

General relativistic heat flow from first order hydrodynamics

Bhera Ram^{1,*} and Bibhas Ranjan Majhi^{1,†}

¹*Department of Physics, Indian Institute of Technology Guwahati, Guwahati 781039, Assam, India.*

Following the recently proposed stable and causal first-order relativistic hydrodynamics by Bemfica, Disconzi, and Noronha, we find the heat flow equation in the presence of gravity for a non-viscous fluid, which suffers heat dissipation. The derivation is confined to static and stationary backgrounds. We find that in the presence of gravity, the heat flux times a redshift factor is conserved. Then for radial heat flow, the temperature profiles are obtained from the heat equation when the gravity is sourced by Schwarzschild, Schwarzschild-dS, Kerr and Kerr-dS black holes, respectively. Consequently, the chemical potential profile is also discussed.

I. INTRODUCTION

It is well-established that Zeroth's law gets modified in the presence of gravity. In a gravity-free environment, two points are at thermal equilibrium when they have the same temperature. However, this equivalence does not hold in the presence of gravity. The constancy of temperature across space-time is modified to $\sqrt{-g_{00}}T(x)$ in thermal equilibrium, which is famously known as the Tolman-Ehrenfest (TE) relation [1, 2]. Here, $T(x)$ denotes the temperature of the fluid as measured locally, while g_{00} represents the time-time component of the static metric. Although Tolman and Ehrenfest derived this for static spacetime, it was later demonstrated to apply for stationary backgrounds as well [3], when the four velocity of fluid is chosen along a specific timelike Killing vector. However, the stationary situations have also been discussed for a generic choice of timelike Killing vector, leading to modification in TE relation (see e.g. [4]). Following the TE relation, Klein demonstrated that μ/T remains constant throughout the spacetime [5]. Here μ is the local chemical potential of the fluid. These interesting results then led to several approaches to derive them. Using maximum entropy condition, Cocker [6] and later followed by several others [7–10] found those aforesaid relations. A further implication was provided by using Noether symmetry formalism [11–13]. Use of an appropriate thermodynamic ensemble has also been done to obtain these results [14–17]. Rovelli and Smerlek [18] introduced a concept of “thermal time” to obtain TE relation for a stationary background. For generalization of TE relation at the quantum level, see [19].

Recently, using the conservation of energy-momentum, entropy and particle number (for an ideal fluid), Lima *et al.* [20, 21] argued that in a static spacetime, when $\mu \neq 0$, TE relation and Klein's law are not satisfied separately. However, using the more viable first-order formalism by Bemfica *et al.*, we in our recent work obtained the definition of local temperature and chemical potential for thermal equilibrium in terms of the acceleration of the fluid. Moreover, it has been observed that further imposition

of the local entropy conservation leads to the standard TE and Klein's relation [22]. Also, see [23] where the TE has been obtained by using equation of state of the fluid.

In the absence of gravity if there is a temperature gradient from one spacetime point to another, there will be heat flux. Correspondingly the temperature profile satisfies the heat flow equation. However, in the presence of gravity, it is the differential of TE relation that dictates the flow of heat from one spacetime point to another. In such a case, we expect an analogous heat equation, governing the flow of heat across two spacetime points. In this particular work, we study the governing heat equation in a gravitational background for which we use the well-established first-order causal and stable formalism of relativistic hydrodynamics. Very recently a few works [24, 25] in this direction appeared, however, we will discuss later that our analysis is different and also the results are more general.

The formulation of a stable and causal theory describing relativistic viscous fluid has been a long quest. The first two important works were done by Landau [26] and Eckart [27] by adding first-order dissipative contribution to the ideal fluid equations. Although they both describe the Navier-Stokes equation in the non-relativistic limit, they suffer from very fundamental issues of being acausal and unstable about an equilibrium state [28]. Later, Israel and Stewart [29, 30] added second-order contributions to formulate a stable and causal theory for relativistic hydrodynamics but introduced more degrees of freedom other than the standard ones. In recent years Kovtun [31, 32] and using Kovtun's idea, F. S. Bemfica, M. M. Disconzi and J. Noronha [33] (BDN) formulated a stable and causal first-order theory of relativistic hydrodynamics.

In this work using BDN formalism, we obtain the heat flow equation in the presence of a gravitational background. The analysis is confined to a non-viscous fluid where the dissipation is governed by heat flux. We find for static and stationary backgrounds that the heat flux current (q^a) times a redshift factor is conserved. Later solving the heat equation for Schwarzschild, Schwarzschild-de Sitter (SdS), Kerr and Kerr-de Sitter (KdS) backgrounds the temperature profiles are obtained in each case. For simplicity, the analysis is done by assuming radial heat flow. It is found that

* bhera.ram@iitg.ac.in

† bibhas.majhi@iitg.ac.in

when the heat flux is zero, all the relations reduce to the well-known TE relation. Moreover, we also work on how Klein's law gets modified.

II. PRE-REQUISITES

In this work, we follow the choice of variables, taken by BDN. Below, we provide a brief overview of the BDN formalism, as outlined in the original work [33]. In this formalism, the fluid description is presented as follows. The constitutive relations that give the baryon current and the energy-momentum tensor are

$$J^a = nu^a, \quad (1)$$

$$T^{ab} = (\varepsilon + \mathcal{A})u^a u^b + (p + \Pi)\Delta^{ab} - 2\eta\sigma^{ab} + u^a q^b + u^b q^a, \quad (2)$$

where the expressions for \mathcal{A} , Π , q^a and σ^{ab} are as follows:

$$\mathcal{A} = \tau_\varepsilon [u^a \nabla_a \varepsilon + (\varepsilon + p)\nabla_a u^a], \quad (3)$$

$$\Pi = -\zeta \nabla_a u^a + \tau_p [u^a \nabla_a \varepsilon + (\varepsilon + p)\nabla_a u^a], \quad (4)$$

$$q_b = \frac{\sigma T(\varepsilon + p)}{n} \Delta_b^a \nabla_a \left(\frac{\mu}{T}\right) + \tau_q [(\varepsilon + p)u^a \nabla_a u_b + \Delta_b^a \nabla_a p], \quad (5)$$

$$\sigma^{ab} = \frac{1}{2} \Delta^{ac} \Delta^{bd} \left(\nabla_c u_d + \nabla_d u_c - \frac{2}{3} \Delta_{cd} \Delta^{ef} \nabla_e u_f \right), \\ = \frac{1}{2} \left(\nabla^a u^b + \nabla^b u^a + u^a \dot{u}^b + u^b \dot{u}^a - \frac{2}{3} \Delta^{ab} \nabla_e u^e \right) \quad (6)$$

In the above expressions ε , s , p , n , T and μ are equilibrium thermodynamic variables – energy density, entropy, pressure, number density, temperature and chemical potential, respectively. These are connected via the Euler relation $\varepsilon + p = Ts + \mu n$ and are consistent with first law of thermodynamics. Further, u^a is a normalized time-like vector (*i.e.* $u_a u^a = -1$), called as the flow or fluid velocity, and $\Delta_{ab} = g_{ab} + u_a u_b$ is a projector onto the space orthogonal to u^a . Additionally, τ_ε , τ_p , and τ_q represent the corrections to the out-of-equilibrium components of the energy-momentum tensor, corresponding to the energy density correction \mathcal{A} , the bulk viscous pressure Π , and the heat flux q^a , respectively. In the above expressions \dot{u}^a is given by, $\dot{u}^a = u^c \nabla_c u^a$. Furthermore, σ_{ab} denotes the traceless shear tensor, with ζ , σ , and η being the coefficients of bulk viscosity, heat conductivity and shear viscosity.

The above description is consistent with the required properties mentioned in [33]. Particularly the local version of the entropy increase theorem is well satisfied upon using the on-shell conditions $\nabla_a J^a = 0$ and $\nabla_a T^{ab} = 0$. The first one gives $u^a \nabla_a n + n \nabla_a u^a = 0$ while the projection of other condition along u^a gives

$$u^a \nabla_a \varepsilon + (\varepsilon + p)\nabla_a u^a = -u^a \nabla_a \mathcal{A} - (\mathcal{A} + \Pi)\nabla_a u^a - \nabla_a q^a - q^a u^b \nabla_b u_a + 2\eta \sigma_{ab} \sigma^{ab}, \quad (7)$$

and its projection on a space perpendicular to u^a provides

$$(\varepsilon + p)u^a \nabla_a u^b + \Delta^{ab} \nabla_a p = -(\mathcal{A} + \Pi)u^a \nabla_a u^b - \Delta^{ab} \nabla_a \Pi + 2\eta \Delta_c^b (\nabla_a \sigma^{ac}) - u^a \nabla_a q^b - (\nabla_a u^a)q^b - (\nabla_a u^b)q^a + u^b q^c u^a \nabla_a u_c. \quad (8)$$

On using these on-shell conditions on (3) – (5) show

$$\mathcal{A} = \mathcal{O}(\partial^2); \quad (9)$$

$$\Pi = -\zeta \nabla_a u^a + \mathcal{O}(\partial^2); \quad (10)$$

$$q_b = \sigma T \frac{(\varepsilon + p)}{n} \Delta_b^a \nabla_a (\mu/T) + \mathcal{O}(\partial^2) \\ = -\kappa \Delta^{ab} (T \dot{u}_b + \nabla_b T) + \mathcal{O}(\partial^2), \quad (11)$$

where κ is the coefficient of thermal conductivity. With this short introduction, let us discuss the heat flow equation in the regime of first-order relativistic hydrodynamics using BDN choice of variables.

III. EQUATIONS GOVERNING NON-VISCOUS HEAT FLOW

Now let us come to discuss the equations governing heat flow in the presence of a gravitational background. In principle the system of equations (7) and (8) along with the baryon current conservation $\nabla_a J^a = 0$ describes the heat convection. However to invoke a specific situation, here we consider a non-viscous fluid (but non-ideal, as there is still dissipation due to heat flux), *i.e.* both shear viscosity and bulk viscosity are vanishing ($\sigma_{ab} = 0 = \Pi$). Since Π contains both first and second order terms (see Eq. (10)), vanishing of Π implies that the terms in each order must vanish individually. Therefore we must have $\nabla_a u^a = 0$. However, there is no thermal equilibrium, *e.g.* $\sqrt{-g_{00}}T(x)$ is not constant at different spacetime points for a static background. Thus, there will be heat flow within the fluid. Moreover, since the present formalism is based on first-order theory, we must retain up to second-order terms in (7) and (8). Therefore use of (9) – (11) in (7) and (8) along with non-viscous condition, one finds

$$u^a \nabla_a \varepsilon = -\nabla_a q^a - q^a \dot{u}_a, \quad (12)$$

and

$$(\varepsilon + p)u^a \nabla_a u^b + \Delta^{ab} \nabla_a p = -u^a \nabla_a q^b - (\nabla_a u^b)q^a + u^b q^c \dot{u}_c, \quad (13)$$

for a non-viscous but out-of-thermal equilibrium fluid. Here (12) has a particular interest. It provides how heat flows across a non-viscous fluid when it is not in thermal equilibrium. In particular if one uses (11) in (12), the obtained equation is the governing equation for the fluid temperature and chemical potential gradient. Therefore this is reminiscent to the heat equation under gravity. This will be more transparent by considering a static fluid on a static background spacetime. Later we will discuss a stationary background situation as well.

A. Static background

A fundamental observer can be assigned whose four-velocity is $u_a = -N\nabla_a t$, where t is the time coordinate and the hypersurface is one of the $t = \text{constant}$ surfaces. Here u^a is normal to the hypersurface, and so $\Delta_{ab} = g_{ab} + u_a u_b$ is the induced metric. Under these circumstances one can show $\dot{u}_a = \Delta_a^b \nabla_b (\ln \sqrt{N^2}) = \mathcal{D}_a (\ln \sqrt{N^2})$. First, we consider a static background metric

$$ds^2 = g_{00} dt^2 + g_{\mu\nu} dx^\mu dx^\nu . \quad (14)$$

In this case, the metric coefficient g_{ab} are time (t) independent and there is no cross term with time differential; *i.e.* $g_{0\mu} = 0$. Also, we have $g^{00} = 1/g_{00}$. Therefore the components of u^a are $u_a = (-N, 0, 0, 0)$ and $u^a = (-Ng^{00}, 0, 0, 0)$ where $N^2 = -g_{00}$. On this static background, the fluid must respect the symmetries of the static background. Therefore in coordinates adapted to the timelike symmetry, the fluid scalar parameters do not change along the flow line; *i.e.* $u^a \partial_a X = 0$, where X stands for any fluid scalar parameter. In this situation, we have $u^a \nabla_a \varepsilon = 0$ and $\dot{u}_a = \Delta_a^b \nabla_b (\ln \sqrt{N^2}) = (\delta_a^b + u^b u_a) \nabla_b (\ln \sqrt{N^2}) = \nabla_a (\ln \sqrt{N^2})$. Then (12) reduces to

$$\nabla_a (\sqrt{N^2} q^a) = 0 . \quad (15)$$

This implies that in the presence of gravity $\sqrt{N^2} q^a$, rather than q^a itself, is the conserved quantity. However, (15) further simplifies to

$$\partial_a (\sqrt{-g} \sqrt{N^2} q^a) = 0 . \quad (16)$$

For the static situation one finds q_a from (11) as

$$q_a = -\kappa T \nabla_a \ln (T \sqrt{N^2}) . \quad (17)$$

Then substitution of this in (15) yields

$$\nabla_a \left[\kappa \nabla^a (T \sqrt{N^2}) \right] = 0 . \quad (18)$$

The above one can be interpreted as the heat flow equation of a static fluid under a static gravitational field. In the case when κ is constant, the above equation reduces to a very nice form $\nabla_a \nabla^a (T \sqrt{N^2}) = 0$, which further reduces to

$$\partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu (T \sqrt{N^2}) \right] = 0 , \quad (19)$$

where μ and ν stand for only space indices.

For simplicity, we consider a static spherically symmetric metric:

$$ds^2 = g_{00} dt^2 + g_{rr} dr^2 + r^2 d\Omega_{(2)}^2 , \quad (20)$$

as an example of a static metric, where, $d\Omega_{(2)}^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Also, assume that the heat flux is just in the radial direction, *i.e.* $q^a = (0, q^r, 0, 0)$. Then Eq. (16) provides the following solution:

$$q^r = -\frac{q_0}{r^2 g_{00} \sqrt{g_{rr}}} , \quad (21)$$

where q_0 is the integration constant. On the other hand, (17) yields

$$q^r = \frac{-\kappa}{g_{rr} \sqrt{-g_{00}}} \partial_r (T \sqrt{-g_{00}}) . \quad (22)$$

In the above, we have used $N^2 = -g_{00}$. Next comparing (21) and (22) we find

$$\frac{d}{dr} (T \sqrt{-g_{00}}) + \frac{q_0 \sqrt{g_{rr}}}{\kappa r^2 \sqrt{-g_{00}}} = 0 . \quad (23)$$

One could obtain the same equation by solving (18). This equation determines the temperature gradient in the absence of thermal equilibrium for a static spherically symmetric background. One can note that when $q_0 = 0$, we have our usual Tolman relation for a static background. Therefore (23) can be interpreted as the generalization of TE relation.

Similarly, use of first equality of (11) yields

$$q^a = \alpha \nabla^a \left(\frac{\mu}{T} \right) , \quad (24)$$

where $\alpha = \sigma T \left(\frac{\varepsilon + p}{n} \right)$. Therefore for radial heat flux, the use of (21) yields

$$\frac{d}{dr} \left(\frac{\mu \sqrt{-g_{00}}}{T \sqrt{-g_{00}}} \right) + \frac{q_0 \sqrt{g_{rr}}}{\alpha r^2 g_{00}} = 0 . \quad (25)$$

In the above, solution of Eq. (23) can be used to find the radial variation of $\mu \sqrt{-g_{00}}$. This can be interpreted as the generalization of Klein's law in the presence of heat flow. In fact use of (23) in (25) yields the radial variation equation for $\mu \sqrt{-g_{00}}$ as

$$\begin{aligned} \frac{d}{dr} (\mu \sqrt{-g_{00}}) &= -\frac{q_0 \sqrt{g_{rr}}}{r^2 \sqrt{-g_{00}}} \left[\frac{\mu}{T} \frac{1}{\kappa} - \frac{T}{\alpha} \right] \\ &= -\left(\frac{q_0^2 g_{rr}(r)}{r^2} \right) \\ &\times \left[\frac{1}{\kappa} \int^r \frac{(g_{rr}(r'))^{3/2}}{\alpha r'^2} dr' + \frac{\sqrt{g_{rr}(r)}}{\alpha} \int^r \frac{g_{rr}(r')}{\kappa r'^2} dr' \right] \end{aligned} \quad (26)$$

In the last step (25) has been used. Although the above equation gives us the radial variation for the quantity $\mu \sqrt{-g_{00}}$, it is easier to obtain the expression for $\mu(r)$ by integrating directly Eq. (25):

$$\frac{\mu}{T} = A - q_0 \int \left(\frac{\sqrt{g_{rr}}}{\alpha r^2 g_{00}} \right) dr , \quad (27)$$

where A is a integration constant. In Appendix A the above equation has been explicitly solved for a few cases assuming α to be constant.

1. Application to static backgrounds

To get a feel of the above results we apply them on a few specific static metrics.

Schwarzschild Geometry: – In this section, we take the Schwarzschild geometry as an example of the static spherically symmetric case. On solving Eq. (23) for the Schwarzschild case with $g_{00} = -1 + \frac{2m}{r}$ and $g_{rr} = \frac{-1}{g_{00}}$, we get

$$T(r) = \frac{1}{\sqrt{1 - \frac{2m}{r}}} \left[T_0 - \frac{q_0}{2m\kappa} \ln \left(1 - \frac{2m}{r} \right) \right], \quad (28)$$

where T_0 is a positive constant. In the above κ has been assumed to be a constant. The integration constant T_0 can be fixed by demanding, under $r \rightarrow \infty$, one should get back the Hawking temperature ($T_H = 1/(8\pi m)$)[34] as seen by a static observer, provided the fluid temperature is controlled by that of the horizon.

Schwarzschild-de Sitter Geometry: – Let us work with Schwarzschild-de Sitter (SdS) spacetime which is a solution of the vacuum Einstein equation with cosmological constant Λ and can have two horizons based on the values of the parameters Λ and m . The geometry for the Schwarzschild-de Sitter is given by the following metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(2)}^2, \quad (29)$$

where $f(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$. For Schwarzschild-de Sitter geometry, we have, $g_{rr} = -1/g_{00}$ and thus (23) becomes

$$\frac{d}{dr} (T\sqrt{-g_{00}}) - \frac{q_0}{\kappa r^2 g_{00}} = 0. \quad (30)$$

Integrating the above equation assuming κ to be a constant, gives

$$T\sqrt{-g_{00}} = T_0 - \frac{q_0}{\kappa} \int \frac{1}{r^2 (1 - 2m/r - \Lambda r^2/3)} dr. \quad (31)$$

The above equation integrates to

$$T(r)\sqrt{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} = T_0 + \frac{q_0}{2m\kappa} \left(\ln|x| - \sum_i \frac{\ln|x - x_i|(\Lambda r_0^2 x_i^2 - 3)}{3(\Lambda r_0^2 x_i^2 - 1)} \right); \quad (32)$$

where $x \equiv r/r_0$, r_0 is the bigger positive root of $f(r) = 0$, x_i are the solutions to the equation below

$$6m - 3r_0 x + \Lambda r_0^3 x^3 = 0. \quad (33)$$

It can be easily seen that the above equation for the de-Sitter case, Eq.(32), reduces to the Schwarzschild case given by Eq.(28) under the limit $\Lambda \rightarrow 0$.

B. Stationary background

Let us consider a stationary background given by the following metric

$$ds^2 = g_{00}dt^2 + g_{t\phi}dtd\phi + g_{r,r}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \quad (34)$$

having both time translation and azimuthal symmetries, meaning g_{ab} is independent of coordinates t and ϕ . Therefore with the choice of $u_a = -N\nabla_a t$, we have $u_a = (-N, 0, 0, 0)$. Further, for the above metric, u^a is given by $u^a = (-N g^{tt}, 0, 0, -N g^{\phi t})$. Demanding $u_a u^a = -1$, we have $N^2 = -1/g^{00}$.

Now since the metric has time translational and azimuthal symmetries, here again we have $\dot{u}_a = \nabla_a (\ln \sqrt{N^2})$. Moreover, the fluid also respects these symmetries and hence $u^a \nabla_a \varepsilon = 0$. In that case a equation, like (15), will follow

$$\nabla_a (\sqrt{N^2} q^a) = \partial_A (\sqrt{-g} \sqrt{N^2} q^A) = 0. \quad (35)$$

In the above, $\sqrt{-g} = \left\{ -g_{rr} g_{\theta\theta} (g_{tt} g_{\phi\phi} - g_{t\phi}^2) \right\}^{1/2}$ and A stands for radial coordinate r and polar coordinate θ .

Further, like the static case, we will have exactly the same heat equation, given by (18). This is because q^a is again given by (17) which provides us with the same conserved quantity $q^a \sqrt{N^2}$ as was in the static case. Furthermore, solving Eq. (35) will provide us with the gradient of the TE relation, which we have explicitly worked out for Kerr and Kerr-de Sitter geometry in the below.

1. Application to stationary metrics

Kerr Geometry: – Let us now turn to an example of a stationary Kerr metric. The metric elements in this case in Boyer-Lindquist coordinates are given by

$$g_{00} = - \left(1 - \frac{2mr}{\rho^2} \right); \quad g_{\phi\phi} = - \frac{2mar \sin^2 \theta}{\rho^2}; \quad g_{rr} = \frac{\rho^2}{\Delta};$$

$$g_{\theta\theta} = \rho^2; \quad g_{\phi\phi} = \left(r^2 + a^2 + \frac{2ma^2 r \sin^2 \theta}{\rho^2} \right) \sin^2 \theta \quad (36)$$

with $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$, $\Delta \equiv r^2 - 2mr + a^2$, and $a \equiv \frac{J}{m}$, where a is the Kerr parameter. On using (36), we obtain $\sqrt{-g} = \rho^2 \sin \theta$. Again, assuming only radial heat flow, the solution to Eq. (35) is given by

$$q^r = \frac{q_0}{\rho^2 \sqrt{N^2}}, \quad (37)$$

where $N^2 = \frac{-1}{g^{00}}$, which for the Kerr metric is given by

$$g^{00} = - \frac{2a^2 m r \sin^2(\theta) + \rho^2 (a^2 + r^2)}{2a^2 m r \sin^2(\theta) - (a^2 + r^2) (2mr - \rho^2)}. \quad (38)$$

Further, on using (17), we find

$$\frac{d}{dr} \left(T\sqrt{N^2} \right) + \frac{q_0}{\kappa g^{rr} \rho^2} = 0, \quad (39)$$

for a constant value of θ . This is the counterpart of (23) for the stationary background.

Similarly on integrating (39) with $g^{rr} = \frac{\Delta_r}{\rho^2}$ gives

$$T\sqrt{N^2} = T_0 - \int \frac{q_0}{\kappa \rho^2 g^{rr}} dr, \quad (40)$$

which with the assumption of κ being a constant yields

$$T\sqrt{N^2} = T_0 - \frac{q_0}{\kappa} \int \left(\frac{1}{r^2 + a^2 - 2mr} \right) dr. \quad (41)$$

On solving (41), we get

$$T\sqrt{N^2} = T_0 - \frac{q_0}{2\kappa} \frac{1}{\sqrt{m^2 - a^2}} \ln \left| \left(\frac{\sqrt{m^2 - a^2} - r + m}{\sqrt{m^2 - a^2} + r - m} \right) \right|. \quad (42)$$

Also, the above equation (42) reduces to the Schwarzschild case given by Eq. (28) under the limit $a \rightarrow 0$.

Kerr-de Sitter Geometry:– The solution to Einstein's equation with cosmological constant describing a black hole with spin a is called the Kerr-de Sitter (KdS) solution. The geometry for the Kerr-de Sitter background in Boyer-Lindquist coordinates is given by the following metric

$$\begin{aligned} ds^2 = & (r^2 + a^2 \cos^2 \theta) \left[\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta} \right] \\ & + \sin^2 \theta \left(\frac{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} \right) \left[\frac{adt - (r^2 + a^2) d\phi}{1 + \frac{\Lambda}{3} a^2} \right]^2 \\ & - \frac{\Delta_r}{(r^2 + a^2 \cos^2 \theta)} \left[\frac{dt - a \sin^2 \theta d\phi}{1 + \frac{\Lambda}{3} a^2} \right]^2, \quad (43) \end{aligned}$$

where

$$\begin{aligned} \rho^2 & \equiv r^2 + a^2 \cos^2 \theta, \\ \Delta_r & \equiv (r^2 + a^2) \left(1 - \frac{\Lambda}{3} r^2 \right) - 2mr, \\ & \equiv r^2 - 2mr + a^2 - \frac{\Lambda r^2}{3} (r^2 + a^2). \quad (44) \end{aligned}$$

On using the above metric, we have $\sqrt{-g} = \rho^2 \sin \theta / \left(1 + \frac{\Lambda a^2}{3} \right)^2$. Similarly, on solving Eq. (35) for the Kerr-de Sitter metric, assuming radial heat flow gives

$$q^r = \frac{q_0 \left(1 + \frac{\Lambda a^2}{3} \right)^2}{\rho^2 \sqrt{N^2}}, \quad (45)$$

where $N^2 = \frac{-1}{g^{00}}$, which for the Kerr-de Sitter metric is given by

$$g^{00} = \frac{- \left(1 + \frac{\Lambda a^2}{3} \right)^2}{\Delta_r \rho^2 \left(1 + \frac{\Lambda a^2 \cos^2 \theta}{3} \right)} \left[\left(a^2 + r^2 \right)^2 \left(1 + \frac{\Lambda a^2 \cos^2 \theta}{3} \right) - \Delta_r a^2 \sin^2 \theta \right]. \quad (46)$$

Further, on using Eq. (17), we have

$$\frac{d}{dr} \left(T\sqrt{N^2} \right) + \frac{q_0 \left(1 + \frac{\Lambda a^2}{3} \right)^2}{\kappa g^{rr} \rho^2} = 0, \quad (47)$$

for a constant value of θ . Integrating (47) with $g^{rr} = \frac{\Delta_r}{\rho^2}$, assuming κ being a constant yields

$$T\sqrt{N^2} = T_0 - \frac{q_0 \left(1 + \frac{\Lambda a^2}{3} \right)^2}{\kappa} \int \left(\frac{1}{\Delta_r} \right) dr, \quad (48)$$

which reduces to

$$T\sqrt{N^2} = T_0 - \frac{q_0 \left(1 + \frac{\Lambda a^2}{3} \right)^2}{\kappa} \int \left(\frac{1}{r^2 - 2mr + a^2 - \frac{\Lambda r^2}{3} (r^2 + a^2)} \right) dr. \quad (49)$$

Integrating the above equation yields

$$\begin{aligned} T\sqrt{N^2} = & T_0 + \frac{q_0 \left(1 + \frac{\Lambda a^2}{3} \right)^2}{\kappa} \\ & \times \frac{3}{2} \sum_i \left(\frac{\log |x - x_i|}{2\Lambda r_0^3 x_i^3 + \Lambda a^2 r_0 x_i - 3r_0 x_i + 3m} \right), \quad (50) \end{aligned}$$

where $x \equiv r/r_0$, r_0 is the bigger positive root of $\Delta_r = 0$, x_i are the solutions to the equation

$$-3a^2 + 6r_0 m x - 3r_0^2 x^2 + \Lambda a^2 r_0^2 x^2 + \Lambda r_0^4 x^4 = 0. \quad (51)$$

Similar to the previous case it reduces to the Schwarzschild-de Sitter and Kerr cases given by the equations (32) and (42) under the limit $a \rightarrow 0$ and $\Lambda \rightarrow 0$, respectively.

IV. FURTHER CONSEQUENCES

For a static background (16) implies that the rate of heat accepted or released by a cross-sectional area perpendicular to the heat flux direction of the fluid volume

is given by

$$\frac{dQ}{dt} = \pm \int_{\partial\mathcal{V}} \sqrt{-g} \sqrt{N^2} q^\mu d^2\Sigma_\mu, \quad (52)$$

where $d^2\Sigma_\mu = d^2x_\perp \hat{n}_\mu$. Here $+$ ($-$) sign signifies heat absorbed (released) by the surface $\partial\mathcal{V}$ and \hat{n}^μ is the unit outward normal on it.

Let us first concentrate on Eq. (52). For the metric (20) with only radial heat flux, it reduces to

$$\frac{dQ}{dt} = \pm \int d\theta d\phi q_0 \sin\theta = \pm 4\pi q_0, \quad (53)$$

on a $r = \text{constant}$ cross-section of the hypersurface. In the first equality we have used $\hat{n}_\mu = (0, 1, 0, 0)$ and Eq. (21). In the second equality q_0 is assumed to be a pure constant. Now if two spacetime points are at different temperatures, say T_1 and T_2 with $T_1 \sqrt{g_{00}(r_1)} > T_2 \sqrt{-g_{00}(r_2)}$, then one with larger TE value will release heat and another one will absorb the same. This situation can arise when a spacetime has two horizons, like in Schwarzschild de-Sitter spacetime. Then the change of entropy of the first point is $dS_1/dt = (1/T_1)(dQ_1/dt) = -(4\pi q_0)/T_1$ and the same for other one is $dS_2/dt = (1/T_2)(dQ_2/dt) = (4\pi q_0)/T_2$. Here q_0 has been chosen to be positive. Note that for both locations, the rate of change of heat is numerically the same because of the conservation equation (16). Then the total entropy change is given by

$$\begin{aligned} \frac{dS}{dt} &= \frac{dS_1}{dt} + \frac{dS_2}{dt} \\ &= 4\pi q_0 \left(\frac{1}{T_2} - \frac{1}{T_1} \right). \end{aligned} \quad (54)$$

However, if the above one is evaluated with respect to the proper time $d\tau = \sqrt{-g_{00}} dt$, then we have

$$\begin{aligned} \Delta_\tau S &= \frac{1}{\sqrt{-g_{00}(r_1)}} \frac{dS_1}{dt} + \frac{1}{\sqrt{-g_{00}(r_2)}} \frac{dS_2}{dt} \\ &= 4\pi q_0 \left(\frac{1}{T_2 \sqrt{-g_{00}(r_2)}} - \frac{1}{T_1 \sqrt{-g_{00}(r_1)}} \right), \end{aligned} \quad (55)$$

where $q_0/(T\sqrt{-g_{00}})$ can be determined from (23). Note that the above one is consistent with the second law of thermodynamics as we have $q_0 > 0$ and $T_1 \sqrt{-g_{00}(r_1)} > T_2 \sqrt{-g_{00}(r_2)}$. Interestingly the total entropy change per unit proper time is determined not only through the local temperatures, but also the redshift factors are playing important role.

We will see that the same also follows for stationary cases. Since here also the conserved quantity is $\sqrt{N^2} q^a$, Eq. (52) retains. Consequently use of (37) and the expression for metric determinant for Kerr, and similarly (45) and the metric determinant for KdS one finds (53) for both the cases. Here the proper time is given by $d\tau = \sqrt{-g_{00}} dt$ (see the metric (34)). Therefore proceeding in a similar way one finds that all the relations (54) – (55) for static case are also valid for stationary ones.

V. CONCLUSIONS

Applying the recently proposed causal and stable first-order relativistic hydrodynamics we discussed the phenomena of heat flow in presence of gravity. Particularly we confined our analysis within static and stationary backgrounds for a specific choice of fluid four-velocity, assigned by a fundamental observer. The fluid has been chosen as non-viscous, in which the heat dissipation is occurring solely due to the gradient of TE relation. We observed that similar to the modification of the conventional Zeroth's law by a redshift factor in the presence of gravity, the conserved quantity due to heat flow is the standard heat flux times the same redshift factor, as described in equation (15). The heat flow equation, describing the temperature profile, has been derived. We explicitly solved it for Schwarzschild, SdS, Kerr and KdS backgrounds providing examples for each case respectively and found the generalization of the Tolman relation under these circumstances. The solutions have been obtained by assuming only radial heat flux. Klein's law has also been addressed. Moreover, we explicitly obtained the temperature profile for each case and found that if the heat flux vanishes, all the expressions reduce to the usual TE relation and Klein's law.

During this work, we came across a similar work [24]. However, their formulation is within the Eckart's first order formalism. Moreover in their analysis, they have worked with the equation of state and have simultaneously used both equations related to the conservation of fluid energy-momentum tensor. On the other hand, our analysis is based on much viable first-order relativistic hydrodynamics, given by BDN, which takes care of all the shortcomings of Eckart's formalism and is stable and causal. Furthermore the calculation does not require the use of equation of state. Additionally, their analysis is confined to static cases, however we included stationary backgrounds as well. Also in [25] the subject of the heat flow equation has been discussed. However, the calculation was done for a charged relativistic gas in the presence of electromagnetic and gravitational fields based on a different approach. On the other hand, we considered a generic fluid, described by first-order BDN formalism.

Although the present analysis features the aspects of heat flow in presence of gravity, the results, as stated earlier, have limited applications due to imposition of various restrictions. It would be interesting and important for practical purpose to investigate the generalization of these analysis. The particular aim would be to check how much restrictions can be lifted to perform an analytical analysis. Our investigations in these directions are going on and hope we will be able to provide more input in future.

ACKNOWLEDGMENTS

The work of BR is supported by the University Grants Commission (UGC), Government of India, under the scheme Junior Research Fellowship (JRF). BRM is supported by Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India, under the scheme Core Research Grant (File no. CRG/2020/000616).

Appendix A: Determining chemical potential

1. Schwarzschild black hole

Solving Eq. (27) for the Schwarzschild case with $g_{00} = \frac{-1}{g_{rr}} = -\left(1 - \frac{2m}{r}\right)$ and assuming α to be constant, we have

$$\frac{\mu}{T} = A + \frac{q_0}{\alpha} \int \frac{1}{r^2 \left(1 - \frac{2m}{r}\right)^{3/2}} dr. \quad (\text{A1})$$

Further, on using Eq. (28), we have

$$\left(\mu(r) \sqrt{1 - \frac{2m}{r}} \right)_{(sch)} = \left(T_0 - \frac{q_0}{2m\kappa} \ln \left(1 - \frac{2m}{r} \right) \right) \times \left(A - \frac{q_0}{m\alpha} \frac{1}{\sqrt{1 - \frac{2m}{r}}} \right), \quad (\text{A2})$$

where A is the integration constant.

2. Schwarzschild-de Sitter black hole

Again for the Schwarzschild-de Sitter case, Eq. (27) assuming α to be constant reduces to,

$$\frac{\mu}{T} = A + \frac{q_0}{\alpha} \int \frac{1}{r^2 \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{3/2}} dr, \quad (\text{A3})$$

which doesn't have a simplified or compact form. However, it can be solved near the horizon, using $r \rightarrow r' + r_b$, where r_b is the black hole horizon. The above equation near the black hole horizon reduces to

$$\frac{\mu}{T} = A + \frac{q_0}{\alpha} \int \frac{1}{(r' + r_b)^2 \left(2\kappa_H r'\right)^{3/2}} dr', \quad (\text{A4})$$

where $f'(r_b) = 2\kappa_H$ with κ_H is the surface gravity. On further simplifications, it yields

$$\frac{\mu}{T} = A + \frac{q_0}{\alpha (2\kappa_H)^{3/2}} \int \frac{1}{(r' + r_b)^2 (r')^{3/2}} dr', \quad (\text{A5})$$

which on integrating, gives

$$\begin{aligned} (\mu\sqrt{-g_{00}})_{SdS} = & \quad (\text{A6}) \\ (T\sqrt{-g_{00}})_{SdS} & \left[A - \frac{q_0}{\alpha (2\kappa_H)^{3/2}} \left(\frac{2r_b + 3r'}{\sqrt{r'}(r_b^3 + r_b^2 r')} \right. \right. \\ & \left. \left. + \frac{3}{r_b^{5/2}} \arctan \sqrt{\frac{r'}{r_b}} \right) \right]. \end{aligned}$$

Note that this is valid near the black hole horizon, where $(T\sqrt{-g_{00}})_{SdS}$ is determined by the near horizon expression of Eq. (32).

Here we prefer to avoid the presentation of same for Kerr and Kerr-dS as the expressions are coming out to be very cumbersome, even within the near-horizon approximation. However, in principle, these can be determined by performing similar integrations.

-
- [1] R. Tolman and P. Ehrenfest, Phys. Rev. **36**, 1791 (1930).
 - [2] R. C. Tolman, Phys. Rev. **35**, 904 (1930).
 - [3] H. A. Buchdahl, Phys. Rev. **76**, 427 (1949), URL <https://link.aps.org/doi/10.1103/PhysRev.76.427.2>.
 - [4] J. Santiago and M. Visser, Phys. Rev. D **98**, 064001 (2018), 1807.02915.
 - [5] O. Klein, Rev. Mod. Phys. **21**, 531 (1949), URL <https://link.aps.org/doi/10.1103/RevModPhys.21.531>.
 - [6] W. J. Cocke, Annales de l'I.H.P. Physique théorique **2**, 283 (1965), URL <http://eudml.org/doc/75503>.
 - [7] J. Katz and Y. Manor, Phys. Rev. D **12**, 956 (1975), URL <https://link.aps.org/doi/10.1103/PhysRevD.12.956>.
 - [8] R. D. Sorkin, R. M. Wald, and Z. J. Zhang, Gen. Rel. Grav. **13**, 1127 (1981).
 - [9] S. Gao, Phys. Rev. D **84**, 104023 (2011), [Addendum: Phys.Rev.D 85, 027503 (2012)], 1109.2804.
 - [10] S. Gao, Phys. Rev. D **85**, 027503 (2012), URL <https://link.aps.org/doi/10.1103/PhysRevD.85.027503>.
 - [11] S. R. Green, J. S. Schiffrin, and R. M. Wald, Class. Quant. Grav. **31**, 035023 (2014), 1309.0177.
 - [12] K. Shi, Y. Tian, X. Wu, H. Zhang, and C. Zhu, Class. Quant. Grav. **39**, 085004 (2022), 2108.08729.
 - [13] K. Shi, Y. Tian, X. Wu, H. Zhang, and J. Zhang (2022), 2211.11574.
 - [14] Z. Roupas, Class. Quant. Grav. **30**, 115018 (2013), [Erratum: Class.Quant.Grav. 32, 119501 (2015)], 1411.0325.
 - [15] Z. Roupas, Class. Quant. Grav. **32**, 135023 (2015), 1411.4267.

- [16] Z. Roupas, *Symmetry* **11**, 1435 (2019), 1809.04408.
- [17] Z. Roupas, *Phys. Rev. D* **91**, 023001 (2015), 1411.5203.
- [18] C. Rovelli and M. Smerlak, *Class. Quant. Grav.* **28**, 075007 (2011), 1005.2985.
- [19] Y. Gim and W. Kim, *Eur. Phys. J. C* **75**, 549 (2015), 1508.00312.
- [20] J. A. S. Lima, A. Del Popolo, and A. R. Plastino, *Phys. Rev. D* **100**, 104042 (2019), 1911.09060.
- [21] J. A. S. Lima and J. Santos, *Phys. Rev. D* **104**, 124089 (2021), 2112.12282.
- [22] B. Ram and B. R. Majhi, *Laws of thermodynamic equilibrium within first order relativistic hydrodynamics* (2024), 2304.11843, URL <https://arxiv.org/abs/2304.11843>.
- [23] M. Xia and S. Gao, *Eur. Phys. J. Plus* **139**, 500 (2024), 2305.17307.
- [24] M. Miranda, M. Rinaldi, and V. Faraoni, *Phys. Rev. D* **110**, 104065 (2024), URL <https://link.aps.org/doi/10.1103/PhysRevD.110.104065>.
- [25] L. Cui, X. Hao, and L. Zhao (2024), 2411.03094.
- [26] L. Landau and E. Lifshitz, 2nd ed. (Butterworth-Heinemann, London) **6**, 552 (1987).
- [27] C. Eckart, *Phys. Rev.* **58**, 919 (1940).
- [28] W. A. Hiscock and L. Lindblom, *Annals Phys.* **151**, 466 (1983).
- [29] W. Israel, *Annals Phys.* **100**, 310 (1976).
- [30] W. Israel and J. M. Stewart, *Annals Phys.* **118**, 341 (1979).
- [31] P. Kovtun, *JHEP* **10**, 034 (2019), 1907.08191.
- [32] R. E. Hoult and P. Kovtun, *JHEP* **06**, 067 (2020), 2004.04102.
- [33] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Phys. Rev. X* **12**, 021044 (2022), URL <https://link.aps.org/doi/10.1103/PhysRevX.12.021044>.
- [34] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975), [Erratum: *Commun.Math.Phys.* 46, 206 (1976)].