A Unified Framework of Unitarily Residual Measures for Quantifying Dissipation

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(Dated: December 4, 2024)

Open quantum systems are governed by both unitary and non-unitary dynamics, with dissipation arising from the latter. Traditional quantum divergence measures, such as quantum relative entropy, fail to account for the non-unitary oriented dissipation as the divergence is positive even between unitarily connected states. We introduce a framework for quantifying the dissipation by isolating the non-unitary components of quantum dynamics. We define equivalence relations among hermitian operators through unitary transformations and characterize the resulting quotient set. By establishing an isomorphism between this quotient set and a set of real vectors with ordered components, we induce divergence measures that are invariant under unitary evolution, which we refer to as the unitarily residual measures. These unitarily residual measures inherit properties such as monotonicity and convexity and, in certain cases, correspond to classical information divergences between sorted eigenvalue distributions. Our results provide a powerful tool for quantifying dissipation in open quantum systems, advancing the understanding of quantum thermodynamics.

Introduction.— The dynamics of isolated quantum systems are governed by unitary transformations, which are reversible and do not result in thermodynamic dissipation. In contrast, open quantum systems cannot be fully described by unitary transformations alone [1]. Naturally, the dissipation in open quantum systems is expected to arise from the non-unitary components, since unitary operations do not contribute to dissipation. In stochastic thermodynamics [2, 3], divergences play a central role in quantifying dissipation. Specifically, the Kullback-Leibler divergence quantifies entropy production by comparing the probabilities of forward and backward trajectories [2, 4, 5]. Similarly, in quantum thermodynamics, quantum divergences, such as quantum relative entropy, are important to quantify dissipation [6–8]. However, conventional quantum divergence measures remain positive even under purely unitary transformations; in other words, the divergence between states that can be transitioned to via a unitary operator does not vanish. Therefore, we need divergence measures that are independent of unitary components; the divergence measure between density operators connected by a unitary transformation should be zero. Regarding the quantum Markov process described by the Lindblad equation, Refs. [9] and [10] introduced the total variation distance and the Kullback-Leibler divergence between sorted eigenvalues of density operators and derived speed limits for the entropy production. Reference [9] showed that the Kullback-Leibler divergence between sorted eigenvalues is equal to the minimum of the quantum relative entropy between a unitary transformed density operator and another density operator. Reference [11] showed that the same relation holds between the total variation distance and the trace distance.

The main aim of this Letter is to provide a unified framework for quantifying the effect of dissipation by non-unitary components. We define equivalence relations between hermitian operators via unitary transformations and its quotient set. By identifying all quantum states that can be transitioned to via unitary transformations as a single point (Fig. 1), we can isolate the effects of non-unitary (dissipative) time evolution in the quotient set. We show that isomorphism exists between the quotient set and a set of real vectors whose components are arranged in non-descending order. We also show that divergence measures are naturally induced in the quotient set from the quantum divergences between density operators (Fig. 1), which we refer to as *unitarily residual measures*. Under the assumption [cf. Eq. (21)], we show that the unitarily residual measure inherits fundamental properties such as monotonicity and convexity of the original quantum divergence. Regarding the monotonicity, when the original quantum divergence is monotonic with respect to completely-positive trace-preserving (CPTP) map, the unitarily residual measure is monotonic with stochastic map of eigenvalues. For certain examples, the unitarily residual measures can be written as the classical information divergence between probability distributions of eigenvalues arranged in non-descending order (Table I). These results allow us to write quantum speed limits on dissipation in semi-classical form. As an application, we show speed limits on unitarily residual measures in general open quantum dynamics and we show speed limits for the purity.

Equivalence classes.— Let $\mathcal{H}$ be a Hilbert space with dimension n, and let $\mathfrak{L}(\mathcal{H})$ be a set of linear operators

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Quantum divergence	Unitarily residual measure
Bures angle $\mathcal{L}_D(\rho,\sigma) = \arccos\left(\sqrt{\mathrm{Fid}(\rho,\sigma)}\right)$	Bhattacharyya (arccos) distance $\widetilde{\mathcal{L}}_D([\rho],[\sigma]) = \arccos\left(\sum_i{}' \sqrt{p_i q_i}\right)$
Trace distance $\mathcal{T}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1$	Total variation distance $\widetilde{\mathcal{T}}([\rho],[\sigma]) = \frac{1}{2}\sum_i{}' \lvert p_i - q_i\rvert$
Petz-Rényi relative entropy $D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\ln\left(\mathrm{Tr}\left[\rho^\alpha\sigma^{1-\alpha}\right]\right)$	Rényi divergence $\widetilde{D}_\alpha([\rho]\,\|\,[\sigma]) = \frac{1}{\alpha-1}\ln\left(\sum_i{}' p_i^\alpha q_i^{1-\alpha}\right)$

TABLE I. Examples of quantum divergences on a set of density operators $\mathfrak{M}_D$ and corresponding unitarily residual measures on the quotient set $\mathfrak{M}_D/\sim$. $\sum'$ denotes the sum of non-decreasing sequences $\{p_i\}$ and $\{q_i\}$, which are eigenvalues of density operators ρ and σ, respectively [cf. Eq. (6)].

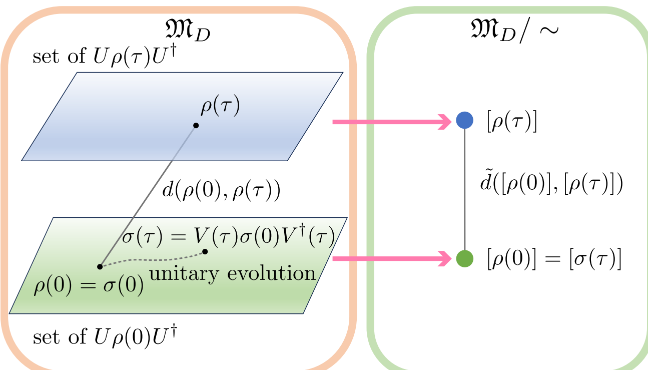

FIG. 1. Illustration of equivalence classes, quotient set $\mathfrak{M}_D/\sim$ and unitarily residual measure $\widetilde{d}$. Time evolution of $\rho(t)$ comprises dissipation and $\sigma(t)$ unitarily evolves by $V(t)$. U is an arbitrary unitary operator. Since $[\sigma(t)]$ stays single point in the quotient set, the unitarily residual measure quantifies the effect of dissipation.

on $\mathcal{H}$. Let $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ be a set of hermitian operators. Since the dimension of Hermitian operators are n^2, we write $\mathfrak{M}_{n^2}$ when we emphasize the dimension. We define an equivalence relation $\sim$ between hermitian operators $A, B \in \mathfrak{M}$ via unitary transformations:

$$A \sim B \text{ if } {}^\exists U \text{ such that } U^\dagger U = \mathbb{I},\ B = UAU^\dagger. \tag{1}$$

Equation (1) shows that any two states connected by a unitary transformation are considered equivalent. For all $A, B, C \in \mathfrak{M}$, this relation satisfies the following three properties:

$$A \sim A, \tag{2}$$
$$A \sim B \text{ if and only if } B \sim A, \tag{3}$$
$$\text{If } A \sim B \text{ and } B \sim C \text{ then } A \sim C. \tag{4}$$

One can easily verify these relations from the definition in Eq. (1). The equivalence relation naturally splits $\mathfrak{M}$ into equivalence classes:

$$[A] := \{B \in \mathfrak{M} :\ B \sim A\}. \tag{5}$$

When $A \sim B$, we write $[A] = [B]$. A set of equivalence classes is called the quotient set, which is denoted by $\mathfrak{M}/\sim$.

We next show that the isomorphism exists between a quotient set and a set of real vectors whose components are arranged in non-descending order. For the sake of simplicity, we introduce the notation before the discussion. Let $\mathbf{x}^\uparrow$ be a sorted vector which is obtained by arranging the components of $\mathbf{x} \in \mathbb{R}^n$ in non-descending order (i.e., $x_1^\uparrow \le x_2^\uparrow \le \cdots \le x_n^\uparrow$). Let $\mathbb{R}^{n\uparrow} := \{\mathbf{x}^\uparrow :\ \mathbf{x} \in \mathbb{R}^n\}$ be a set of sorted vectors. For $\mathbf{a}^\uparrow, \mathbf{b}^\uparrow \in \mathbb{R}^{n\uparrow}$, we use the notation $\sum'$ to define the sum of vectors in non-descending order:

$$\sum_{i=1}^{n}{}' F(a_i, b_i) := \sum_{i=1}^{n} F(a_i^\uparrow, b_i^\uparrow), \tag{6}$$

where F is an arbitrary function. In a similar way, we write the sum $\sum_i' a_i \lvert b_i\rangle\langle b_i\rvert$ and $\sum_i' \lvert b_i\rangle\langle a_i\rvert$ for the non-descending sequences $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$, where $\lvert a_i\rangle$ denotes an eigenvector corresponds to the the i-th eigenvalue a_i ($\lvert b_i\rangle$ is defined analogously). Let $A = \sum_i a_i \lvert a_i\rangle\langle a_i\rvert$ and $B = \sum_i b_i \lvert b_i\rangle\langle b_i\rvert$ be spectral decompositions. Since the sorted vector space $\mathbb{R}^{n\uparrow}$ is closed under addition $\mathbf{a}^\uparrow + \mathbf{b}^\uparrow$ and scalar multiplication $k\mathbf{a}^\uparrow$ for a non-negative real number k, we similarly define addition and scalar multiplication for equivalent classes:

$$[A] + [B] := \left[\sum_i{}' (a_i + b_i)\lvert a_i\rangle\langle a_i\rvert\right], \tag{7}$$

$$k[A] := \left[k\sum_i{}' a_i\lvert a_i\rangle\langle a_i\rvert\right],\ \text{for } k \ge 0. \tag{8}$$

Note that the right-hand side of Eqs. (7) and (8) do not depend on the choice of orthonormal basis since $\sum_i \lvert a_i\rangle\langle v_i\rvert$ is a unitary operator for arbitrary orthonormal basis $\{\lvert v_i\rangle\}$. One can easily check that the definition of Eq. (7) satisfies the fundamental properties of addition such that $[A] + [B] = [B] + [A]$ and $[A] + ([B] + [C]) =$

$([A]+[B])+[C]$. Let $f:\mathfrak{M}_{n^2}/\sim\to\mathbb{R}^{n\uparrow}$ be a map such that

$$f([A]) := \mathbf{a}^{\uparrow}. \tag{9}$$

This map is well-defined because eigenvalues are invariant under unitary transformations. For all $A,B\in\mathfrak{M}_{n^2}$, one can show that

$$f([A]+[B]) = f([A])+f([B]), \tag{10}$$
$$f(k[A]) = kf([A]),\ \text{for } k\geq 0, \tag{11}$$

hold and the map f is bijective. The proof of the properties of the map f is shown in Appendix A 1. This is the first main result in this Letter. The map f preserves the operations defined in Eqs. (7) and (8). As a result, $\mathfrak{M}_{n^2}/\sim$ and $\mathbb{R}^{n\uparrow}$ share the same algebraic structure and there exists one-to-one correspondence between the two sets. This structural equivalence allows us to identify these two mathematical sets as essentially the same. The relationship between these mathematical structures can be expressed as $\mathfrak{M}_{n^2}/\sim\cong\mathbb{R}^{n\uparrow}$. This notation indicates that $\mathfrak{M}_{n^2}/\sim$ and $\mathbb{R}^{n\uparrow}$ are isomorphic, meaning that the two sets have the same properties. By identifying the set of equivalence classes with $\mathbb{R}^{n\uparrow}$, it becomes easier to understand intuitively the structure of $\mathfrak{M}_{n^2}/\sim$.

Unitarily residual measures.— Let $\mathfrak{M}_D\subset\mathfrak{M}$ be a set of density operators. Consider a real-valued function d that satisfies the following axiom. For all $\rho,\sigma\in\mathfrak{M}_D$,

$$d(\rho,\sigma)\geq 0,\ d(\rho,\sigma)=0 \text{ if and only if } \rho=\sigma. \tag{12}$$

The function d is a *metric* when d satisfies additional two axioms. For all $\rho,\sigma,\chi\in\mathfrak{M}_D$,

$$d(\rho,\sigma) = d(\sigma,\rho), \tag{13}$$
$$d(\rho,\sigma) \leq d(\rho,\chi)+d(\chi,\sigma). \tag{14}$$

Equation (14) is the triangle inequality. We impose unitary invariance on d. That is, for an arbitrary unitary operator U,

$$d(U\rho U^\dagger, U\sigma U^\dagger) = d(\rho,\sigma). \tag{15}$$

For instance, the condition is satisfied by trace distance, Bures angle and quantum relative entropy. We define a unitarily residual measure $\widetilde{d}$ on the quotient set $\mathfrak{M}_D/\sim$ as:

$$\begin{aligned}\widetilde{d}([\rho],[\sigma]) &:= \min_{U^\dagger U=V^\dagger V=\mathbb{I}} d(U\rho U^\dagger, V\sigma V^\dagger)\\ &= \min_{U^\dagger U=\mathbb{I}} d(U\rho U^\dagger,\sigma) = \min_{U^\dagger U=\mathbb{I}} d(\rho, U\sigma U^\dagger),\end{aligned} \tag{16}$$

where the minimum is over all possible unitaries U and V, and we use Eq. (15) in the last two equalities. From Eq. (5), the unitarily residual measure satisfies Eq. (12) . If d is a metric, then the unitarily residual measure $\widetilde{d}$ on the quotient set also forms a metric, which we refer to as the *unitarily residual metric*. Equation (13) follows from the symmetry of the definition Eq. (16), and Eq. (14) follows from

$$\begin{aligned}\widetilde{d}([\rho],[\sigma]) &\leq \min_{U^\dagger U=\mathbb{I}} d(U\rho U^\dagger,\chi)+\min_{V^\dagger V=\mathbb{I}} d(\chi, V\sigma V^\dagger)\\ &= \widetilde{d}([\rho],[\chi])+\widetilde{d}([\chi],[\sigma]).\end{aligned} \tag{17}$$

From the definition Eq. (16), the unitarily residual measure satisfies the following property:

$$\widetilde{d}([\rho],[\sigma]) \leq d(\rho,\sigma). \tag{18}$$

Monotonicity and convexity.— Monotonicity and convexity are fundamental properties of quantum divergences. For a CPTP map $\mathcal{E}(\bullet)$ the monotonicity is defined as

$$d(\rho,\sigma) \geq d(\mathcal{E}(\rho),\mathcal{E}(\sigma)). \tag{19}$$

The condition is satisfied by trace distance, Bures angle and quantum relative entropy. For non-negative real numbers $\{\lambda_i\}$ such that $\sum_i\lambda_i=1$, the convexity is defined as

$$\sum_i \lambda_i d(\rho_i,\sigma) \geq d\left(\sum_i\lambda_i\rho_i,\sigma\right). \tag{20}$$

The condition is satisfied by trace distance and quantum relative entropy. Letting $\rho=\sum_i p_i\left|p_i\right\rangle\left\langle p_i\right|$ and $\sigma=\sum_j q_j\left|q_j\right\rangle\left\langle q_j\right|$, we prove that the unitarily residual measures inherit these properties under the additional assumption:

$$\widetilde{d}([\rho],[\sigma]) = d\left({\sum_i}' p_i\left|p_i\right\rangle\left\langle p_i\right|, {\sum_i}' q_i\left|p_i\right\rangle\left\langle p_i\right|\right). \tag{21}$$

Although assumption Eq. (21) seems a strong constraint, it is a reasonable assumption since the right-hand side of Eq.(21) does not depend on the choice of orthonormal basis from unitary invariance of d [Eq. (15)]. One can easily verify that all examples in Table I satisfy Eq. (21). Since $\mathfrak{M}_{n^2}/\sim\cong\mathbb{R}^{n\uparrow}$, we write $\mathbf{p}^\uparrow$ instead of $[{\sum'}_i p_i\left|p_i\right\rangle\left\langle p_i\right|]$ for the sake of simplicity. For $[\rho]=\mathbf{p}^\uparrow$, we define a stochastic map of eigenvalues $\widetilde{\mathcal{E}}:\mathfrak{M}_D/\sim\to\mathfrak{M}'_D/\sim$ as

$$\widetilde{\mathcal{E}}(\mathbf{p}^\uparrow) := (T\mathbf{p})^\uparrow, \tag{22}$$

where T is a stochastic matrix (i.e, $\sum_i T_{ij}=1$ for all i and $\{T_{ij}\}$ are all non-negative). If d satisfies monotonicity Eq. (19), the unitarily residual measure is monotonically decreasing under $\widetilde{\mathcal{E}}$:

$$\widetilde{d}([\rho],[\sigma]) \geq \widetilde{d}(\widetilde{\mathcal{E}}([\rho]),\widetilde{\mathcal{E}}([\sigma])). \tag{23}$$

If d satisfies convexity Eq. (20), $\widetilde{d}$ also satisfies convexity:

$$\sum_i\lambda_i\widetilde{d}([\rho_i],[\sigma]) \geq \widetilde{d}\left(\sum_i\lambda_i[\rho_i],[\sigma]\right), \tag{24}$$

where operations of equivalent classes are defined in Eqs. (7) and (8). The proofs of monotonicity and convexity are shown in Appendices A 2 and A 3. One can prove the convexity with respect to σ and the joint convexity $\lambda d(\rho_1,\sigma_1)+(1-\lambda)d(\rho_2,\sigma_2)\geq d(\lambda\rho_1+(1-\lambda)\rho_2,\lambda\sigma_1+(1-\lambda)\sigma_2)$ are also inherited. These inheritances are the second main results of this Letter.

Examples of unitarily residual measures.— Due to the isomorphism between $[\rho]$ and $\mathbf{p}^\uparrow$, the unitarily residual measure is expected to correspond to the classical information divergence between the probability distributions $\mathbf{p}^\uparrow$ and $\mathbf{q}^\uparrow$. We demonstrate that this correspondence holds in several important examples shown below.

One example of the metric d is the Bures angle, which is widely employed in the literature [12]:

$$\mathcal{L}_D(\rho,\sigma):=\arccos\left[\sqrt{\mathrm{Fid}(\rho,\sigma)}\right], \tag{25}$$

where $\mathrm{Fid}(\rho,\sigma)$ is the quantum fidelity:

$$\mathrm{Fid}(\rho,\sigma):=\left(\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]\right)^2. \tag{26}$$

The unitarily residual metric is the Bhattacharyya (arccos) distance [13] between $\mathbf{p}^\uparrow$ and $\mathbf{q}^\uparrow$:

$$\widetilde{\mathcal{L}}_D([\rho],[\sigma])=\arccos\left({\sum_i}'\sqrt{p_iq_i}\right). \tag{27}$$

The derivation of this relation is shown in Appendix B 1. Another example of the metric d is the trace distance, which is defined by

$$\mathcal{T}(\rho,\sigma):=\frac{1}{2}\|\rho-\sigma\|_1, \tag{28}$$

where $\|X\|_1:=\mathrm{Tr}[\sqrt{X^\dagger X}]$. The unitarily residual metric of the trace distance is given by the total variation distance between $\mathbf{p}^\uparrow$ and $\mathbf{q}^\uparrow$:

$$\widetilde{\mathcal{T}}([\rho],[\sigma])=\frac{1}{2}{\sum_i}'|p_i-q_i|. \tag{29}$$

Equation. (29) was shown in Ref. [11] (see Appendix B 2).

As the last example, we consider the Petz-Rényi relative entropy [14] for $\alpha\in(0,1)\cup(1,\infty)$:

$$D_\alpha(\rho||\sigma):=\frac{1}{\alpha-1}\ln\left(\mathrm{Tr}\left[\rho^\alpha\sigma^{1-\alpha}\right]\right). \tag{30}$$

In the limit $\alpha\to1$, the Petz-Rényi relative entropy reduces to the quantum relative entropy $D(\rho||\sigma):=\mathrm{Tr}\left[\rho\ln\rho-\rho\ln\sigma\right]$:

$$\lim_{\alpha\to1}D_\alpha(\rho||\sigma)=D(\rho||\sigma). \tag{31}$$

The Petz-Rényi relative entropy is not a metric since it does not satisfy Eqs. (13) and (14). The unitarily residual measure of the Petz-Rényi relative entropy is the Rényi divergence [15] between $\mathbf{p}^\uparrow$ and $\mathbf{q}^\uparrow$:

$$\widetilde{D}_\alpha([\rho]\,\|\,[\sigma])=\frac{1}{\alpha-1}\ln\left({\sum_i}'p_i^\alpha q_i^{1-\alpha}\right). \tag{32}$$

The derivation of this relation is shown in Appendix B 3. In limit $\alpha\to1$, we obtain

$$\widetilde{D}([\rho]\,\|\,[\sigma]):=\lim_{\alpha\to1}\widetilde{D}_\alpha([\rho]\,\|\,[\sigma])={\sum_i}'p_i\ln\frac{p_i}{q_i}, \tag{33}$$

where the right-hand side is the Kullback-Leibler divergence. Equation. (33) was shown in Ref. [10]. The Petz-Rényi relative entropy satisfies additivity $D_\alpha(\rho^A\otimes\rho^B\|\sigma^A\otimes\sigma^B)=D_\alpha(\rho^A\|\sigma^A)+D_\alpha(\rho^B\|\sigma^B)$ [14, 15]. On the other hand, the induced unitarily residual measure satisfies superadditivity:

$$\widetilde{D}_\alpha([\rho^A\otimes\rho^B]\,\|\,[\sigma^A\otimes\sigma^B])\geq\widetilde{D}_\alpha([\rho^A]\,\|\,[\sigma^A])+\widetilde{D}_\alpha([\rho^B]\,\|\,[\sigma^B]). \tag{34}$$

The derivation of this relation is shown in Appendix B 4. The correspondences between quantum divergences and unitarily residual measures are summarized in Table I. The first and third columns in Table I are the third main results in this Letter.

General open quantum dynamics.— As an example, we demonstrate speed limits for the entropy production in a general open quantum dynamics comprising a system S and an environment E. The composite system $S+E$ evolves via a joint unitary operator $U(t)$ which acts on $\rho_{SE}(0)$. Then, the density operator of the composite system after the unitary evolution is

$$\rho_{SE}(t)=U(t)\rho_{SE}(0)U^\dagger(t). \tag{35}$$

Let $\rho_S(t):=\mathrm{Tr}_E[\rho_{SE}(t)]$ be a system density operator, where $\mathrm{Tr}_E[\bullet]$ denotes a partial trace with respect to the environment. Let $H_S(t)$ and $H_E:=\sum_r H_{E,r}$ be the Hamiltonian of S and E, where $H_{E,r}$ corresponds to the Hamiltonian of the r-th heat reservoir. It is assumed that H_E is time-independent. Let $H_{SE}(t)$ be the Hamiltonian of the system-environment interaction. The total Hamiltonian $H(t)$ is given by

$$H(t):=H_S(t)\otimes\mathbb{I}_E+\mathbb{I}_S\otimes H_E+H_{SE}(t), \tag{36}$$

where $\mathbb{I}_S$ and $\mathbb{I}_E$ represent the respective identity operators. Letting $\mathbb{T}$ be the time ordered product, the unitary evolution is written as $U(\tau)=e^{-i\mathbb{T}\int_0^\tau H(t)dt}$. Here, we adopt the convention of setting $\hbar=1$. We assume that the initial density matrix can be decomposed into the direct product of the density matrix of S and E:

$$\rho_{SE}(0)=\rho_S(0)\otimes\rho_E, \tag{37}$$

where $\rho_E:=\rho_E(0)$. We additionally assume that

$$\rho_E=\frac{1}{Z_E(\beta)}\prod_r e^{-\beta_rH_{E,r}}, \tag{38}$$

$$Z_E(\beta):=\mathrm{Tr}_E\left[\prod_r e^{-\beta_rH_{E,r}}\right], \tag{39}$$

where β_r denotes r-th inverse temperature at time $t=0$ (Boltzmann's constant k_B is set equal to 1), and $Z_E(\beta)$ denotes the partition function. Under these assumptions, the entropy production from time $t=0$ to τ can be identified as follows [6–8]:

$$\Sigma(\tau) = D(\rho_{SE}(\tau) \| \rho_S(\tau) \otimes \rho_E). \tag{40}$$

We show a speed limit for the unitarily residual measure $\widetilde{D}$. From Eqs. (18) and (40), we obtain

$$\begin{aligned}\Sigma(\tau) &\geq \widetilde{D}([\rho_{SE}(\tau)] \,\|\, [\rho_S(\tau) \otimes \rho_E]) \\ &= \widetilde{D}([\rho_S(0) \otimes \rho_E] \,\|\, [\rho_S(\tau) \otimes \rho_E]),\end{aligned} \tag{41}$$

where we use $[\rho_{SE}(\tau)] = [\rho_S(0) \otimes \rho_E]$. From superadditivity Eq. (34), we obtain a speed limit:

$$\Sigma(\tau) \geq \widetilde{D}([\rho_S(0)] \,\|\, [\rho_S(\tau)]). \tag{42}$$

This is the forth main result in this Letter. As in Eq. (42), eigenvalues of the density operator at time $t=0$ and $t=\tau$ are equal when the entropy production $\Sigma(\tau)$ is zero. The same relation was shown in the case of the quantum Markov process governed by the Lindblad equation [10]. We next derive a speed limit for the system purity $\mathcal{P}_S(t) := \mathrm{Tr}_S[\rho_S^2(t)]$. Since $|p+q-1| \leq |p-1/2| + |q-1/2| \leq 1$ for $p, q \in [0,1]$, it follows that

$$\begin{aligned}{\sum_i}' |p_i - q_i| &\geq {\sum_i}' |(p_i + q_i - 1)(p_i - q_i)| \\ &\geq |\sum_i (p_i^2 - q_i^2)|.\end{aligned} \tag{43}$$

By combining this relation with the the classical Pinsker's inequality [16, 17] $\widetilde{D}([\rho_S(0)] \,\|\, [\rho_S(\tau)]) \geq 2\widetilde{\mathcal{T}}([\rho_S(0)], [\rho_S(\tau)])^2$ and Eq. (42), we obtain a speed limit for the purity:

$$\Sigma(\tau) \geq \frac{|\mathcal{P}_S(\tau) - \mathcal{P}_S(0)|^2}{2}. \tag{44}$$

This is the fifth main result in this Letter. Reference [18] introduced a quantum speed limit for relative purity that incorporates the adjoint Lindblad superoperator. Reference [19] presented a quantum speed limit for purity, where the upper bound is determined by the norm of the Lindblad jump operators. Our result, as shown in Eq. (44), provides an upper bound that relies solely on entropy production.

Here, we have presented the example of open quantum dynamics, but the example of an application to non-Hermitian dynamics is also included in Appendix C.

Conclusion.— This work provides a unified framework for quantifying the effect of dissipation. For this purpose, we introduced the equivalence classes of hermitian operators via unitary transformations and their quotient set. We showed that isomorphism exists between the quotient set and a set of real vectors whose components are in non-descending order, and we showed that unitarily residual measure on the quotient set are naturally induced from quantum divergences between density operators. Under an appropriate assumption, we showed that the unitarily residual measures inherit the monotonicity and convexity of the original quantum divergences. In some examples, the unitarily residual measures can be written as a classical information divergence between probability distributions of sorted eigenvalues of density operators. As an application example, we derived speed limits on the unitarily residual measure and purity for the entropy production in general open quantum dynamics. Further study of the properties of the quotient set and unitarily residual measures would be the focus of our future work.

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grant Number JP23K24915.

Appendix A: Proofs of Properties with respect to quotient set and unitarily residual measures

1. Proof of $\mathfrak{M}_{n^2}/\sim\, \cong \mathbb{R}^{n\uparrow}$

Equations (10) and (11) follow from the definitions of Eqs. (7)–(9) and $(\mathbf{a}^\uparrow + \mathbf{b}^\uparrow)^\uparrow = \mathbf{a}^\uparrow + \mathbf{b}^\uparrow$. Next, we prove that f is bijective by proving it is both surjective and injective. From Eq. (9), it follows that f is surjective because $f(\mathfrak{M}_{n^2}/\sim) = \mathbb{R}^{n\uparrow}$. To establish injectivity, suppose that $\mathbf{a}^\uparrow = \mathbf{b}^\uparrow$. This relation implies $[A]=[B]$, since there exists a unitary operator $W_{BA} := {\sum_i}' |b_i\rangle \langle a_i|$ such that $B = W_{BA} A W_{BA}^\dagger$. Therefore, f is injective. Combining both results, we proved that f is bijective.

2. Proof of monotonicity [Eq. (23)]

From assumption Eq. (21), we obtain

$$\widetilde{d}([\rho],[\sigma]) = \widetilde{d}(\mathbf{p}^{\uparrow},\mathbf{q}^{\uparrow}) = d\left(\sum_j{}' p_j \left|p_j\right\rangle\left\langle p_j\right|, \sum_j{}' q_j \left|p_j\right\rangle\left\langle p_j\right|\right). \tag{A1}$$

Let $\mathbf{r} := T\mathbf{p}$ and $\mathbf{s} := T\mathbf{q}$, and let $K_i := \sum_j \sqrt{T_{ij}}\left|r_i\right\rangle\left\langle p_j\right|$ for $\{|p_j\rangle\} \in \mathcal{H}$ and $\{|r_i\rangle\} \in \mathcal{H}'$. We obtain $\sum_i K_i \left(\sum'_j p_j \left|p_j\right\rangle\left\langle p_j\right|\right) K_i^{\dagger} = \sum_i r_i \left|r_i\right\rangle\left\langle r_i\right|$ and $\sum_i K_i \left(\sum'_j q_j \left|p_j\right\rangle\left\langle p_j\right|\right) K_i^{\dagger} = \sum_i s_i \left|r_i\right\rangle\left\langle r_i\right|$. Since $\sum_i K_i^{\dagger}K_i = \mathbb{I}$, the map $\mathcal{E}_K(\bullet) := \sum_i K_i \bullet K_i^{\dagger}$ is a CPTP map. Here $\mathbb{I}$ is an identity operator in $\mathfrak{L}(\mathcal{H})$. Hence, from the definition Eq. (22) and the monotonicity Eqs. (18) and (19), we obtain

$$d\left(\sum_j{}' p_j \left|p_j\right\rangle\left\langle p_j\right|, \sum_j{}' q_j \left|p_j\right\rangle\left\langle p_j\right|\right) \geq d\left(\sum_i r_i \left|r_i\right\rangle\left\langle r_i\right|, \sum_i s_i \left|r_i\right\rangle\left\langle r_i\right|\right) \geq \widetilde{d}\left(\mathbf{r}^{\uparrow},\mathbf{s}^{\uparrow}\right) = \widetilde{d}(\widetilde{\mathcal{E}}([\rho]),\widetilde{\mathcal{E}}([\sigma])). \tag{A2}$$

Combining this inequality with Eq. (A1), we obtain Eq. (23).

3. Proof of convexity [Eq. (24)]

Recall that $d\left(\sum'_j p_{i,j}\left|p_{i,j}\right\rangle\left\langle p_{i,j}\right|, \sum'_j q_j \left|p_{i,j}\right\rangle\left\langle p_{i,j}\right|\right)$ does not depend on basis from unitary invariance of d [Eq. (15)]. Letting $|p_j\rangle := |p_{1,j}\rangle$, from Eq. (18) and the assumption Eq. (21), we obtain

$$\begin{aligned}\sum_i \lambda_i \widetilde{d}([\rho_i],[\sigma]) &= \sum_i \lambda_i d\left(\sum_j{}' p_{i,j} \left|p_j\right\rangle\left\langle p_j\right|, \sum_j{}' q_j \left|p_j\right\rangle\left\langle p_j\right|\right) \geq d\left(\sum_i \lambda_i \sum_j{}' p_{i,j} \left|p_j\right\rangle\left\langle p_j\right|, \sum_j{}' q_j \left|p_j\right\rangle\left\langle p_j\right|\right)\\ &\geq \widetilde{d}\left(\left(\sum_i \lambda_i \mathbf{p}_i^{\uparrow}\right)^{\uparrow}, \mathbf{q}^{\uparrow}\right) = \widetilde{d}\left(\sum_i \lambda_i \mathbf{p}_i^{\uparrow}, \mathbf{q}^{\uparrow}\right) = \widetilde{d}\left(\sum_i \lambda_i [\rho_i], [\sigma]\right),\end{aligned} \tag{A3}$$

where we use the defintions Eqs. (7) and (8) in the last equality.

Appendix B: Derivations and property of unitarily residual measure examples

1. Derivation of Eq. (27)

Let X and Y be arbitrary operators, and let $\mathfrak{s}_i(X)$ be the i-th singular value of operator X. The von Neumann's trace inequality [20] yields

$$|\mathrm{Tr}[XY]| \leq \sum_i{}' \mathfrak{s}_i(X)\mathfrak{s}_i(Y). \tag{B1}$$

The fidelity can be written as $\sqrt{\mathrm{Fid}(\rho,\sigma)} = \mathrm{Tr}[|\sqrt{\rho}\sqrt{\sigma}|] = \sum_i \mathfrak{s}_i(\sqrt{\rho}\sqrt{\sigma})$. Using the singular value decomposition, there exists a unitary operator V such that $\mathrm{Tr}[|\sqrt{\rho}\sqrt{\sigma}|] = \mathrm{Tr}[\sqrt{\rho}\sqrt{\sigma}V]$. Setting $X = \sqrt{U\rho U^{\dagger}}$ and $Y = \sqrt{\sigma}V$ in Eq. (B1), and using $\mathfrak{s}_i(X) = \sqrt{p_i}$ and $\mathfrak{s}_i(Y) = \sqrt{q_i}$, we obtain

$$\sqrt{\mathrm{Fid}(U\rho U^{\dagger},\sigma)} = \left|\mathrm{Tr}[\sqrt{U\rho U^{\dagger}}\sqrt{\sigma}V]\right| \leq \sum_i{}' \sqrt{p_i q_i}. \tag{B2}$$

Hence, we obtain $\widetilde{\mathcal{L}}_D([\rho],[\sigma]) \geq \arccos\left(\sum'_i \sqrt{p_i q_i}\right)$ from Eqs. (16) and (25). Letting $W := \sum'_i \left|q_i\right\rangle\left\langle p_i\right|$, it follows that W is a unitary operator such that $W\rho W^{\dagger} = \sum'_i p_i \left|q_i\right\rangle\left\langle q_i\right|$. From $\mathcal{L}_D(W\rho W^{\dagger},\sigma) = \arccos\left(\sum'_i \sqrt{p_i q_i}\right)$, we obtain Eq. (27).

2. Derivation of Eq. (29)

From the Mirsky inequality,

$$\|A-B\|_1=\sum_i{}' \mathfrak{s}_i(A-B)\geq \sum_i{}' |\mathfrak{s}_i(A)-\mathfrak{s}_i(B)| \tag{B3}$$

holds for arbitrary Hermitian operator A and B [21, 22]. Since singular values of A and $UAU^\dagger$ is equal for a unitary operator U, from Eqs. (16) and (B3), we obtain

$$\widetilde{\mathcal{T}}([\rho],[\sigma])\geq \frac{1}{2}\sum_i{}' |p_i-q_i|. \tag{B4}$$

From $\mathcal{T}(W\rho W^\dagger,\sigma)=1/2\sum'_i |p_i-q_i|$, we obtain Eq. (29).

3. Derivation of Eq. (32)

Let g and h be monotonically increasing functions. Letting $\rho=\sum_i p_i \left|p_i\right\rangle\left\langle p_i\right|$ and $\sigma=\sum_j q_j\left|q_j\right\rangle\left\langle q_j\right|$, we obtain

$$\mathrm{Tr}[g(U\rho U^\dagger)h(\sigma)]=\sum_{i,j}g(p_i)h(q_j)|\left\langle q_j|U|p_i\right\rangle|^2=\sum_{i,j}C_{ij}g(p_i)h(q_j)=:F(C), \tag{B5}$$

where $C_{ij}:=|\left\langle q_j|U|p_i\right\rangle|^2$. The matrix C_{ij} is the doubly stochastic matrix (i.e., $\sum_i C_{ij}=\sum_j C_{ij}=1$ and $C_{ij}\geq 0$ for all i and j). Since the function $F(C)$ and the constraints of C_{ij} are all linear, the objective function $F(C)$ is maximized when $C_{ij}=1$ or $C_{ij}=0$. By combining these conditions with constraints $\sum_i C_{ij}=\sum_j C_{ij}=1$, we obtain $C_{i\pi(i)}=1$ for all i. Here π is a permutation of subscripts $\{i\}$. Therefore, we obtain

$$\mathrm{Tr}[g(U\rho U^\dagger)h(\sigma)]\leq\sum_i g(p_i)h(q_{\pi(i)})\leq\sum_i{}' g(p_i)h(q_i), \tag{B6}$$

where we use

$$\sum_i a_ib_i\leq\sum_i{}' a_ib_i, \tag{B7}$$

for real numbers $\{a_i\}$ and $\{b_i\}$. When $\alpha>1$, setting $g(x)=x^\alpha$ and $h(x)=-x^{1-\alpha}$ in Eq. (B6) and using Eqs. (16) and (30), we obtain

$$\widetilde{D}_\alpha([\rho]\,\|\,[\sigma])\geq\frac{1}{\alpha-1}\ln\left(\sum_i{}' p_i^\alpha q_i^{1-\alpha}\right). \tag{B8}$$

When $0<\alpha<1$, setting $g(x)=x^\alpha$ and $h(x)=x^{1-\alpha}$ also yields Eq. (B8). From $D_\alpha(W\rho W^\dagger\|\sigma)=(\alpha-1)^{-1}\ln\left(\sum'_i p_i^\alpha q_i^{1-\alpha}\right)$, we obtain Eq. (32).

4. Proof of superadditivity [Eq. (34)]

Let $\{p_i^{AB}\}$ be non-descending eigenvalues of $\rho^{AB}:=\rho^A\otimes\rho^B$ and let $\{p_j^A\}$ and $\{p_k^B\}$ be eigenvalues of ρ^A and ρ^B, respectively. We analogously define $\{q_i^{AB}\}$, $\{q_j^A\}$ and $\{q_k^B\}$ for $\sigma^{AB}:=\sigma^A\otimes\sigma^B$, σ^A and σ^B, respectively. The definition yields $p_i^{AB}=p_j^Ap_k^B$ and $q_i^{AB}=q_{\pi(j)}^Aq_{\pi'(k)}^B$. Here π and π' are permutations of subscripts. Since the classical Rényi divergence satisfies additivity [15], we obtain

$$\widetilde{D}_\alpha([\rho^A\otimes\rho^B]\,\|\,[\sigma^A\otimes\sigma^B])=\frac{1}{\alpha-1}\ln\left(\sum_j (p_j^A)^\alpha(q_{\pi(j)}^A)^{1-\alpha}\right)+\frac{1}{\alpha-1}\ln\left(\sum_k (p_k^B)^\alpha(q_{\pi'(k)}^B)^{1-\alpha}\right). \tag{B9}$$

When $\alpha > 1$, applying Eq. (B7) for $a_i = {p_i}^\alpha$ and $b_i = -{q_{\pi(i)}}^{1-\alpha}$, we obtain

$$\widetilde{D}_\alpha([\rho^A \otimes \rho^B] \,\|\, [\sigma^A \otimes \sigma^B]) \geq \frac{1}{\alpha-1} \ln \left({\sum_i}' (p_i^A)^\alpha (q_i^A)^{1-\alpha} \right) + \frac{1}{\alpha-1} \ln \left({\sum_i}' (p_i^B)^\alpha (q_i^B)^{1-\alpha} \right). \tag{B10}$$

Combining this relation with Eq. (32), we obtain Eq. (34). Similarly, applying Eq. (B7) for $a_i = {p_i}^\alpha$ and $b_i = {q_{\pi(i)}}^{1-\alpha}$ yields Eq. (34) when $0 < \alpha < 1$.

Appendix C: Non-Hermitian dynamics

We derive a speed limit on the induced Bures angle in the system governed by the non-Hermitian Hamiltonian $\mathcal{H}$. In general, $\mathcal{H}$ can be decomposed into

$$\mathcal{H}(t) = H(t) - i\Gamma(t), \tag{C1}$$

where $H(t)$ and $\Gamma(t)$ are Hermitian operators. Consider a density operator $\rho(t)$, whose time evolution is governed by the non-Hermitian Hamiltonian $\mathcal{H}(t)$:

$$\frac{d\rho}{dt}(t) = -i(\mathcal{H}(t)\rho(t) - \rho(t)\mathcal{H}^\dagger(t)). \tag{C2}$$

Equation (C2) reduces to the von Neumann equation when $\mathcal{H}(t)$ is Hermitian. Let $\widehat{\rho}(t)$ be a normalized density operator defined as

$$\widehat{\rho}(t) := \frac{\rho(t)}{\mathrm{Tr}[\rho(t)]}. \tag{C3}$$

In the following, we assume that the density operator is normalized by Eq.(C3), and we write $\widehat{\rho}$ as ρ to simplify the notation. For the normalized density operator, Eq. (C2) is modified as

$$\frac{d\rho}{dt}(t) = -i(\mathcal{H}(t)\rho(t) - \rho(t)\mathcal{H}^\dagger(t)) + 2\langle\Gamma\rangle(t)\rho(t), \tag{C4}$$

where $\langle\Gamma\rangle(t) := \mathrm{Tr}[\Gamma(t)\rho(t)]$ denotes a mean of $\Gamma(t)$. Letting $V(t) := e^{-i\mathbb{T}\int_0^t H(t)dt}$, the upper bound for the Bures angle $\mathcal{L}_D$ is given by

$$\int_0^\tau [\![\Gamma]\!](t)dt \geq \mathcal{L}_D(V(\tau)\rho(0)V(\tau)^\dagger, \rho(\tau)), \tag{C5}$$

where we assume $\int_0^\tau [\![\Gamma]\!](t)dt \leq \pi/2$ and $[\![\Gamma]\!](t) := \sqrt{\langle\Gamma^2\rangle(t) - \langle\Gamma\rangle(t)^2}$ denotes a standard deviation. The details of the derivation are shown in the next subsection. Applying Eq. (18) for Eq. (C5), we obtain a speed limit:

$$\int_0^\tau [\![\Gamma]\!](t)dt \geq \widetilde{\mathcal{L}}_D([\rho(0)], [\rho(\tau)]). \tag{C6}$$

As in Eq. (C6), eigenvalues of the normalized density operator at time $t = 0$ and $t = \tau$ are nearly equal with small standard deviation of $\Gamma(t)$. Let $\mathcal{P}(t) := \mathrm{Tr}[\rho(t)^2]$ be a purity. Combining ${\sum'}_i |p_i - q_i| = {\sum'}_i |\sqrt{p_i} - \sqrt{q_i}||\sqrt{p_i} + \sqrt{q_i}| \leq \sqrt{{\sum'}_i(\sqrt{p_i} - \sqrt{q_i})^2 {\sum'}_i(\sqrt{p_i} + \sqrt{q_i})^2} \leq 2\sqrt{2}\sqrt{1 - {\sum'}_i \sqrt{p_i q_i}} = 2\sqrt{2}\sqrt{1 - \cos\widetilde{\mathcal{L}}_D([\rho],[\sigma])}$ with Eqs. (43) and (C6), we obtain a speed limit for the purity:

$$4\sin\left(\frac{\int_0^\tau [\![\Gamma]\!](t)dt}{2}\right) \geq |\mathcal{P}(\tau) - \mathcal{P}(0)|. \tag{C7}$$

1. Derivation of Eq. (C5)

We purify $\rho(t)$ as follows:

$$|\rho(t)\rangle := e^{-i\mathbb{T}\int_0^t \mathcal{H}(t)dt + \int_0^t \langle\Gamma\rangle(t)dt} \sum_i \sqrt{p_i(0)}\,|p_i(0)\rangle \otimes |a_i\rangle, \tag{C8}$$

where $\{|a_i\rangle\}$ are orthonormal basis in the ancilla and $\rho(0) = \sum_i p_i(0) |p_i(0)\rangle \langle p_i(0)|$. We obtain $\rho(t)$ by taking the trace of $|\rho(t)\rangle \langle \rho(t)|$ with respect to the ancilla. From Eq. (C8), letting $X_I(t) := V(t)^\dagger X(t) V(t)$ and $\delta X(t) := X(t) - \langle X \rangle (t)$, the time evolution of the purified vector in interaction picture $|\rho_I(t)\rangle := V(t)^\dagger |\rho(t)\rangle$ is governed by

$$d_t |\rho_I(t)\rangle = -(\delta\Gamma_I(t) \otimes \mathbb{I}_A) |\rho_I(t)\rangle , \tag{C9}$$

where $\mathbb{I}_A$ denote an identity operator of the ancilla and d_t denotes d/dt. Recall that $|\rho_I(t)\rangle$ is normalized, the Bures angle between $|\rho_I(t)\rangle$ and $|\rho_I(t+dt)\rangle$ can be expanded by

$$\mathcal{L}_D(|\rho_I(t)\rangle , |\rho_I(t+dt)\rangle) = \arccos(|\langle \rho_I(t+dt)|\rho_I(t)\rangle |) = \sqrt{g_{\mathrm{FS}}(t)}dt + O(dt^2), \tag{C10}$$

where $g_{\mathrm{FS}}(t)$ denotes the Fubini-Study metric defined by

$$g_{\mathrm{FS}}(t) := \langle d_t\rho_I(t)|d_t\rho_I(t)\rangle - |\langle d_t\rho_I(t)|\rho_I(t)\rangle |^2. \tag{C11}$$

Substituting Eq. (C9) into Eq. (C10) and using the triangle inequality, we obtain

$$\mathcal{L}_D(|\rho_I(0)\rangle , |\rho_I(\tau)\rangle) \le \int_0^\tau \sqrt{g_{\mathrm{FS}}(t)}dt = \int_0^\tau \sqrt{\langle \rho_I(t)|\delta\Gamma_I(t)^2|\rho_I(t)\rangle}dt = \int_0^\tau [\![\Gamma]\!](t)dt, \tag{C12}$$

where we use $\langle \rho_I(t)|\delta\Gamma_I(t)|\rho_I(t)\rangle = 0$. From $\mathcal{L}_D(|\rho_I(0)\rangle , |\rho_I(\tau)\rangle) = \mathcal{L}_D(V(\tau) |\rho(0)\rangle , |\rho(\tau)\rangle)$, we obtain

$$\int_0^\tau [\![\Gamma]\!](t)dt \ge \mathcal{L}_D(V(\tau) |\rho(0)\rangle , |\rho(\tau)\rangle). \tag{C13}$$

Using the monotonicity of the fidelity, we obtain Eq. (C5).

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