Transition temperature of homogeneous weakly interacting Bose gas in self-consistent Popov approximation

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(Dated: December 4, 2024)

Abstract

Using the Cornwall–Jackiw–Tomboulis effective action framework in conjunction with variational perturbation theory, this study investigates the relative shift in the transition temperature of a homogeneous repulsive weakly interacting Bose gas compared to that of an ideal Bose gas. Employing both one-loop and self-consistent Popov approximations, we derive the universal form of the relative shift in the transition temperature, which is proportional to the *s*-wave scattering length. The results demonstrate excellent agreement with those obtained through precise Monte Carlo simulations. Furthermore, the zero-point energy and various thermodynamic quantities are also analyzed in the condensed phase.

Keywords: Interacting Bose gas, self-consistent Popov approximation, transition temperature, thermodynamic properties

I. INTRODUCTION

The Bose-Einstein condensation (BEC) of a homogeneous repulsive weakly interacting Bose gases remains a highly topical and promising area of research, despite having been theoretically predicted nearly a century ago [1, 2]. It was pointed out that a system of bosonic atoms undergo a condensate phase transition as its temperature is lowered to the critical temperature. Studies of BEC truly blossomed after the condensation of Bose gas was observed experimentally [3] and have yielded numerous significant results. For a perfect Bose gas, the transition temperature is [4, 5],

$$T_C^{(0)} = \frac{2\pi\hbar^2}{mk_B} \left[\frac{\rho}{\zeta(3/2)}\right]^{2/3},$$
(1)

in which \hbar and k_B are reduced Planck and Boltzmann constants, m is the atomic mass and ρ is the particle density. The zeta function is defined as $\zeta(x) = \sum_{n=1}^{\infty} 1/n^x$.

In a system of imperfect Bose gas, the bosonic atoms interact with one another. Various potential models have been proposed to describe these interactions, including two-body and three-body potentials [6]; hard-sphere, soft-sphere, and hard-core square-well models [7] as well as pseudo- and exponential potentials [8]. From these studies, it has been concluded

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that the interatomic interaction in a dilute Bose gas is characterized by the s-wave scattering length a_s . This interaction modifies the transition temperature of the interacting Bose gas compared to that of the ideal Bose gas. To leading order in the s-wave scattering length, the relative shift in the transition temperature of a homogeneous, repulsive, weakly interacting Bose gas, compared to that of an ideal Bose gas, follows a universal form

$$\frac{\Delta T_C}{T_C^{(0)}} \approx c.\alpha_s^a,\tag{2}$$

where the gas parameter $\alpha_s = \rho a_s^3$ is defined in terms of ρ , the particle number density. The determination of the constants a and c has been the subject of prolonged debate and controversy. Most studies suggest a = 1/3 [9–12], though some results indicate a = 1/6[13, 14]. Similarly, the value of the constant c has been contentious: while some calculations yield negative values [9, 15], the majority report positive values over a broad range [10, 11, 16].

To study the dilute Bose gas at finite temperatures, most authors have utilized the Hartree-Fock-Bogoliubov (HFB) theory, as documented in the literature [5, 17, 18]. However, the HFB theory presents certain disadvantages, including issues related to conservation laws and the presence of a gap in the energy spectrum of excitations, which arise due to the anomalous density, as discussed in Ref. [19]. To address these challenges, several approximations have been proposed, such as the Popov, Beliaev-Popov, and T-matrix approximations [20, 21], with the Popov approximation being particularly prevalent. Nevertheless, the Popov approximation results in a first-order phase transition in the condensation phase transition, which is an unphysical outcome. To overcome this inconsistency, Yakulov and collaborators [22] introduced a framework involving Lagrange multipliers with two chemical potentials. In their approach, one chemical potential primarily resolves the conservation problem, while the other ensures the energy spectrum remains gapless. Despite this refinement, their results indicated that the transition temperature for a weakly interacting Bose gas coincides with that of an ideal Bose gas. A similar issue was encountered when Haugset et. al. [23] applied the Cornwall–Jackiw–Tomboulis (CJT) effective action approach in the one-loop approximation to investigate the dilute Bose gas. Fortunately, this conclusion was subsequently disproven and addressed by Kleinert et. al. [13] through the application of variational perturbation theory. While the relative shift in the transition temperature was found to deviate from zero, the authors clarified that their result, characterized by a square-root behavior, was incorrect. In the present study, we integrate the CJT effective action approach within the one-loop approximation with variational perturbation theory in the Popov approximation. This combination aims to refine and extend the findings in Refs. [13, 23], particularly concerning the relative shift in transition temperature and the thermodynamic quantities of a homogeneous repulsive weakly interacting Bose gas.

This paper is organised as follows. In Section II focus on finding the relative shift of transition temperature and thermodynamic quantities of a homogeneous dilute weakly interacting Bose gas by means of the CJT effective action approach at finite temperature. Finally, we present the conclusion and future outlook in Section III.

II. TRANSITION TEMPERATURE AND THERMODYNAMIC QUANTITIES OF A HOMOGENEOUS DILUTE BOSE GAS

A. Self-consistent Popov approximation

In this Section, we will investigate effect of repulsive weakly interatomic interaction to transition temperature of the Bose gas by using CJT effective action in the one-loop approximation. In addition, we investigate it in the Popov approximation with the self-consistency as pointed out in Ref. [13]. To do this, we start with a homogeneous dilute Bose gas described by the Lagrangian density [5],

$$\mathcal{L} = \psi^* \left(-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 \right) \psi - \mu \left| \psi \right|^2 + \frac{g}{2} \left| \psi \right|^4, \tag{3}$$

wherein μ is the chemical potential. The field operator $\psi(\vec{r}, t)$ depends on both the coordinate \vec{r} and time t. The interatomic interaction potential between the atoms can be chosen as the hard-sphere model. In the Born approximation, the strength of the interaction between pairwise atoms is determined via the *s*-wave scattering length a_s as $g = 4\pi\hbar^2 a_s/m$. Now, thermodynamic stability requires that g > 0, i.e., the boson interactions are repulsive [5, 17]. In the tree-approximation, the expectation value ψ_0 of the field operator is independent of both coordinate and time. Therefore, the GP potential can be read off from the Lagrangian density (3),

$$V_{\rm GP} = -\mu \left| \psi_0 \right|^2 + \frac{g}{2} \left| \psi_0 \right|^4.$$
(4)

Without any external fields and macroscopic part of the condensate moving as a whole, the lowest energy solution ψ_0 is real and plays the role of the order parameter. Minimizing the

potential (4) with respect to the order parameter, one arrives at the gap equation

$$\psi_0(-\mu + g |\psi_0|^2) = 0.$$
(5)

The square of the order parameter $\rho_0 \equiv |\psi_0|^2$ is defined as the density of the condensate. Eq. (5) yields the condensed density in the condensed phase

$$\rho_0 = \frac{\mu}{g}.\tag{6}$$

Paying attention to Eq. (5), we can write the propagator in tree-level as follows

$$D_0(k) = \frac{1}{\omega_n^2 + E_{\text{(tree)}}^2(k)} \begin{pmatrix} \varepsilon_k & \omega_n \\ -\omega_n & \varepsilon_k + 2\mu \end{pmatrix},$$
(7)

where $\varepsilon_k = \hbar^2 k^2 / 2m$ is a function of wave vector \vec{k} , the *n*th Matsubara frequency for bosons is defined as $\omega_n = 2\pi n/\beta$ with $n \in \mathbb{Z}$ and $\beta = 1/k_B T$. Examining poles of the propagator (12) gives us the dispersion relation in tree-level

$$E_{\text{(tree)}}(k) = \sqrt{\varepsilon_k \left(\varepsilon_k + 2\mu\right)}.$$
(8)

It is obvious that the energy spectrum of the excitations is gapless, which associates with the spontaneous breaking of the U(1) continuous symmetry. This implies that our system obeys the Goldstone theorem [24]. This fact is referred as the Hugenholtz-Pines theorem [25] at zero temperature and a more general proof for all value of temperature was given by Hohenberg and Martin [26].

Beyond the mean field theory, the field operator is decomposed in form [20]

$$\psi \to \psi_0 + \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2),\tag{9}$$

in which ψ_1 and ψ_2 are associated with fluctuations of the field. Inserting (9) into (3) one obtains the interacting Lagrangian density [27]

$$\mathcal{L}_{\rm int} = \frac{g}{2} \psi_0 \psi_1 (\psi_1^2 + \psi_2^2) + \frac{g}{8} (\psi_1^2 + \psi_2^2)^2.$$
(10)

In the one-loop approximation, the CJT effective potential per unit volume can be read from (10) as follows [28],

$$V_{\beta} = -\mu \left|\psi_{0}\right|^{2} + \frac{g}{2} \left|\psi_{0}\right|^{4} + \frac{1}{2} \int_{\beta} \operatorname{tr} \ln D^{-1}(k), \tag{11}$$

in which the notation \int_β is abbreviated for

$$\int_{\beta} f(k) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3 \vec{k}}{(2\pi)^3} f(k, \omega_n).$$

The propagator in the one-loop approximation D(k) is deduced from (11)

$$D(k) = \frac{1}{\omega_n^2 + E^2(k)} \begin{pmatrix} \varepsilon_k - \mu + g |\psi_0|^2 & \omega_n \\ -\omega_n & \varepsilon_k - \mu + 3g |\psi_0|^2 \end{pmatrix}.$$
 (12)

Analogously, the dispersion relation in the one-loop approximation is

$$E(k) = \sqrt{\left[\varepsilon_k - \mu + 2g \left|\psi_0\right|^2\right]^2 - g^2 \left|\psi_0\right|^4}.$$
(13)

To proceed further we employ the rule for the summation of Matsubara frequencies [29],

$$\frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} \ln\left[\omega_n^2 + E^2(k)\right] = E(k) + \frac{2}{\beta} \ln\left[1 - e^{-\beta E(k)}\right].$$
 (14)

Applying (14) to the last term in right-hand side of (11) the CJT effective potential can be rewritten as

$$V_{\beta} = -\mu \left|\psi_{0}\right|^{2} + \frac{g}{2} \left|\psi_{0}\right|^{4} + \frac{1}{2} \int_{\beta} \frac{d^{3}\vec{k}}{(2\pi)^{3}} E(k) + \frac{1}{\beta} \int \frac{d^{3}\vec{k}}{(2\pi)^{3}} \ln\left[1 - e^{-\beta E(k)}\right].$$
 (15)

By extremizing the CJT effective potential (15) with respect to ψ_0 one arrives at the gap equation in the one-loop approximation

$$-\mu + g |\psi_0|^2 + \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{2g \left[\varepsilon_k - \mu + 2g |\psi_0|^2\right] - g^2 |\psi_0|^2}{E(k)} + \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{2g \left[\varepsilon_k - \mu + 2g |\psi_0|^2\right] - g^2 |\psi_0|^2}{E(k) \left[e^{\beta E(k)} - 1\right]}.$$
(16)

Combining Eqs. (16) and (6) yields a relation the condensed density [13]

$$\rho_0 = \frac{\mu}{g} - \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{2\varepsilon_k + \mu}{\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}} \left[\frac{1}{2} + \frac{1}{e^{\beta\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}} - 1} \right].$$
(17)

We next calculate the pressure of the system, which is defined as the minus of the CJT effective potential (15) taking at its minimum, i.e., satisfying the condition (17). It is easy to find

$$\mathcal{P} = \frac{\mu^2}{2g} - \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k} - \frac{1}{\beta} \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln\left[1 - e^{-\beta\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}}\right].$$
 (18)

The total particle density is defined as derivative of the pressure with respect to the chemical potential

$$\rho = \frac{\partial P}{\partial \mu}.\tag{19}$$

Plugging (18) into (19) yields

$$\rho = \frac{\mu}{g} - \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}} \left[\frac{1}{2} + \frac{1}{e^{\beta\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}} - 1} \right].$$
 (20)

In relevant experiments, the measured quantity is the non-condensed density [30]. Eliminating the chemical potential from (17) and substituting into (20) one attains the non-condensed density

$$\rho - \rho_0 = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k + g\rho_0}{\sqrt{\varepsilon_k^2 + 2g\rho_0 \varepsilon_k}} \left[\frac{1}{2} + \frac{1}{e^{\beta \sqrt{\varepsilon_k^2 + 2g\rho_0 \varepsilon_k}} - 1} \right].$$
 (21)

Eq. (21) represents the non-condensed density in the so-called Popov approximation [20, 21].

To ensure self-consistency, the variational perturbative method introduced by Kleinert *et.al.* [13] is employed. This method involves introducing an expansion parameter η and a variational parameter M defined by the relation $\mu = M + r\eta$ where $r = (\mu - M)/\eta$. These parameters result in a trial CJT effective potential $V_{\beta}^{(\text{trial})}$, replacing V_{β} in Eq. (15). The optimization of $V_{\beta}^{(\text{trial})}$ is performed by requiring that $\partial V_{\beta}^{(\text{trial})}/\partial M = 0$ at $M = M^{(\text{opt})}$ while maintaining a fixed temperature. This yields

$$M^{(\text{opt})} = \mu - g \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + 2M^{(\text{opt})}\varepsilon_k}} \left[\frac{1}{2} \frac{1}{e^{\beta \sqrt{\varepsilon_k^2 + 2M^{(\text{opt})}\varepsilon_k}} - 1} \right]$$
(22)

This condition determines $M = M^{(\text{opt})} = gn$. As a result, the self-consistent pressure is obtained, maintaining the same form as Eq. (18), with the condensed density ρ_0 substituted by the total particle density ρ . For simplicity the self-consistent pressure is still denoted by \mathcal{P}

$$\mathcal{P} = \frac{\left[M^{(\text{opt})}\right]^2}{2g} - \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \sqrt{\varepsilon_k^2 + 2gn\varepsilon_k} - \frac{1}{\beta} \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln\left[1 - e^{-\beta\sqrt{\varepsilon_k^2 + 2gn\varepsilon_k}}\right]$$
$$\equiv \frac{\left[M^{(\text{opt})}\right]^2}{2g} + \mathcal{P}_g^{(0)} + \mathcal{P}_g^{(T)}.$$
(23)

The subsequent steps involve optimizing the CJT effective potential with respect to the condensed density and evaluating the first derivative of the self-consistent pressure with respect to the chemical potential. By combining these results, one obtains

$$\rho - \rho_0 = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k + g\rho}{\sqrt{\varepsilon_k^2 + 2g\rho\varepsilon_k}} \left[\frac{1}{2} + \frac{1}{e^{\beta\sqrt{\varepsilon_k^2 + 2g\rho\varepsilon_k}} - 1} \right].$$
 (24)

A comparison between Eqs. (21) and (24) reaffirms that the results derived under the selfconsistent Popov approximation exhibit the same forms as those within the conventional Popov approximation, with the total particle density ρ replacing the condensed density ρ_0 .

B. Transition temperature

We now calculate the transition temperature of homogeneous repulsive weakly interacting Bose gas. To this end, we first recognize that the first integral in right-hand side of Eq. (24), which does not depend explicitly on temperature. Setting $p^2 = \hbar^2 k^2/2m$, this term can be written in form

$$\frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k + g\rho}{\sqrt{\varepsilon_k^2 + 2g\rho\varepsilon_k}} = \frac{(2m)^{3/2}}{2\hbar^3} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{p^2}{\sqrt{p^2(p^2 + 2g\rho)}} + \frac{(2m)^{3/2}g\rho}{2\hbar^3} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{p^2(p^2 + 2g\rho)}}.$$
(25)

It is obvious that the integrals in right-hand side of (25) are ultraviolet divergent. This divergence is avoidable by means of the dimensional regularization [20] and therefore the integrals can be computed. The integral $I_{m,n}$ is

$$I_{m,n}(\mathcal{M}) = \int \frac{d^d \kappa}{(2\pi)^d} \frac{\kappa^{2m-n}}{(\kappa^2 + \mathcal{M}^2)^{n/2}}$$

= $\frac{\Omega_d}{(2\pi)^d} \Lambda^{2\epsilon} \mathcal{M}^{d+2(m-n)} \frac{\Gamma\left(\frac{d-n}{2} + m\right)\Gamma\left(n - m - \frac{d}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)},$ (26)

where $\Gamma(x)$ is the gamma function, $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a *d*-dimensional sphere. Employing (26) the integral (25) is calculated

$$\frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k + g\rho}{\sqrt{\varepsilon_k^2 + 2g\rho\varepsilon_k}} = \frac{(2m)^{3/2}}{24\pi^2\hbar^3} (2g\rho)^{3/2}.$$
(27)

It is noteworthy that Eq. (27) can be rederived by differentiating Eq. (38) with respect to the chemical potential. Furthermore, by employing the gas parameter, Eq. (27) can be simplified as follows

$$\frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k + g\rho}{\sqrt{\varepsilon_k^2 + 2g\rho\varepsilon_k}} = \frac{8}{3\sqrt{\pi}} \rho \alpha_s^{1/2}.$$
(28)

It is evident that the right-hand side of Eq. (28) precisely corresponds to the quantum fluctuations of a homogeneous, repulsive, weakly interacting Bose gas. This result was originally derived by Bogoliubov [31] and subsequently confirmed by other researchers [6, 32]. Its validity has also been demonstrated through experimental observations [33].

Now we deal with the second integral in the right-hand side of Eq. (24). To do this we introduce an auxiliary variable $q^2 = \beta \varepsilon_k$. Therefore this integral can be rewritten as follows

$$I = \int \frac{d^{3}\vec{k}}{(2\pi)^{3}} \frac{\varepsilon_{k} + g\rho}{\sqrt{\varepsilon_{k}^{2} + 2g\rho\varepsilon_{k}}} \frac{1}{e^{\beta\sqrt{\varepsilon_{k}^{2} + 2g\rho\varepsilon_{k}}} - 1} = \frac{(2m)^{3/2}}{2\pi^{2}\hbar^{3}} \\ \times \int_{0}^{\infty} dq \frac{q^{2}(q^{2} + \beta g\rho)}{\sqrt{q^{2}(q^{2} + 2\beta g\rho)}} \frac{1}{e^{\sqrt{q^{2}(q^{2} + 2\beta g\rho)}} - 1}.$$
(29)

Recall that the system under consideration is the so dilute Bose gas that in region of just below the transition temperature the interaction temperature $T^{(\text{int})} = g\rho/k_B$ is significantly smaller than the transition temperature [13]. Consequently, the parameter $\beta g\rho$ can be regarded as much smaller than unity. To first order in this parameter, the integral (29) can be approximated as follows

$$I \approx \frac{m^{3/2}}{2\sqrt{2}\pi^{3/2}\hbar^3\beta^{3/2}} \left[\zeta(3/2) + \beta g\rho\zeta(1/2)\right].$$
 (30)

Substituting Eqs. (28) and (30) into Eq. (24) yields the expression

$$\frac{\rho_{\rm ex}}{\rho} \equiv \frac{\rho - \rho_0}{\rho} = \frac{8}{3\sqrt{\pi}} \alpha_s^{1/2} + \sqrt{\frac{2m}{\pi\hbar^2}} \frac{\zeta(1/2)a_s}{\beta^{1/2}} + \left(\frac{m}{2\pi}\right)^{3/2} \frac{\zeta(3/2)}{\rho\hbar^3\beta^{3/2}},\tag{31}$$

where ρ_{ex} is defined as the non-condensed density. It is notable that the first term on the right-hand side of Eq. (31) corresponds to quantum fluctuations, which reduce the condensed density due to interaction-induced quantum effects. This contribution was initially identified by Bogoliubov [31] and subsequently corroborated by other studies, such as Refs. [6, 30].

At the transition temperature, the condensed density vanishes ($\rho_0 = 0$). From Eq. (31), the relative shift in the transition temperature, to the lowest order of the *s*-wave scattering length, is given by

$$\frac{\Delta T_C}{T_C^{(0)}} = -\frac{4\zeta\left(\frac{1}{2}\right)}{3\zeta(3/2)^{1/3}}\rho^{1/3}a_s.$$
(32)

It is obvious that result (32) has the same form as in Eq. (2) with

$$c = -\frac{4\zeta(1/2)}{3\zeta(3/2)^{1/3}} \approx 1.413.$$
(33)

This value exhibits a high degree of agreement with our result when compared to those obtained from precise Monte Carlo simulations [34, 35] $c = 1.29 \pm 0.05$. Furthermore, it is in excellent concordance with the theoretical result derived using the nonperturbative linear δ -expansion method [36] $c = 1.32 \pm 0.02$ and aligns closely with other finding in mathematical physics [12] c = 1.49.

C. Zero-point energy and thermodynamic quantities

In this section, we examine the temperature dependence of the zero-point energy and thermodynamic quantities in the condensed phase of the gas. The initial focus is placed on the chemical potential. Inversion of Eq. (20) one has

$$\mu = g\rho + g \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}} \left[\frac{1}{2} + \frac{1}{e^{\beta\sqrt{\varepsilon_k^2 + 2\mu\varepsilon_k}} - 1} \right].$$
(34)

By employing Eqs. (26) and following a procedure analogous to that used for Eq. (29), the chemical potential can be determined and expressed as a function of temperature

$$\mu = g\rho \left(1 + \frac{32}{3\sqrt{\pi}}\alpha_s^{1/2}\right) - \frac{24\pi\hbar^2\zeta(1/2)\rho a_s^2}{m\lambda_B} + \frac{4\pi\hbar^2\zeta(3/2)a_s}{m\lambda_B^3}.$$
(35)

It is evident that the first term on the right-hand side of Eq. (35) represents the contribution from mean-field theory, $g\rho$, as well as beyond-mean-field effects, which have been previously identified [23, 32]. The temperature dependence is encapsulated in the last two terms, which follow a half-integer power law, in contrast to the integer power law described in the findings of Ref. [23]. For the ideal Bose gas, the chemical potential is a function of temperature in form

$$\mu_{\text{ideal}} = k_B T \ln z,\tag{36}$$

with z being the fugacity, which is unity in the condensed phase [37]. It is evident that Eq. (35) simplifies to Eq. (36) when $a_s = 0$ for the ideal Bose gas.

We now estimate the self-pressure, which is shown in (23). The second term taken with a negative sign, represents the zero-point (vacuum) energy, which generates quantum fluctuations in the ground state [32]. In a finite volume, this zero-point energy gives rise to the Casimir effect in BEC(s), a phenomenon that has been extensively investigated (see, for instance, [38, 39])). In the context of the present study, the zero-point energy is given by:

$$V_{\beta}^{(0)} = -\mathcal{P}_{g}^{(0)} = \frac{1}{2} \int \frac{d^{3}\vec{k}}{(2\pi)^{3}} \sqrt{\varepsilon_{k}^{2} + 2\mu\varepsilon_{k}}, \qquad (37)$$

which does not explicitly depend on temperature and is known to exhibit ultraviolet divergence. To obtain a finite physical value, this divergence must be regularized. Fortunately, several regularization methods are available to address this issue [40]. One prominent approach involves using the integral regularization technique developed by 't Hooft and Veltman [41]. Applying this method to the current case yields a finite expression for the zero-point energy:

$$V_{\beta}^{(0)} = -\mathcal{P}_{g}^{(0)} = \frac{8m^{3/2}}{15\pi^{2}\hbar^{3}}\mu^{5/2}.$$
(38)

This result can also be derived using Eq. (26). An alternative approach to resolving the divergence is the momentum-cutoff regularization method. This technique introduces a momentum cutoff Λ , where the divergent terms are absorbed through appropriate counterterms. The results obtained using this method are consistent with Eq. (38), but with the chemical potential μ replaced by its renormalized form $\mu_r = \mu - \frac{g\Lambda^3}{12\pi^2}$. Further details on this approach can be found in Ref. [42]. The result (38) coincides with the original result of Lee and Yang [43].

Similarly to Eq. (30), using condition of the dilute gas and the interacting temperature is much smaller than the transition temperature the last term in right-hand side of Eq. (23) is estimated

$$\mathcal{P}_{g}^{(T)} = \frac{m^{3/2}\zeta(5/2)}{2\sqrt{2}\pi^{3/2}\hbar^{3}\beta^{5/2}} - \frac{m^{3/2}\zeta(3/2)}{2\sqrt{2}\pi^{3/2}\hbar^{3}\beta^{3/2}}\mu.$$
(39)

This result is different from the one in Ref. [23] [cf. Eq. (107)]. Inserting (38) and (39) into (23), using definition of the de Broglie wavelength

$$\lambda_B = \sqrt{\frac{2\pi\hbar^2}{mk_B T}},\tag{40}$$

one can write the self-consistent pressure up to the first order of the scattering length as follows

$$\mathcal{P} = \frac{1}{2}g\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\alpha_s^{1/2}\right) + \frac{4\pi\hbar^2\zeta(3/2)}{m\lambda_B^3}\rho a_s - \frac{6\pi\hbar^2\zeta(3/2)^2 a_s}{m\lambda_B^6} + \frac{2\pi\hbar^2\zeta(5/2)}{m\lambda_B^5}.$$
 (41)

Let us now examine the physical significance of the terms on the right-hand side of Eq. (41). The first term consists of two parts enclosed within the parentheses and their physical interpretation is not difficult to discern: the first part corresponds to the bulk pressure $g\rho^2/2$, while the second part arises from the zero-point energy. This latter contribution constitutes a contribution beyond the mean-field theory, attributable to quantum fluctuations. This term has been found before in Ref. [32] at zero temperature. Two next terms expresses the contribution of both the interatomic interaction and thermal fluctuations on the pressure. The last term is purely attributable to thermal fluctuations. Applying to the perfect Bose gas, i.e., the *s*-wave scattering length vanishes, the right-hand side of Eq. (41) reduces to this last term. It reveals that the pressure of the perfect Bose gas is proportional to $(k_B T)^{5/2}$ [4].

The next thermodynamic quantity to consider is the energy density. At nonzero temperature, the energy density is defined as a Legendre transform of the free-energy density, as given by [32, 37]

$$\mathcal{E} = \mu \rho - \mathcal{P}.\tag{42}$$

Combining this expression with Eq. (41), the energy density becomes

$$\mathcal{E} = \frac{1}{2}g\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\alpha_s^{1/2}\right) - \frac{4\pi\hbar^2\zeta(3/2)\rho a_s}{m\lambda_B^3} + \frac{2\pi\hbar^2\zeta(3/2)^2 a_s}{m\lambda_B^6} - \frac{2\pi\hbar^2\zeta(5/2)}{m\lambda_B^5}.$$
 (43)

Likewise the self-consistent pressure, the first term on the right-hand side of Eq. (43) represents the mean-field and beyond-mean-field contributions to the energy density. These results are consistent with those previously reported in Refs. [6–8, 32, 44] and have been experimentally confirmed [45]. The contributions arising from thermal fluctuations are captured in the last three terms of the expression.

III. CONCLUSION AND OUTLOOK

In the preceding sections, the fundamental properties of a homogeneous repulsive weakly interacting Bose gas have been analyzed. This investigation utilized the CJT effective action method within the one-loop approximation, combined with the variational perturbation theory in the self-consistent Popov approximation. The principal findings of this study are summarized as follows: - The relative shift of the transition temperature of the homogeneous repulsive weakly interacting Bose gas with respect to that of the ideal Bose gas has been found in good agreement with that reported by other groups.

- The zero-point energy in the grand canonical ensemble has been examined. It is demonstrated that the ultraviolet divergence can be effectively eliminated using both dimensional regularization and momentum cutoff regularization. These methods yield the same finite result for the zero-point energy, which accounts for the quantum fluctuations superimposed on the ground state of the system.

- The thermodynamic quantities, namely, the chemical potential, pressure and energy density have been investigated. These results are an improvement of those reported in Refs. [13, 23].

Notably, despite employing the same self-consistent Popov approximation, our findings differ significantly from those presented in Refs. [13, 23]. This discrepancy highlights the potential of this approach for further exploring the thermodynamic properties of the interacting Bose gas.

ACKNOWLEDGMENTS

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 103.01-2023.12.

CONFLICT OF INTEREST

All of the authors declare that we have no conflict of interest.

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