results in [8, 13] that the most difficulties arise when the size of simplex is large with respect to the dimension of the space, for instance, the result in [13] on the number of congruence classes of triangles is only non-trivial if $d \geq 4$.

In [4], Covert et al. addressed the case of triangles in $\mathbb{F}_q^2$, and they established that if $|\mathcal{E}| \gg q^2$, then $\mathcal{E}$ determines at least $c\rho q^3$ congruence classes of triangles. The author of [12] extended this result to the case $d \geq 3$. Formally, he proved that if $|\mathcal{E}| \gg q^{\frac{d+2}{3}}$, then $\mathcal{E}$ determines a positive proportion of all triangles. Using Fourier analytic techniques, Chapman et al. [5] indicated that the threshold $q^{\frac{d+2}{3}}$ on the cardinality of $\mathcal{E}$ in the triangle case can be replaced by $q^{\frac{d+k}{k+1}}$ for the case of $k$-simplices. In a recent result, Bennett et al. [3] improved the threshold $q^{\frac{d+k}{k+1}}$ to $q^{\frac{d-k-1}{k+1}}$. The precise statement is given by the following theorem.
Theorem 1.1 (3). Let $E$ be a subset in $\mathbb{F}_q^d$. Suppose that

$$|E| \gg q^{d-\frac{d+1}{k+1}},$$

then, for $1 \leq k \leq d$, the number of congruence classes of $k$-simplices determined by $E$ is at least $c q^{k+1}$ for some positive constant $c$.

In this paper, by using methods from spectral graph theory and elementary results on group actions, we improve Theorem 1.1 in the case $E$ has Cartesian product structure. In particular, we have the following result.

Theorem 1.2. Let $E = A_1 \times \cdots \times A_d$ be a subset in $\mathbb{F}_q^d$. Suppose that

$$\left( \min_{1 \leq i \leq d} |A_i| \right)^{-1} |E|^{k+1} \gg q^{kd},$$

then for $1 \leq k \leq d$, the number of congruence classes of $k$-simplices determined by $E$ is at least $c q^{k+1}$ for some positive constant $c$.

Corollary 1.3. Let $E = A^d$ be a subset in $\mathbb{F}_q^d$. If $|E| \gg q^{kd}$, then the number of congruence classes of simplices determined by $E$ is at least $c q^{k+1}$ for some positive constant $c$.

As a consequence of Corollary 1.3, we recover the following result in [6].

Theorem 1.4. Let $A$ be a subset in $\mathbb{F}_q$. If $|A| \gg q^{d^2/2}$, then the number of distinct distances determined by points in $A^d \subseteq \mathbb{F}_q^d$ is at least $q$.

On the number of congruence classes of triangles in $\mathbb{F}_q^2$. For the case of triangles in $\mathbb{F}_q^2$, in 2012 Bennett, Iosevich, and Pakianathan [2], using Elekes-Sharir paradigm and an estimate on the number of incidences between points and lines in $\mathbb{F}_q^3$, improved significantly the result in [4]. In particular, they proved that if $|E| \gg q^{7/4}$ and $q \equiv 3 \mod 4$, then the number of triangles determined by $E$ is at least $c q^3$ for some positive constant $c$. The authors of [3] recently improved the exponent $7/4$ to $8/5$ in the following.

Theorem 1.5 (Bennett et al. [3]). Let $E$ be a set of points in $\mathbb{F}_q^2$. If $|E| \gg q^{8/5}$, then $E$ determines a positive proportion of all triangles.

We will give a graph-theoretic proof for this theorem in Section 4. If $E$ has Cartesian product structure of sets with different sizes, as a consequence of Theorem 1.2, we are able to obtain a much stronger result as follows.

Theorem 1.6. Let $\mathcal{A}, \mathcal{B}$ be subsets in $\mathbb{F}_q$. If $|\mathcal{A}| \geq q^{1/2+\epsilon}$ and $|\mathcal{B}| \geq q^{1-\frac{\epsilon}{2}}$ for some $\epsilon \geq 0$, then the number of congruence classes of triangles determined by $\mathcal{A} \times \mathcal{B} \subseteq \mathbb{F}_q^2$ is at least $c q^3$ for some positive constant $c$. 

2
Note that if $A$ and $B$ are arbitrary sets in $\mathbb{F}_q$, then it follows from Theorem 1.6 that in order to prove that there exist at least $cq^3$ congruence classes of triangles, we need the condition $|A|^2|B|^3 \gg q^4$. In particular, if $|A| < q^{1/2}$ then we must have $|B| > q$. In fact, one can not expect to get a positive proportion of congruence of triangles in the set $A \times B$ with arbitrary sets $A$ and $B$ satisfying $|A| = o(q^{1/2})$ and $|B| < q$, since the authors of [3] gave a construction with $|A| = q^{1/2-\epsilon}$ and $|B| = q$, and the number of congruence classes of triangles determined by $E$ is at most $cq^{3-\epsilon'}$ for $\epsilon' > 0$. Therefore the result in the form of Theorem 1.6 is tight up to a factor of $q^{\epsilon/3}$.

**Distinct distance subsets in $\mathbb{F}_q^d$.** Given a subset $E \subset \mathbb{F}_q^d$, a subset $U \subset E$ is called a *distinct distance subset* if there are no four distinct points $x, y, z, t \in U$ such that $||x - y|| = ||z - t||$. In [10], Phuong et al. studied the finite field analogue of this problem. More precisely, the authors of [10] proved that if $|E| \geq 2q^{(2d+1)/3}$, then there exists a distinct distance subset of cardinality $\gg q^{1/3}$. This implies that the result is only non-trivial when $d \geq 3$. In this paper, we fill in this gap. In particular, we prove that if $d = 2$, then the threshold $q^{(2d+1)/3}$ can be improved to $q^{1/3}$. In particular, the statement is in the following.

**Theorem 1.7.** Let $E$ be a subset in $\mathbb{F}_q^d$ with $|E| \gg q^{4/3}$. There exists a distinct distance subset $U \subset E$ satisfying $q^{1/3} \ll |U| \ll q^{1/2}$.

## 2 Tools from spectral graph theory

A graph $G = (V, E)$ is called an $(n, d, \lambda)$-graph if it is $d$-regular, has $n$ vertices, and the second eigenvalue of $G$ is at most $\lambda$. The result below gives an estimate on the number of edges between two multi-sets of vertices in an $(n, d, \lambda)$-graph.

**Lemma 2.1 ([1]).** Let $G = (V, E)$ be an $(n, d, \lambda)$-graph. For any two multi-sets of vertices $B, C$, we have

$$\left| e(B, C) - \frac{d|B||C|}{n} \right| \leq \lambda \left( \sum_{b \in B} m_B(b)^2 \right)^{1/2} \left( \sum_{c \in C} m_C(c)^2 \right)^{1/2},$$

where $e(B, C)$ is the number of edges between $B$ and $C$ in $G$, and $m_X(x)$ is the multiplicity of $x$ in $X$.

Let $PG(q, d)$ denote the projective geometry of dimension $d - 1$ over finite field $\mathbb{F}_q$. The vertices of $PG(q, d)$ correspond to the equivalence classes of the set of all non-zero vectors $[x] = [x_1, \ldots, x_d]$ over $\mathbb{F}_q$, where two vectors are equivalent if one is a multiple of the other by a non-zero element of the field.

In this section, we recall a well-known construction of the Erdős-Rényi graph due to Alon and Krivelevich [1] as follows. Let $\mathcal{ER}(\mathbb{F}_q^d)$ denote the graph whose vertices are the points of $PG(q, d)$, and two (not necessarily distinct) vertices $[x] = [x_1, \ldots, x_d]$ and $[y] = [y_1, \ldots, y_d]$ are adjacent if and only if $x_1y_1 + \ldots + x_dy_d = 0$. Alon and Krivelevich [1] obtained the following result on the spectrum of $\mathcal{ER}(\mathbb{F}_q^d)$. 


Lemma 2.2 (Alon and Krivelevich, [1]). For any odd prime power $q$ and $d \geq 2$, the Erdős-Rényi graph $\mathcal{E}\mathcal{R}(F^d_q)$ is an
\[
\left(\frac{q^d - 1}{q - 1}, \frac{q^{d-1} - 1}{q - 1}, q^{(d-2)/2}\right) - \text{graph}.
\]

The next lemma is useful in the proof of Theorem 1.2, which allows us to reduce $k$-simplices to 2-simplices.

Lemma 2.3 (Bennett et al., [3]). Let $V$ be a finite space and $f : V \to \mathbb{R}_{\geq 0}$ a function. For any $n \geq 2$ we have
\[
\sum_{z \in V} f^n(z) \leq |V| \left(\frac{||f||_1}{|V|}\right)^n + \frac{n(n-1)}{2} ||f||_\infty^{n-2} \sum_{z \in V} \left(f(z) - \frac{||f||_1}{|V|}\right)^2,
\]
where $||f||_1 = \sum_{z \in V} |f(z)|$, and $||f||_\infty = \max_{z \in V} f(z)$.

3 Proof of Theorem 1.2

For $E = A_1 \times \cdots \times A_d \subseteq F^d_q$ and $t \in F_q$, we define $\nu_E(t)$ as the cardinality of the set $\{(x, y) \in E \times E : ||x - y|| = t\}$. In order to prove Theorem 1.2 we first need the following lemma.

Lemma 3.1. For $E = A_1 \times \cdots \times A_d \subseteq F^d_q$ with $|A_d| = \min_{1 \leq i \leq d} |A_i|$, we have
\[
\sum_{t \in F_q} \nu_E(t)^2 < \frac{|E|^d}{q} + 2q^{d-1} |E|^2 |A_d|.
\]

Proof. For a fixed pair $(a, b) \in A_d^2$, let $N(a, b)$ denote the set of quadruples $(x, y, z, t) \in E^4$ with $x = (x_1, \ldots, x_{d-1}, a)$ and $y = (y_1, \ldots, y_{d-1}, b)$ satisfying $||x - z|| = ||y - t||$. Then
\[
\sum_{t \in F_q} \nu_E(t)^2 = \sum_{(a, b) \in A_d^2} |N(a, b)| \leq |A_d|^2 \max_{(a, b) \in A_d^2} |N(a, b)|.
\]

We next show that
\[
|N(a, b)| \leq \frac{|E|^2 |A_d|^{-2}}{q} + q^{d-1} |E|^2 |A_d|^{-1}, \forall (a, b) \in A_d^2.
\]

Indeed, let $U$ and $V$ be, respectively, multi-subsets in $PG(q, 2d)$ defined by
\[
U = \left\{\left[-2x_1, \ldots, -2x_{d-1}, 1, t_1, \ldots, t_{d-1}, -(t_d - b)^2 - \sum_{i=1}^{d-1} t_i^2 + \sum_{i=1}^{d-1} x_i^2\right] : x_i, t_i \in A_i\right\},
\]
\[
V = \left\{ \left[ z_1, \ldots, z_{d-1}, (z_d - a)^2 + \sum_{i=1}^{d-1} z_i^2 - \sum_{i=1}^{d-1} y_i^2; 2y_1, \ldots, 2y_{d-1}, 1 \right] : z_i, t_i \in A_i \right\}.
\]

It is clear that
\[
|U| = |\mathcal{E}| |A_d|^{-1}, |V| = |\mathcal{E}| |A_d|^{-1}, m_U(u) \leq 2, m_V(v) \leq 2, \forall u \in U, v \in V.
\]
and \(|N(a, b)|\) is equal to the number of edges between \(U\) and \(V\) in the Erdős-Rényi graph \(\mathcal{E}\mathcal{R}(\mathbb{F}_q^{2d})\). Thus it follows from Lemmas 2.1 and 2.2 that
\[
|N(a, b)| < \frac{|\mathcal{E}|^4 |A_d|^{-2}}{q} + 2q^{d-1}|\mathcal{E}|^2 |A_d|^{-1},
\]
and this completes the proof of the lemma. \(\square\)

It is convenient to recall the following definition which is given in [3].

**Definition 3.2.** Let \(V\) be the \(\mathbb{F}_q\)-vector space of \((k+1) \times (k+1)\) symmetric matrices \(\mathbb{D}\) which can be viewed as the space of possible ordered \(k\)-simplex distances. For \(E \subset \mathbb{F}_q^{d}\), we define \(\mu: V \to \mathbb{Z}\)
\[
\mu(\mathbb{D}) := \# \{(x_1, \ldots, x_{k+1}) \in \mathcal{E}^{k+1} : ||x_i - x_j|| = d_{i,j}, 1 \leq i < j \leq k + 1\}
\]

**Proof of Theorem 1.2.** Suppose that \(|A_d| = \min_{1 \leq i \leq d} |A_i|\). Let \(T_{k,d}(\mathcal{E})\) denote the set of congruence classes of \(k\)-simplices determined by \(\mathcal{E}\). It follows from the Cauchy-Schwarz inequality that
\[
\sum_{\mathbb{D}} \mu(\mathbb{D}) \leq \left( \sum_{\mathbb{D} \in \text{supp}(\mu)} 1 \right)^{1/2} \left( \sum_{\mathbb{D} \in \text{supp}(\mu)} \mu(\mathbb{D})^2 \right)^{1/2} = \left| T_{k,d}(\mathcal{E}) \right|^{1/2} \left( \sum_{\mathbb{D} \in \text{supp}(\mu)} \mu(\mathbb{D})^2 \right)^{1/2}.
\]
This gives
\[
|T_{k,d}(\mathcal{E})| \geq \frac{\left( \sum_{\mathbb{D}} \mu(\mathbb{D}) \right)^2}{\sum_{\mathbb{D}} \mu(\mathbb{D})^2} \geq \frac{|\mathcal{E}|^{2k+2}}{\sum_{\mathbb{D}} \mu(\mathbb{D})^2}.
\]
For \(\theta \in O(d, \mathbb{F}_q)\) and \(z \in \mathbb{F}_q^d\), we define
\[
w_\theta(z) := \{(u, v) \in \mathcal{E}^2 : \theta(u) + z = v\}.
\]
and denote the common stabilizer size of \(k\)-simplices in the congruence class \(\mathbb{D}\) by \(s(\mathbb{D})\). It has been shown in [3] that \(s(\mathbb{D}) \leq |O(d - k, \mathbb{F}_q)|\), and \(|O(n, \mathbb{F}_q)| = 2(1 + o(1))q^{\binom{n}{2}}\). Furthermore, it is easy to check that
\[
\sum_{\mathbb{D}} s(\mathbb{D}) \mu(\mathbb{D})^2 \leq \sum_{\theta \in O(d, \mathbb{F}_q), \mathbb{z} \in \mathbb{F}_q^d} |w_\theta(z)|^{k+1}, \quad (3.1)
\]
where \( |w_\theta(z)| \) is the cardinality of \( w_\theta(z) \).

For a fixed \( \theta \), it follows from Lemma 2.3 with \( f(z) := |w_\theta(z)|, \|f\|_1 = |\mathcal{E}|^2 \), and \( \|f\|_\infty \leq |\mathcal{E}| \) that

\[
\sum_{z \in \mathbb{F}_q^d} |w_\theta(z)|^{k+1} \leq \frac{|\mathcal{E}|^{2k+2}}{q^{kd}} + \frac{k(k-1)}{2} |\mathcal{E}|^{k-1} \sum_{z \in \mathbb{F}_q^d} \left( |w_\theta(z)| - \frac{|\mathcal{E}|^2}{q^d} \right)^2.
\]

Thus we obtain

\[
\sum_{\theta \in O(d, \mathbb{F}_q), z \in \mathbb{F}_q^d} |w_\theta(z)|^{k+1} \leq |O(d, \mathbb{F}_q)| \left( \frac{|\mathcal{E}|^{2k+2}}{q^{kd}} + \frac{k(k-1)}{2} |\mathcal{E}|^{k-1} \sum_{\theta, z} \left( |w_\theta(z)| - \frac{|\mathcal{E}|^2}{q^d} \right)^2 \right) \tag{3.2}
\]

\[
\leq |O(d, \mathbb{F}_q)| \left( \frac{|\mathcal{E}|^{2k+2}}{q^{kd}} + \frac{k(k-1)}{2} |\mathcal{E}|^{k-1} \left( \sum_{\theta, z} |w_\theta(z)|^2 \right) - \frac{|\mathcal{E}|^4 |O(d, \mathbb{F}_q)|}{q^d} \right).
\]

It follows from the definition of \( w_\theta(z) \) that \( |w_\theta(z)|^2 \) is equal to the number of quadruples \((a, b, c, d)\) \( \in \mathcal{E}^4 \) satisfying \( \theta(a) + z = c \) and \( \theta(b) + z = d \). This implies that \( \theta(a-b) = (c-d) \), and \( |a-b| = |c-d| \). Since the stabilizer of a non-zero element in \( \mathbb{F}_q^d \) is at most \( |O(d-1, \mathbb{F}_q)| \), it follows that each quadruple \((a, b, c, d) \in \mathcal{E}^4 \), which satisfies \( a - b \neq 0 \), \( |a-b| = |c-d| \), and \( \theta(a-b) = (c-d) \) for some \( \theta \), will be counted at most \( |O(d-1, \mathbb{F}_q)| \) times in the sum \( \sum_{\theta, z} |w_\theta(z)|^2 \). If \( a = b \) and \( c = d \), then the quadruples \((a, b, c, d) \) will be counted at most \( |O(d, \mathbb{F}_q)| \) times in the sum \( \sum_{\theta, z} |w_\theta(z)|^2 \).

Let

\[
W := \{ (a, b, c, d) \in \mathcal{E}^4 : |a-b| = |c-d| \}.
\]

It is clear that \( \sum_{t \in \mathbb{F}_q} \nu_\mathcal{E}(t)^2 = |W| \). If \((a, b)\) and \((c, d)\) belong to \( w_\theta(z) \) for some \( \theta \in O(d, \mathbb{F}_q) \) and \( z \in \mathbb{F}_q^d \), then \((a, b, c, d) \in W \) and \((b, a, d, c) \in W \). From this observation, we get the following

\[
\sum_{\theta, z} |w_\theta(z)|^2 \leq \frac{|O(d-1, \mathbb{F}_q)| |W|}{2} + |O(d, \mathbb{F}_q)| |\mathcal{E}|^2 \tag{3.3}
\]

\[
\leq \frac{|O(d-1, \mathbb{F}_q)|}{2} \left( \sum_{t \in \mathbb{F}_q} \nu_\mathcal{E}(t)^2 \right) + |O(d, \mathbb{F}_q)| |\mathcal{E}|^2, \tag{3.4}
\]

where the factor \( |\mathcal{E}|^2 \) comes from the number of quadruples \((a, b, c, d) \) with \( a = b \) and \( c = d \).

Lemma 3.1 together with the inequalities

\[
|O(d-1, \mathbb{F}_q)| \left( \frac{|\mathcal{E}|^4}{2q} \right) \leq \frac{|O(d, \mathbb{F}_q)| |\mathcal{E}|^4}{q^d}
\]

and

\[
|O(d, \mathbb{F}_q)| |\mathcal{E}|^2 \leq |O(d-1, \mathbb{F}_q)| q^{d-1} |\mathcal{E}|^2 A_d
\]

6
leads to
\[
\sum_{\theta, z} |w_\theta(z)|^2 \leq \frac{|O(d - 1, \mathbb{F}_q)|}{2} \left( \frac{|\mathcal{E}|^4}{q} + 2q^{d-1}|\mathcal{E}|^2|A_d| \right) + |O(d, \mathbb{F}_q)||\mathcal{E}|^2
\]
\[
\leq 4|O(d - 1, \mathbb{F}_q)|q^{d-1}|\mathcal{E}|^2|A_d|,
\]

Combining (3.2) with (3.3), we get
\[
\sum_{\theta \in O(d, \mathbb{F}_q), z \in \mathbb{F}_q^d} |w_\theta(z)|^k \leq |O(d, \mathbb{F}_q)||\mathcal{E}|^{k+1} + 2k(k - 1)|\mathcal{E}|^{k-1}|O(d - 1, \mathbb{F}_q)|(q^{d-1}|\mathcal{E}|^2|A_d|)
\]
\[
\leq |O(d, \mathbb{F}_q)||\mathcal{E}|^{2k+2} + 2k(k - 1)q^{d-1}|\mathcal{E}|^{k+1}|A_d||O(d - 1, \mathbb{F}_q)|. \quad (3.5)
\]

It follows from (3.1) and (3.5) that
\[
\sum_D s(D)\mu(D)^2 \leq |O(d, \mathbb{F}_q)||\mathcal{E}|^{2k+2} + 2k(k - 1)q^{d-1}|\mathcal{E}|^{k+1}|A_d||O(d - 1, \mathbb{F}_q)|.
\]

Furthermore we have \( s(D) \leq |O(d - k, \mathbb{F}_q)| \), this implies that
\[
\sum_D \mu(D)^2 \leq \frac{|\mathcal{E}|^{2k+2}}{q^{k+1}} + 2k(k - 1)q^{kd - (k+1)}|\mathcal{E}|^{k+1}|A_d| = (1 + o(1))\frac{|\mathcal{E}|^{2k+2}}{q^{k+1}}
\]
when \( q^{kd} = o(|\mathcal{E}|^{k+1}|A_d|^{-1}) \). In other words, if \( q^{kd} = o(|\mathcal{E}|^{k+1}|A_d|^{-1}) \), then the number of congruence simplices determined by \( \mathcal{E} \) satisfies
\[
|T_{k,d}(\mathcal{E})| = (1 - o(1))q^{k+1},
\]
which ends the proof of the theorem. \( \square \)

4 Proof of Theorem 1.5

First we need the following proposition.

Proposition 4.1. Let \( \mathcal{E} \) be a subset in \( \mathbb{F}_q^2 \) and \( \nu_\mathcal{E}(\lambda) \) be the number of pairs \( (p, q) \in \mathcal{E} \times \mathcal{E} \) such that \( ||p - q|| = \lambda \). If \( |\mathcal{E}| \geq 4q \), then
\[
\sum_{\lambda \in \mathbb{F}_q^*} \nu_\mathcal{E}(\lambda)^2 \ll \frac{|\mathcal{E}|^4}{q} + q|\mathcal{E}|^{5/2}.
\]

The proof of this proposition is based on the following lemma.
Lemma 4.2. Let $\mathcal{E}$ be a subset in $\mathbb{F}_q^2$. For a fixed $\lambda \in \mathbb{F}_q^*$, denote by $H_\lambda(\mathcal{E})$ the number of hinges of the form $(p, q_1, q_2) \in \mathcal{E} \times \mathcal{E} \times \mathcal{E}$ with $||p - q_1|| = ||p - q_2|| = \lambda$. If $|\mathcal{E}| \geq 4q$, then

$$\sum_{\lambda \in \mathbb{F}_q^*} \nu_\mathcal{E}(\lambda)^2 \leq \frac{|\mathcal{E}|}{4} \sum_{\lambda \in \mathbb{F}_q^*} H_\lambda(\mathcal{E}).$$

Proof. By assumption $|\mathcal{E}| \geq 4q$ and note that there are at most $2q$ points on isotropic lines in the case $q \equiv 3 \mod 4$, one may assume that there are no two points $(p, q) \in \mathcal{E} \times \mathcal{E}$ with $||p - q|| = 0$. We now prove that

$$\sum_{\lambda \in \mathbb{F}_q^*} \nu_\mathcal{E}(\lambda)^2 \leq |\mathcal{E}| \sum_{\lambda \in \mathbb{F}_q^*} H_\lambda(\mathcal{E}). \quad (4.1)$$

For each point $p$ in $\mathcal{E}$, let $x_p^\lambda$ be the number of points $q \in \mathcal{E}$ satisfying $||p - q|| = \lambda$. Then one has

$$H_\lambda(\mathcal{E}) = \sum_{p \in \mathcal{E}} (x_p^\lambda)^2.$$

On the other hand, by applying the Cauchy-Schwarz inequality, we obtain

$$\sum_{\lambda \in \mathbb{F}_q^*} \nu_\mathcal{E}(\lambda)^2 = \frac{1}{4} \sum_{\lambda \in \mathbb{F}_q^*} \left( \sum_{p \in \mathcal{E}} x_p^\lambda \right)^2 \leq \frac{1}{4} \sum_{\lambda \in \mathbb{F}_q^*} |\mathcal{E}| H_\lambda(\mathcal{E}) = \frac{1}{4} |\mathcal{E}| \sum_{\lambda \neq 0} H_\lambda(\mathcal{E}),$$

which completes the proof of the lemma. \hfill \Box

A reflection about a point $u \in \mathbb{F}_q^2$ is a map of the form

$$R_u(x) = R(x - u) + u,$$

where $R$ is a matrix of the form

$$R = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad a, b \in \mathbb{F}_q, \quad a^2 + b^2 = 1.$$

For $\lambda \in \mathbb{F}_q^*$, the reflection graph $RF_\lambda(\mathbb{F}_q^2)$ is constructed as

$$V(RF_\lambda(\mathbb{F}_q^2)) = \{ (x, y) \in \mathbb{F}_q^2 \times \mathbb{F}_q^2 : ||x - y|| = \lambda \},$$

and

$$E(RF_\lambda(\mathbb{F}_q^2)) = \{ ((x, y), (z, w)) \in V(RF_\lambda(\mathbb{F}_q^2)) \times V(RF_\lambda(\mathbb{F}_q^2)) : \exists R_u, \; R_u(x) = z, R_u(y) = w \}.$$

Hanson et al. [7] established the $(n, d, \lambda)$ form of this graph as follows.

Lemma 4.3 (Hanson et al., [7]). The reflection graph $RF_\lambda(\mathbb{F}_q^2)$ is

$$(q^2(q + 1), q^2 \pm q, 2(q + 1)) - \text{graph}.$$
The proof of Proposition 4.1 below is quite similar to that of [7, Theorem 1 and Lemma 14], but here we give a straight proof and avoid using the bound \( \sum_{\lambda \in \mathbb{F}_q^*} \nu_\mathcal{E}(\lambda)^2 \ll \frac{\mathcal{E}^4}{q} + q^2|\mathcal{E}|^2 \) which can be proved by employing the distribution of edges between two sets of vertices in the Erdős-Rényi graph with an appropriated setting.

**Proof of Proposition 4.1.** We may again assume that there are no two distinct points \( p \) and \( q \) in \( \mathcal{E} \) satisfying \( ||p - q|| = 0 \) (as in the proof of Lemma 4.2).

For any two distinct points \( q_1 \) and \( q_2 \) in \( \mathbb{F}_q^2 \), the bisector line \( l_{q_1, q_2} \) is defined as

\[
l_{q_1, q_2} := \{ x \in \mathbb{F}_q^2 : ||x - q_1|| = ||x - q_2|| \}.
\]

Let \( \mathcal{L} \) be the multi-set of bisector lines defined as

\[
\mathcal{L} = \bigcup_{(q_1, q_2) \in \mathcal{E}^2} l_{q_1, q_2}.
\]

One can identify each point \((a, b) \in \mathcal{E}\) with a vertex \([a, b, 1]\) in the Erdős-Rényi graph \(PG(q, 3)\), and each line of the form \(cx + dy + e = 0\) in \(\mathcal{L}\) with a vertex \([c, d, e]\) in the Erdős-Rényi graph \(\mathcal{E}R(\mathbb{F}_q^3)\). We denote the corresponding sets of vertices in the Erdős-Rényi graph \(\mathcal{E}R(\mathbb{F}_q^3)\) by \(\mathcal{L}'\) and \(\mathcal{L}'\), respectively.

Hence, we have \( |\mathcal{E}'| = |\mathcal{E}|, |\mathcal{L}'| = |\mathcal{L}| \), and the sum \( \sum_{\lambda \in \mathbb{F}_q^*} H_\lambda(\mathcal{E}) \) is equal to the number of edges between \( \mathcal{E}' \) and \( \mathcal{L}' \) in \( \mathcal{E}R(\mathbb{F}_q^3) \). Therefore, by Lemmas 2.1 and 2.2 we get

\[
\sum_{\lambda \in \mathbb{F}_q^*} H_\lambda(\mathcal{E}) = I(\mathcal{E}', \mathcal{L}') \leq \frac{|\mathcal{E}'||\mathcal{L}|}{q} + q^{1/2}|\mathcal{E}'|^{1/2} \sqrt{\sum_{l \in \mathcal{L}} w(l)^2},
\]

where \( w(l) \) is the multiplicity of \( l \in \mathcal{L} \).

If \( l_{x,z} = l_{y,w} \) and \( ||x - z|| \neq 0 \), then one can check that there exists a unique reflection \( R_u \) such that \( R_u(x) = z, R_u(y) = w, \) and \( ||x - y|| = ||z - w|| \). Thus the sum \( \sum_{l \in \mathcal{L}} w(l)^2 \) is the cardinality of the following set

\[
\mathcal{Q} := \mathcal{Q}(\mathcal{E}) = \{ (x, y, z, w) \in \mathbb{E}^4 : \exists \lambda \in \mathbb{F}_q^*, ((x, y), (z, w)) \in E(RF_\lambda(\mathbb{F}_q^2)) \}.
\]

We set

\[
\mathcal{Q}_\lambda := \{ (x, y, z, w) \in \mathcal{Q} : ||x - y|| = ||z - w|| = \lambda \},
\]

and see that

\[
\sum_{l \in \mathcal{L}} w(l)^2 = \sum_{\lambda \in \mathbb{F}_q^*} |\mathcal{Q}_\lambda|.
\]

For each \( \lambda \neq 0 \), it follows from Lemma 4.3 and Lemma 2.1 that

\[
|\mathcal{Q}_\lambda| \leq \frac{\nu_\mathcal{E}(\lambda)^2}{q} + 2(q - 1)\nu_\mathcal{E}(\lambda).
\]
Hence, we get
\[ \sum_{l \in L} w(l)^2 \leq \frac{\sum_{\lambda \in \mathbb{F}_q^*} \nu_{\mathcal{E}}(\lambda)^2}{q} + 2(q - 1)|\mathcal{E}|^2. \] (4.2)

Combining (4.1) with (4.2), we obtain
\[ \sum_{\lambda \in \mathbb{F}_q^*} \nu_{\mathcal{E}}(\lambda)^2 \leq |\mathcal{E}| \left( \frac{|\mathcal{E}| |\mathcal{L}|}{q} + q^{1/2}|\mathcal{E}|^{1/2} \sqrt{\frac{\sum_{\lambda \in \mathbb{F}_q^*} \nu_{\mathcal{E}}(\lambda)^2}{q}} + 2(q - 1)|\mathcal{E}|^2 \right). \]

Solving this inequality leads to the desired bound, and the proposition follows.

\[ \square \]

Remark 4.4. It follows from the proof of Proposition 4.1 that the number of the number of hinges determined by points in \( \mathcal{E} \) is at most
\[ \frac{|\mathcal{E}|^3}{q} + q^{1/2}|\mathcal{E}|^{1/2} \sqrt{\frac{\sum_{\lambda \in \mathbb{F}_q^*} \nu_{\mathcal{E}}(\lambda)^2}{q}} + 2(q - 1)|\mathcal{E}|^2. \]

On the other hand, we have proved that
\[ \sum_{\lambda \in \mathbb{F}_q^*} \nu_{\mathcal{E}}(\lambda)^2 \ll \frac{|\mathcal{E}|^4}{q} + q|\mathcal{E}|^{5/2}. \]

This leads that the number of hinges is at most \( \ll |\mathcal{E}|^3/q \) when \( |\mathcal{E}| \gg q^{4/3} \).

As an application of Proposition 4.1, we obtain the following result.

Theorem 4.5 (Bennett et al. [3]). Let \( \mathcal{E} \) be a set of points in \( \mathbb{F}_q^2 \). If \( |\mathcal{E}| \gg q^{8/5} \), then \( \mathcal{E} \) determines a positive proportion of all triangles.

Proof. Since \( |\mathcal{E}| \gg q^{8/5} \), without loss of generality, we assume that there are no two points \( x \) and \( y \) in \( \mathcal{E} \) satisfying \( ||x - y|| = 0 \). The proof of Theorem 1.5 is very similar to that of Theorem 1.2 and there is the only one different step. That is, instead of using Lemma 3.1 we use Proposition 4.1 thus we leave the rest to the reader.

\[ \square \]

5 Proof of Theorem 1.7

In order to prove Theorem 1.7, we make use of the following theorem on the cardinality of a maximal independent set of a hypergraph due to Spencer [9].

Theorem 5.1. Let \( H \) be a \( k \)-uniform hypergraph of \( n \) vertices and \( m \) edges with \( m \geq n/k \), and let \( \alpha(H) \) denote the independence number of \( H \). Then
\[ \alpha(H) \geq \left( 1 - \frac{1}{k} \right) \left[ \left( \frac{n^k}{km} \right)^{\frac{k-1}{k}} \right]. \]
Proof of Theorem 1.7. We call a 4-tuple of distinct elements in $E^4$ regular if all six generalized distances determined are distinct. Otherwise, it is called singular. Let $H$ be the 4-uniform hypergraph on the vertex set $V(H) = E$, whose edges are the singular 4-tuples of $E$.

On one hand, it follows from the remark (4.4) that the number of 4-tuples containing a triple induced a hinge is at most $((1 + o(1))|E|^3/q) \cdot |E| = (1 + o(1))|E|^4/q$ when $|E| \gg q^{4/3}$. Thus the number of edges of $H$ containing a triple induced a hinge is at most $(1 + o(1))|E|^4/q$.

On the other hand, according to Proposition 4.1 and from the fact that the number of quadruples with zero-distances is no more than $4q^4$, the number of edges of $H$ that do not contain any hinge is at most $(1 + o(1))|E|^4/q$, when $|E| \gg q^{4/3}$.

In other words, if $|E| \gg q^{4/3}$, we have

$$|E(H)| \leq (1 + o(1)) \frac{|E|^4}{q}.$$

By Theorem 5.1 one has

$$\alpha(H) \geq C \left( \frac{|E|^4}{|E(H)|} \right)^{1/3} = Cq^{1/3},$$

for some positive constant $C$. Since there is no repeated distance determined by the independent set of $H$, there exists a distinct distance subset $U \subseteq E$ satisfying $|U| \geq \alpha(H) \geq Cq^{1/3}$.

Moreover, it is easy to see that there is at least one repeated distance determined by any set of $\sqrt{2}q^{1/2} + 1$ elements since there are only $q = |\mathbb{F}_q|$ distances over $\mathbb{F}_q$. This concludes the proof of the theorem.

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