

Quantum state transfer with sufficient fidelity

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(Dated: December 4, 2024)

We consider an inhomogeneous XX spin chain which interpolates between the Krawtchouk one with perfect state transfer and the homogeneous XX chain. This model can be exploited in order to perform state transfer of a qubit with sufficiently good fidelity. The advantage of this model with respect to the Krawtchouk chain is that while it achieves efficient state transfer, the coupling strengths are capped and do not become excessively large as the number of sites grows.

1. INTRODUCTION

Perfect state transfer (PST) is a protocol that performs with probability one, the transport of a qubit in an unknown state from one location to another. The design, in terms of spin chains, of devices enacting PST without error inducing external controls has been initiated some 20 years ago [3] and is still the object of much attention. One reason for that in the context of noisy intermediate scale devices is that such quantum wires could be used instead of swap gates in circuit routing to adapt for instance to the constrained gate architectures currently available (see [12] for a Reinforcement Learning approach to such issues).

We report in this communication on how to engineer *analytically* a spin chain that generates a transfer that is not perfect but of sufficiently high fidelity and that is free from the excessively large coupling strengths of the lengthy chains with PST.

The basic system that is used to realize such quantum wires is still that of an XX spin chains with nearest-neighbor interactions. The corresponding Hamiltonian H is given by

$$H = \frac{1}{2} \sum_{l=0}^{N-1} J_{l+1} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y) + \frac{1}{2} \sum_{l=0}^N B_l (\sigma_l^z + 1), \quad (1.1)$$

where J_l are the constants coupling the sites $l-1$ and l and B_l are the strengths of the magnetic field at the sites l ($l = 0, 1, \dots, N$). The symbols σ_l^x , σ_l^y , σ_l^z stand for the Pauli matrices which act on the l -th spin.

Each spin at site l has two basic states $|0\rangle_l$ (spin down) and $|1\rangle_l$ (spin up) such that

$$\sigma_l^z |0\rangle_l = -|0\rangle_l, \quad \sigma_l^z |1\rangle_l = |1\rangle_l.$$

Hence the vector space of all the chain states is spanned by the vectors

$$|n_0, n_1, \dots, n_N\rangle = |n_0\rangle_0 |n_1\rangle_1 \dots |n_N\rangle_N,$$

where each n_l can take the values 0 or 1.

It is easily seen that

$$[H, \frac{1}{2} \sum_{l=0}^N (\sigma_l^z + 1)] = 0,$$

which implies that the eigenstates of H split in subspaces labeled by the number of spins over the chain that are up. In order to characterize the chains with PST, it suffices to restrict H to the subspace spanned by the states which contain only one excitation. A natural basis for that subspace is given by the vectors

$$|e_n\rangle = |0, 0, \dots, 1, \dots, 0\rangle, \quad n = 0, 1, 2, \dots, N,$$

where the only “1” occupies the n -th position. The restriction J of H to the 1-excitation subspace acts as follows

$$J|e_n\rangle = J_{n+1}|e_{n+1}\rangle + B_n|e_n\rangle + J_n|e_{n-1}\rangle. \quad (1.2)$$

Note that

$$J_0 = J_{N+1} = 0 \quad (1.3)$$

is assumed.

The goal is to use the chain dynamics to relocate after a time T the quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ from the site $n = 0$ to the site $n = N$. A simple analysis shows that this requires the state $|e_0\rangle = |1, 0, 0, \dots, 0\rangle$ to be unitarily evolved into the state $|e_N\rangle = |0, 0, \dots, 1\rangle$, i.e. to have

$$U(T)|e_0\rangle = e^{i\varphi}|e_N\rangle, \quad (1.4)$$

where $U(t)$ is the evolution operator

$$U(t) = \exp(-itH) \quad (1.5)$$

and φ is a real phase parameter. Condition (1.4) defines PST in XX spin chains.

The initial model proposed in [3] was a uniform XX chain where all the coupling constants are the same $J_l = J$, $l = 1, 2, \dots, N$ and the magnetic fields are absent $B_l = 0$. This model however only yields PST for chains that

contain only 3 or 4 spins. For longer chains there are no times T for which relation (1.4) is verified. It was subsequently shown [1],[7] that inhomogeneous XX spin chains with PST for any number of sites N could be engineered by judiciously picking the coupling constants and the Zeeman terms in a non-uniform fashion. The most celebrated example is that of the Krawtchouk chain with coupling constants

$$J_l = K\sqrt{l(N+1-l)}, \quad B_l = 0, \quad l = 0, 1, 2, \dots, N \quad (1.6)$$

where K is an arbitrary nonzero constant.

This model provides PST at the time $T = \pi/K$. It is seen that this time T does not depend on the length N of the chain. This does not violate the Lieb-Robinson bound [13] nor special relativity for that matter because as N grows, the Hamiltonian is not bounded and becomes singular in view of the parabolic profile of the coupling constants given by (1.6). One can of course rescale the couplings by N , $J_n \rightarrow \frac{J_n}{N}$, to alleviate this issue with the result that the PST time will be $T = \frac{\pi N}{K}$. This fits with the bound of the speed of PST found by Yung [21] (see also [19]). It is appropriate to mention in this connection the protocol developed by Xie, Tamon and Kay [20] that allows in principle to break that bound in the speed of PST.

Renormalizing the coupling constants by N leaves however the problem that the couplings at the extremities of the chain will become very small as N grows meaning that the sites towards the end of the chain are basically uncoupled. In a photonic realization of the chain with waveguides [14] this requires those waveguides to be very far apart, an impractical situation.

In summary, the problem with the Krawtchouk chain that we are stressing lies in the fact that the ratio of the maximal value of squares of J_l (in the center of the chain) and the minimal one (i.e. for $l = 1$) is

$$\frac{J_{max}^2}{J_1^2} = \frac{(N+1)^2}{4N} \quad (1.7)$$

This ratio increases linearly with large N which makes any useful implementation of such a chain difficult or even impossible if N attains large values.

Ways of achieving high fidelity, albeit not perfect, state transfer while avoiding the introduction of couplings that become very large when the size of the chain grows have been explored through variations involving the uniform chain, for instance by using a number of such chains [4], or by modulating only the parameters affecting the end sites of the chain [18], [2]. We here consider the problem from a somewhat different angle by not insisting on the idea that the uniform chain plays a central role but by asking rather the question: is it possible to design an analytic chain which “interpolates” between the Krawtchouk and homogeneous XX chain, so as to be free of the above difficulties and to perform the required transfer to satisfaction?

Such a chain should therefore satisfy the following conditions:

- (i) the transfer form $|e_0\rangle$ to $|e_N\rangle$ is “sufficiently good”,
- (ii) the ratio $\frac{J_{max}^2}{J_{min}^2}$ is smaller than that for the Krawtchouk chain in order to allow for a physically realistic implementation.

We take the point of view that it is not the modulation of the parameters of the chain that is problematic, but rather the fact that these specifications become “unbounded” in the models with PST when the size of the chain grows. Indeed, as the realization with photonic waveguides shows [14], it might not be much more difficult to engineer couplings with specific values at the various sites than making them all exactly equal. In that respect, we suggest that having analytic models is quite useful as this entails exact formulas for the couplings; this is a significant feature of the approach based on spectral surgery initially mentioned in [16] that we shall use in the following.

We shall thus indicate how to construct analytically these interpolating chains that have explicit expressions for couplings J_l and satisfy the conditions (i) and (ii) stated above.

The issue of obtaining sufficiently good transfer was also addressed in [9], where the authors introduced the notion of “pretty good transfer” (PGT) and analyzed it in the context of the uniform chain. However, even with this weaker condition on the fidelity of the transfer, the homogeneous chain still proved to have two significant drawbacks: (i) the set of numbers N for which PGT occurs is rather limited and (ii) the time T for PGT cannot be found by an efficient algorithm.

The same idea of approximately perfect transfer dubbed in this case “almost” perfect state transfer (APST) was applied in [17] to non-uniform chains. In this case, for some models the transfer times can be explicitly computed when APST or PGT happens. These times prove finite but the corresponding models are again plagued with the difficulties already pointed out for the Krawtchouk chain concerning the large values of the couplings.

It is hence indicated to consider a looser definition of efficient transfer to be called *good enough* which is not as stringent as APST/PGT and such that the quality of transfer is empirically established. This leads to less restrictive conditions which are easier to implement in practice. With the adoption of such a definition, our goals will be reached by using the spectral surgery method rooted in the Darboux transformations of orthogonal polynomials [6] to eliminate the most non-linear part of the spectrum of the uniform chain and to thus determine analytically the sought-out chain.

It should be mentioned that, although in a different spirit, the approach to be followed here is not without similarity to the one offered in [10] (see also [5]) where it is proposed to use encoding in the outer parts of a chain to alleviate the presence of spectral points that prevent the standard PST. Taking as middle part the segment of

the uniform chain that has a quasi-linear spectrum bears a resemblance in some sense to the spectral surgery that we shall perform on that chain.

2. TIME EVOLUTION OF A QUBIT IN XX SPIN CHAINS

As shown in [16], the dynamics of the inhomogeneous XX chain can be described by using the orthonormal polynomials $\chi_n(x)$ arising from the recurrence relation

$$J_{n+1}\chi_{n+1}(x) + B_n\chi_n(x) + J_n\chi_{n-1}(x) = x\chi_n(x) \quad (2.1)$$

with

$$\chi_{-1} = 0, \chi_0 = 1. \quad (2.2)$$

It is convenient to introduce the monic orthogonal polynomials $P_n(x)$ through

$$P_n(x) = J_1 J_2 \dots J_n \chi_n(x) = x^n + O(x^{n-1}). \quad (2.3)$$

The orthogonality relations are of the form

$$\sum_{s=0}^N w_s \chi_n(x_s) \chi_m(x_s) = \delta_{nm}, \quad (2.4)$$

or, equivalently,

$$\sum_{s=0}^N w_s P_n(x_s) P_m(x_s) = h_n \delta_{nm}, \quad (2.5)$$

where

$$h_n = J_1^2 J_2^2 \dots J_N^2 \quad (2.6)$$

is the normalization constant. The grid points x_s are the eigenvalues of the tridiagonal matrix J with diagonal entries B_l and off-diagonal entries J_l :

$$J|x_s\rangle = x_s|x_s\rangle, \quad s = 0, 1, \dots, N. \quad (2.7)$$

Note that the x_s are nondegenerate provided that $J_l \neq 0$ for $l = 0, 1, \dots, N$. The discrete weights w_s are given by

$$w_s = \frac{h_N}{P_N(x_s)P'_{N+1}(x_s)}, \quad s = 0, 1, \dots, N, \quad (2.8)$$

where $P_{N+1}(x)$ is the characteristic polynomial of the spectrum:

$$P_{N+1}(x) = (x - x_0)(x - x_1) \dots (x - x_N). \quad (2.9)$$

It is easy to show that the weights are positive $w_s > 0$ and satisfy the normalization condition

$$\sum_{s=0}^N w_s = 1. \quad (2.10)$$

In what follows we shall take the eigenvalues x_s in increasing order

$$x_0 < x_1 < x_2 < \dots < x_N. \quad (2.11)$$

The eigenvectors $|x_s\rangle$ of the tridiagonal matrix J have the expression [16]

$$|x_s\rangle = \sum_{n=0}^N \sqrt{w_s} \chi_n(x_s) |e_n\rangle. \quad (2.12)$$

The tridiagonal matrix is called the *persymmetric* if it is symmetric under reflection with respect to the main antidiagonal, i.e. if

$$J_{N+1-l} = J_l, \quad B_{N-l} = B_l \quad (2.13)$$

The necessary and sufficient conditions for PST are

- (i) the matrix J is persymmetric
- (ii) the spectrum x_s satisfy the conditions

$$x_{s+1} - x_s = \kappa M_s, \quad s = 0, 1, \dots, N-1 \quad (2.14)$$

where M_s are positive odd integers and κ is an arbitrary positive parameter. If conditions (i)-(ii) are fulfilled then the minimal time T for which (1.4) is satisfied is given by

$$T = \frac{\pi}{\kappa d}, \quad (2.15)$$

where d is GCD of the integers M_0, M_1, \dots, M_{N-1} .

The Krawtchouk XX chain has the linear spectrum $x_s = K(s - N/2)$, $s = 0, 1, 2, \dots, N$ which ensures that (2.14) is satisfied with $M_s = 1$ for all s and $\kappa = K$ and the matrix J defined by (1.6) furthermore verifies (2.13). The corresponding polynomials $P_n(x)$ coincide with the Krawtchouk polynomials [1]. Considering some other chain, assume that the matrix J is still persymmetric according to (2.13) but that conditions (2.14) do not hold. In this case PST is impossible. However, we can suppose that the conditions (2.14) are approximately satisfied. We can then expect a state transport with some possibly ‘‘sufficient’’ fidelity. It is convenient to introduce the amplitude

$$A(t) = \langle e_N | U(t) | e_0 \rangle = \langle e_N | \exp(-itH) | e_0 \rangle. \quad (2.16)$$

Obviously for any time t we have the inequality

$$|A(t)| \leq 1$$

The PST condition (1.4) is equivalent to

$$|A(T)| = 1 \quad (2.17)$$

We say that the state transfer is ‘‘good enough’’ if

$$1 - |A(T)| = \delta, \quad (2.18)$$

where δ is a small parameter depending on our requirements for experimental implementation. For example, for practical purposes one might take $\delta \leq 0.05$. The smaller this parameter δ is, the higher the fidelity will be.

3. SURGERED HOMOGENEOUS XX CHAIN

The spectrum x_s of the uniform XX chain with $B_k = 0, k = 0, 1, \dots, M$ and $J_k = 1/2$ is

$$x_s = -2 \cos \omega(s+1), \quad s = 0, 1, \dots, M, \quad (3.1)$$

where

$$\omega = \frac{\pi}{M+2}. \quad (3.2)$$

In Fig. 1 the essentially nonlinear ("red") part of the

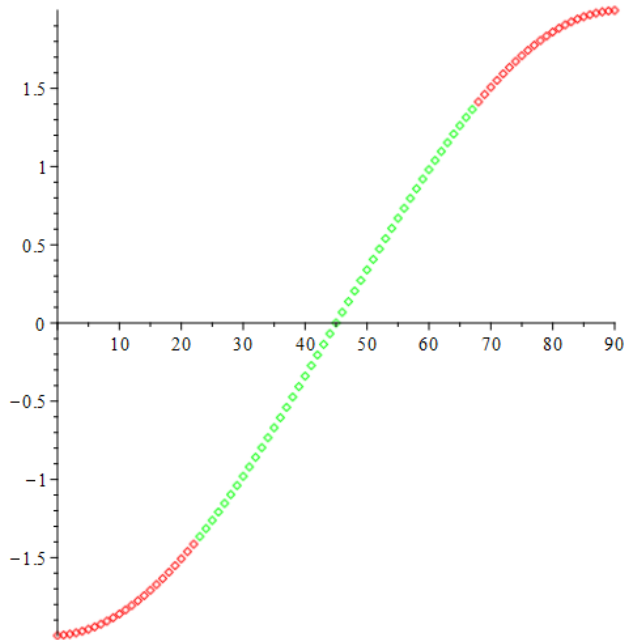


FIG. 1. Spectrum of the uniform XX chain. Green color corresponds to the approximately linear part of the spectrum

spectrum prevents PST because conditions (2.14) cannot be fulfilled. One may hence try to modify the uniform XX chain by "removing" the "red" part of the spectrum. The remaining "green" part of the spectrum will then approximately satisfy conditions (2.14) with $x_{s+1} - x_s \approx \text{const}$.

This procedure is described in [16] and called "spectral surgery". More precisely the idea is the following. Assume that the spectrum of the initial homogeneous (or inhomogeneous) XX chain with $M+1$ sites is $x_0 < x_1 < x_2 < \dots < x_{M-1} < x_M$. Let J_l be the corresponding parameters of this chain. In what follows we assume that magnetic fields are absent $B_l = 0$. Consider a new inhomogeneous XX chain with $M-1$ sites and with spectrum $x_1 < x_2 < \dots < x_{M-2} < x_{M-1}$. That is, the new spectrum is obtained by removing two boundary eigenvalues x_0 and x_M . We denote the new coupling constants as $J_l^{(1)}$ (the magnetic fields remain absent $B_l^{(1)} = 0$). Repeating this procedure step-by-step by removing the boundary eigenvalues, we arrive after j

iterations at the nonhomogeneous XX chain with spectrum $x_j < x_{j+1} < \dots < x_{M-j-1} < x_{M-j}$ and with coupling constants $J_l^{(j)}$ and $B_l^{(j)} = 0$. It is assumed that $l = 0, 1, 2, \dots, N-1, N$, where $N = M-2j$.

In [16] it was demonstrated that the chain with the coupling constants $J_l^{(j)}$ can be obtained from the initial chain by the successive application of j Darboux transformations of the initial Jacobi matrix J . In turn, these transformations are well known as Christoffel transformations and are equivalent to refactorizations of the Jacobi matrix. In general, the coupling constants $J_l^{(j)}$ can be expressed via the initial constants J_l and the values of the orthogonal polynomials $P_n(x)$ at the spectral points x_0, x_1, \dots, x_{j-1} (see [16] for details). It is important to stress that the Jacobi matrix $J^{(j)}$ remains persymmetric for all $j = 0, 1, 2, \dots$. Moreover, if the initial XX chain realizes PST, then all the derived chains with Jacobi matrices $J^{(j)}$ will exhibit PST as well.

Let $P_n^{(j)}(x)$, $n = 0, 1, 2, \dots, N$ be the set of monic orthogonal polynomials corresponding to the "surgered" Jacobi matrix $J^{(j)}$. For $j = 0$ (i.e. for the case of the uniform XX chain), the polynomials $P_n(x)$ coincide with the Chebyshev polynomials of second type $P_n(x) = U_n(x)$. A number of relevant things can now be said about the associated polynomials $P_n^{(j)}(x)$. In [15] it was showed that the Darboux process for the Chebyshev polynomials $U_n(x)$ (which is equivalent to surgering the uniform XX chain) leads to the so-called "q-ultraspherical polynomials" which are well known for q real [11]. In the case of interest here, the parameter q is a root of unity

$$q = \exp\left(\frac{2\pi i}{M+2}\right). \quad (3.3)$$

The corresponding coupling constants are

$$J_l^{(j)2} = K^2 \frac{(1-q^l)(1-q^{l+2j+1})}{(1-q^{l+j})(1-q^{l+j+1})}. \quad (3.4)$$

The positive constant K may be taken to be arbitrary and will depend on the concrete physical implementations.

When $j = 0$ we have the uniform chain, i.e. $J_l = K$. For a positive integer j we have

$$J_0 = J_{N+1} = 0 \quad (3.5)$$

with $N = M-2j$. Condition (3.5) means that the chain consists of $N+1$ sites: $l = 0, 1, \dots, N$.

From the results of [15] it follows on the one hand, that the eigenvalues of the corresponding Jacobi matrix are (putting $K = 1$ for simplicity)

$$x_s = -2 \cos \omega(s+1+j), \quad s = 0, 1, \dots, N. \quad (3.6)$$

On the other hand, the one-excitation spectrum of the uniform XX chain with $M+1$ sites are given by (3.1).

Hence the spectrum (3.6) can be obtained from the spectrum (3.1) by canceling j levels from the top and j levels from the bottom. In other words, this is equivalent to removing the “red” levels in Fig. 1. This corresponds to the spectral surgery procedure described in [16].

Recalling (3.2), one can rewrite (3.4) in the form

$$J_l^2 = K^2 \frac{\sin(\omega l) \sin(\omega(N+1-l))}{\cos(\omega(l-N/2)) \cos(\omega(l-N/2-1))} \quad (3.7)$$

which involves only the integer parameters M and N .

From Fig.2 it is seen that for a fixed number $N = 100$ of spins, the profile J_l^2 of the coupling constants interpolates between the profile of the uniform XX chain when $M = N$, (i.e. when the number of iterations is zero $j = 0$) and that of the inhomogeneous Krawtchouk XX chain (when $M \rightarrow \infty$).

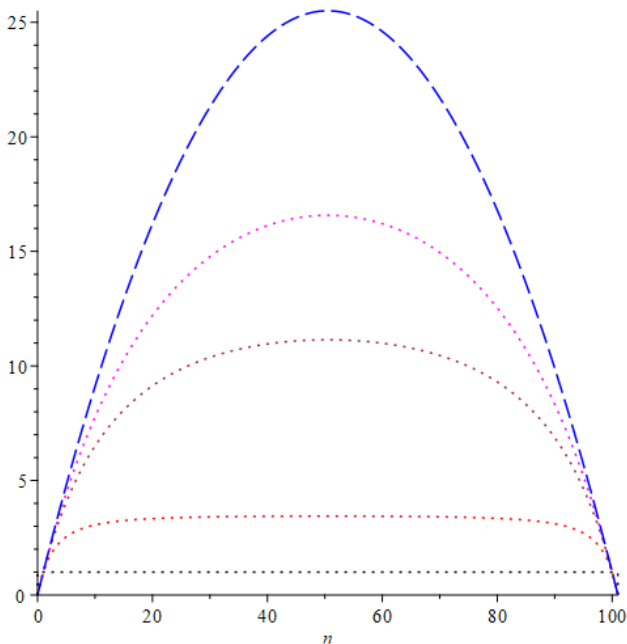


FIG. 2. Profiles (3.7) of the coupling constants of the surgered XX chain with $N = 100$: the blue dash line (parabola) corresponds to the Krawtchouk chain ($M \rightarrow \infty$), the black line to the uniform XX chain ($j = 0$) and the dots depict the surgered chain with $M = 110$ in red, $M = 150$ in brown and $M = 200$ in magenta. All plots are normalized with respect to the value J_1 .

The discrete orthogonality weights are

$$w_s(M, N) = \kappa^{-1} \prod_{k=0}^j \sin(\omega(s+j-k+1)) \sin(\omega(s+j+k+1)) \quad (3.8)$$

where the normalization constant is

$$\kappa = \frac{M+2}{2} \prod_{k=0}^j \frac{\cos(\omega k) \sin(\omega(2k+1))}{2 \sin(\omega(k+1))}. \quad (3.9)$$

These weights are normalized

$$\sum_{s=0}^N w_s = 1. \quad (3.10)$$

The nonnegative integer parameter j is defined as

$$j = (M - N)/2 = 0, 1, 2, \dots \quad (3.11)$$

In particular, for $N = M$ (i.e. for $j = 0$) we have the case of homogeneous XX chain. Then

$$w_s(M; M) = \frac{2}{M+2} \sin^2 \omega(s+1). \quad (3.12)$$

Formulas (3.8) and (3.9) follow from results of [15].

The ratio R_K of the maximal and minimal values of J_l^2 for the Krawtchouk chain is

$$R_K = \frac{J_{(N+1)/2}^2}{J_1^2} = \frac{(N+1)^2}{4N}. \quad (3.13)$$

For the corresponding ratio R_S of the surgered XX homogeneous chain we have

$$R_S = \frac{J_{(N+1)/2}^2}{J_1^2} = \frac{\sin^2(\omega(N+1)/2) \cos^2(\omega(N-2)/2)}{2 \cos(\omega/2) \sin(\omega) \sin(\omega N/2)}. \quad (3.14)$$

Fixing N and increasing M one can obtain a ratio R_S that approaches 1 (i.e. the ratio of the uniform XX chain) which is more suitable as explained before. There remains to determine if the fidelity of the qubit transfer is sufficiently high.

4. FIDELITY ESTIMATION

Because the tridiagonal matrix J is persymmetric, the amplitude $A(t)$ (recall (2.16)) of the quantum signal at the end of the chain can be calculated with the help of the following formula [16]:

$$A(t) = \sum_{s=0}^N w_s (-1)^{N+s} \exp(-ix_s t). \quad (4.1)$$

Note that for $t = 0$ we have

$$A(0) = \sum_{s=0}^N w_s (-1)^{N+s} = 0, \quad (4.2)$$

a consequence of the properties of persymmetric matrices [8]. Formula (4.2) means that at $t = 0$, the quantum signal (qubit) is concentrated at $n = 0$ and that hence the amplitude at $n = N$ is zero. Given the explicit expression of the weights (3.8), one can evaluate the fidelity defined in (2.18)

$$\delta(T) = 1 - |A(T)| \quad (4.3)$$

for different values of N, M, T . The main problem is: find the time T such that $\delta(T)$ has the smallest (necessarily positive) value for the given parameters N and M . Remember that smaller and smaller $\delta(T)$ will amount to higher and higher fidelities that can be defined by $1 - \delta(T)$.

We present several estimates for $N = 100$. It is convenient to normalize the spectrum of the uniform XX chain (and hence its Hamiltonian) according to

$$x_s = -2(M + 2) \cos \omega(s + 1), \quad s = 0, 1, \dots, M \quad (4.4)$$

with $\omega = \pi/(M + 2)$. Then for sufficiently large values of M the levels x_s in the middle part of the spectrum (4.4) have a linear behavior with $x_{s+1} - x_s \approx 2\pi$. We know that the PST time of the Krawtchouk chain (corresponding to $M \rightarrow \infty$) is $T_K = 1/2$. We can then use this T_K as a “zeroth order approximation” for the time T which yields a minimal value of $\delta(T)$, i.e. we will search for a T of the form

$$T = 1/2 + \varepsilon. \quad (4.5)$$

Calculations with formula (4.1) give the following results

(i) for $N = 100$ and $M = 110$ we have $\varepsilon = 10^{-2}$ and $\delta(T) \approx 0.13$. Such a fidelity is better than that of the uniform XX chain but might not be “good enough” from an experimental or engineering point of view.

(ii) for $N = 100$ and $M = 120$ we have $\varepsilon = 10^{-2}$ and $\delta(T) \approx 0.05$. Fidelity with such accuracy could be

considered “good enough” in implementation schemes. Moreover, in this case the ratio R_S between the maximal and minimal values of J_l^2 is approximately 5, while for the Krawtchouk chain this ratio is 25. This means that this chain is much better behaved.

(iii) for $N = 100$ and $M = 150$ we have $\varepsilon = 0.005$ and $\delta(T) \approx 0.008$. This level of fidelity could be considered in some contexts as “perfect enough”.

Finally notice that increasing N with a fixed ratio M/N one can achieve very good fidelity. Consider, for instance the values $M = 550, N = 500$. In this case $\delta(T) \approx 0.07$. For $M = 1100$ and $N = 1000$, $\delta(T) \approx 0.05$ which is already “good enough” and the ratio $R_S = 25$ is 10 times smaller than the ratio $R_K = 250$ for the Krawtchouk chain that generates PST. This means that for long spin chains the surgered chain offers practical candidates as possible registers for quantum computers or as tools to help with circuit routing.

Acknowledgments

AZ thanks the Centre de Recherches Mathématiques (Université de Montréal) for hospitality. The authors would like to thank M.Christandl, M.Derevyagin and A.Filippov for stimulating discussions. The research of LV is supported in part by a research grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

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- [1] C. Albanese, M. Christandl, N. Datta, A. Ekert, *Mirror inversion of quantum states in linear registers*, Physical Review Letters **93** (2004), 230502.
- [2] L. Bianchi, T. J. G. Apollaro, A. Cuccoli, R. Vaia, and P. Verrucchi, *Optimal dynamics for quantum-state and entanglement transfer through homogeneous quantum systems*, Physical Review A **82**, 052321 (2010). arXiv:1006.1217v1.
- [3] S. Bose, *Quantum communication through spin chain dynamics: an introductory overview*, Contemporary Physics, **48**, (2007), 13 – 30.
- [4] D. Burgarth, V. Giovannetti and S. Bose, *Efficient and perfect state transfer in quantum chains*, Journal of Physics A Mathematical and Theoretical, **38**, 6793 (2005). arXiv:quant-ph/0410175v3
- [5] X. Chen, R. Mereau, and D. L. Feder, *Asymptotically perfect efficient quantum state transfer across uniform chains with two impurities*, Physical Review A **93**, 012343 (2016). arXiv:1511.00038v1.
- [6] T. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, NY, 1978.
- [7] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay and A. J. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Physical Review A **71** (2005), 032312. arXiv:quant-ph/0411020
- [8] V. Genest, S. Tsujimoto, L. Vinet and A. Zhedanov, *Persymmetric Jacobi matrices, isospectral deformations and orthogonal polynomials*, Journal of Mathematical Analysis and Applications, **450** (2017), 915–928. arXiv:1605.00708.
- [9] C. Godsil, S. Kirkland, S. Severini, J. Smith, *Number-theoretic nature of communication in quantum spin chains*, Physical Review Letters **109** (2012), 050502; arXiv:1201.4822.
- [10] A. Kay, *Incorporating Encoding into Quantum System Design*, Physical Review A **109**, 042408. arXiv:2207.01954v2.
- [11] R. Koekoek, P. Lesky, R. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their Q-analogues*, Springer-Verlag, 2010.
- [12] D. Kremer, V. Villar, H. Paik, I. Duran, I. Faro, J. Cruz-Benito, *Practical and efficient quantum circuit synthesis and transpiling with Reinforcement Learning*, arXiv:2405.13196, 2024.
- [13] E. H. Lieb and D. W. Robinson, *The Finite Group Velocity of Quantum Spin Systems*, Communications in Mathematical Physics **28** (1972), 251–257.
- [14] A. Perez-Leija, R. Keil, A. Kay, H. Moya-Cessa, S. Nolte, L.-C. Kwek, B. M. Rodriguez-Lara, A. Szameit, D. N. Christodoulides, *Coherent quantum transport in photonic lattices*, Physical Review A **87**, 012309 (2013)
- [15] V. Spiridonov and A. Zhedanov, *q-Ultraspherical poly-*

- nomials for q a root of unity*, Letters in Mathematical Physics **37** (1996), 173–180. arXiv:q-alg/9605033v1
- [16] L. Vinet, A. Zhedanov, *How to construct spin chains with perfect spin transfer*, Physical Review A **85** (2012), 012323.
- [17] , L. Vinet, A. Zhedanov, *Almost perfect state transfer in quantum spin chains*, Physical Review A **86**, 052319 (2012)
- [18] A. Wójcik, T. Luczak, P. Kurzynski, A. Grudka, T. Gdala, and M. Bednarska, *Unmodulated spin chains as universal quantum wires*, Physical Review A **72**, 034303 (2005). arXiv:quant-ph/0505097v1.
- [19] W. Xie, A. Kay, and C. Tamon, *A Note on the Speed of Perfect State Transfer*, arXiv:1609.01854.
- [20] W. Xie, A. Kay, and C. Tamon, *Breaking the Speed Limit for Perfect Quantum State Transfer*, Physical Review A **108**, 012408 (2023)
- [21] M.-H. Yung, *Quantum speed limit for perfect state transfer in one dimension*, Physical Review A **74**, 030303(R) (2006).